Social Choice in the General Spatial Model of Politics

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*Jeff Banks passed away on December 21, 2000. The second and third authors wish to express their respect and admiration for Jeff as a colleague and dear friend. We will miss him.

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Abstract

This paper extends the theory of the core, the uncovered set, and the related undominated set to a general set of alternatives and an arbitrary measure space of voters. We investigate the properties of social preferences generated by simple games, we extend results on generic emptiness of the core, we prove the general nonemptiness of the uncovered and undominated sets, and we prove the upper hemicontinuity of these correspondences when the voters' preferences are such that the core is nonempty and externally stable. Finally, we give conditions under which the undominated set is lower hemicontinuous.

1 Introduction

Since the seminal work of Downs (1957), followed by Davis and Hinich's (1966) introduction of the mathematics of Euclidean preferences, Plott's (1967) treatment of contract curves and symmetry, and Kramer's (1972) adaptation of Farquharson's (1969) analysis of strategic voting in committees with spatial preferences, the spatial theory of elections and committees has occupied a prominent theoretical status within political science. As pointed out by Ordeshook (1993), "The idea of spatial preferences, of representing the set of feasible alternatives as a subset of an m-dimensional Euclidean space, of labelling the dimensions 'issues,' of assuming that people (legislators or voters) have an ideal policy on each issue, and of supposing that each person's preference (utility) decreases as we move away from his or her m-dimensional ideal policy, is now commonplace and broadly accepted as a legitimate basis for modelling electorates and parliaments." However, following the results on the generic emptiness of the majority core and McKelvey's (1976,1979) "chaos" results on intransitive social preferences in multiple dimensions, some scholars have concluded that political decisions represent arbitrary outcomes highly dependent upon the specific details of the particular institutions under consideration (cf. Riker (1980)). This point of view has been challenged by others searching for institution-free properties of social choice to provide bounds on equilibrium predictions and limit the extent of instability.

In that vein of research, the notion of uncovered set occupies a central position. The uncovered set was defined originally by Fishburn (1977) and Miller (1980) and axiomatized by Moulin (1986) in the context of tournaments, i.e., binary relations representing majority preferences of a society over a finite set of policy alternatives. McKelvey (1986) was the first to consider the uncovered set in the standard spatial model, where policy alternatives are modelled as points in a convex subset of Euclidean space and majority preferences over social policies are determined by the continuous, strictly convex preferences of a finite electorate. Under these assumptions, McKelvey demonstrates that the uncovered set is always nonempty and, under the more specific assumption that preferences are Euclidean, he shows that this set is a centrally located region of the policy space. The precise calculation of the uncovered set, or more modestely the search of bounds sharpening those discovered by McKelvey in some specific situations, is the subject of Feld, Grofman, Hartley, Kilgour, and Miller (1987) and Hartley and Kilgour (1987). A more general result on the nonemptiness of the uncovered set is derived by Bordes, Le Breton, and Salles (1992), with a yet more general result derived by Banks, Duggan, and Le Breton (2000). An important conclusion of McKelvey's (1986) is that the uncovered set bounds equilibrium outcomes in several different institutional settings, including sophisticated voting outcomes for an important class of binary trees and mixed strategy equilibria of the two-party competition game. His claim for the latter setting is formally proved and extended by Banks, Duggan, and Le

Breton (2000). Finally, Cox (1987) has shown that, when the number of voters is odd, the uncovered set collapses to the core as voter preferences are aligned to make the core nonempty.

The main objective of this paper is to contribute further to this research program while discarding the standard assumptions on the set of alternatives and nature of the electorate. We impose general topological conditions on the set of alternatives, unifying the cases of a finite set and a convex subset of Euclidean space, and we assume an arbitrary measure space of voters, capturing a finite electorate and a continuous distribution of voters as special cases. For instance, in the Euclidean setting, where preferences are parameterized by ideal points, say in $\Re^m$, we want to consider both the case where the electorate is defined by a finite number of points in $\Re^m$ and the case where the electorate is described by a density with respect to the Lebesgue measure on $\Re^m$. As observed by McKelvey, Ordeshook, and Ungar (1980), though empirical preference distributions are discrete, a substantial literature models them as continuous. For example, Downs (1957) and Tullock (1967) discussed, in very unformalized terms, the question of equilibria for continuous voter distributions. In the Euclidean setting, Davis, Degroot, and Hinich (1972) allowed for a continuum of voters and found that the existence of majority equilibrium was equivalent to the existence of a total median in the distribution of ideal points. McKelvey, Ordeshook, and Ungar (1980) have extended Plott's necessary and sufficient conditions for the existence of majority rule equilibria to the case of a continuous distribution of voters.

There are several reasons why this level of generality is desirable. First, regularity conditions across voter preferences may be more easily formalized and analyzed. This type of concern is transparent in the work of Grandmont (1978) on intermediate preferences and in the work of Caplin and Nalebuff (1988), who have shown in the Euclidean setting that, if a continuous voter distribution is described by a log-concave density and we increase the majority quota to 64% or more, then there exists an undefeated policy. Second, the continuous setting is an important step toward our understanding of finite but large electorates. When we deal with a finite electorate, we do not usually explain how the diversity of preferences among the electorate is generated, i.e., we simply consider as given a finite list of points in the relevant space of preferences. But suppose that the finite list of preferences in the electorate is a sample of independent observations from an underlying continuous distribution on preferences. Then, if the electorate is large, we can deduce from the Glivenko-Cantelli theorem (Hildenbrand (1974)) that the continuous distribution is a good "approximation" of the finite one. Consequently, if continuity results can be established on some sets, e.g., the uncovered set, the sets defined for the limit distribution will "approximate" the sets for large, finite electorates. Third, as long as elections, and not committees, are the main concern of our study, our results should not be too sensitive to specific assumptions on the number of players, e.g., oddness.

In this paper, we offer a theory of the core, the uncovered set, and the related undominated set for the general spatial model. In Section 2, we introduce in abstract terms the above choice sets and derive sufficient conditions for these choice sets to be nonempty, and we establish some additional properties of these sets. Then, in Section 3, we formally define an electorate as a simple game imposed on a measure space of voters, and we investigate the properties of electorates. In Section 4, we show how social preferences can be obtained via a simple game, and we derive sufficient conditions on the electorate to exploit the results of Section 2 on the nonemptiness of the choice sets. Notably, we introduce a dispersion condition on voter preferences that plays a role analogous to the role played by oddness in the finite context. Finally, in Section 5, we present our results on the core, the uncovered set, and the undominated set generated by electoral preferences. We first prove the generic emptiness of the core, and we prove the general nonemptiness of the uncovered set and, under slightly stronger conditions, of the undominated set. These sets are upper hemicontinuous when the voters' preferences are such that the core is nonempty and externally stable. As the supports of mixed strategy equilibria of the two-party competition game are contained in the uncovered set, this gives us upper hemicontinuity of equilibrium outcomes of electoral competition. Finally, we give conditions under which the undominated set is lower hemicontinuous, allowing us, by Michael's selection theorem (Aliprantis and Border (1999), Theorem 16.61) to take a continuous selection from its closed, convex hull. An appendix contains a general analysis of binary relations and proofs of propositions omitted from the text.

2 Choice Sets

We consider an abstract setting in this section, letting P be a strict preference relation and R a weak preference relation over a set A of alternatives, assumed to be a Hausdorff topological space. We assume here that P is irreflexive, that R is reflexive, and that the relations are dual: aRb if and only if not bPa. Note that P is asymmetric if and only if R is complete, in which case P is the asymmetric part of R.[1] We say P is a *tournament* if it is also connected, in the sense that $a \neq b$ implies aPb or bPa. Given an arbitrary relation Q on A, we denote by $Q(a)$ the set $\{b \in A : bQa\}$ and by $Q^{-1}(a)$ the set $\{b \in A : aQb\}$. An alternative a is *Q-maximal* if, for all $b \in A$, bQa implies aQb. If Q is asymmetric, this is equivalent to $Q(a) = \emptyset$. If Q is complete, this is equivalent to $Q^{-1}(a) = X$. For now, we abstract from the details of P and A, though later A will be given the interpretation of a policy space and P will represent strict social preferences, derived from an explicit collection of "winning coalitions."

A central concept in what follows is the *core* of P, defined as the set of

[1]The conditions of asymmetry and completeness are standard, but they are only used for two results at the end of this section. We state our other results without those conditions, to maximize their applicability to the analysis of social choice.

P-maximal alternatives:

$$K = \{a \in A : P(a) = \emptyset\}.$$

We define the *dominance relation* of P, denoted D, as follows: aDb if and only if $P(a) \subseteq P(b)$ and $R(a) \subseteq R(b)$, at least one inclusion strict. The *undominated set* of P consists of the D-maximal alternatives:

$$UD = \{a \in A : D(a) = \emptyset\}.$$

Define the *covering relation* of P, denoted C, as follows: aCb if and only if aPb, $P(a) \subseteq P(b)$, and $R(a) \subseteq R(b)$. Equivalently, aCb if and only if aPb and aDb. The *uncovered set* of P consists of the C-maximal alternatives:

$$UC = \{a \in A : C(a) = \emptyset\}.$$

It is clear from these definitions that $K \cup UD \subseteq UC$. The next proposition gives a condition on preferences sufficient for external stability of the core and for the nesting of these sets. Note that the condition holds if P is a tournament and A is a finite set. Given a set X, $\overline{X}$ denotes the closure of X.

Proposition 1 *Assume $R(a) = \overline{P(a)} \cup \{a\}$ for all $a \in A$. For all $a \in K$ and all $b \in A \setminus \{a\}$, aPb. In particular, K is empty or singleton, and $K \subseteq UD \subseteq UC$.*

Proof: Take any $a \in A$ and any distinct $b \in A$. If not aPb, then $b \in R(a) = \overline{P(a)} \cup \{a\}$. Since $b \neq a$, $P(a) \neq \emptyset$, contradicting $a \in K$. Thus, K cannot contain more than one element. If $a \in K$ and bDa for some $b \in A$, then $b \in R(a) = \overline{P(a)} \cup \{a\}$. Since D is asymmetric, $b \neq a$, and we again arrive at a contradiction. Thus, $K \subseteq UD$. ∎

Though K can be empty in the absence of acyclicity or semi-convexity of P, UD and UC are nonempty under rather weak assumptions. Our results on nonemptiness and external stability of the above sets follow from the general analysis of maximal elements in the appendix.

Proposition 2 *Assume $R(a)$ is compact for some $a \in A$, and $R(b)$ is closed for all $b \in A$. Then $UC \neq \emptyset$.*

The nonemptiness of the undominated set is obtained under stronger assumptions on P.

Proposition 3 *Assume $R(a)$ is compact for some $a \in A$, $R(b)$ is closed for all $b \in A$, and $R(\cdot)$ is lower hemicontinuous as a correspondence. Then $UD \neq \emptyset$.*

We now establish that, under the assumptions of Proposition 3, the undominated set is externally stable.

Proposition 4 *Assume $R(a)$ is compact for all $a \in A$ and $R(\cdot)$ is lower hemicontinuous as a correspondence. If $a \notin UD$, then there exists $b \in UD$ such that bDa.*

Under the same assumptions, the external stability of the uncovered set is also obtained.

Proposition 5 *If $R(a)$ is compact for all $a \in A$, if $R(\cdot)$ is lower hemicontinuous as a correspondence, and if $a \notin UC$, then there exists $b \in UC$ such that bCa*

Proof: Let $c \in A$ be such that cCa. If c is undominated, then it is uncovered and the claim is proved. If c is dominated, then, by Proposition 4, there is some undominated, hence uncovered, b such that bDc. Then $bDcCa$ implies bCa. ∎

We next give conditions that can be used to simplify the definitions of the domination and covering relations. A version of the following result for finite electorates with convex preferences can be found in McKelvey's (1986) Proposition 3.3. Note that the conditions of the proposition hold if P is a tournament and A is finite. Given a set X, X° denotes the interior of X.

