Nash Implementation with a Private Good

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W. Allen Wallis Institute of Political Economy
Working Paper No. 25

January 2002

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First Draft: June 9, 1997
Current Draft: January 23, 2002

Abstract

I construct a general model of social planning problems, including mixed production economies and regulatory problems with negative externalities as special cases, and I give simple mechanisms for Nash implementation under three increasingly general sets of assumptions. I first construct a continuous mechanism to implement the (constrained) Lindahl allocations of an economy, and I then extend this to arbitrary social choice rules based on prices. I end with a mechanism to implement any monotonic social choice rule, assuming only the existence of a private (not necessarily transferable) good. In that general case, each agent simply reports an upper contour set, an outcome, and I need two agents to make binary numerical announcements. I do not require the usual no-veto-power condition.

1 Introduction

The theory of implementation adopts the perspective of a social planner and seeks to characterize the outcomes achievable through the equilibria of a fixed mechanism when the preferences of agents are unknown. If the socially acceptable outcomes match the equilibrium outcomes of a mechanism for all possible preferences of the agents, then the planner's objectives, formalized as a social choice rule, are "implementable." Maskin (1977, 1999) has shown that every social choice rule satisfying two conditions, monotonicity and no-veto-power, is implementable in Nash equilibrium, but his mechanism is quite complex.[1] If some economic structure is imposed on the social planning problem — on the set of alternatives, the possible preferences of the agents, or the objectives of the planner — it is known that simpler mechanisms can work.

In this paper, I construct a model of social planning problems assuming only that there is a private good, generalizing mixed economies, with and without production, and allowing for public "bads." The model also includes, as a special case, regulatory problems in which a social planner seeks to maximize total surplus and activities of the agents impose negative externalities on each other — this type of problem is not covered by most analyses, which typically impose monotonicity on the preferences of the agents. I then give simple mechanisms for Nash implementation under three increasingly general sets of assumptions, beginning with Lindahl allocations in production economies, proceeding to a class of social choice rules based on prices (generalizing the Pareto, Walrasian, Lindahl, and ratio equilibrium social choice rules), and ending with general monotonic social choice rules.

In the first two environments, the outcome functions of the mechanisms are continuous and the message spaces quite small: each agent simply announces a price vector, a social state (i.e., allocation), and I need two agents to announce real numbers. In the third and most general environment, reports of price vectors are replaced by reports of upper contour sets, which can no longer be summarized by price vectors, and the numerical announcements of the two agents are binary. In contrast to the work of Maskin and others in general environments, I do not impose no-veto-power, either explicitly or implicitly.[2]

[1] Also, there must be at least three agents. See Repullo (1987), Saijo (1988), McKelvey (1989), and Moore and Repullo (1990) for other proofs.

[2] Rivalry in the consumption of the private good is not presumed, i.e., the set of feasible allocations of the transferable good may be "rectangular," and so no-veto-power is not automatically satisfied.

Previous work on Nash implementation — even the literature on "simple" mechanisms — relies on constructions, e.g., modulo games and integer games, that induce unwinnable competition among the agents to eradicate socially sub-optimal outcomes. Typically, each agent can preempt the others by reporting a number correctly and, upon doing so, the agent is awarded a most preferred outcome (which may be far from socially optimal). The mechanisms of this paper, in contrast, employ a simple construction similar to matching pennies: one agent tries to match the other, who tries not to match the one, with the payoff differential between winning and losing depending on the reports of all agents. The differential is zero, allowing the agents to match each other in equilibrium, only when the outcome of the mechanism is socially optimal. Unlike integer and modulo games, which typically award agents their best outcomes, incentives are induced by arbitrarily small fines.

Related Literature

A substantial literature has exploited economic structure in specific types of social planning problems. Hurwicz (1979), Schmeidler (1980), and Walker (1981, p.67) offered mechanisms to implement the Walrasian and Lindahl allocations, but these mechanisms suffered from individual feasibility problems. Hurwicz, Maskin, and Postlewaite took up the issue of "feasible" implementation of the "constrained" Walrasian and Lindahl allocations when the endowments of the agents are unknown to the social planner.[3] (See Hurwicz, Maskin, and Postlewaite (1995).) Postlewaite and Wettstein (1989) and Tian (1989) provided continuous mechanisms to implement, respectively, the constrained Walrasian allocations of an exchange economy and the constrained Lindahl allocations of a linear public goods economy. Chakravorti (1991) presented a mechanism to implement the constrained Walrasian allocations of an exchange economy with smaller message spaces, though the mechanism is not generally continuous. Hong (1995) constructed a continuous mechanism to implement the constrained Walrasian allocations of a production economy. Like Hurwicz, Maskin, and Postlewaite (1995), Postlewaite and Wettstein (1989), and Tian (1989), Hong allowed for the possibility that agents' endowments might be subject to misrepresentation (an issue I do not address).

