A Multidimensional Model of Repeated Elections

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Abstract

We analyze a discrete-time, infinite-horizon model of elections. In each period, a challenger is chosen from the electorate to run against an incumbent politician in a majority-rule election, and the winner then selects a policy from a multidimensional policy space. Individuals' policy preferences are private information, whereas policy choices are publicly observable. We prove existence and continuity of equilibria in "simple" voting and policy strategies; we provide examples to show the variety of possible equilibrium patterns in multiple dimensions; we analyze the effects of patience and office-holding benefits on the persistence of policies over time; and we identify relationships between equilibrium policies and the core of the underlying voting game. We compare our results to those of the classic Downsian model, finding similarities in one dimension but differences in higher dimensions.

1 Introduction

Elections occupy a central position in the determination of public policies in representative democracies. By selecting the individuals whose subsequent decisions determine final policy outcomes, elections resolve conflicts among competing majorities and transform the preferences of voters into collective choices. It is well-known that, when the policy space is one-dimensional and voters have single-peaked preferences, a single policy outcome, the ideal point of the median voter, is majority-preferred to all others. In the canonical model of Downs (1957), in which two candidates commit to policy platforms before a single election, this drives the candidates to the median and yields a unique Nash equilibrium of the electoral game. When the policy space is multidimensional, however, majority undominated (or "core") points rarely exist (McKelvey and Schofield 1987; Plott 1967; Schofield 1983). Moreover, in the absence of a core point, results from social choice theory show that the entire space of policy alternatives will be contained in a majority preference cycle (McKelvey 1976, 1979; Austen-Smith and Banks 1999), suggesting to some authors (e.g., Riker 1980) the instability of policies over time. In contrast to that literature, where coalitions are assumed to form fluidly, we explicitly model electoral institutions and the incentives of individuals (in their roles as voters and politicians), which might constrain the formation of coalitions and limit the potential instability of collective choices.

Our objective is not to explain a particular political phenomenon, but rather to improve our general understanding of electoral processes, with special interest in their dynamic and informational aspects. Thus, we consider a model of repeated elections in which politicians determine policies in a multidimensional issue space and in which preferences (modelled quite generally) are private information. Our focus is on foundational issues, such as the formulation of an appropriate equilibrium concept, the existence of equilibria, the stability of policies over time, and the relationship between equilibrium policies and the core. The framework we construct captures the strategic incentives of politicians, whose private preferences and concern for re-election confronts them with a trade-off in choosing policies, and the strategic calculus of voters, who must anticipate the future policy choices of incumbent politicians and challengers. But, because our interest is initially limited to a few topics, and because part of our contribution is to solve some technical difficulties that arise in a multidimensional model of elections, our model omits several important considerations: for example, the role of parties, the entry decision of challengers, and strategic interaction among politicians. Nevertheless, we have sought to provide a solid theoretical foundation from which these issues can be approached in the future.

Most analyses of elections have followed the Downsian tradition in highlighting the pre-election campaign aspects of the competition for the role as representative. In the basic model, each of two otherwise identical candidates simultaneously announces a policy to be implemented if elected, with voters then casting their ballots for the candidate offering their preferred policy. While originally presented as a model of a single election in a one-dimensional policy space with office-motivated candidates and complete information, subsequent research has analyzed repeated elections (e.g., Boylan and McKelvey 1995; Duggan and Fey 2001), multiple dimensions (e.g., Kramer 1978), policy-motivated candidates (e.g., Calvert 1985, Wittman 1983), and incomplete information (e.g., Hinich, Ledyard, and Ordeshook 1972). All of this work, however, has retained the important underlying assumption of the Downsian model that the winning candidate will faithfully carry out her announced policy. This commitment assumption is often rationalized on the grounds that, if a candidate broke a campaign promise, there would be some (unmodelled) electoral punishments inflicted in the future. This maneuver effectively "black boxes" a principle component of the public policy process, namely, why representatives behave as they do while in office.

An alternative approach, beginning with the work of Barro (1973) and Ferejohn (1986) and sometimes referred to as models of "electoral accountability," sheds the commitment assumption and ignores the role of campaign announcements. In any one election voters either re-elect the incumbent, i.e., the representative from the previous period, or else elect a previously untried challenger, with the winning individual then choosing the policy for the current period. In contrast to the Downsian model, voters base their decisions on the past performance of incumbents, rather than their current promises, and, thus, these models are inherently dynamic. In selecting policies, representatives typically care not only about winning, but also about their actions while in office, either through their own policy preferences or else in terms of the "effort" expended on their constituents' behalf. And, with the exception of Barro (1973), incomplete information is present: either the motivations of the representatives are known but their influence over policy, and hence over voter utility, is not (Ferejohn 1986; Austen-Smith and Banks 1989), or their influence over policy is known but their motivations are not (Duggan 2000; Bernhardt, Dubey and Hughson 1998; Reed 1994), or neither is known (Rogoff 1990; Banks and Sundaram 1993, 1998; Coate and Morris 1995; Fearon 1998). To date, however, all of this work has maintained the original Downsian assumption of a unidimensional policy space, conceptualized either as a space of effort levels or (more conventionally) as an ideological dimension. In fact, many of these models are further simplified by the assumption that there is just one voter.[1]

In this paper, we propose a model of electoral accountability in which policies may lie in a subset of any finite-dimensional Euclidean space. Each of a continuum of voters has preferences represented by a continuous and strictly concave utility function. In each period, a challenger is drawn from the electorate to run against the incumbent in a majority-rule election, with the winner choosing the policy for that period. The process then moves to the next period, and the above sequence of events is repeated *ad infinitum.* Voters observe the policies chosen by the representatives but not their preferences. Thus, incomplete information in the form of adverse selection is present, and elections confront voters with a non-trivial problem: they must update their beliefs about the incumbent based on her past policy choices and compare this to the expected policy outcomes upon electing a challenger. Representatives, being chosen from the electorate at large, have well-defined policy preferences of their own and face a trade-off in choice of policy: they have short term incentives to choose policies in their personal interest, but they have long term interests in staying in office. Doing so, a representative may capture certain "non-policy" benefits of office, while obtaining policy outcomes better than expected from a challenger. But pursuit of personal policy interests may reveal information to voters that damages the representative's chances of re-election.

We prove the existence of "simple" equilibria in which voters use strategies that are retrospective (Fiorina 1981) in the following sense: an individual votes for re-

[1]The remaining distinction among papers in this category is whether a finite term limit on the incumbent (or a finite horizon) is imposed (Austen-Smith and Banks 1989; Reed 1994; Coate and Morris 1995; Banks and Sundaram 1998; Bernhardt, Dubey, and Hughson 1998; Fearon 1998) or not (Barro 1973; Ferejohn 1986; Rogoff 1990; Banks and Sundaram 1993; Duggan 2000).

election if and only if her utility in the previous period was at or above a fixed critical level, this level determined endogenously as the expected value of an untried challenger. Thus, in equilibrium, she also votes "prospectively," as though pivotal in the current election. Faced with such voter behavior, individuals in their role as representatives have an incentive to adopt history-independent strategies in which they choose the same policy whenever elected, allowing us to reconcile retrospective and prospective voting. We show through a series of examples how a wide variety of policy and re-election patterns can emerge in equilibrium, particularly in multiple dimensions. It is possible that no representative is ever re-elected, each choosing her ideal policy while in office and failing to gain the support of a majority of voters. With different parameter values, it is possible that all types of representative can and does receive majority support, with the first being re-elected continually over time. It may be that, in such examples, a representative must find some "compromise" policy sufficient to ensure re-election but not too far from her ideal. Or it may be that a representative can win by simply choosing her ideal policy.

In the latter case, the first individual to hold office will remain there, choosing the same policy in every period, demonstrating that an extreme form of stability, or "policy persistence," can occur in the model. We prove that, when non-policy benefits of holding office are sufficiently high or individuals are sufficiently patient (and non-policy benefits are positive) all simple equilibria exhibit such policy persistence. Patience on the part of the voters and representatives can lead to this stability in any number of dimensions, even in the absence of a core point. If non-policy benefits are zero, then it turns out that patience leads to policy persistence *unless* there is a core point, so that the presence of a core point can actually be a destabilizing force. Even then, however, we are able to show that the set of policies acceptable to a majority of voters collapses to the core as patience increases. When patience is great enough, therefore, either policy persistence obtains, or the long run distribution of policies is concentrated arbitrarily closely to the core.

We then examine the connection between simple equilibrium policies and the core, especially in the one-dimensional special case of the model. We first show it is possible that all representatives choose the same policy in equilibrium, a phenomenon we call "policy coincidence," only if non-policy office benefits are sufficiently high, individuals are sufficiently patient, and a core point exists. In that case, all representatives must choose the core point, and we say that the equilibrium exhibits "core equivalence." In one dimension, of course, the core is always non-empty and consists of the median voter's ideal point. We show that, with sufficient benefits of office or sufficient patience (with positive non-policy benefits), there is a *unique* simple equilibrium. In it, all representatives choose the median, giving us full core equivalence and new game-theoretic foundations for the original Downsian median voter theorem — but in a fully dynamic model of elections with asymmetric information and no commitment. If holding office confers no non-policy benefit, core equivalence need not obtain, but we show that, as voters become more patient, the set of policies that ensure re-election, and therefore the long run distribution of equilibrium policies, collapses to the median. Finally, we extend our core equivalence result to multidimensional settings in which voters have quadratic utility functions and the core is non-empty. Equivalence does not hold generally if the assumption of quadratic utilities is dropped, but a result we prove on the continuity of simple equilibria implies that if utilities are "close" to being quadratic then simple equilibrium policies will be "close" to the core. As well, if utilities are close to being quadratic and close to admitting a core point, then continuity implies that all simple equilibrium policies will be close to one another.

In multiple dimensions, where the core is typically empty, it follows that policy coincidence (and therefore core equivalence) will be the exception. Thus, in equilibrium, some representatives choose distinct policies. Then, when voters are sufficiently patient and non-policy benefits of office are sufficiently high, our policy persistence result implies that multiple policies can be sustained in equilibrium. Such a conclusion comes not from a multiplicity of equilibria, but rather from the possibility that representatives with different policy preferences have the willingness and ability to attract and maintain different majority coalitions within a single equilibrium. In this way, when the policy space is multidimensional, two electorates with identical voter preferences can be associated with distinct stable policies.

We end with a discussion of the relationship between the predictions of our model and a repeated version of Downsian competition, finding similarities in one dimension but differences in several dimensions. We examine the relationship between our concept of simple equilibrium and perfect Bayesian equilibrium. And we touch on several extensions of our model.

Before proceeding, the connections between our paper and three others are noteworthy. The structure of our model is similar to that of Duggan (2000), with the key differences being that the latter assumes a one-dimensional policy space and "tent-shaped" Euclidean distance utilities. The existence of simple equilibria is proved, and it is shown that, in all such equilibria, the median voter is decisive: a policy choice by an officeholder secures re-election if and only if it gives the median voter a payoff at least equal to the median's expected payoff from electing a challenger. Alesina (1988) also assumes a one-dimensional policy space in a repeated elections setting, but in a two-candidate, simultaneous-move model without commitment. The preferences of the candidates are known to the voters, and include both policy and non-policy components. He shows that, when discount factors are high enough, a range of policy outcomes can be sustained in equilibrium when voters and candidates employ trigger strategies of a certain form. Finally, Kramer (1977) studies a two-candidate model of repeated elections in multiple dimensions that is, otherwise, dissimilar to ours: in any period, the challenger may commit to a policy, while the incumbent is bound to her previous policy choice. Challengers maximize their margin of victory, and politicians and voters are myopic. He shows that, when voters have Euclidean preferences, equilibrium policies converge to the "minmax" set, a set that coincides with the core when the latter is non-empty.

2 The Model

Let $X \subset \Re^d$ denote a compact and convex set of policies, let $N = [0, 1]$ be a continuum of individuals, and let the possible preferences of voters be indexed by a finite set T of types, denoted t. Each individual i's type t_i is drawn from the distribution $\rho = (\rho_1, \ldots, \rho_{|T|})$, where $\rho_t > 0$ is the probability of type t. We extend the idea of independent types to the current model (which posits a continuum of individuals) as follows: the distribution of an individual's type, conditional on the types of any finite number of other individuals, remains ρ. We assume that the law of large numbers holds, so that, for all $t \in T$, the fraction of type t individuals is ρ_t.[2] Any one

[2] We also assume that, for each type t, the set $\{i \in N : t_i = t\}$ is Lebesgue measurable. Judd (1985) establishes the existence of a joint distribution of voter types for which these conditions

individual's type is private information, but the distribution ρ is common knowledge. The preferences of type t individuals are represented by a utility function u_t on X, assumed to be continuous and strictly concave. We normalize payoffs so that $u_t(x) \geq 0$ for all $t \in T$ and all $x \in X$. Let $x_t = \arg\max\{u_t(x) : x \in X\}$ denote the unique ideal policy for type t individuals. Assume that $x_t \neq x_{t'}$ for all $t, t' \in T$, and note that, by strict concavity, $u_t(x_t) > 0$ for all $t \in T$.

