Roll Call Data and Ideal Points

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Abstract: We show that, in the absence of symmetry or other parametric restrictions on legislators' utility functions, roll call voting records cannot be used to estimate legislators' ideal points unless we complement these data with information on the location of the alternatives being voted upon by the legislature. Without such additional information, the roll-call data cannot differentiate between distinct, arbitrary, sets of ideal points for the legislators no matter how large the roll call record or how low the number of policy dimensions. On the other hand, when the location of voting alternatives is known, we derive simple testable restrictions on the location of legislators' ideal points from the roll call data.

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1. INTRODUCTION

Is it possible to conclude that certain legislators are relatively more or less liberal, left-wing, or conservative on the basis of observations of their voting decisions on a finite number of roll call voting items? A positive answer to this question would have implications that reach well beyond the typical use of such characterizations in the public arena, and for the nature of scrutiny to which citizens can subject individual representatives. For individual preferences lie at the heart of any methodological approach (strategic, behavioral, or other) to the study of politics. Indeed, if we cannot impose any restrictions on individual preferences, the predictive content of modern legislative theories is non-existent. Thus, the estimation of legislators' preferences using roll call data figures prominently among challenges for modern political methodology. It should come as no surprise that a literature with exactly this objective has evolved.

The seminal work on the estimation of legislators' ideal points is the NOMINATE procedure and its variants developed by Poole and Rosenthal, 1985, 1991, 1997. Using a simpler model, Heckman and Snyder, 1997, derive an alternative estimator, establish a number of desirable statistical properties for this estimator, and empirically infer significantly higher dimensionality for the policy space using US Congress data. More recently, Clinton, Jackman, and Rivers, 2004, propose and implement a MCMC estimation strategy. We review this literature in more detail in the sequel.

Despite the considerable progress achieved by ideal point estimation methods, this research program has thus far rested on strong parametric assumptions about legislators' preferences. Without exception, all existing estimators require symmetry of legislators' utility functions around their ideal point. In fact, recent contributions seem to heve converged on the use of the familiar negative quadratic functional form. In this study we investigate whether the results of this literature depend on such functional form restrictions and the possibilities for relaxing these restrictions. Our conclusions have obvious ramifications for the interpretation of these results and, more generally, for the limitations of preference estimation based on roll-call data.

As is standard in the ideal point estimation literature, we maintain a spatial framework so that for each item in the legislative agenda, legislators are confronted with a choice between two alternatives drawn from a finite dimensional policy space. We relax parametric or symmetry restrictions on individual preferences and merely assume that each legislator has a strictly concave utility function over the space of policy alternatives. A legislator with a strictly concave utility function has a unique, well-defined ideal point and her utility decreases at an increasing rate as policies move away from that ideal point. In one policy dimension, strict concavity is a stronger version of the single-peakedness condition familiar from the social choice literature. We then ask what can we learn about legislators' ideal points from a roll call voting record and under these assumptions? It turns out that the answer differs radically depending on whether we assume any restrictions on the location of the alternatives being voted upon by the legislature.

In one scenario, the location of the alternatives being voted upon by the legislature is unknown. This is the relevant situation for most existing political methodology techniques used to recover legislators' ideal points, since in these methods the voting alternatives are unrestricted model parameters for estimation.[4] In this case, we show (theorem 3) that roll-call voting records *contain no information* on legislators' ideal points. In particular, for *any ideal points arbitrarily assigned to each legislator*, we can locate a pair of voting alternatives for each item in the roll call record *and* we can find strictly concave utility functions for each legislator that both (a) have the postulated ideal points, and (b) are perfectly consistent with all the voting decisions of all legislators. This is true for any number of dimensions of the underlying policy space, even for the most restrictive one-dimensional case.

As we already mentioned, most existing techniques for ideal point estimation construe the voting alternatives as unrestricted, unknown parameters. Thus, although these models are statistically identified (*e.g.*, Rivers, 2003), the resultant estimates of ideal points rely critically on parametric and symmetry restrictions imposed on legislators' utility functions.[5] In the absence of these restrictions, and if the location of voting alternatives is unknown, the roll call data do not allow us to empirically discriminate between the ideal points obtained by standard political methodology techniques, or completely different, arbitrary ideal points. All these alternative sets of ideal points are observationally equivalent in that they can be matched with appropriate strictly concave utility functions in order to perfectly explain the voting record (*i.e.*, without even the need

[4]Londregan, 2000, refers to such estimators as "agnostic."

