Rationalizable Voting

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Abstract

We derive necessary and sufficient conditions in order for a finite number of binary voting choices to be consistent with the hypothesis that voters have preferences that admit concave utility representations. When the location of the voting alternatives is known, we apply these conditions in order to derive simple, nontrivial testable restrictions on the location of voters' ideal points, and in order to predict individual voting behavior. If, on the other hand, the location of voting alternatives is unrestricted then voting decisions impose no testable restrictions on the joint location of voter ideal points, even if the space of alternatives is one dimensional. Furthermore, two dimensions are always sufficient to represent or fold the voting records of any number of voters while endowing all these voters with strictly concave preferences and arbitrary ideal points. The analysis readily generalizes to choice situations over any finite sets of alternatives.

1 Introduction

What can we learn about individual voter preferences on the basis of data consisting of a finite number of binary choices? Estimates of voter ideal points are now routinely obtained

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using such records of past voting decisions, e.g., Poole and Rosenthal (1997), Heckman and Snyder (1997). These estimators rely on parametric restrictions on probabilistic choice models and impose symmetry on voters' utility functions around their ideal point, i.e., they require that voter disutility is measured by Euclidean distance from that ideal point. In this paper we take a different route, seeking testable restrictions on voter preferences assuming deterministic choice and without such parametric restrictions.

We maintain a spatial framework so that voters are confronted with a finite number of choices between two alternatives drawn from a finite dimensional Euclidean policy space. We derive necessary and sufficient conditions in order for such voting records to be consistent with voter preferences that admit concave utility representations. While these conditions ensure that the hypothesis that individual preferences are convex is testable using a finite number of binary choices, we show that such data do not allow us to discriminate between the hypotheses that voters have (strictly) concave versus quasi-concave utility representations. On the other hand, if individual voting records are rationalizable in the above sense, then we use these conditions in order to derive nontrivial testable restrictions on the location of voters' ideal points. We also use these rationalizability conditions in order to predict individual voting behavior on new voting items.

The application of the derived necessary and sufficient conditions for the purposes of ideal point estimation and vote prediction requires knowledge of the location of the voting alternatives. In fact, we show that if the location of the voting alternatives is unknown and unrestricted, as in prevalent ideal point estimation techniques from roll call data, then voting decisions alone impose no testable restrictions whatsoever on the joint location of voter ideal points, even if the space of alternatives is one dimensional. For any arbitrary set of ideal points for the voters, and for any record of voting decisions by these voters, we can locate the voting alternatives and find strictly concave utility functions for all voters such that both (i) voters have the prespecified ideal points, and (ii) the utility functions perfectly explain all individual voting decisions. Furthermore, we show that two dimensions are always sufficient in order to represent (or, if the original voting record lies in higher dimensional space, in order to 'fold') any voting records, while at the same time endowing voters with strictly concave utility representations and arbitrary ideal points.

The present study is connected with a branch of the literature on the theory of revealed preferences of the consumer pioneered by Sydney Afriat (1967), in that we seek to make inferences about individual preferences from a finite number of choice observations. Afriat provided necessary and sufficient conditions that must be met by a set of observations of prices and quantity choices of commodities in order for these observations to be consistent with individual maximization of a non-trivial monotone, concave, utility function and, at the same time, constructed the required utility representation. Hal Varian (1982) built on this approach to study the non-parametric estimation of demand. We pursue a similar approach but, unlike the classical theory of demand, in our context we have no observations akin to prices and, once the voting agenda is formed, there is no similar process of individual maximization over a budget set containing an infinite set of alternatives. While we focus the analysis on the case of binary voting choices, as we discuss in section 6, the necessary and sufficient conditions we derive are applicable to more general choice situations over any finite budget sets.

A number of other studies analyze revealed preferences over nonstandard (although not necessarily finite) budget sets, under concavity and/or monotonicity conditions on preferences, e.g., Matzkin (1991), Cox and Chavas (1993), and Forges and Minelli (2006). General finite budget sets are assumed by Chambers and Echenique (2007), who consider testable implications of supermodularity, assuming non-satiated preferences. The present study differs from the above and standard theory of the consumer, in that we do not require monotonicity of preferences. Indeed, individual preferences in political environments are typically assumed to be satiated, with voters that possess well defined ideal points. Non-montonicities (although not necessarily leading to satiation) may also arise naturally in economic models of altruism, as recently studied by, e.g., Cox, Friedman, and Sadiraj (2007). While we do not assume it, we do not rule out monotonicity of preferences so that the present analysis is applicable to economic as well as voting contexts. Indeed, by the generalization we discuss in Theorem 9, the analysis can be applied to the problem of the consumer facing a finite budget set, as is the case in the presence of indivisibilities.

While we shed monotonicity assumptions, we do rely heavily on convexity restrictions on preferences, so that the analysis is intimately related with the literature on the concavifiability of individual preferences. Yakar Kannai (1977) tackled this question for the case of continuous preferences on infinite convex sets. For our purposes, the relevant question is concavifiability of preferences on finite sets, a question that has recently been taken up by Marcel Richter and Kam-Chau Wong (2004) and Kannai (2005), whose results provide a departure point for the present study. Via an application of a Theorem of the alternative, Richter and Wong derive a necessary and sufficient condition for the existence of a (strictly) concave utility function that represents complete and transitive preferences over finite sets. Kannai (2005) discusses various alternative conditions focusing on the construction of the requisite utility function. In the present study we consider a range of possible utility representations from strict concavity to mere quasi-concavity of the rationalizing utility function. The conditions we derive differ from those of Richter and Wong (2004) and Kannai (2005) in that they are applicable to any irreflexive (typically incomplete) revealed preference relation over a finite set of alternatives.

Besides the extensive literature on ideal point estimation using roll call voting records which is reviewed in Kalandrakis (2006), a number of recent studies analyze the consistency of voting choices with specific parametric utility representations for the voters. Bogomolnaia and Laslier (2007) establish bounds on the number of policy dimensions of the policy space that are sufficient in order to represent any voter preferences over a fixed number of alternatives in this space by Eucledian utlity functions. Degan and Merlo (2007) establish conditions on observable choices over multiple elections in order to falsify the hypothesis that voters with Eucledian preferences vote sincerely. Working in a discrete space of alternatives, Schwartz (2007) shows that observed voting histories cannot refute in either direction the hypothesis that a committee's majority rule social preference over the finite number of voting alternatives in the voting record is transitive (respectively, intransitive). He also provides a sufficient condition in order for the committee's preference profile over this finite set to have (respectively, not to have) a single-peaked representation.

We now proceed to the analysis. In the next section, we develop notation and review the question of rationalizability without convexity restrictions. In section 3 we consider the rationalization of voting records by concave utility functions. In section 4 we analyze how or whether the conditions derived in section 3 can be used for the non-parametric estimation of voter ideal points. In section 5 we analyze the use of the voting record for the purposes of prediction. We show how the analysis generalizes to multiple choice situations over a finite number of alternatives in section 6. We conclude in section 7.

2 Rationalizable Voting

Consider a set of n voters $N = \{1, ..., n\}$ who are confronted with a finite number of binary choices over m pairs of alternatives in $\mathbb{R}^d$. We call each pairwise comparison a *voting item*, and denote the set of voting items by $M = \{1, ..., m\}$. Let $z_j, y_j \in \mathbb{R}^d$, $z_j \neq y_j$, $j \in M$, represent the pair of alternatives compared in the j-th voting item. The *voting record* of voter i is given by the collection $\{(y_j, z_j, v_j^i)\}_{j \in M}$ where $v_j^i \in \{yes, no\}$, represents i's decision on the j-th voting item. A decision $v_j^i = yes$ is a vote in favor of alternative y_j over alternative z_j, and vice versa for a decision $v_j^i = no$. We occasionally distinguish the voting decisions of voter i from the entire voting record, in which case we write the former as a vector $v^i \in \{yes, no\}^m$.

Let $X_{M'}$ denote the set of alternatives that are compared in subset $M' \subseteq M$ of the voting items, i.e.,

$$X_{M'} = \bigcup_{j \in M'} \{y_j, z_j\}.$$

We shall find it useful to represent subsets of the voting alternatives, X_M, that correspond to alternatives that voter i voted for or against. Thus, for any subset $M' \subseteq M$ of the voting items we let $N_{M'}^i$ represent the voting alternatives that i voted against, i.e.,

$$N_{M'}^i = \left\{x \in X_{M'} : x = y_j \text{ and } v_j^i = no, \text{ or } x = z_j \text{ and } v_j^i = yes, \text{ for some } j \in M'\right\}.$$

We similarly define $Y^i_{M'}$ as the set of voting alternatives that i voted for in subset M' of voting items, i.e.,

$$Y^i_{M'} = \left\{x \in X_{M'} : x = y_j \text{ and } v^i_j = yes, \text{ or } x = z_j \text{ and } v^i_j = no, \text{ for some } j \in M'\right\}.$$

Before we continue, we recall definitions and notation that will be used extensively in what follows. As usual, $x \succeq_i x'$ reads "i weakly prefers x over x'," $x, x' \in \mathbb{R}^d$, while $\succ_i$ and $\sim_i$ denote strict preference and indifference, respectively. For a finite set $K \subset \mathbb{R}^d$, we write $\mathcal{C}(K)$ to denote the *convex hull* of K. We denote the set of *extreme points* of K by $\mathcal{E}(K)$, which is the set of all the elements of K that cannot be written as a strict convex combination of alternatives in K. The set of extreme points of K, $\mathcal{E}(K)$, is nonempty and coincides with the vertexes of $\mathcal{C}(K)$. We use $|K|$ to indicate the cardinality of the set K, and write the set difference between sets K and K' as $K \setminus K'$.

Given voting record $\{(y_j, z_j, v^i_j)\}_{j\in M}$, a first step in our analysis is to test whether there exists a utility function such that every voting decision of voter i is consistent with utility maximization of that function. A strong formulation of this test is given in the following definition.

Definition 1 *A utility function* $u_i : \mathbb{R}^d \to \mathbb{R}$ strictly rationalizes *voter i's record,* $\{(y_j, z_j, v^i_j)\}_{j\in M}$, *if*

$$v^i_j = \begin{cases} yes & if \quad u_i(y_j) > u_i(z_j) \\ no & if \quad u_i(y_j) < u_i(z_j) \end{cases}, \; j \in M. \tag{1}$$

The above definition rules out the possibility of indifference between any pair of alternatives in any voting item. This is not a particularly stringent requirement if voters have non-trivial preferences over $\mathbb{R}^d$ and the voting alternatives in any particular voting item arise exogenously according to some randomized process. Furthermore, it appears that by requiring any vote to indicate strict preference, we maximize the information on voters' preferences that can be extracted from the voting record. On the other hand, voter indifference arises naturally in many equilibrium models of voting when proposals are determined endogenously by a utility maximizing agenda setter. Thus, a more parsimonious interpretation of the voting record leads to the following weaker criterion.

Definition 2 *A utility function* $u_i : \mathbb{R}^d \to \mathbb{R}$ rationalizes *voter* i*'s record,* $\{(y_j, z_j, v_j^i)\}_{j \in M}$, *if*

$$(2) \qquad v_j^i = \begin{cases} \textit{yes} & \textit{if} \quad u_i(y_j) \geq u_i(z_j) \\ \textit{no} & \textit{if} \quad u_i(y_j) \leq u_i(z_j) \end{cases}, \; j \in M.$$

In accordance with the above definitions, we will say that a voting record is (strictly) rationalizable, if there exists a utility function that (strictly) rationalizes that record. Well known arguments imply that even the strongest of these two criteria places weak restrictions on finite voting records.

Theorem 1 *The voting record* $\{(y_j, z_j, v_j^i)\}_{j \in M}$ *of voter* i *is*

(i) rationalizable.

