Uniqueness of Stationary Equilibria in a One-dimensional Model of Bargaining

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Abstract

We prove uniqueness of stationary equilibria in a one-dimensional model of bargaining with quadratic utilities, for an arbitrary common discount factor. For general concave utilities, we prove existence and uniqueness of a "minimal" stationary equilibrium and of a "maximal" stationary equilibrium. We provide an example of multiple stationary equilibria with concave (non-quadratic) utilities.

1 Introduction

Collective decisionmaking, whether in firms, unions, or households in economics, or whether in committees and legislatures in political science, often takes the form of bargaining. Building on Rubinstein's (1982) model of two-player, alternating-offer bargaining, Baron and Ferejohn (1989) allow for an arbitrary number of individuals who must decide how to allocate a "dollar" among themselves using majority rule. In their model, in any period, one individual is randomly drawn to make a proposal, which is followed by a vote. If the proposal receives a majority of the vote, then the dollar is divided as proposed, and bargaining ends. Otherwise, the individuals consume zero for the current period, and the bargaining process continues in the following period. Jackson and Moselle (2001) consider the problem of dividing

a dollar and simultaneously choosing a point in a one-dimensional policy space. Banks and Duggan (2000) allow for an arbitrary compact, convex set of alternatives, for arbitrary continuous, concave utility functions, and for arbitrary "simple" voting rules, capturing as a special case the traditional one-dimensional spatial model, common in applications. Thus, rather than allocations of a transferable resource, alternatives may represent public policies, such as public good levels, tax rates, or interest rates, or they may index the ideological content of legislation in the liberal-conservative spectrum. Banks and Duggan (2001a) reconsider this model when the status quo is an arbitrary element of the policy space, rather than assuming a status quo payoff of zero, as in earlier models.

Analyses of bargaining typically focus on stationary subgame perfect equilibria, because of their simplicity (which may create a focal effect) and tractability. Of theoretical and practical interest, then, is whether stationary subgame perfect equilibria are unique in these models. In Rubinstein's model with two individuals, there is a unique subgame perfect equilibrium, and it is stationary, a strong result. Merlo and Wilson (1995) have shown that, when an arbitrary number of agents bargain under unanimity rule, there is a unique stationary subgame perfect equilibrium, even when the amount of the resource may vary stochastically over time. While Baron and Ferejohn (1989) prove uniqueness within the class of symmetric equilibria, Eraslan (2001) drops this restriction and establishes uniqueness in the Baron-Ferejohn model. However, Eraslan and Merlo (2001) show that, if the amount to be divided varies stochastically in the Baron-Ferejohn model, where majority rule is used, then multiple stationary subgame perfect equilibria may obtain. Banks and Duggan (2000, 2001a) show that, in the one-dimensional spatial model with majority rule and an odd number of perfectly patient individuals, there is a unique stationary subgame perfect equilibrium. In it, every individual proposes the median of the individuals' ideal points. And as individuals become arbitrarily patient, equilibrium outcomes converge to the median. They show that, in the two-dimensional spatial model, there can exist multiple equilibria.

In this paper, we consider the issue of uniqueness in the one-dimensional spatial model, allowing for an arbitrary common discount factor. Our analysis allows for both models of the status quo used in the literature, either assuming the status quo is an arbitrary element of the set of alternatives, or essentially giving all individuals a zero status quo payoff. For our first result, we allow for any simple voting rule that is "proper" (no two disjoint decisive coalitions) and "strong" (if a coalition is not decisive, then its complement is), capturing majority rule with an odd number of individuals. We prove that there is a unique stationary subgame perfect equilibrium under the standard, but restrictive, assumption of quadratic utilities. The proof uses the observation, apparently not noted before, that, with quadratic utilities, individual preferences over lotteries on the real line are order restricted.

For our second result, we drop the assumptions that the voting rule is proper and strong, and allow for any continuous, concave, strictly quasi-concave utility functions. We prove that any two stationary subgame perfect equilibria will be nested, in the sense that the set of proposals that can be passed (the "social acceptance set") in one equilibrium will form a subset of the social acceptance set of the other equilibrium. This allows us to prove existence and uniqueness of a "minimal" and of a "maximal" stationary subgame perfect equilibrium. We provide an example of multiple (non-nested) stationary subgame perfect equilibria when individuals, odd in number, have concave (non-quadratic) utility functions and majority rule is used.

