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Computation of equilibrium values in the Baron and Ferejohn bargaining model*

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Abstract

Computation of *exact* equilibrium values for n-player divide-the-dollar legislative bargaining games as in Baron and Ferejohn (1989) with general quota voting rules, recognition probabilities, and discount factors, can be achieved by solving at most n bivariate square linear systems of equations. The approach recovers Eraslan's (2002) uniqueness result and relies on a characterization of equilibria in terms of two variables that satisfy a pair of piecewise linear equations.

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Keywords: Computation of equilibrium; Legislative Bargaining; Uniqueness.

1 Introduction

The bargaining model of Baron and Ferejohn (1989) is now a leading framework for the study of legislative decision making. They model the process through which a legislature divides a fixed surplus among its members as a sequential bargaining problem as introduced by Rubinstein (1982). Whereas Rubinstein (1982) assumes players alternate offering proposals, Baron and Ferejohn (1989) assume random recognition of proposers as in Binmore (1987). The legislator recognized in each period offers a proposal and if a majority of the legislators vote to approve it, then the division is implemented and the game ends; otherwise, interaction continues in the next period with a new round of proposal-making and voting. Variants of this model have been used in numerous applications ranging from bicameralism, executive-legislative relations, government and coalition formation, legislative organization, voting and proposal power, etc., and the framework has been extended and generalized in several directions.[1]

While much of the literature spawned by this work has been theoretical in nature, the goal of this paper is to address a more practical problem, namely that of computation of equilibrium. I pursue this question motivated by the belief that computation of equilibrium can prove instrumental for more applied studies aimed at confronting data (observational or experimental). I consider a

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[1]Papers using the divide-the-dollar framework alone include Ansolabehere, Snyder and Ting (2003); Diermeier and Myerson (1999); Diermeier and Feddersen (1998); Kalandrakis (2004, 2006*b*); McCarty (2000*a*,*b*); Montero (2006); Okada (1996); Ray (2007); Snyder, Ting and Ansolabehere (2005); Winter (1996). This is a partial list, and Baron and Ferejohn (1989) has been cited over 1,750 times according to Google scholar. A recent more complete literature review can be found in Eraslan and McLennan (2013).

divide-the-dollar model more general than that originally studied by Baron and Ferejohn, with general quota voting rules, discount factors, and recognition probabilities, admitting the same level of generality as found in Eraslan (2002). This setting is a reasonable starting point to address the computational complexity of bargaining games, given the nascent state of literature specific to computation for this class of games. It is well known that subgame perfect Nash equilibrium admits a severe form of indeterminacy of equilibrium behavior (e.g., Baron and Ferejohn (1989)) and, as is standard in this literature, I focus instead on subgame perfect Nash equilibria in *stationary* strategies. I establish a surprisingly positive result: for any legislature of size n computation of *exact* stationary equilibrium values, that is, legislators' expected payoffs at the beginning of each period, can be achieved by solving at most n bivariate square linear systems of equations, corresponding to the pieces of linearity of a piecewise linear system. Thus, ignoring the cost of evaluating the coefficients of each bivariate system of equations,[2] this problem can be solved at cost that is linear in the size of the legislature. Considerable speed-up is achieved when candidate solutions are prioritized according to a (modified) Newton's method, and in numerical experiments with legislatures of size up to $n = 100,000$, a solution is obtained in no more than four iterations and a tenth of a second on a desktop machine.

As is well known, the computation of Nash equilibria is generally considered a hard problem.[3] There are at least two reasons to expect similar complexity in the bargaining setting studied in this paper. The primary source of concern is the combinatorial explosion in the number of possible optimal (minimum winning) coalitions as the size of the legislature increases, and the concomitant increase in the potential support of proposer mixed strategies. A second and related issue is that equilibrium conditions in this model are not well posed: non-degenerate mixed strategy equilibria are typically consistent with a continuum of equilibrium strategy profiles. The work of Eraslan (2002) partly offsets this latter concern by establishing that equilibrium values are unique. Taking a cue from this uniqueness result, I focus on the computation of equilibrium values and completely avoid the computation of probabilities with which proposers mix among optimal coalitions. Key to the efficiency of the approach is a characterization of equilibrium values as a function of two summary statistics of the equilibrium: the *marginal reservation value*, that is, the largest reservation value in the least costly coalition; and the *maximum proposer surplus* extracted from the least costly coalition. These two quantities fully determine equilibrium values (Lemma 1) by (conditionally) sorting legislators from least to most expensive. This ordering of legislators is central to the main result and is not obviously attainable in such settings where legislators differ in more than one attributes. In particular, because higher recognition probabilities and discount factors each render legislators more expensive, *ceteris paribus*, it is a priori hard to rank legislators with high discount factor and low recognition probability over legislators with low discount factor and high recognition probability, or vice versa.

Armed with this characterization, equilibrium values can be computed if the marginal reservation value and maximum proposer surplus are available. Treating these quantities as a pair of unknowns, I specify an equal number of equations that are necessary in any equilibrium, are continuous and piecewise linear, and admit a unique solution. As a consequence, computation of equilibrium values is reduced to solving this bivariate piecewise linear system of equations. The

[2] In turn, evaluation of the coefficients of each linear system can be achieved at linear cost, so that the overall cost is at most quadratic in the size of the legislature in the worst case scenario.

[3] For example, see Daskalakis, Goldberg and Papadimitriou (2009) for a definitive result for strategic form games, or the recent review by Roughgarden (2010).

advertised efficiency of this approach is due to the fact that a solution must be contained in one of at most n of its pieces of linearity, while a candidate solution within each piece solves a linear system of two equations. There is an extensive literature on solving piecewise linear systems of equations in operations research and economics, with leading algorithms being essentially homotopy continuation methods that systematically search over different pieces of linearity of the system until the piece that contains a solution is visited (e.g., see Eaves and Scarf (1976) for such treatment of the general problem, and the recent review of related methods applied to the problem of computing Nash equilibria by Herings and Peeters (2010)). Due to the special structure of the present problem I instead opt for the speed of a Newton's method that is also guaranteed to converge in this case as it admits a straightforward globalization strategy.

A byproduct of the present approach is an alternative proof of uniqueness of equilibrium values. Eraslan's (2002) proof of uniqueness is by *reductio ad absurdum* whereas I obtain uniqueness by applying a non-smooth generalization of Gale and Nikaido's (1965) theorem due to Kojima and Saigal (1979). At its heart, Kojima and Saigal's (1979) condition is a degree theoretic argument involving the coherent orientation[4] of the Jacobian of the system of equations. In the literature on bargaining games, I state a uniqueness condition relying on a similar argument (Kalandrakis, 2006*a*, Theorem 6, p. 326). A more challenging application of the topological degree appears in Eraslan and McLennan (2013), who considerably strengthen and generalize Eraslan's (2002) result, by showing that for general voting rules there is a unique connected component of equilibria all sharing the same unique continuation values. Finally, uniqueness of equilibrium in such bargaining games is also studied by Cho and Duggan (2003) and Cardona and Ponsati (2011) in the context of one-dimensional agreements spaces.

