Repeated Elections With Asymmetric Information

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Abstract

We model an infinite sequence of elections with no term limits. In each period a challenger with privately known preferences is randomly drawn from the electorate to run against the incumbent, and the winner chooses a policy outcome in a one-dimensional issue space. Our first theorem establishes the existence of an equilibrium in which the median voter is decisive: an incumbent wins reelection if and only if his most recent policy choice gives the median voter a payoff at least as high as he would expect from a challenger. In equilibrium, politicians are divided into three groups, according to their ideal points: centrists, moderates, and extremists. Only centrists and moderates are reelected, though moderates must compromise their ideological preferences to satisfy the median voter. The equilibrium is symmetric, stationary, and the behavior of voters is consistent with both retrospective and prospective voting. Our second theorem is that, in fact, it is the only equilibrium possessing these properties.

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1 Introduction

1.1 The Downsian model of elections rests on two particularly unrealistic assumptions: political candidates have no intrinsic policy preferences (they care only about winning), and they can make binding commitments to policy platforms before taking office.[1] In this paper we view politicians as voters, distinguished only by their choice of profession, and, therefore, they care about policy outcomes as well as the instrinsic rewards of holding office (if any). Furthermore, candidates are unable to commit to their campaign promises. In this setting, the median voter result breaks down completely in single-shot elections: unbound by campaign promises and with no political future, the winner of such an election simply chooses his ideal point, which would coincide with the median voter's only by chance. Of course, elections rarely occur in isolation, and we would expect repeated elections to promote compromise: the promise of reelection should induce some politicians to moderate their policy choices.

We formalize this intuition in a model of infinitely repeated elections with no term limits, thereby maximizing each politician's incentive to compromise. Each voter's preferences over a one-dimensional issue space are given by distance from an ideal point, and ideal points are symmetrically distributed across the continuum. This distribution is known to all, but individual voters' ideal points — in particular, the ideal points of the candidates — are private information. Voters are infinitely-lived and discount future payoffs at a common rate less than one. We establish (Theorem 1) the existence of an equilibrium in which the median voter is decisive in the following sense: incumbents are reelected if and only if their most recent policy choice gives the median voter a payoff at least as high as he would expect from an untried challenger. This equilibrium is symmetric, stationary, and the behavior of voters is consistent with both retrospective and prospective voting. We then show (Theorem 2) that this equilibrium is the *only* one exhibiting the latter four properties — decisiveness of the median voter follows from the others. Thus, perhaps surprisingly, a form of the median voter theorem extends to a model of infinitely repeated elections without term

[1] Downs (1957) and Hotelling (1929,pp.54-55) are the classic applications of the median voter principle to elections. See Black (1958) for an application to committee decision-making. A reference to the median voter principle appears as far back as Galton (1907).

limits, with asymmetrically informed voters, and with political candidates unable to make binding campaign promises.

Our comparative statics formally show that monetary rewards to officeholders, by increasing the opportunity cost of extreme policy choices, can be an effective tool for inducing compromise and increasing the payoff of the median voter. Simulation results suggest that patience also leads to compromise. In electorates with more widely dispersed distributions of ideal points, compromise is more likely but the median voter's expected payoff is lower nonetheless.

1.2 The equilibrium of Theorem 1 is characterized by a partition of the policy space into three regions. The first, called the *win set,* is an interval around zero consisting of the ideal points of "centrists." Policies in the win set secure reelection for encumbents, so centrist officeholders simply pick their ideal points. The second region consists of the ideal points of "moderates" and the third consists of "extremist" ideal points. Unlike centrists, moderates and extremists face a tradeoff: compromise by choosing a policy in the win set, or choose their own ideal points and forgo reelection in the following period. We show that the incentives for politicians to compromise are inversely related to the distance from their ideal points to the median. Thus, moderate politicians are willing to compromise, choosing the policy in the win set closest to their own ideal points, while extremist politicians with are not. These officeholders pick their own ideal points, forgoing reelection.

The partition given by Theorem 1 is consistent with the incentives of voters in the following sense. Given the partition, each voter compares the payoff expected upon reelecting the incumbent to that expected following the election of a challenger, using whatever information is available: namely, the incumbent's record of policy choices and the distribution of the challenger's ideal point. Voters do not directly observe the preferences of political candidates. After making this comparison, the voter votes for the incumbent if and only if his "continuation value" is at least as great as the challenger's. A policy choice by the incumbent leads to reelection if and only if it wins the votes of at least half of the electorate, and these policies comprise the win set — the centrally located interval described above.

An unexpected property of the win set is that it consists precisely of the policies "acceptable" to the median voter. In other words, the expected payoff of an incumbent to the median voter matches that of a challenger exactly when the incumbent's past policies are chosen from the win set. As the equilibrium is played out, policy outcomes may initially lie in the extreme regions of the issue space as left- or right-wing politicians take office, but eventually a centrist or moderate will take office and keep it, giving the median voter a stream of acceptable policy outcomes.

Because challengers are inherently riskier than incumbents, however, they are at a disadvantage in winning votes from a risk averse electorate. This gives incumbents a degree of lattitude in their policy choices: an incumbent whose record indicates a sufficient degree of moderation will offer the median voter a higher payoff in expectation than a challenger, even if the incumbent is not expected to choose exactly the median voter's ideal point. Though long run equilibrium outcomes will be acceptable to the median voter, they won't necessarily be ideal. The median voter theorem does not apply with full force.

1.3 To simplify the analysis we assume that challengers are drawn at random from the electorate. Thus, voters know nothing about challengers except the distribution of ideal points from which they are drawn, which is identical to the distribution across the electorate. In practice this is surely not the case, but our assumption captures the idea that voters are likely to be better informed of an incumbent's policy preferences than those of a challenger untested in the office for which he is running. We also omit the details of electoral campaigns, in which candidates announce the policies they plan to implement upon winning election. In our model, such announcements would be "cheap talk" and can only expand the set of equilibria — Theorems 1 and 2 (appropriately modified) would hold in the model augmented with cheap talk campaigns.

1.4 In the equilibrium of Theorem 1, voters are *prospective* in the sense that each voter formulates expectations of the future and acts accordingly. More precisely, each voter compares the expected payoff of reelecting the incumbent with the expected payoff of electing the challenger and votes for the incumbent if the former is higher than the latter. But voter behavior is also consistent with *retrospective voting.* Informally, we regard a voter as retrospective if his vote is directly determined on the basis of past experience, without recourse to complex calculations of expected payoffs. This is reminescent of both the reward-punishment theory of voting, originating with the writings of Key (1966), and the Downsian model of voting, where the voter uses his past experience (in a simple way) to predict the future. Fiorina (1981) distinguishes between reward-punishment and Downsian retrospective voting, but his distinction involves psychological nuances that are difficult to formalize in a game-theoretic model of elections. We are not so concerned with the voter's frame of mind as we are with the strategy that represents his behavior.

As is standard in the literature, discussed below, we formally define a voter as retrospective if there is some fixed payoff level $\overline{u}$ such that he votes for the incumbent if and only if the incumbent chose a policy outcome in the preceding period that yields the voter a payoff of at least $\overline{u}$. Such a voting rule requires no complex calculations of expected payoffs, as our informal definition stipulates. Downs (1957) saw that in some situations (those marked by stationarity, or in Downs's terminology, "continuity") complex calculations are not needed to derive expected payoffs: past performance of the incumbent indicates his future performance. Theorem 1 formalizes this insight in our game-theoretic setting, showing that retrospective voting can arise in equilibrium among a prospective voting electorate.

1.5 The literature on repeated elections with asymmetric information features two types of models: adverse selection models (politicians have privately known preferences or, more generally, *types*) and moral hazard models (politicians take hidden actions). Campaigns are omitted from the analysis. We now offer a review of work, related to ours, in the adverse selection framework.[2]

Banks and Sundaram (1990) examine the decision problem of a representative voter who must decide whether to reelect incumbents in an infinite sequence of elections. The voter's payoff in each period is stochastically determined by the current officeholder's governing ability (this is privately known to the politician), with more

[2]Ferejohn (1986) and Austen-Smith and Banks (1989) consider moral hazard models in which officeholders choose unobservable levels of effort that affect voter welfare stochastically.

proficient officeholders producing higher expected payoffs — the policy decisions of officeholders are not actually modeled. Banks and Sundaram (1991) prove the existence of a simple equilibrium in a model combining moral hazard with adverse selection. Elections are infinitely repeated, officeholders choose levels of effort, and politicians differ with respect to their disutility of effort. The representative voter's preferences are increasing in an officeholder's expenditure of effort. Reed (1994) analyzes the equilibria of an adverse selection model in which politicians are restricted to at most two terms in office. Again, officeholders pick a level of effort, and a representative voter's preferences are strictly increasing in effort expenditure.