(ii) strictly rationalizable if and only if it satisfies

$$(A) \qquad Y^i_{M'} \neq N^i_{M'}, \textit{ for all non-empty } M' \subseteq M.$$

Part (i) is trivial since a constant function rationalizes any voting record. To see part (ii), note that condition *(A)* is in fact the familiar acyclicity condition. In particular, *(A)* is necessary and sufficient to ensure that there does not exist a set of voting items and corresponding votes that produce a chain of comparisons between voting alternatives of the form $x \succ_i x' \succ_i \ldots \succ_i x$. If *(A)* holds, the choices in the voting record define a strict partial order in X_M, and we can extend this relation to a strict linear order (e.g., Lemma 2 in Richter (1966)). The construction of a (continuous) rationalizing utility function u_i over $\mathbb{R}^d$ is trivial. Condition *(A)* can be traced to general revealed preference analyses by Arrow (1959), Richter (1966), etc., and it amounts to a finite version of Ville-Houthakker SARP in the context of revealed preference theory of the consumer. Nevertheless, this condition

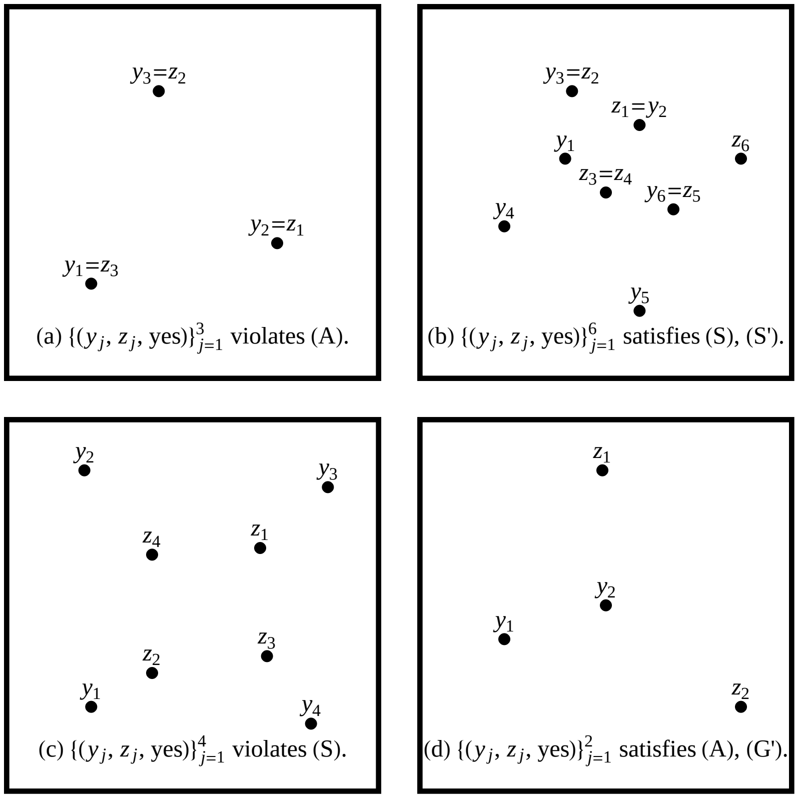


Figure 1: Strict Rationalizability.

has significantly less bite in the context of voting. For example, a sufficient condition on the voting alternatives in order for condition (A) to be satisfied for all voting decisions is:

(N) For all $M' \subseteq M$, there exists $j \in M'$ and $x \in X_{\{j\}}$ such that $x \notin X_{M' \setminus \{j\}}$.

Condition (N) simply requires that for each subset of voting items there exists a voting alternative that appears in only one voting item in that subset. Figure 1 illustrates four voting records in two dimensions, only one of which (Figure 1(a)) violates (N) and (A). Thus, questions of rationalizability of voting choices become interesting only under additional restrictions on voters' preferences. We take up this analysis in the next section.

3 Concave Rationalizations

In this section we consider whether observed voting records are consistent with the hypothesis that voters' decisions are generated by convex preferences. We consider several variants of this restriction, the strongest of which is the existence of a rationalizing utility function, $u_i : \mathbb{R}^d \to \mathbb{R}$, that is strictly concave:

$$(3) \qquad u_i(\lambda x + (1-\lambda)x') > \lambda u_i(x) + (1-\lambda)u_i(x'), \text{ for all } x, x', x \neq x', \text{ and all } \lambda \in (0,1)\,.$$

A weaker restriction is strict quasiconcavity:

$$(4) \qquad u_i(\lambda x + (1-\lambda)x') > \min\{u_i(x), u_i(x')\}, \text{ for all } x, x', x \neq x', \text{ and all } \lambda \in (0,1)\,.$$

When relevant, we also consider mere concavity and quasiconcavity, which are obtained from (3) and (4), respectively, by allowing weak inequality. These restrictions have a natural place in the theory of voting. For example, in a one-dimensional space ($d = 1$) strict quasiconcavity of preferences boils down to the single-peakedness condition familiar from social choice theory.

It turns out that when it comes to strict rationalizability, finite voting records do not allow us to discriminate among these possible utility representations. Nevertheless, not all voting records that are strictly rationalizable can be so rationalized by a (quasi)concave utility function. In the next Theorem we state necessary and sufficient conditions.

Theorem 2 *Given voting record $\{(y_j, z_j, v_j^i)\}_{j\in M}$ of voter i, the following conditions are equivalent:*

(S) *For all $M' \subseteq M$, $|M'| \geq 2$, there exists $x \in \mathcal{E}(X_{M'})$ such that $x \notin Y^i_{M'}$.*

(S') *There exists a nested sequence of subsets $M = M_1 \supset M_2 \supset ... \supset M_k \supset M_{k+1} = \emptyset$, $k \leq m$, such that $N^i_{M_t \setminus M_{t+1}} = \{x_t\} \subset \mathcal{E}(X_{M_t})$ and $x_t \notin X_{M_{t+1}}$, for all $t = 1, ..., k$.*

(S_c) *There exists a strictly concave utility function that strictly rationalizes i's record.*

(S'_c) *There exists a concave utility function that strictly rationalizes i's record.*

(S_q) *There exists a strictly quasiconcave utility function that strictly rationalizes i's record.*

(S'_q) *There exists a quasiconcave utility function that strictly rationalizes i's record.*

Furthermore, if $d = 1$, then (S) is equivalent to:

(S_1) *For all $M' \subseteq M$ with $|M'| = 2$, there exists $x \in \mathcal{E}(X_{M'})$ such that $x \notin Y^i_{M'}$.*

Of course, condition (S) implies condition (A) but it is, in fact, a significant strengthening of that condition. This is in contrast to standard neoclassical theory of the consumer where a finite version of Ville-Houthakker acyclicity in the form of SARP are sufficient for that consumer to have a (strictly) concave utility representation as shown by, e.g., Afriat (1967), Matzkin and Richter (1991), etc. Besides the fact that we analyze choice situations with non-standard, finite budget sets, a major difference in our analysis is the fact that we seek utility representations for possibly satiated preferences, whereas the corresponding analysis of the consumer requires monotonicity of preferences.

The necessity of condition (S) is straightforward. From a practical point of view, condition (S) involves identifying an extreme point with the required property for each of the $\sum_{h=2}^{m}\binom{m}{h}$ subsets $X_{M'} \subseteq X_M$, $|M'| \geq 2$, of the voting alternatives. While this task appears daunting as the number of voting items increases, the equivalent condition[1] (S') of

[1] Yet another equivalent statement of condition (S) that appeared in previous versions is

(S'') For all $M' \subseteq M$, $|M'| \geq 2$, there exists $x \in N^i_{M'}$ such that $x \notin \mathcal{C}(Y^i_{M'})$.

Theorem 2 provides a palatable remedy: it suffices to identify such extreme points for at most m subsets $M' \subseteq M$. As explicitly determined in the proof of Theorem 2, we can construct the sequence of these subsets required by (S') successively shifting over the elements of N_M^i as follows: we must first identify the requisite extreme point, $x_1 \in N_M^i$, from the universe of voting alternatives X_M; we then need to proceed 'inwards' and identify a new extreme point $x_2 \in N_M^i$ with the required properties, by only considering the subset M_2 of voting items, i.e., ignoring voting alternatives involved in voting items such that x_1 is voted against by voter i, etc. For example, for the voting record in Figure 1(b), we have $k = 5 < m = 6$ and the required sequence is $M_1 = \{1, \ldots, 6\}$ with the first extreme point being $x_1 = z_6$, then $M_2 = \{1, 2, 3, 4, 5\}$ with $x_2 = z_5$, $M_3 = \{1, 2, 3, 4\}$ with $x_3 = z_4$, $M_4 = \{1, 2\}$ with $x_4 = z_2$, and finally $M_5 = \{1\}$ with $x_5 = z_1$.

A reversal of the order of the above algorithm yields an inductive proof of the sufficiency of condition (S'). In particular, we can trivially find a concave function that rationalizes revealed preferences over alternatives X_{M_k}. We can then move 'outwards' to extend or modify this function to represent revealed preferences over $X_{M_{k-1}}$ by preserving the existing comparisons among alternatives in X_{M_k} and by assigning a sufficiently lower indifference contour to the extreme point x_{k-1}. Proceeding as above, at the t-th step of the process we can strictly rationalize revealed preferences over the larger set $X_{M_{k-t+1}}$ by assigning a sufficiently lower indifference contour to the extreme point x_{k-t+1}, etc.

A different simplification of condition (S) obtains in the one-dimensional case ($d = 1$). Then, condition (S) is equivalent to (S_1) which only requires the existence of the requisite extreme points for pairs of voting items. In one dimension there can exist at most two extreme points, thus, if condition (S) fails for voter i and a subset $M' \subseteq M$ of three or more voting items, then the condition must also fail for a pair of the voting items $\{j, h\} \subset M'$ such that $Y^i_{\{j,h\}} = \mathcal{E}(Y^i_{M'})$. Intuition may suggest that an analogous weakening of condition (S) is possible in more than one dimensions by requiring that this condition be applied only to subsets comprising at most $d + 1$ voting items when $d > 1$. Unfortunately, this is not the case, as is illustrated in Figure 1(c) in a two-dimensional setting: while condition (S) holds for all triplets ($d+1=3$) of voting items, it fails when we consider all four items in the voting record. In two or more dimensions there is no analogous bound on the number of extreme points such as the one that obtains in one dimension.

It is useful to contrast the above conclusion and condition (S) of Theorem 2 with the following (slightly restated) necessary and sufficient condition of Richter and Wong for the existence of a strictly concave function (Richter and Wong (2004), Theorem 2) that rationalizes a reflexive, transitive, and complete preference relation $\succeq_i$ over a finite set K:

(G') For all $X \subseteq K$ such that $|X| \leq d+1$ and $\mathcal{E}(X) = X$, and for all $x \in K$ such that x is in the interior of $\mathcal{C}(X)$, there exists $x' \in X$ such that $x \succ_i x'$.

Note that, since (G') is necessary and sufficient, if the strict preference relation, say $\succ_i^v$, determined by the voting record can be extended to a total order on X_M that admits a strictly concave utility representation, then condition (G') must hold for that extension. But condition (G') (or its counterpart condition (G) for mere concavity) applied to the incomplete preference relation $\succ_i^v$ defined by the voting record is neither necessary nor sufficient for the existence of such a rationalizing extension. As Richter and Wong point out in their Remark 4, page 344, if $\succ_i^v$ satisfies condition (G') (or (G)) then this condition is sufficient, as long as the voting record also satisfies (A). Such a situation is illustrated in Figure 1(d), where (A) is satisfied and the fact that the voting record reveals that $y_2 \succ_i^v z_2$ ensures that (G') holds for any extension of $\succ_i^v$. But in typical situations condition (G') does not hold on the basis of the information directly or indirectly[2] revealed by the voting record, as is the case in Figures 1(b) and 1(c). Nevertheless, as we have already discussed, a rationalizing strictly concave utility function does exist in the case of Figure 1(b), but not in the case of Figure 1(c).