In what follows, I start with a description of the bargaining model. I then proceed to formulate the bivariate system of equations used for computation and establish it admits a unique solution in section 3. The analysis culminates in section 4, which is devoted to computation of equilibrium. All proofs are relegated to an Appendix.

2 Model and Equilibrium

Consider a legislature comprising n members in set $N = \{1, \ldots, n\}$, $n \geq 2$. They bargain to divide a budget of size one, so that the set of possible agreements takes the form $X = \{x \in \mathbb{R}^n_+ : \sum_{i \in N} x_i = 1\}$. Bargaining takes place sequentially, potentially over a countable infinity of periods $t = 1, 2, \ldots$. If no agreement has been reached at the beginning of period t, legislator i is recognized with probability $p_i \in [0, 1]$ to offer a proposal. Recognition probabilities are constant across periods and sum up to one, $\sum_{i \in N} p_i = 1$. Having observed the proposal, legislators vote in favor or against it, and if at least q, $1 \leq q \leq n$, legislators approve the proposal, then it is implemented and the game ends. Otherwise, the game moves to the next period with a new round of proposal making and voting. Legislator i discounts the future by a factor $\delta_i \in [0, 1)$, so that if agreement x is reached in period t, i's payoff is $\delta_i^{t-1} x_i$.

As is standard in this literature, I focus on subgame perfect equilibria in stationary strategies. A stationary strategy for i is a pair consisting of a (mixed) proposal strategy that takes the form of a probability measure π_i over proposals in X, and a (measurable) voting strategy $\alpha_i : X \to \{1, 0\}$ so that i approves proposal x if $\alpha_i(x) = 1$ and $\alpha_i(x) = 0$ otherwise. Denote a

[4] A piecewise linear map is coherently oriented if the determinant of the Jacobian of all of its pieces of linearity has the same non-zero sign.

profile of strategies by $\sigma = ((\pi_1, \alpha_1), \ldots, (\pi_n, \alpha_n))$ and define for each x the collective acceptance function $\alpha(x;\sigma) = \mathbb{I}_{\{q,\ldots,n\}}\left(\sum_{j\in N} \alpha_j(x)\right)$, which takes the value 1 if proposal x receives the required quota of approvals when legislators use strategy profile σ, and zero otherwise. With such a strategy profile σ fixed, we can recursively define player i's *continuation value* $v_i(\sigma)$, that is, her expected payoff at the beginning of the next period if an agreement is not reached in the current period, specifically:

$$v_i(\sigma) = \sum_{j\in N} p_j \int \left(\alpha(x;\sigma)x_i + (1-\alpha(x;\sigma))\delta_i v_i(\sigma)\right) \pi_j(dx). \tag{1}$$

A stationary subgame perfect equilibrium with stage undominated (see Baron and Kalai (1993)) strategies (henceforth an *equilibrium*) is a strategy profile σ such that proposal strategies are optimal, i.e., for all $y \in X$

$$\int \left(\alpha(x;\sigma)x_i + (1-\alpha(x;\sigma))\delta_i v_i(\sigma)\right) \pi_i(dx) \geq \alpha(y;\sigma)y_i + (1-\alpha(y;\sigma))\delta_i v_i(\sigma),$$

and voting strategies satisfy

$$\alpha_i(x) = \begin{cases} 1 & if \quad x_i > \delta_i v_i(\sigma) \\ 0 & if \quad x_i < \delta_i v_i(\sigma). \end{cases}$$

The following theorem summarizes well known (e.g., Banks and Duggan (2000); Baron and Ferejohn (1989); Eraslan (2002)) properties of equilibrium and is stated without proof.

Theorem 1. *An equilibrium exists. If σ is an equilibrium, then*

1. *There is* no delay, *i.e.,* $\int \alpha(x;\sigma)\pi_i(dx) = 1$ *for all i.*
2. *If i's realized proposal is x, then* $\sum_{j\in N} \mathbb{I}_{(0,1]}(x_j) \leq q$.
3. *If agreement x is reached after a proposal from i and $x_j > 0, j \neq i$, then $x_j = \delta_j v_j(\sigma)$.*

Existence and the no-delay property of equilibrium have been established in a more general setting by Banks and Duggan (2000, Theorem 1, p. 78). Part 2 of Theorem 1 asserts that the largest possible coalitions (as measured by the number of legislators receiving funds) that can prevail in equilibrium are minimum winning coalitions, since no optimal proposal allocates funds to more than q voters. Lastly, by part 3 of Theorem 1, if j is included in i's winning coalition, then she receives an amount exactly equal to her reservation value, i.e., her discounted continuation value. Neither of the last two properties reflect restrictions on strategy profiles legislators may use, but follow from legislators' equilibrium (optimal) behavior.

3 A bivariate characterization

In this section, I shift focus from strategy profiles σ by gently removing mixing probabilities from the analysis and by introducing in their stead two quantities that characterize equilibrium continuation values. The first quantity that summarizes equilibrium is the *marginal reservation value*, that is, the cost of the most expensive legislator in the least expensive coalition. For a

strategy profile σ we can formally define the marginal reservation value as

$$r(\sigma) = \min_i \{\delta_i v_i(\sigma) : \#\{j : \delta_j v_j(\sigma) \leq \delta_i v_i(\sigma)\} \geq q\},$$

which amounts to the q-th lowest reservation value, $\delta_i v_i(\sigma)$. The second quantity is the *maximum proposer surplus* that can be extracted by any proposal that builds the least costly winning coalition, compensating all members of the coalition (including the proposer) with their reservation value. Formally, the maximum proposer surplus can be defined as

$$S(\sigma) = 1 - \sum_{i:\delta_i v_i(\sigma) < r(\sigma)} (\delta_i v_i(\sigma) - r(\sigma)) - qr(\sigma). \tag{2}$$

Note that surplus here is defined as the excess allocation to the proposer over and above her own reservation value. As a result, if a proposer's own reservation value is less or equal to the marginal reservation value, then she extracts a surplus exactly equal to $S(\sigma)$, building a coalition with q members (including herself) with the lowest reservation values. On the other hand, a proposer i with reservation value strictly larger than the marginal reservation value, $\delta_i v_i(\sigma) > r(\sigma)$, extracts a surplus equal to $S(\sigma) + r(\sigma) - \delta_i v_i(\sigma) < S(\sigma)$. In particular, such proposers build a coalition with $q - 1$ other least expensive players.

As shown in Lemma 1, knowledge of these two quantities, $r(\sigma)$ and $S(\sigma)$, suffices in order to recover the full vector of equilibrium continuation values. This result establishes a certain converse to Eraslan's Theorem 4 (Eraslan, 2002, Theorem 4, page 20).