Of interest later is the weighted majority voting game among the types in T, with weights given by the proportions $(\rho_1, \ldots, \rho_{|T|})$ of types present in the electorate. Let

$$\mathcal{D} \;=\; \{C \subseteq T : \sum_{t \in C} \rho_t > 1/2\}$$

denote the decisive coalitions of types. We impose the condition that there is no coalition of types $C \subset T$ such that $\sum_{t \in C} \rho_t = 1/2$, i.e., no coalition of types has precisely half of the population. This implies that the voting game is *strong,* in the following sense: for all $C \subseteq T$, either $C \in \mathcal{D}$ or $T \setminus C \in \mathcal{D}$. The *core* is the set, K, of policies that are undominated in this voting game, i.e.,

$$K \;=\; \left\{ x \in X : \begin{array}{c} \text{there do not exist } y \in X \text{ and } C \in \mathcal{D} \\ \text{such that, for all } t \in C, u_t(y) > u_t(x) \end{array} \right\}.$$

Because X is convex and utility functions are strictly concave, it follows that K, if non-empty, will be a singleton. Denote this core policy by x^c. In addition, $K = \{x^c\}$ satisfies the following *external stability* condition: for all $y \neq x^c$, $\{t \in T : u_t(x^c) > u_t(y)\} \in \mathcal{D}$. It is known that, because $\mathcal{D}$ is strong, the core is typically empty when X is multidimensional,[3] but, in *one* dimension, the core is always non-empty and is equal to the ideal policy of the weighted median type. Defining m as the unique element of T satisfying

$$\{t \in T : x_t \leq x_m\} \in \mathcal{D} \quad \text{and} \quad \{t \in T : x_t \geq x_m\} \in \mathcal{D},$$

we therefore have $x^c = x_m$.

are satisfied for almost all realizations of voter types. See Banks and Duggan (2001) for rigorous foundations of this model.

[3]See Banks (1995) and Saari (1997). When types are equally represented in society, the voting game is simply majority rule, and the core is generically empty when $d \geq 2$. For arbitrary weighted majority voting games, more dimensions are needed for this result.

Elections proceed as follows. In period 1, an individual is arbitrarily chosen as representative and selects a policy in X. In each period $\tau = 2, 3, \ldots$, an individual is selected as representative as follows. A challenger is randomly drawn from the uniform distribution on N to run against the incumbent, the representative from period $\tau - 1$. Individuals observe the "name" of the challenger, but not her type. Once the challenger is determined, each individual casts a vote in $\{In, Ch\}$, where In denotes a vote for the incumbent and Ch a vote for the challenger. If the proportion of individuals voting for the incumbent is at least one-half, then the incumbent wins the election and becomes the period τ representative. Otherwise, the challenger wins. The period τ representative selects any policy in X, this selection is observed by the voters, and the game moves to the next period, where this process is repeated. Note that, since the distribution on N is nonatomic, the probability any given individual is chosen to run as challenger in any given period is zero.

A *public history of length* τ, denoted h^τ, describes the publicly observed events in the first τ periods, namely, the individuals chosen as representatives, those chosen as challengers, vote tallies from elections, and policies selected by winners. An infinite public history, h^∞, is an infinite sequence of these variables. In particular, let $\{i^\tau\}$ denote the corresponding sequence of representatives and $\{x^\tau\}$ the sequence of policies. An individual i's payoff from an infinite public history h^∞ is then defined as

$$(1-\delta)\sum_{\tau=1}^{\infty} \delta^{\tau-1}[u_{t_i}(x^\tau) + \omega_i(i^\tau)\beta],$$

where $\delta \in [0, 1)$ is a common discount factor, $\beta \geq 0$ is a common non-policy benefit from being representative, and ω_i is the indicator function on N taking on the value of one if $i = i^\tau$ and zero otherwise.

A strategy for $i \in N$ describes, for any time period τ, a vote $v_i^\tau \in \{In, Ch\}$ and a policy $p_i^\tau \in X$ if selected as representative, both functions of the public history of length $\tau - 1$. Because types are private information, we follow Harsanyi (1967-68) in modelling votes and policy choices as also depending on an individual's type. We focus on equilibria in which the individuals' strategies are especially simple. First, individuals employ *retrospective* voting rules: for all $i \in N$, there exists $\underline{u}_i: T \rightarrow \Re$ such that, for all $t \in T$, all $\tau > 1$, and all $h^{\tau-1}$,

$$v_i^\tau(h^{\tau-1}, t) = In \quad \text{if and only if} \quad u_t(x^{\tau-1}) \geq \underline{u}_i(t).$$

That is, i votes to retain the incumbent if and only if the incumbent's most recent policy choice satisfied the utility standard, or "cut-off," $\underline{u}_i(t)$. This cut-off is time-invariant, consistent with a "What have you done for me lately?" attitude on the part of the voters. Second, individuals' policy choices are *history-independent:* for all $i \in N$, there exists $p_i: T \rightarrow X$ such that, for all $t \in T$, all $\tau > 1$, and all $h^{\tau-1}$,

$$p_i^\tau(h^{\tau-1}, t) \quad = \quad p_i(t).$$

Thus, i chooses the same policy any time she is elected as representative. Note that these two requirements are mutually re-enforcing: if voter strategies depend on history only through the incumbent's last chosen policy, then an incumbent's policy decision problem looks the same in all periods she is selected. Hence, if an individual has an optimal policy strategy, then she necessarily has an optimal strategy that is history-independent. Similarly, if representatives adopt history-independent policies, then knowledge of the last policy chosen by an individual is sufficient for a voter to accurately predict that individual's policy choices in all future periods.

A *simple strategy* for i consists of a pair $\sigma_i = (p_i, \underline{u}_i)$. A simple strategy *profile,* denoted σ, specifies a simple strategy for every individual with the added restriction of type-symmetry of voting strategies: $\sigma = (p_i, \underline{u}_i)_{i \in N}$, where $\underline{u}_i = \underline{u}_j$ for all $i, j \in N$. Abusing notation slightly, let $\underline{u}_t$ denote the cutoff used by all type t voters. Any strategy profile σ induces a probability distribution over infinite histories from the beginning of the game, and with it an expected utility $v_i(\sigma, t)$ for every $i \in N$ and $t \in T$.[4] Since challengers are drawn from the uniform distribution on N, in almost all histories a challenger will not have held office previously. By our independence assumption, therefore, the voters' beliefs about a challenger's type are given by ρ after almost all histories. By our restriction to simple strategies, then, $v_i(\sigma, t)$ is also i's expected utility, or *continuation value,* of replacing the current incumbent with an untried challenger, after almost every history.[5] Further, since individuals of the same type, say t, have a common per-period utility function, a common discount factor, and common beliefs about challengers, they will have the same continuation value, which we henceforth express as $v_t(\sigma)$. Informally, a profile σ^* constitutes a *simple*

[4] See Banks and Duggan (2001) for an explicit construction of this distribution.

[5] We do not consider the probability zero set of histories in which a challenger has previously held office. After such histories, continuation values would be defined to reflect updating based on all relevant information.

equilibrium if, for all $i \in N$ and all $t \in T$, $p_i^*(t)$ is a "best response" whenever i must make a policy choice and $\underline{u}^*(t)$ is a "best response" in every vote. In the next section, we formalize what it means for policy and voting strategies to be "best responses."

3 Simple Equilibria

One approach to analyzing the model of Section 2 would be to look for perfect Bayesian equilibria of this game, but we define a concept of equilibrium that is tailored to our model of elections.[5] We posit that, in attempting to vote optimally, individuals decide to retain or replace the current incumbent based on which candidate gives the higher expected utility. That is, voters consider themselves *pivotal* in the current election.[6] The expected utility from replacing the incumbent for a type t of individual i is simply $v_t(\sigma)$. As for retaining the incumbent, suppose $x \in X$ is the incumbent's policy choice in the previous period. Since individuals are adopting history-independent policy strategies, the incumbent will continue to select x in the current period if retained.[7] If σ determines that the incumbent subsequently be replaced, then the expected utility to i from retaining the current incumbent is $(1-\delta)u_t(x) + \delta v_t(\sigma)$, which is greater than $v_t(\sigma)$ if and only if $u_t(x)$ is greater than $v_t(\sigma)$. If σ determines that the incumbent be forever retained, then the expected utility to i from retaining the current incumbent is simply $u_t(x)$, and so again retaining the incumbent is preferred by i if and only if $u_t(x)$ is greater than $v_t(\sigma)$. Thus, our best response condition for voting strategies is that, for all $t \in T$,

$$\underline{u}_t \;=\; v_t(\sigma).$$

Note that, while we have described the voters' strategies as "retrospective" because votes are determined by simple cut-off rules, they are actually "prospective" as well in

[5]See Section 9.2 for a discussion of how our concept can be reformulated as a perfect Bayesian equilibrium.

[6]Baron and Kalai (1993), in a model with a finite number of voters, refer to such strategies as "stage-undominated." With a continuum of voters, no voter will ever be pivotal, but our equilibrium condition captures the same intuition.

[7]Strictly speaking, Bayesian updating based on σ can only be applied following histories consistent with σ. In particular, voters will predict that the incumbent will continue to select x only if $p_i = x$ for some $i \in N$ and the incumbent has never chosen a different policy. Otherwise, we are "off the path of play," and beliefs about the incumbent's type must be provided. We return to this issue later.

equilibrium: an individual votes for an incumbent only when retaining the incumbent generates a higher expected future payoff than that generated by replacing her.

An alternative view of the voters' decisions is that they are attempting to solve what might be called a stationary "collective" bandit problem, with an infinite number of arms (i.e., potential representatives) and where the payoff-relevant information about any arm is revealed after one play. Thus, at the beginning of a period, each voter knows with certainty the payoff associated with the incumbent, and chooses to either stay with the incumbent or select a previously untried individual. By history-independence, if a majority finds it optimal to stay with the incumbent in the current period, then they will find it optimal in all future periods as well. And, by the additional assumption of an infinite pool of potential representatives, if a majority finds it optimal to replace the incumbent, then they will find it optimal to replace any future incumbents who have chosen the same policy. Of course, what distinguishes the current environment from the existing bandit literature (cf. Banks and Sundaram 1992) is the *collective* nature of the decision making: a voter's continuation value has contained in it not the optimal future choices of a single individual, but rather future collective decisions on whether to retain or replace the incumbent. That is, while our weak dominance notion essentially means that any one voter considers herself pivotal in the *current* decision to retain or replace the incumbent, she uses the information from σ to predict the *future* collective decisions (in which she is not pivotal).

Given that individuals of the same type adopt common cut-off rules and that ρ_t is the actual proportion of type t voters, the voting stage, from the perspective of the candidates, is simply a weighted voting game among the types in T, with decisive coalitions $\mathcal{D}$. This simplifies the statement of the best response condition on the policy strategies, because an incumbent is retained if and only if the set of types voting for the incumbent is in $\mathcal{D}$. For each $t \in T$, let

$$A_t(\sigma) \;=\; \{x \in X : u_t(x) \geq \underline{u}_t\}$$

denote the *acceptance set* for type t individuals, i.e., those policies satisfying the cut-off $\underline{u}_t$ and inducing all type t individuals to vote for the incumbent. By the compactness and convexity of X and the continuity and concavity of u_t, this set is compact and convex. For each coalition $C \subseteq T$ of types, define the set

$$A_C(\sigma) \;=\; \bigcap_{t \in C} A_t(\sigma)$$

of those policies inducing all types $t \in C$ to vote for the incumbent. As the intersection of compact and convex sets, $A_C(\sigma)$ is compact and convex as well. Finally, define

$$A(\sigma) \;=\; \bigcup_{C \in \mathcal{D}} A_C(\sigma)$$

as those policies that receive majority support and will, therefore, lead to re-election of the incumbent. This *social acceptance set* is compact but not necessarily convex (cf. Example 2 below).

Suppressing for the moment the dependence of the set A on the profile σ, the choice for the type t of individual i when representative is to either select a policy $x \in A$ (if non-empty), in which case she is retained for the next period, or select a policy $x \notin A$ and subsequently be replaced. Clearly, choosing any $x \neq x_t$ from outside of A is dominated by simply choosing x_t, so we will ignore that option. Additionally, if $x_t \in A$, then she will optimally select this as her policy in all periods and remain as incumbent forever, and we will only consider such strategies. Otherwise, i.e., when $x_t \notin A$, the representative faces a trade-off: selecting x_t in the current period and being replaced versus choosing a u_t-maximizing policy from A and being retained. The payoff from choosing $x_t \notin A$ is equal to

$$(1-\delta)[u_t(x_t)+\beta]+\delta v_t(\sigma),$$

reflecting the one-time payoff from the representative's ideal point, followed by the continuation value of an untried challenger thereafter. Further, if choosing from A is optimal in the current period, then it will remain so in all future periods, and any u_t-maximizing policy from A will remain u_t-maximal in all future periods. Our best response requirement on policy strategies is, therefore, that (i) when the representative prefers to remain in office, i.e.,

$$\sup\{u_t(x) : x \in A\} + \beta \;\;>\;\; (1-\delta)[u_t(x_t)+\beta]+\delta v_t(\sigma),$$

we have

$$p_i(t) \in M_t(A) \equiv \arg\max\{u_t(x) : x \in A\},$$

(ii) when the above inequality is reversed, $p_i(t) = x_t$ (and the representative is replaced in the next period), and (iii) when equality holds, $p_i(t) \in M_t(A) \cup \{x_t\}$. Note that, since A_C is compact and convex and u_t is strictly concave, the set $\arg\max\{u_t(x) : x \in A_C\}$ will be a singleton for each coalition C of types. Since $M_t(A)$ is a subset of the (finite) union of these sets over $C \in \mathcal{D}$, the set $M_t(A)$ will be finite for all $t \in T$. This completes our definition of simple equilibrium.

The forgoing shows how representatives, themselves members of the electorate, take into consideration the future policy consequences — even after being removed from office — of their current policy decisions. By choosing her best available policy from the social acceptance set A, a representative can guarantee that this policy remains in effect forever. Alternatively, she can choose from outside of A, with the future policy consequences of such an act summarized by σ. Which of these two options is preferred then depends on the location of her best policy in A relative to her ideal policy (i.e., her best policy in X) and her continuation value, as well as the value of future policies relative to those of the present (represented by the discount factor δ) and the non-policy benefits of remaining in office (β).