Proposition 6

1. *If $R(a) = \overline{P(a)} \cup \{a\}$ and $R(b) = \overline{P(b)} \cup \{b\}$, then $P(a) \subseteq P(b)$ implies $R(a) \subseteq R(b)$.*

2. *If $P(a) \cup \{a\} = R(a)^\circ \cup \{a\}$ and $P(b) \cup \{b\} = R(b)^\circ \cup \{b\}$, then $R(a) \subseteq R(b)$ implies $P(a) \subseteq P(b)$.*

Proof: To prove the first part of the proposition, suppose $P(a) \subseteq P(b)$. Clearly, $\overline{P(a)} \cup \{a\} \subseteq \overline{P(b)} \cup \{a\}$. Note that $b \notin P(a)$ by irreflexivity of P, so $a \in R(b)$. Therefore,

$$R(a) \;=\; \overline{P(a)} \cup \{a\} \;\subseteq\; \overline{P(b)} \cup \{a\} \;\subseteq\; R(b),$$

as required. To prove the second part, suppose $R(a) \subseteq R(b)$. Then

$$P(a) \cup \{a\} \;=\; R(a)^\circ \cup \{a\} \;\subseteq\; R(b)^\circ \cup \{a, b\} \;=\; P(b) \cup \{a\}.$$

Since $a \notin P(a)$ by irreflexivity of P, $P(a) \subseteq P(b)$, as required. ∎

Before continuing, we add some continuity to the condition used in the second part of the previous proposition, and we note an alternative formulation of the augmented condition. It is used in several results to follow.

Lemma 1 *$P(a) \cup \{a\} = R(a)^\circ \cup \{a\}$ and $R^{-1}(a)$ is closed for all $a \in A$ if and only if $R^{-1}(a) = \overline{P^{-1}(a)} \cup \{a\}$ and $P(a)$ is open for all $a \in A$.*

Proof: First assume $P(a) \cup \{a\} = R(a)^\circ \cup \{a\}$ and $R^{-1}(a)$ is closed for all $a \in A$. Since $R^{-1}(a)$ is closed, it follows that $P(a)$, its complement, is open. Note that $b \in P^{-1}(a)$ only if $a \in R(b)^\circ$, which implies $b \in R^{-1}(a)$. Thus, $\overline{P^{-1}(a)} \subseteq R^{-1}(a)$. Since R is reflexive and $R^{-1}(a)$ is closed, we then have $\overline{P^{-1}(a)} \cup \{a\} \subseteq R^{-1}(a)$. Now take any $b \in R^{-1}(a)$ such that $b \neq a$. Suppose that $b \notin \overline{P^{-1}(a)}$, i.e., $b \in R(a)^\circ$. Since $b \neq a$, we then have $b \in P(a)$, a contradiction. Therefore, $b \in \overline{P^{-1}(a)}$, as required.

Now assume $R^{-1}(a) = \overline{P^{-1}(a)} \cup \{a\}$ and $P(a)$ is open for all $a \in A$. Since $P(a)$ is open, it follows that $R^{-1}(a)$, its complement, is closed. Note that $b \in P(a)$ implies $a \in \overline{P^{-1}(b)} \subseteq R^{-1}(b)$, so $b \in R(a)$. Thus, $P(a) \subseteq R(a)$. Since $P(a)$ is open, we then have $P(a) \cup \{a\} \subseteq R(a)^\circ \cup \{a\}$. Now take any $b \in R(a)^\circ$ such that $b \neq a$. Suppose $b \notin P(a)$, i.e., $b \in R^{-1}(a)$. We then have $b \in \overline{P^{-1}(a)}$, contradicting $b \in R(a)^\circ$. Therefore, $b \in P(a)$, as required. ∎

We now study the possibility of deriving a version of the "two-step" principle (Miller (1980)) in our abstract setting. We first prove a simple lemma, a version of which can be found in McKelvey's (1986) Proposition 3.4. For a binary relation Q on A, we denote by $Q^2(a)$ the set $Q(a) \cup \{b \in A : \exists c \in A, bQcQa\}$.

Lemma 2 *Assume R is complete and P is asymmetric.*

1. *If $a \in P^2(b)$, then $P(b) \not\subseteq P(a)$ and $R(b) \not\subseteq R(a)$.*
2. *If $P(b) \not\subseteq P(a)$ or $R(b) \not\subseteq R(a)$, then $a \in R^2(b)$.*
3. *If $R^{-1}(a) = \overline{P^{-1}(a)} \cup \{a\}$ and $P(a)$ is open for all $a \in A$, then $a \in P^2(b)$ if and only if $P(b) \not\subseteq P(a)$.*
4. *If $R(a) = \overline{P(a)} \cup \{a\}$ and $P^{-1}(a)$ is open for all $a \in A$, then $a \in P^2(b)$ if and only if $R(b) \not\subseteq R(a)$.*

Proof: To prove the first part of the lemma, suppose $a \in P^2(b)$, so either aPb or there exists $c \in A$ such that $aPcPb$. In the first case, $b \in R(b) \setminus R(a)$. In the second case, by completeness, we have $c \in R(b) \setminus R(a)$. In both cases, $R(b) \not\subseteq R(a)$ and, by asymmetry of P, $P(b) \not\subseteq P(a)$. To prove the second part of the lemma, note that $P(b) \not\subseteq P(a)$ means $aRcPb$ for some $c \in A$, and that $R(b) \not\subseteq R(a)$ means $aPcRb$. In both cases, by completeness, $a \in R^2(b)$. To prove the third part of the lemma, note that one direction follows from part 1. For the other direction, suppose $P(b) \not\subseteq P(a)$, so that there exists $c \in A$ such that $aRcPb$. If $c = b$, the argument is finished. Otherwise, $c \in \overline{P^{-1}(a)}$. Since $c \in P(b)$, an open set, there exists $d \in P^{-1}(a) \cap P(b)$, as required. To prove the fourth part of the lemma, note that one direction follows from part 1. For the other direction, suppose $R(b) \not\subseteq R(a)$, so there exists $c \in A$ such that $aPcRb$. If $c = b$, the argument is finished. Otherwise, $c \in \overline{P(b)}$. Since $c \in P^{-1}(a)$, an open set, there exists $d \in P^{-1}(a) \cap P(b)$, as required. ∎

We now state a form of the two-step principle. The first part of Proposition 7 is as in McKelvey's (1986) Proposition 4.1. There, however, McKelvey also states that UC is contained in the closure of $\bigcap_{a\in A}(P^2(a) \cup \{a\})$, but his proof contains an error, and we have not been able to verify the result.

Proposition 7 *Assume R is complete and P is asymmetric.*

1. $\bigcap_{a\in A}(P^2(a) \cup \{a\}) \subseteq UD \subseteq UC \subseteq \bigcap_{a\in A} R^2(a)$.
2. *Assume $R^{-1}(a) = \overline{P^{-1}(a)} \cup \{a\}$ and $R(a) = \overline{P(a)} \cup \{a\}$ for all $a \in A$, and P is open. If $a \in UD$ and $b \notin UD$, then $a \in P^2(b)$. If $a \in UC$ and $b \notin UC$, then $a \in P^2(b)$.*
3. *In addition, assume that $P(a) = P(b)$ implies $a = b$ for all $a, b \in A$. If $a \in UD$, then, for all $b \in A \setminus \{a\}$, $a \in P^2(b)$.*

Proof: That $\bigcap_{a\in A}(P^2(a) \cup \{a\}) \subseteq UD$ follows from the first part of Lemma 2. That $UC \subseteq \bigcap_{a\in A} R^2(a)$ follows from the second part of Lemma 2. To prove the second part of the proposition, take $a \in UD$ and $b \notin UD$. Let cDb. Since not cDa, either $P(c) \not\subseteq P(a)$, or $R(c) \not\subseteq R(a)$, or both $P(c) = P(a)$ and $R(c) = R(a)$. In the first two cases, the third and fourth parts of Lemma 2 yield $a \in P^2(c)$ and, therefore, $a \in P^2(b)$. In the last case, aDb, and at least one of $P(a) \subseteq P(b)$ and $R(a) \subseteq R(b)$ holds strictly, and Lemma 2 again implies $a \in P^2(b)$. Now take $a \in UC$ and $b \notin UC$. Let cCb. Since not cCa, there are four possible cases: the three above, which proceed as before, and aRc. If $a = c$, then, since cPb, we are done. If $a \neq c$, there exists $d \in P^{-1}(a) \cap P(b)$, as in the proof of the third part of Lemma 2. To prove the third part of the proposition, take $a \in UD$ and $b \neq a$. Then either $P(b) \not\subseteq P(a)$ or $R(b) \not\subseteq R(a)$, both implying $a \in P^2(b)$, or $P(b) = P(b)$, which implies $a = b$, a contradiction. ∎

3 Electorates

The purpose of this section is to introduce a general framework, describing an electorate as a measurable mapping from an abstract set of voters into the set of continuous weak orderings. The framework is general enough to accommodate a finite number or a continuous distribution of voters. An *electorate* consists of a probability space $(\Omega, \Sigma, \lambda)$, where Ω is a set of voters (or voter "types"), Σ a σ-algebra on Ω, and λ a probability measure, together with a preference profile ρ, formalized as follows. Let $\mathcal{R}$ denote the set of continuous weak orders on A (complete, transitive relations, closed in $A \times A$) endowed with the topology of closed convergence (Hildenbrand (1974)). Endowing $\mathcal{R}$ with the Borel σ-algebra, a *profile* ρ is a measurable mapping $\rho: \Omega \to \mathcal{R}$, where $\rho(\omega)$ is the weak preference relation of voter ω. Let $\pi(\omega)$ denote the asymmetric part of

$\rho(\omega)$, the strict preference relation of voter ω. Note that, given any $a, b \in A$, the set $\{R \in \mathcal{R} : aRb\}$ is closed in the topology of closed convergence, and, since ρ is measurable, the coalitions $\{\omega \in \Omega : a\rho(\omega)b\}$ and $\{\omega \in \Omega : a\pi(\omega)b\}$ are Σ-measurable. If Ω is a topological space and ρ is continuous, then these sets are closed and open, respectively. Except for the regularity imposed by measurability, the notion of electorate is simply the natural extension of the notion of profile used when there is a finite number of voters.

We now turn to our formal representation of the distribution of power in the electorate, a concept that underlies our analysis of social preferences in the next section.

Definition 1 *A simple game is a collection $\mathcal{W} \subseteq \Sigma$ of coalitions such that $\emptyset \notin \mathcal{W}$, $\Omega \in \mathcal{W}$, and, for all $S \in \mathcal{W}$ and all $T \in \Sigma$, $\lambda(S \setminus T) = 0$ implies $T \in \mathcal{W}$.*

The coalitions in $\mathcal{W}$ are *winning coalitions.* Note that, by our definition, winning coalitions can be thought of as equivalence classes: two sets that differ only on a set of λ-measure zero have the same status as winning or not winning. An implication is that the collection of winning coalitions cannot be defined without reference to the distribution of voter types. Furthermore, we incorporate a monotonicity condition into our definition: if $S \in \mathcal{W}$ and $T \in \Sigma$ satisfies $S \subseteq T$, then $T \in \mathcal{W}$.