[3]The issue of "constraining" the demands of agents did not arise in the earlier literature, where individual feasibility constraints were absent.

A separate strand of the implementation literature has examined the problem of simplifying the canonical implementing mechanisms by using "smaller" message spaces. Saijo (1987), working in completely general environments, constructed an implementing mechanism in which each agent reports two preference orderings, an outcome, and a number (used in a modulo game). McKelvey (1989) imposed various types of structure on the implementation problem and provided implementing mechanisms with even smaller message spaces. Among them, his Mechanism III requires agents to report two upper contour sets, an outcome, and a number for the modulo game.[4] Applying the mechanism to implement the constrained Lindahl allocations of a mixed economy, McKelvey showed that announcements of upper contour sets could be replaced by price vectors, yielding a mechanism somewhat more complex than Hurwicz's (1979) but guaranteeing individual feasibility out of equilibrium.[5]

All of the above work, save Saijo (1988) and McKelvey (1989), focused specifically on either the constrained Walrasian or constrained Lindahl social choice rule, but later work has considered generalizations of the Walrasian and Lindahl rules. Corchón and Wilkie (1996) construct a continuous mechanism using reports of prices and allocations to implement the ratio equilibria, defined by Kaneko (1977a,b), in Nash and strong Nash equilibrium. Dutta, Sen, and Vohra (1995) investigate social choice rules satisfying local independence, a condition due to Nagahisa (1991), which dictates that the social optimality of an allocation depend only on the marginal rates of substitution of the agents at that allocation. Saijo, Tatamitani, and Yamato (1996) used a somewhat weaker condition, their Condition M. The latter papers show that, in the context of exchange economies, social choice rules satisfying these conditions can be implemented by well-behaved mechanisms that restrict the announcements of agents to price vectors and allocations. And, for the special case of the constrained Walrasian allocations, messages can be further restricted to price vectors and consumption bundles. Dutta, Sen, and Vohra (1995) also consider generalizations of the constrained Lindahl rule in linear public good environments.

While these message spaces are indeed small, the mechanisms of the latter two papers incorporate a type of modulo game in the reports of allocations (or consumption bundles,

[4]McKelvey also showed that, under weak conditions, the message spaces of the agents could be further reduced by "hiding" the modulo game within the reported upper contour sets.

[5]McKelvey also provides an implementing mechanism for the constrained Walrasian allocations of an exchange economy.

as the case may be). This reduces the size of their message spaces but entails discontinuities for their outcome functions. Furthermore, the analyses of both papers rely on differentiability of the agents' preferences. All three of these papers use the assumption that socially optimal allocations are interior, in the sense that every agent's consumption level of each good is positive. This is true of the constrained Walrasian and Lindahl allocations assuming the agents' indifference curves do not intersect the boundaries of their consumption sets, as with Cobb-Douglas preferences, but it is not true generally.[6]

Results of the Paper

I first implement the constrained Lindahl allocations in a general model of a mixed economy, capturing private good exchange economies and allowing for any convex production technology and negative externalities. Message spaces (each agent reports a price vector and an allocation, and two agents report a number for matching purposes) are as small as or comparable to those of other mechanisms designed specifically for the constrained Lindahl allocations — McKelvey's (1989) and Tian's (1989) mechanisms both require each agent to report an extra price vector. The mechanisms designed specifically to implement the constrained Walrasian allocations have somewhat smaller message spaces than mine, as would be expected.[7] Like Tian's, my mechanism is continuous.

Virtually the same mechanism implements *any* social choice rule based on prices in the sense that the social optimality of an outcome depends only on the "price vectors" supporting the agents' strict upper contour sets at that outcome. This class includes the ratio equilibrium allocations and generalizes those satisfying the local independence condition of Dutta, Sen, and Vohra (1995) and those satisfying Saijo, Tatamitani, and Yamato's (1996) Condition M. As discussed above, the message spaces of their mechanisms are slightly smaller than mine. In contrast to theirs, however, my mechanism is continuous under suitable assumptions regarding the social choice rule. I also drop the assumption of differentiable preferences, made in both papers, and weaken the assumption of interiority: to use the simplest version of the mechanism, I require that, at every socially optimal

[6]See Campbell and Truchon (1988) for a discussion of the importance of boundary Pareto optima in public goods environments.

[7]These mechanisms can have agents report consumption bundles rather than allocations, a luxury not available to me because I do not distinguish private goods from public goods — I model them simply as special types of public goods.

outcome, each agent consumes a positive amount of the numeraire good. Modifying the mechanism somewhat, this can be weakened further: I can assume only that, at every socially optimal outcome, it is not the case that one agent consumes all of the numeraire good. Sacrificing continuity, the mechanism can also be modified to allow for indivisibilities in consumption.