As with the voting strategies, it is convenient to characterize the policy strategies at the type level rather than the individual level, although the fact that the set $M_t(A)$ may contain more than one element means that individuals of the same type, while facing identical decision problems, may be adopting different policies. Thus, we characterize the policies adopted by type t individuals as a probability measure on X. Let $\mathcal{P}(X)$ denote the set of Borel probability measures on X, endowed with the topology of weak convergence. Then, for any measurable set $Y \subseteq X$ and any $\pi_t \in P(X)$, we interpret $\pi_t(Y)$ as the proportion of type t individuals adopting a policy in Y, and we write $\pi = (\pi_1, \ldots, \pi_{|T|})$ as a profile of type-specific policy strategies.[8] The above best response condition can thus be rewritten as follows: (i) when

$$\sup\{u_t(x) : x \in A\} + \beta \quad > \quad (1-\delta)[u_t(x_t) + \beta] + \delta v_t(\sigma),$$

the requirement is that $\pi_t(M_t(A)) = 1$, (ii) when the inequality is reversed, $\pi_t(\{x_t\}) =$

[8]Formally, $\pi_t(Y)$ is the Lebesgue measure of the set $\{i \in N : t_i = t \text{ and } p_i(t) \in Y\}$, which is measurable given our assumptions. Alternatively, we could have imposed *ex ante* type-symmetric policy choice strategies by allowing representatives to mix over policies in the first period of office.

1, and (iii) when equality holds, $\pi_t(M_t(A) \cup \{x_t\}) = 1$. Let $S(\sigma)$ denote the *support* of the policy strategies in σ, i.e., the smallest closed subset of X with probability one under π_t for all $t \in T$. Since $M_t(A)$ is finite for all t and there is a finite number of types, $S(\sigma)$ will be finite in equilibrium.

Given a simple strategy profile σ, a type t individual's continuation value satisfies

$$v_t(\sigma) = \sum_{t' \in T} \rho_{t'} \left[[1 - \pi_{t'}(A(\sigma))][(1-\delta)u_t(x_{t'}) + \delta v_t(\sigma)] + \int_{A(\sigma)} u_t(x)\pi_{t'}(dx) \right].$$

The first term in the brackets is the probability that the current representative chooses from outside $A(\sigma)$ multiplied by t's expected payoff in that case, which is simply one period of the representative's ideal policy followed by her removal and subsequently "starting over." The second (integral) term gives t's expected payoff if the current representative selects from $A(\sigma)$, in which case, by history-independence, the latter will make the same decision and be re-elected in all future periods. Rewriting this equation to get an explicit solution, we have

$$v_t(\sigma) = \frac{\sum_{t' \in T} \rho_{t'} \left[[1 - \pi_{t'}(A(\sigma))](1-\delta)u_t(x_{t'}) + \int_{A(\sigma)} u_t(x)\pi_{t'}(dx). \right]}{1 - \delta \sum_{t' \in T} \rho_{t'}[1 - \pi_{t'}(A(\sigma))]},$$

which is a convex combination of the one-period payoffs to t conditional on representatives choosing from outside $A(\sigma)$ (i.e., $u_t(x_{t'})$) and from inside $A(\sigma)$ (i.e., $\int_{A(\sigma)} u_t(x)\pi_{t'}(dx)/\pi_{t'}(A(\sigma))$). Thus, $v_t(\sigma)$ can be written as the expectation of u_t with respect to a probability distribution over X, where elements in $X \backslash A(\sigma)$ receive relatively less weight (by a factor of $1-\delta$) as these policies are "temporary," whereas policies in $A(\sigma)$ are "permanent."[9] Now define

$$x(\sigma) = \frac{\sum_{t' \in T} \rho_{t'} \left[[1 - \pi_{t'}(A(\sigma))](1-\delta)x_{t'} + \int_{A(\sigma)} x\pi_{t'}(dx) \right]}{1 - \delta \sum_{j \in T} \rho_{t'}[1 - \pi_{t'}(A(\sigma))]},$$

which is a similarly weighted average of equilibrium *policies*. Thus, $x(\sigma)$ is the expected outcome associated with the probability distribution over X induced by σ.

[9] Formally, we define the "continuation" distribution of σ, denoted ψ, as follows: for measurable $Y \subseteq X$,

$$\psi(Y) = \frac{\sum_{t \in T} \rho_t[(1-\pi_t(A))(1-\delta)\mu_{x_t} + \pi_t(Y \cap A)]}{1 - \delta \sum_{t \in T} \rho_t(1 - \pi_t(A))},$$

where μ_{x_t} is the point mass on x_t.

By strict concavity of utility functions, $u_t(x(\sigma)) \geq v_t(\sigma)$ for all $t \in T$, with this inequality strict unless all individuals of all types choose the same policy when in office. Therefore, $x(\sigma) \in A_t(\sigma)$ for all $t \in T$ whenever σ satisfies the best response condition for voters, and so the set $A(\sigma)$ of policies that lead to re-election will be non-empty.

In equilibrium, therefore, there will always exist policies representatives could choose to ensure reelection. The question is whether they find it optimal to do so. With this in mind, given a simple equilibrium σ^*, partition the set T of types into three subsets, W ("winners"), L ("losers") and C ("compromisers") as follows:

$$\begin{aligned} W(\sigma^*) &= \{t \in T : x_t \in A^*\} \\ L(\sigma^*) &= \{t \in T : x_t \notin A^* \text{ and } \pi_t^*(\{x_t\}) > 0\} \\ C(\sigma^*) &= \{t \in T : x_t \notin A^* \text{ and } \pi_t^*(M_t(A^*)) = 1\}, \end{aligned}$$

where $A^* = A(\sigma^*)$ in the above. Thus, winning types find their ideal policy acceptable to a majority, and so implement this policy in all periods. Compromising types are not so fortunate, but they still find *some* acceptable policy as good as choosing their ideal policy and subsequently being replaced, and they always choose such a policy. Finally, losing types have the opposite preference, in that *no* acceptable policy is better than simply choosing their ideal policy and subsequently being replaced, and a positive fraction of these types do choose the latter option. In the next section, we show by way of a series of examples that any one of these sets, or even two, may be empty in equilibrium.

This is of interest because the emptiness or non-emptiness of these sets largely determines the equilibrium dynamics of elections in our model. In particular, if $L(\sigma^*) = \emptyset$, then all representatives choose policies in the social acceptance set. The first individual to hold office is therefore re-elected, and, by history-independence, remains in office forever, implementing the same policy in each period. We refer to this as *perfect policy persistence.* In this case, the voters' continuation values take on a quite simple form, as now everyone knows that, if the incumbent is removed, then whatever policy is chosen next will remain in place forever. Thus, we can rewrite $v_t(\sigma^*)$ as a convex combination of utilities on $S(\sigma^*)$, the (finite) set of policies adopted in equilibrium:

$$v_t(\sigma^*) = \sum_{t' \in T} \rho_{t'} \left[\sum_{x \in S(\sigma^*)} \pi^*_{t'}(\{x\}) u_t(x) \right].$$

One consequence of this is that

$$\max\{u_t(x) : x \in S(\sigma^*)\} \geq v_t(\sigma^*) \geq \min\{u_t(x) : x \in S(\sigma^*)\}. \tag{1}$$

Suppose $|S(\sigma^*)| > 1$. By strict concavity of u_t, we then have $u_t(x(\sigma^*)) > v_t(\sigma^*)$ for all $t \in T$, and, since $x(\sigma^*) \in A^*$, this implies

$$\max\{u_t(x) : x \in A^*\} > v_t(\sigma^*).$$

This, together with the assumption that $\rho_t > 0$, implies both of the inequalities are strict in (1). In particular,

$$v_t(\sigma^*) > \min\{u_t(x) : x \in S(\sigma^*)\},$$

from which we conclude that, for all $t \in T$, there exists $p^* \in S(\sigma^*)$ such that $u_t(p^*) < v_t(\sigma^*)$. That is, even when all types compromise, each type votes against some of the equilibrium policies, and so against some types of incumbent (cf. Example 2). Put differently, "you can't please *any* of the people *all* of the time."[11]

On the other hand, if $L(\sigma^*) \neq \emptyset$, then the first representative and any newly elected challenger will, with positive probability, choose a losing policy and be replaced in the following period. As long as it is not the case that $\pi^*_t(A^*) = 0$ for all types, however, a representative will eventually be elected (with probability one) and choose a policy in the social acceptance set, where again this policy remains in place forever. We call this *eventual policy persistence.* When it obtains, the long run distribution of policy outcomes puts probability one on the social acceptance set, yet, by its nature, this distribution cannot give a measure of how significant the initial periods of policy variability might be. We can construct such a measure as follows. Let

$$\alpha(\sigma^*) = \sum_{t \in T} \rho_t \pi_t(A^*)$$

[11]See Example 2, in the following section. It is important to note that this does not imply that each equilibrium policy receives some negative votes: see Example 3, in which the policy offered by the centrally located type is accepted by all.

denote the probability a challenger chooses an acceptable policy. Given any value of $\alpha \in (0,1)$, the probability of employing exactly n representatives is equal to $\alpha(1-\alpha)^{n-1}$, and so the expected number of representatives, which is equal to the expected number of periods until an acceptable policy is chosen, is given by

$$\sum_{n=1}^{\infty} n\alpha(1-\alpha)^{n-1} = \frac{1}{\alpha}.$$

Therefore $1/\alpha(\sigma^*)$ gives the expected duration of policy variability in the equilibrium σ^*.

4 Examples

Example 1: *"all losers" equilibrium*

Let $d = 2$, $|T| = 3$, $u_t(x) = 1 - (||x_t - x||)^2$, $\beta = 0$, $\rho_t = 1/3$ for all t, $1 = ||x_t - x_{t'}||$ for all $t, t' \in T$. Assuming all individuals propose their ideal policy and subsequently are replaced, the continuation value for any individual is given by $v_t = (1/3)(1) + (2/3)(0) = 1/3$, i.e., in all periods there is a 1/3 chance of having their ideal policy being chosen, generating a utility of 1, and a 2/3 chance of some other type's ideal policy being chosen, generating a utility of 0. What needs to be checked is that individuals in their role as representative prefer this losing strategy to compromising. For a type t individual, the closest point in A^* to x_t is $(1-\sqrt{1/3})$ away, since all individuals have continuation value equal to 1/3 and utility is quadratic. Thus, t can either lose and receive $(1-\delta)(1)+\delta(1/3)$, or compromise and receive $1-(1-\sqrt{1/3})^2$. Grinding through the algebra, we see that losing is preferred as long as $\delta < 2 - 3\sqrt{1/3} \approx .28$. Further, since losing is *strictly* preferred, the equilibrium is unaffected if β is positive and small enough.

Example 2: *"all compromisers" equilibrium*

Let the parameter values be the same as in Example 1, except for δ. Consider Figure 1, from Baron's (1991) model of spatial bargaining.

[Figure 1 here.]

We claim that the following constitutes an equilibrium for δ sufficiently large: all type 1 individuals select policy a and set $\underline{u}_1 = u_1(c)$, all type 2 individuals select policy c

and set $\underline{u}_2 = u_2(e)$, and all type 3 individuals select e and set $\underline{u}_3 = u_3(a)$. Given these cut-offs, each type is optimizing conditional on choosing from A^*, and, further, if individuals adopt these policy strategies, then their cut-offs are indeed equal to their continuation values. Thus, what remains to be checked is whether representatives are optimizing by selecting from A^*, rather than choosing their ideal points. By symmetry, we need only check this condition for one type, say type 1. The relevant comparison is between choosing $p = a$ and remaining in office forever, and choosing $p = x_1$ and being replaced in the following period. The utility of the former is equal to $u_1(a)$, while the utility of the latter is $(1-\delta)(1) + \delta u_1(c) = 1 - \delta(1 - u_1(c))$. Thus, a type 1 individual prefers to compromise whenever

$$u_1(a) \;\geq\; 1 - \delta(1 - u_1(c)),$$

or equivalently,

$$\delta \;\geq\; \frac{1 - u_1(a)}{1 - u_1(c)}.$$

Since $1 > u_1(a) > u_1(c) > 0$, the right-hand side of the above expression lies in $(0, 1)$, and, when δ is above this amount, we have an equilibrium.[12]

Example 3: *"all winners" equilibrium*

Let $d = 2$, $|T| = 5$, $u_t(x) = 16 - (||x_t - x||)^4$, $\rho_t = 1/5$ for all t, $x_1 = (1, 0)$, $x_2 = (0, 1)$, $x_3 = (-1, 0)$, $x_4 = (0, -1)$, $x_5 = (0, 0)$. Note that the ideal point of type 5 is the core point. Assuming all individuals propose their ideal policy and have it accepted, the continuation value for types 1-4 is given by $v_t = (1/5)(16) + (1/5)(15) + (2/5)(12) + (1/5)(0) = 11$, while the continuation value for a type 5 is $v_5 = (1/5)(16) + (4/5)(15) = 15.8$. Hence, type 5 individuals only vote to re-elect their own, and so an individual of, e.g., type 1 must secure the votes of types 2 and 4 to be re-elected. Since $u_2(x_1) = u_4(x_1) = 12 > 11$, type 2 and 4 individuals indeed vote to re-elect type 1 representatives even when they propose their ideal policy. Similarly, types 2 and 4 vote to re-elect type 3 representatives when they propose their ideal policy, and types 1 and 3 vote to re-elect type 2 and type 4 representatives. Finally,

[12]Note that by the symmetry of the environment we actually have, as in Baron (1991), another equilibrium where type 1's select b, type 2's select d and type 3's select f, as well as an equilibrium where half the type 1's select a and the other half b, half the type 2's c and the other half d, and half the type 3's e and the other half f.

$p_5 = x_5$ is acceptable to all types. Since all individuals are implementing their ideal policy when chosen as representative *and* remaining as incumbent forever, the policy strategies are clearly optimal, and we therefore have an equilibrium. A distinguishing feature of this strategy profile is that it constitutes an equilibrium for *every* value of δ and β, regardless of time preferences or non-policy benefits.