Lemma 1. *If σ is an equilibrium, then*

1. *If $\frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma) \geq r(\sigma)$ then $\delta_i v_i(\sigma) \geq r(\sigma)$ and $v_i(\sigma) = p_i(S(\sigma) + r(\sigma))$.*
2. *If $\frac{\delta_i p_i}{1-\delta_i} S(\sigma) \geq r(\sigma) > \frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma)$ then $\delta_i v_i(\sigma) = r(\sigma)$ and $v_i(\sigma) = \delta_i^{-1} r(\sigma)$.*
3. *If $\frac{\delta_i p_i}{1-\delta_i} S(\sigma) < r(\sigma)$ then $\delta_i v_i(\sigma) < r(\sigma)$ and $v_i(\sigma) = \frac{p_i}{1-\delta_i} S(\sigma)$.*

In view of Lemma 1, I will henceforth treat the maximum proposer surplus and the marginal reservation value as a pair of unknowns (S, r),[5] and seek an equal number of necessary equilibrium equations. For that purpose, define a trio of index sets that partition the legislature according to the conditional statements in the three parts of Lemma 1 for each pair (S, r). Specifically, let

$$\begin{aligned} H(S,r) &= \{i : \frac{\delta_i p_i}{1-\delta_i p_i} S \geq r\}, \\ M(S,r) &= \{i : \frac{\delta_i p_i}{1-\delta_i} S \geq r > \frac{\delta_i p_i}{1-\delta_i p_i} S\}, \\ L(S,r) &= \{i : \frac{\delta_i p_i}{1-\delta_i} S < r\}. \end{aligned}$$

Except for ties at the boundary, these sets identify legislators whose reservation value must be higher, equal, or lower, respectively, than the marginal reservation value r when the maximum

[5] In what follows I treat (S, r) as a column vector. Also, whenever the dimension is implied, 1 and 0 represent the 2×1 unit and zero vectors, respectively.

proposer surplus is S. Now define for each i a function $\hat{v}_i : \mathbb{R}^2 \to \mathbb{R}$

$$\hat{v}_i(S,r) = \begin{cases} p_i(S+r) & if \quad i \in H(S,r) \\ \delta_i^{-1} r & if \quad i \in M(S,r) \\ \frac{p_i}{1-\delta_i} S & if \quad i \in L(S,r), \end{cases} \tag{3}$$

which assigns a continuation value to i for each pair S, r according to Lemma 1.

Equation (2) already provides one condition that the pair S, r must satisfy. The no-delay property of equilibrium implies that equilibrium values must also sum up to unity, thus providing a second equation. Combining these two equations, define the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F(S,r) = \begin{pmatrix} \sum_{i \in L(S,r)} (\delta_i \hat{v}_i(S,r) - r) + qr - 1 + S \\ \sum_{i \in N} \hat{v}_i(S,r) - 1 \end{pmatrix}. \tag{4}$$

As shown in Lemma 2, F (and $\hat{v}_i$) is piecewise linear. Additional notation makes this structure more transparent on the relevant portion of F's domain. First, let $\overline{\theta}$ be the q-th lowest ratio $\frac{\delta_i p_i}{1-\delta_i}$, that is,

$$\overline{\theta} \quad = \quad \min_i \{ \frac{\delta_i p_i}{1-\delta_i} : \#\{ j : \frac{\delta_j p_j}{1-\delta_j} \leq \frac{\delta_i p_i}{1-\delta_i} \} \geq q \}.$$

Since it is impossible to have more than $q-1$ legislators with reservation value strictly below $r(\sigma)$ in equilibrium, $\overline{\theta}$ serves as a logical upper bound on the ratio between the marginal reservation value and maximum proposer surplus in equilibrium, that is, $\overline{\theta} S(\sigma) \geq r(\sigma)$ for all equilibria σ (otherwise, if $\overline{\theta} S(\sigma) < r(\sigma)$ then $\#L(S(\sigma), r(\sigma)) \geq q$). A similar lower bound can be deduced as the q-th lowest ratio $\frac{\delta_i p_i}{1-\delta_i p_i}$, that is,

$$\underline{\theta} \quad = \quad \min_i \{ \frac{\delta_i p_i}{1-\delta_i p_i} : \#\{ j : \frac{\delta_j p_j}{1-\delta_j p_j} \leq \frac{\delta_i p_i}{1-\delta_i p_i} \} \geq q \},$$

is such that $r(\sigma) \geq \underline{\theta} S(\sigma)$ for any equilibrium σ, for otherwise more than $n-q$ legislators have reservation value above $r(\sigma)$ (i.e., if $\underline{\theta} S(\sigma) > r(\sigma)$ then $\#H(S(\sigma), r(\sigma)) > n-q$). Because $\frac{\delta_i p_i}{1-\delta_i} \geq \frac{\delta_i p_i}{1-\delta_i p_i}$ for all i, we have $\overline{\theta} \geq \underline{\theta}$. Using these bounds construct the set

$$\Theta = \{ \theta \in (\underline{\theta}, \overline{\theta}) : \theta = \tfrac{\delta_i p_i}{1-\delta_i p_i} \text{ for some } i \} \cup \{ \theta \in (\underline{\theta}, \overline{\theta}) : \theta = \tfrac{\delta_i p_i}{1-\delta_i} \text{ for some } i \}.$$

Let the cardinality of Θ be $\#\Theta = K-1$, enumerate the elements of Θ so that $\theta_1 < \ldots < \theta_{K-1}$, and set $\theta_0 = \underline{\theta}$ and $\theta_K = \overline{\theta}$. Note that by the definition of $\underline{\theta}, \overline{\theta}$, it follows that $K \leq n$. For each $k = 1, \ldots, K$ define the cone $E_k = \{ (S,r) \in \mathbb{R}^2_+ : \theta_k S \geq r \geq \theta_{k-1} S \}, k = 1, \ldots, K$, and define the set $E = \{ (S,r) \in \mathbb{R}^2_+ : \underline{\theta} S \leq r \leq \overline{\theta} S \} = \cup_{k=1}^K E_k$. With this notation in place we can establish:

Lemma 2. *The function F is continuous, it is linear on $E_k, k = 1, \ldots, K$, and if σ is an equilibrium, then $(S(\sigma), r(\sigma)) \in E$ and $F(S(\sigma), r(\sigma)) = 0$.*