The following properties of simple games will be used in the sequel. Given a set X, we denote by X^c its complement. We let $\mathcal{B}$ denote the set $\{S^c : S \notin \mathcal{W}\}$, the elements of which are *blocking coalitions,* and we let $\mathcal{L}$ denote the set $\mathcal{B}^c = \{S \in \Sigma : S^c \in \mathcal{W}\}$, the elements of which are *losing coalitions.* Note that $\mathcal{B}$ is a simple game, but $\mathcal{L}$, because it violates monotonicity, will not be.

Definition 2 *A simple game $\mathcal{W}$ is*

1. *proper if $\mathcal{W} \subseteq \mathcal{B}$.*

2. *open from below if, for all sequences $\{S_n\}$ in Σ and all $S \in \mathcal{W}$, $S_n \uparrow S$ implies there exists m such that, for all $n \geq m$, $S_n \in \mathcal{W}$.*

3. *closed from above if, for all sequences $\{S_n\}$ in $\mathcal{W}$ and all $S \in \Sigma$, $S_\alpha \downarrow S$ implies $S \in \mathcal{W}$.*

4. *λ-continuous if, for all $S \in \mathcal{W}$, there exists $\epsilon > 0$ such that, for all $T \in \Sigma$, $\lambda(S \setminus T) \leq \epsilon$ implies $T \in \mathcal{W}$.*

5. *anonymous if, for all $S \in \mathcal{W}$ and all $T \in \Sigma$, $\lambda(T) = \lambda(S)$ implies $T \in \mathcal{W}$.*

6. *semi-strong if, for all $S \in \mathcal{B}$ and all $T \in \Sigma$, $S \subseteq T$ and $\lambda(T \setminus S) > 0$ implies $T \in \mathcal{W}$.*

7. *strong if $\mathcal{B} \subseteq \mathcal{W}$.*

Note that $\mathcal{W}$ is proper if and only if $S, T \in \mathcal{W}$ implies $\lambda(S \cap T) > 0$. The next proposition establishes connections between some pairs of concepts defined above.

Proposition 8 *$\mathcal{W}$ is open from below if and only if $\mathcal{B}$ is closed from above.*

Proof: By definition, $\mathcal{W}$ is open from below if and only if, for all sequences $\{S_n\}$ in Σ and all $S \in \mathcal{W}$, $S_n \uparrow S$ implies there exists m such that, for all $n \geq m$, $S_n \in \mathcal{W}$. Equivalently: if $S_n \uparrow S$ and there is some subsequence of $\{S_n\}$ (also indexed by n) such that, for all n, $S_n \notin \mathcal{W}$, then $S \notin \mathcal{W}$. Equivalently: if $S_n^c \downarrow S^c$ and there is some subsequence of $\{S_n^c\}$ such that, for all n, $S_n^c \in \mathcal{B}$, then $S^c \in \mathcal{B}$. And the latter means that $\mathcal{B}$ is closed from above. ∎

If Ω is finite, then every simple game is clearly λ-continuous, open from below, and closed from above. The next proposition illustrates a general nesting of some of the former concepts.

Proposition 9 *If $\mathcal{W}$ is λ-continuous, then it is open from below.*

Proof: Suppose $\mathcal{W}$ is λ-continuous, and take a sequence $\{S_n\}$ in Σ and $S \in \Sigma$ such that $S_n \uparrow S$. By continuity of λ as a probability measure, we have $\lambda(S_n) \to \lambda(S)$, and then λ-continuity of $\mathcal{W}$ implies that $S_n \in \mathcal{W}$ for high enough n. ∎

The next proposition establishes an implication of openness from below when voters are continuously distributed. Note that the proof of the proposition does not rely on monotonicity of $\mathcal{W}$ and $\mathcal{B}$, so we can use it to prove that, if $\mathcal{B}$ is closed from above, then $\mathcal{W}$ is not.

Proposition 10 *Assume λ is non-atomic. If $\mathcal{W}$ is open from below, then $\mathcal{B}$ is not open from below.*

Proof: Assume $\mathcal{W}$ is open from below, and suppose $\mathcal{B}$ is also open from below. Let $S_1 \in \mathcal{W}$. Since $\mathcal{W}$ is open from below and λ is non-atomic,

$$E_1 = \{\epsilon > 0 \mid \exists T \in \mathcal{W} : T \subseteq S_1, \lambda(S_1 \setminus T) \geq \epsilon\}$$

is nonempty. Let $\epsilon_1 = \sup E_1 > 0$, and take $S_2 \in \mathcal{W}$ such that $S_2 \subseteq S_1$ and $\lambda(S_1 \setminus S_2) \geq \epsilon_1/2$. Since $\mathcal{W}$ is open from below and λ is non-atomic,

$$E_2 = \{\epsilon > 0 \mid \exists T \in \mathcal{W} : T \subseteq S_2, \lambda(S_2 \setminus T) \geq \epsilon\}$$

is nonempty. Let $\epsilon_2 = \sup E_2 > 0$, and take $S_3 \in \mathcal{W}$ such that $S_3 \subseteq S_2$ and $\lambda(S_2 \setminus S_3) \geq \epsilon_2/2$, and so on. Note that $\epsilon_n \to 0$. Define $S = \bigcap_{n=1}^{\infty} S_n$. If $S \notin \mathcal{W}$, then $S^c \in \mathcal{B}$, and, because $\mathcal{B}$ is open from below, we have $S_n^c \in \mathcal{B}$ for high enough n, contradicting $S_n \in \mathcal{W}$. Thus, $S \in \mathcal{W}$. Note that $\lambda(S) > 0$, for otherwise $\emptyset \in \mathcal{W}$. But then there exists $T \in \Sigma$ such that $T \subseteq S$ and $\lambda(S \setminus T) > 0$. Take n high enough that $\epsilon_n < \lambda(S \setminus T)$, and note that $\lambda(S_n \setminus T) \geq \lambda(S \setminus T) > \epsilon_n$, a contradiction. Therefore, $\mathcal{B}$ is not open from below, as claimed. ∎

If $\mathcal{W}$ is open from below and anonymous, it is easy to see that it is simply defined by a quota $q \in [0,1]$, as follows:

$$S \in \mathcal{W} \text{ if and only if } \lambda(S) > q,$$

where $q \geq 1/2$ if $\mathcal{W}$ is proper. In fact, quota rules of this form are actually λ-continuous. Of course, strong implies semi-strong. The two conditions are distinguished from each other in one important case: if Ω is finite, then majority rule is always semi-strong but not always strong, e.g., when λ is the counting measure and the number of voters is even. Note that $\mathcal{W}$ is strong if and only if $\mathcal{B}$ is proper; and $\mathcal{W}$ is proper if and only if $\mathcal{B}$ is strong.

In models with a finite number of voters, majority rule (i.e., $\mathcal{W} = \{S \in \Sigma : \lambda(S) > 1/2\}$) with n odd and dictatorship (i.e., $\mathcal{W} = \{S \in \Sigma : \omega' \in S\}$ for some $\omega' \in \Omega$) are examples of proper, strong simple games. If the electorate is a continuum and voters are continuously distributed, however, majority rule is not strong and dictatorship is not proper. In the case of majority rule, we can always partition Ω into two sets of exactly equal λ-measure, neither constituting a majority. In the case of dictatorship, $\{\omega'\}$ has λ-measure zero, and so $\emptyset \in \mathcal{W}$, contrary to the definition of a simple game. But there is an interesting alternative to pure dictatorship: define $\mathcal{W}$ to consist of the coalitions containing an open set around ω'. As long as λ has a positive density, so that every open set has positive λ-measure, this simple game is proper. Moreover, given any continuous profile ρ, voter ω' is, in fact, a dictator: if $a\pi(\omega')b$, then, by continuity, an open set around ω' will share that strict preference, so aPb. The collection of winning coalitions in the latter example is not strong, however.

Do proper, strong simple games exist when the electorate is a continuum and voters are continuously distributed? As the next proposition shows, the answer is affirmative. In fact, we prove the existence of a proper and strong simple game such that, for every $S, T \in \mathcal{W}$, we have $S \cap T \in \mathcal{W}$.[2]

Proposition 11 *There exists $\mathcal{W}$ that is proper and strong and such that, for all $S, T \in \mathcal{W}$, $S \cap T \in \mathcal{W}$.*

Proof: We call $\mathcal{F} \subseteq \Sigma$ a measurable filter if (i) $X \in \mathcal{F}$, (ii) $\emptyset \notin \mathcal{F}$, (iii) for all $S \in \mathcal{F}$ and all $T \in \Sigma$, $\lambda(S \setminus T) = 0$ implies $T \in \mathcal{F}$, and (iv) for all $S, T \in \mathcal{F}$, $S \cap T \in \mathcal{F}$. Let $\mathbf{F}$ be the collection of all measurable filters. We claim there is a maximal element $\mathcal{F}^*$ in $\mathbf{F}$. To see this, first note that

$$\{S \in \Sigma \mid \lambda(S) = 1\} \in \mathbf{F},$$

so the collection is nonempty. Take any chain $\mathbf{C}$ of measurable filters, and note that $\bigcup \mathbf{C}$ is itself a measurable filter. Thus, existence of $\mathcal{F}^*$ follows from Zorn's

[2]Technically, the simple game we establish differs from an ultrafilter in that $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$ applies only to measurable sets S. Our simple game is necessarily "free," in the sense that $\bigcap \mathcal{W} = \emptyset$, when λ is non-atomic.

lemma. Now we claim that $\mathcal{F}^*$ is strong, in the sense that, for all $S \in \Sigma$, either $S \in \mathcal{F}^*$ or $S^c \in \mathcal{F}^*$. Suppose not, and define

$$\mathcal{F}^S = \{T \in \Sigma \mid T \in \mathcal{F}^* \text{ or } \lambda((V \cap S) \setminus T) = 0 \text{ for some } V \in \mathcal{F}^*\}.$$

Note the following properties of this collection.

1. $X \in \mathcal{F}^S$.

This is obvious.

2. For all $S' \in \mathcal{F}^S$ and all $T' \in \Sigma$, $\lambda(S' \setminus T') = 0$ implies $T' \in \mathcal{F}^S$.

Take any $S' \in \mathcal{F}^S$ and $T' \in \Sigma$ such that $\lambda(S' \setminus T') = 0$. If $S' \in \mathcal{F}^*$, then $T' \in \mathcal{F}^*$ follows, so $T' \in \mathcal{F}^S$. Otherwise, there exists $V \in \mathcal{F}^*$ such that $\lambda((V \cap S) \setminus S') = 0$. Then $\lambda((V \cap S) \setminus T') = 0$, so $T' \in \mathcal{F}^S$.