Example 4: *"mixed" equilibrium*

Let $X = [-1, 1]$, $|T| = 5$, $\rho_t = 1/5$ for all t, $\beta = 0$, $u_t(x) = 4 - ||x - x_t||^2$, $x_1 = -1$, $x_2 \in (-1, 0)$, $x_3 = 0$, $x_4 \in (0, 1)$, $x_5 = 1$. We construct an equilibrium in which type 1 and 5 individuals lose, type 2 and 4 individuals compromise at $-c$ and c respectively (where $c \in (0, 1)$), and type 3 individuals win. With quadratic utilities and a single dimension, one can show that type 3 individuals are decisive, in the sense that a proposal will satisfy a majority if and only if it satisfies the median voter (see Banks and Duggan 2001, Lemma 2.1), so we only check this continuation value: $v_3 = [(2/5)(1 - \delta)(3) + (1/5)(4) + (2/5)(4 - c^2)]/(1 - (2/5)\delta)$. For a type 3 individual to be indifferent between accepting and rejecting c, we set v_3 equal to $4 - c^2$. Grinding through the algebra, we find the desired value for c is

$$c(\delta) = \sqrt{\frac{2 - 2\delta}{3 - 2\delta}}.$$

Note that $c(1) = 0$, $c(0) = \sqrt{2/3}$, and $c' < 0$. Since $c(\delta)$ is bounded away from 1, there exists a positive δ, say δ^+, for which type 1 and 5 individuals prefer to lose rather than compromise. Now set x_2 slightly to the left of $-c(\delta^+)$ and x_4 slightly to the right of $c(\delta^+)$, so that type 2 and 4 individuals prefer to compromise. See Figure 2.

[Figure 2 here.]

The equilibria in Examples 2 and 3 exhibit perfect policy persistence, in that the first representative remains as incumbent forever by choosing the same acceptable policy in every period. In contrast, the equilibria in Example 4 exhibits eventual policy persistence: only types 2, 3, and 4 choose acceptable policies, and so there will exist policy variability until such a type is elected. Since all types are equally likely, the value of the parameter α discussed in the previous section is simply equal to $\rho_2 + \rho_3 + \rho_4 = 3/5$. Therefore, the expected number of representatives employed, or equivalently the expected time until policy stability is established, is 5/3. Note that the point at which policies eventually come to rest may be the median (with probability 1/3) but may not be (with probability 2/3).

5 Existence and Continuity

Each of the examples above possesses at least one simple equilibrium. Our first result establishes existence of equilibrium generally.

Theorem 1 *There exists a simple equilibrium.*

Proof: We first prove existence of an equilibrium in a modified version of the above game, and we then argue that any equilibrium of the modified game corresponds to an equilibrium in the original game. Augment the set of options available to a representative to include a "shirk" option, s, interpreted as choosing her ideal point and then sitting out the next election. If the current incumbent uses the shirk option, therefore, the voters *must* choose the challenger in the next period. We focus on equilibria in which a representative chooses the shirk option whenever her optimal choice would lose the next election, i.e., if a representative would choose her ideal point and that policy is not in the social acceptance set, then she chooses to shirk. These equilibria are distinguished from others in that representatives foresee the result of choosing $x_t \notin A^*$, taking the initiative by choosing s and declining to run, instead of choosing x_t and forcing voters to replace them.

A policy strategy for type t individuals is now a Borel probability measure $\tilde{\pi}_t$ on $\tilde{X} = X \cup \{s\}$.[13] Given a profile $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_{|T|})$, and assuming that all future representatives who do not shirk are re-elected, the continuation value of electing a challenger for a type t voter can be expressed as a function of $\tilde{\pi}$ only:

$$v_t(\tilde{\pi}) \quad = \quad \sum_{t' \in T} \rho_{t'} \left[\tilde{\pi}_{t'}(\{s\})[(1-\delta)u_t(x_{t'}) + \delta v_t(\tilde{\pi})] + \int_X u_t(x)\tilde{\pi}_{t'}(dx) \right],$$

implying

$$v_t(\tilde{\pi}) \quad = \quad \frac{\sum_{t' \in T} \rho_{t'} \left[\tilde{\pi}_{t'}(\{s\})(1-\delta)u_t(x_{t'}) + \int_X u_t(x)\tilde{\pi}_{t'}(dx) \right]}{1 - \delta \sum_{j \in T} \rho_{t'} \tilde{\pi}_{t'}(\{s\})}.$$

[13]We define $\tilde{Y} \subseteq \tilde{X}$ to be open if $Y \subseteq \tilde{Y} \subseteq Y \cup \{s\}$ for some open $Y \subseteq X$.

Note that v_t is a continuous function of $\tilde{\pi}$ with the topology of weak convergence on $\mathcal{P}(\tilde{X})$, the Borel probability measures on X. We look for an equilibrium in terms of policy strategies only, since individuals vote for the incumbent if and only if the continuation value of the incumbent is at least that of a challenger. That is, a type t individual votes to re-elect if and only if the incumbent chose a policy in the set

$$A_t(\tilde{\pi}) \quad = \quad \{x \in X : u_t(x) \geq v_t(\tilde{\pi})\}.$$

For each $t \in T$, the set $A_t(\tilde{\pi})$ is non-empty, compact, and convex by the continuity and concavity of u_t. If $\tilde{\pi}_t(\{s\}) > 0$, then let

$$y(\tilde{\pi}_t) \quad = \quad \frac{\int_X x \tilde{\pi}_t(dx)}{1 - \tilde{\pi}_t(\{s\})}$$

denote the expected outcome associated with a type t incumbent conditional on the incumbent not shirking. If $\tilde{\pi}_t(\{s\}) = 0$, then let $y(\tilde{\pi}_t)$ be defined arbitrarily. By the concavity of u_t, the policy

$$x(\tilde{\pi}) \quad = \quad \frac{\sum_{t' \in T} \rho_{t'} \left[\tilde{\pi}_{t'}(\{s\})(1-\delta)x_{t'} + (1 - \tilde{\pi}_{t'}(\{s\}))y(\tilde{\pi}_{t'})\right]}{1 - \delta \sum_{t' \in T} \rho_{t'} \tilde{\pi}_{t'}(\{s\})}$$

therefore satisfies $u_t(x(\tilde{\pi})) \geq v_t(\tilde{\pi})$, and hence $x(\tilde{\pi}) \in A_t(\tilde{\pi})$, for all t. As in Section 2, for all $C \in \mathcal{D}$, define $A_C(\tilde{\pi}) = \bigcap_{t \in C} A_t(\tilde{\pi})$, also non-empty, compact, and convex. And define $A(\tilde{\pi}) = \bigcup_{C \in \mathcal{D}} A_C(\tilde{\pi})$, non-empty and compact but not necessarily convex. By an argument similar to that found in Banks and Duggan (2000), one can show that $A(\cdot)$ is a continuous correspondence on $[\mathcal{P}(\tilde{X})]^T$.

Given $\tilde{\pi}$, an incumbent chooses a policy or shirks so as to maximize her discounted expected payoff. Thus, define $U_t(\cdot; \tilde{\pi}): \tilde{X} \to \Re$ by

$$U_t(x; \tilde{\pi}) \quad = \quad \begin{cases} (1-\delta)[u_t(x_t) + \beta] + \delta v_t(\tilde{\pi}) & \text{if } x = s, \\ u_t(x) + \beta & \text{otherwise,} \end{cases}$$

and let

$$M_t(\tilde{\pi}) \quad \equiv \quad \arg\max\{U_t(x; \tilde{\pi}) : x \in A(\tilde{\pi}) \cup \{s\}\}.$$

Because $A(\cdot) \cup \{s\}$ is a continuous correspondence, the Maximum Theorem implies that the correspondence $M_t: [P(\tilde{X})]^T \to\to \tilde{X}$ has non-empty and compact values, and it is upper hemicontinuous. It is not necessarily convex-valued, however, since

$A(\tilde{\pi}) \cup \{s\}$ is not convex. Let $B_t(\tilde{\pi}) = \mathcal{P}(M_t(\tilde{\pi}))$ denote the set of probability measures over optimal choices, which defines a non-empty, compact- and convex-valued, upper hemicontinuous correspondence. Define the correspondence $B\colon [P(\tilde{X})]^T \to\to [P(\tilde{X})]^T$ by

$$B(\tilde{\pi}) \;=\; B_1(\tilde{\pi}) \times B_2(\tilde{\pi}) \times \cdots \times B_{|T|}(\tilde{\pi}),$$

which inherits these properties. Since $[P(\tilde{X})]^T$ is compact and convex, Glicksberg's (1952) theorem yields a fixed point of B, say $\tilde{\pi}^* = (\tilde{\pi}_1^*, \ldots, \tilde{\pi}_{|T|}^*)$. Then $\tilde{\pi}^*$, together with cut-off rules

$$\underline{u}_t^* \;=\; v_t(\tilde{\pi}^*),\; t = 1, \ldots, |T|,$$

constitutes an equilibrium of the augmented game in which individuals either shirk or are re-elected. Finally, it is easy to see how equilibria in the augmented game translate into equilibria of the original game: for all $t \in T$, set $\pi_t^*(\{x_t\}) = \tilde{\pi}_t^*(\{s\})$, and, for all measurable $Y \subseteq A(\tilde{\pi}^*)$, set $\pi_t^*(Y) = \tilde{\pi}_t^*(Y)$. ∎

In proving existence, we need to allow for the possibility that individuals of the same type adopt different policies while in office, i.e., we allow "type-asymmetry" with respect to policy choices. This comes about because, as seen in Example 2, the social acceptance set $A(\sigma)$ need not be convex, and so we may have a situation in which two distinct policies $x, y \in A(\sigma)$ are optimal for type t individuals and yet no convex combination of x and y is in $A(\sigma)$. In addition, even if $A(\sigma)$ *is* convex, type t individuals may be indifferent between choosing (optimally) from $A(\sigma)$ and choosing $x_t \notin A(\sigma)$, with no convex combination giving as high a payoff. Allowing some type t individuals to choose one policy when in office while others choose another, and then having these proportions determined in equilibrium, effectively smooths out, or "convexifies," representative behavior from the perspective of the voters.

We next show that the set of equilibrium policies changes in a nice way as one varies the underlying parameters of the model. So far these parameters include the type distribution ρ, which we assume lies in the set

$$\Delta_\circ \;=\; \{\rho : \forall\, C \subseteq T, \sum_{t \in C} \rho_t \neq 1/2\},$$

the common discount factor $\delta \in [0,1)$, and the common non-policy benefit $\beta \in \Re_+$. To these we add information about the type-specific utility functions, so as to evaluate the effects of changing preferences on the equilibrium policies. We do this by parameterizing the utility functions as $u_t(x) = u_t(x, \lambda)$, where λ lies in $\Lambda \subset \Re^k$. For instance, all types could have quadratic utilities, with $\lambda = (x_1, \ldots, x_{|T|})$ being the vector of ideal points. More generally, λ might consist of ideal points and matrices defining weighted Euclidean distance (cf. Hinich and Munger 1997). We assume that each u_t is jointly continuous in its arguments.

Let $E(\rho, \delta, \beta, \lambda)$ denote the set of simple equilibrium policy strategy profiles given parameters ρ, δ, β, and λ. Thus, we can view E as a correspondence from the parameter space $\Delta_\circ \times [0,1) \times \Re_+ \times \Lambda$ into the space of profiles of probability measures on policies, $[\mathcal{P}(X)]^T$. We say that E is *upper hemicontinuous* if, for every $(\rho, \delta, \beta, \lambda)$ in this space and for every open set $Y \subseteq [\mathcal{P}(X)]^T$ with $E(\rho, \delta, \beta, \lambda) \subseteq Y$, there exists an open set $Z \subseteq \Delta_\circ \times [0,1) \times \Re_+ \times \Lambda$ with $(\rho, \delta, \beta, \lambda) \in Z$ such that, for all $(\rho', \delta', \beta', \lambda') \in Z$, $E(\rho', \delta', \beta', \lambda') \subseteq Y$. In words, "small" variations in the parameters cannot lead the set of equilibrium policies to "blow up."

Theorem 2 *The correspondence E is upper hemicontinuous.*

Proof: We first consider the augmented game defined in the proof of Theorem 1. Given parameters $(\rho, \delta, \beta, \lambda)$, with $\rho \in \Delta_\circ$, and strategy profile $\tilde{\pi} = (\tilde{\pi}_1, \ldots, \tilde{\pi}_{|T|})$, let θ denote the vector $(\rho, \delta, \beta, \lambda, \tilde{\pi})$, and define

$$A_t(\theta) \quad = \quad \{x \in X : u_t(x, \lambda) \geq v_t(\tilde{\pi}, \rho, \delta, \lambda)\}$$

where v_t is a type t individual's continuation value as defined above but using $u_t(\cdot, \lambda)$. By an argument similar to that in Banks and Duggan (2000), we can show that

$$A(\theta) \quad \equiv \quad \bigcup_{C \in \mathcal{D}(\rho)} \left[\bigcap_{t \in C} A_t(\theta) \right]$$

is continuous as a correspondence at θ, where, since $\rho \in \Delta_\circ$, $\mathcal{D}(\rho)$ is constant on an open set containing ρ. Define $U_t(\cdot; \theta) \colon \tilde{X} \to \Re$ by

$$U_t(x; \theta) \quad = \quad \begin{cases} (1-\delta)[u_t(x, \lambda) + \beta] + \delta v_t(\tilde{\pi}, \rho, \delta, \lambda) & \text{if } x = s, \\ u_t(x, \lambda) + \beta & \text{otherwise.} \end{cases}$$

Since $U_t(x;\theta)$ is continuous in (x,θ), the Maximum Theorem implies that

$$M_t(\theta) \equiv \arg\max\{U_t(x;\theta) : x \in A(\theta) \cup \{s\}\}$$

is upper hemicontinuous at θ, and therefore so is $B_t(\theta) \equiv \mathcal{P}(M_t(\theta))$. Since B_t has closed values and regular range as well, it has closed graph at θ (Aliprantis and Border 1994, Theorem 14.17). Now let $(\rho^m, \delta^m, \beta^m, \lambda^m) \to (\rho, \delta, \beta, \lambda) \in \Delta_\circ \times [0,1) \times \Re_+ \times \Lambda$, take any sequence $\{\pi^m\}$ of policy choice profiles in the original game such that $\pi^m \in E(\rho^m, \delta^m, \beta^m, \lambda^m)$ for all m, and suppose $\pi^m \to \pi$. Transform these into policy choice profiles, $\{\tilde{\pi}^m\}$ and $\tilde{\pi}$, in the augmented game in the obvious manner, e.g., if $x_t \notin A(\pi^m)$, then define $\tilde{\pi}_t^m(\{s\}) = \pi_t^m(\{x_t\})$. Thus, $\tilde{\pi}_t^m \in B_t(\theta^m)$ for all m and $\tilde{\pi}^m \to \tilde{\pi}$ weakly. Since B_t has closed graph at θ, we have $\tilde{\pi}_t \in B_t(\theta)$ for all $t \in T$. Therefore, $\pi \in E(\rho, \delta, \beta, \lambda)$, and we conclude that E has closed graph. Since it has compact Hausdorff range as well, it is upper hemicontinuous (Aliprantis and Border 1994, Theorem 14.12). ∎

One of the important consequences of Theorem 2 is the following. If we can solve for all of the equilibria at some particular parameter values, then we know that, for values suitably close to this, all equilibria will be close (in the sense of weak convergence) to the original set: though policies far from the this set may occur with positive probability, that probability must go to zero as we approach the original parameter values of the model. Hence, when we fully characterize the equilibria in specific situations, we can be confident that these results are not "knife-edge" and that they accurately reflect the equilibria in that region of the parameter space.