By Lemma 2, the marginal reservation value and maximum proposer surplus must solve a pair of piecewise linear equations within cone E, delineated by the logical bounds $\overline{\theta}, \underline{\theta}$. By Lemma 1 these bounds imply that any legislator i with ratio $\frac{\delta_i p_i}{1-\delta_i} < \underline{\theta}$ must have lower reservation value than $r(\sigma)$ (i.e., $i \in L(S(\sigma), r(\sigma))$) in every equilibrium σ and, similarly, any legislator i with $\frac{\delta_i p_i}{1-\delta_i p_i} > \overline{\theta}$

must have higher reservation value than $r(\sigma)$ (i.e., $i \in H(S(\sigma), r(\sigma))$) in every equilibrium σ. The partition of the remaining legislators depends on the piece of linearity, E_k, the pair $(S(\sigma), r(\sigma))$ belongs to. Legislators that cannot be consistently classified within the low or high reservation value groups end up with reservation values exactly equal to the marginal reservation value: these legislators become more expensive than the marginal legislator if they are included in all winning coalitions with probability one, *and* they become less expensive than the marginal legislator if they are excluded from any winning coalition they do not propose. The source of this equilibrium tension is manifest in the expression for continuation values in Lemma 1, as the (linear) coefficient on $S(\sigma)$ is larger for the low reservation value legislators compared to high reservation value legislators (because $\frac{p_i}{1-\delta_i} > p_i$ for all i), and underpins the uniqueness proof by contradiction in Eraslan (2002). Of course, this tension is resolved in equilibrium via the use of mixed strategies, which are lurking in the background in this discussion. By invoking a condition of Kojima and Saigal (1979) that generalizes Gale and Nikaido's (1965) univalence theorem, we can produce an alternative proof of the fact that there is a unique resolution of these equilibrating forces, thus recovering Eraslan's (2002) uniqueness result.

Theorem 2. *There exists a unique vector of equilibrium continuation values.*

Theorem 2 is proved by showing that a function $\hat{F}$ that agrees with F on E, as illustrated in Figure 1(a), has a Jacobian that is a P-matrix for all its pieces of linearity. As a consequence, $\hat{F}$ satisfies the uniqueness condition of Kojima and Saigal (1979) which in turn implies that $\hat{F}$ is a homeomorphism of $\mathbb{R}^2$, that is, it is a bijection with a continuous inverse. By the existence of equilibrium (Theorem 1) and Lemma 2 it follows that the unique zero of $\hat{F}$ coincides with the only zero of F in E. Lemma 1 then dictates the unique equilibrium continuation values that correspond to this unique pair of equilibrium marginal reservation value and maximum proposer surplus.

4 Computation of equilibrium values

Using Lemmas 1 and 2, the problem of computing a vector of equilibrium continuation values has been reduced to that of solving the pair of equations $F(S, r) = 0$ for $(S, r) \in E$. Focusing on the hard case $\overline{\theta} \neq \underline{\theta}$,[6] denote the Jacobian of F at points of differentiability of $(S, r) \in E_k$ by A_k, so that $F(S, r) = A_k \cdot (S, r) - 1$ on $E_k, k = 1, \ldots, K$. Since a pair $(S, r) \in E_k$ solves $F(S, r) = 0$ if and only if $(S, r) = A_k^{-1} \cdot 1 \in E_k$ and there are at most K candidate solutions, we can compute equilibrium values via an exhaustive search of all K possible solutions, that is, solving at most $K \leq n$ bivariate linear systems of equations.

In fact, we can speed up this search by using the current candidate solution $(S, r) = A_k^{-1} \cdot 1$ in order to determine the next candidate solution if $(S, r) \notin E_k$. In particular, if $(S, r) \in E_{k'}, k' \neq k$, then we may set the new candidate solution at $(S', r') = A_{k'}^{-1} \cdot 1$. This rule for selection of the new candidate piece of linearity $E_{k'}$ of F that contains the solution may appear arbitrary at first, but it is actually an implementation of Newton's method as becomes obvious by executing the Newton step (assuming differentiability at the current iterate $(S, r) \in E_{k'}$):

$$(S', r') \quad = \quad (S, r) - [DF(S, r)]^{-1} \cdot F(S, r) = (S, r) - A_{k'}^{-1} \cdot (A_{k'} \cdot (S, r) - 1) = A_{k'}^{-1} \cdot 1.$$

[6] Equation (9) provides a closed form solution for the case $\overline{\theta} = \underline{\theta}$.

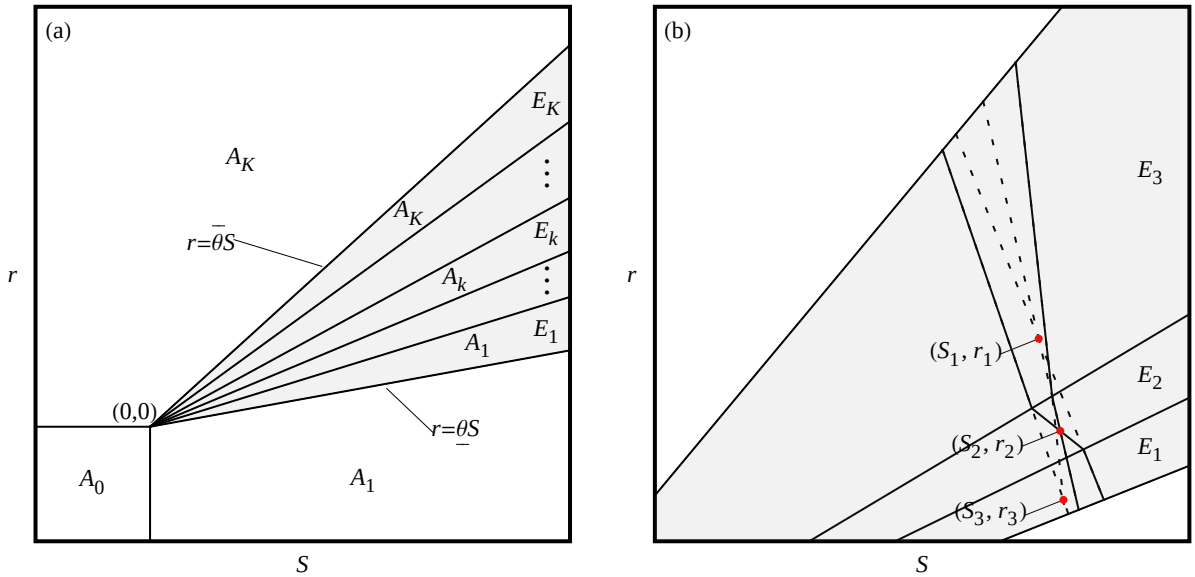

Figure 1: (a) Pieces of linearity of the piecewise linear function $\hat{F} : \mathbb{R}^2 \to \mathbb{R}^2$ that agrees with F on E and satisfies the homeomorphism condition of Kojima and Saigal (1979). The set E is highlighted in gray. The Jacobian of each piece of linearity of $\hat{F}$ is indicated by the 2×2 matrix $A_k, k = 0, \ldots, K$, where A_0 is constructed from the columns of A_1 and A_K to ensure continuity of $\hat{F}$. (b) An illustration of the zeros of F on E for Example 1 (aspect ratio is 1:8). The solid lines represent the zeros of each of the two coordinates of F. The unique (joint) zero of F is at the intersection of the two solid lines, (S_2, r_2) in E_2. The zeros of the linear functions of F in E_1 and E_3 are extended outside the corresponding piece of linearity and are shown with dashed lines. Newton's method initiated from E_1 or E_3 oscillates between iterates (S_1, r_1) and (S_3, r_3).