Finally, I construct a mechanism to implement monotonic social choice rules in general environments with a private good for each agent — no other structure is imposed. The message space of the mechanism is quite small (each agent reports an upper contour set and an outcome, and two agents report either -1 or 1), indeed smaller than McKelvey's (1989) Mechanism III and comparable to his Mechanism II, which has the smallest message space of the three he proposes. In contrast to Mechanism II, which threatens the agents with a universally bad outcome, my mechanism uses a matching construction that imposes arbitrarily small fines on the agents. And in contrast to the canonical mechanisms of Maskin, Saijo, McKelvey, and others, my construction relies only on monotonicity of the social choice rule to be implemented — no-veto-power is unneeded.

2 The Social Planning Problem

Consider the following social planning problem. There are n agents, indexed by i and j, and a set X of *feasible outcomes,* denoted x. I assume that $X \subseteq S \times T$, where S is a set of *social states,* denoted s, and T is a set of allocations, denoted t, of a private good. Thus, each outcome x can be decomposed as (s,t), x' as (s',t'), and so on. An allocation $t = (t_1, \ldots, t_n) \in \Re^n$ lists the consumption level of each agent. The set T is defined by an arbitrary lower bound, $\underline{t}_i \in [-\infty, 0]$, on consumption of private good for agent i, representing individual feasibility constraints.[8] Thus, $T = \prod_{i=1}^{n}[\underline{t}_i, \infty)$. In all but the last section of the paper, I will assume the following condition on social states.

(S1) $S = \Re^m_+$.

Under (S1), a state consists of m non-negative real-valued *state variables,* which may represent allocations of private or public goods, outputs of firms, the levels of m activities available to the agents, etc. A state is then a vector $s = (s_1, \ldots, s_m) \in \Re^m_+$ listing the value of each state variable.

[8]Of course, $\underline{t}_i = -\infty$ corresponds to the case in which there is no feasibility constraint for agent i. In that case, we should write i's feasible consumptions as $(-\infty, \infty)$, rather than $[-\infty, \infty)$.

Technological feasibility constraints are reflected in the set X of feasible outcomes. I will employ, at times, the following assumptions on feasible outcomes, where $\underline{t} = (\underline{t}_1, \ldots, \underline{t}_n)$. I will always assume (X1). Note that (X2) is predicated on (S1).[9]

(X1) There exists $x \in X$ such that $t \gg \underline{t}$. For all $x \in X$ and all $t' \in T$, if $t \geq t' \geq \underline{t}$, then $(s, t') \in X$.

(X2) X is closed and convex and, for all $s \in S$ and all $t, t' \in T$, if $(s,t) \in X$ and $\sum_{i=1}^n t_i = \sum_{i=1}^n t'_i$, then $(s, t') \in X$.

Condition (X1) says that it is possible to place each agent above its lower bound on consumption of the private good and that the private good can be freely disposed of. Condition (X2) captures the possibility of a convex production technology and implies that the private good is actually transferable.

Each agent i has preferences over X represented by a utility function $u_i \colon X \to \Re$, where the profile $u = (u_1, \ldots, u_n)$ of utility functions is unknown to the planner but known to lie in a set $\mathcal{U}$ of possible utility profiles. The utility functions of the agents will always be restricted by (U1) and (U2). Note that (U3) is predicated on (S1) and (X2).

(U1) For all $i \in N$, all $x \in X$, and all $t' \in T$, if $(s, t') \in X$ and $t_i = t'_i$, then $u_i(x) = u_i(s, t')$.

(U2) For all $i \in N$ and all $x, x' \in X$, if $s = s'$ and $t'_i > t_i$, then $u_i(x') > u_i(x)$.

(U3) For all $i \in N$, u_i is semi-strictly quasi-concave, i.e., for all $x, x' \in X$ and all $\alpha \in (0,1)$, if $u_i(x') > u_i(x)$, then $u_i(\alpha x' + (1-\alpha)x) > u_i(x)$.

(U4) For all $i \in N$ and all $x, x' \in X$, if $t_i = \underline{t}_i$ and $t'_i > t_i$, then $u_i(x') > u_i(x)$.

Condition (U1) merely formalizes the idea that the private good is consumed privately, and (U2) formalizes the idea that it is indeed a "good." (U3) is self-explanatory. (U4) means that the private good is "necessary," in that being at the feasibility constraint is arbitrarily worse for an agent than being above it. It is vacuously satisfied if $\underline{t}_i = -\infty$.

The objectives of the social planner are formulated as a social choice rule, $F \colon \mathcal{U} \to 2^X$, where $F(u)$ represents the set of social optima at utility profile u. I consider classes of social choice rules isolated by the following conditions. Let $U_i(x) = \{x' \in X | u_i(x') > u_i(x)\}$ denote i's strict upper contour set at x. Note that (F3) is predicated on (S1).

[9] As is standard, the relation $\gg$ between vectors denotes component-wise $>$, and $\geq$ denotes component-wise $\geq$.

