Genericity of Minority Governments : The Role of Policy and Office

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Working Paper No. 39
September 2004

W. ALLEN
WALLIS
Institute of
POLITICAL
ECONOMY

UNIVERSITY OF
ROCHESTER

Genericity of Minority Governments: The Role of Policy and Office[1]

by

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Comments Welcomed.

Abstract: We develop a general theory for the emergence of minority governments in multi-party parliamentary systems using a bargaining model in the tradition of Baron and Ferejohn, 1989. We show that generically (*i.e.* except for a set of Lebesgue measure zero in the space of the model's parameters) minority governments form with strictly positive probability when office utility from cabinet positions is small relative to political disagreement. The result holds for policy spaces of arbitrary finite dimension and a general class of preferences over the government agreements space.

JEL Classification Numbers: C72, C73, C78.

Keywords: Coalition theory, Minority Governments, Parliamentary systems.

[1] Earlier versions of this paper have benefited from the comments of participants in the 2002 APSA conference, and seminars at Yale University, and NYU. I also thank Daniel Diermeier for interesting discussions on minority governments as well as John Duggan, Hein Goemans, John Roemer, Ken Shepsle, and conference participants at the University of Washington, St. Louis for helpful comments. All errors are mine.

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1. INTRODUCTION

The formation of minority governments in parliamentary systems constitutes one of the most intriguing paradoxes in the study of coalition building. Parliamentarism operates on the fundamental principle that the executive's survival in office hinges on the (tacit) support of a majority in parliament. Yet, by definition, minority governments obtain majority support by allocating cabinets to a set of parties with only a minority of seats in parliament. Why do political parties support (or tolerate) minority governments without receiving cabinet office?

The nature of this paradox has been elegantly articulated in game theoretic analyses dating to William Riker's, 1962, pioneering study of coalition formation. Since Riker's work there has been a plethora of contributions focused on parliamentary government formation, including theories for the formation of minority governments. A starting point for all extant explanations of minority administrations is the assumption that political parties participating in coalition negotiations have heterogeneous preferences over the public policies to be pursued by the prospective government. Casual empiricism would leave little doubt as to the validity of this assumption.

But, aside from public policies, coalition negotiations also determine the distribution of cabinet posts (number as well as responsibilities of portfolios, etc.) among bargaining parties. The attainment of cabinet office is the apex of a successful political career for most political actors involved directly or indirectly in coalition negotiations. Thus, it is natural to assume that bargaining parties prefer larger shares of cabinet positions. Indeed, minority governments are "paradoxical" only if we entertain the latter assumption. Yet, there is no comprehensive theory of minority governments that reconciles the incidence of such governments with the foundations of the paradox, *i.e.* the fact that cabinet office to be distributed in government negotiations is desirable. Our goal in the sequel is to develop such a theory.

Can we explain minority governments while (a) making the allocation of cabinet office among political parties an explicit choice during government negotiations, (b) assuming that political parties' utility increases with more cabinet office, and (c) without undue restrictions on the policy space over which political parties bargain? We show that the answer to this question is *almost always* in the affirmative. Minority governments emerge with positive probability when political disagreement or policy polarization among bargaining parties is marked relative to the 'spoils' of office associated with holding cabinet positions. On the other hand, when utility from holding cabinet office is significant relative to policy disagreement, only majority governments form. The result is independent of the number of political parties represented in parliament (assuming none controls a majority of seats) or the number of dimensions of the underlying policy space.

We qualify the statement of our theory by *almost always* because it is possible to construct otherwise unspectacular examples in which the stated comparative static does not hold. In particular, in these examples minority governments do not form independent of the importance of utility from cabinet positions relative to political disagreement. Although these examples do not prove detrimental to our further theorizing, they make it plain that the puzzle of minority governments is not automatically resolved simply by admitting a mix of office and policy motivations for the parliamentary parties that bargain over governments.[3]

Our analysis is not foreclosed by these counter-examples because we are able to show that they are not generic. Thus, by *almost always* we mean that the set of cases in which the stated comparative static does not hold has measure zero in the space of parameters. To put it otherwise, if we imagine different realizations of the world are drawn probabilistically from this space of parameters, there is probability one that the advertised property holds. In effect, rather than being paradoxical, minority governments are a regular equilibrium phenomenon.

Our theory is consonant with one of the earliest accounts (subsequently neglected) of the phenomenon to appear in the comparative politics literature (*e.g.* Dodd, 1976), according to which minority governments emerge when bargaining parties are too polarized. In these explanations, though, the connection between policy polarization and minority governments is almost assumed. It amounts to an inability of polarized parties to participate in the same cabinet. Furthermore, minority governments of that flavor are expected to be of short duration (e.g. Powell, 1982, page 142). While there are numerous conceivable mechanisms to link policy polarization with minority governments, our result is premised on the following.

Imagine government negotiations such that, due to exogenous restrictions, parties bargain

[3]The penalty we pay for these counter-examples is that we are forced to pursue abstract arguments from differential topology and global analysis.

over public policies but cannot distribute any of the cabinet posts among them. At the heart of our result is the fact that the policy compromises reached in this counter-factual situation are (almost always) different from those reached by majority coalitions. The latter policy compromises differ because the tenure of cabinet posts among all participating parties allows efficient trades between policies and office. By invoking a continuity property of the associated bargaining game (Banks and Duggan, 2000), we show that even when cabinet office is available for distribution among parties but utility from holding this office is relatively unimportant, equilibrium policy compromises must be different from the efficient compromises of majority coalitions. As a consequence, we deduce that when cabinet office utility is small relative to policy disagreements, equilibrium minority governments emerge.

Our finding is consistent with both systematic empirical evidence and stylized facts about the incidence of minority governments. For example, minority governments are more likely in Scandinavian countries where political disagreement is marked, a series of norms and institutional restrictions limit the spoils that political parties can extract from the tenure of cabinet positions, and ministerial office is relatively less significant. If we compare systems with similar levels of party fractionalization, minority governments were significantly more common in Denmark than pre-reform Italy, where cabinet office has been associated with the accrual of significant spoils. Lastly, in a large n study, Warwick, 1998, shows that minority governments are more likely to form when policy polarization increases.

Of course, ours is not the only theory of minority governments, and other theories may operate in conjunction or to complement our arguments. Before we move to the presentation of our analysis, we review this literature by highlighting the aspects in which these alternative theories differ from the present study. Kaare Strom in a series of contributions (Strom, 1984, 1986, 1990) provides an explanation of minority governments based on the inter-temporal trade-offs parties face when considering their options for government participation. According to Strom, parties care about both policies and office but gaining office immediately may not be optimal. In particular, by "deferring gratification" of their office aspirations, parties may avoid costly electoral consequences. Thus, it may be rational to allow a minority government to form – particularly if parties are patient, have opportunities to influence policy in the legislature even if not present in the cabinet, and face competitive elections.

There is also a diverse formal literature on minority governments that can be broadly classified according to the approach (cooperative vs non-cooperative) and assumptions regarding parties' preferences. Under the assumption that parties only care about the division of cabinet office, both approaches converge to the conclusions of Riker, 1962, that only minimum winning coalitions form and minority governments are impossible. Interestingly, this conclusion is no longer true in a dynamic game with endogenous status quo for which there exist equilibria such that all cabinet office eventually goes to a single party in each period (see Kalandrakis, 2003, 2004a).

Laver and Schofield, 1990,[4] and Laver and Shepsle, 1996, deduce minority governments in essentially cooperative frameworks by assuming, instead, that parties only care about policies and are indifferent about the spoils of office. In both accounts the emergence of minority governments depends on the presence of either core or "strong parties," which may or may not exist depending on the dimensionality of the policy space and the configuration of partisan preferences. For instance, when non-trivial preferences are defined over both the public policies pursued by the government as well as over the division of the spoils of cabinet office, the number of dimensions becomes prohibitively large for minority governments to emerge. Lastly, in the same cooperative mode of analysis, Sened, 1995, assumes parties care about the spoils of office, but their utility varies with policy outcomes only if they participate in the government, while they are indifferent about policies otherwise. This discontinuity produces instances when an (iv-)core exists that amounts to a minority government.

Diermeier and Merlo, 2000, arrive at equilibrium minority governments in a rich model that also addresses the stability of these governments. They work with three parties and a two-dimensional policy space. Diermeier and Merlo assume utility is transferable which is an atypical assumption given the public goods character of government policies. Utility transfers in their model are not construed as the division of cabinet positions among parties and there is no such explicit division. Thus, instead of identifying the cabinet by the observed government proposal, Diermeier

[4]See also Schofield, 1993, 1995.

and Merlo define the cabinet coalition as the 'proto-coalition' that eventually offers the final government proposal. Baron and Diermeier, 2001, use a similar definition but restrict transfers only among parties in the proto-coalition and do not obtain equilibrium minority governments.

Finally, we discuss models with the same assumptions as ours when it comes to parties' payoffs such as Austen-Smith and Banks, 1988, Crombez, 1996, Kalandrakis, 2000, and Cho, 2003. All involve a single policy dimension, three parties, and finite period bargaining protocols at the government formation stage. In the first of these models by Austen-Smith and Banks, minority governments are not obtained in equilibrium because the authors assume in the outset that office utility is large.[5] Crombez, 1996, obtains equilibrium minority governments and associates the phenomenon with the size of the median party. Kalandrakis, 2000, allows office utility to vary and obtains equilibrium minority governments when the latter is small. Similarly, minority governments emerge in the long-run in the dynamic model with an endogenous status quo and elections of Seok-Ju Cho. We also mention the work of Jackson and Moselle, 2002, who study a version of the same bargaining model as in our analysis with a single policy dimension, focusing on party rather than government formation.