[5]In addition to the identifying role of parametric assumptions on the random utility term.

to assume a random utility term).

The obvious next question is whether or how these conclusions are qualified if we assume that information on the location of the voting alternatives is available. We provide a complete answer to this question in the ideal scenario when the exact location of voting alternatives is known. We obtain this answer in two steps. First, our analysis yields a simple necessary and sufficient condition (theorem 1) for the existence of preferences satisfying our assumptions that are consistent with the voting decisions of an individual legislator over the given alternatives. This condition requires that for each subset of the votes or items included in the roll call record, there must exist at least one alternative that this legislator votes against, that cannot be written as a convex combination of the alternatives in this subset that this legislator votes for.[6] If (and only if) this condition is met for a particular legislator, then this legislator's preferences can be represented by a strictly concave utility function. In this case, we say that the voting record is *rationalizable*.

We then proceed to show that such rationalizable voting records along with the restriction of strict concavity on individual preferences imply testable restrictions on the location of legislators' ideal points (theorem 2). The associated restrictions have a simple statement, and essentially constitute generalizations of the condition stated in the previous paragraph. We also show that in the one dimensional case these necessary and sufficient conditions simplify considerably. These restrictions imply that there exist candidate ideal points for individual legislators that are ruled out by the roll call data.

In sum, when we know the location of voting alternatives, there exist testable restrictions on the location of legislators' ideal points. These testable restrictions arise solely from finite roll call voting data and the non-parametric assumption of strict concavity of legislators' utility functions. By implication, these testable restrictions open the possibility for recovering information about legislators' ideal points or relative ideological location from roll call data using either the logical conditions we derive in the present study or via statistical models built on our assumptions on legislators' utility functions. Inevitably, a price we have to pay in order to obtain these estimates is that we must first complement the roll-call data with additional information on the location of

[6] We further discuss this condition and provide motivation via graphical methods in section 3.

the voting alternatives.

Before we move to the main body of our analysis, we review the literature on ideal point estimation in greater detail as well as contributions related to our study from microeconomic theory.

Related Literature

The literature on ideal point estimation using roll call data has grown considerably in recent years. With the exception of NOMINATE, existing estimators are based on linear/quadratic utility functions. A one-dimensional model of that form is estimated by Ladha, 1991. Londregan, 2000, incorporates a valence utility component for each item/bill in the roll call record. His estimator, as well as those of Bailey, 2001, and Lewis, 2001, produces estimates of ideal points or, in the latter case, their distributions, when either committee size is small and/or the number of roll call items is small. Based on ideas that originate in Coombs, 1964, Poole, 2000, develops an optimal classification estimation scheme. Clinton and Meirowitz, 2001, 2003, obtain estimators that incorporate constraints emanating from the agenda formation process. Martin and Quinn, 2002, put the Bayesian framework to use in order to estimate ideal points for members of the Supreme Court that change over time.

There also exists a large related literature from statistics and psychology, which we do not survey here. The interested reader will find Keith Poole's recent monograph (Poole, 2005) informative on this, as indeed in so many other aspects of ideal point estimation. We do mention that at least part of the statistical literature with connections to roll call analysis is atheoretical (e.g., Gifi, 1990), in that the associated techniques are not necessarily motivated from or apparently related to the spatial model of voting. Some of these techniques obtain strikingly similar data reduction pictures with estimators built on the spatial model (see De Leeuw, 2006). Quinn and Spirling, 2005, also avoid ideal point estimation but, instead, propose a method to cluster legislators into groups with similar voting patterns.

