On Participation Games with Complete Information

Tasos Kalandrakis

Working Paper No. 40
October 2005

W. ALLEN
WALLIS
Institute of
POLITICAL
ECONOMY

UNIVERSITY OF
ROCHESTER

On Participation Games with Complete Information[1]

Tasos Kalandrakis[2]

Abstract: We analyze a class of two-candidate voter participation games under complete information that encompasses as special cases certain public good provision games. We characterize the Nash equilibria of these games as stationary points of a non-linear programming problem, the objective function of which is a Morse function (one that does not admit degenerate critical points) for almost all costs of participation. We use this fact to establish that, outside a closed set of measure zero of participation costs, all equilibria of these games are regular (an alternative to the result of De Sinopoli and Iannantuoni, 2005). One consequence of regularity is that the equilibria of these games are robust to the introduction of (mild) incomplete information. Finally, we establish the existence of monotone Nash equilibria, such that players with higher participation cost abstain with (weakly) higher probability.

JEL Classification Numbers: C72, D72.

Keywords: Turnout, Public Goods, Regular Equilibrium, Monotone Equilibrium.

[1] First draft: September 15, 2005; this draft: September 22, 2005.

[2] Assistant Professor, Department of Political Science, University of Rochester. Correspondence Address: Department of Political Science, University of Rochester, Harkness Hall 333, Rochester, NY 14627-0146. E-mail: akalandr@mail.rochester.edu. Web: http://mail.rochester.edu/~akalandr/.

1. INTRODUCTION

We analyze a class of complete information, two-candidate voter participation games in which players in two groups decide whether to vote for their favorite candidate or abstain. We consider a generalized version of plurality rule according to which one group must exceed the votes cast by the other group by a certain (possibly negative) number in order to win the election. Ties are resolved by a (possibly biased) coin toss. Participation is costly, and the costs of voting may vary across players.

In the special case when one of the two groups is empty, the game becomes a discrete public good provision game. In particular, we can interpret players' decision to participate as a contribution of one unit towards the public good. The number of contributions must exceed a threshold in order for the good to be provided, and players receive no refunds if that threshold is not reached.

Thus, the class of games we consider encompasses those analyzed by Palfrey and Rosenthal, 1983, as well as games analyzed by Palfrey and Rosenthal, 1984. In the former, the authors studied rational turnout under complete information, while in the latter they analyzed the incentives for the collective provision of public goods. Unlike our analysis, both of these studies are executed under the restriction that players' participation costs are identical.

We have three main results. First, we characterize the Nash equilibria of these games as stationary points of a minmax optimization problem. Although this result may have independent value for the purposes of computation of equilibria in these games, its importance in our analysis stems from the fact that the objective function of this optimization problem is a *Morse function* for almost all participation costs. The Hessian of a Morse function is not singular at all the critical points of that function.

We use the above to establish that, except for a closed set of measure zero of participation costs, all the Nash equilibria of the participation games we consider are *regular*. In the games we analyze, an equilibrium is regular if players that use a pure strategy strictly prefer that strategy, and the Jacobian associated with the indifference conditions of the players that use non-degenerate mixed strategies is not singular. Our earlier characterization immediately yields that all totally mixed Nash equilibria of a participation game are regular, for almost all participation costs. A simple additional step allows us to extend this conclusion to equilibria that also involve pure strategy choices by players.

Regular equilibria are isolated, and are locally expressible as continuous functions of parameters. Furthermore, these equilibria can be "purified," *i.e.* approximated by Bayesian Nash equilibria of nearby games of incomplete information, as shown originally by Harsanyi, 1973a, and, with a much shorter proof, by Govindan, Reny, and Robson, 2003. As we discuss in the concluding section of our analysis, these properties imply that the received dichotomy in the literature on the possibility of rational turnout between games of complete and incomplete information merits further qualification. In particular, if a (regular) voter participation game of complete information admits equilibria with high turnout, then these equilibria essentially survive in nearby games of incomplete information.[3]

We emphasize that regularity of the Nash equilibrium set for the games we study does not

[3] Aldrich, 1993, and, more recently, Feddersen, 2004, provide a review of the literature on rational turnout.

follow from the corresponding theorem of Harsanyi, 1973b, for finite games in normal form. The reason is that the latter theorem is obtained by, in principle, perturbing the payoffs associated with each possible outcome of the game. Yet, the anonymity built into our game implies that players assign the same payoff for large sets of outcomes. Indeed, although a game with n players has 2^n possible outcomes in our analysis, each player can receive only four distinct possible payoffs from these outcomes. Nevertheless, we can show that these games are regular by only perturbing the n parameters reflecting these players' cost of participation.

De Sinopoli and Iannantuoni, 2005, also show generic regularity of k-candidate ($k \geq 2$) plurality voter participation games.[4] Their theorem is obtained by following a line of proof similar to that used by Harsanyi, 1973b, and van Damme, 1987. The proof proceeds by creating a mapping the range of which is the set of payoff parameters, and concludes by application of a version of Sard's theorem.

De Sinopoli and Iannantuoni show generic regularity in the space of games defined over the $n(k+1)$ parameters corresponding to the n players' k payoffs associated with the victory of each of the k candidates, as well as each player's cost of participation. This involves some overparameterization, as the authors discuss in their remark 6, page 486, to the effect that their conclusion holds even if participation costs are assumed identical across players, *i.e.* for a space of games with a total of only $nk+1$, instead of $n(k+1)$, parameters.

Lastly, motivated by an example to the contrary, we consider whether the games we analyze admit equilibria in which, within each group, players with higher participation cost abstain with weakly higher probability. We call such equilibria monotone, and establish their existence for all games.

The analysis in the sequel is organized as follows. In section 2 we formally define the games analyzed and establish some preliminary results. In section 3 we provide the characterization of equilibria in these games. We show that regular games are generic in section 4, and establish existence of monotone equilibria in section 5. We conclude in section 6.

2. MODEL & PRELIMINARIES

We shall be concerned with a participation game played by two sets of players, $N = \{1, ..., n\}$ and $M = \{n+1, ..., n+\mu\}$, with $\mu \leq n$.[5] We denote a generic player in N by i, and reserve j for a generic player in M. Each group N and M have a favorite candidate in a two-candidate election. Players receive a payoff equal to 1 if their favorite candidate wins the election, and 0 if the other group's favorite candidate wins the election.

Player i's strategy is whether to vote in favor of group N's candidate at a cost $c_i \in \mathbf{R}_{++}$, $i \in N$, or abstain.[6] Similarly the strategy of player j is whether to vote in favor of group M's

[4] Although De Sinopoli and Iannantuoni consider simple plurality rule games with ties resolved by a fair lottery in the main analysis, their results extend to a more general class of games when it comes to the voting rule, as they discuss in remark 1, page 484.