In the following section we present the theoretical model. In section 3, we elaborate on the comparative statics result we wish to establish and present two counter-examples in which majority governments form with probability one, independent of the level of utility from cabinet portfolios. In section 4 we establish the advertised result. We conclude with section 5.

2. BARGAINING OVER GOVERNMENTS

In this section we present the framework for the analysis of coalition bargaining. We assume a parliament consisting of $n \geq 3$ parties and denote the set of these parties by $N \equiv \{1, ..., n\}$. Each party $i \in N$ has a positive share of seats in parliament equal to $s_i > 0$, with $\sum_{i=1}^{n} s_i = 1$. We assume that no single party controls a parliamentary majority, *i.e.* we have $s_i \leq \frac{1}{2}$ for all $i \in N$.

During government formation negotiations parties must decide on a policy $\mathbf{x} \in X$. We

[5] In that analysis minority governments form in subsequent rounds of bargaining that are not reached in equilibrium.

assume $X \subset \mathbb{R}^d$ is a convex, compact subset of a d-dimensional Euclidean policy space, where the number of policy dimensions $d \geq 1$ can be arbitrarily large. The policy set X encompasses all agreements reached by the government *except* the allocation of cabinet portfolios. Since it is exactly the allocation of cabinet positions that allows us to distinguish between different types of governments in reality, we make this allocation an explicit choice in the model.

We represent cabinet allocations as divisions of a total amount G of cabinet portfolios among the n parties.[6] Thus, these allocations take the form of a vector $\mathbf{g} = (g_1, ..., g_n) \in \mathbf{G}$, where $\mathbf{G} \equiv \left\{\mathbf{g} \in \mathbb{R}^n_+ : \sum_{i=1}^n g_i = G\right\}$. According to the above, we have the following definition of a *government*:

Definition 1 *A government is a pair* $(\mathbf{x}, \mathbf{g})$ *consisting of a policy* $\mathbf{x} \in X$*, and an allocation of cabinet portfolios* $\mathbf{g} \in \mathbf{G}$*.*

We shall assume that parties have preferences over governments given by a function $U_i : X \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}$. We assume that $U_i(\mathbf{x}, \mathbf{g})$ is additively separable in the policy and office components and takes the following form:

$$U_i(\mathbf{x}, \mathbf{g}) = u_i(\mathbf{x}) + [m_i(g_i) + c_i g_i]$$

Although standard in this literature, the assumption of additive separability is not essential for our arguments and can be significantly relaxed as we discuss at the end of section 4.

We admit a wide class of preferences over policy decisions $\mathbf{x} \in X$. In particular, we assume that u_i is smooth with negative definite second derivative, $D^2 u_i(\mathbf{x})$, for all $i \in N$, and that $u_i(\mathbf{x}) > 0$, $\mathbf{x} \in X$. For reasons that will become immediate from our discussion in section 3, we also assume that parties' ideal policies differ, *i.e.* there exists $\mathbf{x}^i \in X$ for each $i \in N$ (unique by strict concavity) that maximizes u_i over choices in X and $\mathbf{x}^i \neq \mathbf{x}^j$ for all $j \neq i$, $j \in N$. Finally, we make the assumption that the set of Pareto preferred policies, $P(X)$ belongs in the interior of X, $P(X) \subset intX$. This last assumption is merely for convenience and can be relaxed.

[6] Although in reality the number of portfolios is finite, a continuous approximation is legitimate since the government agreement can, and often does, alter the jurisdiction or responsibilities of each cabinet portfolio, effectively inducing continuous divisions.

Minority governments are paradoxical if cabinet allocations are desirable. Thus, when it comes to preferences over cabinet portfolios we require m_i to be smooth, concave, and strictly increasing with g_i so that $m_i' > 0$ for all $i \in N$. So, parties prefer larger share of cabinet positions allocated to them and are indifferent about the share of cabinets received by the remaining parties. We set $m_i(0) = 0$ and require $m_i'(0) < +\infty$, *i.e.* the marginal utility from cabinet shares is bounded from above at zero share of cabinets.

Preferences over cabinets also involve the additive linear term $c_i g_i$ and we assume $c_i \in \mathbb{R}_{++}$, $i \in N$. Clearly, we could have incorporated the term $c_i g_i$ into the function m_i.[7] Thus, our functional specification is more general than it appears. We make c_i (c for cabinet) an explicit parameter that enables us to "perturb" parties' cabinet preferences. These preference parameters c_i, $i \in N$ will serve to make precise the notion that minority governments occur generically.

To be invested, governments must receive the assent of a set of parties with a majority of seats in parliament. We thus define:

Definition 2 *A coalition* $C \subseteq N$ *is a* winning coalition *if the sum of seat shares of the constituent parties exceeds one-half, i.e.* $\sum_{j \in C} s_j > \frac{1}{2}$.

We assume a standard Baron & Ferejohn, 1989, bargaining process. In each period $t = 1, 2, ...$ before the attainment of an agreement party i becomes the *formateur* with constant probability π_i. When party i is the formateur in period t, it proposes a government. If this proposal is accepted by a winning coalition $C \subseteq N$, the game ends with the formation of that government. Otherwise the game moves to the next period and continues as above until an agreement is reached. We will assume that parties discount the future with a discount factor $\delta_i \in (0, 1]$, $i \in N$. Thus, if a government $(\mathbf{x}, \mathbf{g})$ is invested in period t, parties' payoffs are given by $\delta_i^{t-1} U_i(\mathbf{x}, \mathbf{g})$.

The fact that the allocation of cabinet office is explicit in the model, allows us to distinguish minority governments by the default empirical criterion, *i.e.* whether cabinet portfolios are allocated only among parties with a minority of seats in parliament:

Definition 3 *A government* $(\mathbf{x}, \mathbf{g}) \in X \times \mathbf{G}$ *is a* minority government *if the set of parties that*

[7] In particular, every concave, strictly monotonic function $\widehat{m}(g)$ can be represented by a function $m(g) + cg$ satisfying our assumptions (and vice versa).

receive positive share of cabinets is not a winning coalition, i.e.

$$\sum_{h=1}^{n} s_h I_{\mathbb{R}_{++}}(g_h) \leq \tfrac{1}{2}.$$

Of course, if a government $(\mathbf{x}, \mathbf{g})$ is not a minority government, then it is a majority government. Trivially, if there are no cabinets to be allocated ($G = 0$) then all governments are minority governments. The distinction becomes non-trivial when $G > 0$.

To complete the description of the model, we shall impose (minimal) requirements with regard to recognition probabilities. We preface these restrictions with a definition. In particular, among parliamentary parties we single out the class of *dummy parties*.

Definition 4 *Party $i \in N$ is a* dummy party *if for all winning coalitions C with $i \in C$, $C \setminus \{i\}$ is also a winning coalition.*

Not all parliaments contain dummy parties,[8] but if they do these parties are not created (and certainly will not be treated) equal in what follows. In particular, we set $\pi_i = 0$ for all dummy parties. On the other hand, we assume all parties that are not dummy parties have some positive probability $\pi_i > 0$, maybe arbitrarily small, of becoming formateurs to lead coalition negotiations. Of course, we have these probabilities satisfy $\sum_{i=1}^{n} \pi_i = 1$.

We now have specified the model which is summarized by a vector of seat shares for the parties $\mathbf{s}$, a vector of recognition probabilities $\boldsymbol{\pi}$, utility functions u_i and m_i for each $i \in N$, the level of cabinet spoils G, and preference parameters δ_i and c_i, $i \in N$. We represent the latter with vectors $\boldsymbol{\delta} \in (0,1]^n$ and $\mathbf{c} \in \mathbb{R}^n_{++}$, respectively. Our arguments in what follows trace the effect of changes on cabinet spoils G, and rely on perturbations of parameters $\boldsymbol{\delta}, \mathbf{c}$. For notational convenience, we denote games satisfying our assumptions by $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$. This notation highlights our focus on changes of parameters $G, \boldsymbol{\delta}, \mathbf{c}$, while implicitly holding the remaining aspects of the model fixed.

We shall restrict our analysis to the study of *stationary subgame perfect* (SSP) equilibria. Since SSP equilibria form a subset of the set of subgame perfect equilibria, minority governments can

[8] One example is a four-party parliament with three parties having 0.3 share of seats. The fourth party is a dummy party.

certainly emerge in a subgame perfect equilibrium if they can emerge in a stationary equilibrium. Thus, the restriction to stationary strategies makes our task harder in what follows.

A *stationary proposal strategy* for party i is a probability distribution over governments in $X \times \mathbf{G}$. A *stationary voting strategy* for party i is a set of governments $A_i \subset X \times \mathbf{G}$ which this party approves. Given stationary proposal and voting strategies, we calculate the *continuation value* of party i, v_i, as the *expected utility if government negotiations continue in the next period*. We restrict voting strategies, so that parties accept governments if and only if they weakly prefer them over their discounted continuation value.[9] An SSP equilibrium is *no delay* if all proposal strategies are such that all proposed governments are invested.

We can readily check that this government formation model satisfies the assumptions of Banks and Duggan, 2000. As a consequence, no-delay SSP equilibria exist. Furthermore, the set of these equilibria changes upper-hemicontinuously[10] with the model's parameters.