Two features distinguish the above models of adverse selection from ours. First, officeholders do not determine policy outcomes in the conventional sense of points in an issue space, but rather they choose levels of effort or make no choice at all. Though effort choice is appropriate in economic models of agency and must play some part in political decision making, it is doubtful whether the complexity of political choices is captured in this one variable. Second, voters are either explicitly assumed to have identical preferences or they are modeled using a single, representative voter. In other words, voters are *homogeneous.* This simplifies the analysis of voting behavior, but perhaps too much. Two important determinants of elections — diversity of political opinion across the electorate and strategic interaction within the electorate — cannot be captured in such models.

As we do in this paper, Reed (1989) and Bernhardt, Dubey, and Hughson (1995) consider models in which officeholders choose policy outcomes in a one-dimensional issue space and voters are *heterogeneous.* Reed shows it is possible that the median voter is worse off with more information about incumbents when politicians are limited to two terms of office, but his analysis relies on an asymmetry between the preferences of voters and politicians. Bernhardt *et al.* construct a model of infinitely repeated elections similar to ours but imposing arbitrary finite term limits. They show there exists a unique mixed strategy equilibrium when the rewards to holding office are large, but, because of "end game" effects created by term limits, there does not exist an equilibrium in pure strategies. Thus, the imposition of term limits can significantly affect the analysis. The authors show that a simple, stationary equilibrium exists when rewards of holding office are sufficiently low.

1.6 In Section 2 we present the model. In Section 3 we formalize retrospective and prospective voting in our game-theoretic framework. In Section 4 we present the main theorems of the paper. In Section 5 we present comparative statics and simulation results for our model under various specifications of discount rates, rewards for holding office, and distributions of voter ideal points. In Section 6 we discuss some directions in which our analysis might be extended. Proofs of the main theorems are supplied in two appendices.

2 The Model

The set of political outcomes is modeled as the interval $I = [-1, 1]$, and each voter's preferences are given by Euclidean distance from an ideal point. Specifically, the payoff of outcome x to a voter with ideal point $\overline{x}$ is $u_{\overline{x}}(x) = -|x - \overline{x}|$. The distribution of voter ideal points is given by a strictly positive, continuous density function f, which is assumed to be symmetric around zero. Thus, there is a continuum of voters, and the proportion of voters with ideal points in $Y \subseteq I$ is $\int_Y f(x)\,dx$. The location of a voter's ideal point is private information, but the distribution of ideal points across the electorate is public knowledge.

Before each of an infinite sequence of elections, a challenger is randomly selected from the electorate to run against the incumbent in a majority rule election.[3] No term limits are imposed on officeholders. With a continuum of voters, majority rule is defined as follows: all voters cast ballots for or against the incumbent, and the incumbent wins the election if the proportion of voters voting for the incumbent is at least one half.[4] The winner chooses a point in I, which determines payoffs to voters for that period according to the utility function defined above. If a winner with ideal point $\overline{x}$ chooses x, he receives a payoff of $u_{\overline{x}}(x) + \rho$, where ρ is a non-negative monetary reward for holding office.

Voters are infinitely-lived and payoffs from future political outcomes are dis-

[3]In the first period, a randomly chosen voter may be placed in office.

[4]Note that no abstention is allowed and a tie-breaking rule in favor of incumbents is used.

counted at the common rate $\delta < 1$. Thus, the payoff to a voter with ideal point $\bar{x}$ of a sequence $\{x_k\}$ of outcomes is

$$(1-\delta)\sum_{k=1}^{\infty}\delta^{k-1}u_{\bar{x}}(x_k),$$

where, for convenience, we have normalized payoffs by $(1-\delta)$. If the voter holds office at any time, the discounted reward for holding office should be added to this payoff.

The strategies of voters must specify a vote for or against the incumbent in each period, and the winner's strategy must also specify a policy outcome to be imposed. In equilibrium these policy choices will certainly depend on the winner's ideal point. They may also be conditioned on the history of electoral returns and policy choices, but they need not be. If all officeholders with the same ideal point use the same history independent, or *stationary,* policy choice rule, the rule can be represented by a function $p_{\bar{x}}$, which gives the policy choice of an officeholder with ideal point $\bar{x}$.

3 Voting Behavior

3.1 In practice, voters commonly condition their decisions on the history of electoral returns and policy outcomes — an incumbent's past policy choices may contain information particularly useful to voters. An especially simple history-dependent voting strategy is a *retrospective voting rule,* which specifies a vote for the incumbent if and only if the voter's payoff from the incumbent's immediately preceding policy choice meets a fixed utility standard. If all voters with the same ideal point use the same retrospective voting rule, it can be conveniently summarized by the function

$$v_{\bar{x}}(x) = \begin{cases} 1 & \text{if } u_{\bar{x}}(x) \geq \overline{U}_{\bar{x}} \\ 0 & \text{else,} \end{cases}$$

where x is the incumbent's policy choice in the previous period, $\overline{U}_{\bar{x}}$ is a fixed utility standard, 1 is a vote for the incumbent, and 0 is a vote for the challenger.

A retrospective voting rule determines the set (possibly empty) of policy outcomes that secure reelection for the incumbent. This is the *win set,* denoted W and given by

$$W = \left\{x \in I \,\middle|\, \int v_{\bar{x}}(x) f(\bar{x})\, d\bar{x} \geq 1/2\right\}.$$

Thus, W is just the set of policies that satisfy some winning coalition of voters. Since the standard $\overline{U}_{\overline{x}}$ is fixed through time, the win set induced by a retrospective voting rule will be constant as well. It can also be shown that W is necessarily a closed subset of I.

Under our assumptions, a retrospective voting rule is equivalent to the specification of an interval $R_{\overline{x}} = [\overline{x} - r_{\overline{x}}, \overline{x} + r_{\overline{x}}]$ around $\overline{x}$ of radius $r_{\overline{x}}$, where

$$r_{\overline{x}} = -\overline{U}_{\overline{x}}$$

and

$$v_{\overline{x}}(x) = \begin{cases} 1 & \text{if } x \in R_{\overline{x}} \\ 0 & \text{else.} \end{cases}$$

That is, a voter following a retrospective voting rule will vote for the incumbent if and only if the incumbent's last policy choice lies within a certain closed interval around the voter's ideal point. Call this interval the voter's ***retrospective set.*** We reserve the special notation R_m for the retrospective set of the median voter.

3.2 An arbitrary retrospective voting rule may be inconsistent in the following sense. Suppose $\overline{U}_{\overline{x}} = 0$ for all $\overline{x}$, i.e., the voter with ideal point $\overline{x}$ votes for the incumbent if and only if the incumbent picked $\overline{x}$ in the previous period. Clearly, $W = \emptyset$, so that incumbents are never reelected. In any equilibrium in which voting behavior is described by this rule, officeholders will pick their own ideal points. Now consider a voter with ideal point $\overline{x}$ who observes policy outcome x close, but not equal, to $\overline{x}$. This reveals the incumbent's ideal point and indicates the policy he would pick if reelected. The retrospective voting rule prescribes a vote for the challenger even if the expected outcome upon doing so is much further away from $\overline{x}$ than x. In other words, it is possible that the voter's normalized expected discounted payoff, or *continuation value,* from reelecting the incumbent is higher than his continuation value from electing a randomly drawn challenger. Intuitively, however, we would expect the voter to vote for the incumbent.

A *prospective voting rule* is one which dictates a vote in favor of the incumbent if and only if his continuation value is at least as high as that of electing a randomly selected challenger to office. Prospective voting rules, because they require the calculation of continuation values of the candidates, can be computationally quite complex. When policy choices are stationary and voters are retrospective, however, the calculation of continuation values is significantly simplified by the observation that the continuation value from electing a randomly drawn challenger is constant through time. It will depend on the win set, W, and the *compromise set,* which consists of the ideal points of politicians who are willing to compromise (if necessary) to win reelection. Formally, it is denoted by C and defined as

$$C = \{\overline{x} \in I | p_{\overline{x}} \in W\}.$$

We let $U_{\overline{x}}(W, C)$ denote the continuation value of electing a randomly chosen challenger for a voter with ideal point $\overline{x}$, given W and C.