3. For all $S', T' \in \mathcal{F}^S$, $S' \cap T' \in \mathcal{F}^S$.

Take any $S', T' \in \mathcal{F}^S$. If both coalitions are in $\mathcal{F}^*$, then $S' \cap T' \in \mathcal{F}^*$, so $S' \cap T' \in \mathcal{F}^S$. If $S' \in \mathcal{F}^*$ and $T' \notin \mathcal{F}^*$, then there exists $V \in \mathcal{F}^*$ such that $\lambda((V \cap S) \setminus T') = 0$. Then $S' \cap V \in \mathcal{F}^*$, and $\lambda((S' \cap V \cap S) \setminus (S' \cap T')) = 0$, so $S' \cap T' \in \mathcal{F}^S$. If $S', T' \notin \mathcal{F}^*$, then there exist $V, W \in \mathcal{F}^*$ such that $\lambda((V \cap S) \setminus S') = \lambda((W \cap S) \setminus T') = 0$. Then $V \cap W \in \mathcal{F}^*$, and $\lambda((V \cap W \cap S) \setminus (S' \cap T')) = 0$, so $S' \cap T' \in \mathcal{F}^S$. Of course, conditions $1 - 3$ hold for

$$\mathcal{F}^{S^c} = \{T \in \Sigma \mid T \in \mathcal{F}^* \text{ or } \lambda((V \cap S^c) \setminus T) = 0 \text{ for some } V \in \mathcal{F}^*\}$$

as well. Finally, suppose $\emptyset \in \mathcal{F}^S \cap \mathcal{F}^{S^c}$. Since $\emptyset \in \mathcal{F}^S \setminus \mathcal{F}^*$, there exists $V \in \mathcal{F}^*$ such that $\lambda((V \cap S) \setminus \emptyset) = 0$, i.e., $\lambda(V) = \lambda(V \cap S^c)$. Similarly, there exists $W \in \mathcal{F}^*$ such that $\lambda(W) = \lambda(W \cap S)$. Then $\lambda(V \cap W) = 0$, but $V \cap W \in \mathcal{F}^*$, so $\emptyset \in \mathcal{F}^*$, a contradiction. Therefore, assume without loss of generality that $\emptyset \notin \mathcal{F}^S$, which implies $\mathcal{F}^S \in \mathbf{F}$, contradicting maximality of $\mathcal{F}^*$. Therefore, $\mathcal{F}^*$ is strong, and we may set $\mathcal{W} = \mathcal{F}^*$. ∎

As we formalize at the end of the next section, proper and strong simple games are "rare" when the preferences of voters are sufficiently heterogeneous: such simple games are inconsistent with a fundamental continuity property of social preferences.

4 Social Preferences

Given an electorate and a proper simple game, social preferences are determined as follows. For $S \in \Sigma$, let

$$P_S = \bigcap_{\omega \in S} \pi(\omega) \text{ and } R_S = \bigcap_{\omega \in S} \rho(\omega),$$

and define strict social preference, P, and weak social preferences, R, as

$$P = \bigcup_{S \in \mathcal{W}} P_S \text{ and } R = \bigcup_{S \in \mathcal{B}} R_S.$$

Equivalently, aPb if the set of voters who strictly prefer a to b is a winning coalition; and aRb if the set of voters who strictly prefer a to be is not a winning coalition. Thus, $\emptyset \notin \mathcal{W}$ implies that P is irreflexive, and $\Omega \in \mathcal{B}$ implies R is reflexive. Moreover, R is complete and P is the asymmetric part of R if $\mathcal{W}$ is proper (i.e., $\mathcal{B}$ is strong). When we wish to emphasize the dependence of P and R on the preference profile ρ, we write $P[\rho]$ and $R[\rho]$. In this section, we analyze the continuity properties of strict and weak social preferences in terms of primitive assumptions on electorates. We first prove our main result on continuity of strict social preferences.

Proposition 12 *Assume A is first countable. If $\mathcal{W}$ is open from below, then P is open and R is closed.*

Proof: Suppose aPb, or equivalently, $(a,b) \in P$. Thus, there exists $S \in \mathcal{W}$ such that $(a,b) \in \bigcap_{\omega \in S} \pi(\omega)$. Let $\{G_n\}$ be a countable neighborhood base of (a,b) in $A \times A$, and assume without loss of generality that it is decreasing. Since $\rho(\omega)$ is continuous for each ω, $\pi(\omega)$ is open for each ω. Therefore, letting $S^n = \{\omega \in S : G_n \subseteq \pi(\omega)\}$, we have $S^n \uparrow S$. Since Since $\mathcal{W}$ is open from below, there exists m such that, for all $n \geq m$, $S_n \in \mathcal{W}$. Therefore, $G_n \subseteq P$ for high enough n, so P is open. That R is closed follows by definition. ∎

From Proposition 8, it follows that P is open and R is closed if $\mathcal{B}$ is closed from above. Adding compactness of A, it follows that $R(\cdot)$ is upper hemicontinuous as a correspondence. Though Proposition 12 yields openness of P, it cannot be simultaneously be applied to $\bigcup_{S \in \mathcal{B}} P_S$, the social preference generated by blocking coalitions: Proposition 10 has shown that $\mathcal{W}$ and $\mathcal{B}$ cannot both be open from below. We can, however, derive continuity properties of strict social preferences in simple games that are not open from below, if we impose topological conditions on the electorate and a condition restricting shared weak preferences across voters. Assuming Ω is a topological space, we denote by S^* the support of λ, i.e., the smallest closed set with λ-measure one.

Definition 3 *Limited shared weak preference (LSWP) holds if, for all $a, b \in A$ with $a \neq b$, and for all $S \in \Sigma$, aR_Sb implies $a \in \overline{P_T(b) \setminus \{a\}}$, where $T = S \cap S^*$.*

Thus, if every member of S weakly prefers a to $b \neq a$, then we can approximate a by alternatives strictly preferred to b by λ-almost every member of S. One condition sufficient for LSWP is evident. Assuming A is a vector space, we say voter ω's preferences are *strictly convex* if the following condition is satisfied: for all $a \in A$, all $b \in \rho(\omega)(a)$, and all $\alpha \in (0,1)$, $\alpha a + (1-\alpha)b \in \pi(\omega)(a)$. LSWP holds if λ puts measure one on some closed subset of the voters with strictly convex preferences. Banks and Duggan (1999,2000) give several examples of other environments satisfying LSWP in the finite-voter framework. As discussed in the earlier of these papers, an important class of environment captured by the present analysis, but in which strict convexity does not hold, is private good economies.[3] Note that LSWP implies that A is infinite.

The next result is stated for an arbitrary simple game $\mathcal{S}$, and so it applies when $\mathcal{S} = \mathcal{W}$, giving us lower hemicontinuity of strict social preferences, and when $\mathcal{S} = \mathcal{B}$.

Lemma 3 *Assume A is second countable and locally compact; Ω is a compact topological space and ρ is continuous; and LSWP holds. Let $\mathcal{S}$ be an arbitrary simple game. Then $\bigcup_{S \in \mathcal{S}} P_S(\cdot)$ is lower hemicontinuous as a correspondence.*

Proof: Take any $a \in A$ and any open set $G \subseteq A$ such that $G \cap \bigcup_{S \in \mathcal{S}} P_S(a) \neq \emptyset$, i.e., $b \in G \cap \bigcup_{S \in \mathcal{S}} P_S(a)$ for some $b \in A$. Thus, $\{\omega \in \Omega : b\pi(\omega)a\} \in \mathcal{S}$, and therefore $S = \{\omega \in \Omega : b\rho(\omega)a\} \in \mathcal{S}$. Since $\{R \in \mathcal{R} : bRa\}$ is closed in the closed convergence topology, and since ρ is continuous, it follows that S is closed. Since S^* has λ-measure one, $T = S \cap S^* \in \mathcal{S}$. Since Ω is compact, and since S and S^* are closed, T is compact. By LSWP, there exists a sequence $\{c_n\}$ in $P_T(a)$ such that $c_n \to b$. Mas-Colell (1977) establishes the existence of a jointly continuous function $U: \mathcal{R} \times A \to \Re$ such that, for all $R \in \mathcal{R}$, $U(R, \cdot)$ is a utility representation of R. By construction, $U(\rho(\omega), c_n) - U(\rho(\omega), a) > 0$ for all $\omega \in T$. Since this function is continuous in ω and T is compact, it follows that $\min_{\omega \in T} U(\rho(\omega), c_n) - U(\rho(\omega), a) > 0$. By continuity, there is an open set $X_n \subseteq A$ with $a \in X_n$ such that $\min_{\omega \in T} U(\rho(\omega), c_n) - U(\rho(\omega), x) > 0$ for all $x \in X_n$. Picking n high enough, we have $c_n \in G$ and, by the preceding discussion, $c_n \in \bigcup_{S \in \mathcal{S}} P_S(x)$ for all $x \in X_n$, implying $G \cap \bigcup_{S \in \mathcal{S}} P_S(x) \neq \emptyset$ over an open set around a, as required. ∎

Lower hemicontinuity of weak social preferences, critical for the nonemptiness of the undominated set, holds under the conditions of Lemma 3. Since those conditions are satisfied if the set of voters is finite with strictly convex preferences, the next result generalizes McKelvey's (1986) Lemma 4.

Proposition 13 *Assume A is second countable and locally compact; Ω is a compact topological space and ρ is continuous; and LSWP holds. Then $P(\cdot)$ and $R(\cdot)$ are lower hemicontinuous as correspondences.*

Proof: Lower hemicontinuity of $P(\cdot)$ follows directly from Lemma 3 by setting $\mathcal{S} = \mathcal{W}$. Now take any $a \in A$ and any $b \in R(a)$. If $b \neq a$, let $S \in \mathcal{B}$ be such that $b \in R_S(a)$. By LSWP, there exists a sequence $\{c_n\}$ in $P_T(a)$ such that $c_n \to b$,

[3] A consumer's preferences may be strictly convex in his own consumption but not in others'. Since an alternative must specify bundles for all consumers, strict convexity is not satisfied.

where $T = S \cap S^* \in \mathcal{B}$. Thus, $b \in \{a\} \cup \overline{P_T(a)}$. Since $b \in R(a)$ was arbitrary, we have

$$\{a\} \cup \bigcup_{S \in \mathcal{B}} P_S(a) \subseteq R(a) \subseteq \{a\} \cup \bigcup_{S \in \mathcal{B}} \overline{P_S(a)} \subseteq \{a\} \cup \overline{\bigcup_{S \in \mathcal{B}} P_S(a)}.$$

The correspondence defined by $a \mapsto \{a\}$ is clearly lower hemicontinuous, as is $\bigcup_{S \in \mathcal{B}} P_S(\cdot)$, by Lemma 3. The union of these two correspondences is lower hemicontinuous, so $R(\cdot)$ differs from a lower hemicontinuous correspondence only at points of closure, implying that it is lower hemicontinuous. ∎

The result gives conditions under which upper sections of R are compact. The assumptions on the set of alternatives are satisfied if A is compact or if all voters have compact weak upper sections and A is a subset of finite-dimensional Euclidean space. More general sufficient conditions follow the proposition.

Proposition 14 *Assume that, for all $a \in A$ and all $\epsilon > 0$, there exists a compact set K_ϵ such that $\lambda(\{\omega \in \Omega \mid \rho(\omega)(a) \subseteq K_\epsilon\}) > 1 - \epsilon$; and assume that $\mathcal{W}$ is open from below. Then, for all $a \in A$, $R(a)$ is compact.*

Proof: Take any $a \in A$, and suppose $R(a)$ is not compact, implying that, for each n, there exists $a_n \in R(a) \setminus K_{1/n}$. Let S_n satisfy $S_n \in \mathcal{B}$ and, for all $\omega \in S_n$, $a_n \rho(\omega) a$. Note that $S_n \subseteq T_n = \{\omega \in \Omega \mid \rho(\omega)(a) \not\subseteq K_{1/n}\}$, implying $T_n \in \mathcal{B}$, for all n. We may assume, without loss of generality, that $\{K_{1/n}\}$ is an increasing sequence, implying that $\{T_n\}$ is decreasing. Note that $\lambda(T_n) \leq 1/n$ for each n. Since $\mathcal{W}$ is open from below, it follows that $\mathcal{B}$ is closed from above, implying that $T = \bigcap_{n=1} T_n \in \mathcal{B}$. But $\lambda(T) = 0$, implying $\emptyset \in \mathcal{B}$, which implies $\Omega \notin \mathcal{W}$, a contradiction. ∎

The next proposition shows that compactness of voters' weak upper sections is essentially sufficient for the condition of Proposition 14.