[5] As a mnemonic rule, we shall reserve Latin characters for variables or parameters pertaining to players in N, and Greek characters for players in M.

[6] We follow the standard approach of eliminating players' dominated strategy to vote in favor of the candidate of the opposing group.

candidate at a cost $\kappa_j \in \mathbf{R}_{++}$, $j \in M$, or abstain. We denote the vector of costs for players $i \in N$ by $\mathbf{c} = (c_1, ..., c_n) \in \widetilde{C} \equiv \mathbf{R}^n_{++}$, and that for players $j \in M$ by $\boldsymbol{\kappa} = (\kappa_{n+1}, ..., \kappa_{n+\mu}) \in \widetilde{K} \equiv \mathbf{R}^{\mu}_{++}$.

We represent a strategy for player $i \in N$ by the probability of participation $p_i \in [0,1]$. We denote the vector of strategies for players in N by $\mathbf{p} = (p_1, ..., p_n) \in [0,1]^n$. We similarly define the strategy of player $j \in M$ by $\pi_j \in [0,1]$, and collect the strategies of players in M in vector $\boldsymbol{\pi} = (\pi_{n+1}, ..., \pi_{n+\mu}) \in [0,1]^{\mu}$. We use the standard notation $\mathbf{p}^{-i}$ ($\boldsymbol{\pi}^{-j}$) to represents the strategies of all players in N except i (M except j).

Denote the actual number of votes cast by group N by l, and the corresponding number cast by group M by λ. Group N wins if $l - \lambda > w$, $w \in \mathbf{Z}$. Group M wins if $l - \lambda < w$. In cases of a "tie," $l - \lambda = w$, group N wins with probability $\widehat{w} \in [0,1]$, and group M wins with probability $1 - \widehat{w}$.

Observe that $w = 0$, $\widehat{w} = \frac{1}{2}$ corresponds to the familiar plurality rule with ties resolved by coin-toss. The case when $w \neq 0$, or $w = 0$, $\widehat{w} \in \{0,1\}$, corresponds to a generalized version of what Palfrey and Rosenthal, 1983, refer to as the *status quo rule*. Finally, the case when $\mu = 0$, $w \in \{1, ..., n\}$, $\widehat{w} = 1$ corresponds to the public good provision games analyzed by Palfrey and Rosenthal, 1984, assuming no refunds when the good is not provided. We represent a game satisfying the above assumptions by $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$.

We now define the probability that exactly l players from set $N \backslash C$ vote, $C \subset N$, as the following function of players' strategies:

$$f(\mathbf{p}, l, C) \equiv \begin{cases} \sum_{K \in L^l(C)} \left(\prod_{i \in K} p_i \prod_{i \in N \backslash (K \cup C)} (1 - p_i) \right) & \text{if} \quad l \leq |N \backslash C| \\ 0 & \text{if} \quad l > |N \backslash C|, \, l < 0 \end{cases},$$

where $L^l(C) \equiv \{K \subseteq N \backslash C : |K| = l\}$. We similarly define the corresponding function for group M as:

$$\phi(\boldsymbol{\pi}, \lambda, C) \equiv \begin{cases} \sum_{K \in \Lambda^{\lambda}(C)} \left(\prod_{j \in K} \pi_j \prod_{j \in M \backslash (K \cup C)} (1 - \pi_j) \right) & \text{if} \quad \lambda \leq |M \backslash C| \\ 0 & \text{if} \quad \lambda > |M \backslash C|, \, \lambda < 0 \end{cases},$$

where $\Lambda^{\lambda}(C) \equiv \{K \subseteq M \backslash C : |K| = \lambda\}$. Now $\phi(\boldsymbol{\pi}, \lambda, C)$ represents the probability that exactly λ players from set $M \backslash C$ vote, where $C \subset M$.

Given the above, the probability of victory of group N is a function:

$$F(\mathbf{p}, \boldsymbol{\pi}) \equiv \sum_{\lambda=0}^{\mu} \sum_{l=\lambda+w+1}^{n} f(\mathbf{p}, l, \emptyset) \, \phi(\boldsymbol{\pi}, \lambda, \emptyset) + \widehat{w} \sum_{\lambda=0}^{\mu} f(\mathbf{p}, \lambda + w, \emptyset) \, \phi(\boldsymbol{\pi}, \lambda, \emptyset).$$

The probability of victory of group M is simply

$$\Phi(\mathbf{p}, \boldsymbol{\pi}) \equiv 1 - F(\mathbf{p}, \boldsymbol{\pi}).$$

Denote partial derivatives of F, Φ by F_i, Φ_j. We establish the following lemma, which we will use in the remainder of the analysis:

Lemma 1 *Assume $i \in N$, $j \in M$. Then:*

$$F_i(\mathbf{p}, \boldsymbol{\pi}) = F\left(\left(\mathbf{p}^{-i}, 1\right), \boldsymbol{\pi}\right) - F\left(\left(\mathbf{p}^{-i}, 0\right), \boldsymbol{\pi}\right), \text{ and} \tag{1}$$

$$\Phi_j(\mathbf{p}, \boldsymbol{\pi}) = \Phi\left(\mathbf{p}, \left(\boldsymbol{\pi}^{-j}, 1\right)\right) - \Phi\left(\mathbf{p}, \left(\boldsymbol{\pi}^{-j}, 0\right)\right). \tag{2}$$

Proof. See the appendix. ■

In the appendix, we provide explicit expressions for the quantities in (1) (and (2)), which can be interpreted in terms of the probability that player i (respectively, j) is "pivotal." By inspection of the right hand side of (1) and (2), it is immediate that these quantities appear in the conditions for a Nash equilibrium of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$.

In the next section we use lemma 1 to give a characterization of Nash equilibrium as a stationary point of a non-linear programming problem.