Indeed, Newton's method is theoretically justified (e.g., Qi (1993); Qi and Sun (1993)) in this setting even if the current iterate is *not* a point of differentiability of F, since $A_{k'}$ is a generalized derivative of $F(S, r)$. While such generalized Newton's methods for non-smooth equations recover convergence properties of the method in the smooth setting, global convergence is not guaranteed (as in the smooth setting) and globalization strategies are typically needed to ensure it. In Example 1 I present a legislature of size $n = 7$ with simple majority rule for which Newton's method exhibits non-convergence, as illustrated in Figure 1(b). But such non-convergence can be easily detected in the piecewise linear case, as it takes the form of a cycle over the pieces of linearity of F. We can thus guarantee convergence by moving to a yet not visited piece of linearity $E_{k''}$ when the current Newton iterate completes a cycle $k_1 \to k_2 \to \ldots \to k' = k_1$ revisiting $E_{k'}$, (or when the iterate $A_{k'}^{-1} \cdot 1 \notin E$). While there are well documented globalization strategies for Newton's method in general, a simple and efficient rule in our setting is to visit a new piece of linearity $E_{k''}$ that is closest (in a suitable metric) to the unadjusted Newton step. Algorithm 1 executes this globalized Newton method and is guaranteed to terminate successfully.

Theorem 3. *Algorithm 1 converges with an exact solution in at most $K \leq n$ iterations.*

Input: $q, \delta_1, \ldots, \delta_n, p_1, \ldots, p_n$;
Output: Equilibrium values $v_1, \ldots, v_n$;
if $\overline{\theta} = \underline{\theta} (= \theta^*)$ **then**
  $S \leftarrow \frac{1-\sum_{i \in L(1,\theta^*)} \frac{\delta_i p_i}{1-\delta_i}}{1+(q-\#L(1,\theta^*))\theta^*}$; $r \leftarrow \frac{\theta^* \left(1-\sum_{i \in L(1,\theta^*)} \frac{\delta_i p_i}{1-\delta_i}\right)}{1+(q-\#L(1,\theta^*))\theta^*}$;
else
  $\mathcal{K} \leftarrow \{1, \ldots, K\}$;
  $k \leftarrow \lceil \frac{K}{2} \rceil$;
  $(S, r) \leftarrow A_k^{-1} \cdot 1$;
  while $(S, r) \notin E_k$ **do**
    $\mathcal{K} \leftarrow \mathcal{K} \setminus \{k\}$;
    if $(S, r) \in \cup_{k \in \mathcal{K}} E_k$ **then**
      $k \leftarrow \min_{k' \in \mathcal{K}} \{k' : S\theta_{k'-1} \leq r \leq S\theta_{k'}\}$;
    else
      $k \leftarrow \min\{k' : k' \in \arg\min_{k'' \in \mathcal{K}} ||r - \frac{\theta_{k''}+\theta_{k''-1}}{2} S||\}$;
    end
    $(S, r) \leftarrow A_k^{-1} \cdot 1$;
  end
end
for *i=1* **to** n **do**
  $v_i \leftarrow \hat{v}_i(S, r)$;
end

Algorithm 1: Computation of equilibrium values.

As demonstrated in Example 1, the bound on the number of iterations of Algorithm 1 established in Theorem 3 is tight, in the sense that there exists an initial piece of linearity from which Algorithm 1 is executed that requires the maximum number of $K = 3 < 7 = n$ iterations to converge (including an adjusted/globalized Newton step). Nevertheless, this bound is unlikely to be binding as the legislature size n increases. Numerical experiments with random legislatures of sizes up to $n = 100,000$ exhibit fast convergence in no more than four iterations.

Example 1. *Consider a legislature with $n = 7$, simple majority rule ($q = 4$), and parameters p_i, δ_i as given in the first two rows of Table 1. We compute $\overline{\theta} = \theta_3 = \frac{\delta_4 p_4}{1-\delta_4} = \frac{3}{20}, \theta_2 = \frac{\delta_5 p_5}{1-\delta_5} = \frac{23}{308}, \theta_1 = \frac{\delta_4 p_4}{1-\delta_4 p_4} = \frac{23}{377}, \underline{\theta} = \theta_0 = \frac{\delta_3 p_3}{1-\delta_3 p_3} = \frac{19}{381}$. The solution of $F(S, r) = 0$ is $(S_2, r_2) = (\frac{24,975,860}{30,059,109}, \frac{556,738}{10,019,703})$ and lies in E_2. If the non-globalized version of Newton's method is initiated from the first or third piece of linearity of F, then the algorithm cycles between points $(S_1, r_1) = (\frac{1,924}{2,397}, \frac{1,976}{27,965}) \in E_1$ and $(S_3, r_3) = (\frac{1,076,075}{1,288,577}, \frac{57,057}{1,288,577}) \in E_3$. This is illustrated in Figure 1(b). The computed equilibrium and reservation values are displayed in the third and fourth row of Table 1. The equilibrium involves mixed proposal strategies that render legislators 3, 4, and 5 equally expensive coalition partners. Ex ante probabilities of inclusion in the winning coalition (over all, $\mu_i(\sigma) + p_i$, and by players $j \neq i$, $\mu_i(\sigma)$) are computed according to equation (5) in the last two rows of Table 1.*

A few additional remarks are in order concerning Algorithm 1. First, termination of the algorithm does not involve a numerical criterion that must be satisfied approximately at some

i	1	2	3	4	5	6	7
p_i	0.05	0.1	0.05	0.05	0.25	0.25	0.25
δ_i	0.35	0.30	0.95	0.75	0.23	0.70	0.80
$v_i(\sigma)$	0.0639	0.1187	0.0585	0.0741	0.2416	0.2216	0.2216
$\delta_i v_i(\sigma)$	0.0224	0.0356	0.0556	0.0556	0.0556	0.1551	0.1773
$\mu_i(\sigma)$	0.95	0.9	0.2549	0.5356	0.3594	0	0
$\mu_i(\sigma)+p_i$	1	1	0.3049	0.5856	0.6094	0.25	0.25

Table 1: Equilibrium continuation values and probabilities of inclusion in the winning coalition in Example 1. Probability $\mu_i(\sigma)$ is computed using equation (5) and the output of Algorithm 1. All computed quantities are rounded to the fourth decimal digit.