Theorem 1 *For every government formation game* $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$*:*

(a) a no-delay SSP equilibrium exists, and

(b) the set of no-delay SSP equilibria is an upper hemicontinuous correspondence of G*.*

Proof. (a) Banks & Duggan, 2000, theorem 1, page 78. (b) Banks & Duggan, 2000, theorem 3, page 81. In particular, G, is a preference parameter of the type assumed in that theorem. To see this, consider an equivalent game where parties split a fixed dollar of size 1. Denote possible divisions by y such that $\sum_{i=1}^{n} y_i = 1$ and assume utility, U_i^*, given by $U_i^*(\mathbf{x}, \mathbf{y}) = U_i(\mathbf{x}, G\mathbf{y})$. ■

Note that the game may (and in general does) admit multiple no-delay SSP equilibria.

3. COMPARATIVE STATICS & COUNTER-EXAMPLES

As we outlined in the introduction, we wish to study the incidence of equilibrium minority governments for game $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$ in relation to the magnitude of cabinet spoils parameter G. This

[9] Formally, in equilibrium, we require $(\mathbf{x}, \mathbf{g}) \in A_i \Longleftrightarrow U_i(\mathbf{x}, \mathbf{g}) \geq \delta_i v_i$, $i \in N$.

[10] Upper-hemicontinuity is one possible generalization of the continuity property of functions to correspondences. A correspondence $\varphi : X \rightrightarrows Y$, Y compact, is upper-hemicontinuous at $x \in X$ if for every pair of convergent sequences $x_k \longrightarrow x \in X$ and $y_k \to y$ with $y_k \in \varphi(x_k)$, we have $y \in \varphi(x)$.

would amount to a typical comparative statics exercise except for the fact that we do not analyze departures from a specific (or unique) equilibrium of the game $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$. The potential multiplicity of equilibria of game $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$ requires that we make statements about the manner the entire equilibrium set behaves as we change G.[11]

In view of the need to cast our comparative statics statements in terms of the entire equilibrium set, we formalize the effect of changes in cabinet spoils, G, as follows. First, we shall show that there exists some level of cabinet spoils $\overline{G} > 0$ such that only majority governments form in all no-delay SSP equilibria of game $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$ for all $G > \overline{G}$. Conversely, when it comes to minority governments, we wish to show that there exists some $\underline{G} > 0$ with $\underline{G} \leq \overline{G}$ such that for all $G < \underline{G}$ minority governments form with positive probability in all no-delay SSP equilibria of the game.

The above formalization implies that larger utility from office (or relatively smaller political disagreement) leads to majority governments, while minority governments are guaranteed to emerge when the opposite holds. Figure 1 displays a graphical illustration of the result we aim to establish.

[insert figure 1 here]

The significant complication arises, though, that the statement that minority governments are guaranteed to emerge for sufficiently low $G > 0$ cannot be true for all versions of the model we described in section 2. In particular, we shall provide two examples of government formation games that have the property that for all levels of $G > 0$ there exist stationary equilibria in which majority governments form with probability one.

The first of the two examples is consistent with and highlights intuition for the proposed theory:

Example 1 *Let parties' preferences, U_i, satisfy the assumptions of the model except assume that ideal policy points coincide, i.e. $\mathbf{x}^i = \mathbf{x}^*$ for all $i \in N$. Let $\delta_i = 1$ for all $i \in N$.*

[11] Another alternative would be to analyze the local behavior of specific equilibria as we change G; but this presumes that equilibria are locally unique. Unfortunately, this has been established to be true (generically) only for the subset of pure strategy equilibria of such games (Kalandrakis, 2004b).

Equilibrium: *Since ideal policy points coincide, in every SSP equilibrium all governments implement policy* $\mathbf{x}^* \in X$*. Thus, for every level of* $G > 0$*, parties effectively bargain only over the division of a dollar. Since* $\pi_i > 0$ *for all non-dummy parties, the reservation value,* v_i*, for these parties is always larger than the utility from the policy* $\mathbf{x}^*$*, i.e.* $v_i > u_i(\mathbf{x}^*)$ *when* $G > 0$*. As a result, all non-dummy parties that approve governments must be receiving cabinet portfolios. Thus, all governments are majority governments for all SSP equilibria and all levels of* $G > 0$*.*

Since policy disagreement is absent in example 1, bargaining revolves exclusively around the allocation of cabinet portfolios and the game becomes a "divide-the-dollar" game similar to Baron and Ferejohn, 1989. Thus, the equilibrium outcome should come as no surprise. In some sense, any size of cabinet spoils G is "large" in the absence of policy disagreement and, as the theory suggests, minority governments are impossible in equilibrium.

In the next example we shall show that, even if political disagreement is present, it is possible to obtain equilibria without minority governments for all positive levels of cabinet office, $G > 0$. Since this example meets all of our assumptions in section 2, it forces us to pursue the genericity arguments we elaborate in section 4.

Example 2 *Let the space of policies be of dimension two* ($d = 2$) *with* $X = [-1, 1]^2$*. Assume four parties* ($n = 4$) *with equal share of seats in parliament* $s_i = \frac{1}{4}$, $i = 1, ..., 4$ *and preferences given by:*

$$U_i(\mathbf{x}, \mathbf{g}) = k - \left(x_1 - x_1^i\right)^2 - \left(x_2 - x_2^i\right)^2 + g_i, \; i = 1, ..., 4$$

where k *is a sufficiently large positive constant. Parties' ideal policy points,* $\mathbf{x}^i$*, are given by* $\mathbf{x}^1 = (0, 1)$*,* $\mathbf{x}^2 = (1, 0)$*,* $\mathbf{x}^3 = (0, -1)$*, and* $\mathbf{x}^4 = (-1, 0)$*. Probabilities of recognition and discount factors are identical and given by* $\pi_i = \frac{1}{4}$ *and* $\delta_i = 1$ *for all* $i \in N$*, respectively.*[12]

Equilibrium, $G = 0$***:*** *There exists a continuum of SSP equilibria such that party 1 proposes* $\mathbf{x} = (0, \alpha)$*, party 2* $\mathbf{x} = (\alpha, 0)$*, party 3* $\mathbf{x} = (0, -\alpha)$*, and party 4* $\mathbf{x} = (-\alpha, 0)$*, for every* $\alpha \in [0, 1]$ *(see Banks and Duggan, 2000, example 6, page 83).*

[12] The term $c_i g_i$ is implicitly incorporated in the linear term $.... + g_i$. This counter-example can be appropriately modified to allow for discount factors less than unity, in which case the equilibrium when $G = 0$ is locally unique.

Equilibrium, $G > 0$***:*** *For every level of* $G > 0$ *there exists a SSP equilibrium such that party 1 proposes* $\mathbf{x} = \left(0, \frac{1}{3}\right)$ *and* $\mathbf{g} = \left(\frac{G}{2}, \frac{G}{4}, 0, \frac{G}{4}\right)$*, party 2* $\mathbf{x} = \left(\frac{1}{3}, 0\right)$ *and* $\mathbf{g} = \left(\frac{G}{4}, \frac{G}{2}, \frac{G}{4}, 0\right)$*, party 3* $\mathbf{x} = \left(0, -\frac{1}{3}\right)$ *and* $\mathbf{g} = \left(0, \frac{G}{4}, \frac{G}{2}, \frac{G}{4}\right)$*, and party 4* $\mathbf{x} = \left(-\frac{1}{3}, 0\right)$ *and* $\mathbf{g} = \left(\frac{G}{4}, 0, \frac{G}{4}, \frac{G}{2}\right)$*. Indeed, with these proposal strategies, parties' continuation values are given by:*

$$v_i = k - 1^2 - \left(\tfrac{1}{3}\right)^2 + \tfrac{G}{4},\ i = 1, ..., 4$$

By symmetry, to verify that the above proposals form an SSP, it suffices to check the optimality of the government proposed by party 1. Given that party 3 is the most expensive coalition partner, party 1's optimization problem is equivalent to:

$$\max_{\mathbf{x},\mathbf{g}} \left(k - (x_1)^2 - (x_2 - 1)^2 + G - g_2 - g_4\right) \ s.t.$$

$$k - (x_1 - 1)^2 - (x_2)^2 + g_2 \geq v_2$$

$$k - (x_1 + 1)^2 - (x_2)^2 + g_4 \geq v_4$$

$$g_i \geq 0, i = 2, 4$$

$$G \geq g_2 + g_4$$

From the Kuhn-Tucker conditions we can verify that for cabinet allocation $\mathbf{g} = \left(\frac{G}{2}, \frac{G}{4}, 0, \frac{G}{4}\right)$ *the optimal policy proposal must satisfy*

$$-2x_1 - 2(x_1 - 1) - 2(x_1 + 1) = 0 \quad (1)$$

$$-2(x_2 - 1) - 2x_2 - 2x_2 = 0 \quad (2)$$

Hence, $x_1 = 0$*,* $x_2 = \frac{1}{3}$*, and* $\mathbf{g} = \left(\frac{G}{2}, \frac{G}{4}, 0, \frac{G}{4}\right)$ *is indeed an optimal government proposal and the equilibrium holds.*

In view of example 2, our task in what follows is considerably harder than typical. In particular, the claim that low cabinet utility G results to equilibrium minority governments can be true only in a subset of the model's parameters, not for all possible versions of the model. At best we can hope to show that the advertised comparative static holds *generically*, *i.e.* models that fail the desired property have (Lebesgue) measure zero in the space of parameters.

An insight as to why example 2 may not be generic can be obtained by noticing the following feature of the equilibrium correspondence with respect to $G \geq 0$. For all majority governments when $G > 0$ in the model, equilibrium policies are determined by equations analogous to those in (1) and (2). These equations produce unique policy compromises that are weighted averages of coalition parties' ideal points, where the weights depend on the marginal utility from extra cabinet portfolios. This fact follows from efficiency considerations peculiar to majority governments and is made precise in lemma 3 of section 4.