Thus, uniqueness holds in an important specification of the model, demonstrating its tractability and its potential usefulness in applications in economics and political science. A partial uniqueness result holds more generally, but we have not investigated bounds on the extent of multiplicity, i.e., the gap between the minimal and maximal stationary equilibria. We leave that as an open question.

2 The Model

Let $X \subseteq \Re$ denote a nonempty, compact, convex set of alternatives, i.e., X is a closed, bounded interval. Let $N = \{1, \ldots, n\}$ denote a set of individuals with $n \geq 2$, who play an infinite-horizon bargaining game over the set of alternatives. The bargaining in every period is described as follows. If no alternative has been accepted prior to period t, then (1) individual $i \in N$ is recognized with probability ρ_i, where $\rho = (\rho_1, \ldots, \rho_n) \in \Delta$, the unit simplex in $\Re^n$. These *recognition probabilities* are exogenously fixed throughout the game. (2) If recognized, i makes a proposal $p_i \in X$. (3) After observing p_i, all $j \in N$ simultaneously vote to either accept or reject the proposal. Let $\mathcal{D} \subseteq 2^N \backslash \{\emptyset\}$ denote an exogenously fixed collection of *decisive* coalitions. (4) If $\{j \in N \mid j \text{ accepts }\} \in \mathcal{D}$, then proposal p_i is chosen and bargaining ends with outcome p_i in period t and in every subsequent period. Otherwise, each $i \in N$ gets utility $\bar{u}_i$ from the status quo in period t, and steps 1-4 are repeated for period $t + 1$.

Each individual i's preferences over sequences of outcomes, and lotteries over them, are described by a von Neumann-Morgenstern utility representation $u_i : X \rightarrow \Re$ and a common discount factor $\delta \in [0, 1)$ as follows. If $x \in X$ is accepted in period t, then i's payoff is

$$(1 - \delta^{t-1})\bar{u}_i + \delta^{t-1} u_i(x),$$

which represents i's discounted utility from the status quo for the first $t-1$ periods and from x every subsequent period. If no alternative is ever accepted, then each individual simply receives $\bar{u}_i$. We assume throughout that each u_i is continuous, concave, and strictly quasi-concave. Banks and Duggan (2000, 2001a) assume the condition of *limited shared weak preference* (LSWP), which is generally weaker than strict quasi-concavity. However, in one dimension, the conditions are equivalent.

Throughout the paper, we assume that $\mathcal{D}$ is nonempty and *monotonic:* $C \in \mathcal{D}$ and $C \subseteq C'$ imply $C' \in \mathcal{D}$. For some of our results, we impose additional restrictions on the voting rule. Say that $\mathcal{D}$ is *proper* if $C \in \mathcal{D}$ implies $N \backslash C \notin \mathcal{D}$; and *strong* if $C \notin \mathcal{D}$ implies $N \backslash C \in \mathcal{D}$. The *core*, denoted K, consists of the alternatives $x \in X$ that are weakly preferred to all others according to the voting rule: for all $y \in X$ and all $C \in \mathcal{D}$, there exists $i \in C$ such that $u_i(x) \geq u_i(y)$. If $\mathcal{D}$ is proper, then, because X is one-dimensional and individual preferences are "single-peaked," K is nonempty. If $\mathcal{D}$ is also strong, then K is actually a singleton and consists of the ideal point $\tilde{x}_k$ of some individual k, the "core voter."

So far we have let the payoff from status quo be arbitrary. Now we define two alternative assumptions on status quo payoffs.

- (A1) There exists $q \in X$ such that, for every $i \in N$, $\bar{u}_i = u_i(q)$.
- (A2) There exists $\bar{u} \in \Re$ such that, for every $i \in N$ and $x \in X$, $\bar{u} = \bar{u}_i \leq u_i(x)$.

A1 assumes that the status quo is a point in the policy space, as in Banks and Duggan (2001a). A2 follows Rubinstein (1982) and Baron and Ferejohn (1989) in assuming a "bad" status quo and imposing the additional normalization that the status quo payoff is the same for all individuals.

Since our focus is only on equilibria in stationary strategies, we provide a formal definition only of such strategies. A (pure) *stationary strategy* for $i \in N$ consists of a proposal $p_i \in X$ to be offered anytime i is recognized, and a measurable voting rule $r_i : X \rightarrow \{\text{accept, reject}\}$. For the latter, we will use the more convenient representation of an *acceptance set*, $A_i = r_i^{-1}(\text{accept})$, i.e., the set of proposals that i would vote to accept. Thus, a stationary strategy for i is a pair $\sigma_i = (p_i, A_i)$, and we let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a profile of stationary strategies. Since Banks and Duggan (2000, 2001a) prove that every no-delay stationary equilibrium is in pure strategies, there is no loss of generality in focusing on such equilibria. Given a profile σ and given $C \subseteq N$, define the set

$$A_C = \bigcap_{i \in C} A_i$$

of proposals acceptable to all members of C, and define the *social acceptance set*

$$A = \bigcup_{C \in \mathcal{D}} A_C,$$

consisting of proposals that could be passed in any and all periods. The profile is a *no-delay* profile if $\rho_i > 0$ implies $p_i \in A$.