3. NASH EQUILIBRIA

The fact that the change in the probability of victory from a players' participation decision is obtained as the partial derivatives of the probability of victory of the respective group in lemma 1, allows us to characterize Nash equilibria of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ as stationary points of a minmax problem:

Lemma 2 $(\mathbf{p}, \boldsymbol{\pi})$ *is a Nash equilibrium of* $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ *if and only if it is a stationary point of the following minmax problem*

$$\min_{\boldsymbol{\pi}} \max_{\mathbf{p}} \left(F(\mathbf{p}, \boldsymbol{\pi}) - \sum_{i \in N} p_i c_i + \sum_{j \in M} \pi_j \kappa_j \right) \text{ s.t.} \tag{3}$$
$$\mathbf{p} \in [0,1]^n, \boldsymbol{\pi} \in [0,1]^\mu.$$

Proof. We proceed by formulating the Langrangean for the maximization problem

$$L^p(\mathbf{p}, \mathbf{b}, \mathbf{d}; \boldsymbol{\pi}) = F(\mathbf{p}, \boldsymbol{\pi}) - \sum_{i \in N} p_i c_i + \sum_{j \in M} \pi_j \kappa_j + \sum_{i \in N} b_i p_i + \sum_{i \in N} d_i (1 - p_i)$$

from which we obtain the following maximization conditions, for all $i \in N$:

$$F_i(\mathbf{p}, \boldsymbol{\pi}) - c_i + b_i - d_i = 0$$
$$b_i p_i = d_i (1 - p_i) = 0$$
$$b_i, d_i \geq 0, \; p_i \in [0,1]$$

We deduce from (1) that these conditions are equivalent to:

$$p_i \begin{cases} = 1 & \text{if} \quad F\left(\left(\mathbf{p}^{-i}, 1\right), \boldsymbol{\pi}\right) - c_i > F\left(\left(\mathbf{p}^{-i}, 0\right), \boldsymbol{\pi}\right) \\ \in [0,1] & \text{if} \quad F\left(\left(\mathbf{p}^{-i}, 1\right), \boldsymbol{\pi}\right) - c_i = F\left(\left(\mathbf{p}^{-i}, 0\right), \boldsymbol{\pi}\right) \\ = 0 & \text{if} \quad F\left(\left(\mathbf{p}^{-i}, 1\right), \boldsymbol{\pi}\right) - c_i < F\left(\left(\mathbf{p}^{-i}, 0\right), \boldsymbol{\pi}\right) \end{cases} \tag{4}$$

We similarly proceed by transforming the minimization problem of (3) into a maximization one and writing the Langrangean

$$L^{\pi}\left(\boldsymbol{\pi},\boldsymbol{\beta},\boldsymbol{\delta};\mathbf{p}\right)=-F\left(\mathbf{p},\boldsymbol{\pi}\right)+\sum_{i\in N}p_ic_i-\sum_{j\in M}\pi_j\kappa_j+\sum_{j\in M}\beta_j\pi_j+\sum_{j\in M}\delta_j\left(1-\pi_j\right).$$

We now obtain the analogous conditions

$$\begin{gathered}\Phi_j\left(\mathbf{p},\boldsymbol{\pi}\right)-\kappa_j+\beta_j-\delta_j=0\\ \beta_j\pi_j=\delta_j\left(1-\pi_j\right)=0\\ \beta_j,\delta_j\geq 0,\ \pi_j\in[0,1].\end{gathered}$$

which, again by lemma 1, are equivalent to

$$\pi_j\begin{cases}=1 & \text{if}\quad \Phi\left(\mathbf{p},\left(\boldsymbol{\pi}^{-j},1\right)\right)-\kappa_i>\Phi\left(\mathbf{p},\left(\boldsymbol{\pi}^{-j},0\right)\right)\\ \in[0,1] & \text{if}\quad \Phi\left(\mathbf{p},\left(\boldsymbol{\pi}^{-j},1\right)\right)-\kappa_i=\Phi\left(\mathbf{p},\left(\boldsymbol{\pi}^{-j},0\right)\right)\\ =0 & \text{if}\quad \Phi\left(\mathbf{p},\left(\boldsymbol{\pi}^{-j},1\right)\right)-\kappa_i<\Phi\left(\mathbf{p},\left(\boldsymbol{\pi}^{-j},0\right)\right)\end{cases}. \tag{5}$$

But conditions (4) and (5) are necessary and sufficient for $(\mathbf{p},\boldsymbol{\pi})$ to constitute a Nash equilibrium. ■

The characterization in lemma 2 is an immediate consequence of lemma 1. A somewhat more subtle implication follows by the remark, on which we shall elaborate shortly, that the objective function in the programming problem (3) is a Morse function, for almost all costs of participation $\mathbf{c}$, $\boldsymbol{\kappa}$. In the next section we shall use this fact to show that game $\Gamma\left(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w}\right)$ is regular for almost all $\mathbf{c}$, $\boldsymbol{\kappa}$.

4. REGULAR EQUILIBRIA

Before we proceed to study regularity of the equilibria of game $\Gamma\left(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w}\right)$, we clarify some terminology. A *critical point* of a smooth function $f:\mathbf{R}^k\longrightarrow\mathbf{R}$ is a point $\mathbf{x}\in\mathbf{R}^k$ such that all the partial derivatives of f vanish at $\mathbf{x}$: $f_h\left(\mathbf{x}\right)=0, h=1,...,k$. A critical point is *degenerate* if the Hessian of f at $\mathbf{x}$ is singular, $\det\left[Df_{\mathbf{x}}\right]=0$; it is not degenerate if $\det\left[Df_{\mathbf{x}}\right]\neq 0$. Function f is a *Morse function* if it does not have degenerate critical points.

An application of Sard's theorem demonstrates that Morse functions abound. In particular, given function $f:\mathbf{R}^k\longrightarrow\mathbf{R}$:

Theorem 1 *(Guillemin and Pollack, 1974, page 43) The function $f_{\mathbf{a}}=f+\sum_{h=1}^{k}a_hx_h$ is a Morse function for almost every $\mathbf{a}\in\mathbf{R}^k$.*

We now develop notation in order to define regular equilibria of game $\Gamma\left(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w}\right)$. Consider an equilibrium of game $\Gamma\left(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w}\right)$ in which players in $N_1\subseteq N$ and $M_1\subseteq M$ are indifferent between participating and abstaining, players in N_2 and M_2 strictly prefer to abstain, and players in N_3 and M_3 strictly prefer to vote. Let $|N_h|=n_h$ and $|M_h|=\mu_h$, $h=1,2,3$, denote strategies and participation costs of players in the respective sets by $\mathbf{p}_h,\boldsymbol{\pi}_h$, and $\mathbf{c}_h,\boldsymbol{\kappa}_h$, $h=1,2,3$, respectively, and denote the set of possible participation costs by $\widetilde{C}_h,\ \widetilde{K}_h,\ h = 1, 2, 3$. Finally, denote the set of equilibria of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ with the above properties by $E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$, and the set of all equilibria of $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ by $E(\mathbf{c}, \boldsymbol{\kappa})$.