Yet, if we study the version of the game without cabinet portfolios ($G = 0$) there is no reason to expect that equilibrium policies will coincide with these efficient policy compromises of majority coalitions. Indeed, without cabinet spoils ($G = 0$) there are no transfers in the form of an exchange of portfolio positions to induce efficient trades among parties in the remaining government decisions. Certainly, equilibrium policies when $G = 0$ are independent of parties' marginal utility from cabinet positions.

[insert figure 2 about here]

Example 2 is not generic because in it we have an unlikely coincidence between the efficient policy compromises of majority coalitions and equilibrium policy compromises when $G = 0$. Specifically, policies at $\left(0, \frac{1}{3}\right)$, $\left(\frac{1}{3}, 0\right)$, $\left(0, -\frac{1}{3}\right)$, and $\left(-\frac{1}{3}, 0\right)$ form *both* optimal majority coalition policies *and* SSP equilibrium policies when $G = 0$. This coincidence makes it possible for the equilibrium correspondence, which is depicted in figure 2, to be (upper hemi)continuous at $G = 0$ without minority governments forming for any $G > 0$.

But if we appropriately perturb model parameters in this example, we can ensure that efficient majority policies do not coincide with policies proposed in any SSP equilibrium when $G = 0$. For instance, if party 1 had smaller marginal utility from cabinet positions (lower c_1), then the optimal majority policy for minimum winning coalition $\{1, 2, 3\}$[13] would be closer to that party's ideal policy point at some $\left(\frac{1}{3} + \varepsilon, \eta\right)$, $\varepsilon, \eta \neq 0$. Obviously, such a change in parameter c_1 does not affect the set of SSP equilibria when $G = 0$. But, with $\left(\frac{1}{3} + \varepsilon, \eta\right) \neq \left(\frac{1}{3}, 0\right)$ being the efficient majority

[13] Obtained by equations similar to (1) and (2).

policy compromise of coalition $\{1,2,3\}$, either party 2 must cease proposing majority governments as G tends to zero, or the set of SSP equilibria must change discontinuously at $G = 0$. Since the latter contradicts the continuity of the equilibrium (theorem 1, part b) minority governments must form with positive probability at some level of $G > 0$.

In light of the above discussion, our result relies on the *generic non-coincidence between the efficient majority government policies and the equilibrium policies when cabinet portfolios are absent* ($G = 0$). To facilitate the demonstration of the disparity between these two sets of policies, it is useful (not necessary) to know that the version of the game without cabinet portfolios (when $G = 0$) does not produce manifold equilibrium points. This is shown to be true generically[14] in the space of discount factors for the pure strategy equilibria of these games by Kalandrakis, 2003b. As we shall show, focusing on the pure strategy SSP equilibria at $G = 0$ is sufficient to prove the desired comparative static. We pursue these arguments more rigorously in the next section.

4. GENERICITY OF MINORITY GOVERNMENTS

We shall start our analysis by showing that we can ensure that majority governments occur with probability one in all SSP equilibria by increasing the size of cabinet spoils, G. Loosely speaking, the argument relies on the fact that parties' expected utility if a government proposal is rejected, *i.e.* their continuation value, is increasing with G. Since utility from policies $\mathbf{x} \in X$, is bounded from above, there exists some level of continuation value above which parties must receive cabinets in order to approve a government. Thus, for large enough G, all parties that are approving governments must be receiving strictly positive fractions of cabinets.

We state this and the remaining formal results in this section and move all proofs in the Appendix.

Proposition 1 *Consider government formation game* $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$. *There exists* $\overline{G}$ *such that for every* $G > \overline{G}$, *all equilibrium governments are majority governments in all no-delay, SSP equilibria of game* $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$.

[14] Incidentally, not in the case of example 2.

Proof. See the appendix. ■

The hard task in what follows will be to show that there exist strictly positive levels of G such that minority governments are guaranteed to occur with positive probability in all SSP equilibria. Given example 2, our objective is to show that this is true for a subset of the model's parameters that has full (Lebesgue) measure in the space of parameters.

We start with a characterization of the policies that are implemented by majority governments in equilibrium. This result follows simply from the optimization considerations of formateurs.

Lemma 1 *If party i proposes a majority government $(\mathbf{x}, \mathbf{g}) \in X \times \mathbf{G}$ in an SSP equilibrium with equilibrium continuation values given by $(v_1, ..., v_n) \in \mathbb{R}^n$, and the set of parties receiving cabinets for government $(\mathbf{x}, \mathbf{g})$ is $C \equiv \{j \in N : g_j > 0\}$, then $i \in C$ and*

(i) the associated policy $\mathbf{x} \in intX$ uniquely solves

$$\sum_{j \in C} \left(m'_j \left(g_j \right) + c_j \right)^{-1} Du_j \left(\mathbf{x} \right) = \mathbf{0}, \tag{3}$$

(ii) for all $j \in C \setminus \{i\}$

$$U_j \left(\mathbf{x}, \mathbf{g} \right) = \delta_j v_j, \; j \in C \setminus \{i\} \tag{4}$$

Proof. See the appendix. ■

The equations[15] in (3) are generalizations of equations (1) and (2) in example 2. Note that the unique solution of equation (3) depends (besides the majority coalition C) on the allocation of cabinets $\mathbf{g} \in \mathbf{G}$ which is implicitly determined by parties' equilibrium continuation values. Denote the solution to the equations in (3) that correspond to cabinet allocation $\mathbf{g}$ and a coalition of portfolio recipient parties C by $\mathbf{x}_{\mathbf{g}}^{C} \in intX$.

With the above we can outline more concretely the basic argument in the main proposition. Suppose, to obtain a contradiction, that for all $G > 0$ there exists some SSP equilibrium such that majority governments form with probability one. For each such SSP equilibrium, proposed policies satisfy equation (3). But, as G goes to zero, cabinet proposals $\mathbf{g}$ in these equilibria also go to $\mathbf{0} \in \mathbb{R}^n$. By continuity of the equilibrium (theorem 1, part b), we deduce that at $G = 0$ there

[15] The number of equations is $d \geq 1$.

exists an equilibrium with all proposed governments taking the form $\left(\mathbf{x}_{\mathbf{0}}^{C}, \mathbf{0}\right) \in X \times \mathbf{G}$, and policy proposals $\mathbf{x}_{\mathbf{0}}^{C}$ satisfy equations (3) and (4).

We use lemma 2 below to show that for almost all parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$ such a limit SSP equilibrium must be in pure strategies.

Lemma 2 *Consider distinct majority coalitions C, C', and a vector of continuation values $(v_1, ..., v_n) \in \mathbb{R}^n$. If, for these continuation values, distinct policies $\mathbf{x}_{\mathbf{0}}^{C}, \mathbf{x}_{\mathbf{0}}^{C'} \in X$ and $\mathbf{g} = \mathbf{0}$ solve equations (3) and (4) for formateur $i \in C \cap C'$, then $u_i\left(\mathbf{x}_{\mathbf{0}}^{C}\right) \neq u_i\left(\mathbf{x}_{\mathbf{0}}^{C'}\right)$ except for a set of Lebesgue measure zero in the space of parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$.*

Proof. See the appendix. ■

The argument in lemma 2 is illustrated in figure 3. This figure displays a two-dimensional policy space and the ideal policy points of members of two winning coalitions with party i belonging in both coalitions. The two highlighted policy points display the efficient majority policies that satisfy equation (3) for these two coalitions and $\mathbf{g} = \mathbf{0}$. If for some version of the model these policies fall on the indifference contour representing party i's policy preferences, u_i, then party i is indifferent between these two policies when $\mathbf{g} = \mathbf{0}$, since $U_i(\mathbf{x}, \mathbf{0}) = u_i(\mathbf{x})$, $i \in N$. Yet, there is a perturbation of a subset of preference parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$ that ensures that one of the two policies is strictly preferred by party i, instead. Thus, for almost all parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$, party i cannot be mixing between these policies in an equilibrium with $G = 0$.

[insert figure 3 about here]

Now, the equilibria of the game when $G = 0$ do not depend on parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$. Also, the number of pure SSP equilibria when $G = 0$ is finite for almost all discount factors $\boldsymbol{\delta} \in (0, 1]^n$ (Kalandrakis, 2004). On the other hand, policies $\mathbf{x}_{\mathbf{0}}^{C}$ that solve equations (3) depend on (at least one) of the parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$. Thus, except for a set of measure zero of parameters $\mathbf{c} \in \mathbb{R}_{++}^{n}$, we can ensure that all possible equilibrium policies in the set of pure strategy SSP equilibria when $G = 0$ do not coincide with any of the policies of the form $\mathbf{x}_{\mathbf{0}}^{C}$.

From the above, for almost all parameters $(\boldsymbol{\delta}, \mathbf{c}) \in (0, 1]^n \times \mathbb{R}_{++}^{n}$, the limit equilibrium in pure strategies that we deduce exists at $G = 0$ by entertaining the hypothesis that only majority governments prevail for some equilibrium of game $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$ for every $G > 0$, *does not coincide with any of the pure strategy equilibria of this game when* $G = 0$. This implies that at $G = 0$ we must have a discontinuity of the equilibrium set, contradicting theorem 1, part b, hence we obtain the desired result.

[insert figure 4 about here]

A graphic illustration of this argument is depicted in Figure 4. We state the result formally as follows:

Proposition 2 *Except for a set of Lebesgue measure zero of parameters* $(\boldsymbol{\delta}, \mathbf{c}) \in (0,1]^n \times \mathbb{R}^n_{++}$, *there exists* $\underline{G}$ *with* $\underline{G} > 0$ *such that for all* $G < \underline{G}$ *in all no-delay, SSP equilibria of government formation game* $\Gamma(G, \boldsymbol{\delta}, \mathbf{c})$ *minority governments form with strictly positive probability.*

Proof. See the appendix. ■

We emphasize the generality of the result in Propositions 1 and 2, which hold for all admissible specifications of parameters $\mathbf{s}$, $\boldsymbol{\pi}$, and utility functions u_i, m_i.