The voter's continuation value must satisfy

$$\begin{aligned} U_{\overline{x}}(W,C) &= \Pr(\overline{z} \in C)E[u_{\overline{x}}(p_{\overline{z}})|\overline{z} \in C] + \Pr(\overline{z} \notin C)\{(1-\delta)E[u_{\overline{x}}(p_{\overline{z}})|\overline{z} \notin C] \\ &\quad +\delta U_{\overline{x}}(W,C)\}, \end{aligned}$$

where we interpret this equation is interpreted as follows. With probability $\Pr(\overline{z} \in C)$, the newly elected challenger will pick a policy $p_{\overline{z}}$ in W. He will be reelected, and since he uses a stationary policy choice rule, he will pick the same policy in the following period. Since voters use retrospective voting rules, the win set will be unchanged, and the politician will be reelected. This process continues, and the incumbent's continuation value to a voter with ideal point $\overline{x}$ is $u_{\overline{x}}(p_{\overline{z}})$. With probability $\Pr(\overline{z} \notin C)$, the newly elected challenger will pick a policy outside W, failing to win a majority in the following election. Instead, a randomly selected challenger will take office with continuation value $U_{\overline{x}}(W, C)$. This expectation is discounted by δ, since the anticipated stream of payoffs starts one period in the future. Solving, we get an expression,

$$U_{\overline{x}}(W,C) = \frac{\Pr(\overline{z} \in C)E[u_{\overline{x}}(p_{\overline{z}})|\overline{z} \in C] + \Pr(\overline{z} \notin C)(1-\delta)E[u_{\overline{x}}(p_{\overline{z}})|\overline{z} \notin C]}{1-\delta\Pr(\overline{z} \notin C)},$$

for $U_{\overline{x}}(W, C)$ that can be calculated directly.

3.3 Theorem 1, stated in Section 4, establishes that voter behavior can be consistent with both retrospective and prospective voting. To see how, consider an equilibrium in which political candidates use stationary policy choice rules and the win set W is fixed through time. It follows that the compromise set C and continuation value of a randomly selected challenger, $U_{\overline{x}}(W, C)$, are constant as well. The continuation value, denoted $V_{\overline{x}}(W, C|x)$, of an incumbent who chooses x in the previous period is

$$V_{\overline{x}}(W,C|x) = \begin{cases} u_{\overline{x}}(x) & \text{if } x \in W \\ (1-\delta)u_{\overline{x}}(x) & \text{else,} \\ +\delta U_{\overline{x}}(W,C) & \end{cases}$$

where we interpret this expression as follows. Because each politician uses a stationary policy choice rule, every voter can infer that an incumbent who chose x in period $t-1$ will choose x again if reelected in period t. If $x \in W$, the incumbent will also be reelected in period $t+1$, choose x again, and so on. The continuation value of the period t incumbent to the voter is then $u_{\overline{x}}(x)$. If $x \notin W$, the incumbent will lose the period $t+1$ election period to be replaced by a randomly selected challenger. The continuation value of the incumbent is therefore $(1-\delta)u_{\overline{x}}(x) + \delta U_{\overline{x}}(W, C)$.

By definition, a prospective voter votes for an incumbent if and only if

$$V_{\overline{x}}(W,C|x) \geq U_{\overline{x}}(W,C),$$

and from the above expression for $V_{\overline{x}}(W, C|x)$ it is readily verified that this holds if and only if

$$u_{\overline{x}}(x) \geq U_{\overline{x}}(W,C).$$

Thus, in this case, prospective voting is equivalent to a retrospective voting rule with standard $U_{\overline{x}}(W, C)$. If we calculate the win set for this retrospective voting rule, we may get back W but we may not. The problem of finding a win set that "generates itself," in this sense, is addressed next.

4 Main Results

We require politicians and other voters to adopt strategies that maximize their expected payoffs at every point in this repeated game. Because there is a continuum of voters, however, no voter will ever be pivotal in an election, so the "inconsistent" retrospective voting rule of Section 3.2 fulfills this minimal requirement. The interesting equilibria, however, are those in which voters use simple yet *intuitive* voting rules. Theorem 1 establishes the existence of such an equilibrium: officeholders use symmetric and stationary policy choice rules, and voting behavior is consistent with both retrospective and prospective voting. Furthermore, the median voter is decisive in the sense that an incumbent is reelected if and only if his continuation value to the median voter meets or exceeds the continuation value of a randomly selected challenger. Theorem 2 establishes the uniqueness of this equilibrium.

Because voters have incomplete information about the preferences of other voters and elections are repeated through time, the appropriate equilibrium concept for the analysis is perfect Bayesian equilibrium. A specification of strategies for the voters is a *perfect Bayesian equilibrium* if voters (including officeholders) are maximizing their expected payoffs at every point in time, where the expectation is with respect to the voters' beliefs, updated by Bayes rule when possible. If Bayes rule cannot be applied, it must be possible to assign beliefs to political candidates and other voters in such a way that they continue to maximize their expected payoffs. Appendix A contains the proof of Theorem 1, along with a precise treatment of strategies and beliefs "off the equilibrium path."

Theorem 1 (Existence) *There exists a perfect Bayesian equilibrium possessing the following properties.*

Policy Stationarity. *The policy choices of officeholders along the path of play depend only on their ideal points.*

Policy Symmetry. *Officeholders with ideal point* $\overline{x}$ *choose* $p_{\overline{x}}$ *if and only if those with ideal point* $-\overline{x}$ *choose* $-p_{\overline{x}}$.

Retrospective Voting. *Voters use retrospective voting rules along the path of play, and voters sharing ideal points use the same rule.*

Prospective Voting. *Voters vote for an incumbent if and only if his continuation value to the voter is at least as high as that of a randomly selected challenger.*

Median Decisiveness. *The win set coincides with the median voter's retrospec-*

tive set.

The equilibrium isolated in Theorem 1 is depicted graphically in Figure 1. The median voter's payoffs are drawn as a function, u_m, of policy outcomes, and the median voter's continuation value of a randomly selected challenger, denoted U_m, is given by the dashed line. The win set $W = [-w, w]$ is just the set of points that give the median voter a payoff of at least U_m, and the compromise set is indicated by $C = [-c, c]$. Centrist politicians, with ideal points in the win set, simply pick their ideal points. Moderates, with ideal points in $[-c, -w) \cup (c, w]$, choose the closest of $-w$ and w to their ideal points. And extremists, with ideal points in $[-1, -c) \cup (c, 1]$, choose their own ideal points, forgoing reelection.

[Figure 1 about here.]

The tendency of moderate politicians to be more compromising than extremists is not immediately obvious. The cost of compromise will be less for such politicians, for the endpoints of the win set are closer to their ideal points, but they also generally enjoy higher continuation values from challengers — their inclinations are ambiguous. We show in Appendix A that moderation does indeed lead to compromise.

Theorem 2 provides a uniqueness result within the class of equilibria defined by the conditions of Theorem 1, save median decisiveness. This property follows from the others, providing a version of the median voter theorem for infinitely repeated elections with asymmetric information: in the "simplest" equilibria of our infinitely repeated game, the median voter is decisive. In Theorem 2, we view as identical any two equilibria that coincide along the equilibrium path.

Theorem 2 (Uniqueness) *There is at most one perfect Bayesian equilibrium exhibiting Policy Stationarity, Policy Symmetry, Retrospective Voting, and Prospective Voting.*

In Appendix B we show that corresponding to any equilibrium of the sort specified in Theorem 2 is a win set $[-w', w']$ and compromise set $[-c', c']$ for which w' and c' solve a system of two equations. These are

$$\phi_1(w', c') = 0 \quad \phi_2(w', c') = 0.$$

An analysis of the partial derivatives of ϕ_1 and ϕ_2 reveals that these equations are solved by no other pair (w'', c''). This result strengthens the comparative statics of Section 5 and simplifies the task of numerical simulation of equilibrium, as multiple solutions are not an issue.