For equilibrium $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$ we construct a function $H : \mathbf{R}^{n+\mu} \to \mathbf{R}^{n+\mu}$ given by:

$$H_g(\mathbf{p}, \boldsymbol{\pi}) = \begin{cases} p_g \left[F_g(\mathbf{p}, \boldsymbol{\pi}) - c_g\right] & \text{for } g \in N \text{ with } \quad p_g < 1 \\ \pi_g \left[\Phi_g(\mathbf{p}, \boldsymbol{\pi}) - \kappa_g\right] & \text{for } g \in M \text{ with } \quad \pi_g < 1 \\ (1 - p_g)\left[c_g - F_g(\mathbf{p}, \boldsymbol{\pi})\right] & \text{for } g \in N \text{ with } \quad p_g = 1 \\ (1 - \pi_g)\left[\kappa_g - \Phi_g(\mathbf{p}, \boldsymbol{\pi})\right] & \text{for } g \in M \text{ with } \quad \pi_g = 1 \end{cases} \tag{6}$$

Obviously, if $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$, then this equilibrium satisfies $H(\mathbf{p}, \boldsymbol{\pi}) = \mathbf{0}$. Essentially[7] following the definition of van Damme, 1987, page 38-39, the Nash equilibrium $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$ is *regular* if the Jacobian of H does not vanish at $(\mathbf{p}, \boldsymbol{\pi})$. The *game* $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ is regular if *all* of its Nash equilibria are regular.

The determinant of the Jacobian of H, $J(\mathbf{p}, \boldsymbol{\pi})$, calculated at $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$ is given as:

$$\det\left[J(\mathbf{p}, \boldsymbol{\pi})\right] = \det\left[\widetilde{J}(\mathbf{p}_1, \boldsymbol{\pi}_1)\right] \prod_{i \in N_2 \cup N_3} \left(F_i(\mathbf{p}, \boldsymbol{\pi}) - c_i\right) \prod_{j \in M_2 \cup M_3} \left(\Phi_j(\mathbf{p}, \boldsymbol{\pi}) - \kappa_j\right) \tag{7}$$

where $\widetilde{J}(\mathbf{p}, \boldsymbol{\pi})$ is the $(n_1 + \mu_1) \times (n_1 + \mu_1)$ matrix obtained by the rows and columns of $J(\mathbf{p}, \boldsymbol{\pi})$ that correspond to the players in $N_1 \cup M_1$.

Lemma 2 and theorem 1 immediately yield:

Lemma 3 *All equilibria* $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$ *of game* $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ *such that* $(\mathbf{p}_1, \boldsymbol{\pi}_1) \in (0, 1)^{n_1 + \mu_1}$ *are regular for almost all* $(\mathbf{c}, \boldsymbol{\kappa}) \in \widetilde{C} \times \widetilde{K}$.

Proof. By assumption, the term $\prod_{i \in N_2 \cup N_3} (F_i(\mathbf{p}, \boldsymbol{\pi}) - c_i) \prod_{j \in M_2 \cup M_3} (\Phi_j(\mathbf{p}, \boldsymbol{\pi}) - \kappa_j)$ in (7) is different than zero. As a consequence, an equilibrium $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$ is regular if and only if $\det\left[\widetilde{J}(\mathbf{p}, \boldsymbol{\pi})\right] \neq 0$. The latter will be shown to hold because, by theorem 1, the following function of $(\mathbf{p}_1, \boldsymbol{\pi}_1)$

$$\widetilde{H}(\mathbf{p}_1, \boldsymbol{\pi}_1; \mathbf{p}_2, \mathbf{p}_3, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3) \equiv F(\mathbf{p}, \boldsymbol{\pi}) - \sum_{i \in N} p_i c_i + \sum_{j \in M} \pi_j \kappa_j, \tag{8}$$

with $\mathbf{p}_2 = \mathbf{0}$, $\boldsymbol{\pi}_2 = \mathbf{0}$, and $\mathbf{p}_3 = \mathbf{1}$, $\boldsymbol{\pi}_3 = \mathbf{1}$, is a Morse function for almost all $(\mathbf{c}_1, \boldsymbol{\kappa}_1) \in \widetilde{C}_1 \times \widetilde{K}_1$.

Note that the set of critical points of this function $\widetilde{H}$ is independent of the value of $(\mathbf{c}_2, \mathbf{c}_3, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_3) \in \widetilde{C}_2 \times \widetilde{C}_3 \times \widetilde{K}_2 \times \widetilde{K}_3$, which only shift $\widetilde{H}$ by a constant. Thus, for almost all $(\mathbf{c}, \boldsymbol{\kappa}) \in \widetilde{C} \times \widetilde{K}$,

[7] Our formulation differs slightly from van Damme's in that we omit the $n + \mu$ linear equations that ensure that probabilities of voting and abstaining for each player sum up to one. Instead, we substitute from these equations directly into $H(\mathbf{p}, \boldsymbol{\pi})$ by representing the probability of abstention by $1 - p_i$, $1 - \pi_j$.

the critical points of $\widetilde{H}$ are not degenerate, *i.e.* $\det\left[D\widetilde{H}_{(\mathbf{p}_1,\boldsymbol{\pi}_1)}\right] \neq 0$ at all points $(\mathbf{p}_1,\boldsymbol{\pi}_1)$ such that $\widetilde{H}_g(\mathbf{p}_1,\boldsymbol{\pi}_1) = 0$, $g \in N_1 \cup M_1$. Furthermore, $\widetilde{H}_g(\mathbf{p}_1,\boldsymbol{\pi}_1) = 0$, $g \in N_1 \cup M_1$ are necessary conditions for $(\mathbf{p},\boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^3, \mathbf{c}, \boldsymbol{\kappa}\right)$, by lemma 2. We complete the proof by observing that the determinant of the Hessian of $\widetilde{H}$ is related with $\widetilde{J}(\mathbf{p},\boldsymbol{\pi})$ in (7) according to $\det\left[\widetilde{J}(\mathbf{p},\boldsymbol{\pi})\right] = \det\left[D\widetilde{H}_{(\mathbf{p}_1,\boldsymbol{\pi}_1)}\right]\prod_{i\in N_1} p_i \prod_{j\in M_1}\pi_j \neq 0$. ■

Lemma 3 covers all possible Nash equilibria of game $\Gamma(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w})$, except those that involve players using a pure strategy *and* are indifferent between participating and abstaining.[8] Such equilibria correspond to those among the critical points of function $\widetilde{H}$ in (8) that lie on the boundary of $[0,1]^{n_1+\mu_1}$. But such critical points can be ruled out for almost all costs of participation. Indeed, we are ready to show:

Theorem 2 *Except for a closed set of measure zero in $\widetilde{C} \times \widetilde{K}$, game $\Gamma(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w})$ is regular.*

Proof. That the set of participation costs for which game $\Gamma(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w})$ is not regular is closed follows from standard arguments involving the upper-hemicontinuity of the equilibrium correspondence with respect to $\mathbf{c},\boldsymbol{\kappa}$, and the smoothness of the mapping H in (6) (see van Damme, 1987, page 42). Thus it remains to show that this set has measure zero.