(F1) For all $u \in \mathcal{U}$, if $x \in F(u)$, then $t \gg \underline{t}$.

(F2) There exists a mapping Φ: $\left(\prod_{i=1}^{n} 2^X\right) \times X \to \Re_+$ such that, for all $u \in \mathcal{U}$, (i) $x \in F(u)$ if and only if

$$\Phi(U_1(x), \ldots, U_n(x), x) \quad = \quad 0,$$

and (ii) if $\Phi(U_1, \ldots, U_n, x) = 0$ and $U_i' \subseteq U_i$ for all i, then $\Phi(U_1', \ldots, U_n', x) = 0$.

(F3) There exists a mapping ϕ: $\Re^{mn} \times X \to \Re_+$ such that $x \in F(u)$ if and only if, for all $i \in N$, there exist $p^i \in \Re^m$ such that

$$\phi(p^1, \ldots, p^n, x) = 0$$

and, for all $i \in N$ and all $x' \in U_i(x)$, $p^i \cdot (s' - s) + (t_i' - t_i) > 0$.

Condition (F1) says that it is not socially optimal to place any agent at its individual feasibility constraint.

I claim that (F2) is equivalent to Maskin's (1977, 1999) condition of *monotonicity:* for all $x \in X$ and all profiles $u, u' \in \mathcal{U}$, if $x \in F(u)$ and, for all i, $U_i'(x) \subseteq U_i(x)$, then $x \in F(u')$. Every social choice rule satisfying (F2) is clearly monotononic. For the converse, let F be monotonic and define $\Phi(U_1, \ldots, U_n, x) = 0$ if there exists $u \in \mathcal{U}$ such that $x \in F(u)$ and, for all $i \in N$, $U_i \subseteq U_i(x)$; otherwise, $\Phi(U_1, \ldots, U_n, x) = 1$. Thus, (F2) is satisfied by every Nash implementable social choice rule. The Pareto social choice rule, for example, has a particularly simple representation fulfilling (F2): set $\Phi(U_1, \ldots, U_n, x) = 0$ if $\bigcap_{i=1}^{n} U_i = \emptyset$, and set it equal to one otherwise.

Condition (F3) parallels (F2) but it is somewhat stronger than monotonicity: it says that the social optimality of an outcome is determined exclusively by price vectors supporting the strict upper contour sets of the agents. For an example of such a rule, assume (S1) and (X2), and define property rights by a system of endowments and profit shares: agent i's endowment and profit share are denoted $(\sigma^i, \tau_i) \in S \times (\underline{t}_i, \infty)$ and $\theta_i \in [0, 1]$, respectively, and the "social endowment" is denoted (σ, τ), where $\sigma = \sum_{i=1}^{n} \sigma^i$ and $\tau = (\tau_1, \ldots, \tau_n)$. I assume $(\sigma, \tau) \in X$ and $\sum_{i=1}^{n} \theta_i = 1$. In this case, the normative and positive properties of a particular class of outcomes has generated much interest:

Definition: The outcome $x \in X$ is a *constrained Lindahl outcome* relative to (σ^i, τ_i), $i = 1, \ldots, n$, if there are price vectors, $p^i \in \Re^m, i = 1, \ldots, n$, such that

(i) $p^i \cdot s + t_i = (\sum_{j=1}^n p^j) \cdot \sigma^i + \tau_i + \theta_i \pi(\sum_{j=1}^n p^j, x)$ for all i, where

$$\pi(p, x) = p \cdot (s - \sigma) + \sum_{j=1}^{n} (t_j - \tau_j),$$

(ii) for all $x' \in X$, $u_i(x') > u_i(x)$ implies $p^i \cdot s' + t_i' > p^i \cdot s + t_i$,

(iii) for all $x' \in X$, $\pi(\sum_{i=1}^n p^i, x) \geq \pi(\sum_{i=1}^n p^i, x')$.

Prices satisfying conditions (i)-(iii) are called *Lindahl prices*. Since I always assume (U2), the price of the transferable good is always positive at a constrained Lindahl outcome, so it is without loss of generality that I normalize it to one. This also justifies defining the agent i's budget constraint with an equality in condition (i). Defining $\lambda\colon \Re^m \times X \to \Re$ by

$$\lambda(p, x) = \min\{1, \sup_{x' \in X} \pi(p, x') - \pi(p, x)\},$$

condition (iii) may be replaced by $\lambda(\sum_{i=1}^n p^i, x) = 0$. The constrained Lindahl social choice rule, which chooses the set of Lindahl outcomes at every profile u, satisfies (F3) and actually has a continuous representation in terms of prices,

$$\phi_L(p^1, \ldots, p^n, x) = \lambda(\sum_{j=1}^n p^j, x) + \sum_{i=1}^n |(\sum_{j=1}^n p^j) \cdot \sigma^i + \tau_i + \theta_i \pi(\sum_{j=1}^n p^j, x) - p^i \cdot s - t_i|,$$

which I will use in the next section.