Separability between Policies and Cabinet Office

Perhaps the only substantively important restriction we have imposed has to do with the additive separability of preferences for policies and cabinet allocations, $U_i(\mathbf{x}, \mathbf{g})$. This restriction may be important, for example, if we entertain the possibility that cabinet spoils tied to particular cabinet positions imply some type of direct or automatic public goods policy implication. Another example of a situation in which separability may be restrictive is the case when cabinet allocations to specific coalitions or combinations of parties affect or interact with the utility these parties receive from specific policies $\mathbf{x} \in X$.[16]

Fortunately, our argument can be generalized considerably, relaxing additive separability to admit substantive interactions such as the ones we describe in the above examples. For instance,

[16] Interactions of the type we describe above are implicit in some of the arguments of, for instance, Baron and Diermeier, 2001, Diermeier and Merlo, 2000, or Laver and Shepsle, 1996, etc.

the proof in proposition 2 holds without modification if we allow a general functional form $U_i(\mathbf{x},\mathbf{g})$ and simply require that $\frac{\partial U_i(\mathbf{x},\mathbf{g})}{\partial g_i}$ is independent of $\mathbf{x} \in X$ at $\mathbf{g} = \mathbf{0}$. In other words, our proof relies on a very limited form of separability that holds only at the level of cabinet allocations $\mathbf{g} = \mathbf{0}$. Without even being necessary, this much weaker assumption allows us to accommodate a wealth of interaction effects between policies $\mathbf{x}$, and positive cabinet allocations $\mathbf{g} \neq \mathbf{0}$.

5. CONCLUSIONS

With considerable generality we have derived a theory for the emergence of minority governments in multi-party parliamentary systems using a sequential bargaining model of coalition formation in the tradition of Baron and Ferejohn, 1989. We derived a comparative static to the effect that minority governments are (for almost all parameters) guaranteed to emerge when utility from cabinet posts is small, or when policy disagreement or polarization is significant. On the other hand, only majority governments form when the opposite is true, *ceteris paribus.* Besides anecdotal evidence or traditional intuition in comparative politics (*e.g.* Dodd, 1976) that supports our finding, Warwick, 1998, provides systematic evidence to the effect that the probability that minority governments form increases with policy polarization.

At the core of the mechanism we propose is the fact that the efficient policy compromises of majority governments differ from policy compromises that would emerge in a counter-factual situation when (due, for example, to constitutional or other restrictions) cabinet office is not available to be distributed among parliamentary parties. It follows, as a result of this disparity, that when office utility is small it is impossible for all parties in the winning coalition to be compensated with cabinet positions in order for this winning coalition to reach the efficient majority government compromise. The configuration of parties' bargaining power in equilibrium is such that some parties are willing to approve proposed governments and policies without receiving cabinet portfolios. As a result, minority governments emerge.

Besides being general, our theory of minority governments is also parsimonious in some regards. For example, we do not need to assume that there exist parties that are either located at the core of the policy space or that are similarly centrally located 'strong' parties (e.g. Laver &

Schofield, 1990, Laver and Shepsle, 1996, etc.). Importantly, our argument holds for policy spaces of arbitrarily large dimension. Furthermore, we do not introduce inter-temporal calculations such as the presence of future electoral costs from participation in government, or considerations about the ability of extra-cabinet parties to influence policies outside the cabinet as in the theory of Strom. To the degree that such additional assumptions are valid in actual parliaments, they form the basis for complementary, alternative explanations to the one we provide.

The essence of our argument admits further generalization. In particular, our result follows from two equilibrium properties: (a) the disparity between bargaining compromises when cabinet office is absent and the corresponding policy compromises when majority cabinets form, and (b) the fact that the equilibrium set changes continuously with the size of office utility. Our conclusions extend directly to alternative government formation bargaining models, that satisfy these two properties.

One straightforward generalization involves the related model of Banks and Duggan, 2003, who relax the assumption that agreements are desirable by adding a status quo policy that is implemented in each period coalition negotiations fail.[17] Focusing on alternative bargaining protocols, Baron and Diermeier, 2001, and Diermeier and Merlo, 2000, propose a bargaining game that allows formateur parties to select proto-coalitions which negotiate over agreements prior to the resultant government being presented for an overt or tacit investiture vote. It seems likely that for this and similar extensive forms the necessary continuity of the equilibrium correspondence holds. As a consequence, our conclusions may follow directly for such and other alternative bargaining protocols.

On a methodological note, our study presents an instance of a (possibly) intuitive theory that is surprisingly hard to prove in view of the counter-examples we provide in section 3. Indeed, our analysis required certain 'deep,' abstract theoretical results about the behavior of the equilibrium set of bargaining games of government formation. We believe that the epistemological significance of theorems about the continuity or local uniqueness of equilibria of such games should alone warrant them a place in the modern study of politics. But our (unanticipated) application of these results

[17] Note that the two models are identical in the case discount factors $\delta_i = 1$, for all $i \in N$.

also demonstrates their 'usefulness' even to sceptics that demand immediate applications from such theoretical studies.

APPENDIX

In this appendix we provide the proofs of the lemmas and propositions from section 4. We start with proposition 1:

Proof of Proposition 1. Define $\overline{u}_i \equiv \max\{u_i(\mathbf{x}) : \mathbf{x} \in X\}$. We first claim that:

(1) *If $\delta_i v_i$ is the discounted continuation value of party $i \in N$ in an SSP equilibrium and $\delta_i v_i > \overline{u}_i$ for all $i \in N$, then all equilibrium governments are majority governments.* Suppose not. Then there exists equilibrium proposal $(\mathbf{y}, \mathbf{g}) \in X \times \mathbf{G}$ with $g_j = 0$ for some $j \in C \subset N$, where C is the set of parties approving government $(\mathbf{y}, \mathbf{g})$. Then $U_j(\mathbf{y}, \mathbf{g}) = u_j(\mathbf{y}) \geq \delta_j v_j > \overline{u}_j$, a contradiction.

We shall next show that:

(2) *For each $i \in N$ with $\pi_i > 0$, there exists $\overline{G}_i$ such that $G > \overline{G}_i \Longrightarrow \delta_i v_i > \overline{u}_i$ in every SSP equilibrium.* Let $\underline{u}_i \equiv \min\{u_i(\mathbf{x}) : \mathbf{x} \in X\}$. If $(\overline{\mathbf{y}}, \overline{\mathbf{g}}) \in X \times \mathbf{G}$ is the expected value calculated from the lottery over proposals in an SSP equilibrium, we have $\delta_j v_j \leq U_j(\overline{\mathbf{y}}, \overline{\mathbf{g}})$, for all $j \in N$, due to the concavity of u_j and m_j. For government $(\overline{\mathbf{y}}, \overline{\mathbf{g}})$ there exists $h \neq i$ such that $\overline{g}_h \geq \overline{g}_j$ for all $j \in N \backslash \{i\}$, *i.e.* h is the party with the highest expected cabinet allocation among parties other than i. Clearly $\sum_{j \neq h,i} \overline{g}_j \leq \dfrac{n-2}{n-1} G$. Thus, proposal $(\overline{\mathbf{y}}, \mathbf{w}) \in X \times \mathbf{G}$ with $w_j = \overline{g}_j$ if $j \neq i, h$, $w_h = 0$, and $w_i = G - \sum_{j \neq h,i} \overline{g}_j \geq \dfrac{1}{n-1} G$ is approved by all legislators but h. Hence, in every SSP i can guarantee herself utility level $\underline{u}_i + m_i\left(\frac{G}{n-1}\right) + c_i \frac{G}{n-1}$, when proposing. Thus, in every SSP, i's continuation value must satisfy $v_i \geq (1-\pi_i)\underline{u}_i + \pi_i\left(\underline{u}_i + m_i\left(\frac{G}{n-1}\right) + c_i \frac{G}{n-1}\right) = \underline{u}_i + \pi_i\left(m_i\left(\frac{1}{n-1}G\right) + \frac{c_i}{n-1}G\right)$. Since $\pi_i > 0$, $\delta_i \in (0,1]$, and $m_i' > 0$, $c_i > 0$, there exists $\overline{G}_i > 0$ such that $\delta_i\left(\underline{u}_i + \pi_i\left(m_i\left(\frac{1}{n-1}G\right) + \frac{c_i}{n-1}G\right)\right) > \overline{u}_i$ for all $G > \overline{G}_i$. As a result, $G > \overline{G}_i \Longrightarrow \delta_i v_i > \overline{u}_i$ and we have completed the proof of step (2).

Set $\overline{G} = \max\left\{\overline{G}_i \mid i \in N\right\}$. We now have $G > \overline{G} \Longrightarrow \delta_i v_i > \overline{u}_i$ for all $i \in N$, by step (2). But in every SSP equilibrium, only majority governments form for $G > \overline{G}$, by step (1), and the proof of the proposition is complete. ■

Next we prove lemma 1.

Proof of Lemma 1. Part (ii) follows from the fact that $m'_h(g_h) + c_h > 0,\ h \in N$. Specifically, if $U_j(\mathbf{x}, \mathbf{g}) > \delta_j v_j,\ j \in C \setminus \{i\}$, then by the continuity of $U_j(\mathbf{x}, \mathbf{g})$ it is possible to reduce g_j and increase g_i (and party i's utility) with the new government still being invested. Similarly, if $U_j(\mathbf{x}, \mathbf{g}) < \delta_j v_j$, $j \in C \setminus \{i\}$ then it is possible to set $g_j = 0$ and increase g_i (and party i's utility) with the new government still being invested.

