

Electoral Competition with Policy-Motivated Candidates*

John Duggan
Department of Political Science
and Department of Economics
University of Rochester

Mark Fey
Department of Political Science
University of Rochester

February 22, 2001

*The authors thank Jeff Banks for helpful discussions of the paper.

1 Introduction

In the tradition of spatial modeling in positive political theory, majority-rule elections are often conceptualized as competition between two candidates who stake out positions in a space of policies, followed by the votes of voters who observe those policy platforms. If the candidates are purely office-motivated, if the policy space is one-dimensional, and if the preferences of voters are single-peaked, then the median voter theorem applies: there is a unique equilibrium, and in it both candidates adopt the median ideal point of the voting population as their platforms (cf. Downs (1957), Black (1958)). The property of the median ideal point behind this result is that it is defeated by no other platforms in majority voting. In higher dimensional spaces, such an undominated policy position is referred to as a “core point.” In fact, regardless of dimensionality, office-motivated candidates must locate at core points in equilibrium: if one candidate were to locate at a beatable position, the other would move to exploit that opportunity. We refer to this phenomenon as “core equivalence.” An implication is that, in the absence of a core point, there will be no (pure strategy) equilibrium of the game between the candidates.

As is well-known, the existence of a core point when the number of voters is odd entails a symmetry condition on voter preferences that is extremely restrictive in two or more dimensions: Plott (1967) shows that a core point must be the ideal point of some voter, and the gradients of the other voters’ utility functions must be paired so that, for every voter with a gradient pointing in one direction, there is exactly one voter whose gradient points in the opposite direction (see also McKelvey and Schofield (1987)). This necessary condition suggests that, for “most” specifications of voter preferences, core points — and therefore electoral equilibria — will fail to exist. When existence does obtain, it will be vulnerable to even slight variations in preferences (cf. Rubinstein (1979), Schofield (1983), Cox (1984), Le Breton (1987)). When the number of voters is even, the results are not so negative: a core point need not be the ideal point of a voter, the symmetry condition is no longer necessary, and the existence of core points may be robust to variations in preferences.

Clearly, aside from the assumption of an odd number of voters in the negative results on equilibrium existence, the assumption that candidates are office-motivated plays an important role: since candidates care only about winning the election, if one’s position can be beaten by any other

preferences near the equilibrium point. We give an example in which three voters have non-Euclidean preferences, there is a unique core point, and there is an equilibrium in which the candidates do not locate at the core point. Thus, the core equivalence result of Calvert does not generalize to arbitrary differentiable, strictly quasi-concave voter utility functions. In that example, however, the candidates locate at one voter's ideal point and a type of symmetry on the voters' gradients holds: for every voter whose gradient lies between the candidates' gradients, there must be exactly one voter whose gradient points in exactly the opposite direction. We prove that those properties are in fact necessary conditions for equilibrium when the number of voters is odd. Though potentially restrictive, the symmetry condition does not imply the existence of a core point nor, generally, the kind of fragility of equilibria seen in models with office-motivated candidates. Indeed, we give a three-voter, two-dimensional example in which the core is empty; there exists an electoral equilibrium with policy-motivated candidates; and the equilibrium is robust to small changes in the preferences of voters and candidates. Thus, the negative conclusions drawn for office-motivated candidates do not carry over with full force.

In three or more dimensions, the existence of equilibria is considerably more precarious. Given any equilibrium with an odd number of voters, we show that, for every voter whose gradient does not lie on the plane spanned by the candidates' gradients, there must be exactly one voter whose gradient points in the opposite direction. In other words, if we remove the voters whose gradients lie on that plane, the equilibrium platform must be a core point of the modified majority voting game. Because the plane is a lower-dimensional subspace, we would not expect it to contain the gradients of all voters. Typically, therefore, we must have some pairs of voters with diametrically opposed gradients, a result suggesting that electoral equilibria will be rare and that, when existence does obtain, it will be vulnerable to even slight variations of voter or candidate preferences. Thus, with three or more dimensions and a finite, odd number of voters, the equilibria conjectured by Calvert (1985) can exist only in knife-edge situations.

When the number of voters is even, the optimistic case in models of office-motivated candidates, we show that equilibria in which the candidates' gradients point in different directions *never* exist, even in the one-dimensional case. Though the generality of this result may be unexpected, it is due to a feature of policy-motivation that is quite intuitive: a policy-motivated candidate may have an incentive to move from one platform to another, even if the

2 The Model

We consider two candidates, A and B , competing for the votes of an electorate, N , containing a number n of voters. We use the notation C for an arbitrary candidate and i, j, k , etc., for an arbitrary voter. Let X be a convex subset of d -dimensional Euclidean space, \mathbb{R}^d . The candidates simultaneously choose policy platforms from X , with candidate C 's platform denoted x_C . We use the notation x, y, z , etc., for arbitrary policies. Each voter i has a preference relation on X represented by a strictly quasi-concave, differentiable utility function $u_i: X \rightarrow \mathbb{R}$. Thus, if voter i has a utility-maximizing platform, it is unique. We call such a policy i 's *ideal point* and denote it \tilde{x}_i . For simplicity, we assume that $\nabla u_i(x) = 0$ if and only if x is voter i 's ideal point. We assume that no two voters have the same ideal point: $\nabla u_i(x) = \nabla u_j(x) = 0$ for no x, i , and $j \neq i$. We say voter i 's preferences are *Euclidean* if i has an ideal point \tilde{x}_i and, for some strictly decreasing function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$, $u_i(x) = f(\|x - \tilde{x}_i\|)$, i.e., voter i has circular indifference curves.