Lastly, note that Theorem 2 establishes that if there exists a quasiconcave utility function that strictly rationalizes a voting record, then there also exists a (strictly) concave function that strictly rationalizes that voting record. In contrast, the equivalence between

[2] I.e., even if we consider the transitive closure of the directly revealed preferences, $\succ_i^v$.

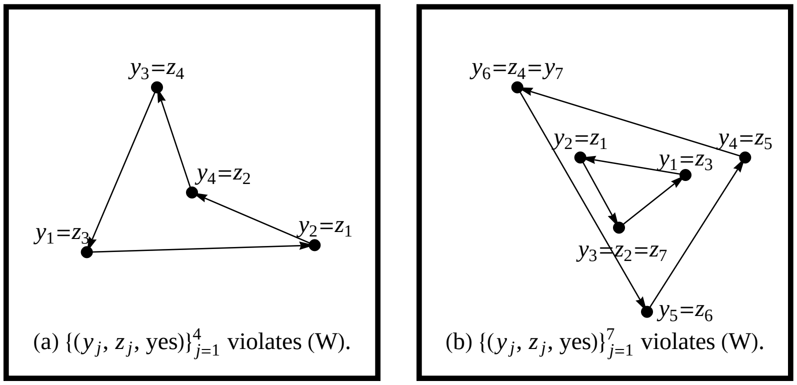


Figure 2: Rationalizability and Admissible Cycles.

concave and quasiconcave rationalizations ((S'_c) and (S'_q)) does not obtain for general complete preferences over finite sets. In particular, Richter and Wong (2004) provide an example of preferences over a set K of three alternatives that admit a quasiconcave utility representation, yet do not admit a concave representation. That example, though, requires indifference and such situations with indifference are ruled out when the available information is obtained from binary voting indicating strict preference.

We now turn to the case when individual votes may indicate weak preference. Obviously, condition (S) (or (S')) of Theorem 2 is now sufficient for concave rationalizability but it is not necessary. In fact, the gap between these two notions of rationalizability is significant when we merely require (quasi)concave rationalizing functions, since a constant function rationalizes every voting record. But when it comes to rationalizability by strictly (quasi)concave functions, Theorem 3 establishes a necessary and sufficient condition that turns out to be only mildly weaker than the corresponding condition of Theorem 2. Furthermore, when this necessary and sufficient condition obtains, we can assign strict preferences to all pairwise comparisons in the voting record except those that are entangled in an individual preference voting cycle, in accordance to the following intermediate criterion for rationalizability.

Definition 3 *A utility function* $u_i : \mathbb{R}^d \to \mathbb{R}$ virtually rationalizes *the voting record*

$\{(x_j, y_j, v^i_j)\}_{j\in M}$ *if it rationalizes that record and, in addition, it strictly rationalizes the record* $\{(x_j, y_j, v^i_j)\}_{j\in M_a}$ *that comprises all voting items in the subset*

$$M_a = \{j \in M : \nexists M' \subseteq M \text{ such that } j \in M' \text{ and } Y^i_{M'} = N^i_{M'}\}. \tag{5}$$

When a voting record is virtually rationalized, indifference between any pair of alternatives is imputed by the rationalizing function in a minimal way. As we show, we can virtually rationalize a voting record by a strictly (quasi)concave utility function whenever we can rationalize this record by such a function.

Theorem 3 *Given voting record* $\{(y_j, z_j, v^i_j)\}_{j\in M}$ *of voter* i*, the following conditions are equivalent:*

- (W) *For all* $M' \subseteq M$, $|M'| \geq 2$, *there exists* $x \in \mathcal{E}(X_{M'})$ *such that* $x \notin Y^i_{M'}$, *or there exists non-empty* $M'' \subseteq M'$ *such that* $N^i_{M''} = Y^i_{M''} \subseteq \mathcal{E}(X_{M'})$ *and* $Y^i_{M''} \cap Y^i_{M'\setminus M''} = \emptyset$.
- (W') *There exists a nested sequence of subsets* $M = M_1 \supset ... \supset M_k \supset M_{k+1} = \emptyset, k \leq m$, *such that* $N^i_{M_t\setminus M_{t+1}} \subseteq \mathcal{E}(X_{M_t})$, *either* $N^i_{M_t\setminus M_{t+1}} = \{x_t\}$ *or* $N^i_{M_t\setminus M_{t+1}} = N^i_{M'_t} = Y^i_{M'_t}$, $M'_t \subseteq M_t \setminus M_{t+1}$, *and* $N^i_{M_t\setminus M_{t+1}} \cap X_{M_{t+1}} = \emptyset$, *for all* $t = 1, ..., k$.
- (W'_c) *There exists a strictly concave utility function that virtually rationalizes* i*'s record.*
- (W'_q) *There exists a strictly quasiconcave utility function that virtually rationalizes* i*'s record.*
- (W_c) *There exists a strictly concave utility function that rationalizes* i*'s record.*
- (W_q) *There exists a strictly quasiconcave utility function that rationalizes* i*'s record.*

Furthermore, if $d = 1$, *then* (W) *is equivalent to:*

- (W_1) *For all* $M' \subseteq M$, $|M'| = 2$, *there exists* $x \in \mathcal{E}(X_{M'})$ *such that* $x \notin Y^i_{M'}$, *or* $N^i_{M'} = Y^i_{M'}$.

The arguments that prove Theorem 3 are analogous to those we outlined for the proof of Theorem 2.[3] An inspection of conditions (W) and (W') reveals that the gap between strict rationalizability and mere rationalizability is quite narrow under the requirement that the rationalizing utility function is strictly (quasi)concave. In particular, voting records that cannot be strictly rationalized but can be rationalized exhibit a particular type of violation of acyclicity, (A). In order to rationalize voting records that violate (A), we must assign all alternatives that are entangled in the revealed voting cycle to the same indifference contour. While this is possible in the case of Figure 1(a), Figures 2(a) and 2(b) make it plain that not all individual voter preference cycles can be rationalized by strictly (quasi) concave utility functions. In the case of Figure 2(a) this is because the required indifference contour cannot delineate a convex set, and in the case of Figure 2(b) because nested indifference contours that rationalize cycles must be ranked in ascending order, and this is impossible for the voting record depicted in that figure since we must have $y_7 \succeq_i z_7$. Clearly, if violations of acyclicity are ruled out, such as is the case when (N) holds, then conditions (S) and (W) are equivalent.

Corollary 1 *If the voting record $\{(x_j, y_j, v_j^i)\}_{j \in M}$ satisfies (A), then $(S) \Leftrightarrow (W)$.*

Despite the fact that the two conditions are virtually identical (barring revelations of individual voting cycles), in section 5 we shall show that the weaker premises of Theorem 3 yield much stronger payoffs when it comes to using the voting record in order to predict voter i's voting decisions.

In this section we derived necessary and sufficient conditions that must be satisfied by a voting record in order for it to be strictly rationalized by a (quasi)concave function, and we have shown that these conditions are identical whether we require strict (quasi)concavity or not. If we require the rationalizing utility function to be strictly quasiconcave, then mildly weaker conditions are necessary and sufficient to (merely) rationalize a voting record. These conclusions are summarized in Table 1.

[3]In fact, to avoid duplication of these arguments, we rely on the proof of Theorem 3 in the statement of the proof of Theorem 2.

	u_i is unrestricted	u_i is (quasi)concave	u_i is strictly (quasi)concave
u_i rationalizes i's voting record	$\emptyset$	$\emptyset$	(W)
u_i strictly rationalizes i's voting record	(A)	(S)	(S)

Table 1: Necessary and Sufficient Conditions for Rationalizability.

4 Ideal Points

If i's voting record is (strictly) rationalizable, then voter i may have an *ideal point*, i.e., there may exist an alternative $\hat{x} \in \mathbb{R}^d$ such that i prefers $\hat{x}$ over all other alternatives. In particular, the evidence from the voting record of i cannot refute the existence of such an ideal point $\hat{x}$ whenever i's voting record can be rationalized by a utility function that is uniquely maximized at $\hat{x}$:

Definition 4 *A utility function $u_i : \mathbb{R}^d \to \mathbb{R}$ rationalizes i's voting record $\{(y_j, z_j, v_j^i)\}_{j \in M}$ with ideal point $\hat{x}_i$ if (2) holds and*

(6) $$u_i(\hat{x}_i) > u_i(x), \text{ for all } x \in \mathbb{R}^d, x \neq \hat{x}_i.$$

It strictly rationalizes i's record with ideal point $\hat{x}_i$ if both (1) and (6) hold.

Armed with the above criterion, we may then inquire whether i's voting record places any testable restrictions on the location of her ideal point? Obviously, this question has a trivial answer if we do not impose any restrictions on i's preferences: if we can rationalize i's voting record, then we can do so with any ideal point $\hat{x}_i \notin N_M^i$. On the other hand, under convexity restrictions on preferences, the results of our investigation in the previous section provide a more promising approach to the problem. In fact, as we will explain shortly, the following Lemma reduces the question on the nature of testable restrictions on a voter's ideal point from her voting record to a question of rationalizability of an augmented voting record.

Lemma 1 *Consider a finite set $K \subset \mathbb{R}^d$ and strictly concave $u_i : \mathbb{R}^d \to \mathbb{R}$ that represents i's preferences over K. If $\hat{x} \in K$ is such that $\hat{x} \succeq_i x$ for all $x \in K$, $x \neq \hat{x}$, then there exists another strictly concave $\tilde{u}_i : \mathbb{R}^d \to \mathbb{R}$ that represents i's preferences over $K \setminus \{\hat{x}\}$ such that $\tilde{u}_i(\hat{x}) > \tilde{u}_i(x)$ for all $x \in \mathbb{R}^d$, $x \neq \hat{x}$.*

Thus, if we can rationalize the preferences of a voter over a finite set with a strictly concave function, and there exists an alternative $\hat{x}$ in that finite set that is (weakly) preferred to every other alternative in that set, then we cannot reject the hypothesis that this voter has a strictly concave utility function with ideal point $\hat{x}$. As a consequence, Lemma 1 suggests a straightforward test for the hypothesis that the voting record $\{(y_j, z_j, v_j^i)\}_{j \in M}$ of voter i can be strictly rationalized by a strictly concave utility function with ideal point $\hat{x}_i$. We can construct an augmented voting record that includes $\hat{m} = |X_M \setminus \{\hat{x}_i\}|$ additional voting items of the form $(\hat{x}_i, z, yes)$, for each of the alternatives $z \in X_M \setminus \{\hat{x}_i\}$. Specifically,

Definition 5 *Given voting record $\{(y_j, z_j, v_j^i)\}_{j \in M}$ for voter i and alternative $\hat{x}_i$, the augmented voting record of voter i, $\{(y_j, z_j, v_j^i)\}_{j \in M \cup \hat{M}}$, is such that $\hat{M} = \{m+1, ..., m+\hat{m}\}$, $y_j = \hat{x}_i$, $z_j \in X_M \setminus \{\hat{x}_i\}$, and $v_j^i = yes$ for all $j \in \hat{M}$, and $N_{\hat{M}}^i = X_M \setminus \{\hat{x}_i\}$.*

By Lemma 1 and Theorem 2, there exists a strictly concave utility function that strictly rationalizes i's voting record $\{(y_j, z_j, v_j^i)\}_{j \in M}$ with ideal point $\hat{x}_i$ if and only if the augmented voting record $\{(y_j, z_j, v_j^i)\}_{j \in M \cup \hat{M}}$ satisfies (S). In Theorem 4 we state this necessary and sufficient condition as $(\widehat{S'})$ and show that, in fact, it is equivalent to the apparently weaker condition $(\widehat{S})$ of that Theorem.

Theorem 4 *Given voting record* $\{(y_j, z_j, v_j^i)\}_{j\in M}$ *of voter* i*, and an alternative* $\hat{x}_i \in \mathbb{R}^d$*, the following conditions are equivalent:*

($\widehat{S}$) *For all* $M' \subseteq M, |M'| \geq 1$*, there exists* $x \in \mathcal{E}(X_{M'} \cup \{\hat{x}_i\})$ *such that* $x \notin Y_{M'}^i \cup \{\hat{x}_i\}$.