Informally, a profile σ constitutes a stationary equilibrium if, for all $i \in N$, p_i is optimal given the acceptance sets of the other individuals, and A_i is optimal given that σ describes what would happen if a current proposal is rejected. Any strategy profile σ defines in an obvious manner a probability distribution over sequences of outcomes and, with it, an expected utility $v_i(\sigma)$ for each $i \in N$ as evaluated at the beginning of the game. By stationarity, this is also i's *continuation value* throughout the game, i.e., i's expected utility evaluated at the beginning of next period if the current period's proposal is rejected. If σ is a no-delay profile, then i's continuation value has the simple form $v_i(\sigma) = \sum_{j \in N} \rho_j u_i(p_j)$.

Formally, σ is a *stationary equilibrium* if two conditions hold. First, we require that the individuals' acceptance sets satisfy *weak dominance*, i.e., individual i votes for proposal x if and only if the utility from x is at least as great as that of rejecting the proposal and continuing to the next period: for all $i \in N$, we require that

$$A_i = \{x \in X \mid u_i(x) \geq (1-\delta)\bar{u}_i + \delta v_i(\sigma)\}.$$

This condition ensures that the individuals' votes are best responses, and that they are weakly undominated in the voting stage. Second, we require that the individuals' proposals satisfy *sequential rationality*, i.e., individual i, when recognized as proposer, either chooses utility maximizing outcomes from within A or chooses an outcome that will be rejected, depending on which yields the higher payoff: for all $i \in N$, we require that

$$p_i \in \arg\max\{u_i(y) \mid y \in A\}$$

when $\sup\{u_i(y) \mid y \in A\} > (1-\delta)\bar{u}_i + \delta v_i(\sigma)$; that $p_i \in X \backslash A$ when the inequality is reversed; and that either of these two conditions is satisfied when equality holds. In a no-delay stationary equilibrium, we have $p_i \in A$ and, consequently, $u_i(p_i) \geq (1-\delta)\bar{u}_i + \delta v_i(\sigma)$ for all $i \in N$ with $\rho_i > 0$.

Allowing mixed proposal strategies, Banks and Duggan (2000, 2001a) establish the existence of no-delay stationary equilibria under A1 and A2, assuming X is a compact, convex subset of finite-dimensional Euclidean space. Those papers also prove that all no-delay stationary equilibria are in pure strategies when X is one-dimensional, establishing existence of no-delay stationary equilibria as defined in this paper. We next turn to the issue of uniqueness.

3 Results

Banks and Duggan (2000) prove that "core equivalence" holds when $X \subseteq \Re$, $\delta = 1$, $\rho_i > 0$ for all $i \in N$, and $\mathcal{D}$ is proper: the social acceptance sets corresponding to no-delay stationary equilibria are singletons and each consists of the same core point. If $\mathcal{D}$ is strong, so the core is a singleton, then there is a unique no-delay stationary equilibrium. The question of uniqueness when $\delta < 1$, which we take up here, is not addressed. Banks and Duggan (2000) prove that, under the above conditions and $\delta < 1$, all stationary equilibria are no-delay, whereas Banks and Duggan (2001a) prove this holds unless $q \in K$. Thus, the results below, which are stated in terms of no-delay equilibria, can be strengthened somewhat: under A2 we can drop "no-delay," while under A1 we can drop it unless $q \in K$.

We first examine the issue of uniqueness for the important special case of quadratic utility, in which each individual i has an "ideal point" $\tilde{x}_i \in \Re$ such that $u_i(x) = -(x-\tilde{x}_i)^2$. With the assumption of quadratic preferences, our first proposition establishes uniqueness when the status quo is a point in the policy space or when the status quo is bad.