Proposition 15 *Assume A is second countable and locally compact, and, for all $a \in A$, $\lambda(\{\omega \in \Omega \mid \rho(\omega)(a)$ is compact$\}) = 1$. Then, for all $a \in A$ and all $\epsilon > 0$, there exists a compact set K_ϵ such that $\lambda(\{\omega \in \Omega \mid \rho(\omega)(a) \subseteq K_\epsilon\}) > 1-\epsilon$.*

Proof: By Aliprantis and Border's (1999) Lemma 2.69, A is σ-compact. By Aliprantis and Border's (1999) Corollary 2.70, A is hemi-compact, i.e., there exist compact subsets $K_1, K_2, \ldots$, such that $A = \bigcup_{n=1}^{\infty} K_n$, and for every compact $K \subseteq A$, there exists n such that $K \subseteq \bigcup_{m=1}^{n} K_m$. Let

$$S_n = \{\omega \in \Omega \mid \rho(\omega)(a) \subseteq \bigcup_{m=1}^{n} K_m\},$$

and note that $\lambda(\bigcup_{n=1}^{\infty} S_n) = 1$. Therefore, for all $\epsilon > 0$, there exists n such that $\lambda(S_n) > 1 - \epsilon$. Setting $K_\epsilon = \bigcup_{m=1}^{n} K_m$, the condition of the proposition is fulfilled. ∎

We next turn to conditions under which $R(a) = \{a\} \cup \overline{P(a)}$ for all a. An easily verified sufficient condition is that $\mathcal{W}$ is strong and all but a λ-measure zero set of voters have anti-symmetric weak preferences. As these assumptions seem quite restrictive, we seek a condition that is more intuitive, in combination with the assumption of a continuum of voters. We use a lemma that gives conditions under which $P_S(a)$ is open for all compact coalitions S.

Lemma 4 *Assume A is second countable and locally compact, Ω is a topological space, and ρ is continuous. Let S be compact. Then P_S is open.*

Proof: Let $U: \mathcal{R} \times A \to \Re$ be a jointly continuous function such that, for all $R \in \mathcal{R}$, $U(R, \cdot)$ is a utility representation of R. Take $(a, b) \in P_S$, so $U(\rho(\omega), a) > U(\rho(\omega), b)$ for all $\omega \in S$. By continuity and compactness, $\min_{\omega \in S} U(\rho(\omega), a) - U(\rho(\omega), b) > 0$. By continuity, there exists an open set $G \subseteq A \times A$ such that $\min_{\omega \in S} U(\rho(\omega), c) - U(\rho(\omega), d) > 0$ for all $(c, d) \in G$. ∎

The next condition formalizes the notion that the preferences of voters are widely distributed.

Definition 4 *Dispersion holds if, for all distinct $a, b \in A$, for all $c \in A \setminus \{a, b\}$, and for every neighborhood G of c, there exists $d \in G$ such that $\lambda(\{\omega \in \Omega: d\pi(\omega)a\pi(\omega)b\}) > 0$.*

Dispersion is satisfied, for example, if voter preferences are Euclidean with ideal points distributed over Euclidean space by a strictly positive density. In that case, take any distinct $a, b \in A$, and take any other c. Given any neighborhood G of c, we can find $d \in G$ such that a, b, d are not collinear. Then there exists e such that $||e - d|| < ||e - a|| < ||e - b||$, and these strict inequalities will hold for some open set around e. The set of voters with ideal points in this set has positive measure, fulfilling the condition. If A is finite, dispersion is satisfied if all linear orders of A are present in the preferences of the electorate. Because we use LSWP in the next proposition, however, we preclude the finite A case. In fact, if A is finite, if $\mathcal{W}$ is majority rule, and if the number of voters is even, then $R(a) = \{a\} \cup \overline{P(a)} = \{a\} \cup P(a)$ will not hold generally, even if voter preferences are dispersed.

Proposition 16 *Assume that A is second countable and locally compact; Ω is a compact topological space and ρ is continuous; LSWP holds; dispersion holds; and $\mathcal{W}$ is proper, open from below, and semi-strong. Then $R(a) = \{a\} \cup \overline{P(a)}$ for all $a \in A$.*

Proof: Because $\mathcal{W}$ is proper, P is the asymmetric part of R, so $P(a) \subseteq R(a)$. Then, by Proposition 12, we have $\{a\} \cup \overline{P(a)} \subseteq R(a)$. Now take any $b \in R(a)$ with $b \neq a$, so $S = \{\omega \in \Omega \mid b\rho(\omega)a\} \in \mathcal{B}$. Take any open set G around b. By LSWP, there is a sequence $\{c_n\}$ converging to b such that $c_n \neq b$ for all n

and $c_n\pi(\omega)a$ for all $\omega \in T$, where $T = S \cap S^* \in \mathcal{B}$. Take any $c_n \in G$. Since ρ is continuous, T is closed, in fact compact, and therefore Lemma 4 implies that $P_T(a)$ is open. Thus, $G' = P_T(a) \cap G$ is an open set around c_n, so, by dispersion, there exists $d \in G'$ such that $\lambda(\{\omega \in \Omega \mid d\pi(\omega)a\pi(\omega)b\}) > 0$. Let $T' = T \cup \{\omega \in \Omega \mid d\pi(\omega)a\pi(\omega)b\}$. Since $\{\omega \in \Omega \mid d\pi(\omega)a\pi(\omega)b\} \cap T = \emptyset$, we have $\lambda(T') > \lambda(T)$, and our assumption that $\mathcal{W}$ is semi-strong implies $T' \in \mathcal{W}$. Therefore, $d \in P(a)$. Since G was arbitrary, we have $b \in \overline{P(a)}$. ∎

McKelvey (1986) proves that, if Ω is finite, if voters have strictly convex preferences over a subset of Euclidean space, and if $\mathcal{W}$ is strong, then $R(a) = \{a\} \cup \overline{P(a)}$ for all $a \in A$. The conclusion of Proposition 16 would hold under these assumptions even if Ω and ρ were allowed to be completely general, and even without imposing LSWP. Proposition 11 establishes the existence of a strong, proper simple game, so this extension of McKelvey's result is not vacuous. Its applicability is limited, however, by the following result, which shows that, when voters' preferences over a continuous set of alternatives are sufficiently rich, all proper, strong simple games fail to generate continuous social preferences.

Proposition 17 *Assume A is path-connected; for all $a, b \in A$, $\lambda(\{\omega \in \Omega : \rho(\omega)(a) = \rho(\omega)(b)\}) = 0$; and there exists $a \in A$ with $P(a) \neq \emptyset$ and $P^{-1}(a) \neq \emptyset$. If R is closed, then $\mathcal{W}$ is not both proper and strong.*

Proof: Let bPa and aPc. Let $f\colon [0,2] \to A$ be a continuous function satisfying $f(0) = a$, $f(1) = c$, and $f(2) = b$. Let $s = \sup\{x \in [0,2] : aPf(x)\}$, and note that $1 \leq s \leq 2$. By construction, there exists an increasing sequence $\{r_n\}$ in $[0,2]$ such that $r_n \to s$ and, for all n, $aPf(r_n)$. Thus, $aRf(r_n)$ for all n. By continuity of f, $f(r_n) \to f(s)$, and, since R is closed, $aRf(s)$. Similarly, we may take a decreasing sequence $\{t_n\}$ in $[0,2]$ such that $t_n \to s$ and, for all n, $f(t_n)Ra$. Again, $f(t_n) \to f(s)$, and, since R is closed, $f(s)Ra$. So there exist $S, T \in \mathcal{B}$ such that $aR_Sf(s)$ and $f(s)R_Ta$. By assumption, $S' = \{\omega \in S : a\pi(\omega)f(s)\}$ is λ-equivalent to S, so $S' \in \mathcal{B}$, and similarly $T' = \{\omega \in T : f(s)\pi(\omega)a\} \in \mathcal{B}$. If $\mathcal{W}$ is strong, then $\mathcal{B} \subseteq \mathcal{W}$, so $S', T' \in \mathcal{W}$. But $S' \cap T' = \emptyset$, so $\mathcal{W}$ is not proper. ∎

5 Electoral Competition

Given a policy space A, and an electorate Ω with winning coalitions $\mathcal{W}$, consider the competition between two office-motivated parties where the two parties choose strategically their platform in the set A in order to maximize their chances to win the election. Precisely, the two parties play a symmetric zero-sum game where A is their common set of pure strategies and the payoff of, say party 1, for the strategy profile (a, b) is 1 if aPb, -1 if bPa, and 0 otherwise. It is easy to see that a^* is an optimal play in this game if and only if a^* is not defeated by a majority, i.e., if $a^* \in K(P)$. This means that the existence of a Nash equilibrium in pure strategies is equivalent to the nonemptiness of the core. On the other hand, the set of strategies obtained after deletion of the weakly dominated strategies is precisely the undominated set. The uncovered set, which is a superset (sometimes proper) of the undominated set, does not have a direct game-theoretic interpretation. But as demonstrated by Banks, Duggan, and Le Breton (2000) for a class of games including the two-party competition game described above, the support of every Nash equilibrium in mixed strategies is contained in the uncovered set.

In this section, we analyze properties of these sets as electoral preferences vary. Fixing the winning coalitions $\mathcal{W}$, we write $P[\rho]$ and $R[\rho]$ for the strict and weak social preferences determined by profile ρ. We write $K[\rho]$, $UC[\rho]$, and $UD[\rho]$ for the core, uncovered set, and undominated set of the social preference relations $P[\rho]$ and $R[\rho]$. We extend previous results on generic emptiness of the core to the general spatial model, and we establish nonemptiness of the uncovered and undominated sets in the general model. We then show that the three above correspondences are upper hemicontinuous at profiles with a non-empty and externally stable core. Thus, though small perturbations of preferences may (and usually will) lead to a non-empty core, this result, with our above observations on electoral competition, suggests that electoral outcomes will change continuously. Finally, we provide conditions under which the undominated set correspondence is lower hemicontinuous. Thus, the uncovered set correspondence contains a lower hemicontinuous correspondence. The next lemma on continuity of social preferences is essential to the analysis.

Assume that Ω is a Polish space, i.e., a complete and separable metric space, and that σ is the corresponding Borel σ-algebra. Let d be the metric on Ω, let $\tilde{d}$ be the metric of closed convergence on $\mathcal{R}$, let $\mathbf{P}$ be the space of profiles, and define the metric Δ on $\mathbf{P}$ as follows:

$$\Delta(\rho, \rho') \quad = \quad \int_{\Omega} \tilde{d}(\rho(\omega), \rho'(\omega))\, \lambda(d\omega),$$

for $\rho, \rho' \in \mathbf{P}$. We say a sequence $\{\rho_n\}$ of profiles converges to profile ρ if $\Delta(\rho_n, \rho) \to 0$.