($\widehat{S}'$) *The augmented voting record* $\{(y_j, z_j, v_j^i)\}_{j\in M\cup\hat{M}}$ *satisfies* (S) *or* (S').

($\widehat{S}_c$) *There exists a strictly concave utility function that strictly rationalizes i's record with ideal point* $\hat{x}_i$.

($\widehat{S}_q$) *There exists a strictly quasiconcave utility function that strictly rationalizes i's record with ideal point* $\hat{x}_i$.

($\widehat{S}'_c$) *There exists a concave utility function that strictly rationalizes i's record with ideal point* $\hat{x}_i$.

($\widehat{S}'_q$) *There exists a quasiconcave utility function that strictly rationalizes i's record with ideal point* $\hat{x}_i$.

If $d = 1$*, then* ($\widehat{S}$) *is equivalent to:*

($\widehat{S}_1$) *For all* $M' \subseteq M$*,* $1 \leq |M'| \leq 2$*, there exists* $x \in \mathcal{E}(X_{M'} \cup \{\hat{x}_i\})$ *such that* $x \notin Y_{M'}^i \cup \{\hat{x}_i\}$.

Condition ($\widehat{S}$), provides a precise set of testable restrictions on the location of voter i's ideal point arising from her voting record, assuming that i has a (strictly quasi)concave utility function. Of course, condition ($\widehat{S}$) implies condition (S). Furthermore, as is true for Theorem 2, the one-dimensional case admits a further simplification of condition ($\widehat{S}$). We provide a graphical illustration of the implications of Theorem 4 in Figure 3, where we depict five voting alternatives associated with four voting items ($m = 4$) in a two-dimensional space. Application of condition ($\widehat{S}$) restricts voter i's ideal point, $\hat{x}_i$, to lie outside the areas marked gray in Figure 3(b).

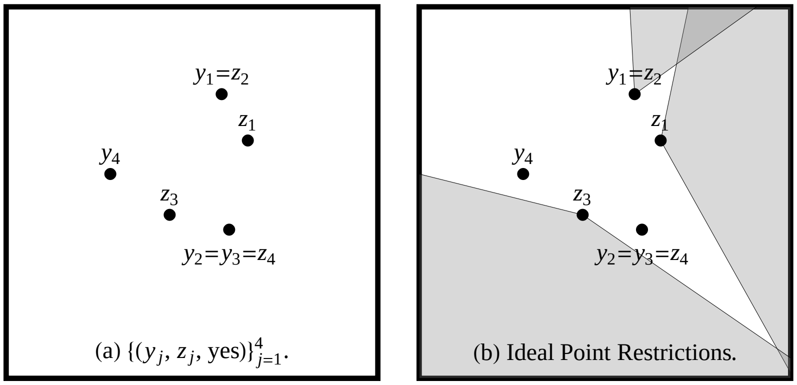


Figure 3: Voter i cannot have a (strictly) (quasi)concave utility function with ideal point that lies in the gray areas.

Because Lemma 1 allows the candidate ideal point to be weakly preferred over the remaining alternatives in finite set K, virtually identical arguments lead to the following Theorem when we consider mere rationalizability of the voting record.

Theorem 5 *Given voting record $\{(y_j, z_j, v_j^i)\}_{j\in M}$ of voter i, and an alternative $\hat{x}_i \in \mathbb{R}^d$, the following conditions are equivalent:*

($\widehat{W}$) *For all non-empty $M' \subseteq M$ there exists $x \in \mathcal{E}(X_{M'} \cup \{\hat{x}_i\})$ such that $x \notin Y^i_{M'} \cup \{\hat{x}_i\}$, or $N^i_{M''} = Y^i_{M''} \subseteq \mathcal{E}(X_{M'} \cup \{\hat{x}_i\})$ and $Y^i_{M''} \cap Y^i_{M'\setminus M''} = \emptyset$, for some non-empty $M'' \subseteq M'$.*

($\widehat{W'}$) *The augmented voting record $\{(y_j, z_j, v_j^i)\}_{j\in M\cup\hat{M}}$ satisfies (W) or (W').*

($\widehat{W}_c$) *There exists a strictly concave utility function that (virtually) rationalizes i's record with ideal point $\hat{x}_i$.*

($\widehat{W}_q$) *There exists a strictly quasiconcave utility function that (virtually) rationalizes i's record with ideal point $\hat{x}_i$.*

If $d = 1$, then $(\widehat{W})$ is equivalent to:

$(\widehat{W}_1)$ *For all $M' \subseteq M$, $1 \leq |M'| \leq 2$, there exists $x \in \mathcal{E}(X^i_{M'} \cup \{\hat{x}_i\})$ such that $x \notin Y^i_{M'} \cup \{\hat{x}_i\}$, or $N^i_{M'} = Y^i_{M'} \subseteq \mathcal{E}(Y^i_{M'} \cup \{\hat{x}_i\})$.*

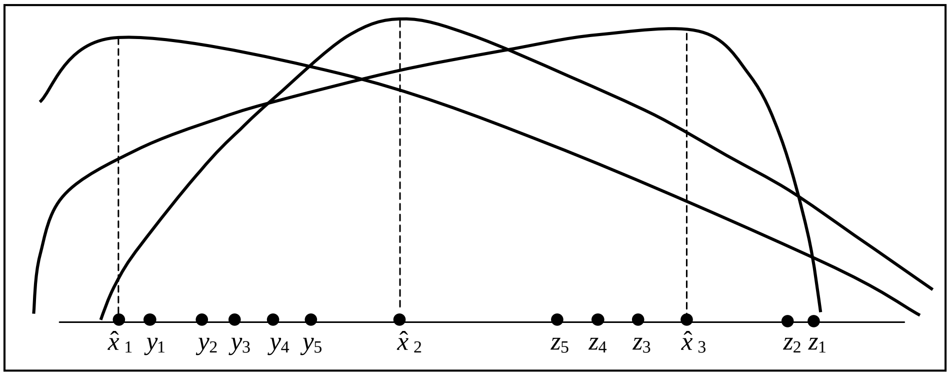
Jointly, Theorems 4 and 5 establish that finite voting records impose nontrivial testable restrictions on voters' ideal points. Compared to existing parametric methods for the estimation of voters' ideal points, though, the non-parametric tests suggested by Theorems 4 and 5 impose a significant burden on the analyst, as they require the availability of the voters' entire voting record. On the contrary, most existing techniques rely only on partial information on the voting record that typically reduces to mere knowledge of voters' vector of voting decisions.[4] Thus, it is important to ask whether the testable restrictions on ideal points we have derived so far have any bearing if we relax the assumption that the location of the voting alternatives z_j, y_j is known. We devote the rest of this section to this question.

First, we show that the conditions of Theorem 4 and 5 are vacuously met for all voters and for every number of issue dimensions $d \geq 1$, if the location of the voting alternatives is unrestricted. Specifically, we show:

Theorem 6 *Consider any voting decisions $(v^1, ..., v^n) \in \{yes, no\}^{nm}$. For every $d \geq 1$ and every n-tuple of points $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}^d$, there exist voting alternatives z_j, $y_j \in \mathbb{R}^d$, $j = 1, \ldots, m$, and n strictly concave utility functions u_i, each with ideal point $\hat{x}_i$, such that for every voter i u_i strictly rationalizes the voting record $\{(y_j, z_j, v^i_j)\}_{j \in M}$.*

Note that Theorem 6 states that all possible voting decisions and all possible ideal points for the n voters can *all* be rationalized by appropriately choosing the location of the voting alternatives. That is, one choice of the location of the voting alternatives works

[4] It does not follow that additional information cannot be acquired. Financial legislation disbursing funds in different policy areas readily supplies such information. If we embed the voting in the committee within a larger process in which proposals emerge endogenously, then such information may arise structurally from the assumption that the sponsors of the proposals optimize.



An example of the construction in the proof of Theorem 6 for $n = 3$ *voters,* $m = 5$ *voting items, and ideal points that satisfy* $\hat{x}_1 < \hat{x}_2 < \hat{x}_3$. *The voting decisions are given by* $v_j^1 = yes$ *for all* $j = 1, \dots, 5$, *for voter* 1, $v_1^2 = v_3^2 = v_4^2 = no$ *and* $v_2^2 = v_5^2 = yes$, *for voter 2, and* $v_3^3 = v_4^3 = v_5^3 = no$ *and* $v_1^3 = v_2^3 = yes$, *for voter* 3.

Figure 4: Illustration of Theorem 6.

for all voters at the same time. Obviously, Theorem 6 is valid a fortiori if we impose weaker requirements on voters' utility functions, for instance, if we relax strict concavity to quasiconcavity. In the one-dimensional case, Theorem 6 is shown by construction. An illustration is provided in Figure 4. In essence, the result stems from the fact that there exists a way to arrange the voting alternatives z_j, y_j, such that all voting records necessarily satisfy condition (S_1) of Theorem 2, for any voting decisions. This arrangement amounts to locating one of the two voting alternatives in each voting item in some arbitrary order, then locating the remaining voting alternatives in a non-overlapping interval, in the reverse order of voting items. It is then a simple additional step to translate the above arrangement in the space of alternatives where the given ideal points have already been located in order to ensure that the added restrictions of condition $(\widehat{S}_1)$ of Theorem 4 are not violated. In the generic case when voters' ideal points are distinct, this construction can be achieved while at the same time ensuring that at least one of the voting alternatives z_j, y_j, lies in the Pareto set for each voting item $j \in M$.

Theorem 6 forecloses any possibility for the nonparametric estimation of agnostic (Londregan (2000)) models of legislator ideal points, i.e., models that assume no information on the location of the voting alternatives. Barring knowledge of the voting alternatives, non-parametric estimation of voter preferences requires at least some restrictions on their location for identification purposes. One such extra identification restriction in the context of a parametric probabilistic voting model is used by Clinton and Meirowitz (2001) who require that the victorious voting alternative from voting item j become the status quo voting alternative in voting item $j+1$. If we impose this extra condition, then it is easy to see using condition (S_1) of Theorem 2 that the conclusion of Theorem 6 no longer obtains.[5] There exist voting records that cannot be strictly rationalized for all voters, opening the possibility for the non-parametric estimation of the one-dimensional probabilistic voting model. Yet, as the following Theorem shows, the identifying role of this additional restriction, while possibly strong in one dimension, has no bite in higher dimensions.

Theorem 7 (Folding) *Consider a space of voting alternatives of any dimension $d \geq 1$, any voting alternatives z_j, $y_j \in \mathbb{R}^d$, $j \in M$, and any voting decisions $v^i \in \{yes, no\}^m$ for the n voters. For every $d' \geq 2$, and for every n-tuple of points $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}^{d'}$, there exists a one-to-one function $f : X_M \to \mathbb{R}^{d'}$ such that for every voter i:*

(i) there exists a strictly concave utility function $u_i : \mathbb{R}^{d'} \to \mathbb{R}$ with ideal point $\hat{x}_i$ that virtually rationalizes the voting record $\{(f(y_j), f(z_j), v^i_j)\}_{j \in M}$.

(ii) if the voting record $\{(y_j, z_j, v^i_j)\}_{j \in M}$ satisfies (A), then there exists a strictly concave utility function $u_i : \mathbb{R}^{d'} \to \mathbb{R}$ with ideal point $\hat{x}_i$, that strictly rationalizes the voting record $\{(f(y_j), f(z_j), v^i_j)\}_{j \in M}$.