Proposition 1. *Assume either A1 or A2. Assume $\mathcal{D}$ is proper and strong. If u_i is quadratic for all $i \in N$, then there is a unique no-delay stationary equilibrium.*

Proof. Let σ and σ' be no-delay stationary equilibria, and suppose $\sigma \neq \sigma'$. Since $\mathcal{D}$ is proper and strong, $K = \{\tilde{x}_k\}$ for some $k \in N$. Let $u = (1-\delta)\bar{u}_k + \delta v_k(\sigma)$, and $u' = (1-\delta)\bar{u}_k + \delta v_k(\sigma')$. By Lemma 1 and Lemma 2 in the Appendix, we have $A = \{x \in X \mid u_k(x) \geq u\}$ and $A' = \{x \in X \mid u_k(x) \geq u'\}$. If $u = u'$, then $A = A'$. But then, by no-delay and sequential rationality, for all $i \in N$, $p_i = p'_i$, which contradicts $\sigma \neq \sigma'$. Thus, $u \neq u'$. Without loss of generality, assume $u > u'$, implying $A \subseteq A'$. Then we have $u - u' = \delta(v_k(\sigma) - v_k(\sigma')) > 0$. Since $\delta < 1$,

$$u - u' \;<\; v_k(\sigma) - v_k(\sigma'). \tag{1}$$

Let $C = \{i \in N \mid \rho_i > 0 \text{ and } p_i \neq p'_i\}$. Note that $u \neq u'$ implies $C \neq \emptyset$. We claim that, for all $i \in C$, $u_k(p_i) = u$. Suppose not. Then there exists an $i \in C$ such that $p_i \in \text{int}(A)$, which implies $p_i = \tilde{x}_i$ by sequential rationality. Then since $A \subseteq A'$, sequential rationality implies $p_i = p'_i$, contradicting $i \in C$. We can write

$$v_k(\sigma) \;=\; \sum_{i \in C} \rho_i u + \sum_{i \in N \setminus C} \rho_i u_k(p_i), \tag{2}$$

and

$$v_k(\sigma') \;=\; \sum_{i \in C} \rho_i u_k(p'_i) + \sum_{i \in N \setminus C} \rho_i u_k(p'_i). \tag{3}$$

Note that $\sum_{i \in N \setminus C} \rho_i u_k(p_i) = \sum_{i \in N \setminus C} \rho_i u_k(p'_i)$ since, for all $i \in N \setminus C$, either $p_i = p'_i$ or $\rho_i = 0$. By no-delay, for all $i \in N$, $u_k(p'_i) \geq u'$. Note that $\sum_{i \in C} \rho_i \leq 1$. Subtracting (3) from (2), we then have

$$\begin{aligned} v_k(\sigma) - v_k(\sigma') &= \sum_{i \in C} \rho_i (u - u_k(p'_i)) \\ &\leq \sum_{i \in C} \rho_i (u - u') \\ &\leq u - u', \end{aligned}$$

which contradicts (1). □

We next consider the extent to which uniqueness carries over to the general case of continuous, concave, and strictly quasi-concave preferences. The next result establishes that every pair of no-delay stationary equilibria are nested. As before, we use $\tilde{x}_i$ to denote the unique maximal alternative of individual i.

Proposition 2. *Let σ and σ' be no-delay stationary equilibria, and let A and A' be the social acceptance sets corresponding to σ and σ' respectively. Then $A \subseteq A'$ or $A' \subseteq A$.*

Proof. Suppose that $A \not\subseteq A'$ and $A' \not\subseteq A$. Let $A = [\underline{x}, \overline{x}]$ and $A' = [\underline{y}, \overline{y}]$. Without loss of generality, assume that $\underline{x} < \underline{y}$ and $\overline{x} < \overline{y}$. Let $C_{\overline{y}} = \{i \in N \mid \overline{y} \in A'_i\}$. Since $\overline{y} \in A'$, $C_{\overline{y}} \in \mathcal{D}$. Note that there is some $i \in C_{\overline{y}}$ such that $\tilde{x}_i \leq \overline{x}$ and

$$u_i(\overline{x}) \quad \leq \quad (1-\delta)\bar{u}_i + \delta v_i(\sigma). \tag{4}$$

If not, then we can find $\epsilon > 0$ satisfying $\overline{x} + \epsilon \in A_j$ for every $j \in C_{\overline{y}}$. But then $\overline{x} + \epsilon \in A$, which contradicts $A = [\underline{x}, \overline{x}]$. Now we will show that the existence of such an i implies $\overline{y} - \overline{x} < \underline{y} - \underline{x}$. Since $i \in C_{\overline{y}}$,

$$(1-\delta)\bar{u}_i + \delta v_i(\sigma') \quad \leq \quad u_i(\overline{y}). \tag{5}$$