prespecified tolerance level. If the inclusion condition $(S, r) = A_k^{-1} \cdot 1 \in E_k$ is satisfied, then (S, r) is an *exact* solution. Second, the globalization strategy of adjusting the Newton step to the nearest piece of linearity minimizes the distance of the unadjusted Newton step from the ray that splits cone E_k in half. Alternative metrics are certainly possible, but are not likely to have appreciable effect on the speed of the algorithm. Third, the cost per iteration of Algorithm 1 increases with n only up to the formulation of the matrix A_k; the size of the linear system to be solved is constant at two for all iterations and independent of n. It is easily verified that evaluation of A_k can be performed at cost linear in n, while an initial sorting of the ratios $\frac{\delta_i p_i}{1-\delta_i p_i}$ and $\frac{\delta_i p_i}{1-\delta_i}$ is achieved at cost of order $n \log n$. As a consequence, factoring these costs in, the overall cost of computation in the worst case scenario is *at most* quadratic in the size of the legislature. Fourth, I have focused discussion on computation of equilibrium values, but other equilibrium quantities are easily obtained once these values are available. For example, probabilities of inclusion in the winning coalition $\mu_i(\sigma)$ are directly available from equation (5), as illustrated in Example 1. It is less trivial, to recover a complete equilibrium profile, σ. On the one hand, all equilibrium proposal mixed strategies consistent with values $v_i(\sigma)$ and probabilities $\mu_i(\sigma)$ are characterized by a system of linear equalities and inequalities which can be solved by linear programming techniques. On the other hand, the size of this linear system may increase dramatically with n because the number of *ex post* (i.e., once equilibrium values are known) optimal coalitions may also increase. Nevertheless, the present approach is likely to economize considerably over a naive alternative that attempts to solve for an equilibrium profile σ directly. Besides the fact that the marginal probabilities $\mu_i(\sigma)$ are fixed and known once continuation values are at hand, further economization stems from a (potentially significant) reduction in the number of possible optimal coalitions: at a solution (S^*, r^*) all proposers include legislators in $L(S^*, r^*)$ in the winning coalition, while any legislator $i \in H(S^*, r^*)$ is excluded from all coalitions proposed by $j \neq i$.

5 Conclusion

I have developed an algorithm to compute equilibrium values in the model of Baron and Ferejohn (1989) with general quota voting rules. The approach bypasses the main stumbling block to efficient computation of a solution for this problem, that is, the combinatorial explosion in the number of possible winning coalitions as the size of the legislature increases. The characterization

used recovers the uniqueness result of Eraslan (2002). The cost of finding an equilibrium, in terms of the number of Newton's iterations needed in the worst case scenario is bounded by the size of the legislature, and the computation cost of each iteration amounts to that of evaluating and solving a square bivariate linear system, independent of the size of the legislature. It is an open question whether a generalization of this approach can be fruitfully applied to the study of legislatures with more general voting rules.

Appendix

Lemma 1 (Restated)**.** *If σ is an equilibrium, then*

1. *If $\frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma) \geq r(\sigma)$ then $\delta_i v_i(\sigma) \geq r(\sigma)$ and $v_i(\sigma) = p_i(S(\sigma) + r(\sigma))$.*
2. *If $\frac{\delta_i p_i}{1-\delta_i} S(\sigma) \geq r(\sigma) > \frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma)$ then $\delta_i v_i(\sigma) = r(\sigma)$ and $v_i(\sigma) = \delta_i^{-1} r(\sigma)$.*
3. *If $\frac{\delta_i p_i}{1-\delta_i} S(\sigma) < r(\sigma)$ then $\delta_i v_i(\sigma) < r(\sigma)$ and $v_i(\sigma) = \frac{p_i}{1-\delta_i} S(\sigma)$.*

Proof. For the sake of completeness, I start with certain equilibrium properties that are also established by Eraslan (2002). In particular, except for variations in definition and notation, my equations (5), (6), (7), and (8) match her equations (8), (10), (9), and (11). By equation (1) and the equilibrium properties of Theorem 1, the continuation value of i in any equilibrium σ can be expressed as

$$v_i(\sigma) = p_i(1 - \min_{C \subseteq N \setminus \{i\} : |C| = q-1} \sum_{j \in C} \delta_j v_j(\sigma)) + \mu_i(\sigma) \delta_i v_i(\sigma), \tag{5}$$

where $\mu_i(\sigma) = \sum_{j \neq i} p_j \int_{x_i > 0} \pi_j(dx) \in [0, 1 - p_i]$ is the probability i receives a strictly positive allocation (by necessity her reservation value, $\delta_i v_i(\sigma)$) by players other than i. Note that

$$1 - \min_{C \subseteq N \setminus \{i\} : |C| = q-1} \sum_{j \in C} \delta_j v_j(\sigma) = \begin{cases} S(\sigma) + r(\sigma) & if \quad \delta_i v_i(\sigma) > r(\sigma) \\ S(\sigma) + \delta_i v_i(\sigma) & if \quad \delta_i v_i(\sigma) \leq r(\sigma), \end{cases} \tag{6}$$

and that optimality of proposals by $j \neq i$ implies

$$\mu_i(\sigma) = \begin{cases} 0 & if \quad \delta_i v_i(\sigma) > r(\sigma) \\ 1 - p_i & if \quad \delta_i v_i(\sigma) < r(\sigma). \end{cases} \tag{7}$$

Substituting from (6) and (7) into (5) and solving for $v_i(\sigma)$ we obtain

$$v_i(\sigma) = \begin{cases} p_i(S(\sigma) + r(\sigma)) & if \quad \delta_i v_i(\sigma) > r(\sigma) \\ \frac{p_i}{1-\delta_i(p_i + \mu_i(\sigma))} S(\sigma) & if \quad \delta_i v_i(\sigma) = r(\sigma) \\ \frac{p_i}{1-\delta_i} S(\sigma) & if \quad \delta_i v_i(\sigma) < r(\sigma). \end{cases} \tag{8}$$

We can now show the three parts of the Lemma:

Part 1. We first show $\delta_i v_i(\sigma) \geq r(\sigma)$ by showing that $\delta_i v_i(\sigma) < r(\sigma)$ leads to a contradiction. Indeed, from (8), we conclude that $\delta_i v_i(\sigma) = \frac{\delta_i p_i}{1-\delta_i} S(\sigma) < r(\sigma) \leq \frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma)$, a contradiction. It follows that $\delta_i v_i(\sigma) \geq r(\sigma)$. By (8), $v_i(\sigma) = p_i(S(\sigma) + r(\sigma))$ if $\delta_i v_i(\sigma) > r(\sigma)$, so it remains to show

that $v_i(\sigma) = p_i(S(\sigma)+r(\sigma))$ also when $\delta_i v_i(\sigma) = r(\sigma)$. To that end, first observe that (8) in this case yields $\delta_i v_i(\sigma) = \frac{\delta_i p_i}{1-\delta_i(p_i+\mu_i(\sigma))}S(\sigma) = r(\sigma)$, which implies $r(\sigma) = \delta_i p_i(S(\sigma)+r(\sigma)) + \delta_i\mu_i(\sigma)r(\sigma)$. Furthermore, $r(\sigma) \leq \frac{\delta_i p_i}{1-\delta_i p_i}S(\sigma)$ implies $r(\sigma) \leq \delta_i p_i(S(\sigma)+r(\sigma))$. Since $\delta_i\mu_i(\sigma)r(\sigma) \geq 0$, it follows that $\delta_i v_i(\sigma) = r(\sigma) = \delta_i p_i(S(\sigma)+r(\sigma))$. If $\delta_i > 0$, then $v_i(\sigma) = p_i(S(\sigma)+r(\sigma))$ as we wished to show. If $\delta_i = 0$, then $r(\sigma) = 0$ and, once more invoking (8), $v_i(\sigma) = p_i S(\sigma) = p_i(S(\sigma)+r(\sigma))$, as we wished to show.