We use the notation R for weak majority preference, P for strict preference, and I for indifference: xRy if and only if $u_i(x) \geq u_i(y)$ for at least half of the voters; xPy if and only if $u_i(x) > u_i(y)$ for more than half of the voters (i.e., not yRx); and xIy if and only if xRy and yRx . In the appendix, we state a lemma on the "star-shapedness" of majority preferences: if xRy , then any point between x and y will be weakly majority-preferred to y , strictly so if the number of voters is odd. We define the *core* as the set of platforms x weakly majority-preferred to all other platforms: for all $y \in X$, xRy . If the number of voters is odd, then a standard result under our assumptions is that the core, when non-empty, consists of a single point, say x^* , and that, for all $y \neq x^*$, x^*Py . Moreover, x^* is the ideal point of some voter, say i^* . In case all voters have Euclidean preferences, it is known that the majority preference relation coincides with the preferences of the "core voter" i^* , i.e., xRy if and only if $u_{i^*}(x) \geq u_{i^*}(y)$ (cf. Davis, DeGroot, and Hinich (1972)). Thus, in that case, the majority weak preference relation is complete and transitive, with circular indifference curves. None of these conclusions holds generally when n is even.

We assume each candidate C has a preference relation on X represented by a strictly quasi-concave, differentiable utility function $u_C: X \rightarrow \mathbb{R}$. If candidate C has an ideal point, we denote it by \tilde{x}_C . Again for simplicity, we assume $\nabla u_C(x) = 0$ if and only if x is candidate C 's ideal point. We assume that the candidates are policy-motivated, which means that a candidate may

The preceding conditions characterize situations in which a candidate has an incentive to change his/her position. At times, we will want to use three additional conditions that are sufficient for the candidates to *not* have profitable deviations.

- (C5) If yPz or $u_A(z) \leq u_A(y)$, then not $(z, y) \succ_A (y, y)$ (and likewise for B).
- (C6) If candidate A has ideal point \tilde{x}_A and if \tilde{x}_APy , then not $(z, y) \succ_A (\tilde{x}_A, y)$ (and likewise for B).
- (C7) If yPx , and if yPz or $u_A(z) \leq u_A(y)$, then not $(z, y) \succ_A (x, y)$ (and likewise for B).

Suppose that both candidates adopt platform y , which is then the outcome with probability one. If candidate A adopts a platform that loses to y , and therefore does not affect the policy outcome, or if A moves to some platform no more desirable than y , then that move should not be profitable, as stipulated in condition (C5). For condition (C6), if \tilde{x}_APy , so that candidate A wins by locating at his/her ideal point, then it is clear that no move can be profitable for A , as in condition (C6). Finally, if candidate A 's platform x loses to y , and if A moves to a platform z that also loses to y (and so does not change the policy outcome) or that is less desirable than y , then that move should not be profitable, as stated in condition (C7).

Conditions (C1)-(C7) hold, for example, in the following environment: voters vote sincerely (eliminating weakly dominated strategies), flipping coins when indifferent; the candidate with the majority of votes wins and implements his/her policy platform, with ties broken by a coin flip; and candidates evaluate lotteries over policy outcomes according to expected utility. In fact, we could allow voter i to randomize between the candidates with any positive probabilities when $u_i(x_A) = u_i(x_B)$. Conditions (C1)-(C3) hold — and so, therefore, does the analysis for n odd — even if these probabilities vary arbitrarily with the particular platforms over which i is indifferent. The conditions are general enough that we could even allow indifferent voters to abstain from voting with any probability (possibly one), as long as the winner in case of a tie is determined randomly with each candidate receiving positive probability.

We say (x_A, x_B) is an *equilibrium* if neither candidate C can deviate to a different platform to produce a preferred pair: there does not exist $x'_A \in X$ such that $(x'_A, x_B) \succ_A (x_A, x_B)$ (and likewise for B). We say that (x_A, x_B) is a

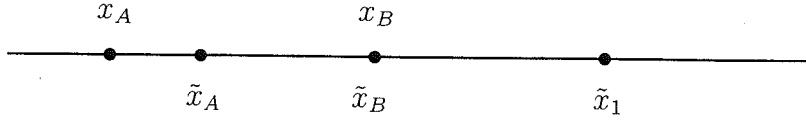


Figure 1: A non-competitive equilibrium without policy coincidence

$p, q \in \mathbb{R}^d$, we use the notation $\text{cone}^\circ\{p, q\} = \{\alpha p + \beta q \mid \alpha, \beta > 0\}$ to denote the open cone generated by p and q .

Theorem 2 *Assume n is odd, and assume (C1)-(C3). If (x_A, x_B) is a polarized equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . For $p \in \mathbb{R}^d$,*

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| = |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|,$$

if either $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly dependent or $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly independent and $p \in \text{cone}^\circ\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$.