Lemma 5 *Assume A is a complete, separable metric space. Let $\rho_n \to \rho$, let $a_n \to a$, and let $b_n \to b$. If W is λ-continuous and $a_n R[\rho_n] b_n$ for all n, then $aR[\rho]b$.*

Proof: From the Polish version of Lusin's theorem (Aliprantis and Border (1999), Theorem 10.8), for all $\epsilon > 0$, there exists a compact subset K_ϵ of Ω such that $\lambda(K_\epsilon) \geq 1 - \epsilon$ and the profile ρ, restricted to K_ϵ, is continuous. Since $\tilde{d}(\rho_n, \rho) \to 0$ in the L_1-norm, Aliprantis and Border's (1999) Theorem 12.6 yields a subsequence $\{\rho_{n_k}\}$ such that $\tilde{d}(\rho_{n_k}, \rho) \to 0$ λ-almost surely. Now let $U\colon \mathcal{R} \times A \to \Re$ be a jointly continuous mapping such that, for all $R \in \mathcal{R}$, $U(R, \cdot)$

is a utility representation of R. For all $\omega \in \Omega$ and all k, define

$$\begin{aligned}\Phi_k^a(\omega) &= U(\rho_{n_k}(\omega), a_{n_k}) \\ \Phi_k^b(\omega) &= U(\rho_{n_k}(\omega), b_{n_k})\end{aligned}$$

and

$$\begin{aligned}\Phi^a(\omega) &= U(\rho(\omega), a) \\ \Phi^b(\omega) &= U(\rho(\omega), b).\end{aligned}$$

Since U is jointly continuous, we deduce that $\Phi_k^a \to \Phi^a$ and $\Phi_k^b \to \Phi^b$ λ-almost surely. From Egoroff's theorem (Aliprantis and Border (1999), Theorem 9.37), there are compact subsets K_ϵ^a and K_ϵ^b of Ω such that

- $\lambda(K_\epsilon^a) \geq 1-\epsilon$ and $\lambda(K_\epsilon^b) \geq 1-\epsilon$
- $\Phi_k^a \to \Phi^a$ uniformly on K_ϵ^a
- $\Phi_k^b \to \Phi^b$ uniformly on K_ϵ^b.

Now let $S = \{\omega \in \Omega \mid b\pi(\omega)a\}$, and suppose that $S \in \mathcal{W}$. Since Ω is Polish, Aliprantis and Border's (1999) Theorem 10.7 implies the existence of a compact subset $K_\epsilon' \subseteq S$ such that $\lambda(K_\epsilon') \geq \lambda(S) - \epsilon$. Let $K_\epsilon'' = K_\epsilon \cap K_\epsilon' \cap K_\epsilon^a \cap K_\epsilon^b$. Since $\lambda(K_\epsilon'') \geq \lambda(S) - 4\epsilon$ and $\mathcal{W}$ is λ-continuous, we have $K_\epsilon'' \in \mathcal{W}$ for small enough ϵ. Furthermore, since ρ is continuous on K_ϵ'', it follows that $\Phi^b - \Phi^a$ is continuous on K_ϵ''. And since K_ϵ'' is compact and $\Phi^b(\omega) - \Phi^a(\omega) > 0$ for all $\omega \in K_\epsilon''$, there exists $\delta > 0$ such that,

$$\Phi^b(\omega) - \Phi^a(\omega) \quad > \quad \delta \text{ for all} \omega \in K_\epsilon''.$$

But by uniform convergence on K_ϵ^a and K_ϵ^b, we deduce that, for k large enough,

$$|\Phi_k^a(\omega) - \Phi^a(\omega)| \quad \leq \quad \frac{\delta}{2} \text{ for all } \omega \in K_\epsilon''$$

and

$$\left|\Phi_k^b(\omega) - \Phi^b(\omega)\right| \quad \leq \quad \frac{\delta}{2} \text{ for all } \omega \in K_\epsilon''.$$

Therefore, for k large enough,

$$\Phi_k^b(\omega) - \Phi_k^a(\omega) > 0 \text{ for all } \omega \in K_\epsilon''.$$

Since $K_\epsilon'' \in \mathcal{W}$, this implies that, for k large enough, $b_{n_k} P[\rho_{n_k}] a_{n_k}$, a contradiction. Therefore, $S \notin \mathcal{W}$, implying $aR[\rho]b$, as required. ∎

When there is a finite number of voters, different formulations of the assertion that the core is generically empty have been provided. Some authors, in the vein of Plott's (1967) seminal early contribution, assume that voter preferences are differentiable and perhaps convex. Their results provide characterizations of core points in terms of differentiability properties, which are evidently quite difficult to satisfy when the dimensionality of the space of alternatives is sufficiently high (at least two, for the case of majority rule with an odd number of voters). Other authors do not assume differentiability or convexity of voter preferences and directly prove the generic emptiness of the core, without any preliminary characterization of core points (Rubinstein (1979), Schofield (1983), Cox (1984), Le Breton (1987)). In the latter work, because the space of voter preferences is richer, no dimensionality restrictions are needed. McKelvey, Ordeshook, and Ungar (1980) have proved that Plott's characterization in terms of symmetry of voter gradients at core points holds true while allowing for a measure space of voters. The following proposition plays the complementary role for the above-cited papers by explicitly establishing the generic emptiness of the core without differentiability or convexity assumptions.

When Ω is finite, we say $\mathcal{W}$ is non-collegial if $\bigcap \mathcal{W} = \emptyset$. When there is a continuum of massless voters, however, every simple game is non-collegial according to this definition: for each $\omega \in \Omega$, $\Omega \setminus \{\omega\} \in \mathcal{W}$, so $\mathcal{W}$ has empty intersection. We extend the usual definition as follows. We say $\mathcal{W}$ is *non-collegial* if, for every $S \in \Sigma$, there exists a finite partition, $\{S_1, \ldots, S_K\}$, of S such that, for all k, $S_k \notin \mathcal{B}$. Given a non-collegial simple game, note that, if $S \in \Sigma$ is an atom, then $S \notin \mathcal{B}$. On the other hand, if λ is atomless, then, by Aliprantis and Border's (1999) Theorem 12.34, for all $\epsilon > 0$, there exists a finite partition, $\{S_1, \ldots, S_K\}$, of Ω such that $\lambda(S_k) < \epsilon$ for all k. Thus, if λ is atomless and there exists $\epsilon > 0$ such that $\lambda(S) > 1 - \epsilon$ implies $S \in \mathcal{W}$, then $\mathcal{W}$ is non-collegial.

Proposition 18 *Assume A is a compact and convex subset of some Euclidean space with $|A| > 2$ and $\mathcal{W}$ is non-collegial and λ-continuous. Then the set*

$$\mathbf{K} \;=\; \{\rho \in \mathbf{P} \mid K[\rho] \neq \emptyset\}$$

is nowhere dense in $\mathbf{P}$ with the Δ metric.

Proof: We first prove $\mathbf{K}$ is closed. Let $\rho_n \to \rho$ with $\rho_n \in \mathbf{K}$ for all n. Let $a_n \in K[\rho_n]$ for each n. Since A is compact, there is a subsequence $\{a_{n_k}\}$ converging to some limit a. Take any $b \in A$ and note that $a_{n_k} R[\rho_{n_k}] b$ for all k. Then Lemma 5 implies $aR[\rho]b$. Since b is arbitrary here, we have $a \in K[\rho] \neq \emptyset$.

To prove that $\mathbf{K}$ has empty interior, take $\rho \in \mathbf{K}$ and $\epsilon > 0$. We will show that there exists ρ_ϵ such that $\Delta(\rho_\epsilon, \rho) \leq \epsilon$ and $\rho_\epsilon \notin \mathbf{K}$. Let $c = \sup\{\tilde{d}(R, R') \mid R, R' \in \mathcal{R}\}$, which is finite. From Lusin's theorem (Aliprantis and Border (1999), Theorem 10.8), there exists a compact subset K_ϵ of Ω such that $\lambda(K_\epsilon) \geq 1 - (\epsilon/2c)$ and ρ, restricted to K_ϵ, is continuous. Since K_ϵ is compact, ρ is uniformly continuous on K_ϵ. Let $\delta > 0$ be such that $d(\omega, \omega') \leq \delta$ implies $\tilde{d}(\rho(\omega), \rho(\omega')) \leq \epsilon/2$. Letting $B_\delta(\omega)$ denote the open d-ball with radius δ centered at ω, $\{B_\delta(\omega) \mid \omega \in K_\epsilon\}$ is an open cover of K_ϵ. By compactness, it has a finite subcover, say $\{B_1, \ldots, B_K\}$. Let $S_1 = B_1$, let

$$S_k \;=\; B_k \setminus \bigcup_{j=1}^{k-1} B_j$$

for $k = 2, \ldots, K$, and let $S_{K+1} = \Omega \setminus K_\epsilon$. The family $\{S_k\}$ is a measurable partition of Ω. Since $\mathcal{W}$ is non-collegial, for each k there is a finite partition $\{S_k^j \mid j = 1, \ldots, J_k\}$ such that, for all j, $S_k^j \notin \mathcal{B}$. Now let $\mathcal{T}$ be the finite partition

$$\mathcal{T} \;=\; \{S_k^j \mid j = 1, \ldots, J_k, k = 1, \ldots, K+1\},$$

and index the elements of $\mathcal{T}$ as T_n, $n = 1, \ldots, N$. For each n, let ρ_n be an arbitrary element of $\{\rho(\omega) \mid \omega \in T_n\}$, and define the profile ρ'_ϵ as $\rho'_\epsilon(\omega) = \rho_n$ for $\omega \in T_n$. Since

$$\Delta(\rho, \rho'_\epsilon) \;=\; \int_{\Omega \setminus K_\epsilon} \tilde{d}(\rho(\omega), \rho'_\epsilon(\omega))\, \lambda(d\omega) + \sum_{n=1}^{N} \int_{T_n \cap (\Omega \setminus K_\epsilon)} \tilde{d}(\rho(\omega), \rho'_\epsilon(\omega))\, \lambda(d\omega),$$

it follows that $\Delta(\rho, \rho'_\epsilon) \leq \epsilon/2$.

Now define a finite simple game $\mathcal{W}^N$ on the set $\{1, \ldots, N\}$ as follows: for $S \subseteq \{1, \ldots, N\}$, let

$$S \in \mathcal{W}^N \text{ if and only if } \bigcup_{n \in S} T_n \in \mathcal{W}.$$

Thus, by choice of $\{T_n\}$, $\mathcal{W}^N$ is non-collegial. Given a finite profile $(R_1, \ldots, R_N)$, define the core of $\mathcal{W}^N$ as $K(P^N)$, where $P^N = \bigcup_{S \in \mathcal{W}^N} \bigcap_{n \in S} P_n$. It is straightforward to show that $a \in A$ is in the core of $\mathcal{W}^N$ for $(R_1, \ldots, R_N)$ if and only if a is in the core of $\mathcal{W}$ for the profile $\hat{\rho}$, where $\hat{\rho}: \Omega \to \mathcal{R}$ is defined by $\hat{\rho}(\omega) = R_n$ for all $\omega \in T_n$ and all $n = 1, \ldots, N$. But since $\mathcal{W}^N$ is non-collegial, it follows from Le Breton (1987) that there exists a finite profile $(R'_1, \ldots, R'_N)$ such that the core of $\mathcal{W}^N$ for $(R'_1, \ldots, R'_N)$ is empty and $\tilde{d}(R_n, R'_n) \leq \epsilon/2$ for all $n = 1, \ldots, N$. Define the profile $\rho_\epsilon: \Omega \to \mathcal{R}$ by

$$\rho_\epsilon(\omega) \;=\; R'_n,$$

for all $\omega \in T_n$ and for all $n = 1, \ldots, N$. It follows from above that $\rho_\epsilon \notin \mathbf{K}$ and $\Delta(\rho, \rho_\epsilon) \leq \epsilon$, as required. ∎

In contrast, the next results show that the uncovered set and undominated sets are non-empty quite generally.

Proposition 19 *Assume that, for all $a \in A$ and all $\epsilon > 0$, there exists a compact set K_ϵ such that $\lambda(\{\omega \in \Omega \mid \rho(\omega)(a) \subseteq K_\epsilon\}) > 1 - \epsilon$; and assume that $\mathcal{W}$ is open from below. Then $UC[\rho] \neq \emptyset$.*

Proof: By Proposition 14, each $R[\rho](a)$ is compact, and $R[\rho](b)$ is therefore closed for all $b \in A$. By Proposition 2, we have $UC[\rho] = UC(R[\rho]) \neq \emptyset$. ∎

Nonemptiness of the undominated set follows if we impose topological conditions on the electorate. We write $D[\rho]$ and $C[\rho]$ for the dominance and covering relations determined by $R[\rho]$ and $P[\rho]$.