A special case of the planning problem is that of implementing the constrained Walrasian allocations of a private good economy with k goods. Such an environment may be formulated in the above framework by designating one good, say good 1, as numeraire, consumed in amount t_i by agent i, and letting a state vector $s \in \Re_+^{n(k-1)}$ list each agent's consumption of other goods. In the case of pure exchange, for example, we let X be the allocations distributing no more than the total endowment of each good. The constrained Lindahl allocations, as defined above, are then equivalent to the constrained Walrasian allocations.

3 Implementing Lindahl Outcomes

I construct a mechanism to implement the constrained Lindahl social choice rule with convex production technology, with individual feasibility constraints, without assuming differentiability. The mechanism is continuous, and its message space is comparable to or smaller than other mechanisms implementing the constrained Lindahl outcomes. Because private good economies are a special case of the model, the mechanism also implements the constrained Walrasian outcomes. I assume (S1) and (X2) and fix endowments (σ^i, τ_i) and profit shares θ_i, $i = 1, \ldots, n$, as in the definition of constrained Lindahl outcome. As stated, the theorem below does not address the issue of existence of Lindahl outcomes — see Foley (1970) for conditions under which Lindahl, and therefore constrained Lindahl, outcomes exist.

I will have each agent i announce a price vector $p^i \in \Re^m$ and an outcome $x^i \in X$, the latter used for two purposes: to define a "reference" outcome, $\overline{x}$, and to allow the agent to choose from a "budget set" of outcomes. Each agent i's budget set will be defined by the price vector p^{i-1}, announced by agent $i-1$, and by the reference outcome, $\overline{x} = (\overline{s}, \overline{t})$, defined next. Let

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} \overline{x}^i,$$

where

$$\begin{aligned} \overline{x}^i &= \alpha_i x^i + \frac{1-\alpha_i}{n-1}\sum_{j \neq i} x^j \\ \alpha_i &= \min\{1, \sum_{j,k \neq i} \|x^j - x^k\|\}. \end{aligned}$$

Agent i's reported outcome, $x^i = (s^i, t^i)$, will be restricted so that $t^i \gg \underline{t}$, implying $\overline{t} \gg \underline{t}$. Note that $\overline{x}$ is a continuous function of the agents' reports, and, when all agents announce the same outcome, no agent can change $\overline{x}$ unilaterally.

A complication, if we also interpret x^i as a requested outcome, is that it may be outside the agent's budget set. In this case, x^i must be replaced with another outcome, $\tilde{x}^i = (\tilde{s}^i, \tilde{t}^i)$, in a way that satisfies four criteria: (i) $\tilde{x}^i$ must be in i's budget set, (ii) it must be equal to x^i when x^i is in the budget set, (iii) it must be a continuous function of the agents' reports, and (iv) we will need $\tilde{t}^i \gg \underline{t}$. Simply taking $\tilde{x}^i$ to be the point in i's budget set closest to x^i may inadvertently place some agents at their individual feasibility constraints. Multiplying x^i by a scalar will not work generally, because I do not require $p^{i-1} \gg 0$, which means it may not be possible to scale x^i to reach agent i's budget set. I modify the former approach. Let $\xi^i \in X$ denote the solution to

$$\min_{x \in X} ||x^i - x||$$

$$\text{s.t. } p^{i-1} \cdot s + t_i \leq p^{i-1} \cdot \overline{s} + \overline{t}_i,$$

which exists, since $\overline{x}$ satisfies the constraint, and is unique, by (X2). I will define $\tilde{x}^i$ as a weighted combination of ξ^i and $\overline{x}$, namely,

$$\tilde{x}^i = \beta_i \overline{x} + (1 - \beta_i)\xi^i,$$

where

$$\beta_i = \min\{1, ||x^i - \xi^i||\}.$$

Note that, since $\overline{t} \gg \underline{t}$ and $t^i \gg \underline{t}$, this specification meets the criteria outlined above.

Formally, assuming (X1) and (X2), define the mechanism μ_1 as follows. Agent 1 reports $(p^1, x^1, z^1) \in \Re^m \times X \times \Re$ with $t^1 \gg \underline{t}$; agent 2 reports $(p^2, x^2, z^2) \in \Re^m \times X \times \Re$ with $t^2 \gg \underline{t}$; and, for $i \geq 3$, agent i reports $(p^i, x^i) \in \Re^m \times X$ with $t^i \gg \underline{t}$. Given such reports, let $\hat{x} = \frac{1}{n}\sum_{i=1}^n \tilde{x}^i$. The outcome of the mechanism is $x = (s, t)$, where $s = \hat{s}$ and the private good is allocated as follows:

$$\begin{aligned} t_1 &= \max\{\underline{t}_1, \hat{t}_1 - |z^1 - z^2 + \phi_L(p^n, p^1, \ldots, p^{n-1}, \hat{x}) + \sum_{j=1}^n ||x^j - \hat{x}|| \,|\} \\ t_2 &= \max\{\underline{t}_2, \hat{t}_2 - |z^2 - z^1|\} \\ t_i &= \hat{t}_i, \end{aligned}$$

with i ranging over $i \geq 3$. That is, the allocation of private good is as in $\hat{x}$, but we modify the consumption of agents 1 and 2 depending on the outcome of a "matching game."