5 Welfare Analysis

The equilibrium of Theorem 1 is characterized by two variables, w and c, which measure the *ex ante* expected welfare of the median voter and the degree of compromise by politicians, respectively.[5] Since the equilibrium values of these variables solve a system of two equations, we can use the Implicit Function Theorem to investigate how small changes in the values of parameters affect the equilibrium win set and compromise set. It is an exercise in comparative statics to check that

$$\frac{\partial w}{\partial \rho} \leq 0 \quad \frac{\partial c}{\partial \rho} \geq 0,$$

so that an increase in the reward for holding office cannot lead to an expansion of the win set or a contraction of the compromise set: higher rewards for holding office promote compromise and increase the welfare of the median voter.[6] This provides a strong argument that monetary rewards for officeholders, by increasing the opportunity cost of extreme policy choices, can be an effective tool for inducing compromise and increasing the expected payoff to the median voter.

The partials of w and c with respect to δ are difficult to sign. An officeholder's discount rate is an important determinant of his expected payoff upon choosing his ideal point and forgoing reelection, which will be

$$(1 - \delta)\rho + \delta U_{\overline{x}}(W, C)$$

for an officeholder with ideal point $\overline{x}$. Though it is suppressed in our notation, $U_{\overline{x}}(W, C)$ itself depends on δ. Differentiating the above payoff with respect to δ,

[5]Note that, because w represents the radius of the median voter's retrospective set, an increase in w means the median voter is worse off.

[6]A change in ρ does not directly affect the continuation value of a challenger to the median voter, but, because an increase in ρ leads to a larger compromise set, the median voter's continuation value of a challenger increases with ρ.

we have

$$-\rho + U_{\bar{x}}(W,C) + \delta \frac{\partial}{\partial \delta} U_{\bar{x}}(W,C).$$

The first term in this expression represents the monetary rewards forgone as a result of losing reelection, and the second term represents the (relatively poor) policy outcomes the officeholder can expect after leaving office. Both of these terms are negative. The last term, which can be shown to be positive, represents the advantage of patience in the presence of risk aversion. An impatient voter puts relatively more weight on the possibly extreme outcomes that might follow initially upon electing an untried challenger, while the more patient voter is better able to enjoy the stream of moderate policy outcomes that will eventually occur with probability one. Though we might expect the first effects to outweigh the last, leading to an increase in c, the net effect of an increase in δ is ambiguous.

A weaker comparative statics result can be shown: for high enough $\rho \geq 0$,

$$\tfrac{\partial w}{\partial \delta} \leq 0 \quad \tfrac{\partial c}{\partial \delta} \geq 0.$$

When the rewards to holding office are high enough, greater patience increases the value of an infinite stream of rewards — at a faster rate than the officeholder's continuation value of a challenger increases. The net effect is that forgoing election becomes less attractive as δ increases, enlarging the compromise set and increasing the median voter's welfare.

[Figure 2 about here.]

For several settings of our parameters, we used numerical methods to solve for the equilibrium of Theorem 1. The simulation results in Figure 2 bear out the partial derivatives with respect to ρ and suggest that, as we suspected,

$$\tfrac{\partial w}{\partial \delta} \leq 0 \quad \tfrac{\partial c}{\partial \delta} \geq 0.$$

For these parameter settings, patience leads to a higher level of welfare for the median voter. In the extreme case, when δ is close to one and $\rho > 0$, we see that the median voter can actually expect his ideal point to be chosen by a randomly selected challenger.

Several distributions of voter ideal points were considered, including the uniform distribution and truncated normals with different variances. As the variance of the truncated normals is increased, treating the uniform distribution as the limiting case, values of w and c increase without exception. This indicates that compromise will be more frequent, but the median voter's welfare lower, in societies with more widely dispersed preferences.

6 Extensions

Our model incorporates more detail than the canonical Downsian model and fulfills its purpose as a benchmark, but it is simple in four important respects that should be addressed in future research: (*i*) the process of candidate selection, (*ii*) the powers of political office, (*iii*) voter preferences, and (*iv*) the policy space. We end with short remarks on each of these points.

(*i*) If challengers incur a positive cost by running for election, some potential candidates won't run: the voter whose ideal point is very close to the incumbent's policy choice stands little to gain by running against the incumbent. This introduces an asymmetry in the distribution of challengers and could result in a bias away from the median, because potential candidates with ideal points at the extremes of the policy space have the most to gain by running for office. Voters, knowing this, will expect lower payoffs from challengers, resulting in a larger win set and outcomes further from the median.

(*ii*) In our model, politicians are temporary dictators who can pick any policy outcome for society. A more realistic model would partition society into districts, each electing a representative, these representatives then playing a game among themselves to determine an outcome for society. In this context, we can still suppose an officeholder chooses points in I but we now interpret I as the officeholder's strategy set in the "political game." For example, if representatives $i = 1, \dots, n$ pick policies $x_1, \dots, x_n$, the outcome of the political game may be the median of the n policies. If reelection were not an issue, this game would have a pure strategy equilibrium in which politicians simply pick their ideal points. However, forward-looking politicians may have incentives to manipulate the game in order to gain reelection.

(*iii*) We conjecture that, as long as voter utility functions are weakly concave, there exist equilibria satisfying all the conditions of Theorem 1, save possibly Median Decisiveness. All that is needed to prove existence is that the win set $W(w, c)$, defined in Appendix A, varies continuously in its arguments. This is true when the median voter is decisive, as proved in Lemma 3, but should be true in general, as long as the density f is continuous. The sharp characterization of the win set as the median voter's retrospective set could fail when voters are arbitrarily risk averse, for the continuation value of a challenger could decrease too quickly as we consider voters further from the median. (See Lemmas 1-3.) As a result, the win set could be a superset of the median voter's retrospective set.

(*iv*) Finally, an extension to multi-dimensional spaces is promising but not immediate. The proof of equilibrium existence in Appendix A is greatly simplified by the simple structure of win sets and compromise sets in our one-dimensional policy space: they are intervals centered around zero, so they can be characterized by their righthand endpoints. This allows us to use Brower's Fixed Point Theorem. In higher dimensions, it may be sufficient, with appropriate symmetry in the model, to look for win sets and compromise sets among the hyperspheres centered at the median, in which case we can still characterize these sets by real numbers (their diameters) and still use Brower's theorem. However, an extension of our technique to the case of non-symmetric distributions of ideal points is not likely to be so straightforward.

A Proof of Theorem 1

Let $U_{\overline{x}}(w, c)$ denote the continuation value to the voter with ideal point $\overline{x}$ of electing a randomly selected challenger, when the win set is an interval of the form $[-w, w]$ and the compromise set is an interval of the form $[-c, c]$. Now, for arbitrary w and c with $c \geq w$, define strategies as follows. An officeholder with ideal point $\overline{x}$ chooses policy $p_{\overline{x}}$ defined by

$$p_{\overline{x}} = \begin{cases} \arg\max_{x \in W} u_{\overline{x}}(x) & \text{if } \overline{x} \in C \\ \overline{x} & \text{else,} \end{cases}$$

if he has not chosen a policy outcome outside $[-w, w]$ in the previous period. Otherwise, the officeholder chooses $\overline{x}$. Letting x denote the incumbent's most recent policy choice, the voter with ideal point $\overline{x}$ votes for the incumbent if and only if $u_{\overline{x}}(x) \geq U_{\overline{x}}(w, c)$. Voters update using Bayes rule along the path of play. Following "impossible" histories, we must supply voters with (reasonable) beliefs about the incumbent's preferences: if an officeholder's decisions are inconsistent with $p_{\overline{x}}$, voters believe the officeholder's last policy choice is his ideal point; inconsistencies exhibited by earlier officeholders or election outcomes are ignored.

Strategies of this form clearly exhibit Policy Symmetry and Retrospective Voting. They exhibit Policy Stationarity as well, since, along the path of play, an officeholder with ideal point $\overline{x}$ simply picks $p_{\overline{x}}$. Median Decisiveness will follow after a sequence of three lemmas. Once this is done, we will find a specification of w and c (the fixed points of a certain mapping) for which the corresponding strategies possess the Prospective Voting property and constitute a perfect Bayesian equilibrium.

Lemma 1 shows that the continuation value of a randomly selected challenger is highest for the median voter and decreases monotonically with the distance of a voter's ideal point from the median. The rate of decrease is bounded below by -1. To facilitate the proof of Theorem 2 in Appendix B, we allow C to be any set centered around zero in each of the lemmas — it need not be connected.