Questions pertaining to the identification of standard ideal point estimation models are taken up by Rivers, 2003. Issues with the consistency of various estimators have been discussed by a number of scholars (e.g., Brady, 1989, Poole and Rosenthal, 1991, Heckman and Snyder, 1997, and

Londregan, 2000). Lewis and Poole, 2004, obtain standard errors for inference for the NOMINATE estimators, using the parametric bootstrap. Jackman, 2000, 2001, and Bafumi et al., 2005, provide illuminating discussions of some estimation and inference aspects in the context of sampling based Bayesian estimation methods.

Closest to our analysis, in that he explicitly identifies the need to incorporate information on the location of voting alternatives in ideal point estimation, is the work of Londregan, 2000. Londregan, 2000, considers a statistical framework in which individual utilities are subjected to random shocks, and shows that ideal points are not identified without restrictions on the distribution of voting alternatives. He also shows that that distribution is inconsistently estimated using mere roll call data. Londregan moves in the direction of improving on sources of information on voting alternatives by specifying a structural model of the agenda formation. More distant from our analysis is an early paper by Niemi and Weisberg, 1974, on the connection between Guttman scalable roll-call voting records and ideal points. Their main objective is to show that Guttman scalable roll-call records (a 'well behaved' subset of possible roll-call records) do not imply single-peakedness of preferences in one dimension. Niemi and Weisberg also provide graphical examples[7] of Guttman scalable roll-call records that admit ideal points which are different from the corresponding Guttman scores.

Our analysis originates from a different tradition, the literature on the rationalizability of individual demand in economic theory. The seminal paper in this literature is by Sydney Afriat, 1967, who provided necessary and sufficient conditions that must be met by a set of observations of prices and quantity choices of commodities in order for these observations to be consistent with individual maximization of a non-trivial monotone, concave, utility function. Hal Varian, 1982, provides a lucid elaboration on this question and the implications of individual maximization of a classical utility function for the non-parametric estimation of demand.

In the voting context we consider, individuals are faced with a binary choice between two pre-specified alternatives. We have no observations akin to prices and, once the agenda is formed, there is no similar process of individual maximization subject to a budget constraint among an

[7]See figures 4 and 5 in Niemi and Weisberg, 1974, pages 41-42. Some of the utility functions in these plots are evidently not concave.

infinite set of alternatives. Furthermore, unlike the classical theory of demand, we do not require monotonicity of individual preferences. Indeed, preferences in political environments are typically assumed to be satiated, *i.e.,* to possess a well defined ideal point.

The literature most directly related to our analysis focuses on the question of concavifiability of individual preferences. Yakar Kannai, 1977, tackled this question for the case of continuous preferences on infinite convex sets. The relevant case for our purposes is concavifiability of preferences on finite sets. A complete answer in this case is attained in an incisive paper by Marcel Richter and Kam-Chau Wong, 2004. Via an application of the theorem of the alternative they provide an elegant necessary and sufficient condition for the existence of a (strictly) concave utility function that represents given preferences over finite sets. Kannai, 2005, discusses the actual construction of such utility functions.

Our analysis differs from that of Richter and Wong, 2004, in that in our case the preferences among the finite set of alternatives are incompletely specified. In particular, a roll-call record assigns preferences only between the pairs of alternatives directly compared in each item of the record. As a result, our conditions relate the location of certain groups of alternatives (those least preferred between the pair of alternatives compared in each vote) with the remaining alternatives.

We now proceed to our analysis. In the next section, we develop notation and state some definitions. All our results are contained in section 3. Our analysis proceeds in reverse order, compared to the order of presentation of our findings in the introduction. We first derive conditions for rationalizability of individual choices and ideal points when voting alternatives are known. We then show that these conditions are vacuously met for all roll call records and ideal points when the location of voting alternatives is unrestricted. In section 4 we discuss these findings. We conclude in section 5.

2. PRELIMINARIES

Consider a set of n legislators $N = \{1, ..., n\}$. We take as given a roll call voting record of these legislators' "yea" or "nay" decisions on m pairs of voting alternatives. We denote the pair of voting alternatives compared in the j-th voting item of the roll call record by $\mathbf{z}^j, \mathbf{y}^j \in \mathbb{R}^d$,

$j = 1, ..., m$, with $\mathbf{z}^j$, $\mathbf{y}^j$ distinct. The decision of legislator i on the j-th item of the roll call record is given by $v_i^j \in \{yea, nay\}$, where $v_i^j = yea$ indicates a preference for alternative $\mathbf{z}^j$. The preferences of legislator i over voting alternatives are represented by an unobservable utility function $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$.