From (4) and (5), we have $u_i(\overline{x}) - u_i(\overline{y}) \leq \delta(v_i(\sigma) - v_i(\sigma'))$. By strict quasi-concavity, $u_i(\overline{x}) > u_i(\overline{y})$. Then, because $\delta < 1$, we have $u_i(\overline{x}) - u_i(\overline{y}) < v_i(\sigma) - v_i(\sigma')$. Note that the righthand side of the latter inequality is equal to $\sum_{j \in N} \rho_j(u_i(p_j) - u_i(p'_j))$, which is less than or equal to $\max\{u_i(p_j) - u_i(p'_j) \mid j \in N\}$. Thus,

$$u_i(\overline{x}) - u_i(\overline{y}) \quad < \quad \max\{u_i(p_j) - u_i(p'_j) \mid j \in N\}. \tag{6}$$

We claim that $\tilde{x}_i < \underline{y}$. If $\overline{x} < \underline{y}$, then the claim is true because $\tilde{x}_i \leq \overline{x}$, so suppose $\overline{x} \geq \underline{y}$ and $\tilde{x}_i \geq \underline{y}$. Then, however, $\max\{u_i(p_j) - u_i(p'_j) \mid j \in N\} \leq u_i(\overline{y}) - u_i(\overline{x})$, contradicting (6) and proving the claim. Thus, we have two cases. Case 1: $\tilde{x}_i \leq \underline{x}$. Then (6) implies $\max\{u_i(p_j) - u_i(p'_j) \mid j \in N\} \leq u_i(\underline{x}) - u_i(\underline{y})$, so we have $u_i(\overline{x}) - u_i(\overline{y}) < u_i(\underline{x}) - u_i(\underline{y})$. Since $\underline{x} < \underline{y}$ and $\overline{x} < \overline{y}$, concavity of u_i implies, $\overline{y} - \overline{x} < \underline{y} - \underline{x}$. Case 2: $\underline{x} < \tilde{x}_i < \underline{y}$. Then (6) implies $\max\{u_i(p_j) - u_i(p_j) \mid j \in N\} \leq u_i(\tilde{x}_i) - u_i(\underline{y})$. Then $u_i(\overline{x}) - u_i(\overline{y}) < u_i(\tilde{x}_i) - u_i(\underline{y})$. By concavity, $\overline{y} - \overline{x} < \underline{y} - \tilde{x}_i < \underline{y} - \underline{x}$. Therefore, we've shown that

$$\overline{y} - \overline{x} \quad < \quad \underline{y} - \underline{x}.$$

Now, let $C_{\underline{x}} = \{i \in N \mid \underline{x} \in A_i\} \in \mathcal{D}$. Again, there is some $i \in C_{\underline{x}}$ such that $\tilde{x}_i \geq \underline{y}$ and $u_i(\underline{y}) \leq (1-\delta)u_i(q) + \delta v_i(\sigma')$, and a symmetric argument shows $\overline{y} - \overline{x} > \underline{y} - \underline{x}$, a contradiction. □

Proposition 2 shows that when the policy space is one-dimensional, all no-delay stationary equilibria are comparable with each other in terms of set inclusion of the corresponding social acceptance sets. A no-delay stationary equilibrium σ is *minimal* if $A(\sigma) \subseteq A(\sigma')$ for every no-delay stationary equilibrium σ', and it is *maximal* if $A(\sigma') \subseteq A(\sigma)$ for every no-delay stationary equilibrium σ'. Under A1 and A2, Banks and Duggan (2000, 2001a) prove upper hemicontinuity as well as existence of no-delay stationary equilibria. With these results, Proposition 2 yields the following corollary.

Corollary 1. *Assume A1 or A2. A minimal and a maximal no-delay stationary equilibrium exist, and they are unique.*