To show part (i) note that the above arguments and the fact that i is not a dummy party ensure that $i \in C$, *i.e.* the proposing party is included among the parties receiving cabinets. Thus, i's government proposal must solve the program

$$\max_{\mathbf{x},\mathbf{g}} U_i(\mathbf{x}, \mathbf{g}) \text{ subject to}$$

$$U_j(\mathbf{x}, \mathbf{g}) \geq \delta_j v_j, j \in C \setminus \{i\}$$

$$g_j \geq 0, j \in C$$

$$\sum_{j \in C} g_j = G$$

$$\mathbf{x} \in X$$

Since the Pareto set with regard to policies is a subset of the interior of X, $P(X) \subset int(X)$, we can ignore the constraint $\mathbf{x} \in X$. We now form the Langrangean accounting for the remaining constraints:

$$L(\mathbf{x}, \mathbf{g}) = U_i(\mathbf{x}, \mathbf{g}) + \sum_{j \in C \setminus \{i\}} \left(\theta_j \left(U_j(\mathbf{x}, \mathbf{g}) - \delta_j v_j\right) + \mu_j g_j\right) + \mu_i g_i - \theta \left(\sum_{j \in C} g_j - G\right).$$

Substituting $U_h(\mathbf{x},\mathbf{g}) = u_h(\mathbf{x}) + m_h(g_h) + c_h g_h$, $h \in N$, we obtain from the first order conditions:

$$\begin{aligned}
\frac{\partial L(\mathbf{x},\mathbf{g})}{\partial \mathbf{x}} &= Du_i(\mathbf{x}) + \sum_{j \in C \setminus \{i\}} \theta_j Du_j(\mathbf{x}) = 0 \\
\frac{\partial L(\mathbf{x},\mathbf{g})}{\partial g_i} &= m_i'(g_i) + c_i + \mu_i - \theta = 0 \\
\frac{\partial L(\mathbf{x},\mathbf{g})}{\partial g_j} &= \theta_j \left(m_j'(g_j) + c_j\right) + \mu_j - \theta = 0, j \in C \setminus \{i\} \\
\theta_j \left(u_j(\mathbf{x}) + m_j(g_j) + c_j g_j - \delta_j v_j\right) &= 0, j \in C \setminus \{i\} \\
\mu_j g_j &= 0, j \in C \\
\sum_{j \in C} g_j &= G
\end{aligned}$$

By part (ii) and the fact that $g_j > 0$, $j \in C$, the Kuhn-Tucker conditions reduce to:

$$\begin{aligned}
\sum_{j \in C} \left(m_j'(g_j) + c_j\right)^{-1} Du_j(\mathbf{x}) &= \mathbf{0} \\
\theta &= m_i'(g_i) + c_i \\
\theta_j &= \frac{m_i'(g_i) + c_i}{m_j'(g_j) + c_j}, j \in C \\
u_j(\mathbf{x}) + m_j(g_j) + c_j g_j - \delta_j v_j &= 0, j \in C \setminus \{i\} \\
\mu_j &= 0, j \in C \\
\sum_{j \in C} g_j &= G
\end{aligned}$$

Thus, given that $m_j'(g_j) + c_j > 0$, the optimal policy $\mathbf{x}$ maximizes the strictly concave function $\sum_{j \in C} \left(m_j'(g_j) + c_j\right)^{-1} u_j(\mathbf{x})$ which is a weighted sum of players' utilities. Hence, $\mathbf{x}$ is unique and belongs in the interior of X. ■

We now prove lemma 2:

Proof of lemma 2. Since $\mathbf{x}_\mathbf{0}^C$, $\mathbf{x}_\mathbf{0}^{C'}$ are distinct, we have $\left(\mathbf{x}_\mathbf{0}^C, \mathbf{x}_\mathbf{0}^{C'}\right) \in (intX \times intX) - \Delta$, where Δ is the diagonal of $intX \times intX$. Since $(intX \times intX) - \Delta$ is an open set, it is a smooth manifold of dimension twice the dimension of manifold $intX$. By equation (4) and the fact that $U_k(\mathbf{x},\mathbf{0}) = u_k(\mathbf{x})$, $k \in N$ we must have $u_j\left(\mathbf{x}_\mathbf{0}^C\right) = u_j\left(\mathbf{x}_\mathbf{0}^{C'}\right)$ for all $j \in (C \cup C' - \{i\})$. We wish to show that $u_i\left(\mathbf{x}_\mathbf{0}^C\right) = u_i\left(\mathbf{x}_\mathbf{0}^{C'}\right)$ can be true only for a set of Lebesgue measure zero in the space of parameters $\mathbf{c} \in \mathbb{R}^n_{++}$. For that purpose, it suffices to show that the $2d + |C \cup C'| - 1$ equations

$$\begin{aligned}
\sum_{j \in C} \left(m'_j(0) + c_j\right)^{-1} Du_j\left(\mathbf{x}_{\mathbf{0}}^{C}\right) &= \mathbf{0} \\
\sum_{j \in C'} \left(m'_j(0) + c_j\right)^{-1} Du_j\left(\mathbf{x}_{\mathbf{0}}^{C'}\right) &= \mathbf{0} \\
u_j\left(\mathbf{x}_{\mathbf{0}}^{C}\right) - u_j\left(\mathbf{x}_{\mathbf{0}}^{C'}\right) &= 0, \text{ for all } j \in \left(C \cup C' - \{i\}\right)
\end{aligned}$$

that $\mathbf{x}_{\mathbf{0}}^{C}$, $\mathbf{x}_{\mathbf{0}}^{C'}$ must satisfy, along with $u_i\left(\mathbf{x}_{\mathbf{0}}^{C}\right) - u_i\left(\mathbf{x}_{\mathbf{0}}^{C'}\right) = 0$ are inconsistent outside a set of Lebesgue measure zero in the space of parameters $\mathbf{c} \in \mathbb{R}^n_{++}$.

Of course, *a fortiori*, it is sufficient to show that a subset of these equations are generically inconsistent. We shall work with $|C \bigcap C'|$ possible subsets of the following form:

$$\begin{aligned}
\sum_{j \in C} \left(m'_j(0) + c_j\right)^{-1} Du_j\left(\mathbf{x}_{\mathbf{0}}^{C}\right) &= \mathbf{0} \\
\sum_{j \in C'} \left(m'_j(0) + c_j\right)^{-1} Du_j\left(\mathbf{x}_{\mathbf{0}}^{C'}\right) &= \mathbf{0} \\
u_j\left(\mathbf{x}_{\mathbf{0}}^{C}\right) - u_j\left(\mathbf{x}_{\mathbf{0}}^{C'}\right) &= 0, \text{ for one } j \in C \bigcap C'
\end{aligned}$$

We view each of these $|C \bigcap C'|$ subsets as a set of $2d + 1$ equations with 'unknowns' the policies $\mathbf{x}_{\mathbf{0}}^{C}$, $\mathbf{x}_{\mathbf{0}}^{C'}$ and the $|C \cup C'|$ parameters c_h, $h \in C \cup C'$ that belong in $\mathbb{R}^{|C \cup C'|}_{++}$.

Define the mapping $F_j : (intX \times intX - \Delta) \times \mathbb{R}^{|C \cup C'|}_{++} \longrightarrow \mathbb{R}^{2d+1}$, to be the left-hand side of the above equations. F_j, $j \in C \bigcap C'$, is a smooth mapping between smooth manifolds since m_h, u_h are smooth. We shall show that $\mathbf{0} \in \mathbb{R}^{2d+1}$ is a *regular value* of F_j, for at least one $j \in C \bigcap C'$. As a result, the pre-image of $\mathbf{0}$, $F_j^{-1}(\mathbf{0})$, that constitutes the set of solutions to the equations $F_j(\cdot) = \mathbf{0}$ is a $(2d + |C \cup C'|) - (2d + 1) = (|C \cup C'| - 1)$-dimensional manifold by the Preimage theorem (Guilemin and Pollack, 1974, page 21). This is one dimension smaller than the space of parameters $\{c_i\}_{i \in C \cup C'} \in \mathbb{R}^{|C \cup C'|}_{++}$ and, as a consequence, the equations $F_j(\cdot) = \mathbf{0}$ (and supersets of these equations) are consistent only for a set of Lebesgue measure zero of the parameters c_h, $h \in C \cup C'$.