Figure 2 depicts depicts a situation in which the necessary conditions given in the theorem are satisfied. Here, the candidates locate at voter 5's ideal point. The gradients of voters 1 and 3 point in opposite directions. The gradients of voters 2 and 4 are not matched in this way, but, because neither gradient (or its opposite) lies in the open cone generated by the candidates' gradients, the symmetry condition of the theorem is preserved.

By the first part of the theorem, the candidates must locate at some ideal point, say \hat{x} , in a polarized equilibrium. The proof of the second part of the theorem is largely concerned with the case in which the candidates' gradients are linearly independent. We show that the set of platforms weakly majority-preferred to \hat{x} , the region described by hash marks in Figure 3, lies below the hyperplanes defined by the gradients of the candidates. This implies a kind of "kink" in the boundary of that set, one that is not possible when the core is non-empty and the preferences of the voters are Euclidean. Under those conditions, the majority preference relation would coincide with the preference relation of the core voter, so the majority indifference curves would simply be circles and obviously could not have kinks. Thus, in Calvert's (1985) model, the only platform weakly preferred to \hat{x} is \hat{x} itself, i.e., the candidates must locate at the core point, and then symmetry of the voters' gradients follows from Plott's (1967) theorem. In the proof of Theorem 2,

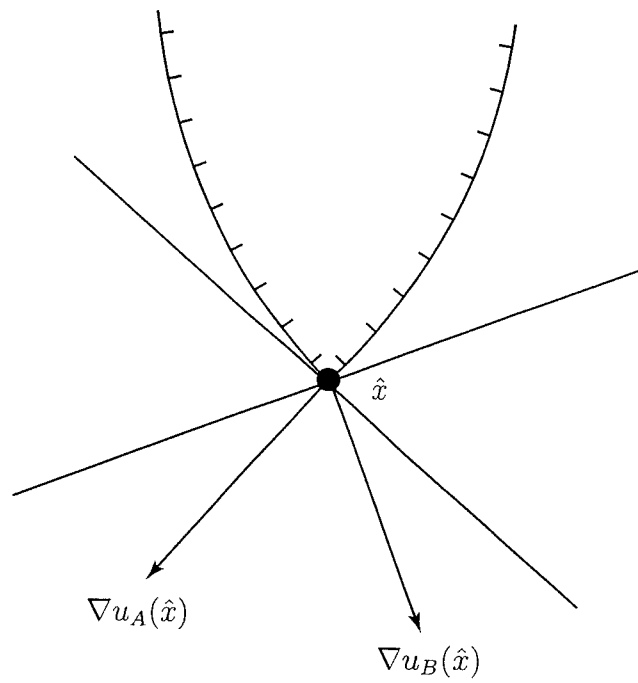


Figure 3: A kink in the boundary of the majority-preferred-to set

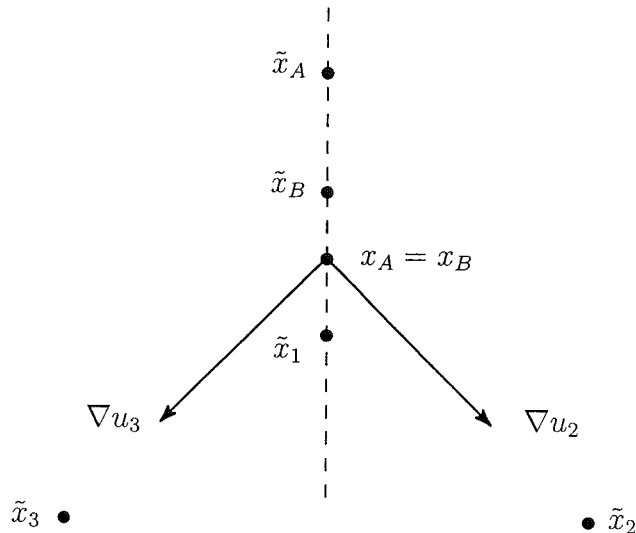


Figure 4: A competitive non-polarized equilibrium violating the conditions of Theorem 3

The proof the corollary is simple. Theorem 3 tells us that, given a polarized equilibrium (x_A, x_B) , the candidates must locate at the ideal point of some voter, say i . Since $\text{span}\{\nabla u_A(\tilde{x}_i), \nabla u_B(\tilde{x}_i)\}$ is a two-dimensional space and the dimension of $\text{span}\{\nabla u_j(\tilde{x}_i) \mid j \in N\}$ is at least three, there is some voter j such that $\nabla u_j(\tilde{x}_i) \notin \text{span}\{\nabla u_A(\tilde{x}_i), \nabla u_B(\tilde{x}_i)\}$. But, under the assumptions of the corollary, there is no voter whose gradient points in the direction opposite that of voter j 's, a contradiction.

The implications of Theorem 2 when the number of voters is even are more striking, as that result allows us to prove the following.