Proposition 20 *Assume A is second countable and locally compact; Ω is a compact topological space and ρ is continuous; LSWP holds; for all $a \in A$, $\lambda(\{\omega \in \Omega : \rho(\omega)(a)$ is compact$\}) = 1$; and $\mathcal{W}$ is open from below. Then (i) $UD[\rho] \neq \emptyset$, (ii) if $a \notin UD[\rho]$, then there exists $b \in UD[\rho]$ such that $bD[\rho]a$, (iii) if $a \notin UC[\rho]$, then there exists $b \in UC[\rho]$ such that $bC[\rho]a$*

Proof: By Proposition 15, for every $\epsilon > 0$, there exists a compact set K_ϵ such that $\lambda(\{\omega \in \Omega \mid \rho(\omega)(a) \subseteq K_\epsilon\}) > 1 - \epsilon$, so Proposition 14 implies that each $R(a)$ is compact. Then $UD[\rho] = UD(R[\rho]) \neq \emptyset$ follows from Proposition 13. External stability of $UD[\rho]$ and $UC[\rho]$ follow from Propositions 4 and 5. ∎

We now examine the continuity properties of the uncovered set correspondence. We demonstrate that, at profiles ρ such that $K[\rho]$ is nonempty and is strongly externally stable, the correspondence $UC[\cdot]$ is upper hemicontinuous at ρ. If $UC[\rho]$ is a singleton, then, because the uncovered set is non-empty for all profiles under the assumptions of the proposition, it follows that $UC[\cdot]$ is actually continuous at ρ. Note that conditions for external stability of the core used here are given in Propositions 1 and 16.

Proposition 21 *Assume A is compact, $\mathcal{W}$ is λ-continuous, $K[\rho] \neq \emptyset$, and, for all $a \in K[\rho]$ and all $b \notin K[\rho]$, $aP[\rho]b$. Then $UC[\rho] = UD[\rho] = K[\rho]$ and $UC[\cdot]$ is upper hemicontinuous at ρ.*

Proof: That $UC[\rho] = K[\rho]$ is immediate. Suppose that $UC[\cdot]$ is not upper hemicontinuous at ρ, so that there exists an open set $V \supseteq UC[\rho]$ such that, for all open neighborhoods U of ρ, there exists some $\rho_U \in U$ such that $UC[\rho_U] \setminus V \neq \emptyset$. View $\{\rho_U\}$ as a net directed by set-inclusion converging to ρ. Since the space of profiles is a metric space, there is a subsequence $\{\rho_n\}$ converging to ρ. For each ρ_n, let $a_n \in UC[\rho_n] \setminus V$. Since A is compact, we may assume with loss of generality that $\{a_n\}$ converges to some $a \in A \setminus V$. Therefore, $a \notin UC[\rho] = K[\rho]$. By assumption, for any $b^* \in K[\rho]$, we have $b^* P[\rho]a$. We claim that there is some m such that $b^* C[\rho_m]a_m$, a contradiction.

By Lemma 5, there exists m such that for all $n \geq m$, $b^* P[\rho_n]a_n$. Suppose that, for all m, there exists $n \geq m$ and $b_n \in P[\rho_n](b^*) \setminus P[\rho_n](a_n)$. Let $\{b_{n_k}\}$

be a subsequence converging to some b. Since $aR[\rho_{n_k}]b_{n_k}$ for all k, Lemma 5 implies $aR[\rho]b$. By our assumption of external stability, $b \notin K[\rho]$ and $b^*P[\rho]b$. By Lemma 5 again, $b^*P[\rho_{n_k}]b_{n_k}$ for k large enough, a contradiction. Therefore, there exists m such that, for all $n \geq m$, $P[\rho_n](b^*) \subseteq P[\rho_n](a_n)$.

Suppose now that, for all m, there exists $n \geq m$ and $c_n \in R[\rho_n](b^*) \setminus R[\rho_n](a_n)$. Let $\{c_{n_k}\}$ be a subsequence converging to some c. By Lemma 5, $cR[\rho]b^*$ which implies $c \in K[\rho]$ and $cP[\rho]a$. By Lemma 5 again, $c_{n_k}P[\rho_{n_k}]a_{n_k}$ for k large enough, a contradiction. Therefore, there exists m such that, for all $n \geq m$, $R[\rho_n](b^*) \subseteq R[\rho_n](a_n)$. Therefore, $b^*C[\rho_m]a_m$ for high enough, completing the proof. ∎

Proposition 21 states that upper hemicontinuity of $UC[\cdot]$ holds at ρ whenever, among other things, the core at ρ is non-empty. The correspondence $UC[\cdot]$ is not generally upper hemicontinuous, as demonstrated by the following example. Let A be the convex hull of a_1, a_2, and a_3 in $\Re^2$, let $\Omega = \{1, 2, 3\}$, and assume preferences (R_1, R_2, R_3) are strictly convex with indifference contours depicted in Figure 1. There, the point a_i is the bliss point of preference R_i. For this electorate, bCc and $c \notin UC$.

[Figure 1 here.]

Now perturb preferences as depicted in Figure 2. One can check that, for arbitrarily small perturbations, c is uncovered, violating upper hemicontinuity.

[Figure 2 here.]

Finally, we turn to lower hemicontinuity of the undominated set. For the next proposition, assume Ω is a topological space, and let $\mathbf{P}^*$ be the subset of profiles ρ such that

- ρ is continuous,
- LSWP holds,
- for all $a \in A$, $R[\rho](a) = \{a\} \cup \overline{P[\rho](a)}$,
- for all $a \in A$, $R^{-1}[\rho](a) = \{a\} \cup \overline{P^{-1}[\rho](a)}$,
- for all $a, b \in A$, if $P[\rho](a) = P[\rho](b)$ and $R[\rho](a) = R[\rho](b)$, then $a = b$.

Note that Proposition 16 gives sufficient conditions, involving dispersion of voter preferences, for the third requirement above. Assuming $\mathcal{W}$ is open from below, the fourth is equivalent, by Lemma 1, to the condition that $P[\rho](a) \cup \{a\} = R[\rho](a)^\circ \cup \{a\}$ for all $a \in A$. While we do not give sufficient conditions for the fourth and fifth requirements, we conjecture that they are fairly unrestrictive. Thus, $\mathbf{P}^*$ should not be too "sparse."

Proposition 22 *Assume A is a compact metric space, Ω is compact, and $\mathcal{W}$ is proper and λ-continuous. Then $UD[\cdot]$ is lower hemicontinuous on $\mathbf{P}^*$.*

The proof of this result is contained in the appendix. An implication of Michael's selection theorem (Aliprantis and Border (1999), Theorem 16.61) is that, when A is a compact subset of a Banach space, the correspondence consisting of the closed, convex hull of $UD[\rho]$ has a continuous selection. Thus, so does the closed, convex hull of $UC[\rho]$.

A Maximal Elements

Given a set X, with elements x, y, z, etc., a relation Q is a *preorder* if it is reflexive and transitive. An element x is *Q-maximal in $Y \subseteq X$* if $x \in Y$ and, for all $y \in Y$, yQx implies xQy. We say x is *Q-maximal* if it is Q-maximal in X. As above, define the *upper* and *lower sections* of Q as

$$\begin{aligned} Q(x) &= \{y \in X \mid yQx\} \\ Q^{-1}(x) &= \{y \in X \mid xQy\}, \end{aligned}$$

respectively. We say Q is *upper semicontinuous* if $Q(x)$ is closed for all x. An implication of the next proposition is that, under weak conditions, the set of Q-maximal elements is non-empty and externally stable.

Proposition A1 If Q is transitive and upper semicontinuous, and if $Q(x)$ is compact for some x, then $Q(x)$ contains a Q-maximal element.

Proof: Take any Q-chain, E, in $Q(x)$. By transitivity and upper semiconintuity, $\{Q(y) \mid y \in E\}$ is a collection of compact sets with the finite intersection property, so there exists $z \in \bigcap_{y \in E} Q(y)$. Thus, z is a Q-upper bound for E. By Zorn's lemma, Q has a maximal element, say x^*, in $Q(x)$. If x^* is not maximal in X, then there exists $w \in X$ such that wQx^* and not x^*Qw. By transitivity, wQx^*Qy implies $w \in Q(y)$, a contradiction. ∎

If we strengthen our compactness assumption in Proposition A1, we can deduce the external stability of the Q-maximal elements.

Proposition A2 Assume Q is transitive and upper semicontinuous, and $Q(x)$ is compact for all x. If x is not Q-maximal, then there exists Q-maximal x^* such that x^*Qx and not xQx^*.

Proof: Suppose x is not Q-maximal. Since $Q(x)$ is compact, it contains a Q-maximal element, say x^*. Thus, x^*Qx. Suppose xQx^*. Since x is not Q-maximal, there exists y such that yQx and not xQy. By transitivity, $yQxQx^*$ implies yQx^*. Since x^* is Q-maximal, x^*Qy. Then, by transitivity, xQx^*Qy implies xQy, a contradiction. Therefore, not xQx^*. ∎

Define the relation Q^* as follows: xQ^*y if and only if $Q(x) \subseteq Q(y)$. Note that Q^* is reflexive and transitive.

Proposition A3 If $Q^{-1}(x)$ is open for all x, then Q^* is upper semicontinuous.

Proof: Take any x and a net $\{x_\alpha\}$ in $Q^*(x)$ with $x_\alpha \to y$. We need to show yQ^*x, i.e., $Q(y) \subseteq Q(x)$. Take any $z \in Q(y)$, i.e., $y \in Q^{-1}(z)$. Since $Q^{-1}(z)$ is open, there exists α' such that, for all $\alpha \geq \alpha'$, $x_\alpha \in Q^{-1}(z)$, i.e., $z \in Q(x_\alpha)$. Since $x_\alpha Q^* x$, we have $z \in Q(x)$, as desired. ∎

To prove Proposition 2, assume R is upper semicontinuous and $R(a)$ compact for some $a \in A$, and define $Q = R \cap P^*$, i.e., bQc if and only if $b \in R(c)$ and $P(b) \subseteq P(c)$. Since $Q(b) = R(b) \cap P^*(b)$, Proposition A3 implies that Q is upper semicontinuous, and, by irreflexivity of P, it is transitive. Finally, note that $Q(a) = R(a) \cap P^*(a)$, so $Q(a)$ is compact. Applying Proposition A1, there exists a Q-maximal element, which must belong to $UC(R)$. Therefore, $UC(R) \neq \emptyset$.

Proposition A4 If Q is upper semicontinuous, and if $Q(\cdot)$ is lower hemicontinuous as a correspondence, then Q^* is upper semicontinuous.

Proof: Take any x and any net $\{x_\alpha\}$ in $Q^*(x)$ converging to some y. Take any $z \in Q(y)$, and suppose $z \notin Q(x)$. By upper semicontinuity, $G = X \setminus Q(x)$ is open. Of course, $Q(y) \cap G \neq \emptyset$. By lower hemicontinuity, there exists α such that $Q(x_\alpha) \cap G \neq \emptyset$, but then $Q(x_\alpha) \not\subseteq Q(x)$, a contradiction. Therefore, $z \in Q(x)$, and we conclude $y \in Q^*(x)$, as desired. ∎

To prove Proposition 3, assume that R is upper semicontinuous, that $R(a)$ is compact for some $a \in A$, and that, viewed as a correspondence, $R(\cdot)$ is lower hemicontinuous. By Proposition A2, P^* is upper semicontinuous, and, by Proposition A3, R^* is upper semicontinuous as well. Thus, $P^* \cap R^*$ is upper semicontinuous and is clearly transitive. Also note that $R^*(a) \subseteq R(a)$, so $R^*(a)$ is compact. Applying Proposition A1 yields a $(R^* \cap P^*)$-maximal element, which must belong to $UD(R)$. Therefore, $UD(R) \neq \emptyset$.