Part 2. We first argue that it cannot be that $\delta_i v_i(\sigma) \neq r(\sigma)$. If $\delta_i v_i(\sigma) > r(\sigma)$, then substituting for $v_i(\sigma)$ from (8) we obtain $\delta_i p_i(S(\sigma)+r(\sigma)) > r(\sigma)$ which implies $\frac{\delta_i p_i}{1-\delta_i p_i}S(\sigma) > r(\sigma)$, a contradiction. If instead $\delta_i v_i(\sigma) < r(\sigma)$, then once more substituting for $v_i(\sigma)$ from (8) we obtain $\frac{\delta_i p_i}{1-\delta_i}S(\sigma) < r(\sigma)$, also a contradiction. Thus $\delta_i v_i(\sigma) = r(\sigma)$ and, since $\frac{\delta_i p_i}{1-\delta_i}S(\sigma) \geq r(\sigma) > \frac{\delta_i p_i}{1-\delta_i p_i}S(\sigma)$ implies $\delta_i > 0$, we conclude $v_i(\sigma) = \delta_i^{-1}r(\sigma)$.

Part 3. By (8) it suffices to show that $\delta_i v_i(\sigma) < r(\sigma)$. Suppose $\delta_i v_i(\sigma) \geq r(\sigma)$ instead, to get a contradiction. If $\delta_i v_i(\sigma) > r(\sigma)$ then, after substituting from (8) we obtain

$$\delta_i v_i(\sigma) = \delta_i p_i(S(\sigma)+r(\sigma)) > r(\sigma) \Rightarrow \frac{\delta_i p_i}{1-\delta_i p_i}S(\sigma) > r(\sigma),$$

which is impossible since $r(\sigma) > \frac{\delta_i p_i}{1-\delta_i}S(\sigma) \geq \frac{\delta_i p_i}{1-\delta_i p_i}S(\sigma)$. If $\delta_i v_i(\sigma) = r(\sigma)$, we conclude using (8) that

$$\delta_i v_i(\sigma) = \frac{\delta_i p_i}{1-\delta_i(p_i+\mu_i(\sigma))}S(\sigma) = r(\sigma) > \frac{\delta_i p_i}{1-\delta_i}S(\sigma) \Rightarrow p_i + \mu_i(\sigma) > 1,$$

another contradiction. Thus, $\delta_i v_i(\sigma) < r(\sigma)$. □

Lemma 2 (Restated)**.** *Function F is continuous, it is linear on $E_k, k = 1, \ldots, K$, and if σ is an equilibrium, then $(S(\sigma), r(\sigma)) \in E$ and $F(S(\sigma), r(\sigma)) = 0$.*

Proof. To establish continuity of F it suffices to show that $\hat{v}_i$ is continuous, by considering in turn the two possible cases for discontinuity. First, let $(S_\ell, r_\ell) \to (S, r)$ such that $\frac{\delta_i p_i}{1-\delta_i}S_\ell < r_\ell$ for all ℓ, and $\frac{\delta_i p_i}{1-\delta_i}S = r$. If $r > \frac{\delta_i p_i}{1-\delta_i p_i}S$ then it follows that $\delta_i > 0$ and $\hat{v}_i(S_\ell, r_\ell) = \frac{p_i}{1-\delta_i}S_\ell \to \delta_i^{-1}r = \hat{v}_i(S, r)$. If $r = \frac{\delta_i p_i}{1-\delta_i p_i}S$ instead, then $\hat{v}_i(S_\ell, r_\ell) \to \frac{p_i}{1-\delta_i}S = p_i S + \frac{\delta_i p_i}{1-\delta_i}S = p_i S + r$. Now further distinguish two subcases. If $r = 0$, then $\hat{v}_i(S_\ell, r_\ell) \to p_i S = \hat{v}_i(S, r)$. If $r > 0$, then $\delta_i > 0$ and $p_i = 1$, hence $\hat{v}_i(S_\ell, r_\ell) \to S + r = \hat{v}_i(S, r)$. Second, consider a sequence $(S_\ell, r_\ell) \to (S, r)$ such that $\frac{\delta_i p_i}{1-\delta_i}S_\ell \geq r_\ell > \frac{\delta_i p_i}{1-\delta_i p_i}S_\ell$ for all ℓ, and $\frac{\delta_i p_i}{1-\delta_i p_i}S = r \Leftrightarrow r = \delta_i p_i(S+r)$. Then $\hat{v}_i(S_\ell, r_\ell) = \delta_i^{-1}r_\ell \to \delta_i^{-1}r = p_i(S+r) = \hat{v}_i(S, r)$. This completes the proof of continuity. Linearity of F on E_k follows from the fact that the partition $\{L(S,r), M(S,r), H(S,r)\}$ is constant in the interior of E_k and by the linearity of $\hat{v}_i$. By Lemma 1, $\hat{v}_i(S(\sigma), r(\sigma)) = v_i(\sigma)$ for any equilibrium σ and all i, so necessity of $F(S(\sigma), r(\sigma)) = 0$ for equilibrium σ follows by definition (2) for the first of the two equations, and the fact that no equilibrium admits delay for the second equation. To show $(S(\sigma), r(\sigma)) \in E$, we first show that it cannot be that $\overline{\theta}S(\sigma) < r(\sigma)$, for in that case we obtain the contradiction

$$q \leq \#\{i : \frac{\delta_i p_i}{1-\delta_i} \leq \overline{\theta}\} \leq \#\{i : \frac{\delta_i p_i}{1-\delta_i}S(\sigma) < r(\sigma)\} = \#\{i : \delta_i v_i(\sigma) < r(\sigma)\} < q.$$

From left to right, the first inequality above follows from the definition of $\overline{\theta}$, the second by the implication $\frac{\delta_i p_i}{1-\delta_i} \leq \overline{\theta} \Rightarrow \frac{\delta_i p_i}{1-\delta_i}S(\sigma) \leq \overline{\theta}S(\sigma) < r(\sigma)$ when $\overline{\theta}S(\sigma) < r(\sigma)$ is true, the equality holds by Lemma 1, and the final inequality by the definition of $r(\sigma)$ as the q-th lowest reservation value.