Theorem 7 provides a new twist on the common finding of many parametric ideal point estimation techniques that two dimensional representations are sufficient to capture

[5] Recently, Schwartz (2007), Theorem 4, working in a discrete space of alternatives, gave such a sufficient condition on the voting record that guarantees violation of single-peakedness of the preferences of at least one voter over the voting alternatives, X_M.

voting patterns as is the case in, e.g., Poole and Rosenthal's approach to the analysis of US Congressional roll call votes.[6] According to Theorem 7, two dimensions are sufficient to represent any voting record and any ideal points for all voters, while at the same time endowing each voter with a strictly concave rationalizing utility function. Note that the voting records may satisfy the condition that the victorious alternative becomes the status quo in successive voting items, or any possible recurrence of voting alternatives across voting items. In fact, the Theorem places no other restrictions on the location of the original voting alternatives, so that if that location is not known, we may place the voting alternatives arbitrarily in the space before Theorem 7 can be applied. As long as the original voting record does not reveal any individual preference cycles, we can achieve this representation while at the same time ensuring that every voting record is *strictly* rationalized by part (ii) of the Theorem. Independently, Bogomolnaia and Laslier have recently shown a related result (Bogomolnaia and Laslier (2007), Theorem 16) that any individual's preferences over a finite set of alternatives, $X \subset \mathbb{R}^2$, can be rationalized by (possibly discontinuous) convex preferences in $\mathbb{R}^2$ if and only if $X = \mathcal{E}(X)$. Theorem 7 on the other hand ensures that the rationalizing convex preferences are representable by continuous strictly concave functions and, in addition, that the revealed preference relation is jointly rationalized with arbitrarily prespecified ideal points.

5 Vote Prediction

In this section we turn to the question of predicting the future voting behavior of an individual voter on the basis of past observations of that individual's voting choices. Theorems 2 and 3 suggest a straightforward strategy for the task. Suppose that voter i has preferences represented by an unobserved (strictly) (quasi)concave utility function $u_i : \mathbb{R}^d \to \mathbb{R}$, that the record $\{(y_j, z_j, v_j^i)\}_{j \in M}$ of past votes indicating strict preference is available, and that voter i is faced with a decision between an alternative $x \in \mathbb{R}^d$, and some

[6] But see Heckman and Snyder (1997) for different conclusions on the dimensionality of the policy space in US Congressional voting.

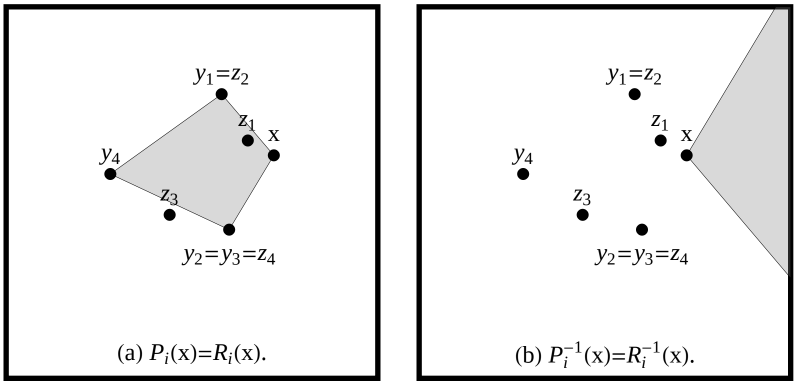


Figure 5: Voter i with voting record $\{(y_j, z_j, yes)\}_{j=1}^4$ must (strictly) prefer any alternative in the gray area of Figure 5(a) over x, while x must be (strictly) preferred over every alternative in the gray area of 5(b).

alternative $x' \in \mathbb{R}^d$. Then, by Theorem 2 we deduce that voter i must weakly prefer x' over x ($u_i(x') \geq u_i(x)$) if x' belongs in the set:[7]

$$R_i(x) = \{x' \in \mathbb{R}^d \setminus \{x\} : \text{The record } \{(y_j, z_j, v_j^i)\}_{j=0}^m, y_0 = x, z_0 = x', v_0^i = yes, \text{ violates } (S)\}.$$

In particular, if $u_i(x) > u_i(x')$, instead, then the voting record $\{(y_j, z_j, v_j^i)\}_{j=0}^m$ is strictly rationalized by i's utility function u_i, which is impossible since that voting record violates (S). An identical argument ensures that we must have $u_i(x) \geq u_i(x')$ if x' belongs in the set:

$$R_i^{-1}(x) = \{x' \in \mathbb{R}^d \setminus \{x\} : \text{The record } \{(y_j, z_j, v_j^i)\}_{j=0}^m, y_0 = x, z_0 = x', v_0^i = no, \text{ violates } (S)\}.$$

In fact, stronger conclusions obtain if we relax the assumption that i's voting decisions indicate strict preference, while strengthening the assumption on i's unobserved utility function. In particular, we now assume that i has a strictly (quasi)concave utility function $u_i : \mathbb{R}^d \to \mathbb{R}$ and that the record $\{(y_j, z_j, v_j^i)\}_{j \in M}$ of past votes reveals weak preference with each voting decision. Then, if voter i is faced with a decision between an alternative $x \in \mathbb{R}^d$,

[7] Of course, we can be more explicit defining $R_i(x) = \bigcup_{M' \subseteq M} \{x' \in \mathbb{R}^d \setminus \{x\} : \mathcal{E}(X_{M'} \cup \{x, x'\}) \subseteq Y_{M'}^i \cup \{x\}\}$.

and some alternative $x' \in \mathbb{R}^d$, it must be that $u_i(x') > u_i(x)$ if x' belongs in the set:

$$P_i(x) = \{x' \in \mathbb{R}^d \setminus \{x\} : \text{The record } \{(y_j, z_j, v_j^i)\}_{j=0}^m, y_0 = x, z_0 = x', v_0^i = yes, \text{ violates } (W)\}.$$

The stronger conclusion obtains because now it suffices to have $u_i(x) \geq u_i(x')$, in order for u_i to rationalize the voting record $\{(y_j, z_j, v_j^i)\}_{j=0}^m$ in contradiction of Theorem 3. Analogously, we obtain that $u_i(x) > u_i(x')$ if x' belongs in:

$$P_i^{-1}(x) = \{x' \in \mathbb{R}^d \setminus \{x\} : \text{The record } \{(y_j, z_j, v_j^i)\}_{j=0}^m, y_0 = x, z_0 = x', v_0^i = no, \text{ violates } (W)\}.$$

Hence, our penultimate Theorem is:

Theorem 8 *Assume voter i has preferences represented by a utility function $u_i : \mathbb{R}^d \to \mathbb{R}$. Consider a voting record $\{(y_j, z_j, v_j^i)\}_{j \in M}$, and any $x \in \mathbb{R}^d$.*

(i) If u_i is (strictly) (quasi)concave and strictly rationalizes i's voting record, then $u_i(x') \geq u_i(x)$ for all $x' \in R_i(x)$, and $u_i(x') \leq u_i(x)$ for all $x' \in R_i^{-1}(x)$.

(ii) If u_i is strictly (quasi)concave and rationalizes i's voting record, then $u_i(x') > u_i(x)$ for all $x' \in P_i(x)$, and $u_i(x') < u_i(x)$ for all $x' \in P_i^{-1}(x)$.

(iii) If $\{(y_j, z_j, v_j^i)\}_{j \in M}$ satisfies (A), then $R_i(x) \setminus Y_M^i \subseteq P_i(x)$ and $R_i^{-1}(x) \setminus N_M^i \subseteq P_i^{-1}(x)$.

Figure 5 depicts the same voting record as the one depicted in Figure 3, and displays the set of alternatives $P_i(x)$ that must be strictly preferred over alternative x by voter i, given that voter's observed voting behavior. Similarly, i must strictly prefer x over all alternatives in the set $P_i^{-1}(x)$ of Figure 5. In view of part (iii) of Theorem 8, we conclude that as long as we are willing to assume strict (quasi-)concavity of voters' utility representations, then the added parsimony in the interpretation of the voting record in the analysis leading to Theorem 3 has a significant payoff when it comes to predicting future decisions of individual voters. In particular, assuming the observed voting record does not violate (A), then the domain of possible pairs of alternatives for which we can predict voter i's voting decision using Theorem 8, is only slightly meager if we assume past choices reveal weak preference as in part (ii) versus strict preference as in part (i).

6 Multiple Choice Data

We have focused the analysis on preference revelation from binary voting choices but our results readily generalize to arbitrary choice situations over a finite set of alternatives. To be concrete, suppose we observe individual i make a choice $x_j \in B_j$, in each of m choice situations $j = 1, ..., m$, where $B_j \subset \mathbb{R}^d$ is a finite budget set with $|B_j| \geq 2$. Cast in that language, a voting record $\{(y_j, z_j, v_j^i)\}_{j=1}^m$ is represented by a collection $\{(B_j, x_j)\}_{j=1}^m$ where $B_j = \{y_j, z_j\}$ and $x_j = y_j$ if $v_j^i = yes$ or $x_j = z_j$ if $v_j^i = no$. Conversely, if we are given data $\{(B_j, x_j)\}_{j=1}^m$ with budget sets of arbitrary finite cardinality, we can equivalently represent the information in these data in the form of a voting record with $\sum_{j=1}^m (|B_j| - 1)$ voting items: for each choice instance j, we simply create $|B_j| - 1$ voting items (x_j, z, yes), one for each $z \in B_j \backslash \{x_j\}$. Given the equivalence of these representations, the rationalizability conditions established in Theorems 2 and 3 are also necessary and sufficient when we consider multiple choice data $\{(B_j, x_j)\}_{j=1}^m$ and finite budget sets.

In general, we may consider a finite set $K \subset \mathbb{R}^d$ where $K = \cup_{j=1}^m B_j$. The observed multiple choice data $\{(B_j, x_j)\}_{j=1}^m$ now define an irreflexive preference relation $R \subset K \times K$, such that $(y, z) \in R$ if and only if there exists $j = 1, ..., m$ such that $x_j = y$ and $z \in B_j \backslash \{x_j\}$. For any $R' \subseteq R$, we may define the analogues of $N_{M'}^i$, $Y_{M'}^i$, and $X_{M'}$, respectively as

$$N(R') = \{x : (y, x) \in R'\},$$

$$Y(R') = \{x : (x, z) \in R'\}, \text{ and}$$

$$X(R') = N(R') \cup Y(R').$$

Furthermore, say that the revealed preference relation R is strictly rationalized if there exists a utility function $u : \mathbb{R}^d \to \mathbb{R}$ such that $u(y) > u(z)$ for all $(y, z) \in R$, merely rationalized if $u(y) \geq u(z)$, instead, and virtually rationalized if it is rationalized and the subrelation

$$R_a = \{x \in R : \nexists R' \subseteq R \text{ such that } x \in R' \text{ and } N(R') = Y(R')\}, \tag{7}$$

is strictly rationalized. Then, we have the following restatement of Theorems 2 and 3:

Theorem 9 *Consider any irreflexive preference relation $R \subset K \times K$, where $K \subset \mathbb{R}^d$ is a finite set of alternatives.*

(i) There exists a (strictly) (quasi)cocnave utility function $u : \mathbb{R}^d \to \mathbb{R}$ that strictly rationalizes R if and only if there exists a nested sequence $R = R_1 \supset R_2 \supset ... \supset R_k \supset R_{k+1} = \emptyset$ such that $N(R_t \setminus R_{t+1}) = \{x\} \subseteq \mathcal{E}(X(R_t))$, and $N(R_t \setminus R_{t+1}) \cap X(R_{t+1}) = \emptyset$ for all $t = 1, ..., k$.

(ii) There exists a strictly (quasi)cocnave utility function $u : \mathbb{R}^d \to \mathbb{R}$ that (virtually) rationalizes R if and only if there exists a nested sequence $R = R_1 \supset R_2 \supset ... \supset R_k \supset R_{k+1} = \emptyset$ such that $N(R_t \setminus R_{t+1}) \subseteq \mathcal{E}(X(R_t))$, either $N(R_t \setminus R_{t+1}) = \{x\}$ or $N(R_t \setminus R_{t+1}) = N(R'_t) = Y(R'_t)$, $R'_t \subseteq R_t \setminus R_{t+1}$, and $N(R_t \setminus R_{t+1}) \cap X(R_{t+1}) = \emptyset$, for all $t = 1, ..., k$.