Theorem 4 *Assume n is even, and assume (C1)-(C4). There does not exist a polarized equilibrium.*

In the proof of the theorem, we first verify that, as in Theorem 2, the candidates would have to locate at the ideal point, say \hat{x} , of some voter, say i . Deleting that voter from N , we are left with an electorate, N' , with an odd number of voters. Furthermore, there is no voter in N' with ideal point \hat{x} , violating a necessary condition in Theorem 2 for equilibrium in the reduced model. Thus, one of the candidates can move to a better platform, say x' , preferred by a majority of voters in N' to \hat{x} . Adding i back to the electorate,

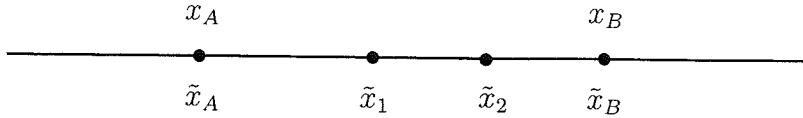


Figure 5: A non-competitive equilibrium with n even, as in Proposition 1

must have a profitable deviation in this situation — that will depend on the exact specification of the candidates' strategic preferences in the model.

The next proposition gives a condition, strengthening that of Proposition 1, under which all interior equilibria are polarized. Once again, the condition extends the familiar one from one-dimensional models that the candidates' ideal points are on opposite sides of the median. We will say that an interior platform x “fails the polarization condition” if $\alpha \nabla u_A(x) = \beta \nabla u_B(x)$ for some $\alpha, \beta \geq 0$, at least one non-zero.

Proposition 2 *Assume (C1)-(C4). Assume that, for each $x \in X$ failing the polarization condition, there exists a platform $y \in X$ such that yPx and, for some candidate C , $u_C(y) > u_C(x)$. If (x_A, x_B) is an interior competitive equilibrium, then it is polarized.*

The proof is trivial, owing to condition (C1), Theorem 1, and the strength of the condition stated in the proposition. To see that this condition is indeed stronger than that of Proposition 1, set $x = \tilde{x}_A$; then the condition of Proposition 2 yields C and y such that $u_C(y) > u_C(x)$; and then, of course, we must have $C = B$, fulfilling the condition of Proposition 1.

The condition of Proposition 2 is not completely transparent, and so it is of interest to understand when it (and, therefore, the condition of Proposition 1) might hold. Suppose that $d \geq 2$, that n is odd, and that voter preferences are Euclidean. Let $Y \subseteq X$ denote the *yolk*, the smallest closed ball intersecting all median hyperplanes (cf. McKelvey (1986)). Thus, if the hyperplane

$$H_{x,y} = \{z \in \mathbb{R}^d \mid 2z \cdot (x - y) = (x + y) \cdot (x - y)\}$$

bisecting two platforms, x and y , does not intersect Y , majority indifference between x and y cannot hold. Whether xPy or yPx depends on whether Y is on the x -side or y -side of $H_{x,y}$. Suppose further that there exists $t \in \mathbb{R}^d$ such that, for all $y \in Y$,

$$t \cdot \tilde{x}_A < t \cdot y < t \cdot \tilde{x}_B,$$

for $\alpha \geq 1$ or $\alpha \leq -1$. If $\alpha \geq 1$, then

$$\begin{aligned} t \cdot x &= t \cdot \tilde{x}_B + t \cdot (\tilde{x}_A - \tilde{x}_B) + (1 - \alpha)t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &= t \cdot \tilde{x}_A + (1 - \alpha)t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &\leq t \cdot \tilde{x}_A. \end{aligned}$$

Similarly, $t \cdot x \geq t \cdot \tilde{x}_B$ if $\alpha \leq -1$. (See Figure 6 for $\alpha \geq 1$.) Suppose, without loss of generality, that $\alpha \geq 1$. Define $x_\epsilon = x + \epsilon t$ for $\epsilon > 0$, and pick ϵ smaller than the lesser of

$$(\min t \cdot Y) - t \cdot \tilde{x}_A \quad \text{and} \quad t \cdot \tilde{x}_B - (\max t \cdot Y),$$

as in Figure 6. Then we have

$$\begin{aligned} t \cdot x_\epsilon &= t \cdot x + \epsilon \\ &\leq t \cdot \tilde{x}_A + \epsilon \\ &< \min t \cdot Y, \end{aligned}$$

which implies that the bisecting hyperplane H_{x, x_ϵ} does not intersect the yolk. And since the yolk is on the x_ϵ -side of the hyperplane, we have $x_\epsilon P x$. Finally, note that

$$\begin{aligned} t \cdot (\tilde{x}_B - x) &= \alpha t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &> 0, \end{aligned}$$

which implies that $u_B(x_\epsilon) > u_B(x)$ for small enough ϵ , as required.

5 Discussion

We have shown that policy coincidence and a type of Plott symmetry are necessary conditions for equilibrium. In Figure 7, we verify that they are not sufficient for equilibrium. In this example, the voters have Euclidean preferences and the candidates are located at voter 3's ideal point. With candidate gradients as depicted, the conditions of Theorem 2 are satisfied, but either candidate can move to a more desirable platform preferred by voters 1 and 2, a profitable deviation by condition (C1). In this example, voter 2's ideal point is the core, a likely candidate for equilibrium. In fact, if the number of voters is odd, if condition (C5) holds, and if there is a core

Theorem 1: without it, we might have xIy and y the outcome of the election with probability one; but then we would expect candidate A to be indifferent between (x, y) and (y, y) , creating the possibility of competitive equilibria violating policy coincidence. Suppose, for example, that $d = 1$, that there is one voter with Euclidean preferences and ideal point at zero, and that the candidates' ideal points are both at -1 . In this setting, it would be a competitive equilibrium for candidate A to locate at $-1/2$ and B at $1/2$, if the voter votes for candidate B with zero probability: all of the platforms preferred by the voter to candidate A 's are less desirable, from candidate B 's perspective, than A 's platform.