The set of Nash equilibria of $\Gamma(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w})$ is obtained as

$$E(\mathbf{c},\boldsymbol{\kappa}) = \bigcup_{\{N_h,M_h\}_{h=1}^3} E\left(\{N_h, M_h\}_{h=1}^3, \mathbf{c}, \boldsymbol{\kappa}\right).$$

There only exist a finite number of possible combinations of sets $\{N_h, M_h\}_{h=1}^3$. Thus, since finite unions of sets of measure zero have measure zero, it will suffice to show that for almost all $(\mathbf{c},\boldsymbol{\kappa}) \in \widetilde{C} \times \widetilde{K}$ all the critical points $(\mathbf{p}_1,\boldsymbol{\pi}_1) \in [0,1]^{n_1+\mu_1}$ of function (8) lie in the interior of $[0,1]^{n_1+\mu_1}$. Then the theorem follows from lemma 3.

We proceed in the same fashion as above, by observing that there exist only a finite number of possible combinations of the coordinates of $(\mathbf{p}_1,\boldsymbol{\pi}_1)$ that can be equal to zero or one. Thus, assume $p_i = \pi_j = 0$ for all $i \in N_1', j \in M_1'$ and $p_i = \pi_j = 1$ for all $i \in N_1'', j \in M_1''$, where $N_1', N_1'' \subseteq N_1$, $M_1', M_1'' \subseteq M_1$, and $|N_1'| + |M_1'| + |N_1''| + |M_1''| = q \leq n_1 + \mu_1$. We define a function $\widehat{H} : \mathbf{R}^{n_1+\mu_1} \times \widetilde{C}_1 \times \widetilde{K}_1 \longrightarrow \mathbf{R}^{n_1+\mu_1+q}$, the first n_1 coordinates of which are given by $c_i - F_i(\mathbf{p},\boldsymbol{\pi})$, one for each $i \in N_1$, and the next μ_1 by $\kappa_j - \Phi_j(\mathbf{p},\boldsymbol{\pi})$, one for each $j \in M_1$. The remaining q coordinates, we set to either p_i or π_j corresponding to each $i \in N_1'$, $j \in M_1'$, or $p_i - 1$ or $\pi_j - 1$ corresponding to each $i \in N_1''$, $j \in M_1''$. We shall show that $\mathbf{0}$ is a regular value of the smooth function $\widehat{H}$. Indeed, by appropriate arrangement of its columns, the Jacobian of $\widehat{H}$ is obtained as:

$$D\widehat{H} = \begin{bmatrix} * & * & I_{n_1+\mu_1} \\ \mathbf{0} & I_q & \mathbf{0} \end{bmatrix}$$

which has full rank, so that $\mathbf{0}$ is a regular value of $\widehat{H}$.

[8] *i.e.* it covers all quasi-strict equilibria.

Hence, by the Preimage theorem (Guillemin and Pollack, 1974, page 21), $\widehat{H}^{-1}(\mathbf{0})$ is a submanifold of $\mathbf{R}^{n_1+\mu_1} \times \widetilde{C}_1 \times \widetilde{K}_1$ of dimension $n_1+\mu_1-q$. Thus, if $q \geq 1$, $\widehat{H}(\mathbf{p}_1, \boldsymbol{\pi}_1, \mathbf{c}_1, \boldsymbol{\kappa}_1) = \mathbf{0}$ can be true only for a subset of $\widetilde{C}_1 \times \widetilde{K}_1$ with measure zero. Outside that set, there cannot exist an equilibrium $(\mathbf{p}, \boldsymbol{\pi}) \in E\left(\{N_h, M_h\}_{h=1}^{3}, \mathbf{c}, \boldsymbol{\kappa}\right)$ with $p_i = \pi_j = 0$ for all $i \in N_1'$, $j \in M_1'$ and $p_i = \pi_j = 1$ for all $i \in N_1''$, $j \in M_1''$. We conclude, as we set out to show, that the critical points $(\mathbf{p}_1, \boldsymbol{\pi}_1) \in [0,1]^{n_1+\mu_1}$ of the function $\widetilde{H}$ in (8) lie on the interior of $[0,1]^{n_1+\mu_1}$ for almost all $(\mathbf{c}_1, \boldsymbol{\kappa}_1) \in \widetilde{C}_1 \times \widetilde{K}_1$. This completes the proof. ■

We have shown that non-regular games $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ are exceptional. One implication of this conclusion obtains from theorem 1 in Harsanyi, 1973b: for almost all participation costs $\mathbf{c}, \boldsymbol{\kappa}$, game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ has an odd (finite) number of equilibria. As an illustration, consider the following example:

Example 1 *Assume $n = 2$, $\mu = 1$, $w = 0$, and $\widehat{w} = \frac{1}{2}$. Let $c_1 < c_2 < \frac{1}{2}$.*
If $\frac{1}{2} > \kappa_1 > \frac{c_2 - c_1}{1 - 2c_1}$, this game has three equilibria:

1. *$p_1 = 1 > p_2 = 1 - 2\kappa_1$, and $\pi_1 = 2c_2$,*
2. *$p_1 = 1 - 2\kappa_1 < p_2 = 1$, and $\pi_1 = 2c_1$, and*
3. *$p_1 = \sqrt{\frac{(1-2\kappa_2)^2(1-2c_2)}{1-2c_1(1-\kappa_1)-2\kappa_1(1-c_1)}} < p_2 = \sqrt{\frac{1-2c_1(1-\kappa_1)-2\kappa_1(1-c_1)}{1-2c_2}}$, and $\pi_1 = 1 - \sqrt{\frac{(1-2c_2)^3}{1-2c_1(1-\kappa_1)-2\kappa_1(1-c_1)}}$.*

If $\kappa_1 < \frac{c_2 - c_1}{1 - 2c_1}$, then only the first equilibrium survives.
Finally, if $\kappa_1 = \frac{c_2 - c_1}{1 - 2c_1}$ then only the first and the second equilibrium obtain, and the latter is not regular.