Recall that $\mathbf{0}$ is a regular value of F_j if and only if the Jacobian of F_j evaluated at $\mathbf{x}$, $DF_j(\mathbf{x})$, has full rank for every $\mathbf{x} \in F_j^{-1}(\mathbf{0})$. In what follows we index parties using the convention $q \in C \backslash C'$,

$h \in C' \setminus C$, and $l \in (C \cup C' - \{j\})$. Calculating the Jacobian $DF_j(\mathbf{x})$ using the order of variables implied by $\mathbf{x} = \left(\mathbf{x}_{\mathbf{g}}^{C}, \mathbf{x}_{\mathbf{g}}^{C'}, c_j, c_l, ..., c_q, ..., c_h\right)$, we get:

$$DF_j(\mathbf{x}) = \begin{bmatrix} A & \mathbf{0} & -w_j^{-2}\mathbf{y}_j & -w_l^{-2}\mathbf{y}_l & ... & -w_q^{-2}\mathbf{y}_q & ... & \mathbf{0} & ... \\ \mathbf{0} & B & -w_j^{-2}\mathbf{z}_j & -w_l^{-2}\mathbf{z}_l & ... & \mathbf{0} & ... & -w_h^{-2}\mathbf{z}_h & ... \\ \mathbf{y}_j^T & -\mathbf{z}_j^T & 0 & 0 & ... & 0 & ... & 0 & ... \end{bmatrix}$$

where $A = \sum_{k \in C} (m_k'(0) + c_k)^{-1} D^2 u_k\left(\mathbf{x}_{\mathbf{0}}^{C}\right)$, $B = \sum_{k \in C'} (m_k'(0) + c_k)^{-1} D^2 u_k\left(\mathbf{x}_{\mathbf{0}}^{C'}\right)$, $\mathbf{y}_k = Du_k\left(\mathbf{x}_{\mathbf{0}}^{C}\right)$, $\mathbf{z}_k = Du_k\left(\mathbf{x}_{\mathbf{0}}^{C'}\right)$, and $w_k = m_k'(0) + c_k$, $k \in C \cup C'$. Note that by our assumptions $w_k > 0$, $k \in C \cup C'$. We must also have that equations (3) are valid for $\mathbf{x}_{\mathbf{0}}^{C}, \mathbf{x}_{\mathbf{0}}^{C'}$. From these equations we get:

$$\sum_{k \in C} w_k^{-1}\mathbf{y}_k = \mathbf{0} \Longrightarrow \sum_{j \in C \bigcap C'} w_j^{-1}\mathbf{y}_j = -\sum_{q \in C \setminus C'} w_q^{-1}\mathbf{y}_q \tag{5}$$

and

$$\sum_{k \in C} w_k^{-1}\mathbf{z}_k = \mathbf{0} \Longrightarrow \sum_{j \in C \bigcap C'} w_j^{-1}\mathbf{z}_j = -\sum_{h \in C' \setminus C} w_h^{-1}\mathbf{z}_h. \tag{6}$$

Performing a few equivalence operations on $DF_j(\cdot)$ we get a new matrix with the same rank:

$$\begin{bmatrix} A & \mathbf{0} & w_j^{-1}\mathbf{y}_j & w_l^{-1}\mathbf{y}_l & ... & w_q^{-1}\mathbf{y}_q & ... & \mathbf{0} & ... \\ \mathbf{0} & B & w_j^{-1}\mathbf{z}_j & w_l^{-1}\mathbf{z}_l & ... & \mathbf{0} & ... & w_h^{-1}\mathbf{z}_h & ... \\ \mathbf{y}_j^T & -\mathbf{z}_j^T & 0 & 0 & ... & 0 & ... & 0 & ... \end{bmatrix}$$

Since A, B are negative definite, this matrix has full rank if and only if the $1 \times |C \cup C'|$ matrix

$$M_j = \mathbf{0} - \begin{bmatrix} \mathbf{y}_j^T & -\mathbf{z}_j^T \end{bmatrix} \begin{bmatrix} A^{-1} & \mathbf{0} \\ \mathbf{0} & B^{-1} \end{bmatrix} \begin{bmatrix} w_j^{-1}\mathbf{y}_j & w_l^{-1}\mathbf{y}_l & ... & w_q^{-1}\mathbf{y}_q & ... & \mathbf{0} & ... \\ w_j^{-1}\mathbf{z}_j & w_l^{-1}\mathbf{z}_l & ... & \mathbf{0} & ... & w_h^{-1}\mathbf{z}_h & ... \end{bmatrix} =$$

$$\begin{bmatrix} w_j^{-1}\mathbf{y}_j^T A^{-1}\mathbf{y}_j - w_j^{-1}\mathbf{z}_j^T B^{-1}\mathbf{z}_j & w_l^{-1}\mathbf{y}_j^T A^{-1}\mathbf{y}_l - w_l^{-1}\mathbf{z}_j^T B^{-1}\mathbf{z}_l & ... & w_q^{-1}\mathbf{y}_j^T A^{-1}\mathbf{y}_q & ... & -w_h^{-1}\mathbf{z}_j^T B^{-1}\mathbf{z}_h & ... \end{bmatrix}$$

has rank 1. Thus to prove the lemma it suffices to show that there exists at least one $j \in C \bigcap C'$ such that at least one of the $|C \cup C'|$ elements of the matrix M_j is non-zero.

Suppose $M_j = \mathbf{0}$ for all $j \in C \bigcap C'$ instead. Then we have $w_q^{-1}\mathbf{y}_j^T A^{-1}\mathbf{y}_q = 0$ for all $q \in C \setminus C'$ and all $j \in C \bigcap C'$, and we can obtain (by summing equations $w_q^{-1}\mathbf{y}_j^T A^{-1}\mathbf{y}_q = 0$ and collecting terms):

$$\left(\sum_{j \in C \bigcap C'} w_j^{-1}\mathbf{y}_j^T\right) A^{-1} \left(\sum_{q \in C \setminus C'} w_q^{-1}\mathbf{y}_q\right) = 0$$

Using (5) and the fact that A^{-1} is negative definite, we deduce

$$\sum_{j\in C\bigcap C'} w_j^{-1}\mathbf{y}_j = \sum_{j\in C\bigcap C'} \left(m_j'(0)+c_j\right)^{-1} Du_j\left(\mathbf{x}_{\mathbf{0}}^{C}\right) = \mathbf{0}. \tag{7}$$

An identical argument using (6) gives us

$$\sum_{j\in C\bigcap C'} w_j^{-1}\mathbf{z}_j = \sum_{j\in C\bigcap C'} \left(m_j'(0)+c_j\right)^{-1} Du_j\left(\mathbf{x}_{\mathbf{0}}^{C'}\right) = \mathbf{0}. \tag{8}$$

Since (7) and (8) imply that $\mathbf{x}_{\mathbf{0}}^{C}$, $\mathbf{x}_{\mathbf{0}}^{C'}$ both maximize the strictly concave function $\sum_{j\in C\bigcap C'}\left(m_j'(0)+c_j\right)^{-1} u_j(\mathbf{x})$, we must have $\mathbf{x}_{\mathbf{0}}^{C}=\mathbf{x}_{\mathbf{0}}^{C'}$, a contradiction emanating from the working hypothesis that $M_j=\mathbf{0}$ for all $j\in C\bigcap C'$. This completes the proof of the lemma. ■

Lastly, in order to prove proposition 2, we make use of the following result:

Theorem 2 *(Kalandrakis, 2004b) For almost all discount factors* $\boldsymbol{\delta}\in[0,1]^n$ *the number of pure strategy, stationary, no-delay equilibria of game* $\Gamma(0,\boldsymbol{\delta},\mathbf{c})$ *is finite (possibly zero).*

We now state the proof of proposition 2:

Proof of Proposition 2. Fix any $\mathbf{s},\boldsymbol{\pi},u_i,m_i$, $i\in N$, consistent with our assumptions. Assume (to show a contradiction for almost all $\boldsymbol{\delta},\mathbf{c}$) that for some game $\Gamma(G,\boldsymbol{\delta},\mathbf{c})$ there is no $\underline{G}>0$ such that minority governments form with positive probability in all SSP equilibria when $0\leq G<\underline{G}$. In other words, the working hypothesis is that for each $G>0$ there exists some G' with $G>G'>0$ for which an SSP equilibrium exists with all proposed governments being majority coalitions. Then, we can construct a sequence G_k with $G_k>0$, $G_k\longrightarrow 0$ and a sequence of associated SSP equilibria e_k with $e_k\longrightarrow e$, such that all governments proposed in each e_k are majority governments. We can immediately deduce:

(1) *For all* $\mathbf{c}\in\mathbb{R}^n_{++}$, $\boldsymbol{\delta}\in(0,1]^n$, *the limit* e *of the sequence of SSP equilibria* e_k *is a no delay, SSP equilibrium of the game* $\Gamma(0,\boldsymbol{\delta},\mathbf{c})$. This follows immediately from the upper-hemicontinuity of the equilibrium correspondence (theorem 1, part b).

(2) *For almost all* $\mathbf{c}\in\mathbb{R}^n_{++}$, *the equilibrium* e *in step 1 is in pure strategies.* This follows from lemma 2, the fact that for each $i\in N$ there exists a finite number of possible pairs of distinct minimum winning coalitions C,C' with $i\in C\bigcap C'$, and the fact that finite unions of sets of measure zero have measure zero.

(3) *The proposal offered by non-dummy party $i \in N$ in pure strategy SSP equilibrium e of step 2 is a policy $\mathbf{x}_{\mathbf{0}}^{C}$ that satisfies (3) for one of the finite number of minimum winning coalitions $C \subset N$, $i \in C$.* This follows from the working hypothesis that in the sequence of equilibria e_k only majority governments are proposed and from lemma 1.

(4) *For almost all discount factors $\boldsymbol{\delta} \in (0,1]^n$, the possible proposals offered by non-dummy party $i \in N$ in all the pure strategy SSP equilibria of game $\Gamma(0, \boldsymbol{\delta}, \mathbf{c})$, is a finite set $\left\{\mathbf{x}_1^i, ..., \mathbf{x}_k^i\right\}$ (k possibly zero).* This follows from theorem 2.