An alternative specification of candidate preferences is the "mixed" model: candidate C receives a utility of $u_C(x)$ if the platform x is implemented, plus a positive utility, say β , if C is the winner of the election. In this model, condition (C2) would not hold, but the arguments of Theorem 1 could be modified to obtain the result that, in equilibrium, the candidates must adopt the same platform, say \hat{x} . Then it is easy to see that, when the number of voters is odd, for example, \hat{x} must be a core point. If not, there is some y majority-preferred to it. That platform may be a worse policy outcome from a candidate's point of view, but every platform between \hat{x} and y is also majority-preferred to \hat{x} . By picking such a platform close enough to \hat{x} , the candidate can make the disutility of the policy change less than β , the utility from winning, a contradiction. Then the symmetry of the voters' gradients follows from Plott's (1967) theorem. Clearly, driving this argument is a discontinuity in the candidates preferences, one introduced by the positive reward for winning in the mixed model. This raises the question: Are the strong necessary conditions for equilibrium existence merely an artifact of the discontinuity introduced by office motivation? Our results yield the answer: When the dimension of the policy space is at least three or the number of voters is even, those restrictive conditions are inherent in the strategic incentives of electoral competition, even with purely policy motivated candidates.

Finally, we have assumed a finite number of voters, whereas Calvert (1985) assumed a continuum of voters. This was mainly for convenience, though it also allowed us to highlight a subtlety of the n even case and to emphasize a distinction from models of office-motivated candidates. There are results suggesting that key properties of finite, n odd, electorate models carry over to continuum models. For example, Banks, Duggan, and Le Breton (1999) give a condition on the dispersion of voter ideal points, in a model of a continuous electorate, under which the second part of Lemma 1