The only equilibrium that is not regular in example 1 is not quasi-strict. Thus, we also give an example of a quasi-strict equilibrium that is not regular:

Example 2 *Assume $n = 4$, $\mu = 2$, $w = 0$, and $\widehat{w} = \frac{1}{2}$. Let $c_i = \frac{3}{8}$, $i \in N$, and $\kappa_j < \frac{5}{16}$, $j \in M$. Then, $p_i = \frac{1}{2}$, $i \in N$, and $\pi_j = 1$, $j \in M$ is a quasi-strict equilibrium that is not regular. It follows that $p_i = \frac{1}{2}$, $i \in N$ is an irregular equilibrium for the public goods game with $n = 4$, $\mu = 0$, $c_i = \frac{3}{8}$, $i \in N$, $w = 2$, and $\widehat{w} = \frac{1}{2}$.*

A second implication of theorem 2 also follows from Harsanyi's work (namely his purification theorem in Harsanyi, 1973a). For almost all participation costs $\mathbf{c}, \boldsymbol{\kappa}$, the equilibria of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ can be purified, *i.e.* can be obtained as (essentially) pure strategy Bayesian Nash equilibria of nearby games of incomplete information. We further discuss the implications of this fact in the concluding section. In the penultimate section that follows, we address a couple of extant questions raised by our analysis.

5. MONOTONE EQUILIBRIA

The minmax programming problem in (3) of lemma 2, suggests the possibility of a connection between the Nash equilibria of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ and the pure strategy Nash equilibria of an artificial two-player zero-sum game where the strategy of (fictitious) player 1 is $\mathbf{p} \in [0,1]^n$, that of (fictitious) player 2 is $\boldsymbol{\pi} \in [0,1]^\mu$, and the payoff of player 1 (respectively 2) is the (negative of the) objective function in (3).

Example 1 refutes the existence of such a connection. In particular, the third (totally mixed) equilibrium in this example makes it plain that the fictitious player 1 of the above described zero-sum game is not playing a best response at a Nash equilibrium of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$. This is because $p_1 < p_2$ in the third equilibrium of the example, while $c_1 < c_2$ requires $p_1 \geq p_2$ at an optimum for this fictitious player.[9]

Motivated by this discussion and example 1, we define the following refinement of the Nash equilibria of game $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$:

Definition 1 *A Nash equilibrium* $(\mathbf{p}, \mathbf{q})$ *of game* $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ *is monotone if*

$$\begin{aligned} c_i &> c_h \Longrightarrow p_i \leq p_h, \text{ all } i, h \in N, \text{ and} \\ \kappa_j &> \kappa_h \Longrightarrow \pi_j \leq \pi_h, \text{ all } j, h \in M. \end{aligned}$$

In a monotone equilibrium, within groups, players with higher participation cost abstain with (weakly) higher probability. Note that a monotone equilibrium exists for all cost parameters in example 1. In the last result of our analysis we show this is not an accident.

Theorem 3 *Every game* $\Gamma(\mathbf{c}, \boldsymbol{\kappa}, w, \widehat{w})$ *has at least one monotone Nash equilibrium.*

Proof. Assume (without loss of generality) that players' costs within groups are ranked so that:

$$\begin{aligned} c_i &\leq c_{i+1}, \; i = 1, ..., n-1, \text{ and} \\ \kappa_j &\leq \kappa_{j+1}, \; j = n+1, ..., n+\mu-1. \end{aligned}$$

Consider the sets

$$S_N = \{\mathbf{p} \in [0,1]^n : p_{i+1} \leq p_i, i = 1, ..., n-1\}, \text{ and}$$

$$S_M = \{\boldsymbol{\pi} \in [0,1]^\mu : \pi_{j+1} \leq \pi_j, j = 1, ..., \mu - 1\}$$

Let the continuous function $[]^+ : \mathbf{R} \to \mathbf{R}_+$ be defined as $[x]^+ \equiv \max\{x, 0\}$. Consider the mapping $G : S_N \times S_M \to S_N \times S_M$ where the 1-st and $(n+1)$-th coordinates of G are given by

$$G_1(\mathbf{p}, \mathbf{q}) \equiv \frac{p_1 + [F_1(\mathbf{p}, \boldsymbol{\pi}) - c_1]^+}{1 + |F_1(\mathbf{p}, \boldsymbol{\pi}) - c_1|}$$

$$G_{n+1}(\mathbf{p}, \mathbf{q}) \equiv \frac{\pi_{n+1} + [\Phi_{n+1}(\mathbf{p}, \boldsymbol{\pi}) - \kappa_{n+1}]^+}{1 + |\Phi_{n+1}(\mathbf{p}, \boldsymbol{\pi}) - \kappa_{n+1}|}$$

[9] In fact, the three Nash equilibria of example 1 are not even local Nash equilibria of the associated 2-player zero-sum game implied by (3).

while the remaining coordinates are inductively defined as:

$$\begin{aligned} G_i(\mathbf{p},\boldsymbol{\pi}) &\equiv \min\left\{\frac{p_i+[F_i(\mathbf{p},\boldsymbol{\pi})-c_i]^+}{1+|F_i(\mathbf{p},\boldsymbol{\pi})-c_i|}, G_{i-1}(\mathbf{p},\boldsymbol{\pi})\right\}, i=2,...,n, \\ G_j(\mathbf{p},\boldsymbol{\pi}) &\equiv \min\left\{\frac{\pi_j+[\Phi_j(\mathbf{p},\boldsymbol{\pi})-\kappa_j]^+}{1+|\Phi_j(\mathbf{p},\boldsymbol{\pi})-\kappa_j|}, G_{j-1}(\mathbf{p},\boldsymbol{\pi})\right\}, j=n+2,...,n+\mu \end{aligned}$$

Clearly, G is continuous in $\mathbf{p}$,$\boldsymbol{\pi}$. G maps the convex compact set $S_N \times S_M$ into itself. Thus, it has a fixed point by Brouwer's theorem. We now claim that if $G(\mathbf{p}^*,\boldsymbol{\pi}^*)=(\mathbf{p}^*,\boldsymbol{\pi}^*)$ then $\mathbf{p}^*,\boldsymbol{\pi}^*$ is a (monotone) Nash equilibrium of $\Gamma(\mathbf{c},\boldsymbol{\kappa},w,\widehat{w})$.

We use the following lemma:

Lemma 4 *Consider $\mathbf{p},\boldsymbol{\pi} \in [0,1]^{n+m}$ with $p_i = p_h$, $i,h \in N$ (or $\pi_j = \pi_h$, $j,h \in M$). Then $F_i(\mathbf{p},\boldsymbol{\pi}) = F_h(\mathbf{p},\boldsymbol{\pi})$ ($\Phi_j(\mathbf{p},\boldsymbol{\pi}) = \Phi_h(\mathbf{p},\boldsymbol{\pi})$).*

Proof. *Obvious since we have $f(\mathbf{p},l,i) = f(\mathbf{p},l,h)$ (or $\phi(\boldsymbol{\pi},\lambda,j) = \phi(\boldsymbol{\pi},\lambda,h)$) for all l, λ.*
■

We will successively consider coordinates of the fixed point $(\mathbf{p}^*,\boldsymbol{\pi}^*)$ such that $p_i^*=0$, $p_i^*=1$, or $p_i^*\in(0,1)$ and verify that the corresponding players play best responses. Let $h=\min\{i\in N : p_i^*=0\}$. Then, $p_i^*=0$, $i=h,...,n$. Also, if $h\geq 2$ we have $p_{h-1}^*>0$. Hence $G_h(\mathbf{p}^*,\boldsymbol{\pi}^*)=0 \Longrightarrow [F_h(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_h]^+=0 \Longrightarrow F_h(\mathbf{p}^*,\boldsymbol{\pi}^*)\leq c_h$. But, by lemma 4 and the fact that $c_{i+1}\geq c_i$, we have $F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)\leq c_i$ for all $i=h{+}1,...,n$. Similarly, we conclude $\Phi_j(\mathbf{p}^*,\boldsymbol{\pi}^*)\leq\kappa_j$ for all $j\in M$ such that $\pi_j^*=0$.