6 Policy Persistence

Example 1 above showed the possibility of "all losers" in multiple dimensions when δ and β are sufficiently small. The first result of this section shows that for sufficiently large δ or β, assuming both are positive, there cannot exist *any* losers. In that case, we have perfect policy persistence, i.e., the first representative remains as incumbent forever by choosing the same policy from the social acceptance set A^* in every period. This analysis also identifies a weaker constraint on δ and β under which not all types are losers, implying eventual policy persistence, i.e., a type of representative who selects a policy from A^*, and implements this in all remaining periods, is eventually chosen. Our last result of the section is that, even when non-policy benefits of office are zero, sufficiently high δ implies perfect policy persistence, *unless* the core is non-empty — and in that case, the social acceptance set, and therefore the long run distribution of policies, must collapse to the core.

As argued earlier, in any simple equilibrium σ^* we must have $u_t(x(\sigma^*)) \geq v_t(\sigma^*)$ for all $t \in T$, and, since $x(\sigma^*) \in A^*$, it follows that $\hat{u}_t(\sigma^*) \equiv \max\{u_t(x) : x \in A^*\}$ satisfies $\hat{u}_t(\sigma^*) \geq v_t(\sigma^*)$. A type t incumbent will prefer to compromise whenever

$$\hat{u}_t(\sigma^*) + \beta \quad > \quad (1-\delta)[u_t(x_t) + \beta] + \delta v_t(\sigma^*).$$

For every $\delta < 1$, the first term on the left-hand side is strictly greater than the last term on the right-hand side (recall utilities are non-negative), and, therefore, if $\beta \geq (1-\delta)[u_t(x_t) + \beta]$, then it must be that $t \notin L(\sigma^*)$. Rewriting this inequality,

$$u_t(x_t) \quad \leq \quad \frac{\beta\delta}{1-\delta}.$$

Now define

$$\begin{aligned} \overline{\gamma} &= \max\{u_t(x_t) : t \in T\} \\ \underline{\gamma} &= \min\{u_t(x_t) : t \in T\}, \end{aligned}$$

which are well-defined and positive, since each $u_t(x_t)$ is strictly positive and T is finite. The next result is an immediate consequence of the foregoing observations.

Theorem 3 *(i) If $\beta\delta/(1-\delta) \geq \overline{\gamma}$, then $L(\sigma^*) = \emptyset$ in every simple equilibrium σ^*. (ii) If $\beta\delta/(1-\delta) \geq \underline{\gamma}$ then $T \backslash L(\sigma^*) \neq \emptyset$ in every simple equilibrium σ^*.*

The theorem has two implications. First, (i) implies that, for every $\beta > 0$, there exists $\overline{\delta} \in [0, 1)$ such that, when $\delta \geq \overline{\delta}$, all simple equilibria exhibit perfect policy persistence. And, for every $\delta > 0$, there exists $\overline{\beta} > 0$ such that, when $\beta \geq \overline{\beta}$, every equilibrium exhibits perfect policy persistence. Second, using (ii), we can get lower bounds than these, at the cost of replacing "perfect" with "eventual."

Since $\underline{\gamma}$ is positive, the above result is silent when either β or δ (or both) are equal to zero. Indeed, it is clear that, when δ equals zero, representatives will simply choose their ideal policy while in office regardless of the value of β, implying that, in certain situations (e.g., Example 1), no representative is ever re-elected. Hence, eventual policy persistence fails to hold. On the other hand, even when β equals zero, we can show that, for sufficiently high values of δ, eventual policy persistence must hold in every equilibrium (e.g., Example 2). To see this, let $\beta = 0$ and suppose (to the contrary) that we can find a sequence $\{\delta^k\}$ with $\delta^k \to 1$ and a corresponding sequence $\{\sigma^k\}$ of simple equilibria with acceptance sets $\{A^k\}$ such that $\pi_t^k(A^k) = 0$ for all $t \in T$ and all k. Hence, given any k, each type of officeholder chooses her ideal point and fails to gain re-election, and so $v_t(\sigma^k)$ is simply $\sum_{t' \in T} \rho_{t'} u_t(x_{t'})$, which is independent of k. Denote this amount $\hat{v}_t$. Thus, the equilibrium social acceptance sets, A^k, are also independent of k, which implies $\max\{u_t(x) : x \in A^k\}$ is independent of k. Denote this amount $\hat{u}_t$. Since ideal points are distinct and utilities are strictly concave, $\hat{u}_t > \hat{v}_t$. Therefore, for all $t \in T$ and for k high enough, we have

$$\hat{u}_t \quad > \quad (1-\delta^k)u_t(x_t) + \delta^k \hat{v}_t.$$

But then the optimal policy choice for type t representatives is to compromise by choosing a point in A^k, a contradiction. Therefore, even when there are no non-policy benefits to office, all simple equilibria exhibit eventual policy persistence, if the discount factor is sufficiently high.

Theorem 4 *There exists $\overline{\delta} \in (0,1)$ such that, for all $\delta \in [\overline{\delta}, 1)$, if σ^* is a simple equilibrium with discount factor δ and social acceptance set A^*, then $\pi_t^*(A^*) > 0$ for some $t \in T$.*

We can give another condition sufficient for eventual policy persistence that is demonstrated in Examples 3 and 4 and that anticipates our results on core equivalence in the next section. In those examples, the core was non-empty, located at some type's ideal point, and in the social acceptance set. So when a representative of that type is elected, she simply chooses that policy and remains in office. To see that this generalizes, suppose the core is non-empty, so $K = \{x^c\}$, and fix an arbitrary equilibrium σ^*. We first claim that $x^c \in A^*$. This follows since $x(\sigma^*) \in A_t^*$ for all $t \in T$ (by concavity), and $u_t(x^c) \geq u_t(x(\sigma^*))$ for a weighted majority of types (by external stability), implying $x^c \in A_t^*$ for a weighted majority of types. Thus, the core x^c is an acceptable policy in every equilibrium. In one dimension, we know that the core point is the ideal point of the weighted median type, i.e., $x^c = x_m$, so we know that type m individuals will always select this as their policy (and subsequently remain in office for all remaining periods by doing so), implying eventual policy persistence for all δ and β. In contrast, in multiple dimensions, the core x^c need not in general be equal to *any* type's ideal point, even with strictly concave utilities. See Figure 3 for a simple three-type example.

[Figure 3 here.]

But if we add the assumptions that x^c is interior to X and that individual utility functions are differentiable, eliminating the "kinks" in Figure 3, then it must be that $x^c = x_t$ for some $t \in T$. The argument is as follows. Take any point y interior to X and such that $y = x_t$ for no $t \in T$. Since T is finite we can find a hyperplane H, with normal p, through zero and containing none of the gradient vectors $\{\nabla u_1(y), \ldots, \nabla u_{|T|}(y)\}$. Since $\mathcal{D}$ is strong, either

$$\{t \in T : \nabla u_t(y) \cdot p > 0\} \quad \text{or} \quad \{t \in T : \nabla u_t(y) \cdot p < 0\}$$

is decisive. Without loss of generality suppose the former. Since $y \in intX$, there exists $\epsilon > 0$ such that $y + \epsilon p \in X$. Taking ϵ small enough, $u_t(y + \epsilon p) > u_t(y)$ for all t in the first coalition. Therefore, any interior point which is not some type's ideal point cannot be in the core, and, hence, if x^c exists and is interior, it must coincide with some type's ideal point. Just as in the one-dimensional, case this "core" type will always select x^c as her policy, and remain in office forever.

Theorem 5 *If $d = 1$, or if u_t is differentiable for all $t \in T$ and there exists a core point $x^c \in intX$, then there exists $t \in T$ such that $\pi_t^*(\{x^c\}) = 1$ and $t \in W(\sigma^*)$ in every simple equilibrium σ^*.*

Thus, in every simple equilibrium, the core point x^c always has a positive probability of being selected by a randomly chosen challenger, and, when it is selected, it remains as the policy in all subsequent periods. When the core is non-empty, therefore, every equilibrium exhibits eventual policy persistence: with probability one a policy will be selected that remains in place in all subsequent periods. On the other hand, we know from Examples 3 and 4 that x^c need not be the only policy exhibiting such persistence.

The final result of this section examines properties of simple equilibria as the discount factor δ approaches one. From Theorems 3 and 4, we already have two results on this score: perfect policy persistence must occur if β is positive, and eventual policy persistence must occur even if β is zero. It turns out, however, that we can say more: in any environment where $\beta = 0$ and where high δ does not imply perfect policy persistence, the core must be non-empty. Further, in the absence of perfect policy persistence, the social acceptance sets must converge to the core point, x^c. Convergence here is with respect to the *Hausdorff* metric (cf. Aliprantis and Border 1994), which for our purposes can be simplified to the following: for any compact set $Y \subseteq X$ and element $x \in X$, define the Hausdorff distance between Y and x as $h(Y, x) = \max\{\|y - x\| : y \in Y\}$. Then a sequence $\{Y^k\}$ of compact sets is said to "converge to x" if the sequence $\{h(Y^k, x)\}$ converges to zero.

Theorem 6 *Let $\{\delta^k\}$ converge to one. If there exists a corresponding sequence of simple equilibria $\{\sigma^k\}$ with social acceptance sets $\{A^k\}$ such that $\min_{t \in T} \pi_t^k(A^k) < 1$ for all k, then the core is non-empty, and $\{A^k\}$ converges to x^c.*

Proof: Take any sequence $\{\sigma^k\}$ of simple equilibria such that $\min_{t \in T} \pi_t^k(A^k) < 1$ for all k. We first show that the core is non-empty. From Theorem 3, we know that β must equal zero. And since T is finite, there must exist a type $t' \in T$ and a subsequence (also indexed by k) such that $\pi_{t'}^k(A^k) < 1$ for all k. It follows that representatives of type t' are willing to shirk for all k:

$$(1 - \delta^k)u_{t'}(x_{t'}) + \delta^k v_{t'}(\sigma^k) \quad \geq \quad \hat{u}_{t'}^k,$$

where $\hat{u}_{t'}^k = \max\{u_{t'}(x) : x \in A^k\}$. Since $\{v_{t'}(\sigma^k)\}$ and $\{\hat{u}_{t'}^k\}$ lie in compact sets, we may go to a subsequence (also indexed by k) along which these sequences converge. It follows that

$$\lim v_{t'}(\sigma^k) \quad \geq \quad \lim \hat{u}_{t'}^k. \tag{2}$$

Since $\hat{u}_{t'}^k \geq v_{t'}(\sigma^k)$ for all k, (2) actually holds with equality. Now let ψ^k be the distribution on X associated with the kth equilibrium, so that

$$v_{t'}(\sigma^k) \quad = \quad \int u_{t'}(x)\psi^k(dx)$$

for all k. Since X is compact, $\{\psi^k\}$ has a subsequence (also indexed by k) that converges weakly to some probability measure ψ on X. By weak convergence,

$$\lim v_{t'}(\sigma^k) \;=\; \int u_{t'}(x)\psi(dx). \tag{3}$$

Let

$$\overline{x}(\psi^k) \;=\; \int x\psi^k(dx) \;\text{ and }\; \overline{x}(\psi) \;=\; \int x\psi(dx),$$

and note that $\overline{x}(\psi^k) \in A^k$ for all k by concavity of voter utility functions. Hence,

$$\hat{u}^k_{t'} \;\geq\; u_{t'}(\overline{x}(\psi^k)) \tag{4}$$

for all k. Using $\overline{x}(\psi^k) \to \overline{x}(\psi)$ and the continuity of $u_{t'}$, (2), (3) and (4) yield

$$\int u_{t'}(x)\psi(dx) \;=\; \lim v_{t'}(\sigma^k) \;\geq\; \lim \hat{u}^k_{t'} \;\geq\; \lim u_{t'}(\overline{x}(\psi^k)) \;=\; u_{t'}(\overline{x}(\psi)).$$