Let $\widehat{\mathbf{x}}_i \in \mathbb{R}^d$ stand for legislator i's *ideal point* (if it exists), *i.e.*, alternative $\widehat{\mathbf{x}}_i$ is such that

$$u_i(\widehat{\mathbf{x}}_i) > u_i(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{R}^d. \tag{1}$$

Without any restrictions on the shape of legislator i's preferences, we can construct a (continuous) utility function u_i with given ideal point in order to "account" for the roll call voting data in a trivial manner. By this we mean that we can assign values $u_i(\mathbf{z}^j)$, $u_i(\mathbf{y}^j)$, and $u_i(\widehat{\mathbf{x}}_i)$ so that both

$$v_i^j = \begin{cases} yea & \text{if} \quad u_i(\mathbf{z}^j) > u_i(\mathbf{y}^j) \\ nay & \text{if} \quad u_i(\mathbf{z}^j) < u_i(\mathbf{y}^j) \end{cases}, \ j = 1, ..., m, \tag{2}$$

and (1) are true for all legislators. We illustrate this graphically in figure 1 for a roll call record with six voting items ($m = 6$), on all of which legislator i votes *yea*. The utility function represented with the solid line in figure 1 is consistent with these voting decisions and with legislator i having an ideal point $\widehat{\mathbf{x}}_i$.

Indeed, except for 'knife-edge' situations,[8] a construction similar to that represented in figure 1 is possible for every ideal point $\widehat{\mathbf{x}}_i$. For example, we could easily represent legislator i's ideal point to be to the left of alternative z^1 in figure 1, or we could similarly place legislator i's ideal point to the right of z^6, or between y^3 and y^1, etc. Therefore, in order to use the roll call voting data to extract information about legislator i's preferences, such as the location of legislator i's ideal point, we must impose restrictions on the 'shape' of these preferences.

[insert figure 1 about here]

It is exactly via restrictions of that nature that various techniques in political methodology produce estimates of legislators' ideal points. These techniques posit a particular functional form

[8] e.g., the ideal point of legislator i cannot coincide with an alternative that legislator i votes against.

for the utility function u_i, in fact parameterized by legislator i's ideal point, $\widehat{\mathbf{x}}_i$. Invariably, these assumptions require symmetry of the utility function u_i around the ideal point $\widehat{\mathbf{x}}_i$.

Without assuming symmetry or particular parametric forms for legislators' utility functions, the strongest condition we may impose is strict concavity. A function is strictly concave if the (weighted) average of the values of the function evaluated at two points is smaller than the value of the function evaluated at the (weighted) average of these points. Specifically, a utility function u is strictly concave if for all distinct $\mathbf{x}$, $\mathbf{x}' \in \mathbb{R}^d$, and for all $\lambda \in (0, 1)$

$$u(\lambda \mathbf{x} + (1 - \lambda)\mathbf{x}') > \lambda u(\mathbf{x}) + (1 - \lambda)u(\mathbf{x}'). \tag{3}$$

For example, the utility function represented with a dashed line in figure 1 is strictly concave, while those represented with a solid and dotted lines, respectively, are not.

Concavity implies diminishing returns in that the 'marginal' utility gain of a legislator, moving from any particular alternative toward this legislator's ideal point, is decreasing. Strict concavity also implies that legislators have a unique ideal point, with utility decreasing in every direction from this ideal point. Thus, with strictly concave preferences, it makes sense to label legislators as, *e.g.*, pro-leftist, liberal or conservative, anti-environmentalist, etc. These notions become ill-defined for utility functions without any restrictions, such as the utility function represented with a solid line in figure 1. In one dimension, strict concavity implies the single-peakedness condition familiar from the social choice literature. In fact, it is a stronger assumption than single-peakedness: the utility represented with a dotted line in figure 1 is single peaked but not concave. Thus, our conclusions in the sequel, in particular those in subsection 3.3, are valid with weaker alternative assumptions.