Next, consider $i\in N$ such that $p_i^*=1$. Since $G_i(\mathbf{p}^*,\boldsymbol{\pi}^*)=1$, we must have $\frac{1+[F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_i]^+}{1+|F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_i|}=1 \Longrightarrow [F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_i]^+=|F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_i| \Longrightarrow F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)\geq c_i$. By the same argument we deduce $\pi_j^*=1\Longrightarrow\Phi_j(\mathbf{p}^*,\boldsymbol{\pi}^*)\geq\kappa_j$.

Finally, consider $i\in N$ such that $p_i^*\in(0,1)$. We shall prove by induction that $F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_i$ for all i such that $p_i^*\in(0,1)$.

1. $F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_i$ if $i=\min\{h\in N : p_h^*\in(0,1)\}$. Either $i=1$, or $p_i^*<G_{i-1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=p_{i-1}^*=1$. In either case we must have $G_i(\mathbf{p}^*,\boldsymbol{\pi}^*)=\frac{p_i+[F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_i]^+}{1+|F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_i|}=p_i^*\Longrightarrow F_i(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_i$.

   Based on the inductive hypothesis that $F_h(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_h$, $p_h^*\in(0,1)$, we shall show:

2. if $p_{h+1}^*\in(0,1)$, $F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_{h+1}$. We have two possibilities: (a) $p_{h+1}^*=p_h^*$, in which case $F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=F_h(\mathbf{p}^*,\boldsymbol{\pi}^*)$ by lemma 4. Since $c_{h+1}\geq c_h$, and $F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=F_h(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_h$, we have $[F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}]^+=0$. Thence, we deduce from the fact that $G_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=\min\left\{\frac{p_{h+1}^*}{1+|F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}|},p_h^*\right\}=p_{h+1}^*$ that $|F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}|=0\Longleftrightarrow F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_{h+1}$, as desired. The second possibility is (b) $p_{h+1}^*<p_h^*$. Then, since $G_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=\min\left\{\frac{p_{h+1}^*+[F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}]^+}{1+|F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}|},p_h^*\right\}$ we deduce $\frac{p_{h+1}^*+[F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}]^+}{1+|F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}|}=p_{h+1}^*\Longleftrightarrow$ $[F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}]^+=|F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)-c_{h+1}|\,p_{h+1}^*$, from which we conclude that $F_{h+1}(\mathbf{p}^*,\boldsymbol{\pi}^*)=c_{h+1}$ as desired. This completes the induction proof.

The same induction argument applies for the case $j \in M$ with $\pi_j^* \in (0,1)$. In summary, we have shown that if $\mathbf{p}^*, \boldsymbol{\pi}^*$ is a fixed point of G, then it satisfies conditions (4) and (5) for a Nash equilibrium, which concludes the proof. ■

6. CONCLUSIONS

We have analyzed participation games with heterogeneous participation costs in the context of two-candidate elections as well as the provision of a discrete public good. We characterized the equilibria of these games as stationary points of a non-linear programming problem. We showed that, outside a closed set of measure zero of participation costs, these games are regular. Finally, we established the existence of monotone equilibria.

In this concluding section, we take the opportunity to relate our analysis in section 4 with the literature on the possibility of rational turnout when voting is costly. The possibility of high turnout in large electorates with costly voting has been established by Palfrey and Rosenthal, 1983, in games of complete information with equal participation costs. On the other hand, Palfrey and Rosenthal, 1985, show that if players have incomplete information about each others' costs, equilibrium turnout rate approaches zero in large electorates. Palfrey and Rosenthal, 1985, fix the level of uncertainty regarding players' participation cost, and perform a limit calculation as the number of players increases. Thus, if the level of uncertainty is large *relative* to the size of the electorate, only low turnout equilibria prevail.

The latter qualification is often omitted in the literature, where a prevalent rendition of these results is that incomplete information eliminates high turnout equilibria. Our analysis shows that this omission is not warranted since, for regular participation games with arbitrarily large electorates, equilibria are robust to the introduction of (mild) incomplete information. As a consequence, *if* a regular complete information game admits equilibria with high turnout, then these high turnout equilibria also survive in nearby games of incomplete information.

An additional caveat for the comparison between the games analyzed by Palfrey and Rosenthal, 1983, 1985, is that it is unclear whether the complete information games with equal participation costs from which the incomplete information versions depart are regular for all sizes of the electorate. An obvious direction for further research, which is the subject of our current investigation, is to establish (or refute) the possibility of large equilibrium turnout in complete information games with heterogeneous participation costs.

APPENDIX: PROOF OF LEMMA 1

We use the fact that $f(\mathbf{p},l,\emptyset) = p_i f(\mathbf{p},l-1,i) + (1-p_i) f(\mathbf{p},l,i)$ in order to re-write $F(\mathbf{p},\boldsymbol{\pi})$ as:

$$\begin{aligned} F(\mathbf{p},\boldsymbol{\pi}) &= \sum_{\lambda=0}^{\mu}\sum_{l=\lambda+w+1}^{n}\left(p_i f(\mathbf{p},l-1,i) + (1-p_i) f(\mathbf{p},l,i)\right)\phi(\boldsymbol{\pi},\lambda,\emptyset) \\ &\quad +\widehat{w}\sum_{\lambda=0}^{\mu}\left(p_i f(\mathbf{p},\lambda+w-1,i) + (1-p_i) f(\mathbf{p},\lambda+w,i)\right)\phi(\boldsymbol{\pi},\lambda,\emptyset) \end{aligned} \tag{9}$$