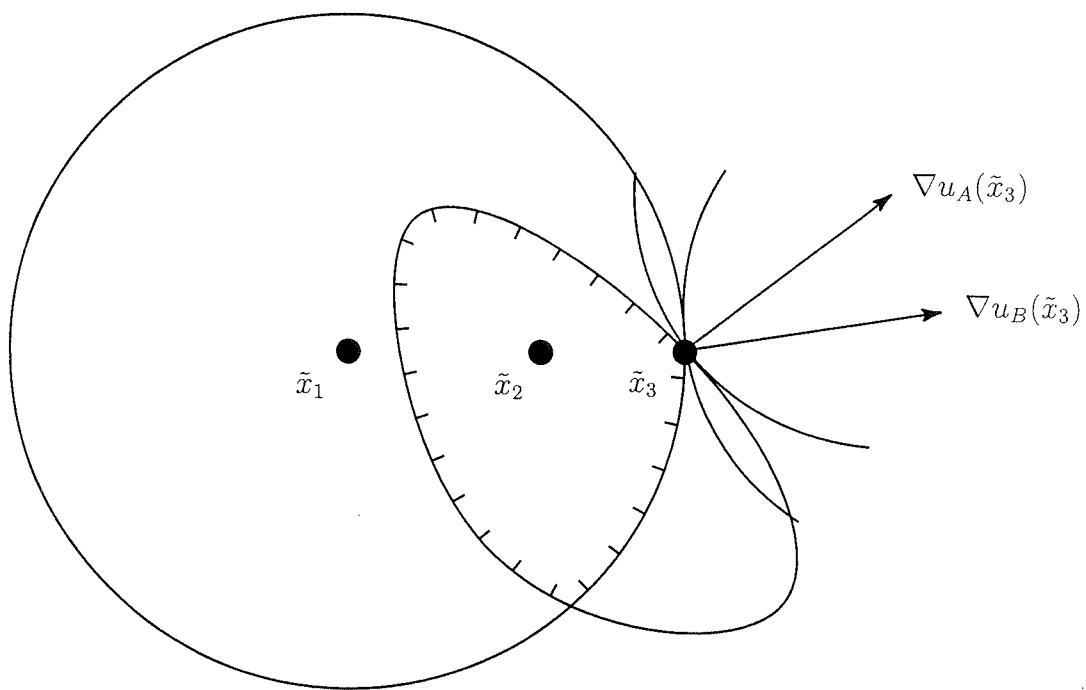


Figure 8: A polarized equilibrium not at the core

$u_A(x_A) = u_A(x_B)$, then, as above, let $x' = (1/2)x_A + (1/2)x_B$. If $x'Px_B$, then (C1) yields a contradiction. In the n even case, however, Lemma 1 implies only that $x'Rx_B$, so suppose $x'Ix_B$. Since the u_i 's are strictly quasi-concave, this means that $u_i(x_B) \geq u_i(x_A)$ for exactly half of the voters. Consequently, $u_i(x_A) > u_i(x_B)$ for half of the voters. Therefore, by continuity, there is an open set $Y \subseteq X$ containing x_A such that, for all $x \in Y$, xRx_B . Since (x_A, x_B) is competitive, $\nabla u_A(x_A) \neq 0$. Defining x_ϵ as above, we can choose $\epsilon > 0$ small enough that $u_A(x_\epsilon) > u_A(x_A)$ and $x_\epsilon Rx_B$. Then, by condition (C4), we have $(x_\epsilon, x_B) \succ_A (x_A, x_B)$, a contradiction. Therefore, $u_A(x_A) > u_A(x_B)$ holds. As above, x_APx_B , for suppose otherwise. Defining x_m as above, Lemma 1 now implies $x_m Rx_B$ for all m . If this majority preference holds strictly, (C3) again yields a contradiction. If x_mIx_B for any m , we repeat the preceding argument, using (C4) to establish a contradiction. Therefore, x_APx_B , and, as above, (C1) yields a final contradiction. ■

Theorem 2 *Assume n is odd, and assume (C1)-(C3). If (x_A, x_B) is a polarized equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . For $p \in \mathbb{R}^d$,*

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| = |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|,$$

if either $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly dependent or $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly independent and $p \in \text{cone}^\circ\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$.

Proof: Consider any polarized equilibrium (x_A, x_B) . As every polarized equilibrium is competitive, we know from Theorem 1 that $x_A = x_B = \hat{x}$ for some $\hat{x} \in X$. To simplify notation, let $p_A = \nabla u_A(\hat{x})$ and $p_B = \nabla u_B(\hat{x})$, and normalize both vectors so that $\|p_A\| = \|p_B\| = 1$. We first claim that, for both candidates C , we must have $p_C \cdot y < p_C \cdot \hat{x}$ for all platforms $y \neq \hat{x}$ such that yRx . Otherwise, we would have $p_C \cdot y \geq p_C \cdot \hat{x}$ for some $y \neq \hat{x}$ such that yRx . It follows from Lemma 1 that $x_\alpha = \alpha \hat{x} + (1 - \alpha)yP\hat{x}$ for all $\alpha \in (0, 1)$. Also, $p_C \cdot x_\alpha \geq p_C \cdot \hat{x}$. Using the assumption that \hat{x} is interior to X , we take α close enough to one that x_α is also interior to X . Since the u_i 's are continuous, there is an open set $Y \subseteq X$ containing x_α such that, for all $z \in Y$, zPx . Defining $z_\beta = x_\alpha + \beta p_C$, we take β small enough that $z_\beta \in Y$, and therefore $z_\beta Px$. By construction,

$$p_C \cdot (z_\beta - \hat{x}) = p_C \cdot (x_\alpha - \hat{x}) + \beta p_C \cdot p_C > 0.$$

close enough to zero. But then, by condition (C1), we have $(x_\epsilon, \hat{x}) \succ_A (\hat{x}, \hat{x})$, a contradiction. Therefore, $\nabla u_k(\hat{x}) = 0$ for some voter k .

Now take any $p \in \text{cone}^\circ\{p_A, p_B\}$, and suppose the symmetry condition of the theorem is violated. We will show that one of the candidates has a profitable deviation, a contradiction. Let $\sigma = 1$ if

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| > |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|,$$

and let $\sigma = -1$ if the opposite inequality holds. As above, pick $q \in \mathbb{R}^d$ such that $p \cdot q = 0$, $p_A \cdot q > 0$, and $p_B \cdot q < 0$. Let Q be an open set on which these strict inequalities hold, and let $Q' = \{s \in Q \mid p \cdot s = 0\}$ be the elements of that set orthogonal to p . For $r \in Q'$, let $O(r) = \{s \in \mathbb{R}^d \mid s \cdot r = 0\}$ denote the subspace orthogonal to r . We claim that $\bigcap_{r \in Q'} O(r) = \text{span}\{p\}$. To see this, let $\{b_1, \dots, b_{d-1}\}$ be a basis for the $(d-1)$ -dimensional subspace orthogonal to p , and take $r \in Q'$ and $\epsilon > 0$ such that $\{r + \epsilon b_1, \dots, r + \epsilon b_{d-1}\}$ is linearly independent and contained in Q' . By linear independence, the dimension of

$$\bigcap_{h=1}^{d-1} O(r + \epsilon b_h)$$

is one. Of course, $p \in O(r)$ for all $r \in Q'$, establishing the claim.

Then, since N is finite and k is the only voter with ideal point \hat{x} , choose $r \in Q'$ so that $r \cdot \nabla u_i(\hat{x}) = 0$ if and only if $i = k$ or, for some $\alpha \neq 0$, $\nabla u_i(\hat{x}) = \alpha p$. Partition $N \setminus \{k\}$ into four sets,

$$\begin{aligned} I &= \{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\} \\ J &= \{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\} \\ K &= \{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \sigma \alpha p\} \\ L &= \{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \sigma \alpha p\}, \end{aligned}$$