In Theorem 9 we have chosen to restate the necessary and sufficient conditions (S') and (W'), respectively, of Theorems 2 and 3, but obviously the equivalent conditions (S) and (W) could be stated, instead. In view of Theorem 9, the applications of these conditions on the location of ideal points and choice prediction that we developed in sections 4 and 5 can be replicated with general multiple choice data $\{(B_j, x_j)\}_{j=1}^m$.

The fact that part (ii) of Theorem 9 provides a necessary and sufficient condition for the relation R to be virtually rationalized makes the Theorem applicable even outside a revealed preference context. In particular, suppose we are given a finite, irreflexive, and symmetric 'indifference' relation $I_i \subseteq \mathbb{R}^d \times \mathbb{R}^d$ and a finite, irreflexive, and asymmetric 'strict' preference relation $R_i \subseteq \mathbb{R}^d \times \mathbb{R}^d$. We may then construct an irreflexive relation $R = I_i \cup R_i$. As long as $R_i = R_a$, as defined in (7) for this relation R, then according to part (ii) of Theorem 9 there exists a strictly (quasi)concave utility function that represents the indifference and strict preference relations I_i and R_i if and only if R satisfies the stated condition of the Theorem.

7 Conclusions

We have derived necessary and sufficient conditions in order for observed binary voting choices to be consistent with the hypothesis that the voters making these choices have preferences that admit concave utility representations. These conditions imply simple testable restrictions on the location of voters' ideal points from their voting record, and can be used to predict individual voting behavior. If the location of voting alternatives is unrestricted (as is assumed in prevalent political methodology techniques for the estimation of legislators' ideal points) then the derived conditions are vacuously satisfied for arbitrary ideal points for the voters, even if we restrict the space of alternatives in one dimension. The analysis is readily applicable to the nonparametric study of general deterministic choice situations over finite budget sets with only convexity restrictions on individual preferences.

APPENDIX

In this appendix we prove Theorems 2 to 8 and Lemma 1. We start with two Lemmas.

Lemma 2 *Consider disjoint finite sets $K, K' \subset \mathbb{R}^d$ (K possibly empty) such that $K' \subseteq \mathcal{E}(K \cup K')$, and a strictly concave function $u_i : \mathbb{R}^d \to \mathbb{R}$ that represents i's preferences over K. If $x \sim_i x'$ for all $x, x' \in K'$, and $x \succ_i x'$ for all $x \in K$, $x' \in K'$, then there exists another strictly concave $u'_i : \mathbb{R}^d \to \mathbb{R}$ that represents i's preferences over $K \cup K'$.*

Proof. Since u_i is strictly concave, condition (G') of Richter and Wong (2004) holds for K. Consider any $X \subseteq K \cup K'$, such that $|X| \leq d+1$ and $X \cap K' \neq \emptyset$. For every $x \in \mathcal{C}(X) \setminus X$, we have $x \notin K'$ since $K' \subseteq \mathcal{E}(K \cup K')$. Furthermore, there exists $x' \in X \cap K'$ since $X \cap K' \neq \emptyset$, and we have $x \succ_i x'$. Thus, (G') holds for $K \cup K'$ ensuring the existence of the required function u'_i. ■

The second Lemma is:

Lemma 3 *Consider a finite set $X \subset \mathbb{R}^d$ and a quasiconcave function $u_i : \mathbb{R}^d \to \mathbb{R}$.*

(i) There exists $x \in \mathcal{E}(X)$ that minimizes u_i over $\mathcal{C}(X)$.

(ii) If u_i is strictly quasiconcave, then $\arg\min\{u_i(x) : x \in \mathcal{C}(X)\} \subseteq \mathcal{E}(X)$.

Proof. Let the set of extreme points of X be given by $\mathcal{E}(X) = \{x_1, \ldots, x_e\} \subseteq X$, which is nonempty by Lemma 7.76, page 301, in Aliprantis and Border, (2006). Without loss of generality assume that $x_1 \in \arg\min\{u_i(x) : x \in \mathcal{E}(X)\}$. Every $y \in \mathcal{C}(X)$ with $y \notin \mathcal{E}(X)$ can be written as a non-trivial convex combination of the elements of $\mathcal{E}(X)$, i.e., $y = \sum_{h\in I} \lambda_h x_h$, where $I \subseteq \{1, ..., e\}$, $\lambda_h \in (0, 1)$ for all $h \in I$, and $\sum_{h\in I} \lambda_h = 1$. If u_i is quasiconcave we have that

$$u_i(y) = u_i(\lambda_j x_j + (1 - \lambda_j)y') \geq \min\{u_i(x_j), u_i(y')\},$$

where $j \in I$, $y' = \sum_{h\in I\setminus\{j\}} \lambda'_h x_h$, and $\lambda'_h = \frac{\lambda_h}{1-\lambda_j}, h \in I \setminus \{j\}$. In turn, if $|I| > 2$, we deduce that

$$u_i(y') = u_i\left(\lambda'_l x_l + (1 - \lambda'_l)y''\right) \geq \min\{u_i(x_l), u_i(y'')\},$$

where now $l \in I \setminus \{j\}$, $y'' = \sum_{h\in I\setminus\{j,l\}} \lambda''_h x_h$, and $\lambda''_h = \frac{\lambda'_h}{1-\lambda'_l}, h \in I \setminus \{j, l\}$. Repeatedly invoking the definition of quasiconcavity as above, we obtain that

$$u_i(y) \geq \min\{u_i(x_h) : h \in I\} \geq u_i(x_1).$$

Since this is true for arbitrary $y \in \mathcal{C}(X) \setminus \mathcal{E}(X)$, we conclude that $x_1 \in \arg\min\{u_i(x) : x \in \mathcal{C}(X)\}$. To show part (ii), note that if u_i is strictly quasiconcave, then the above arguments, using definition (4) instead of weak inequality, ensure that $u_i(y) > u_i(x_1)$ for all $y \in \mathcal{C}(X) \setminus \mathcal{E}(X)$. Thus, we conclude that $\arg\min\{u_i(x) : x \in \mathcal{C}(X)\} \subseteq \mathcal{E}(X)$, as desired. ∎

Proof of Theorem 2

We have (S_c)⇒(S'_c)⇒(S'_q) and (S_c)⇒(S_q)⇒(S'_q). Thus, in order to show (S)⇔(S')⇔(S_c) ⇔(S_q)⇔ (S'_c)⇔(S'_q) it suffices to show (S'_q)⇒(S), (S)⇒(S'), and (S')⇒(S_c).

(S'_q)⇒(S): Let quasiconcave $u_i : \mathbb{R}^d \to \mathbb{R}$ strictly rationalize i's record. By Lemma 3, part (i), there exists an alternative $x \in \mathcal{E}(X_{M'})$ such that $u_i(x) \leq u_i(y)$ for all $y \in \mathcal{C}(X_{M'})$.

If $x \in Y^i_{M'}$ then, since u_i strictly rationalizes i's record, there exists $x' \in X_{M'}$ such that $u_i(x) > u_i(x')$, a contradiction. Thus, $x \notin Y^i_{M'}$, as we wished to show.

(S)⇒(S'): We have (S)⇒(W), so (S)⇒(W') by Theorem 3. We also have (S)⇒(A), so that (S)⇒[(W') and (A)]⇒(S').

(S')⇒(S_c): We will first show that (S')⇒(A). Suppose not to get a contradiction, i.e., suppose (S') holds and there exists nonempty $M' \subseteq M$ such that $N^i_{M'} = Y^i_{M'}$. By (S') we have that $N^i_M = \{x_1, \ldots, x_t, \ldots, x_k\}$ where x_t is such that $\{x_t\} = N^i_{M_t \setminus M_{t+1}}$. Let $t' = \min\{t : x_t \in N^i_{M'}\}$, so that we have $M' \subseteq M_{t'}$. By (S') we have $x_{t'} \notin X_{M_{t'+1}}$. Furthermore, $x_{t'} \notin Y^i_{\{j\}}$ for all $j \in N^i_{M_{t'} \setminus M_{t'+1}}$, since $y_h \neq z_h$ for all $h \in M$. We conclude that $x_{t'} \notin Y_{M_{t'}} \Rightarrow x_{t'} \notin Y^i_{M'} = N^i_{M'}$, a contradiction. Now we have (S')⇒[(A) and (W')]⇒[(A) and (W'_c)] by Theorem 3. But, [(A) and (W'_c)]⇒(S_c) since we have $M = M_a$ defined in (5), when (A) is true.

We have established the equivalence $(S) \Leftrightarrow (S') \Leftrightarrow (S_c) \Leftrightarrow (S_q) \Leftrightarrow (S'_c) \Leftrightarrow (S'_q)$. Since $(S) \Rightarrow (S_1)$, it remains to show:

[$d = 1$ and (S_1)] $\Rightarrow$ (S): Assume $d = 1$ and (S_1) holds, and suppose (S) fails, in order to get a contradiction. Then there exists $M' \subseteq M$ with $|M'| > 2$ for which $\mathcal{E}(X_{M'}) \subseteq Y^i_{M'}$. Since $d = 1$ and $|X_{M'}| \geq 2$, we have $\mathcal{E}(X_{M'}) = \{x, x'\}$ for some distinct $x, x' \in Y^i_{M'}$. Let $x \in Y^i_{\{j\}}$, and $x' \in Y^i_{\{h\}}$. Then, $\mathcal{E}(X_{\{j,h\}}) = Y^i_{\{j,h\}}$ which contradicts our assumption that condition (S_1) holds. ■

Proof of Theorem 3

Since (W'_c)⇒(W_c)⇒(W_q), and (W'_c)⇒(W'_q)⇒(W_q), in order to show (W)⇔(W')⇔(W'_c) ⇔(W'_q)⇔(W_c)⇔(W_q), we will show (W_q)⇒(W), (W)⇒(W'), and (W')⇒(W'_c).

(W_q)⇒(W): Let u_i be a strictly quasiconcave function that rationalizes i's voting record. Consider any $M' \subseteq M$. If there does not exist $x \in \mathcal{E}(X_{M'})$ such that $x \notin Y^i_{M'}$, then $\mathcal{E}(X_{M'}) \subseteq Y^i_{M'}$. Furthermore, by part (ii) of Lemma 3 we have $\arg\min\{u_i(x) : x \in \mathcal{C}(X_{M'})\} \subseteq \mathcal{E}(X_{M'}) \subseteq Y^i_{M'}$. We shall now show that there exists non-empty $M'' \subseteq M'$ such that $Y^i_{M''} = N^i_{M''} \subseteq \mathcal{E}(X_{M'})$ and $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$. Set $K = \arg\min\{u_i(x) : x \in \mathcal{C}(X_{M'})\}$ and define

$$M_m = \{j \in M' : Y^i_{\{j\}} \subseteq K\}.$$

Since $K \subseteq Y^i_{M'}$, we must have $Y^i_{M_m} = K$ and M_m is non-empty. We also claim that we must have $N^i_{M_m} \subseteq Y^i_{M_m}$. If not, then there exists $x \in N^i_{M_m}$ such that $x \in X_{M'} \setminus K$. But then $u_i(x) > u_i(y)$, for all $y \in Y^i_{M_m} = K$, contradicting the assumption that u_i rationalizes i's voting record. We thus indeed have $N^i_{M_m} \subseteq Y^i_{M_m} = K$. We now inductively define a nested sequence of nonempty subsets of M_m by setting $M^0_m = M_m$, and

$$M^{t+1}_m = \{j \in M^t_m : Y^i_{\{j\}} \subseteq N^i_{M^t_m}\}.$$

Note that $Y^i_{M^t_m} \supseteq N^i_{M^t_m} = Y^i_{M^{t+1}_m} \supseteq N^i_{M^{t+1}_m}$ for all t and, since M_m is finite, there exists integer k such that $M^{k-1}_m = M^k_m = M'' \neq \emptyset$.[8] We thus have $N^i_{M''} = Y^i_{M''} \subseteq \mathcal{E}(X^*)$. It remains to show that $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$. If not, then there exists $j \in M' \setminus M''$ such that $Y^i_{\{j\}} \subseteq Y^i_{M''} = N^i_{M''} \subseteq K$. It cannot be that $j \in M_m$ as in that case $j \in M''$ since $Y^i_{\{j\}} \subseteq N^i_{M''} \subseteq N^i_{M^t_m}$ for all t. Thus, it must be that $j \in M' \setminus M_m$, but this is impossible since $Y^i_{\{j\}} \cap K = \emptyset$ by the definition of M_m, a contradiction proving that $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$.