We will say that the roll call data are *rationalizable* for legislator i if there exists a strictly concave utility function so that all of i's voting decisions are consistent with that utility function. More precisely:

Definition 1 *A roll call voting record is* rationalizable *for legislator i if there exists a strictly concave utility function u_i such that (2) holds.*

Accordingly, we shall say that a utility function u_i *rationalizes* legislator i's voting decisions if i's roll call record is rationalizable with this function. Observe that our definition of rationalizability requires i's vote in favor of voting alternative $\mathbf{z}^j$ ($\mathbf{y}^j$) to imply a *strict* preference for this alternative over $\mathbf{y}^j$ ($\mathbf{z}^j$). This is the relevant case, since it represents the most restrictive information contained in a particular roll call voting record. The analysis is trivially extended if a vote in favor or against an alternative represents weak preference. For the same reason, the analysis is easily extended if we allow abstention in the roll call data.

We also give the following definition concerning rationalizability of legislator i's voting decisions by a utility function with a specific ideal point.

Definition 2 *A roll call voting record is* rationalizable with ideal point $\widehat{\mathbf{x}}_i$ *for legislator i, if there exists a strictly concave utility function u_i such that (1) and (2) hold.*

If the roll call voting record of legislator i is rationalizable with some ideal point $\widehat{\mathbf{x}}_i$, then on the basis of the evidence in this roll call record we cannot reject the hypothesis that legislator i's ideal point is indeed $\widehat{\mathbf{x}}_i$. As we will see in the next section, there exist testable restrictions on the location of the voting alternatives $\mathbf{z}^j, \mathbf{y}^j$ that must be satisfied in order for a particular voting record to be rationalizable. In turn, for given voting alternatives $\mathbf{z}^j, \mathbf{y}^j$, there exist testable restrictions on the location of legislators' ideal points, $\widehat{\mathbf{x}}_i$. On the other hand, if the voting alternatives, $\mathbf{z}^j, \mathbf{y}^j$, are unknown, then on the basis of the evidence in the roll call voting record we cannot reject any hypothesis regarding the location of legislators' ideal points.

3. TESTABLE RESTRICTIONS

In the first two subsections of this section, we assume that the exact location of the m pairs of voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$ is known. In subsection 3.1 we seek conditions that render an individual legislator's voting record rationalizable. In subsection 3.2 we derive conditions on the location of the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$ and ideal point $\widehat{\mathbf{x}}_i$ in order for the roll call record of legislator i to be rationalizable with this ideal point. In the last subsection, 3.3, we pursue the same analysis for the case in which the location of the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$, is unknown. We state all of our results in this section without proofs, which we have relegated to the appendix.

Before we proceed, we introduce some necessary notation. For a subset $M \subseteq \{1, ..., m\}$ of the voting items in the roll call record, we let $N_i(M)$ represent the voting alternatives that i votes against in this subset of voting items, *i.e.*,

$$N_i(M) = \left\{ \mathbf{x} : \mathbf{x} = \mathbf{z}^j \text{ if } v_i^j = nay, \text{ or } \mathbf{x} = \mathbf{y}^j \text{ if } v_i^j = yea, j \in M \right\}.$$

For example, consider a roll call record with $m = 5$ items and let the voting decisions of legislator i be $v_i^1 = yea$, $v_i^2 = nay$, $v_i^3 = yea$, $v_i^4 = nay$, and $v_i^5 = nay$. In this case, for the subset of voting items $M = \{1, 2, 5\}$ we have $N_i(M) = \left\{\mathbf{y}^1, \mathbf{z}^2, \mathbf{z}^5\right\}$; for $M = \{2, 3, 4, 5\}$, we have $N_i(M) = \left\{\mathbf{z}^2, \mathbf{y}^3, \mathbf{z}^4, \mathbf{z}^5\right\}$. We similarly define $Y_i(M)$ as the set of voting alternatives that i votes for in the roll call voting items M, *i.e.*,

$$Y_i(M) = \left\{ \mathbf{x} : \mathbf{x} = \mathbf{z}^j \text{ if } v_i^j = yea, \text{ or } \mathbf{x} = \mathbf{y}^j \text{ if } v_i^j = nay, j \in M \right\}.$$

Continuing our previous example, we now have $Y_i(M) = \left\{\mathbf{z}^1, \mathbf{y}^2, \mathbf{z}^3, \mathbf{y}^4, \mathbf{y}^5\right\}$ for $M = \{1, 2, 3, 4, 5\}$, $Y_i(M) = \left\{\mathbf{y}^2, \mathbf{z}^3, \mathbf{y}^5\right\}$ for $M = \{2, 3, 5\}$, etc.