From strict concavity, we conclude that ψ is concentrated on some point $\hat{x}$, i.e., $\psi(\{\hat{x}\}) = 1$. We claim that $\hat{x}$ is a core point, i.e., $\hat{x} = x^c$. If not, then there exist $y \in X$ and $C \in \mathcal{D}$ such that, for all $t \in C$, $u_t(y) > u_t(\hat{x})$. Since

$$\lim v_t(\sigma^k) \;=\; \int u_t(x)\psi(dx) \;=\; u_t(\hat{x}),$$

we have $y \in A^k$ for high enough k. Also, we have

$$u_t(y) \;>\; (1-\delta^k)u_t(x_t) + \delta^k v_t(\sigma^k)$$

for all $t \in C$ when k is high enough. This is implies that, for all $t \in C$, $\pi^k_t(A^k) = 1$ when k is high enough. Let $Y \subseteq X$ be any open set such that $\hat{x} \in Y$ and, for all $t \in C$ and all $z \in Y$, $u_t(y) > u_t(z)$. For each $t \in C$, clearly $\pi^k_t(A^k) = 1$ implies $\pi^k_t(A^k \backslash Y) = 1$. But then ψ^k does not converge weakly to the point mass at $\hat{x}$, contradicting the above result. We now show that $\{A^k\}$ converges to the core. If not, then there is an open set Y with $x^c \in Y$ and a subsequence (also indexed by k) $\{x^k\}$ such that, for all k, $x^k \in A^k \cap (X\backslash Y)$. Since $X\backslash Y$ is compact, there is a subsequence (also indexed by k) and a policy $\tilde{x} \in X\backslash Y$ such that $x^k \to \tilde{x}$. Since $\mathcal{D}$ is finite, we may suppose (going to a subsequence if necessary) there is some $C \in \mathcal{D}$ such that, for all k, $x^k \in A_C(\sigma^k)$. Thus, for all $t \in C$, $u_t(x^k) \geq v_t(\sigma^k)$. Now, by our first argument, we may choose a subsequence (also indexed by k) with continuation distributions $\{\psi_k\}$ converging to the point mass on x^c, and, therefore, $v_t(\sigma^k) \to u_t(x^c)$

for all $t \in T$. Then, by continuity, we have $u_t(\tilde{x}) \geq u_t(x^c)$ for all $t \in C$. But then strict concavity implies $u_t((1/2)\tilde{x} + (1/2)x^c) > u_t(x^c)$ for all $t \in C$, a contradiction. Therefore $\{A^k\} \to \{x^c\}$. ∎

From Theorem 5, we conclude that greater patience leads to perfect policy persistence, except under rather specific conditions. When the core is empty, the typical case in two or more dimensions, perfect policy persistence necessarily obtains for discount factors close enough to one. When the core is non-empty, perfect policy persistence may not obtain: for discount factors arbitrarily close to one, there may be equilibria in which some types choose their ideal points and fail to be re-elected. In this case, however, the social acceptance sets corresponding to these equilibria, and the long run distribution of policies, must converge to the core.

7 Core Equivalence

We have seen that, under weak background conditions, if the core is non-empty, then there is some type of representative that chooses the core policy and is continually re-elected. Thus, the long run distribution of policy outcomes puts positive probability on the core point. As in Example 3, however, there may be other policy outcomes that occur with positive probability. In this section, we investigate the conditions under which the core point is the *only* policy selected in equilibrium, i.e., $\pi_t(\{x^c\})$ for all $t \in T$, a phenomenon we call "core equivalence." Note the implication, in particular, that all representatives must choose the same policy, which we call "policy coincidence." Our first result shows that policy coincidence, while conceptually weaker, is actually equivalent to core equivalence in equilibrium, and it gives a necessary and sufficient condition for this equivalence to hold. Going to one dimension, we show that, assuming sufficient patience or non-policy benefits of office (and $\delta > 0$ and $\beta > 0$), core equivalence obtains in *every* simple equilibrium, providing a strong version of the median voter theorem for repeated elections. When non-policy benefits are zero, we prove an asymptotic median voter result for patient electorates.

Suppose that, in a simple strategy profile σ, all representatives choose the same policy, say, $\pi_t(\{\hat{x}\}) = 1$ for all $t \in T$. In this case, $v_t(\sigma)$ is simply equal to $u_t(\hat{x})$, and so individuals always vote to retain the incumbent and unanimity prevails. Clearly, it cannot be an equilibrium for all individuals to adopt a common policy $\hat{x}$ other than the core point: there would then be a policy y and a decisive coalition C of types such that $u_t(y) > u_t(\hat{x}) = v_t(\sigma^*)$ for all $t \in C$, and, hence, any time a member of C is elected, she would not select $\hat{x}$ as her policy. Conversely, if $\hat{x} = x^c$, then we may have an equilibrium, depending on the values of δ and β. Since x^c is the unique core point and $v_t(\sigma) = u_t(x^c)$, it follows from external stability that $A^* = \{x^c\}$, and, therefore, we need only check whether representatives prefer compromising at x^c to choosing their ideal points. If

$$u_t(x^c) + \beta \;\; \geq \;\; (1-\delta)(u_t(x_t) + \beta) + \delta u_t(x^c) \qquad (5)$$

for all $t \in T$, then $\pi_t^*(\{x^c\}) = 1$ for all $t \in T$ is an equilibrium. If this inequality fails to hold for some $t \in T$, then this is *not* an equilibrium. Re-arranging (5), we have

$$\frac{\delta\beta}{1-\delta} \;\; \geq \;\; u_t(x_t) - u_t(x^c).$$

Define

$$\gamma^c \;\; = \;\; \max\{u_t(x_t) - u_t(x^c) : t \in T\},$$

and note that, because $x_t \neq x_{t'}$ for all $t, t' \in T$, we have $\gamma^c > 0$. Note also that γ^c is only defined when x^c exists, whereas $\overline{\gamma}$ and $\underline{\gamma}$ are always defined. Since $u_t(x^c) \geq 0$ for all $t \in T$, it must be that $\gamma^c \leq \overline{\gamma}$ when the core is non-empty.

Theorem 7 *There is a simple equilibrium with $\pi_t^*(\{\hat{x}\}) = 1$ for all $t \in T$ if and only if the core is non-empty, $\hat{x} = x^c$, and $\delta\beta/(1-\delta) \geq \gamma^c$.*

As an application, return to Example 3, and note that x^c exists and is at the origin. For all $t \in \{1, 2, 3, 4\}$, we have $u_t(x_t) - u_t(x^c) = 16 - 15 = 1$, and, since $x_5 = x^c$, this difference for $t = 5$ is zero. Thus, $\gamma^c = 1$. By Theorem 7, therefore, whenever $\beta \geq (1-\delta)/\delta$, we will have a second equilibrium in which all representatives select the core point $(0, 0)$. An immediate consequence of Theorem 7 is that, when non-policy benefits of holding office are zero, policy coincidence cannot occur in equilibrium. Another consequence is that, if the core is empty, then policy coincidence, and therefore core equivalence, cannot occur in any simple equilibrium. Since the core is generically empty in two or more dimensions, we have a negative result for policy coincidence in multiple dimensions. Of course, the core is non-empty and equal to the ideal point of the weighted median type whenever the policy space is one-dimensional, and Theorem 7 has the following implication.

Corollary 1 *If $d = 1$, then $\pi_t^*(\{x_m\}) = 1$ for all $t \in T$ is a simple equilibrium if and only if $\delta\beta/(1-\delta) \geq \gamma^c$.*

Thus, as long as the individuals are sufficiently patient and non-policy benefits are sufficiently high, we can support the Downsian prediction of convergence to the median in *one* simple equilibrium of the one-dimensional model. In that case, clearly all types but the median compromise, and the first representative chosen remains as incumbent forever, continually implementing x_m. We next take up the issue of when core equivalence obtains in *all* simple equilibria. Though Example 3 shows that, in multiple dimensions, other equilibria may exist for all β and δ, we will show that a strengthening of the condition in Corollary 1 is sufficient for a unique equilibrium outcome at the core in one dimension. As a step in that direction, our next result shows that, in one dimension, we can partition the set of equilibria into two distinct classes: either all representatives choose the median, or else some representatives do not compromise at all. Thus, in one dimension, perfect policy persistence implies core equivalence.

Lemma 1 *If $d = 1$, then, in every simple equilibrium σ^*, $L(\sigma^*) = \emptyset$ implies $\pi_t^*(\{x_m\}) = 1$ for all $t \in T$.*

Proof: Suppose $L(\sigma^*) = \emptyset$, i.e., all types propose policies in A^* and, hence, are re-elected. Define

$$\underline{p} = \min_{\leq} S(\sigma^*) \quad \text{and} \quad \overline{p} = \max_{\leq} S(\sigma^*).$$

If $\underline{p} = \overline{p}$, then we know from Theorem 7 that $S(\sigma^*) = \{x_m\}$, and so it must be that $\pi_t^*(\{x_m\}) = 1$ for all $t \in T$. If $\underline{p} < \overline{p}$, then, by strict concavity of u_t

$$\arg\min\{u_t(x) : x \in S(\sigma^*)\} \quad \subseteq \quad \{\underline{p}, \overline{p}\}$$

for all $t \in T$. So define $\underline{T} \subseteq T$ as those types for which this minimizer is unique and equal to $\underline{p}$, $\overline{T} \subseteq T$ as the types for which this is unique and equal to $\overline{p}$, and $I \subseteq T$

as the types for which $u_t(\underline{p}) = u_t(\overline{p})$. As argued at the end of Section 3, $L(\sigma^*) = \emptyset$ and $|S(\sigma^*)| > 1$ imply $v_t(\sigma^*) > u_t(\underline{p}) = u_t(\overline{p})$ for all $t \in I$. Therefore, all $t \in I$ vote against incumbents choosing policies $\underline{p}$ and $\overline{p}$, and so the only types voting in favor of the policy $\underline{p}$ are those in $\overline{T}$, and the only types voting in favor of $\overline{p}$ are those in $\underline{T}$. Each of these policies is accepted, so it must be that $\overline{T} \in \mathcal{D}$ and $\underline{T} \in \mathcal{D}$, i.e., $\sum_{t \in \overline{T}} \rho_t > 1/2$ and $\sum_{t \in \underline{T}} \rho_t > 1/2$. However, $\overline{T} \cap \underline{T} = \emptyset$, contradiction. ∎

This result is consistent with viewing the voting half of the one-dimensional model as akin to a single person stationary bandit problem, with this decision maker being of the weighted median type: as described in Section 3, the "arms" of the bandit are the potential representatives (i.e., the members of the population), with types belonging to a finite set. Each type generates a constant reward stream (by history-independence), and, thus, all of the payoff-relevant information about an arm is revealed after a single play. Different types may, however, generate different reward streams, but one, the weighted median, surely generates the highest possible reward. Upon observing this reward, therefore, the decision-maker uses the arm forever. Now if *all* types generate this highest reward, then the decision-maker is clearly optimizing when she retains any arm forever. Conversely, if there is *any* variation in the reward streams, then the decision-maker will certainly replace an arm any time the lowest reward is observed. These "lowest reward" types are the "loser" types in the electoral game, with the conclusion that, if these types do not exist, then *all* types must be giving the weighted median her highest possible reward, i.e., they must be choosing policy $x_m = x^c$.

We can now state our sufficient condition, a direct consequence of Theorem 3 and Lemma 1, for core equivalence in every simple equilibrium.

Theorem 8 *If $d = 1$ and $\delta\beta/(1-\delta) \geq \overline{\gamma}$, then $\pi_t^*(\{x_m\}) = 1$ for all $t \in T$ is the unique simple equilibrium.*

Recall from Corollary 1 that policy coincidence at the core constitutes *an* equilibrium in one dimension when $\delta\beta/(1-\delta) \geq \gamma^c$. Theorem 8 establishes that policy coincidence at the core constitutes *the* equilibrium when δ and β satisfy the stronger restriction that $\delta\beta/(1-\delta) \geq \overline{\gamma}$. Thus, when individuals are sufficiently patient (and

$\beta > 0$) or non-policy benefits from incumbency are sufficiently high (and $\delta > 0$), we obtain equivalence in one dimension between the core of the underlying voting game and we obtain the well-known median voter result from considerably different microfoundations than the Downsian model of elections: even with incomplete information, and even when politicians cannot commit to policy platforms, electoral incentives lead to policy outcomes at the median.

The next result, which follows immediately from Theorem 7 and Lemma 8, shows that, if non-policy benefits from office or the discount factor are low enough, then a positive fraction of at least one type of representative does not compromise. In particular, if $\beta = 0$, then, regardless of how patient the individuals are, there will always exist some uncompromising types. This implies that, with positive probability, the search for an acceptable representative will last more than one period. We know from Theorem 5, however, that, with probability one, this search will not last forever.

Corollary 2 *If $d = 1$ and $\delta\beta/(1-\delta) < \gamma^c$, then $L(\sigma^*) \neq \emptyset$ in every simple equilibrium σ^*.*

An implication of Theorem 8 is that, when $\beta > 0$, the social acceptance set A^* will be equal to the median voter's ideal point x_m for sufficiently large values of δ. On the other hand, Corollary 2 proves the existence of non-compromising types for *every* value of δ when β equals zero, with one implication being that A^* is always a strict superset of x_m. Our next result shows, however, that, while not necessarily equal to x_m, the social acceptance set A^* does indeed converge to x_m as δ approaches one.

Theorem 9 *Let $d = 1$, let $\{\delta^k\}_{k=1}^{\infty}$ be a sequence converging to 1, and for each k let σ^k be a simple equilibrium, with social acceptance set A^k. Then $\{A^k\}$ converges to x_m.*

Proof: Suppose to the contrary that there exists a subsequence of $\{A^k\}$ (also indexed by k) and some $\varepsilon > 0$ such that, for all k, $d(A^k, \{x^c\}) \geq \varepsilon$. It follows from Theorem 8 that $\beta = 0$, and it follows from Theorem 6 that there exists $\overline{k}$ such that, for all $k \geq \overline{k}$ and all $t \in T$, $\pi_t^k(A^k) = 1$. But this implies $L(\sigma^k) = \emptyset$ for all $k \geq \overline{k}$, contradicting Corollary 2. ∎

Therefore, even when there are no non-policy benefits from holding office, if the policy space is one-dimensional and individuals are sufficiently patient, then the only acceptable policies will be those close to the weighted median type's ideal policy. In sum, Corollary 2 shows that there will always exist non-compromising types when $\beta = 0$. Theorem 5 shows that, even so, eventually an acceptable policy will be chosen. And now Theorem 9 shows that, as individuals become increasingly patient, these acceptable policies will be arbitrarily close to the core, and the long run distribution of policies will, therefore, be close to the median.