(W)$\Rightarrow$(W′): We have $M_1 = M$, and we will successively define M_2, M_3, etc., up to M_{k+1}. In order to determine M_{t+1} at the $(t+1)$-th step when M_t has been defined, note that by (W) either there exists $x_t \in \mathcal{E}(X_{M_t})$ such that $x_t \notin Y^i_{M_t}$, or (if such x_t does not exist) there exists $M'' \subseteq M_t$ such that $N^i_{M''} = Y^i_{M''} \subseteq \mathcal{E}(X_{M_t})$ and $Y^i_{M''} \cap Y^i_{M_t \setminus M''} = \emptyset$. Let $M_{t+1} = \{j \in M_t : N^i_{\{j\}} \neq \{x_t\}\}$ in the former case, or $M_{t+1} = \{j \in M_t : N^i_{\{j\}} \cap N^i_{M''} = \emptyset\}$, in the latter case. Obviously, $N^i_{M_t \setminus M_{t+1}} \cap N^i_{M_{t+1}} = \emptyset$ in either case. In addition, $N^i_{M_t \setminus M_{t+1}} \cap Y^i_{M_{t+1}} = \emptyset$, since either $N^i_{M_t \setminus M_{t+1}} = \{x_t\}$ and $x_t \notin Y^i_{M_t} \supseteq Y^i_{M_{t+1}}$, or because $N^i_{M_t \setminus M_{t+1}} = N^i_{M''} = Y^i_{M''}$, $Y^i_{M''} \cap Y^i_{M_t \setminus M''} = \emptyset$ by (W), and $Y^i_{M_{t+1}} \subseteq Y^i_{M_t \setminus M''}$. As a result, $N^i_{M_t \setminus M_{t+1}} \cap X_{M_{t+1}} = \emptyset$, in both cases. Furthermore, the set M'_t required by (W′) is given by $M'_t = M'' \subseteq M_t \setminus M_{t+1}$ and

[8] Indeed, M'' is the greatest fixed point of the function $T : 2^{M_m} \to 2^{M_m}$ defined as $T(A) = \{j \in A : Y^i_{\{j\}} \subseteq N^i_A\}$. Such a fixed point exists by the Knaster-Tarski fixed point Theorem, since 2^{M_m} is a complete lattice ordered by set inclusion, and T is monotone.

$N^i_{M_t \setminus M_{t+1}} = Y^i_{M'_t}$. Proceeding as above, we obtain a sequence $M_1, ..., M_{k+1}$ at the $(k+1)$-th step when $M_{k+1} = \emptyset$. Thus we must have $k \leq |N^i_M|$ and the sequence satisfies (W').

$(W') \Rightarrow (W'_c)$: We use induction, first establishing the existence of the required function for the record $\{(y_j, z_j, v^i_j)\}_{j \in M_k}$. Consider any strictly concave function $u_i : \mathbb{R}^d \to \mathbb{R}$. From (W'), we have $N^i_{M_k} \subseteq \mathcal{E}(X_{M_k})$, so now Lemma 2 (applied on $K = X_{M_k} \setminus N^i_{M_k}$, $K' = N^i_{M_k}$) ensures the existence of a strictly concave u^k_i that strictly rationalizes the voting record $\{(y_j, z_j, v^i_j)\}_{j \in M_k}$ if $|N^i_{M_k}| = 1$, and virtually rationalizes this record if $|N^i_{M_k}| > 1$, since we have $u^k(x) > u^k(x')$ for all $x \in X_{M_k} \setminus N^i_{M_k} = Y^i_{M_k \setminus M'_k}$, $x' \in N^i_{M_k} = N^i_{M'_k}$. Now, suppose there exists such a strictly concave function $u^t_i : \mathbb{R}^d \to \mathbb{R}$ that virtually rationalizes the record $\{(y_j, z_j, v^i_j)\}_{j \in M_t}$, $t > 1, t \leq k$. We wish to show that there also exists such a function u^{t-1} for the record $\{(y_j, z_j, v^i_j)\}_{j \in M_{t-1}}$. By (W') we have $N^i_{M_{t-1} \setminus M_t} \subseteq \mathcal{E}(X_{M_{t-1}})$ and $N^i_{M_{t-1} \setminus M_t} \cap X_{M_t} = \emptyset$. Set $K = X_{M_{t-1}} \setminus N^i_{M_{t-1} \setminus M_t}$ and $K' = N^i_{M_{t-1} \setminus M_t}$. Assuming that $x \succ_i x'$ for all $x \in K$, $x' \in K'$ then, since $K' \subseteq \mathcal{E}(K \cup K')$, by Lemma 2 there exists a strictly concave $u^{t-1}_i : \mathbb{R}^d \to \mathbb{R}$ that satisfies

$$u^{t-1}_i(x) > u^{t-1}_i(x') \Leftrightarrow u^t_i(x) > u^t_i(x'), \text{ for all } x, x' \in K,$$

$$u^{t-1}_i(x) > u^{t-1}_i(x'), \text{ for all } x \in K, x' \in K', \text{ and}$$

$$u^{t-1}_i(x) = u^{t-1}_i(x'), \text{ for all } x, x' \in K'.$$

Thus, u^{t-1} virtually rationalizes the voting record $\{(y_j, z_j, v^i_j)\}_{j \in M_{t-1}}$, since it imputes indifference only to those among the added voting items $j \in M_{t-1} \setminus M_t$ such that $j \in M'_{t-1}$ with $Y^i_{M'_{t-1}} = N^i_{M'_{t-1}} = K'$.

To complete the proof, we need to show [$d = 1$ and (W_1)]$\Rightarrow$$(W)$. In particular, we already have $(W) \Rightarrow$ (W_1): if (W) holds and $\mathcal{E}(X_{M'}) \subseteq Y^i_{M'}$ for some $|M'| = 2$, we must have $N^i_{M'} = Y^i_{M'}$, since $N^i_{\{j\}} \neq Y^i_{\{j\}}$ for all $j \in M$. Thus, to show that $(W) \Leftrightarrow (W_1)$ when $d = 1$, consider arbitrary $M' \subseteq M$ with $|M'| > 2$. If there does not exist $x \in \mathcal{E}(X_{M'})$ such that $x \notin Y^i_{M'}$, then $\mathcal{E}(X_{M'}) \subseteq Y^i_{M'}$ and we need to show that there exists non-empty $M'' \subseteq M'$ such that $N^i_{M''} = Y^i_{M''} \subseteq \mathcal{E}(X_{M''})$ and $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$. Since $\mathcal{E}(X_{M'}) \subseteq Y^i_{M'}$, we have $\mathcal{E}(X_{M'}) = \{x_L, x_R\}$ for some $x_L, x_R \in Y^i_{M'}$. Let $M'' \subseteq M'$ be the largest subset of M' such that $Y^i_{M''} = \{x_L, x_R\}$, so that $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$. It suffices to show that

$N^i_{M''} = Y^i_{M''} = \{x_L, x_R\}$. First, it cannot be that $N^i_{M''} = \{x\} \subseteq \{x_L, x_R\}$ as in that case $y_j = z_j$ for some voting item $j \in M''$ which is impossible. Thus, the proof is complete if we can show that there does not exist $x \in N^i_{M''}$ such that $x \notin \{x_L, x_R\}$. Suppose otherwise to get a contradiction. Without loss of generality, let $x \in N^i_{\{j\}}$, $x_L \in Y^i_{\{j\}}$, and $x_R \in Y^i_{\{h\}}$. Then we have both $\mathcal{E}(X_{\{j,h\}}) \subseteq Y^i_{\{j,h\}}$ and $Y^i_{\{j,h\}} \neq N^i_{\{j,h\}}$ contradicting the assumption that (W_1) holds. ∎

Proof of Lemma 1

For unknowns $u_i^z \in \mathbb{R}$, and $d^z \in \mathbb{R}^d$, one for each $z \in K$, consider the following set of equalities and inequalities:

$$\begin{aligned} u_i^z - u_i^y &> 0, \text{ for all } z, y \in K \text{ with } z \succ_i y, \\ u_i^z - u_i^y &= 0, \text{ for all } z, y \in K \text{ with } z \sim_i y, \\ u_i^z - u_i^y - (d^z)^T(z - y) &> 0, \text{ for all } z \in K, \text{ all } y \in K,\ y \neq z. \end{aligned}$$

Since preferences over K can be represented by strictly concave u_i, there exists a solution to this system (by setting $u_i^z = u_i(z)$, and d^z equal to a supergradient of u_i at z). If we set $d^{\hat{x}} = 0$, $u_i^{\hat{x}} = u_i(\hat{x}) + \eta$ for small enough $\eta > 0$, and maintain the remaining values of the original solution, we obtain a solution to the modified system

$$\begin{aligned} u_i^z - u_i^y &> 0, \text{ for all } z, y \in K \setminus \{\hat{x}\} \text{ with } z \succ_i y, \\ u_i^{\hat{x}} - u_i^z &> 0, \text{ for all } z \in K \setminus \{\hat{x}\}, \\ u_i^z - u_i^y &= 0, \text{ for all } z, y \in K \setminus \{\hat{x}\} \text{ with } z \sim_i y, \\ u_i^z - u_i^y - (d^z)^T(z - y) &> 0, \text{ for all } z \in K, \text{ all } y \in K,\ y \neq z. \end{aligned}$$

This solution to the latter system produces a strictly concave utility function $\tilde{u}_i : \mathbb{R}^d \to \mathbb{R}$ (as in Richter and Matzkin, 1991, or Richter and Wong (2004)) defined as

$$\tilde{u}_i(x) = \min_{z \in K} \left\{ u_i^z + (d^z)^T(x - z) - \varepsilon (x - z)^T(x - z) \right\},$$

which, for small enough $\varepsilon > 0$, represents i's preferences over $K \setminus \{\hat{x}\}$ with ideal point $\hat{x}$. ∎

Proof of Theorem 4

We have $(\widehat{S}'_q)$⇒$[\{(y_j, z_j, v^i_j)\}_{j\in M\cup\hat{M}}$ satisfies $(S'_q)]$⇔$(\widehat{S}')$, the latter equivalence by Theorem 2. Furthermore, by Lemma 1, we also have $(\widehat{S}')$⇒$(\widehat{S}_c)$. Since $(\widehat{S}_c)$⇒$(\widehat{S}_q)$⇒$(\widehat{S}'_q)$ and $(\widehat{S}_c)$⇒$(\widehat{S}'_c)$⇒$(\widehat{S}'_q)$, we conclude $(\widehat{S}')$⇔$(\widehat{S}_c)$⇔$(\widehat{S}_q)$⇔$(\widehat{S}'_c)$⇔$(\widehat{S}'_q)$⇔$(\widehat{S}')$. Since $(\widehat{S}')$⇒$(\widehat{S})$, it remains to show $(\widehat{S})$⇒$(\widehat{S}')$, which follows from cases 1 and 2 in the proof of Theorem 5.