[insert figure 2 about here]

Lastly, we introduce the notion of the *convex hull* of a finite set of alternatives $K = \{\mathbf{x}_1, ..., \mathbf{x}_k\} \subset \mathbb{R}^d$. The *convex hull* of K, denoted $C(K)$, is the set of points that can be expressed as convex combinations of elements of K:

$$C(K) = \left\{ \sum_{h=1}^{k} \lambda_h \mathbf{x}_h : \lambda_h \geq 0 \text{ for all } h, \text{ and} \sum_{h=1}^{k} \lambda_h = 1 \right\}.$$

This is illustrated graphically in figure 2 for a set K consisting of nine alternatives in a two dimensional space. For example, point $\mathbf{z}$ in figure 2 belongs in the convex hull of K, since it can be written as a convex combination of elements $\mathbf{x}_4$ and $\mathbf{x}_5$ of K. Thus, the convex hull of set K, $C(K)$, in figure 2 consists of all the points in gray in that figure.

3.1 Legislators with Concave Utility Functions

Before we even contemplate using the restriction that utility functions are concave in order to extract information from the roll call data about the ideal point of legislator i, we must verify that this legislators' preferences can be represented by such a concave utility function. Thus, in this subsection we ask: what must be true about the location of voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$, in order for the voting decisions of legislator i to be consistent with the hypothesis that this legislator has a strictly concave utility function? The following theorem provides a complete answer:

Theorem 1 *Fix the location of the voting alternatives. The roll call voting record of legislator i is rationalizable if and only if for every subset of the roll call items $M \subseteq \{1, ..., m\}$, there exists at least one alternative that i votes against in that subset which cannot be written as a convex combination of alternatives that i votes for in that subset, i.e.,*

$$N_i(M) \setminus C(Y_i(M)) \neq \emptyset, \text{ for all } M \subseteq \{1, ..., m\}. \tag{4}$$

Condition (4) states that in order for the roll call data to be consistent with the hypothesis that legislator i has a concave utility function, the voting alternatives that legislator i votes against must be arranged in a particular manner with respect to the alternatives that i votes for. The necessity of condition (4) becomes obvious by graphical arguments. As usual, the one-dimensional case is easiest to illustrate. This is done in figure 3 for four possible configurations of the voting alternatives $\{z^j, y^j\}$, $j = 1, 2$, where the voting record requires $v_i^1 = v_i^2 = yea$. While cases (a) to (c) of figure 3 satisfy condition (4), case (d) does not. In that case we have both $y^1, y^2 > z^1$ and $y^1, y^2 < z^2$. Thus, any utility function u_i that rationalizes legislator i's preferences must reverse monotonicity, initially strictly decreasing, then strictly increasing. This is inconsistent with the concavity of any function u_i.

[insert figure 3 about here]

Intuitively, the argument for the sufficiency of condition (4) relies on the fact that the roll call voting record only pins down legislator i's preferences between the two alternatives $\mathbf{z}^j, \mathbf{y}^j$. In other words, the data in the roll call record do no constrain legislator i's utility comparisons between distinct alternatives in different voting items. As a consequence, since we can certainly construct strictly concave preferences for a given pair of voting alternatives $\mathbf{z}^j, \mathbf{y}^j$, we can ensure these preferences can be extended to an additional pair by requiring that the least preferred between the alternatives in the new pair is least preferred to all the preceding voting alternatives. In order to be able to add the j-th pair with, say, $v_i^j{=}yea$, it suffices that $\mathbf{y}^j$ is "far away" from $\mathbf{z}^j$ and the alternatives that have preceded in this construction (*i.e.,* it suffices that $\mathbf{y}^j$ cannot be written as a convex combination of these alternatives). Theorem 1 asserts that if condition (4) is met, then we can choose the order of the voting items appropriately in order to construct a consistent preference ordering of all voting alternatives in the above described fashion.