and note that $|K| > |L|$. Without loss of generality, suppose $|I| \geq |J|$. Since $N \setminus \{k\}$ contains $n-1$ voters, we have $|K| + |I| > (n-1)/2$, and this implies $|K| + |I| \geq (n+1)/2 > n/2$. We will use r to construct a profitable deviation for candidate A . (If the inequality $|I| < |J|$ held instead, we would use $-r$ to construct a profitable deviation for B .) Let $x_\delta = \hat{x} + \delta r$ for $\delta > 0$. Then $\nabla u_i(\hat{x}) \cdot (x_\delta - \hat{x}) = \delta \nabla u_i(\hat{x}) \cdot r > 0$ for all $i \in I$, and $p_A \cdot (x_\delta - \hat{x}) = \delta \nabla p_A \cdot r > 0$. Choose δ close enough to zero that x_δ is interior

denote the two-dimensional subspace spanned by the projections of the candidates' gradients onto the space orthogonal to p . Given $p, q \in \mathbb{R}^d$, let

$$q(p) = \text{proj}_{S(p)} q$$

denote the projection of q onto that plane. Note that, since $\text{proj}_{O(p)} p_C \in S(p)$ and $S(p) \subseteq O(p)$, we have

$$p_C(p) = \text{proj}_{S(p)} p_C = \text{proj}_{O(p)} p_C,$$

so $p_C(p)$ is just the gradient of candidate C projected onto the subspace orthogonal to p . That, in turn, implies $S(p_A(p), p_B(p)) = S(p)$. Finally, note the further implication that $q(p) = q(p_A(p), p_B(p))$.

Let $q, r \in \mathbb{R}^d$ be vectors such that the gradients of the candidates, projected onto the plane $S(q, r)$, point in different directions, i.e., there do not exist $\alpha, \beta \geq 0$, at least one non-zero, such that $\alpha p_A(q, r) = \beta p_B(q, r)$. Thus, if we restrict the candidates' platforms to the two-dimensional space $\hat{x} + S(q, r)$, then the pair (\hat{x}, \hat{x}) is a polarized equilibrium of the restricted game. Take any $p \in \text{cone}^\circ\{p_A(q, r), p_B(q, r)\}$ in the open cone generated by the candidates' projected gradients, so that the antecedent conditions of Theorem 2 hold in the restricted game. We claim that

$$\begin{aligned} & |\{i \in N \mid \exists \alpha > 0 : p_i(q, r) = \alpha p(q, r)\}| \\ &= |\{i \in N \mid \exists \alpha < 0 : p_i(q, r) = \alpha p(q, r)\}|. \end{aligned}$$

If not, then, by Theorem 2, one of the candidates has a profitable deviation in the restricted game, and therefore the candidate has a profitable deviation in the original game, a contradiction. This establishes the claim.

To prove the theorem, take any $p \notin \text{span}\{p_A, p_B\}$, normalize so $\|p\| = 1$, let

$$\begin{aligned} I &= \{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\} \\ J &= \{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}, \end{aligned}$$

and suppose that $|I| \neq |J|$. Without loss of generality, suppose $|I| > |J|$. In light of the above claim, a contradiction is proved if we find vectors q and r satisfying three conditions:

- (1) there do not exist $\alpha, \beta \geq 0$, at least one non-zero, such that $\alpha p_A(q, r) = \beta p_B(q, r)$;

Thus, $p(s_\epsilon) \in \text{cone}^\circ\{p_A(s_\epsilon), p_B(s_\epsilon)\}$. Taking ϵ close enough to zero that $p_A(s_\epsilon)$ and $p_B(s_\epsilon)$ are linearly independent, we set $s = s_\epsilon$ for the desired perturbation.

We now wish to find perturbations, q and r , of $p_A(s)$ and $p_B(s)$ that satisfy condition (3) as well as (1) and (2). Let voter j satisfy $p_j(s) = \alpha p(s)$ for some $\alpha < 0$ but $p_j \neq \alpha p$. That is, although the voter's gradient appears to point in the $-p$ direction when projected, the voter is not a member of J . Note the immediate implication that p_j and p are linearly independent. We will find arbitrarily close vectors v and w such that $p_j(v, w) = \alpha' p(v, w)$ for no $\alpha' < 0$. Note that

$$\begin{aligned} p_j \cdot p_A(s) &= (p_j - p_j(s)) \cdot p_A(s) + p_j(s) \cdot p_A(s) \\ &= p_j(s) \cdot p_A(s) \\ &= \alpha p(s) \cdot p_A(s) \\ &= \alpha(p(s) - p) \cdot p_A(s) + \alpha p \cdot p_A(s) \\ &= \alpha p \cdot p_A(s), \end{aligned}$$

where the second equality follows from $(p_j - p_j(s)) \cdot p_A(s) = 0$ and the fourth equality from $(p(s) - p) \cdot p_A(s) = 0$. Similarly, $p_j \cdot p_B(s) = \alpha p \cdot p_B(s)$. These equalities imply

$$\frac{p_j \cdot p_A(s)}{p_j \cdot p_B(s)} = \frac{p \cdot p_A(s)}{p \cdot p_B(s)}.$$

Since p_j and p are linearly independent, there exists $t \in \mathbb{R}^d$ such that $p_j \cdot t > 0$ and $p \cdot t < 0$. Define $v_\epsilon = p_A(s) + \epsilon t$ and $w_\epsilon = p_B(s) - \epsilon t$ for $\epsilon > 0$, and note that

$$\frac{p_j \cdot v_\epsilon}{p_j \cdot w_\epsilon} > \frac{p \cdot v_\epsilon}{p \cdot w_\epsilon}.$$