8 Quadratic Utilities

As mentioned above, in multiple dimensions the core point x^c rarely exists, and, when it does not, we know from our examples that nothing similar to the one dimensional results need hold. In Example 2, for instance, we saw that, in the absence of losing types, we need not achieve the policy coincidence found in Lemma 1. And Example 3 showed that, even when a core point exists and δ and β are arbitrarily large, we do not necessarily get the core equivalence found in Theorem 8. On the other hand, for a restricted class of preferences, we *can* generate analogous results, namely, when utility functions are quadratic: $u_t(x) = k_t - ||x - x_t||^2$. Since these functions are differentiable, we know from Theorem 5 that, if the core point x^c exists, then it is the ideal policy of some type, which we assume without loss of generality is type 1. Furthermore, by the continuity result of Theorem 2, we know that if utilities are "close" to admitting a core point then equilibrium behavior will be "close" to that occurring when a core point does exist, and, thus, the characteristics of such behavior when x^c exists will be robust to (small) deviations in preferences.

The results of this section rely on Lemma 2.1 of Banks and Duggan (2001), which states that, with quadratic utilities and a non-empty core, the type 1 voters are "decisive," in the following sense: a majority prefers one lottery over X to another if and only if the type 1 voters do. To see the usefulness of this lemma, note that, in identifying acceptable policies, a voter compares the expected utility from a "degenerate" lottery over X, i.e., the policy chosen by the incumbent, to her continuation value. As seen in Section 3, because voters share a common discount factor, we can express this value as the expected utility associated with a lottery over X as well. Invoking Lemma 2.1 from Banks and Duggan (2001), we conclude that a policy is socially acceptable if and only if it is acceptable to the "core voter" type, type 1. That is, in any equilibrium with quadratic utilities and a core point at x_1, we must have $A^* = A_1^*$. An immediate implication of this is the following analogue to Lemma 1.

Lemma 2 *If there exists a core point $x^c \in intX$ and u_t is quadratic for all $t \in T$, then, in every simple equilibrium σ^*, $L(\sigma^*) = \emptyset$ implies $\pi_t^*(\{x^c\}) = 1$ for all $t \in T$.*

Proof: Let σ^* be an equilibrium in which $L(\sigma^*) = \emptyset$. We know from Theorem 5 that x^c is the ideal point of some type, say $t = 1$, and that $\pi_1^*(\{x^c\}) = 1$, which implies $x^c \in S(\sigma^*)$. Let $\underline{x} \in \arg\min\{u_1(x) : x \in S(\sigma^*)\}$ and suppose that $\underline{x} \neq x^c$, i.e., $|S(\sigma^*)| > 1$. Then $u_1(\underline{x}) < v_1(\sigma^*)$, since $\pi_1^*(\{x^c\}) = 1$ and $\rho_1 > 0$. This implies $\underline{x} \notin A_1^* = A^*$. Because $\underline{x}$ is, by definition, proposed with positive probability, this contradicts the assumption that $L(\sigma^*) = \emptyset$. ∎

Combining Theorem 3 and Lemma 2, we have the following.

Theorem 10 *If there exists a core point $x^c \in intX$, if u_t is quadratic for all $t \in T$, and if $\delta\beta/(1-\delta) \geq \overline{\gamma}$, then $\pi_t^*(\{x^c\}) = 1$ for all $t \in T$ is the unique simple equilibrium.*

Thus, we once again get core equivalence for all simple equilibria when individuals are sufficiently patient or non-policy benefits are sufficiently high, but now under the additional assumption of quadratic utilities. Note that the quadratic function form is needed here, for Example 3 shows that Theorem 10 does not extend even to all utility functions based on Euclidean distance.

Theorem 10 requires a core point to exist, yet we know from Plott (1967) that the conditions on the ideal points $(x_1, \ldots, x_{|T|})$ required for core existence with quadratic utilities are quite severe: a core point can only exist at some type's ideal point, here x_1, and then only if the remaining types can be paired up in such a way as to make the ideal points of the members of each pair line up on opposite sides of x_1. But then slightly shifting x_1 in *any* direction will render the core empty. On the other hand, we can use the continuity result found in Theorem 2 to say something about equilibrium behavior when ideal points are "close" to admitting a core. Specifically, suppose $\delta\beta/(1-\delta) \geq \overline{\gamma}$ and x^c exists when utility functions are quadratic, with ideal points $(x_1, \ldots, x_{|T|})$. By Theorem 10, there is a unique equilibrium, and this exhibits policy coincidence at x^c. Now take any sequence of ideal point profiles $\{(x_1^k, \ldots, x_{|T|}^k)\}_{k=1}^{\infty}$ converging to $(x_1, \ldots, x_{|T|})$. By Theorem 1, there exists at least one equilibrium at each k. By Theorem 2, since the equilibrium at $(x_1, \ldots, x_{|T|})$ is unique, it must be that the equilibrium policies converge (in the sense of weak convergence) to x^c. That is, though equilibria with ideal points near $(x_1, \ldots, x_{|T|})$ need exhibit neither uniqueness nor policy coincidence, and though policies away from the core may have positive probability, this probability must go to zero: with arbitrarily high probability, equilibrium policies will be arbitrarily close to x^c. In this sense, then, the "core equilibrium" identified in Theorem 10 is robust to perturbations of the individuals' preferences.

Similarly, while Theorem 10 requires utility functions to be quadratic, Theorem 2 can again be employed to show that, if x^c exists and utilities are "close" to being quadratic, then the equilibrium policies will be "close" to x^c. For instance, when $u_t(x) = k_t - ||x - x_t||^{\lambda}$, we would conclude that, for λ close to 2, all equilibrium policies would be close (in the weak convergence sense) to the core point. From this, we see that the "non-core equilibrium" constructed in Example 3, where $u_t(x) = 16 - ||x - x_t||^4$, relies on the fact that 4 is sufficiently different from 2.

Of course, we also have the following analogue to Corollary 2. As a consequence, even when a core point exists and individuals are arbitrarily patient, if non-policy benefits are zero, then some types will not compromise in equilibrium.

Corollary 3 *If there exists a core point $x^c \in intX$, if u_t is quadratic for all $t \in T$, and if $\delta\beta/(1-\delta) < \gamma^c$, then $L(\sigma^*) \neq \emptyset$ in every simple equilibrium σ^*.*

Finally, we get a convergence result analogous to that found in Theorem 9 for one dimension. The proof mimics that for Theorem 9, with Theorem 10 and Corollary 3 replacing Theorem 8 and Corollary 2, respectively.

Theorem 11 *Assume there exists a core point $x^c \in intX$ and u_t is quadratic for all $t \in T$. Let $\{\delta^k\}_{k=1}^{\infty}$ be a sequence converging to 1, and for each k let σ^k be a simple equilibrium, with social acceptance set A^k. Then $\{A^k\}$ converges to x^c.*

The results of Theorems 9 and 11 can be interpreted in terms of the long run distribution of policy outcomes: because the long run distribution is concentrated on the social acceptance set, the results imply that the long run distribution converges to the core as voters become arbitrarily patient. When $\beta = 0$, however, Corollaries 2 and 3 show that there must be some losing types, some positive fraction of representatives who choose policies outside the social acceptance set. Thus, the short run distribution of policies chosen by a challenger is *not* concentrated on A^*, and Theorems 9 and 11 do *not* imply that the short run distribution of challenger policies converges to the core.

The next example illustrates the structure of equilibria as discount factors go to one and shows that, in at least some environments, the distribution of challenger policies will in fact converge to the core point.

Example 5: *convergence to core in multiple dimensions*

Suppose that $\beta = 0$ and that the ideal points of the different types are symmetrically distributed about $x_1 = x^c$, i.e., for every $x \in X$,

$$|\{t \in T : x_t = x\}| \quad = \quad |\{t \in T : x_t = 2x_1 - x\}| \,.$$

In words, the ideal points of the different voter types can be paired up so that, for each pair $r, s \in T$, the ideal points $\{x_r, x_s, x_1\}$ are collinear, with x_r and x_s equal distances to either side of x_1. We further assume that the distribution $\rho = (\rho_1, \ldots, \rho_{|T|})$ is symmetric: if types r and s are paired together, then $\rho_r = \rho_s$. Thus, x_1, in addition to being the unique core point, is also the mean of the ideal points with respect to the recognition probabilities.

Let r and s be paired types, "extreme" in the sense that they maximize $||x_t - x_1||$ over $t \in T$. We will show that, for high enough δ, there exists $\gamma > 0$ for which the following is an equilibrium: a fraction $1 - \gamma$ of r and s type representatives select their ideal points and subsequently lose the next election, while all other representatives compromise by selecting the best policy acceptable to type 1 voters. Furthermore, γ approaches zero as δ approaches one. Letting y_t denote the best compromise policy for type t representatives, it is evident that, if t and t' are paired together, then y_t and $y_{t'}$ are, as with their ideal points, an equal distance from, and on either side of, x_1. Therefore, the continuation value of the type t voters, v_t, is the expected utility associated with a certain distribution over X with mean $x_1 = x^c$. Since u_t is quadratic, we can express v_t as

$$v_t = -\|x_t - x^c\|^2 - \text{var}.$$

In particular, $v_1 = -\text{var}$. The variance term, var, is

$$\text{var} = \left(\sum_{t\neq r,s} \rho_t \|y_t - x^c\|^2\right) + 2\rho_r\gamma \|y_r - x^c\|^2 + 2\rho_r(1-\gamma)(1-\delta)\|x_r - x^c\|^2,$$

where we use the assumption that $\rho_r = \rho_s$ and $\|x_r - x^c\| = \|x_s - x^c\|$. From Theorem 11, we can take δ close enough to one so that the "compromise constraint" is binding for all non-core types, i.e., $t \neq 1$ implies $x_t \notin A_1$. By decisiveness of the type 1 voters, y_t will be such that, for all $t \neq 1$,

$$\|y_t - x^c\| = \sqrt{\text{var}},$$

which implies

$$\|x_t - y_t\| = \|x_t - x^c\| - \sqrt{\text{var}}.$$

Substituting into our expression for var and solving,

$$\text{var} = \frac{2\rho_r(1-\gamma)(1-\delta)\|x_r - x^c\|^2}{\rho_1 + 2\rho_r(1-\gamma)}.$$

Now a strategy profile of the form described above is an equilibrium if and only if type r and s voters are indifferent between losing and compromising, with all other voters weakly prefering to compromise. The indifference condition is

$$\delta v_r = -\|x_r - y_r\|,$$

or, after manipulating,

$$\sqrt{\frac{2\rho_r(1-\gamma)}{(1-\delta)[\rho_1 + 2\rho_r(1-\gamma)]}} = \frac{1}{2} + \frac{\rho_r(1-\gamma)(1-\delta)}{\rho_1 + 2\rho_r(1-\gamma)}. \tag{6}$$

To see that (6) has a solution, say $\hat{\gamma}$, satisfying $0 < \hat{\gamma} < 1$ when δ is high enough, substitute z for $\rho_r(1-\gamma)/[\rho_1 + 2\rho_r(1-\gamma)]$ and rewrite (6) as

$$\sqrt{\frac{2z}{1-\delta}} = \frac{1}{2} + (1-\delta)z. \tag{7}$$

Any solution to (7) corresponds immediately to a solution to (6), and, indeed, (7) clearly has a unique solution, $\hat{z}$, when δ is high enough. Furthermore, $\hat{z} > 0$ and $\hat{z} \to 0$ as $\delta \to 1$. Thus, the corresponding solution to (6) satisfies $\hat{\gamma} < 1$ and $\hat{\gamma} \to 1$ as $\delta \to 1$. Combining the latter observation with our expression for var, we see that var $\to 0$ as $\delta \to 1$.

Lastly, we verify that, given $\hat{\gamma} \in (0,1)$ solving (6), voter types other than r and s weakly prefer to compromise. That is, for type $t \neq r, s$ voters, we need

$$\delta v_t \;\leq\; -\|x_t - y_t\|^2,$$

or, after manipulating,

$$\sqrt{\frac{2\rho_r(1-\hat{\gamma})}{(1-\delta)[\rho_1 + 2\rho_r(1-\hat{\gamma})]}} \;\geq\; \frac{\|x_t - x^c\|}{\|x_r - x^c\|} + \frac{\text{var}}{\|x_t - x^c\| \cdot \|x_r - x^c\|}$$

for high enough δ. If $\|x_t - x^c\| = \|x_r - x^c\|$ then, as in (6), this holds with equality: type t voters are indifferent between losing and compromising, and so compromising is a best response. Suppose $\|x_t - x^c\| < \|x_r - x^c\|$. Since var $\to 0$, the right-hand side converges to $\|x_t - x^c\| \,/\, \|x_r - x^c\| < 1/2$. By (6), the left-hand side converges to $1/2$, and we conclude that the inequality holds for δ close enough to one.

Therefore, we have an equilibrium for δ high enough, and, since (6) has a unique solution, it is the unique equilibrium of this form. Since $\hat{\gamma} \to 1$ as $\delta \to 1$, we see that the fraction of compromising types goes to one, as claimed. And since var $\to 0$ as $\delta \to 1$, the set of policies acceptable to type 1 voters, and hence socially acceptable, converges to $x_1 = x^c$, as required by Theorem 11.