[insert figure 4 about here]

In fact, the proof in the appendix informally outlines an algorithm to determine the ordering of the pairs $\{\mathbf{z}^j, \mathbf{y}^j\}$, which we successively rank by assigning them to indifference contours, backwards from last to first. As a graphical example, in figure 4 we illustrate the construction of one possible utility function (specifically the indifference contours of this function) for a rationalizable voting record with $m = 5$ voting items in $d = 2$ dimensions. As a practical matter, verifying the validity of condition (4) as well as the actual construction of the desired utility functions can be achieved efficiently by solving a linear programming problem similar to that specified in lemma 2 in the appendix.

In principle, the necessary and sufficient condition (4) requires checking all possible subsets $M \subseteq \{1, ..., m\}$, of the m voting items in the roll-call record. This amounts to checking $\sum_{h=2}^{m} \binom{m}{h}$ distinct possible subsets $M \subseteq \{1, ..., m\}$.[9] In the one-dimensional case, this burden can be simplified considerably, as it suffices to verify the condition only for pairs of voting items, not for subsets M of higher cardinality. Thus, we have the following corollary:

Corollary 1 *Suppose one issue dimension ($d = 1$) and fix the location of the voting alternatives. A roll call voting record is rationalizable for legislator i if and only if for all pairs of voting items in the roll call record, the two alternatives that legislator i votes against are not both contained in the interval defined by the two alternatives that legislator i votes for in this pair of voting items, i.e.,*

$$N_i(\{h, j\}) \setminus C(Y_i(\{h, j\})) \neq \emptyset, \text{ for all distinct } h, j \in \{1, ..., m\}\,. \tag{5}$$

[9] We need not check single items, $M = \{j\}$, since we have assumed in the outset that $z^j \neq y^j$.

Corollary 1 follows from the fact that a convex hull in one dimension ($d = 1$) is an interval, hence it is characterized by two points. Thus, if condition (4) fails for legislator i and a subset $M \subseteq \{1, ..., m\}$ of three or more voting items, then the condition must also fail for a pair of the voting items $\{j, h\} \subset M$. In particular, these voting items j, $h \in M$ are obtained by identifying the voting alternatives that are approved by i and are located at the extremes of the interval $C(Y_i(M))$. This is illustrated graphically in figure 5(a). In that figure, condition (4) fails for the set of voting items $M = \{1, 2, 3, 4\}$ and a roll call record satisfying $Y_i(M) = \left\{z^1, z^2, z^3, z^4\right\}$. Note that, in this case, the convex hull of the alternatives that are approved by i is given by the interval $[z^1, z^4]$, *i.e.*, $C(Y_i(M)) = [z^1, z^4]$. But then, if we consider the pair of voting items $M = \{1, 4\}$, we have $Y_i(M) = \left\{z^1, z^4\right\}$ and the convex hull remains the same, *i.e.*, $C(Y_i(\{1, 4\})) = [z^1, z^4]$. Thus, if condition (5) holds in one dimension, so does condition (4).

[insert figure 5 about here]

Intuition might suggest that a generalization of corollary 1 obtains in more than one dimensions ($d > 1$) by requiring that only subsets of $d + 1$ voting items need to be checked to verify condition (4). As is illustrated in figure 5(b) for a two dimensional space ($d = 2$), this is not true. In that figure, condition (4) holds for all triplets ($d + 1 = 3$) of voting items, but fails when we consider all four items in the roll call record. As already alluded above, this difference arises from the fact that, while a convex hull in one dimension ($d = 1$) has at most two distinct vertex points, there is no such limit on the number of vertices in $d = 2$ or more dimensions.[10]

3.2 Restrictions on Ideal Points

Maintaining the assumption that the location of the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$ is known, suppose we are given a voting record that is rationalizable for legislator i, i.e., the roll call record satisfies condition (4) of theorem 1 so that there exists a concave utility function for this legislator that represents her recorded preferences over the voting alternatives. In this section we ask, what do these roll call voting data imply about the ideal point of legislator i? Put differently, what must