Theorem 1 *Assume $n \geq 3$. Under (S1), (X1), (X2), and (U1)-(U3), if x^* is a Nash equilibrium outcome of μ_1, then it is a constrained Lindahl outcome. Conversely, under (S1), (X1), (X2), (U1), (U2), and (U4) if x^* is a constrained Lindahl outcome, then it is a Nash equilibrium outcome of μ_1.*

Suppose x^* is a Nash equilibrium outcome of μ_1. By (U2), in equilibrium, agent 2 sets $z^2 = z^1$, and agent 1 sets

$$z^1 = z^2 - \phi_L(p^n, p^1, \ldots, p^{n-1}, \hat{x}) - \sum_{j=1}^n ||x^j - \hat{x}||,$$

which implies

$$\phi_L(p^n, p^1, \dots, p^{n-1}, \hat{x}) + \sum_{j=1}^{n} ||x^j - \hat{x}|| = 0.$$

Recalling that $\phi_L(p^n, p^1, \dots, p^{n-1}, \hat{x}) \geq 0$, this implies that each term in the sum is equal to zero. In particular, each agent i reports $x^i = \hat{x}$, which is therefore equal to $\overline{x}$, which is equal to $\tilde{x}^i$ for all i, which is therefore equal to x^*. As a consequence,

$$\phi_L(p^n, p^1, \dots, p^{n-1}, x^*) = 0,$$

so it remains only to be shown that each agent i's strict upper contour set at x^* is supported by p^{i-1}. Suppose there is some $i \in N$ and some $x' \in X$ such that $u_i(x') > u_i(x)$ and $p^{i-1} \cdot s' + t_i' \leq p^{i-1} \cdot s^* + t_i^*$. If i deviates by reporting $x^i = x'$, then $\overline{x}$ is unaffected, but since

$$p^{i-1} \cdot s' + t_i' \leq p^{i-1} \cdot s^* + t_i^* = p^{i-1} \cdot \overline{s} + \overline{t}_i,$$

we have $\tilde{x}^i = x'$. Thus, i's deviation changes the outcome of the mechanism to $\frac{1}{n}x' + \frac{n-1}{n}x^*$.[10] By (U3), this outcome is preferred to x^*, contradicting our assumption that x is an equilibrium outcome.

Conversely, if x^* is a constrained Lindahl outcome, then (U4) implies $t^* \gg \underline{t}$. Therefore, it is supported as a Nash equilibrium by the strategy profile in which each agent i reports x^* and agent $i+1$'s Lindahl price at x^*, and agents 1 and 2 report $z^1 = z^2 = 0$. To see that this strategy profile is an equilibrium, note that $\overline{x} = x^*$ and that no agent can change $\overline{x}$ unilaterally. Therefore, because agent $i+1$ announces $x^{i+1} = x^* = \overline{x}$, deviating from p^i does not affect $\tilde{x}^{i+1}$ or the outcome of the mechanism (except that agent 1's fine may be increased). Of course, deviating from $x^i = x^*$ can only move $\tilde{x}^i$ to other outcomes in agent i's budget set (and possibly increase agent 1's fine), which is unprofitable. Clearly, agents 1 and 2 have no incentives to change their numerical reports.

Remark: The penalties imposed on agents 1 and 2 can be bounded by arbitrarily low $\epsilon > 0$. For example, agent 2's private good level can be defined as

$$t_2 = \max\{\underline{t}_2, \hat{t}_2 - \min\{\epsilon, |z^2 - z^1|\}\},$$

and agent 1's can be defined similarly.

[10]The deviation could also change the value of ϕ_L, so, if $i = 1$, then an offsetting change in z^1 could be needed.

Remark: Condition (F1), which requires that no agent be at its individual feasibility constraint, can be relaxed — substantially, if a more complex matching construction is used. The restriction is only used to induce matching incentives for agents 1 and 2, so it is enough if neither agents 1 nor 2 receive their lower bounds at a social optimum. A much weaker condition is the following: it is never the case that $n-1$ agents receive their lower bounds at a social optimum. I can use this condition if I give every agent an incentive to match every other agent, i.e., if each agent i reports $z^i \in \Re^n$ and i's private good level is

$$\begin{aligned} t_i &= \max\{\underline{t}_i, \hat{t}_1 - \sum_{j>i} |z^i_j - z^j_i + \phi_L(p^n, p^1, \ldots, p^{n-1}, \hat{x}) + \sum_{k=1}^{n} \|x^k - \hat{x}\| \,| \\ &\quad - \sum_{j<i} |z^i_j - z^j_i|\}. \end{aligned}$$

Thus, every pair of agents plays the matching game. If $\hat{t}_i > \underline{t}_i$ and $\hat{t}_j > \underline{t}_j$ for any two agents, then these agents will have incentives to match, ensuring that the terms in question are equal to zero in equilibrium.