Proof. Let Σ be the set of no-delay stationary equilibria, which is nonempty by existence theorems in Banks and Duggan (2000, 2001a). Let $A^* = \bigcap_{\sigma \in \Sigma} A(\sigma)$, where $A(\sigma)$ denotes the social acceptance set corresponding to a given equilibrium σ, and define $\underline{x} = \min A^*$ and $\overline{x} = \max A^*$. Construct a sequence of equilibria $\{\sigma^m\}_{m=1}^{\infty}$ satisfying $A(\sigma^m) \subseteq [\underline{x} - 1/m, \overline{x} + 1/m]$ for each m in the following way. Since $\underline{x} - 1/m \notin A^*$ and $\overline{x} + 1/m \notin A^*$, we can find σ', $\sigma'' \in \Sigma$ such that $\underline{x} - 1/m \notin A(\sigma')$ and $\overline{x} + 1/m \notin A(\sigma'')$. By Proposition 2, either $A(\sigma') \subseteq A(\sigma'')$ or $A(\sigma'') \subseteq A(\sigma')$. If $A(\sigma') \subseteq A(\sigma'')$, then index σ' as σ^m; and if $A(\sigma'') \subseteq A(\sigma')$, then index σ'' as such. Now consider a sequence of profiles of proposal strategies $\{p^m\}_{m=1}^{\infty}$, where $p^m = (p_1^m, \ldots, p_n^m)$ is the profile of proposal strategies corresponding to σ^m. Since the sequence $\{p^m\}_{m=1}^{\infty}$ lies in the compact set X^n, there is a subsequence that has a limit point, say $p^* = (p_1^*, \ldots, p_n^*)$. Let $\sigma^* = (p^*, A^*)$. By upper hemicontinuity of no-delay stationary equilibria, σ^* is a no-delay stationary equilibrium, and by construction, it is minimal. Furthermore, by strict quasi-concavity, the proposals satisfying sequential rationality, given A^*, are unique, implying uniqueness of minimal no-delay stationary equilibrium. Existence and uniqueness of maximal no-delay stationary equilibrium is proved similarly, letting $A^* = \bigcup_{\sigma \in \Sigma} A(\sigma)$ and defining $\underline{x} = \inf A^*$ and $\overline{x} = \sup A^*$. □

The preceding proposition leaves open the question of uniqueness, achieved in Proposition 1 by the decisiveness of the core voter, which is guaranteed by quadratic preferences. The following example shows that multiple equilibria are possible when such a decisive voter does not exist.

Example 1. *Multiple equilibria*

Let $X = [0, 13]$, $n = 5$, $q = 0$, and $\delta = 5/6$. Let $\mathcal{D}$ be majority rule, and let $\rho_i = 1/5$ for every i. Assume utility functions of individuals are as follows.

- $u_1(x) = -|x - 1|$
- $u_2(x) = \begin{cases} -|x - 6| & \text{if } x \geq 4 \\ 5x - 22 & \text{if } x < 4 \end{cases}$

- $u_3(x) = -|x-8|$
- $u_4(x) = \begin{cases} -|x-10| & \text{if } x \in [3,12] \\ 5x-22 & \text{if } x < 3 \\ -5x+58 & \text{if } x > 12 \end{cases}$
- $u_5(x) = -|x-13|$

Consider a strategy profile $\sigma = ((p_1, A_1), \ldots, (p_5, A_5))$ in which proposals and acceptance sets are as follows.

- $p_1 = 4$, $p_2 = 6$, $p_3 = 8$,$p_4 = 10$, and $p_5 = 12$.
- $A_1 = \{x \in X \mid u_1(x) \geq -6\} = [0,7]$,
- $A_2 = \{x \in X \mid u_2(x) \geq -6\} = [16/5, 12]$,
- $A_3 = \{x \in X \mid u_3(x) \geq -10/3\} = [14/3, 34/3]$,
- $A_4 = \{x \in X \mid u_4(x) \geq -6\} = [4, 64/5]$, and
- $A_5 = \{x \in X \mid u_5(x) \geq -19/3\} = [20/3, 13]$.

Given the profile of acceptance sets, the social acceptance set is $A = [4,12]$, so σ is a no-delay profile. Note that the social acceptance set is different from the acceptance set of individual 3, who is the core voter in this setting. Here, the continuation lottery corresponding to σ places probability 1/6 on each proposal, and it can be checked that every A_i satisfies weak dominance. For example, A_3 described above satisfies weak dominance, since the expected utility of rejection for individual 3 is

$$\begin{aligned} & (1-\delta)u_3(q) + \delta \sum_{i=1}^{5} \rho_i u_3(p_i) \\ = \; & \frac{1}{6}(u_3(0) + u_3(6) + u_3(8) + u_3(10) + u_3(12)) \\ = \; & \frac{1}{6}(-8-4-2+0-2-4) \\ = \; & -\frac{10}{3}. \end{aligned}$$

Also, given that $A = [4,12]$, each proposal is sequentially rational. Thus, σ is a no-delay stationary equilibrium.

Now consider another strategy profile $\sigma' = ((p'_1, A'_1), \ldots, (p'_5, A'_5))$ in which proposals and acceptance sets are as follows.