To prove Proposition 4, that assume R is upper semicontinuous, that $R(a)$ compact for all $a \in A$, and that, viewed as a correspondence, $R(\cdot)$ is lower hemicontinuous. Note that $R^*(a)$, and therefore $(P^* \cap R^*)(a)$, is compact for all $a \in A$. Take a and b such that aDb, implying $a(R^* \cap P^*)b$. By Proposition A2, there exists a $(R^* \cap P^*)$-maximal element c, therefore belonging to $UD(R)$, such that $c(R^* \cap P^*)a$. Therefore, $P(c) \subseteq P(a) \subseteq P(b)$ and $R(c) \subseteq R(a) \subseteq R(b)$. Since aDb, we have cDb, and external stability of $UD(R)$ follows.

Let $\mathcal{Q}$ denote the set of closed relations on X, endowed with the topology of closed convergence. Let Θ be a topological space, and consider a mapping with domain Θ, range $\mathcal{Q}$, and values $Q[\theta]$. We say the mapping $Q[\cdot]$ is *outer continuous* if, for all nets $\{\theta_\alpha\}$ in Θ converging to some θ and $\{(x_\alpha, y_\alpha)\}$ in

$X \times X$ converging to some (x, y), $x_\alpha Q[\theta_\alpha] y_\alpha$ for all α implies $xQ[\theta]y$. Let UQ denote the Q-maximal elements, and let $UQ[\theta]$ denote the $Q[\theta]$-maximal elements. The next proposition provides conditions for lower hemi-continuity of the maximal element correspondence.

Proposition A5 Assume X is compact; for all $\theta \in \Theta$, $Q[\theta]$ is anti-symmetric, $UQ[\theta] \neq \emptyset$, and, for all $x \notin UQ[\theta]$, there exists $y \in UQ[\theta]$ such that yQx and not xQy; and $Q[\cdot]$ is outer continuous. Then $UQ[\cdot]$ is lower hemi-continuous.

Proof: Take any $\theta \in \Theta$ and any open set G such that $G \cap UQ[\theta] \neq \emptyset$. Let $x \in G \cap UQ[\theta]$. Let $\{\theta_\alpha\}$ be a net converging to θ, and suppose that, for each α, $UQ[\theta_\alpha] \cap G = \emptyset$. Thus, for each α, $x \notin UQ[\theta_\alpha]$. By external stability, there exists $x_\alpha \in UQ[\theta_\alpha]$ such that $x_\alpha Q[\theta_\alpha] x$ and not $xQ[\theta_\alpha]x_\alpha$. Moreover, $x_\alpha \notin G$ for each α. By compactness, $\{x_\alpha\}$ has a convergent subnet with limit, say, z. By outer continuity, $zQ[\theta]x$. Because $x_\alpha \notin G$ for all α, we have $z \neq x$, and then anti-symmetry implies not $xQ[\theta]z$, which implies $x \notin UQ[\theta]$, a contradiction. ∎

Now consider a mapping from Θ to $\mathcal{Q}$ with values $R[\theta]$. Following the above convention, write $R[\theta](x)$ for the upper section of $R[\theta]$ at x, and say $xR^*[\theta]y$ if and only if $R[\theta](x) \subseteq R[\theta](y)$. Write $xP[\theta]y$ if and only if not $xR[\theta]y$.

Proposition A6 Assume for all $\theta \in \Theta$ and all $x \in X$, $R[\theta](x) = \{x\} \cup \overline{P[\theta](x)}$, and $R[\cdot]$ is outer continuous. Then $R^*[\cdot]$ is outer continuous.

Proof: Take any nets $\{\theta_\alpha\}$ converging to θ and $\{(x_\alpha, y_\alpha)\}$ converging to (x, y) such that, for each α, $x_\alpha R^*[\theta_\alpha] y_\alpha$. Take any $z \in R[\theta](x)$. If not $z \in R[\theta](y)$, then, because $R[\theta](y)$ is closed, $G = P^{-1}[\theta](y)$ is an open set around z such that $G \cap R[\theta](y) = \emptyset$. Pick $w \in P[\theta](x) \cap G$, so $yP[\theta]w$. It then follows from outer continuity that, for some subnet, also indexed by α, $y_\alpha P[\theta_\alpha]w$ for all α. Then, by $x_\alpha R^*[\theta_\alpha]y_\alpha$, we have $x_\alpha P[\theta_\alpha]w$ for all α. This implies $x_\alpha R[\theta_\alpha]w$ for all α, and outer continuity implies $xR[\theta]w$, a contradiction. Therefore, $z \in R[\theta](y)$, and we conclude that $xR^*[\theta]y$. ∎

To prove Proposition 22, let $\Theta = \mathbf{P}^*$. Note that $R[\rho](a) = \{a\} \cup \overline{P[\rho](a)}$ for all $\rho \in \mathbf{P}^*$, and that, by Lemma 5, $R[\cdot]$ is outer continuous on $\mathbf{P}^*$. Therefore, by Proposition A6, $R^*[\cdot]$ is outer continuous. To see that $R^*[\rho]$ is anti-symmetric for all $\rho \in \mathbf{P}^*$, suppose $aR^*[\rho]b$, i.e., $R(a) \subseteq R(b)$. By assumption, we have $R^{-1}[\rho](a) = \{a\} \cup \overline{P^{-1}[\rho](a)}$ and $R^{-1}[\rho](b) = \{b\} \cup \overline{P^{-1}[\rho](b)}$. From Lemma 1, this implies $P[\rho](a) \cup \{a\} = R[\rho](a)^\circ \cup \{a\}$ and $P[\rho](b) \cup \{b\} = R[\rho](b)^\circ \cup \{b\}$, so Proposition 6 yields $P[\rho](a) \subseteq P[\rho](b)$. Then our assumptions on $\mathbf{P}^*$ imply $a = b$. By Proposition 13, $R[\rho](\cdot)$ is lower hemicontinuous, so Proposition A4 implies that $R^*[\rho]$ is upper semicontinuous. Because A is compact, Propositions A1 and A2 then imply non-emptiness and external stability of the $R^*[\rho]$-maximal elements, denoted $UR^*[\rho]$, and Proposition A5 implies that $UR^*[\cdot]$ is lower hemicontinuous on $\mathbf{P}^*$. Finally, note that $UR^*[\rho] = UD[\rho]$ for all $\rho \in \mathbf{P}^*$.

To prove the claimed equality, take $a \in UR^*[\rho]$, and suppose there exists $b \in A$ such that $bD[\rho]a$, i.e., $R[\rho](b) \subseteq R[\rho](a)$ and $P[\rho](b) \subseteq P[\rho](a)$, at least one inclusion strict. In fact, because $\rho \in \mathbf{P}^*$, the first must be strict, but then $a \notin UR^*[\rho]$, a contradiction. Therefore, $UR^*[\rho] \subseteq UD[\rho]$. For the remaining inclusion, suppose $a \in A \setminus UR^*[\rho]$, so there is some $b \in A$ such that $bR^*[\rho]a$ and not $aR^*[\rho]b$. Note that, by reflexivity of $R^*[\rho]$, we have $b \neq a$. By definition, $R[\rho](b) \subseteq R[\rho](a)$. By Lemma 1 and Proposition 6, we also have $P[\rho](b) \subseteq P[\rho](a)$. At least one inclusion must be strict, for otherwise $\rho \in \mathbf{P}^*$ implies $a = b$. Therefore, $bD[\rho]a$, implying $a \in A \setminus UD[\rho]$. We conclude that $UD[\rho] \subseteq UR^*[\rho]$, as required.

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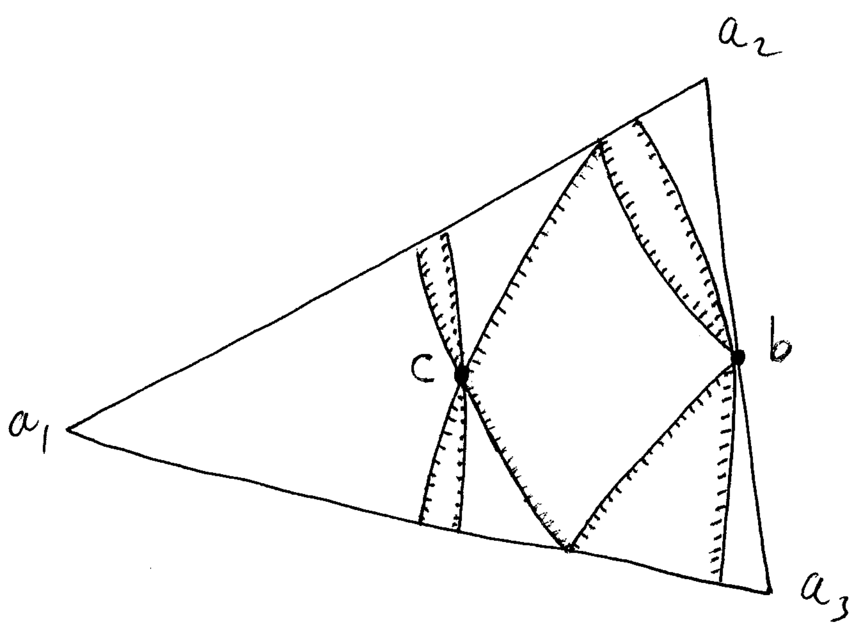

Fig 1: b covers c

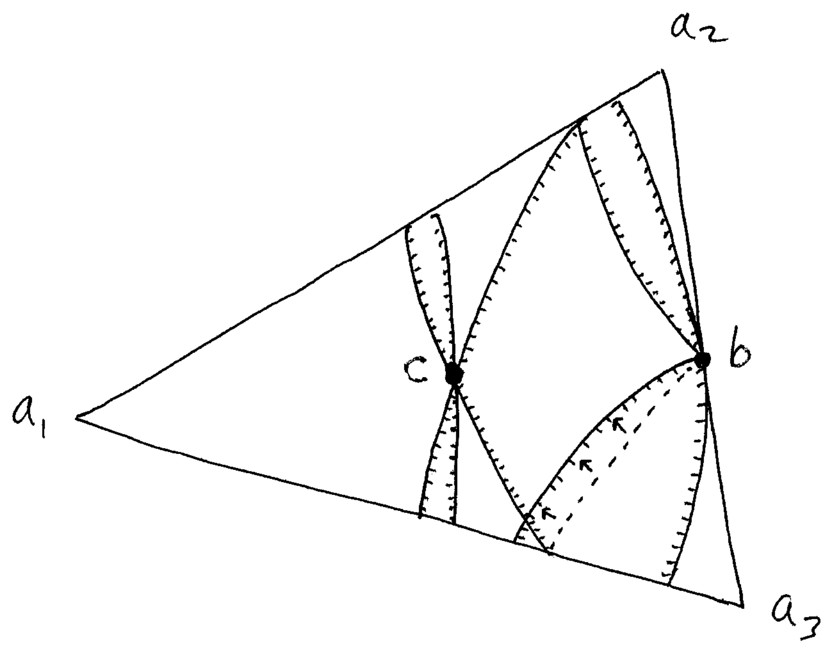

Fig 2: C uncovered

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