4 Social Choice Rules Based on Prices

In this section, I consider the implementation of a class of social choice rules generalizing the constrained Lindahl rule: those based on prices, in the sense of (F3). It should be clear from the role played by ϕ_L in Section 3 that exactly the same techniques used there can be applied to any social choice rule satisfying (F3), a class that also includes Kaneko's (1977a,b) ratio equilibria, generalized by Diamantras and Wilkie (1994), and the cost share equilibria of Mas-Colell and Silvestre (1989). As long as the ϕ representation is continuous, this construction gives us a continuous mechanism implementing a wider class of social choice rules than those considered by Dutta, Sen, and Vohra (1995) and Saijo, Tatamitani, and Yamato (1996), in more general social planning problems — differentiability is dropped and their interiority assumptions are weakened. Moreover, with a minor modification of the mechanism, condition (X2) can be relaxed to allow for indivisibilities of social states.

Assuming (X1), (X2), and (F3), the mechanism μ_2 is defined in the fashion of μ_1. Agent 1 reports $(p^1, x^1, z^1) \in \Re^m \times X \times \Re$ with $t^1 \gg \underline{t}$; agent 2 reports $(p^2, x^2, z^2) \in \Re^m \times X \times \Re$ with $t^2 \gg \underline{t}$; and, for $i \geq 3$, agent i reports $(p^i, x^i) \in \Re^m \times X$ with $t^i \gg \underline{t}$. Given such reports, define $\overline{x}$, $\tilde{x}^i$, $i = 1, \ldots, n$, and $\hat{x}$ as in Section 3. The outcome of the mechanism is $x = (s, t)$, where $s = \hat{s}$ and private good allocated as follows:

$$\begin{aligned} t_1 &= \max\{\underline{t}_1, \hat{t}_1 - |z^1 - z^2 + \phi(p^n, p^1, \ldots, p^{n-1}, \hat{x}) + \sum_{j=1}^{n} ||x^i - \hat{x}|| \,|\} \\ t_2 &= \max\{\underline{t}_2, \hat{t}_2 - |z^2 - z^1|\} \\ t_i &= \hat{t}_i, \end{aligned}$$

with i ranging over $i \geq 3$. Thus, the only difference between μ_2 and μ_1 is the use of an arbitrary ϕ rather than ϕ_L.

Theorem 2 *Assume $n \geq 3$. Under (S1), (X1), (X2), (U1)-(U3), and (F3), if x^* is a Nash equilibrium outcome of μ_2 then $x \in F(u)$. Conversely, under (S1), (X1), (X2), (U1), (U2), (F1), and (F3), if $x^* \in F(u)$ then it is a Nash equilibrium outcome of μ_2.*

The proof is identical to that of Theorem 3, with ϕ in place of ϕ_L. Similar remarks apply, in addition to the following.

Remark: Condition (X2) is easily dropped to allow for indivisibilities in social states if continuity of the mechanism can be sacrificed. The main change needed is the redefinition of $\overline{x}$ and $\hat{x}$. Instead, let $\overline{x}$ be arbitrary and define $\hat{x}$ as

$$\hat{x} = \begin{cases} x & \text{if } x^i = x \text{ for all } i \in N \text{, or there exists } j \in N \text{ such that} \\ & p^{j-1} \cdot (s^j - s) + (t^j_j - t_j) > 0 \text{ and, for all } i \neq j,\ x^i = x, \\ x^j & \text{if there exist } x \in X \text{ and } j \in N \text{ such that } x^i = x \text{ for} \\ & \text{all } i \neq j \text{ and } p^{j-1} \cdot (s^j - s) + (t^j_j - t_j) \leq 0, \\ \overline{x} & \text{else.} \end{cases}$$

If x^* is a Nash equilibrium outcome of the modified mechanism, then, by the arguments of the previous section, each agent i announces outcome $x^i = \hat{x} = x^*$, and $\phi(p^n, p^1, \ldots, p^{n-1}, x^*) = 0$. Clearly, $p^{i-1} \cdot (x' - x^*) > 0$ for all $i \in N$ and all $x' \in U_i(x^*)$, for otherwise a deviation to $x^i = x'$ makes agent i better off. By (F3), $x^* \in F(u)$. If $x^* \in F(u)$, then it is supported as a Nash equilibrium by the strategy profile in which agent i reports x^* and agent $i + 1$'s price, given by (F3), and agents 1 and 2 report zero.