Cases 1 and 2 in the proof of Theorem 5 applied on subsets $M' \subseteq M \cup \hat{M}$ with $|M'| = 2$ also prove that $(\widehat{S}_1)$⇒$[\{(y_j, z_j, v^i_j)\}_{j\in M\cup\hat{M}}$ satisfies $(S_1)]$. Thus, if $d = 1$, (S_1)⇔(S) by Theorem 2, hence $(\widehat{S}_1)$⇒$(\widehat{S})$, since we have shown (S)⇔$(\widehat{S})$ when $d = 1$. Thus, we have shown that if $d = 1$, $(\widehat{S}_1)$⇔$(\widehat{S})$. ■

Proof of Theorem 5

We have $(\widehat{W}_q)$⇒$[\{(y_j, z_j, v^i_j)\}_{j\in M\cup\hat{M}}$ satisfies $(W_q)]$⇔$(\widehat{W'})$, the latter by Theorem 3. Furthermore, by Lemma 1, we also have $(\widehat{W'})$⇒$(\widehat{W}_c)$. Thus, $(\widehat{W'})$⇔$(\widehat{W}_c)$⇔$(\widehat{W}_q)$. Since $(\widehat{W'})$⇒$(\widehat{W})$, it remains to show $(\widehat{W})$⇒$(\widehat{W'})$, i.e., our goal is to show that $\{(y_j, z_j, v^i_j)\}_{j\in M\cup\hat{M}}$ satisfies condition (W) when $(\widehat{W})$ holds. Consider any $M' \subseteq M \cup \hat{M}$ and distinguish three possibilities:

Case 1, $M' \subseteq \hat{M}$: Then $Y^i_{M'} = \{\hat{x}_i\}$ and $\hat{x}_i \notin N^i_{M'}$ so that there exists $x \in \mathcal{E}(X_{M'})$ such that $x \notin Y^i_{M'}$.

Case 2, $M' \cap M \neq \emptyset$, and there exists $x \in \mathcal{E}(X_{M'\cap M} \cup \{\hat{x}_i\})$ such that $x \notin Y^i_{M'\cap M} \cup \{\hat{x}_i\}$: Since $x \neq \hat{x}_i$, we must have $x \in X_{M'\cap M} \setminus \mathcal{C}(Y^i_{M'\cap M} \cup \{\hat{x}_i\})$. We also have $X_{M'} \supseteq X_{M'\cap M}$ and $Y^i_{M'} \subseteq Y^i_{M'\cap M} \cup \{\hat{x}_i\}$, since $Y^i_{\{j\}} = \{\hat{x}_i\}$ for all $j \in M' \setminus M$. Hence we have $x \in X_{M'} \setminus \mathcal{C}(Y^i_{M'})$. We conclude that there exists $x' \in \mathcal{E}(X_{M'})$ such that $x' \notin Y^i_{M'}$.

Case 3, $M' \cap M \neq \emptyset$, and there exists non-empty $M'' \subseteq M' \cap M$ such that $Y^i_{M''} = N^i_{M''} \subseteq \mathcal{E}(X_{M'\cap M} \cup \{\hat{x}_i\})$, and $Y^i_{M''} \cap Y^i_{(M'\cap M)\setminus M''} = \emptyset$: We distinguish two subcases. First, if $N^i_{M'\setminus M} \setminus \mathcal{C}(X_{M'\cap M} \cup \{\hat{x}_i\}) \neq \emptyset$ then, since $Y^i_{M'\setminus M} = \{\hat{x}_i\}$ given that $M' \setminus M \neq \emptyset$, there exists $x \in \mathcal{E}(X_{M'}) \cap N^i_{M'\setminus M}$ such that $x \notin Y^i_{M'}$. Second, if $N^i_{M'\setminus M} \subset \mathcal{C}(X_{M'\cap M} \cup \{\hat{x}_i\})$, then $\mathcal{E}(X_{M'} \cup \{\hat{x}_i\}) = \mathcal{E}(X_{M'\cap M} \cup \{\hat{x}_i\})$. Note that we must have $\hat{x}_i \notin N^i_{M''}$, otherwise $N^i_{\{j\}} = \{\hat{x}_i\}$ for some $j \in M'' \subseteq M$ for which $Y^i_{\{j\}} \neq N^i_{\{j\}}$ and $\mathcal{E}(X_{\{j\}} \cup \{\hat{x}_i\}) = Y^i_{\{j\}} \cup \{\hat{x}_i\}$

violating condition $(\widehat{W})$. Thus, we conclude that for $M'' \subseteq M'$ we have $Y^i_{M''} = N^i_{M''} \subseteq \mathcal{E}(X_{M' \cap M} \cup \{\hat{x}_i\}) \backslash \{\hat{x}_i\} \subseteq \mathcal{E}(X_{M'})$. Furthermore, since $\hat{x}_i \notin Y^i_{M''}$, $Y^i_{M' \backslash M''} \subseteq Y^i_{(M' \cap M) \backslash M''} \cup \{\hat{x}_i\}$ and $Y^i_{M''} \cap Y^i_{(M' \cap M) \backslash M''} = \emptyset$, we conclude that $Y^i_{M''} \cap Y^i_{M' \backslash M''} = \emptyset$.

In sum, in all three cases, either there exists $x \in \mathcal{E}(X_{M'})$ such that $x \notin Y^i_{M'}$, or there exists non-empty $M'' \subseteq M'$ such that $Y^i_{M''} = N^i_{M''} \subseteq \mathcal{E}(X_{M'})$ and $Y^i_{M''} \cap Y^i_{M' \backslash M''} = \emptyset$, i.e., condition (W) is satisfied for the augmented voting record $\{(y_j, z_j, v^i_j)\}_{j \in M \cup \hat{M}}$ as we wished to show.

We also conclude from cases 1 to 3 above applied to subsets $M' \subseteq M \cup \hat{M}$ with $|M'| = 2$ that $(\widehat{W}_1) \Rightarrow [\{(y_j, z_j, v^i_j)\}_{j \in M \cup \hat{M}}$ satisfies $(W_1)]$. Thus, if $d = 1$, $(W_1) \Leftrightarrow (W)$ by Theorem 3, hence $(\widehat{W}_1) \Rightarrow (\widehat{W})$, since we have shown $(W) \Leftrightarrow (\widehat{W})$. ∎

Proof of Theorem 6

Assume $d = 1$. Without loss of generality, let $\hat{x}_1 \leq \hat{x}_2 \leq ... \leq \hat{x}_n$ and (by swapping alternatives y_j, z_j if necessary) let $v^1_j = yes$ for all $j \in M$. The proof is by construction. If $\hat{x}_1 < \hat{x}_2$, position alternatives $y_1, ..., y_m$ in the interval $[\hat{x}_1, \hat{x}_2)$, so that $\hat{x}_1 \leq y_1 < ... < y_m < \hat{x}_2$, otherwise set $y_1 < ... < y_m < \hat{x}_1$. Position alternatives $z_1, ..., z_m$ in $(\hat{x}_n, +\infty)$ so that $\hat{x}_n < z_m < ... < z_1$. Now, for every pair of voting items $h, j \in M$, and for every voter i, $N^i_{\{j\}} \cap (Y^i_{\{j,h\}} \cup \{\hat{x}_i\}) = \emptyset$. Furthermore, if $h > j$, we have $y_j < y_h < z_h < z_j$. Thus, for every voter i and for every $h, j \in M$, $x \in N^i_{\{j\}}$ (where $x = z_j$ in the case of voter 1) is such that $x \in \mathcal{E}(X_{\{j\}} \cup \{\hat{x}_i\})$ and $x \notin Y^i_{\{j\}} \cup \{\hat{x}_i\}$, and, if $h > j$, $x \in \mathcal{E}(X_{\{j,h\}} \cup \{\hat{x}_i\})$ and $x \notin Y^i_{\{j,h\}} \cup \{\hat{x}_i\}$. We conclude that condition $(\widehat{S}_1)$ is satisfied and, by Theorem 4, for every voter i the voting record $\{(y_j, z_j, v^i_j)\}_{j \in M}$ is rationalizable by a strictly concave utility function u_i with ideal point $\hat{x}_i$.

To see that the Theorem also obtains in $d' > 1$ dimensions, note that the constructed voting records, $\{(y_j, z_j, v^i_j)\}_{j \in M}$, in $d = 1$ dimension are strictly rationalizable satisfying (N). Then, by part (ii) of Theorem 7, for every ideal points $\hat{x}_i \in \mathbb{R}^{d'}$, $i \in N$, $d' > 1$, there exist voting alternatives $y'_j, z'_j \in \mathbb{R}^{d'}$ such that for every voter i, voting record $\{(y'_j, z'_j, v^i_j)\}_{j \in M}$ is strictly rationalizable by a strictly concave utility function u_i with ideal point $\hat{x}_i$. ∎

Proof of Theorem 7

Let $\hat{X} = \{\hat{x}_1, ..., \hat{x}_n\}$ and construct a finite set $X'_M \subset \mathbb{R}^{d'}$ such that $|X'_M| = |X_M|$ and $X'_M = \mathcal{E}(X'_M \cup \hat{X})$. Since $d' > 1$, such an X'_M trivially exists. Consider any onto function $f : X_M \to X'_M$, and for every $M' \subseteq M$, denote the image of $X_{M'}$ under f by $X'_{M'} = f(X_{M'})$. Fix arbitrary voter i, and consider the voting record $\{f(y_j), f(z_j), v^i_j\}_{j\in M}$. We shall show:

Claim: *If $\mathcal{E}(X'_{M'} \cup \{\hat{x}_i\}) \subseteq Y^i_{M'} \cup \{\hat{x}_i\}$ for some non-empty $M' \subseteq M$, then there exists non-empty $M'' \subseteq M'$ such that $N^i_{M''} = Y^i_{M''} \subseteq \mathcal{E}(X'_{M'} \cup \{\hat{x}_i\})$ and $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$.*

Since $X'_M = \mathcal{E}(X'_M \cup \hat{X})$, we have $X'_{M'} \subseteq \mathcal{E}(X'_{M'} \cup \{\hat{x}_i\})$ and $X'_{M'} \cap \{\hat{x}_i\} = \emptyset$ for all voters i. Thus, we have $N^i_{M'} \subseteq Y^i_{M'}$. As in the proof that (W_q)⇒(W) for Theorem 3, inductively define a nested sequence of nonempty subsets of M' by setting $M'_0 = M'$, and $M'_{t+1} = \{j \in M'_t : Y^i_{\{j\}} \subseteq N^i_{M'_t}\}$. We analogously conclude that there exists integer k such that $M'_{k-1} = M'_k = M'' \neq \emptyset$. Furthermore, $N^i_{M''} = Y^i_{M''}$ and $Y^i_{M''} \cap Y^i_{M' \setminus M''} = \emptyset$, the latter because otherwise there exists $j \in M' \setminus M''$ such that $Y^i_{\{j\}} \subseteq Y^i_{M''} = N^i_{M''}$, which is impossible by the definition of the sequence $\{M'_t\}$.

From the Claim we conclude that the voting record $\{(f(y_j), f(z_j), v^i_j)\}_{j\in M}$ satisfies $(\widehat{W})$. Since the Claim obtains for every voter i, the conclusion in part (i) follows by Theorem 5. Under the additional assumption of part (ii), the voting record $\{(f(y_j), f(z_j), v^i_j)\}_{j\in M}$ satisfies (A), since the original record $\{(y_j, z_j, v^i_j)\}_{j\in M}$ does and f is one to one. Thus, since the voting record $\{(f(y_j), f(z_j), v^i_j)\}_{j\in M}$ satisfies $(\widehat{W})$ and (A), it also satisfies $(\widehat{S})$, and part (ii) now follows from Theorem 4. ■

Proof of Theorem 8

We have already shown parts (i) and (ii), so it remains to show part (iii). By Corollary 1, a voting record $\{(y_j, z_j, v^i_j)\}^m_{j=0}$, with $y_0 = x$ violates (W) if it violates (S) and $\{(y_j, z_j, v^i_j)\}^m_{j=0}$ satisfies (A). But, since $\{(y_j, z_j, v^i_j)\}_{j\in M}$ satisfies (A), $\{(y_j, z_j, v^i_j)\}^m_{j=0}$ must satisfy (A) if $v^i_0 = yes$ and $z_0 \notin Y^i_M$, or if $v^i_0 = no$ and $z_0 \notin N^i_M$.■

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