Similarly, if $\underline{\theta}S(\sigma) > r(\sigma)$ then we deduce another contradiction

$$n - q < \#\{i : \frac{\delta_i p_i}{1-\delta_i p_i} \geq \underline{\theta}\} \leq \#\{i : \frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma) > r(\sigma)\} = \#\{i : \delta_i v_i(\sigma) > r(\sigma)\} \leq n - q.$$

This time, the first inequality follows because $\#\{i : \frac{\delta_i p_i}{1-\delta_i p_i} \geq \underline{\theta}\} + \#\{i : \frac{\delta_i p_i}{1-\delta_i p_i} < \underline{\theta}\} = n$ and $q > \#\{i : \frac{\delta_i p_i}{1-\delta_i p_i} < \underline{\theta}\}$ from the definition of $\underline{\theta}$ as the q-th lowest ratio $\frac{\delta_i p_i}{1-\delta_i p_i}$, the second inequality because $\frac{\delta_i p_i}{1-\delta_i p_i} \geq \underline{\theta} \Rightarrow \frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma) \geq \underline{\theta}S(\sigma) > r(\sigma)$ when $\underline{\theta}S(\sigma) > r(\sigma)$, the equality holds by Lemma 1 and the fact that $\frac{\delta_i p_i}{1-\delta_i p_i} S(\sigma) > r(\sigma) \Leftrightarrow \delta_i p_i (S(\sigma) + r(\sigma)) > r(\sigma)$ and, finally, the last inequality holds because $\#\{i : \delta_i v_i(\sigma) \leq r(\sigma)\} \geq q$. □

Theorem 2 (Restated)**.** *There exists a unique vector of equilibrium continuation values.*

Proof. Since an equilibrium exists (Theorem 1), by Lemma 2 it suffices to show that there is at most one $(S, r) \in E$ that solves $F(S, r) = 0$. We distinguish two cases:

Case 1: $\overline{\theta} = \underline{\theta} = \theta^*$. In this case any solution must satisfy $r = \theta^* S$. This equation along with the first of the two equations $F(S, \theta^* S) = 0$ yield the unique solution

$$S = \frac{1 - \sum_{i \in L(1,\theta^*)} \frac{\delta_i p_i}{1-\delta_i}}{1 + (q - \#L(1,\theta^*))\,\theta^*} \qquad r = \frac{\theta^* \left(1 - \sum_{i \in L(1,\theta^*)} \frac{\delta_i p_i}{1-\delta_i}\right)}{1 + (q - \#L(1,\theta^*))\,\theta^*}. \tag{9}$$

Case 2: $\overline{\theta} > \underline{\theta}$. Let $\theta_k^* = \frac{\theta_k + \theta_{k+1}}{2}$ and set $A_k = DF(1, \theta_k^*)$, which coincides with the Jacobian of F in the interior of every piece of linearity E_k, $k = 1, \ldots, K$. Denote $A_k = \begin{pmatrix} a_{11}^k & a_{12}^k \\ a_{21}^k & a_{22}^k \end{pmatrix}$ and define $A_0 = \begin{pmatrix} a_{11}^K & a_{12}^1 \\ a_{21}^K & a_{22}^1 \end{pmatrix}$. We now define another continuous piecewise linear function $\hat{F} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{F}(S, r) = F(S, r)$ for all $(S, r) \in E$ (Figure 1(a) illustrates), specifically

$$\hat{F}(S,r) = \begin{cases} A_K \cdot (S,r) - 1 & if \quad r > \overline{\theta}S \geq 0 \text{ or } r \geq 0, S \leq 0 \\ A_k \cdot (S,r) - 1 & if \quad (S,r) \in E_k, k = 1, \ldots, K \\ A_1 \cdot (S,r) - 1 & if \quad 0 \leq r < \underline{\theta}S \text{ or } r \leq 0, S \geq 0 \\ A_0 \cdot (S,r) - 1 & if \quad r \leq 0, S \leq 0. \end{cases}$$

We will show that $\hat{F}$ is a homeomorphism of $\mathbb{R}^2$ (hence $\hat{F}(S, r) = 0$ has a unique solution) by verifying the condition of Theorem 5.1 of Kojima and Saigal (1979). In particular, we will show that A_k is a P-matrix for all $k = 0, 1, \ldots, K$, which implies that $Det[\lambda A_k + (1-\lambda)\mathbf{I}_2] > 0$ for all $\lambda \in [0, 1]$, by Lemma 3.1.1 of Saigal (1976). Before we proceed, note that $M(1, \theta_k^*) \neq \emptyset$ for all k since $\theta_k^* \in (\overline{\theta}, \underline{\theta})$, and $\#L(1, \theta_k^*) < q$ since $\theta_k^* < \overline{\theta}$. To show A_k, $k = 0, 1, \ldots, K$, is a P-matrix we first show that $a_{11}^k = 1 + \sum_{i \in L(1,\theta_k^*)} \frac{\delta_i p_i}{1-\delta_i} > 0$ and $a_{22}^k = \sum_{i \in M(1,\theta_k^*)} \delta_i^{-1} + \sum_{i \in H(1,\theta_k^*)} p_i > 0$ for all $k = 1, \ldots, K$. Furthermore, we will show that $Det[A_k] > 0$ for all $k = 0, 1, \ldots, K$ (including A_0 which shares its first column with A_1 and its second column with A_K) by showing that $a_{11}^k > a_{21}^k \geq 0$ and $a_{22}^k > a_{12}^k \geq 0$ for all $k = 1, \ldots, K$. Indeed, $a_{21}^k = \sum_{i \in L(1,\theta_k^*)} \frac{p_i}{1-\delta_i} + \sum_{i \in H(1,\theta_k^*)} p_i \geq 0$ and

$$a_{11}^k > a_{21}^k \Leftrightarrow 1 + \sum_{i \in L(1,\theta_k^*)} \frac{\delta_i p_i}{1-\delta_i} > \sum_{i \in L(1,\theta_k^*)} \frac{p_i}{1-\delta_i} + \sum_{i \in H(1,\theta_k^*)} p_i \Leftrightarrow 1 > \sum_{i \in L(1,\theta_k^*) \cup H(1,\theta_k^*)} p_i,$$

which is true since $p_i > 0$ for all $i \in M(1, \theta_k^*) \neq \emptyset$ (else $0 \geq \theta_k^* > 0$). Furthermore, $a_{12}^k = q - \#L(1, \theta_k^*) > 0$ and

$$a_{22}^k > a_{12}^k \Leftrightarrow \#L(1, \theta_k^*) + \sum_{i \in M(1,\theta_k^*)} \delta_i^{-1} + \sum_{i \in H(1,\theta_k^*)} p_i > q,$$

which once more is true because $M(1, \theta_k^*) \neq \emptyset$ and $\#L(1, \theta_k^*) + \#M(1, \theta_k^*) \geq q$ since $\theta_k^* > \underline{\theta}$. To summarize, for all $k = 0, 1, \ldots, K$, all the principal minors of A_k are positive, establishing that A_k is a P-matrix, as we wished to show. □

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