- $p'_1 = 3$, $p'_2 = 6$, $p'_3 = 8$, $p'_4 = 10$, and $p'_5 = 13$.
- $A'_1 = \{x \in X \mid u_1(x) \geq -6\} = [0, 7]$,
- $A'_2 = \{x \in X \mid u_2(x) \geq -7\} = [3, 13]$,
- $A'_3 = \{x \in X \mid u_3(x) \geq -11/3\} = [13/3, 35/3]$,
- $A'_4 = \{x \in X \mid u_4(x) \geq -7\} = [3, 13]$, and
- $A'_5 = \{x \in X \mid u_5(x) \geq -19/3\} = [20/3, 13]$.

Given the profile of acceptance sets, the social acceptance set is $A = [3, 13]$, so σ' also is a no-delay profile. It can be easily verified that σ' is also a no-delay stationary equilibrium.

A Appendix

In the following two lemmas, we assume that $\mathcal{D}$ is proper and strong, and we show that, in any no-delay stationary equilibrium, the social acceptance set equals the acceptance set of the core voter k, assuming quadratic preferences and either A1 or A2. The first result is proved by establishing that the individuals' preferences over lotteries on X are *order restricted,* in the sense that they can be ordered, $i_1, i_2, \ldots, i_n$ (where i_j is the jth individual in the ordering), so that, for every pair of lotteries λ and λ',

$$\begin{aligned}\{j \mid \textstyle\int u_{i_j} d\lambda > \int u_{i_j} d\lambda'\} \; &< \; \{j \mid \textstyle\int u_{i_j} d\lambda = \int u_{i_j} d\lambda'\} \\ &< \; \{j \mid \textstyle\int u_{i_j} d\lambda < \int u_{i_j} d\lambda'\}\end{aligned}$$

or

$$\begin{aligned}\{j \mid \textstyle\int u_{i_j} d\lambda < \int u_{i_j} d\lambda'\} \; &< \; \{j \mid \textstyle\int u_{i_j} d\lambda = \int u_{i_j} d\lambda'\} \\ &< \; \{j \mid \textstyle\int u_{i_j} d\lambda > \int u_{i_j} d\lambda'\},\end{aligned}$$

where $I < J$ indicates that $i < j$ for all $i \in I$ and all $j \in J$ (cf. Rothstein, 1990). As a consequence, the core voter k is "decisive," i.e., k prefers a lottery λ to a lottery λ' if and only if all members of some decisive coalition prefer λ to λ'. The lemma extends Lemma 1 of Banks and Duggan (2001b) to all proper, strong voting rules, whereas their Lemma 1 considered only majority rule (and allowed for a continuum of voters). The key property of quadratic utilities used is mean-variance analysis: we can write i's expected utility from a lottery λ as $u_i(m) - v$, where m is the mean of λ and v is the variance.

In the proofs of both lemmas, let $C_L = \{i \in N \mid \tilde{x}_i < \tilde{x}_k\}$ and $C_R = \{i \in N \mid \tilde{x}_i > \tilde{x}_k\}$. Note that $C_L \notin \mathcal{D}$ and $C_R \notin \mathcal{D}$, and since $\mathcal{D}$ is strong, $C_L \cup \{k\} \in \mathcal{D}$, and $C_R \cup \{k\} \in \mathcal{D}$.

Lemma 1. *Assume A1. Assume $\mathcal{D}$ is proper and strong. If u_i is quadratic for all $i \in N$, then in every no-delay stationary equilibrium σ, $A = A_k$.*

Proof. We claim that individual preferences over lotteries on X are order restricted when individuals are ordered in the order of their ideal points. For any distinct pair of lotteries λ and λ', let

$$\begin{aligned} C_1 &= \{i \mid \int u_i(z)\lambda(dz) > \int u_i(z)\lambda'(dz)\} \\ C_2 &= \{i \mid \int u_i(z)\lambda(dz) = \int u_i(z)\lambda'(dz)\} \\ C_3 &= \{i \mid \int u_i(z)\lambda(dz) < \int u_i(z)\lambda'(dz)\}. \end{aligned}$$