Lemma 1 *Fix $W = [-w, w]$ and $C \supseteq W$ centered around zero. If $\overline{x} \in [-1, 0]$ then*

$$0 \leq \frac{\partial}{\partial x}\bigg|_{\overline{x}} U_x(W, C) \leq 1,$$

and if $\overline{x} \in [0, 1]$ then

$$-1 \leq \frac{\partial}{\partial x}\bigg|_{\overline{x}} U_x(W, C) \leq 0.$$

Proof: It will be convenient to work with the notation

$$\begin{aligned} E_{\overline{x}}(W, C|\overline{z} \in C) &= \Pr(\overline{z} \in C) E[u_{\overline{x}}(p_{\overline{z}})|\overline{z} \in C] \\ E_{\overline{x}}(W, C|\overline{z} \notin C) &= \Pr(\overline{z} \in C) E[u_{\overline{x}}(p_{\overline{z}})|\overline{z} \notin C], \end{aligned}$$

which brings out the dependence of these expectations on the parameters W and C. Then

$$U_{\overline{x}}(W,C) \;=\; \frac{E_{\overline{x}}(W,C|\overline{z}\in C)+(1-\delta)E_{\overline{x}}(W,C|\overline{z}\notin C)}{1-\delta\Pr(\overline{z}\notin C)},$$

and differentiating $U_{\overline{x}}(w,c)$ with respect to $\overline{x}$ yields

$$\left.\frac{\partial}{\partial x}\right|_{\overline{x}} U_x(W,C) \;=\; \frac{\left.\frac{\partial}{\partial x}\right|_{\overline{x}} E_x(W,C|\overline{z}\in C)+(1-\delta)\left.\frac{\partial}{\partial x}\right|_{\overline{x}} E_x(W,C|\overline{z}\notin C)}{1-\delta\Pr(\overline{z}\notin C)}.$$

The exact form of this partial derivative depends on which of three regions of I contains the voter's ideal point $\overline{x}$: $\overline{C}$, $C\setminus W$, or W. ($\overline{C}$ denotes the complement of C in I.) We consider these cases for $\overline{x}\leq 0$, establishing the first part of the lemma. The second follows by symmetric arguments.

Case 1: Suppose $\overline{x}\in[-1,0]\cap\overline{C}$. Then

$$\begin{aligned} E_{\overline{x}}(W,C|\overline{z}\in C) \;&=\; \int_{[-1,0]\cap(C\setminus W)}(w+\overline{x})f(\overline{z})\,d\overline{z}+\int_W -(\overline{z}-\overline{x})f(\overline{z})\,d\overline{z} \\ &\quad+\int_{[0,1]\cap(C\setminus W)} -(w-\overline{x})f(\overline{z})\,d\overline{z} \end{aligned}$$

$$E_{\overline{x}}(W,C|\overline{z}\notin C) \;=\; \int_{\overline{C}\cap[-1,\overline{x}]} -(\overline{x}-\overline{z})f(\overline{z})\,d\overline{z}+\int_{\overline{C}\cap[\overline{x},1]} -(\overline{z}-\overline{x})f(\overline{z})\,d\overline{z}$$

Differentiating these expressions with respect to $\overline{x}$ yields

$$\begin{aligned} \left.\frac{\partial}{\partial x}\right|_{\overline{x}} E_x(W,C|\overline{z}\in C) \;&=\; \int_{[-1,0]\cap(C\setminus W)} f(\overline{z})\,d\overline{z}+\int_W f(\overline{z})\,d\overline{z}+\int_{[0,1]\cap(C\setminus W)} f(\overline{z})\,d\overline{z} \\ &=\; \Pr(\overline{z}\in C) \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial}{\partial x}\right|_{\overline{x}} E_x(W,C|\overline{z}\notin C) \;&=\; \int_{\overline{C}\cap[-1,\overline{x}]} -f(\overline{z})\,d\overline{z}+\int_{\overline{C}\cap[\overline{x},1]} f(\overline{z})\,d\overline{z} \\ &=\; \Pr(\overline{z}\in\overline{C}\cap[\overline{x},-\overline{x}]), \end{aligned}$$

where the last equality follows by the symmetry of C and f. Substituting into the above expression for $\left.\frac{\partial}{\partial x}\right|_{\overline{x}} U_x(W,C)$ yields

$$\left.\frac{\partial}{\partial x}\right|_{\overline{x}} U_x(W,C)=\frac{\Pr(\overline{z}\in C)+(1-\delta)\Pr(\overline{z}\in\overline{C}\cap[\overline{x},-\overline{x}])}{1-\delta\Pr(\overline{z}\notin C)},$$

Differentiating these expressions with respect to $\overline{x}$ yields

$$\begin{aligned}\left.\frac{\partial}{\partial x}\right|_{\overline{x}} E_x(W,C|\overline{z}\in C) &= \int_{[-1,0]\cap(C\setminus W)} -f(\overline{z})\,d\overline{z} + \int_{[-1,\overline{x}]\cap W} -f(\overline{z})\,d\overline{z} \\ &\quad + \int_{[\overline{x},1]\cap W} f(\overline{z})\,d\overline{z} + \int_{[0,1]\cap(C\setminus W)} f(\overline{z})\,d\overline{z} \\ &= \Pr(\overline{z}\in[\overline{x},1]\cap C) - \Pr(\overline{z}\in[-1,\overline{x}]\cap C)\end{aligned}$$

$$\begin{aligned}\left.\frac{\partial}{\partial x}\right|_{\overline{x}} E_x(W,C|\overline{z}\notin C) &= \int_{[-1,0]\cap\overline{C}} -f(\overline{z})\,d\overline{z} + \int_{[0,1]\cap\overline{C}} f(\overline{z})\,d\overline{z} \\ &= 0,\end{aligned}$$

where the last equality follows from the symmetry of C and f. Substituting into the expression for $\left.\frac{\partial}{\partial x}\right|_{\overline{x}} U_x(W,C)$ yields

$$\left.\frac{\partial}{\partial x}\right|_{\overline{x}} U_x(W,C) = \frac{\Pr(\overline{z}\in[\overline{x},1]\cap C) - \Pr(\overline{z}\in[-1,\overline{x}]\cap C)}{1-\delta\Pr(\overline{z}\notin C)},$$

which, by symmetry of C and f, satisfies the desired inequalities. ∎

Let $R_{\overline{x}}(W,C)$ be the retrospective set of the voter with ideal point $\overline{x}$ with utility standard $U_{\overline{x}}(W,C)$. That is,

$$R_{\overline{x}}(W,C) = [\overline{x} - r_{\overline{x}}(W,C), \overline{x} + r_{\overline{x}}(W,C)],$$

where $r_{\overline{x}}(W,C) = -U_{\overline{x}}(W,C)$. Lemma 2 builds on Lemma 1, exploring the centrality of the median voter.

Lemma 2 *Fix $W=[-w,w]$ and $C\supseteq W$ centered around zero. If $\overline{x}\in[-1,0]$ then*

$$\overline{x} + r_{\overline{x}}(W,C) \le r_m(W,C),$$

and if $\overline{x}\in[0,1]$ then

$$\overline{x} - r_{\overline{x}}(W,C) \ge -r_m(W,C).$$

Proof: We prove the lemma for $\overline{x}\le 0$, the remaining case following by a similar argument. Note that the voter with ideal point -1 is risk neutral with respect to policy outcomes, so that $U_{-1}(W,C)=-1$. Then

$$r_{\overline{x}}(W,C) = -U_{\overline{x}}(W,C) = 1 - \int_{-1}^{\overline{x}} \left.\frac{\partial}{\partial x}\right|_{\overline{z}} U_x(W,C)\,d\overline{z},$$

after appropriate manipulation. Then Lemma 1 implies

$$r_{\overline{x}}(W,C) \geq 1 - \int_{-1}^{\overline{x}} d\overline{z} = -\overline{x},$$

and $\overline{x} + r_{\overline{x}}(W,C) \geq 0$. To see the second part of the claim, note that

$$\begin{aligned}\overline{x} + r_{\overline{x}}(W,C) - r_m(W,C) &= \overline{x} - U_{\overline{x}}(W,C) + U_m(W,C) \\ &= -\int_{\overline{x}}^{0} 1 - \frac{\partial}{\partial x}\bigg|_{\overline{z}} U_x(W,C)\, d\overline{z},\end{aligned}$$

after appropriate manipulation. Then Lemma 1 implies $\overline{x}+r_{\overline{x}}(W,C)-r_m(W,C) \leq 0$.