Now, differentiation with respect to p_i gives us:

$$\begin{aligned} F_i(\mathbf{p},\boldsymbol{\pi}) &= \sum_{\lambda=0}^{\mu}\sum_{l=\lambda+w+1}^{n} f(\mathbf{p},l-1,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) - \sum_{\lambda=0}^{\mu}\sum_{l=\lambda+w+1}^{n} f(\mathbf{p},l,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) \\ &\quad +\widehat{w}\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w-1,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) - \widehat{w}\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) \\ &= \sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) \\ &\quad -\widehat{w}\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) + \widehat{w}\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w-1,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) \\ &= (1-\widehat{w})\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) + \widehat{w}\sum_{\lambda=1}^{\mu} f(\mathbf{p},\lambda+w-1,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) \end{aligned}$$

By substituting in (9) for $p_i = 1$ and $p_i = 0$, respectively, we also obtain

$$F\left(\left(\mathbf{p}^{-i},1\right),\boldsymbol{\pi}\right) = \sum_{\lambda=0}^{\mu}\sum_{l=\lambda+w+1}^{n} f(\mathbf{p},l-1,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) + \widehat{w}\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w-1,i)\phi(\boldsymbol{\pi},\lambda,\emptyset)$$

and

$$F\left(\left(\mathbf{p}^{-i},0\right),\boldsymbol{\pi}\right) = \sum_{\lambda=0}^{\mu}\sum_{l=\lambda+w+1}^{n} f(\mathbf{p},l,i)\phi(\boldsymbol{\pi},\lambda,\emptyset) + \widehat{w}\sum_{\lambda=0}^{\mu} f(\mathbf{p},\lambda+w,i)\phi(\boldsymbol{\pi},\lambda,\emptyset).$$

Subtracting the second expression from the first we get

$$F_i(\mathbf{p},\boldsymbol{\pi}) = F\left(\left(\mathbf{p}^{-i},1\right),\boldsymbol{\pi}\right) - F\left(\left(\mathbf{p}^{-i},0\right),\boldsymbol{\pi}\right)$$

as desired.

Similarly, we use the fact that $\phi(\boldsymbol{\pi}, \lambda, \emptyset) = \pi_j \phi(\boldsymbol{\pi}, \lambda - 1, j) + (1 - \pi_j) \phi(\boldsymbol{\pi}, \lambda, j)$ in order to re-write $F(\mathbf{p}, \boldsymbol{\pi})$ as:

$$\begin{aligned} F(\mathbf{p}, \boldsymbol{\pi}) &= \sum_{\lambda=0}^{\mu} \sum_{l=\lambda+w+1}^{n} f(\mathbf{p}, l, \emptyset) \left(\pi_j \phi(\boldsymbol{\pi}, \lambda - 1, j) + (1 - \pi_j) \phi(\boldsymbol{\pi}, \lambda, j)\right) \\ &\quad + \widehat{w} \sum_{\lambda=0}^{\mu} f(\mathbf{p}, \lambda + w, \emptyset) \left(\pi_j \phi(\boldsymbol{\pi}, \lambda - 1, j) + (1 - \pi_j) \phi(\boldsymbol{\pi}, \lambda, j)\right) \end{aligned} \tag{10}$$

Now, differentiation with respect to π_j gives us:

$$\begin{aligned} F_j(\mathbf{p}, \boldsymbol{\pi}) &= \sum_{\lambda=0}^{\mu} \sum_{l=\lambda+w+1}^{n} f(\mathbf{p}, l, \emptyset) \left(\phi(\boldsymbol{\pi}, \lambda - 1, j) - \phi(\boldsymbol{\pi}, \lambda, j)\right) \\ &\quad + \widehat{w} \sum_{\lambda=0}^{\mu} f(\mathbf{p}, \lambda + w, \emptyset) \left(\phi(\boldsymbol{\pi}, \lambda - 1, j) - \phi(\boldsymbol{\pi}, \lambda, j)\right) \\ &= -\sum_{\lambda=1}^{\mu} f(\mathbf{p}, \lambda + w, \emptyset) \phi(\boldsymbol{\pi}, \lambda - 1, j) + \widehat{w} \sum_{\lambda=0}^{\mu} f(\mathbf{p}, \lambda + w, \emptyset) \left(\phi(\boldsymbol{\pi}, \lambda - 1, j) - \phi(\boldsymbol{\pi}, \lambda, j)\right) \\ &= -(1 - \widehat{w}) \sum_{\lambda=1}^{\mu} f(\mathbf{p}, \lambda + w, \emptyset) \phi(\boldsymbol{\pi}, \lambda - 1, j) - \widehat{w} \sum_{\lambda=0}^{\mu-1} f(\mathbf{p}, \lambda + w, \emptyset) \phi(\boldsymbol{\pi}, \lambda, j) \end{aligned}$$

In identical manner, by substituting in (10) for $\pi_j = 1$ and $\pi_j = 0$, respectively, we also obtain

$$\Phi_j(\mathbf{p}, \boldsymbol{\pi}) = \Phi\left(\mathbf{p}, \left(\boldsymbol{\pi}^{-j}, 1\right)\right) - \Phi\left(\mathbf{p}, \left(\boldsymbol{\pi}^{-j}, 0\right)\right).$$

This completes the proof of the lemma.

REFERENCES

1. Aldrich, J. 1993. "Rational Choice and Turnout," *American Journal of Political Science*, 37(1), 246-79.

2. De Sinopoli F., and Iannantuoni G. 2005. "On the generic strategic stability of Nash equilibria if voting is costly," *Economic Theory* 25(2): 477-486.

3. Feddersen, Timothy. 2004. "Rational Choice Theory and the Paradox of Not Voting," *Journal of Economic Perspectives* 18(1): 99-112.

4. Govindan S., Reny P. J., and Robson A. J. 2003. "A short proof of Harsanyi's purification theorem" *Games and Economic Behavior* 45(2): 369-374.

5. Guillemin, Victor and Alan Pollack. 1974. *Differential Topology.* Prentice-Hall: Englewood Cliffs, New Jersey.

6. Harsanyi J. C. 1973a. "Games with randomly disturbed payoffs: a new rationale for mixed-strategy equilibrium points" *International Journal of Game Theory* 2: 1-23.

7. Harsanyi, J. C. 1973b. "Oddness of the number of equilibrium points: a new proof" *International Journal of Game Theory* 2: 235-250.

8. Palfrey, T., Rosenthal, H. 1983. "A Strategic Calculus of Voting," *Public Choice* 41(1): 7-53.

9. Palfrey, T., Rosenthal, H. 1984. "Participation and the Provision of Public Goods: A Strategic Analysis," *Journal of Public Economics* 24(2): 171-193.

10. Palfrey, T., Rosenthal, H. 1985. "Voter participation and strategic uncertainty," *American Political Science Review*, 79(1): 62–78.

11. van Damme, E. 1987. *Stability and perfection of Nash equilibria.* Springer-Verlag: Berlin Heidelberg.