Thus, we claim that if $h \in C_1$, $i \in C_2$, and $j \in C_3$, then either $\tilde{x}_h > \tilde{x}_i > \tilde{x}_j$, or $\tilde{x}_h < \tilde{x}_i < \tilde{x}_j$. Let m and v denote the mean and variance of λ, and let m' and v' denote the mean and variance of λ'. If $m = m'$, then every individual prefers the lottery with smaller variance, so the claim is vacuously satisfied. Suppose $m \neq m'$. Since $h \in C_1$, we have $u_h(m) - v > u_h(m') - v'$, or equivalently

$$u_h(m) - u_h(m') \quad > \quad v - v'.$$

Similarly,

$$u_i(m) - u_i(m') \quad = \quad v - v',$$

and

$$u_j(m) - u_j(m') \quad < \quad v - v'.$$

Thus, we have

$$u_h(m) - u_h(m') \; > \; u_i(m) - u_i(m') \; > \; u_j(m) - u_j(m')$$

If $m > m'$, then this implies $\tilde{x}_h > \tilde{x}_i > \tilde{x}_j$; if $m < m'$, then it implies $\tilde{x}_h < \tilde{x}_i < \tilde{x}_j$. This establishes the claim. Consider any no-delay stationary equilibrium σ. Take any $x \in A_k$, i.e., $u_k(x) \geq (1-\delta)u_k(q) + \delta v_k(\sigma)$. This means that individual k weakly prefers the point mass on x to the lottery corresponding σ, namely, the lottery that puts probability $1-\delta$ on q and, for each $i \in N$, probability $\delta\rho_i$ on p_i. By order-restriction, either $x \in A_{C_L \cup \{k\}}$ or $x \in A_{C_R \cup \{k\}}$. This establishes $A_k \subseteq A$. Now take any $x \in A$, and suppose $x \notin A_k$. Since k weakly prefers the lottery corresponding to σ to the point mass on x, order-restriction implies either $\{i \mid x \in A_i\} \subseteq C_L$ or $\{i \mid x \in A_i\} \subseteq C_R$. But since $C_L \notin \mathcal{D}$ and $C_R \notin \mathcal{D}$, we have $x \notin A$. This contradiction establishes $A \subseteq A_k$. □

Lemma 2. *Assume A2. Assume $\mathcal{D}$ is proper and strong. If u_i is quadratic for all $i \in N$, then in every no-delay stationary equilibrium σ, $A = A_k$.*

Proof. Consider any no-delay stationary equilibrium σ, and let m and v denote the mean and variance of the lottery with probability ρ_i on p_i for each $i \in N$. Take any $x \in X$. If $x = m$, then every individual weakly prefers x, so, trivially, $x \in A$ if and only if $x \in A_k$. Suppose $x < m$, let $u_z(x) = -(x-z)^2$ be the quadratic utility function with ideal point z, define

$$f(z) \quad = \quad u_z(x) - \delta u_z(m),$$

and note that $f(\tilde{x}_i) \geq (1-\delta)\bar{u} - \delta v$ if and only if $x \in A_i$. We claim that there exists $\overline{x} \in \overline{\Re}$ such that $x \in A_i$ if and only if $\tilde{x}_i \leq \overline{x}$. Note that, because $\bar{u} \leq u_i(x)$, every individual with ideal point $\tilde{x}_i \leq x$ weakly prefers x to any lottery with mean m and variance v, i.e.,

$$u_i(x) \quad \geq \quad (1-\delta)\bar{u} + \delta(u_i(m) - v).$$

Thus, $f(z) \geq (1-\delta)\bar{u} - \delta v$ for all $z \leq x$. Take any $z > x$, and note that

$$\begin{aligned} f'(z) &= -2(1-\delta)z + 2(x - \delta m) \\ &= -2(1-\delta)z + 2(1-\delta)m + 2(x-m) \\ &= 2(1-\delta)(m-z) + 2(x-m). \end{aligned}$$

This is clearly negative if $m \leq z$. If $x < z < m$, then

$$\begin{aligned} f'(z) &\leq 2(m-z) + 2(x-m) \\ &\leq 2(x-z) \\ &< 0. \end{aligned}$$

Thus, $f'(z) < 0$ for all $z > x$, which establishes the claim. Now note that $x \in A$ if and only if $\{i \mid x \in A_i\} \in \mathcal{D}$, which, by the above argument, holds if and only if $C^* = \{i \mid \tilde{x}_i \leq \overline{x}\} \in \mathcal{D}$. If $x \in A_k$, then $\tilde{x}_k \leq \overline{x}$, which implies that $C_L \cup \{k\}$ is a subset of C^*, so C^* is decisive, so $x \in A$. If $x \notin A_k$, then $\tilde{x}_i > \overline{x}$, which implies that C^* is a subset of C_L, so it is not decisive, so $x \notin A$. A symmetric argument can be made for the $x > m$ case. □

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