9 Discussion

9.1 The Downsian Approach

Many of our results may best be understood in comparison to results from a repeated version of the Downsian model, in which at the beginning of a period each of two purely office-motivated candidates announces a policy to be faithfully implemented in that period if elected. While this infinitely-repeated game may have a large set of subgame perfect equilibria, to maintain consistency with the "simple" equilibrium selection employed above, we restrict attention to stationary equilibria of the Downsian game, which reduce to one-shot Nash behavior (Duggan and Fey 2001). We will assume that $\beta\delta/(1-\delta) \geq \overline{\gamma}$ in our model, implying that non-policy benefits from office are high enough so as to be roughly consistent with the assumption of office-motivated candidates in the Downsian model.

In one dimension, the Downsian model and our model give identical and unique policy predictions, namely, that the median voter's ideal policy is implemented in every period. In the Downsian model, this is generated by the explicit competition of the two candidates for the role as representative. In our model, in contrast, electoral competition is implicit, as the potential challengers to the incumbent (i.e.,the voters at large) are not making any campaign promises whatsoever but are rather passively waiting to be chosen. However, their presence, and their expected behavior if chosen, provide the necessary incentives for the incumbent to locate at the core policy. In terms of the predicted identity of the representative the Downsian model is silent, as either candidate could be chosen in any period. In our model the identity of the first period incumbent is obviously random. In all remaining periods, however, this individual is re-elected, thereby generating an extreme form of incumbency advantage. It is important to note that this advantage does not have any policy consequences, as the incumbent is re-elected only because she continually selects the core policy.

Turning to multiple dimensions, assume that there does not exist a majority rule core point, as is generically the case. In the Downsian model, no pure strategy equilibria will exist, leaving only the possibility of mixed strategy equilibria. Assuming these exist, there will be uncertainty within each period about what the policy will be as well as who will be the incumbent. Furthermore, across periods policies will appear to be independent and identically distributed random variables from some non-trivial distribution over the policy space. In our model, on the other hand, equilibria will always exist. And while the *first* policy will be uncertain, there will be perfect policy persistence, in that whatever policy occurs in the first period will also be the policy in all remaining periods. As well, while the identity of the representative in the first period is random, this individual remains as incumbent for all periods. In contrast to the one dimensional model, however, here this incumbency advantage does have a policy consequence, as different types of representatives may choose distinctly different policies.

When the core is empty, there necessarily exists a policy that a majority of the voters actually prefer to the policy continually implemented by the incumbent in the current model. In a Downsian world this cannot happen, as a challenger could replace the incumbent by simply adopting this majority-preferred alternative. Thus, the source of the multidimensional policy persistence identified above is two-fold: (i) the inability of individuals to promise policies they would not find optimal to adopt once in office, and (ii) the inability of individuals who would optimally adopt any majority-preferred alternative to distinguish themselves from the other potential challengers. In this way we see how the current approach and the Downsian approach stake out extreme and opposing positions regarding these two modeling assumptions, and arrive at extreme and opposing predictions regarding policy persistence in multiple dimensions.

That said, the policy predictions of the two models in any given period can be almost indistinguishable, even in multiple dimensions. In the Downsian model, the support of any mixed strategy equilibrium will lie in the uncovered set (Banks, Duggan, and Le Breton 2000), a set which collapses to the core as the preferences of the voters converge to a profile which admits a core point. That is, when viewed as a correspondence, the uncovered set is upper hemicontinuous at the core (Cox 1987, Banks, Duggan, and Le Breton 2001). Similarly, in our model the set of equilibrium policies is upper hemicontinuous, and is equal to the core when utility functions are quadratic. Therefore, in this environment the predictions of both models are close to core outcomes (and so each other) when voter preferences are close to admitting a core.

One final point has to do with a comparison of our model with one versus multiple dimensions, again assuming $\beta\delta/(1-\delta) \geq \overline{\gamma}$. We know from Theorem 8 that we have a unique prediction in one dimension, namely, policy coincidence (and hence policy persistence) at the core. In multiple dimensions when the core is empty, we know from Theorem 7 that policy coincidence does not obtain, implying some representatives choosing different policies than others. What remains is the policy persistence found in Theorem 3: as in the one-dimensional environment, in multiple dimensions the same policy will be implemented in every period. In this sense, then, the stability of the political process, as exemplified by policy persistence, is independent of the dimensionality of the policy space, though the uniqueness of this stable policy is certainly not.

9.2 Perfect Bayesian Equilibrium

At the cost of departing from history-independence of policy choice strategies, our concept of simple equilibrium can be re-cast in terms of perfect Bayesian equilibrium. The technical problem is to specify policy choices and voter beliefs in a consistent way following out-of-equilibrium histories of play. To be more explicit, fix a simple equilibrium σ^*. Suppose that a representative $i \in N$ is a compromising type, say t, and compromises at $p_i^*(t)$ in her first period in office, but then chooses her ideal point $x_t \notin A^*$ in the second period. Because this path of play has probability zero under σ^*, voters cannot use Bayes rule to update about the representative's type, so we must assign beliefs arbitrarily. If we let voters simply ignore the deviation, then, given that the representative is indeed using the history-independent policy strategy specified by σ^*, voters assume the representative will resume compromising in the future, choosing the policy $p_i^*(t)$ in every term of office. By construction, a majority of voters would then vote to re-elect the incumbent, and the deviation would be profitable. The problem, then, is to specify voter beliefs and policy strategies so as to make such deviations unprofitable.

One possibility is the following. If some type, say t', of that representative chooses a policy outside the social acceptance set, i.e., $p_i^*(t') = x_{t'}$, then we can simply assign voter beliefs so that, after deviations as above, the voters believe the representative is type t' with probability one. Then, because $x_{t'} \notin A^*$, the challenger will be elected and the above deviation is not profitable. This approach does not involve any departure from full history-independence, but, since the construction may not be possible for all $i \in N$ and all simple equilibria, it is not fully general.

More generally, define voting and policy choice strategies as in σ^* along the path of play. However, if a representative has ever chosen a policy $x \notin A^*$, then specify that the representative choose x_t in all future terms of office, and specify that all voters believe the representative is a type t such that $x_t \notin A^*$.[14] Thus, in the above

[14]If there is no such type, as in Example 3, we do not need to consider deviations from policy

example, when the representative chooses x_t in her second term of office, all voters assume that the representative will continue to choose some policy outside the social acceptance set. Then the challenger will be elected, and the representative will be appropriately punished. This maneuver weakens history-independence in a way that punishes politicians who shirk and yet preserves much of the simplicity of σ^*.

9.3 Extensions

The analysis of our model would be complicated, but our results largely unaffected, by any of several generalizations and extensions. First, Theorems 1-4, on existence and policy persistence, would continue to hold for arbitrary voting rules, such as a quota rule or even an arbitrary collection of winning coalitions of types would suffice. This would change the definition of $A(\sigma)$ slightly without changing its fundamental continuity properties.[15] Our other results would continue to hold as long as the voting rule, represented by $\mathcal{D}$, were "strong," in the sense we have defined.

Second, Theorems 1-4 would also hold if the set T of types were a continuum, rather than a finite: though more difficult to verify, the continuity properties of $A(\sigma)$ are quite general with respect to the set of types (cf. Banks, Duggan, and Le Breton 2001, Proposition 13). With a continuum of types, the assumption that $\mathcal{D}$ is strong becomes quite restrictive, but our other results would hold as long as the distribution of preferences among the electorate were sufficiently "dispersed." If, for example, voter utilities were quadratic with ideal points distributed over $\Re^d$ according to some positive density function, then the core (if non-empty) would be a singleton and would be the ideal point of some voter type. Our core equivalence results would then go through unchanged.

Third, our results would hold if we added an exogenous and time-invariant positive probability of an incumbent being removed from office (through death, impeachment, etc.), though the results on policy persistence would obviously have to be re-interpreted in terms of expected duration of tenure in office. In such a model, even winners and compromisers, through no fault of their own, would eventually be

choices designated by σ^*.

[15]See Banks, Duggan, and Le Breton (2001) for an analysis of arbitrary collections of winning coalitions with a continuum of voters.

replaced, generating richer dynamics for the model. The current formulation, which admittedly leads to very stark dynamics, was chosen for its simplicity and because the possibility of turnover does not essentially change our message about voter patience and non-policy benefits of office.

Fourth, the distribution from which challengers are drawn was assumed equal to ρ in the above analysis, but all of our results would go through if challengers were drawn from an arbitrarily fixed distribution, as long as all types had positive probability. This would simply change the weights on different types in our expression for a voter's continuation value, without changing any of the analysis. Theorems 1 and 2, on existence and continuity, would hold even if this distribution varied over time. We could, for example, have a challenger drawn from the side of a one-dimensional space opposite that of the incumbent, to capture some notion of party. This would lower the value of a challenger to an incumbent, yielding weaker conditions for policy persistence and, therefore, for core equivalence.

Finally, all of our results would hold in a version of the model with a finite number of voters and a separate, countably infinite pool of potential challengers (who do not vote) with types identically and independently distributed according to ρ. Suppose a new challenger is drawn in every period to run against the incumbent. The continuation values of voters and representatives would be unchanged: what is essential is that no voter perceives a chance that she will be drawn as a challenger, and no representative perceives a chance that she will be re-drawn as a challenger after losing an election. The main advantage of this reformulation, aside from avoiding some technical complexities that arise in a model with a continuum of players, is that voters are now conceivably pivotal in elections, so that our equilibrium condition on voting strategies can be justified directly in terms of weak dominance. The obvious disadvantage is that we must treat voters as essentially different from candidates (who cannot vote). Our philosophical preference, given this trade-off, is to model candidates exactly as voters.

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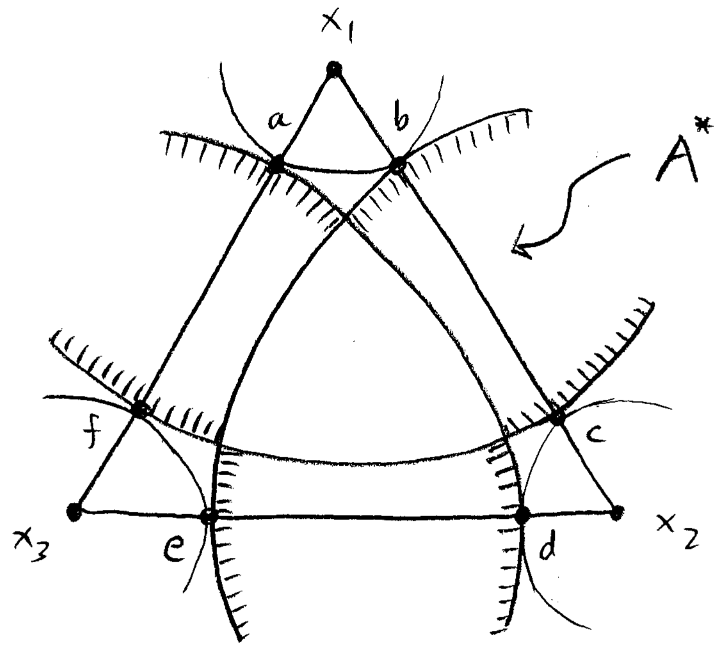

Figure 1

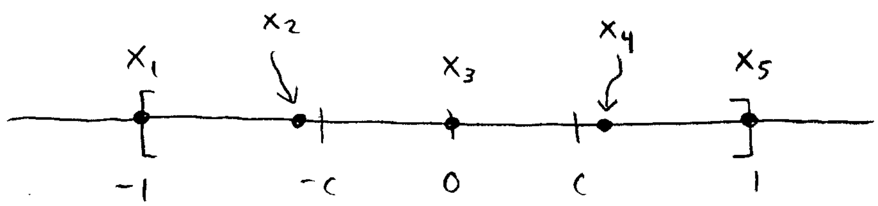

Figure 2

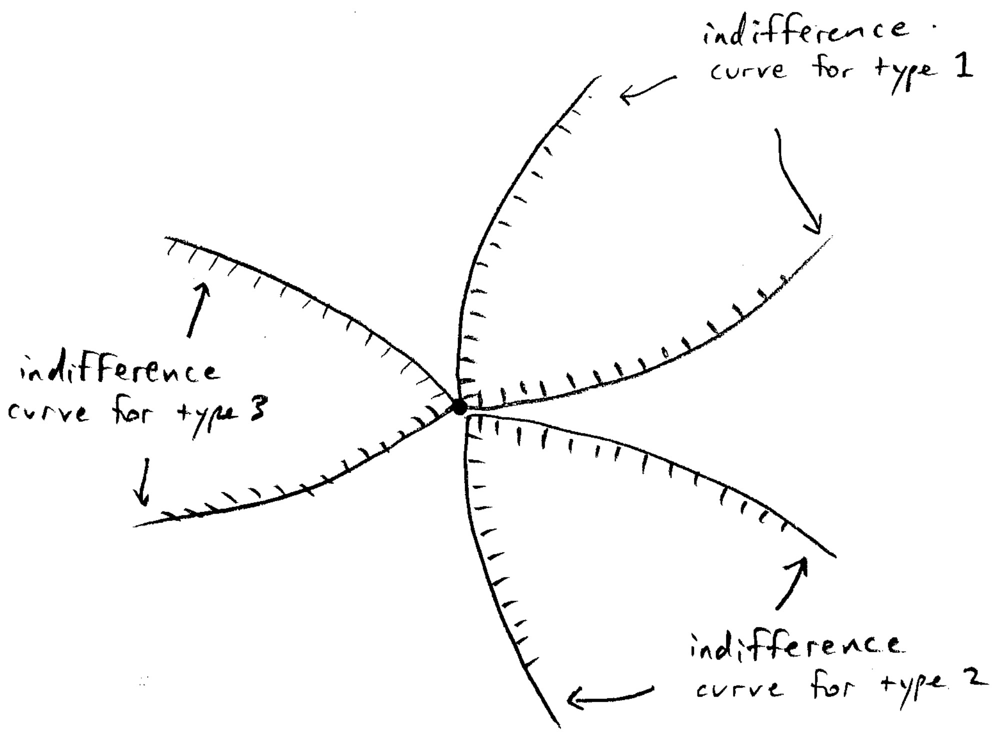

Figure 3

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