Note that the equilibria of the game when $G = 0$, hence policies $\left\{\mathbf{x}_1^i, ..., \mathbf{x}_k^i\right\}$ in step 4, do not depend on parameters $\mathbf{c} \in \mathbb{R}_{++}^n$. But by lemma 1, the finite number of policies $\mathbf{x}_{\mathbf{0}}^{C}$ in step 3, do depend on $\mathbf{c} \in \mathbb{R}_{++}^n$. In particular, since parties' ideal policy points differ, for each minimum winning coalition C that includes i, the solutions to (3), $\mathbf{x}_{\mathbf{0}}^{C}$, can be perturbed by at least $|C| - 1$ parameters. Then for almost all $\mathbf{c} \in \mathbb{R}_{++}^n$ there exist no $\mathbf{x} \in \left\{\mathbf{x}_1^i, ..., \mathbf{x}_k^i\right\}$ such that $\mathbf{x} = \mathbf{x}_{\mathbf{0}}^{C}$ for all minimum winning coalitions C such that $i \in C$. Thus, except for a set of Lebesgue measure zero of parameters $\mathbf{c} \in \mathbb{R}_{++}^n$, $\boldsymbol{\delta} \in (0,1]^n$, we have a contradiction of step 1 emanating from the working hypothesis. ■

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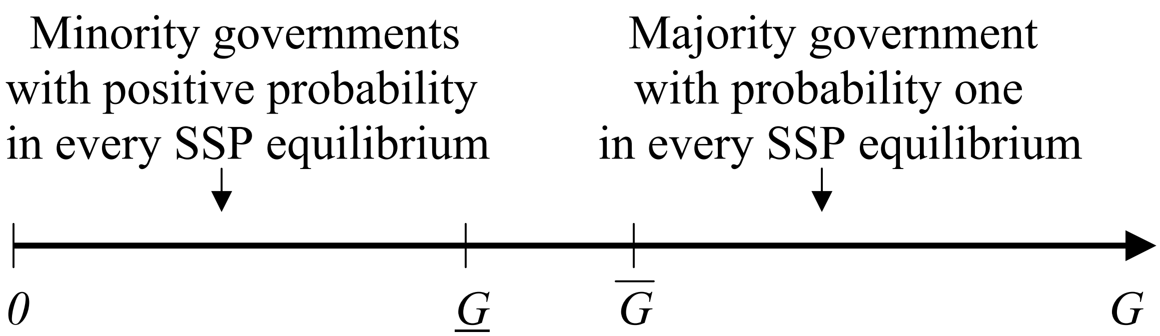
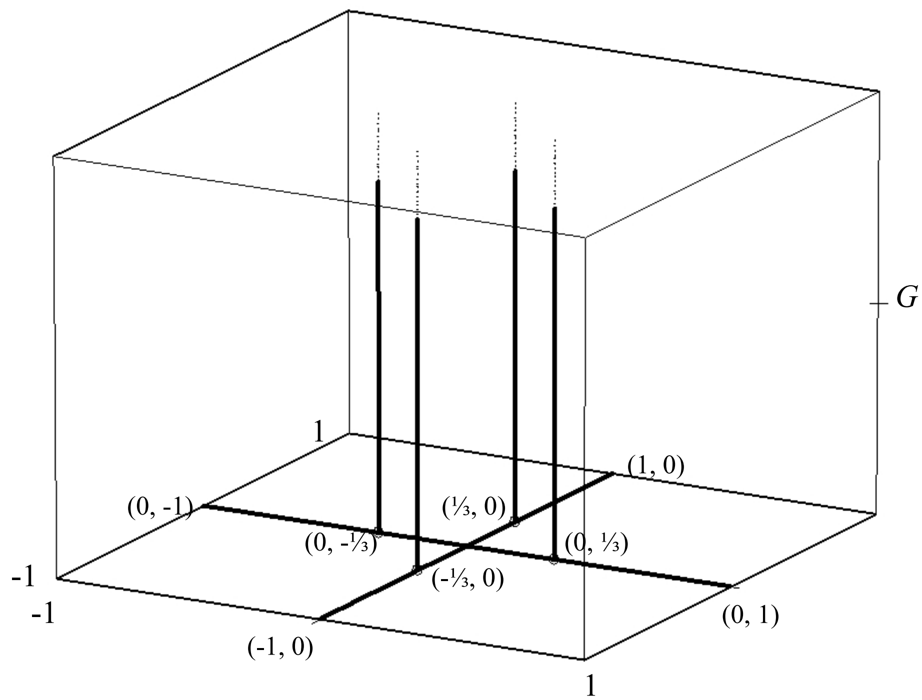
Figure 1: Comparative Statics



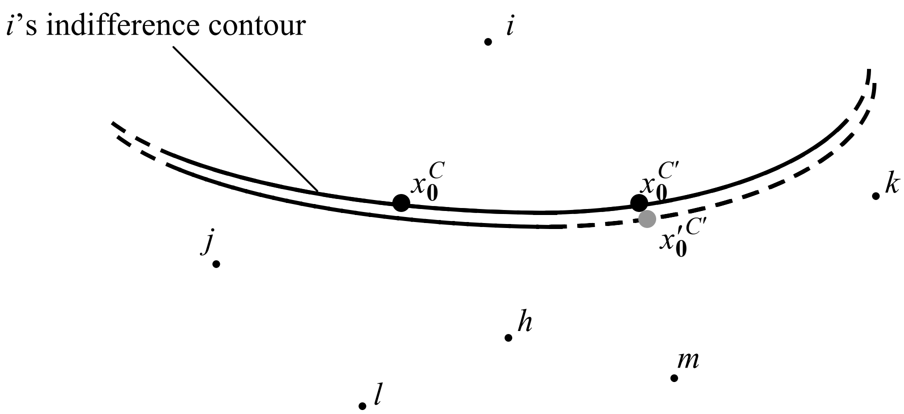
Key: Since the bargaining game may admit multiple equilibria, it is possible that in some intermediate range of cabinet spoils ($\underline{G}$, $\overline{G}$) a subset of equilibria involve minority governments and the rest do not.

Figure 2: Equilibrium Policies vs Cabinet Spoils *G* in Example 2



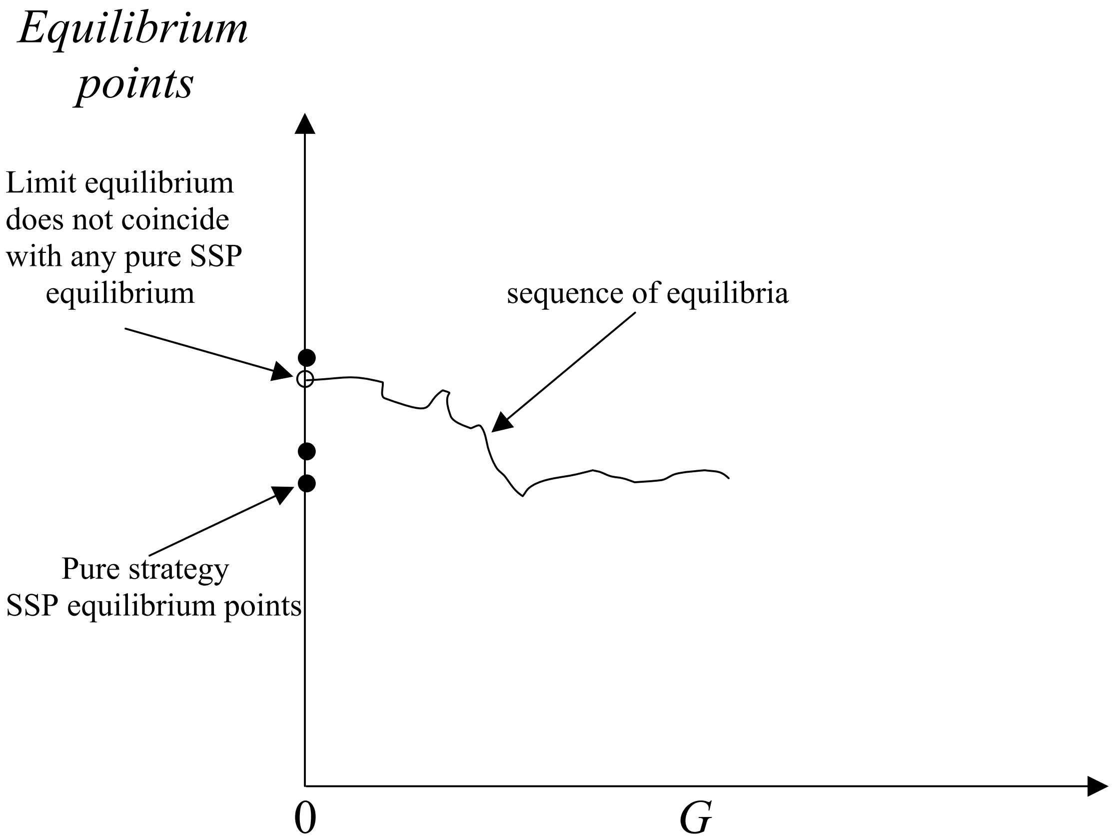
Key: *For all G > 0, there is an equilibrium in which all four parties propose majority governments. At G = 0 there is a continuum of pure strategy SSP equilibria, including one in which proposed policies coincide with the policies proposed in the SSP equilibria when G > 0.*

Figure 3: Graphic Illustration of Lemma 2



Key: Policy $x_{\mathbf{0}}^{C}$ represents the efficient majority compromise for coalition $C = \{i, j, l, h\}$ and cabinet allocation $\mathbf{g} = \mathbf{0}$, while policy $x_{\mathbf{0}}^{C'}$ represents the respective compromise for coalition $C = \{i, h, m, k\}$. There exists a perturbation of parties' preference parameters (hence of $x_{\mathbf{0}}^{C'}$ to $x_{\mathbf{0}}^{\prime C'}$) that ensures that the two policies do not fall on the same indifference contour of party $i \in C \bigcap C'$.

Figure 4: Minority Governments & Proposition 2.



Key: If majority governments form with probability one in a sequence of equilbria as the size of G goes to zero, then we deduce a failure of upper-hemicontinuity at $G = 0$ for almost all parameters of the model.