∎

Finally, let $\mathcal{W}(W,C)$ denote the win set induced by retrospective voting with utility standards $U_{\overline{x}}(W,C)$ and the policy choice rule $p_{\overline{x}}$ defined above. Lemma 3 shows that, as long as $W = [-w,w]$ for some $w > 0$ and C is centered around zero, the median voter is decisive in the sense of Theorem 1. This establishes Median Decisiveness of the strategies specified above.

Lemma 3 *Fix $W = [-w,w]$ and $C \supseteq W$ centered around zero. Then $\mathcal{W}(W,C) = R_m(W,C)$.*

Proof: Take any $x \in R_m(W,C)$, and assume without loss of generality that $x < 0$. From Lemma 2, $0 \in R_{\overline{x}}(W,C)$ for all $\overline{x} \in [-1,x]$, which implies that $x \in R_{\overline{x}}(W,C)$ for all such voters. Now consider a voter with ideal point $\overline{x} \in [x,0]$. Lemma 1 gives us $r_{\overline{x}}(W,C) \geq r_m(W,C)$, which implies $\overline{x} - r_{\overline{x}}(W,C) \leq -r_m(W,C) \leq x$, and $x \in R_{\overline{x}}(W,C)$. Therefore, x is in the retrospective sets of at least half the electorate, and $x \in \mathcal{W}(W,C)$.

Now take any $x \notin R_m(W,C)$, and assume without loss of generality that $x > 0$. That is, $x > r_m(W,C)$. From Lemma 2, $\overline{x} + r_{\overline{x}}(W,C) \leq r_m(W,C) < x$ for all $\overline{x} \in [-1,0]$. Continuity of $U_{\overline{x}}(W,C)$ with respect to $\overline{x}$ then implies that this statement is true for all $\overline{x} \in [-1,\epsilon]$ for some $\epsilon > 0$. Since the density f is strictly positive, x is in the retrospective sets of strictly less than half the electorate, and $x \notin \mathcal{W}(W,C)$. ∎

Of course, the lemmas hold when $C = [-c, c]$ for $c \geq w \geq 0$, in which case we denote $R_{\overline{x}}(W, C)$ by $R_{\overline{x}}(w, c)$ and $W(W, C)$ by $W(w, c)$. Note that each $\overline{x}$ defines a mapping $U_{\overline{x}} : S \to [-1, 0]$, where $S = \{(w, c) \in [0, 1] \times [0, 1] | w \leq c\}$. Use this map to define $\psi = (\psi_1, \psi_2) : S \to S$ as

$$\begin{aligned}
\psi_1(w, c) &= -U_m(w, c) \\
\psi_2(w, c) &= \min\{\max\{w + \delta\rho - \delta U_c(w, c), -U_m(w, c)\}, 1\}.
\end{aligned}$$

Since S is compact and convex and ψ is continuous, Brower's Theorem yields a fixed point $(w^*, c^*) \in S$. We claim that the strategies corresponding to w^* and c^* possess the Prospective Voting property. By construction, the voter with ideal point $\overline{x}$ votes for the incumbent if and only if $u_{\overline{x}}(x) \geq U_{\overline{x}}(w^*, c^*)$. Since (w^*, c^*) is a fixed point, $w^* = -U_m(w^*, c^*)$, so Lemma 3 implies that $W(w^*, x^*) = [-w^*, w^*]$. Therefore, each voter's utility standard, $U_{\overline{x}}(w^*, c^*)$, is set at the continuation value of an untried challenger.

These strategies constitute a perfect Bayesian equilibrium, then, if the strategies for officeholders are optimal at every point in the repeated game. If an officeholder has chosen a point outside $[-w^*, w^*]$ in the previous period and finds himself in office (this does not happen on the path of play), he knows from the strategies used by voters that he will not be reelected again. Therefore, it is optimal for the officeholder to choose his ideal point, as specified above. Otherwise, the officeholder's policy choice is given by $p_{\overline{x}}$ (parameterized by w^* and c^*). We must verify that, as dictated by the strategies under consideration, (1) an officeholder with ideal point $\overline{x} \in (c^*, 1]$ optimally chooses his own ideal point, (2) an officeholder with ideal point $\overline{x} \in (w^*, c^*]$ optimally compromises, and (3) symmetric conditions hold for officeholders with negative ideal points.

We prove (1) and (2) for $\overline{x} \geq 0$, leaving the remaining case for a symmetric argument. Let $\gamma_{\overline{x}}$ denote the continuation value to an officeholder with ideal point $\overline{x} > w^*$ from compromise, defined by

$$\gamma_{\overline{x}} = w^* - \overline{x} + \rho,$$

and let $\sigma_{\overline{x}}$ denote the continuation value to the officeholder from choosing his own ideal point and forgoing election, defined by

$$\sigma_{\overline{x}} = (1-\delta)\rho + \delta U_{\overline{x}}(w^*, c^*).$$

There are two cases: $c^* < 1$ and $c^* = 1$. In the first case, note that $\gamma_{c^*} - \sigma_{c^*} = 0$, and

$$\begin{aligned} \gamma_{\overline{x}} - \sigma_{\overline{x}} &= (\gamma_{c^*} - \sigma_{c^*}) + \int_{c^*}^{\overline{x}} \frac{\partial}{\partial x}\bigg|_{\overline{z}} (\gamma_x - \sigma_x)\, d\overline{z} \\ &= \int_{c^*}^{\overline{x}} -1 - \delta \frac{\partial}{\partial x}\bigg|_{\overline{z}} U_x(w^*, c^*)\, d\overline{z}. \end{aligned}$$

By Lemma 1 and $\delta < 1$, the integrand above is strictly negative, as required. In the second case, $\gamma_{c^*} - \sigma_{c^*} \geq 0$, and the above argument shows that $\gamma_{\overline{x}} - \sigma_{\overline{x}} \geq 0$ for all $\overline{x} < 1$. Thus, the proof is complete.

B Proof of Theorem 2

Consider any perfect Bayesian equilibrium exhibiting the properties of Theorem 2: Policy Stationarity, Policy Symmetry, Retrospective Voting, and Prospective Voting. We noted in Section 3 that the win set is closed. Because officeholders' policy choices are stationary and symmetric, the win set and compromise set must be symmetric. Furthermore, W is connected. To see this, suppose $x \in W$ with $x > 0$, take any y with $x > y > 0$, and partition the set of voters who find x acceptable as follows:

$$\begin{aligned} V_x^1 &= \{\overline{x} \in [y,1] | u_{\overline{x}}(x) \geq U_{\overline{x}}(W,C)\} \\ V_x^2 &= \{\overline{x} \in [-1,y] | u_{\overline{x}}(x) \geq U_{\overline{x}}(W,C)\}. \end{aligned}$$

We can also partition the set of voters who find y acceptable:

$$\begin{aligned} V_y^1 &= \{\overline{x} \in [y,1] | u_{\overline{x}}(x) \geq U_{\overline{x}}(W,C)\} \\ V_y^2 &= \{\overline{x} \in [-1,y] | u_{\overline{x}}(x) \geq U_{\overline{x}}(W,C)\}. \end{aligned}$$

Clearly, $V_x^2 \subseteq V_y^2$, since y is preferred to x for these voters. Claim: $0 \in R_{\overline{x}}(W,C)$ for all $\overline{x}$. This follows because the voter with ideal point $\overline{x}$ is risk averse, so the continuation value of a challenger must be no higher than the voter's utility of the expected outcome, which is zero in a symmetric equilibrium. Therefore, $V_y^1 = [y,1] \supseteq V_x^1$, and $y \in W$. We conclude that $W = [-w', w']$ for some $w' \geq 0$.

Lemma 3 then implies Median Decisiveness of this equilibrium. We also claim that C is connected, for in equilibrium $\overline{x} \in C$ implies $(1-\delta)\rho + \delta U_{\overline{x}}(W,C) \leq u_{\overline{x}}(w) + \rho$. Inspection of the expression for $\frac{\partial}{\partial x}\Big|_{\overline{x}} U_x(W,C)$ in the proof of Lemma 1 shows that this expression is strictly negative as long as $\Pr(\overline{z} \in C) > 0$, which must hold if the median voter is decisive: $\Pr(\overline{z} \in C) = 0$ implies that $U_m(W,C) = 0$, or, that the median voter's continuation value from a challenger is zero. But then, because f is strictly positive, $C = [-1,1]$, a contradiction. Therefore, $(1-\delta)\rho + \delta U_{\overline{z}}(W,C) < u_{\overline{z}}(w) + \rho$ for all $\overline{z} < \overline{x}$, which implies $\overline{z} \in C$. We conclude that $C = [-c', c']$ for some $c' \geq 0$.