Thus, $p_j(v_\epsilon, w_\epsilon) = \alpha' p(v_\epsilon, w_\epsilon)$ for no $\alpha' < 0$. That is, the gradient of voter j , projected onto the plane spanned by v_ϵ and w_ϵ , no longer appears to point in the $-p$ direction. Since conditions (1) and (2) hold on open sets around $p_A(s)$ and $p_B(s)$, we can choose ϵ small enough that (1) and (2) hold for v_ϵ and w_ϵ . Since N is finite, we can perturb v_ϵ and w_ϵ a finite number of times, if needed, so that the only voters whose projected gradients point in the $-p(v_\epsilon, w_\epsilon)$ direction are the members of J . By a similar argument, we can perturb v_ϵ and w_ϵ so that the only voters whose projected gradients point in the $p(v_\epsilon, w_\epsilon)$ direction are the members of I , fulfilling condition (3). ■

also adopts his/her ideal point. We first assume n is odd. Suppose (\tilde{x}_A, x_B) is an interior equilibrium. There are three cases to check. First, $\tilde{x}_A P x_B$. Letting $x P \tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$, condition (C1) implies that $(\tilde{x}_A, x) \succ_B (\tilde{x}_A, x_B)$, a contradiction. Second, $\tilde{x}_A I x_B$. As in the proof of Theorem 1, $u_B(x_B) > u_B(\tilde{x}_A)$. Then, by condition (C3) and Lemma 1, B can gain by moving toward \tilde{x}_A a small amount, a contradiction. Third, $x_B P \tilde{x}_A$. By continuity of the u_i 's, there is an open set of platforms containing x_B that are majority-preferred to \tilde{x}_A . Then, by condition (C1) and our assumption that (\tilde{x}_A, x_B) is an equilibrium, we have $\nabla u_B(x_B) = 0$, i.e., $x_B = \tilde{x}_B$. Then, as B could in the first case, candidate A can gain by moving to a platform y such that $y P x_B$ and $u_A(y) > u_A(x_B)$, a contradiction. If n is even, then we need modify the above argument only in the second case. Then $\tilde{x}_A I x_B$, and Lemma 1 implies that moving slightly toward \tilde{x}_A leads to a platform majority-preferred or -indifferent to \tilde{x}_A . If the latter, then, as in the proof of Theorem 1, there is an open set of platforms containing x_B majority-indifferent to \tilde{x}_A . If $\nabla u_B(x_B) \neq 0$, then (C4) yields a contradiction. Otherwise, $x_B = \tilde{x}_B$, and both candidates are at their ideal points. If the former, then the argument in Theorem 1 yields a contradiction. ■

References

- [1] J. Banks and J. Duggan (1999) "The Theory of Probabilistic Voting in the Spatial Model of Elections," mimeo., University of Rochester.
- [2] J. Banks and J. Duggan (2000) "A Multidimensional Model of Repeated Elections," mimeo., University of Rochester.
- [3] J. Banks, J. Duggan, and M. Le Breton (1999) "The Core and Uncovered Set with Arbitrary Distributions of Voters," mimeo., University of Rochester.
- [4] D. Black (1958) *The Theory of Committees and Elections*, Cambridge University Press: Cambridge.
- [5] T. Besley and S. Coate (1997) "An Economic Model of Representative Democracy," *Quarterly Journal of Economics*, 112: 85-114.

- [18] A. Rubinstein (1979) "A Note on the Nowhere Denseness of Societies Having an Equilibrium under Majority Rule," *Econometrica*, 47: 511-514.
- [19] N. Schofield (1983) "Generic Instability of Majority Rule," *Review of Economic Studies*, 50: 695-705.
- [20] D. Wittman (1977) "Candidates with Policy Preferences: A Dynamic Model," *Journal of Economic Theory*, 14: 180-189.
- [21] D. Wittman (1983) "Candidate Motivation: A Synthesis of Alternatives," *American Political Science Review*, 77: 142-157.

W. Allen Wallis Institute for Political Economy

Working Paper Series

To order copies of the working papers, complete the attached invoice and return to:

Mrs. Terry Fisher
W. Allen Wallis Institute of Political Economy
107 Harkness Hall
University of Rochester
Rochester, NY 14627.

Three (3) papers per year will be provided free of charge as requested below. Each additional paper will require a \$5.00 service fee which must be enclosed with your order.

An invoice is provided below in order that you may request payment from your institution as necessary. Please make your check payable to the W. Allen Wallis Institute of Political Economy.

OFFICIAL INVOICE

Requestor's Name:

Requestor's Address:

Please send me the following papers free of charge:
(Limit: 3 free per year)

WP# _____ WP# _____ WP# _____

I understand there is a \$5.00 fee for each additional paper.
Enclosed is my check or money order in the amount of \$ _____.
Please send me the following papers.

WP# _____ WP# _____ WP# _____
WP# _____ WP# _____ WP# _____
WP# _____ WP# _____ WP# _____
WP# _____ WP# _____ WP# _____