5 Monotonic Social Choice Rules

In this section, I extend the approach of Sections 3 and 4 to general social planning problems, imposing only monotonicity on the social choice rule and assuming only the existence of a private good consumed by each agent. In this context, announcements of price vectors are replaced by announcements of strict upper contour sets, yielding a mechanism with a message space comparable to that of McKelvey's (1989) Mechanism II and smaller than that of his Mechanism III, which he uses to implement the constrained Lindahl allocations. Because transferability of the private good, condition (X2), is not assumed, the general mechanism constructed below can be used to implement social choice rules violating no-veto-power.

Assuming (X1) and (F2), the mechanism μ_3 is defined as follows. Agent 1 reports $(U^1, x^1, z^1) \in 2^X \times X \times \{-1, 1\}$ with $t^1 \gg \underline{t}$; agent 2 reports $(U^2, x^2, z^2) \in 2^X \times X \times \{-1, 1\}$ with $t^2 \gg \underline{t}$; and, for $i \geq 3$, agent i reports $(U^i, x^i) \in 2^X \times X$ with $t^i \gg \underline{t}$. Given such reports, define $\hat{x}$, following the construction in the last remark, by

$$\hat{x} = \begin{cases} x & \text{if } x^i = x \text{ for all } i \in N, \text{ or there exists } j \in N \\ & \text{such that } x^j \in U^{j-1} \text{ and, for all } i \neq j,\ x^i = x, \\ x^j & \text{if there exist } x \in X \text{ and } j \in N \text{ such} \\ & \text{that } x^i = x \text{ for all } i \neq j \text{ and } x^j \notin U^{j-1}, \\ \overline{x} & \text{else,} \end{cases}$$

where $\overline{x}$ is arbitrary. The outcome of the mechanism is $x = (s, t)$, where $s = \hat{s}$ and the private good is allocated as follows:

$$\begin{aligned} t_1 &= \max\{\underline{t}_1, \hat{t}_1 - |z^1 - z^2 + \Delta|\} \\ t_2 &= \max\{\underline{t}_2, \hat{t}_2 - |z^2 - z^1|\} \\ t_i &= \hat{t}_i, \end{aligned}$$

where $\Delta = 0$ if

$$\Phi(U^n, U^1, \ldots, U^{n-1}, \hat{x}) + \sum_{j=1}^{n} d(x^j, \hat{x}) = 0,$$

and it equals $2z^2$ otherwise; $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ else; and i ranges over $i \geq 3$.

Theorem 3 *Assume $n \geq 3$. Under (X1), (U1), (U2), and (F2), if x^* is a Nash equilibrium outcome of μ_3 then $x \in F(u)$. Conversely, under (X1), (U1), (U2), (F1), and (F2), if $x^* \in F(u)$ then it is a Nash equilibrium outcome of μ_3.*

Suppose x^* is a Nash equilibrium outcome of μ_3. By (U2), in equilibrium, agent 2 sets $z^2 = z^1$, and agent 1 sets $z^1 = z^2 - \Delta$, which implies $\Delta = 0$, i.e.,

$$\Phi(U^n, U^1, \ldots, U^{n-1}, \hat{x}) + \sum_{j=1}^{n} d(x^j, \hat{x}) = 0.$$

Recalling that $\Phi(U^n, U^1, \ldots, U^{n-1}, \hat{x}) \geq 0$, this implies that each term in the sum is equal to zero. That is, each agent reports $x^i = \hat{x} = x^*$ and x^* is socially optimal when the strict upper contour set of each agent i at x^* is U^{i-1}. I claim that, in fact, $U_i(x^*) \subseteq U^i$ for all i, for suppose there exist i and $x' \in U_i(x^*) \setminus U^{i-1}$. If i deviates by reporting $x^i = x'$ then the outcome of the mechanism is x',[11] contradicting our assumption that x^* is an equilibriumoutcome. Therefore,

$$\Phi(U_1(x^*), \ldots, U_n(x^*), x^*) = 0,$$

or equivalently, $x^* \in F(u)$.

Conversely, if $x^* \in F(u)$ then, by (F1), $t^* \gg \underline{t}$. Therefore, it is supported as a Nash equilibrium by the strategy profile in which agent i reports $U^i = U_{i+1}(x^*)$ and $x^i = x^*$, and agents 1 and 2 report $z^1 = z^2 = 1$. To see this, note that no agent can change $\hat{x}$ profitably, and deviations in other reports can only increase agent 1's fine.

Remark: The agents' message spaces can be restricted to reflect the domain of F: agent i can be limited to reporting U^i and x^i such that there exists $(u_1, \ldots, u_n) \in \mathcal{U}$ with $x^i \in F(u_1, \ldots, u_n)$ and $U^i = U_{i+1}(x^i)$.

As in Section 3, condition (F1) can be weakened by having every agent submit reports of -1 or 1.

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