Define $\phi = (\phi_1, \phi_2) : S \to \Re$ as

$$\begin{aligned}\phi_1(w,c) &= w + U_m(w,c)\\ \phi_2(w,c) &= c - w - \delta\rho + \delta U_c(w,c),\end{aligned}$$

and note that $\phi_1(w',c') = 0$ follows directly from Median Decisiveness. Also, because f is strictly positive, it must be that $w' < 1$. The marginal compromising officeholder must be indifferent in equilibrium between compromise and choosing his own ideal point, so $\phi_2(w',c') = 0$ follows as long as $0 < c' < 1$. This indifference condition need not hold in the boundary cases $c' = 0$ and $c' = 1$, but we already noted that $c' = 0$ is impossible. If $c' = 1$, the officeholder with ideal point $\overline{x} = 1$ does not strictly prefer to compromise, so $\phi_2(w',c') \leq 0$.

Now consider any other specification of strategies possessing the properties of Theorem 2 and constituting a perfect Bayesian equilibrium. They must correspond to a pair $(w'',c'') \neq (w',c')$ such that $\phi_1(w'',c'') = 0$ and, if $c'' < 1$, $\phi_2(w'',c'') = 0$. An analysis of the partial derivatives of ϕ will show this entails a contradiction. First we will prove a lemma on the marginal compromising officeholder's continuation value of electing an untried challenger.

Lemma 4 *Fix w and $c \geq w$. Then $U_c(w,c) \leq -c$.*

Proof: Policy Symmetry and inspection of u_c reveal that

$$\begin{aligned}E[u_c(p_{\overline{z}})|\overline{z} \in C] &\leq -c\\ E[u_c(p_{\overline{z}})|\overline{z} \notin C] &\leq -c,\end{aligned}$$

where $p_{\overline{z}}$ is the policy choice rule corresponding to w and c. Therefore, using the expression for $U_c(w,c)$ from Section 3,

$$\begin{aligned} U_c(w,c) &\leq \frac{-c\Pr(\overline{z}\in C)-c\Pr(\overline{z}\notin C)(1-\delta)}{1-\delta\Pr(\overline{z}\notin C)} \\ &= \frac{-c+c\delta\Pr(\overline{z}\notin C)}{1-\delta\Pr(\overline{z}\notin C)} \\ &= -c, \end{aligned}$$

as desired. ∎

In the next lemma, note that, by continuity, inequalities (1) through (4) hold weakly when $c=1$.

Lemma 5 *Fix w and c with $1>c\geq w$. Then*

(1) $$\left.\frac{\partial}{\partial a}\right|_w \phi_1(a,c) > 0$$

(2) $$\left.\frac{\partial}{\partial b}\right|_c \phi_1(w,b) > 0$$

(3) $$\left.\frac{\partial}{\partial a}\right|_w \phi_2(a,c) = -1$$

(4) $$\left.\frac{\partial}{\partial b}\right|_c \phi_2(w,b) > 0,$$

where equation (2) assumes $\phi_2(w,c)\geq 0$.

Proof: We now denote $E_{\overline{x}}(W,C|\overline{z}\in C)$ by $E_{\overline{x}}(w,c|\overline{z}\in C)$, and $E_{\overline{x}}(W,C|\overline{z}\notin C)$ by $E_{\overline{x}}(w,c|\overline{z}\notin C)$. Differentiating the expression for $U_{\overline{x}}(w,c)$ in the proof of Lemma 1, we have

$$\left.\frac{\partial}{\partial a}\right|_w U_m(a,c) = \frac{\left.\frac{\partial}{\partial a}\right|_w E_m(a,c|\overline{z}\in C)+(1-\delta)\left.\frac{\partial}{\partial a}\right|_w E_m(a,c|\overline{z}\notin C)}{1-\delta\Pr(\overline{z}\notin C)}$$

$$\begin{aligned}\left.\frac{\partial}{\partial b}\right|_c U_m(w,b) &= \frac{\left.\frac{\partial}{\partial b}\right|_c E_m(w,b|\overline{z}\in[-b,b])+(1-\delta)\left.\frac{\partial}{\partial b}\right|_c E_m(w,b|\overline{z}\notin[-b,b])}{1-\delta\Pr(\overline{z}\notin C)} \\ &\quad -\frac{U_m(w,c)\left.\frac{\partial}{\partial b}\right|_c \Pr(\overline{z}\notin[-b,b])}{1-\delta\Pr(\overline{z}\notin C)}.\end{aligned}$$

We now verify the desired inequalities.

To verify (1), note that

$$E_m(w,c|\bar{z}\in C) = \int_{-c}^{-w} -wf(\bar{z})\,d\bar{z} + \int_{-w}^{w} -\bar{z}f(\bar{z})\,d\bar{z} + \int_{w}^{c} -wf(\bar{z})\,d\bar{z}$$

$$E_m(w,c|\bar{z}\notin C) = \int_{-1}^{-c} -\bar{z}f(\bar{z})\,d\bar{z} + \int_{c}^{1} -\bar{z}f(\bar{z})\,d\bar{z}.$$

Differentiating with respect to w yields

$$\begin{aligned} \left.\frac{\partial}{\partial a}\right|_w E_m(a,c|\bar{z}\in C) &= -2\Pr(\bar{z}\in[w,c]) \\ \left.\frac{\partial}{\partial a}\right|_w E_m(a,c|\bar{z}\notin C) &= 0, \end{aligned}$$

where we use the symmetry of f. After substitution,

$$\left.\frac{\partial}{\partial a}\right|_w U_m(a,c) = \frac{-2\Pr(\bar{z}\in[w,c])}{1-\delta\Pr(\bar{z}\notin C)}.$$

This quantity is strictly greater than -1 if and only if

$$1 > \delta\Pr(\bar{z}\notin C) + 2\Pr(\bar{z}\in[w,c]),$$

which follows, since f is strictly positive, if and only if $c<1$. Then

$$\left.\frac{\partial}{\partial a}\right|_w \phi_1(a,c) = 1 + \left.\frac{\partial}{\partial a}\right|_w U_m(a,c)$$

yields (1).

To verify (2), note that

$$\begin{aligned} \left.\frac{\partial}{\partial b}\right|_c E_m(w,b|\bar{z}\in[-b,b]) &= -2wf(c) \\ \left.\frac{\partial}{\partial b}\right|_c E_m(w,b|\bar{z}\notin[-b,b]) &= 2cf(c) \\ \left.\frac{\partial}{\partial b}\right|_c \Pr(\bar{z}\notin[-b,b]) &= -2f(c). \end{aligned}$$

After substitution,

$$\left.\frac{\partial}{\partial b}\right|_c U_m(w,b) = \frac{-2wf(c)+(1-\delta)2cf(c)-2\delta f(c)U_m(w,c)}{1-\delta\Pr(\bar{z}\notin C)},$$

which is strictly greater than zero if and only if

$$c - w - \delta c > \delta U_m(w, c).$$

Since $\phi_2(w, c) \geq 0$, this is implied by

$$\delta\rho - \delta U_c(w, c) - \delta c > \delta U_m(w, c),$$

which, by Lemma 4, is implied by $\delta\rho > \delta U_m(w, c)$. The latter condition follows when $c < 1$, for then, because f is strictly positive, $U_m(w, c) < 0$. Then

$$\left.\frac{\partial}{\partial b}\right|_c \phi_1(w, b) = \left.\frac{\partial}{\partial b}\right|_c U_m(w, b)$$

yields (2).

Differentiating $U_c(w, c)$, we have

$$\begin{aligned}
\left.\frac{\partial}{\partial a}\right|_w U_c(a, c) &= \frac{\left.\frac{\partial}{\partial a}\right|_w E_c(a, c|\overline{z} \in C) + (1 - \delta) \left.\frac{\partial}{\partial a}\right|_w E_c(a, c|\overline{z} \notin C)}{1 - \delta \Pr(\overline{z} \notin C)} \\
\left.\frac{\partial}{\partial b}\right|_c U_b(w, b) &= \frac{\left.\frac{\partial}{\partial b}\right|_c E_b(w, b|\overline{z} \in [-b, b]) + (1 - \delta) \left.\frac{\partial}{\partial b}\right|_c E_b(w, b|\overline{z} \notin [-b, b])}{1 - \delta \Pr(\overline{z} \notin C)} \\
&\quad - \frac{U_c(w, c) \left.\frac{\partial}{\partial b}\right|_c \Pr(\overline{z} \notin [-b, b])}{1 - \delta \Pr(\overline{z} \notin C)}.
\end{aligned}$$

Note that the partial derivative of $U_c(w, c)$ with respect to c is taken through both the subscript and the second argument.

To verify (3), note that

$$\begin{aligned}
E_c(w, c|\overline{z} \in C) &= \int_{-c}^{-w} (-w - c) f(\overline{z})\, d\overline{z} + \int_{-w}^{w} (-c + \overline{z}) f(\overline{z})\, d\overline{z} + \int_{w}^{c} (w - c) f(\overline{z})\, d\overline{z} \\
E_c(w, c|\overline{z} \notin C) &= \int_{-1}^{-c} (-c + \overline{z}) f(\overline{z})\, d\overline{z} + \int_{c}^{1} (c - \overline{z}) f(\overline{z})\, d\overline{z}
\end{aligned}$$

Differentiating with respect to w yields

$$\begin{aligned}
\left.\frac{\partial}{\partial a}\right|_w E_c(a, c|\overline{z} \in C) &= 0 \\
\left.\frac{\partial}{\partial a}\right|_w E_c(a, c|\overline{z} \notin C) &= 0.
\end{aligned}$$

After substitution, we have $\frac{\partial}{\partial a}\Big|_w U_c(a,c) = 0$, and

$$\frac{\partial}{\partial a}\bigg|_w \phi_2(a,c) \;=\; -1 + \delta \frac{\partial}{\partial a}\bigg|_w U_c(a,c)$$

yields (3).

Note that (4) holds trivially if $\delta = 0$. To verify (4) in case $\delta > 0$, note that

$$\begin{aligned}\frac{\partial}{\partial b}\bigg|_c E_b(w,b|\overline{z} \in [-b,b]) &= -2cf(c) - \Pr(\overline{z} \in C)\\ \frac{\partial}{\partial b}\bigg|_c E_b(w,b|\overline{z} \notin [-b,b]) &= 2cf(c)\\ \frac{\partial}{\partial b}\bigg|_c \Pr(\overline{z} \notin [-b,b]) &= -2f(c).\end{aligned}$$

Then, after simplification,

$$\frac{\partial}{\partial b}\bigg|_c U_b(w,b) > -\frac{1}{\delta}$$

if and only if

$$-2\delta f(c)(U_c(w,c)+c) > 1 - \frac{1}{\delta}.$$

This holds by Lemma 4 and $\delta < 1$, and then

$$\frac{\partial}{\partial b}\bigg|_c \phi_2(w,b) \;=\; 1 + \delta\frac{\partial}{\partial c}U_c(w,c)$$

yields (4). ∎

The proof of Theorem 2 consists of several cases. *Case 1:* Suppose that $w' \neq w''$, and without loss of generality let $w' < w''$.

Case 1.1: Suppose $c'' \geq c'$. Note that $c' < 1$, for otherwise $c' = c'' = 1$ would imply $w' = w'' = 0$. Thus, $\phi_2(w',c') = 0$ and part (4) of Lemma 5 implies $\phi_2(w',c) \geq 0$ for all $c \in [c',c'')$. Then (2) implies $\frac{\partial}{\partial b}\Big|_c \phi_1(w',b) > 0$ for all $c \in (c',c'')$, and it follows that

$$\phi_1(w',c'') - \phi_1(w',c') = \int_{c'}^{c''} \frac{\partial}{\partial b}\bigg|_c \phi_1(w',b)\,dc > 0.$$

Furthermore, (1) implies

$$\phi_1(w'',c'') - \phi_1(w',c'') = \int_{w'}^{w''} \frac{\partial}{\partial a}\bigg|_w \phi_1(a,c'')\,dw \geq 0,$$

which yields

$$\begin{aligned}\phi_1(w'',c'') &= \phi_1(w'',c'') - \phi_1(w',c') \\ &= \phi_1(w'',c'') - \phi_1(w',c'') + \phi_1(w',c'') - \phi_1(w',c') \\ &> 0,\end{aligned}$$

a contradiction.

Case 1.2: Suppose $c'' < c'$. Note that, by (4), $\phi_2(w',c') - \phi_2(w',c'') > 0$. Also

$$\begin{aligned}\phi_2(w'',c'') - \phi_2(w',c'') &= \int_{w'}^{w''} \frac{\partial}{\partial a}\bigg|_w \phi_2(a,c'')\,dw \\ &= w' - w'' \\ &> 0.\end{aligned}$$

Then, since $\phi_2(w',c') \leq 0$,

$$\begin{aligned}\phi_2(w'',c'') &\leq \phi_2(w'',c'') - \phi_2(w',c') \\ &= \phi_1(w'',c'') - \phi_1(w',c'') + \phi_1(w',c'') - \phi_1(w',c') \\ &< 0,\end{aligned}$$

a contradiction.

Case 2: Suppose $w' = w''$ and $c' \neq c''$. Since the arguments are symmetric in the two possible subcases, suppose $c' < c''$. Then, as in Case 1.1, we can use (2) to show that

$$\begin{aligned}\phi_1(w'',c'') &= \phi_1(w',c'') \\ &= \phi_1(w',c'') - \phi_1(w',c') \\ &> 0,\end{aligned}$$

a contradiction. This establishes contradictions in all possible cases and completes the proof of Theorem 2.

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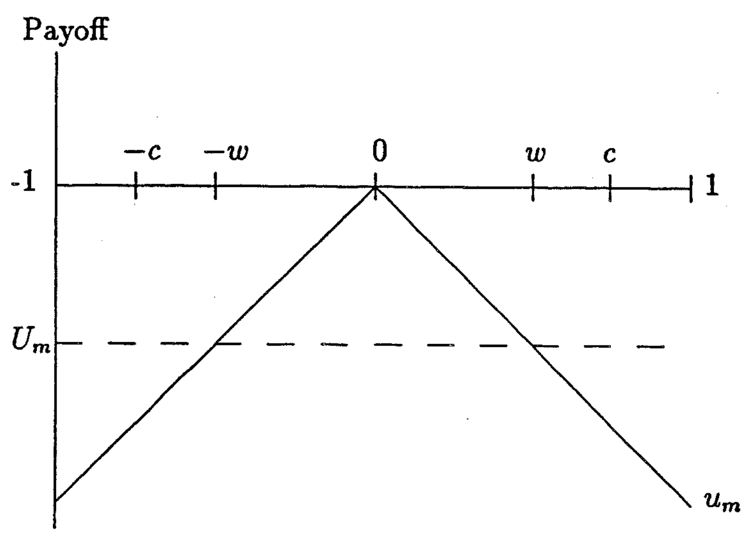

Figure 1: The Perfect Bayesian Equilibrium

Uniform Distribution

	$\delta = .1$	$\delta = .5$	$\delta = .95$
$\rho = 0$	$w = .485$ $c = .549$	$w = .362$ $c = .743$	$w = .049$ $c = .975$
$\rho = .1$	$w = .473$ $c = .643$	$w = .314$ $c = .836$	$w = 0$ $c = 1$
$\rho = .2$	$w = .451$ $c = .727$	$w = .253$ $c = .909$	$w = 0$ $c = 1$

Truncated Normal Distribution

		$\delta = .1$	$\delta = .5$	$\delta = .95$
$\sigma = .5$	$\rho = 0$	$w = .349$ $c = .397$	$w = .261$ $c = .546$	$w = .044$ $c = .877$
	$\rho = .2$	$w = .317$ $c = .578$	$w = .184$ $c = .771$	$w = 0$ $c = 1$

		$\delta = .1$	$\delta = .5$	$\delta = .95$
$\sigma = 1$	$\rho = 0$	$w = .445$ $c = .505$	$w = .333$ $c = .688$	$w = .048$ $c = .96$
	$\rho = .2$	$w = .413$ $c = .685$	$w = .237$ $c = .876$	$w = 0$ $c = 1$

		$\delta = .1$	$\delta = .5$	$\delta = .95$
$\sigma = 5$	$\rho = 0$	$w = .483$ $c = .548$	$w = .361$ $c = .741$	$w = .049$ $c = .974$
	$\rho = .2$	$w = .449$ $c = .725$	$w = .253$ $c = .907$	$w = 0$ $c = 1$

Figure 2: Simulation Results