[10] Contrast this with the necessary and sufficient condition of Richter and Wong, 2004, for the case of completely specified preferences.

be true about the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$, in order for the roll call data to reject the hypothesis that legislator i's ideal point is $\widehat{\mathbf{x}}_i$? The answer is a generalization of condition (4):

Theorem 2 *Fix the location of the voting alternatives. The roll-call voting record of legislator i is rationalizable with ideal point* $\widehat{\mathbf{x}}_i$ *if and only if for every subset of the roll call items,* $M \subseteq \{1, ..., m\}$*, there exists at least one voting alternative that i votes against in that subset which cannot be written as a convex combination of i's ideal point,* $\widehat{\mathbf{x}}_i$*, and the alternatives that i votes for in that subset, i.e.,*

$$N_i(M) \setminus C(Y_i(M) \cup \{\widehat{\mathbf{x}}_i\}) \neq \emptyset, \text{ for all } M \subseteq \{1, ..., m\}\,. \tag{6}$$

Condition (6) provides a precise set of testable restrictions on the location of legislator i's ideal point arising from a given roll call voting record, assuming that the voting alternatives are known, and that this legislator has a strictly concave utility function. Notice that condition (6) subsumes condition (4): a roll-call voting record that is not rationalizable cannot be rationalizable with any ideal point.

[insert figure 6 about here]

We provide a graphical illustration of the implications of theorem 2 in figure 6. In this figure we depict eight voting alternatives associated with four voting items ($m = 4$) in a two-dimensional space of voting alternatives ($d = 2$). It is straightforward to verify that, for these alternatives, the roll call record summarized by $Y_i(\{1, 2, 3, 4\}) = \{\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4\}$ is rationalizable for legislator i, *i.e.,* condition (4) is satisfied. Now, application of condition (6) restricts legislator i's ideal point, $\widehat{\mathbf{x}}_i$, to lie outside the areas marked gray in figure 6. Thus, the roll call voting record contains information about i's ideal point solely due to the non-parametric restriction of strict concavity of i's utility function.

As is true for theorem 1, the one-dimensional case admits a further simplification of the condition derived in theorem 2:

Corollary 2 *Suppose one issue dimension* ($d = 1$) *and fix the location of the voting alternatives. The roll call voting record of legislator i is rationalizable with ideal point* $\widehat{x}_i$ *if and only if condition*

(5) holds, and, for each of the m voting items, the alternative that legislator i votes against is not contained in the interval defined by $\widehat{x}_i$ and the alternative that i votes for, i.e.,

$$N_i(\{j\}) \cap C(Y_i(\{j\}) \cup \{\widehat{x}_i\}) = \emptyset, \text{ for all } j \in \{1, ..., m\}. \tag{7}$$

Of course, condition (5), which is necessary and sufficient for rationalizability in one dimension, is necessary for rationalizability with a given ideal point. In addition to that condition, it is sufficient to verify condition (7), that requires that for each pair $\{z^j, y^j\}$ the interval defined by legislator i's ideal point and the alternative preferred by legislator i between z^j, y^j does not contain the other one of these two alternatives. For example, when we apply this condition in the case of the roll call record and voting alternatives represented in figure 3, we deduce that the ideal point of legislator i must lie between alternatives y_1 and y_2, in the case of figure 3(a), and anywhere below alternative y_1 in cases (b) and (c) of figure 3.

In the next section, we ask whether such testable restrictions have any bearing when the location of the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$ is not known.

3.3 Ideal Points with Unknown Voting Alternatives

In this subsection we show that the conditions derived in the previous subsection are vacuously met for all legislators and for every number of issue dimensions $d \geq 1$, if the location of the voting alternatives is unknown. By that we mean that for *all* roll call voting records and *all* ideal points $\widehat{\mathbf{x}}_i$ for the n legislators, we can locate the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$, *and* construct n strictly concave utility functions u_i for the n legislators that both: (a) have the required ideal points $\widehat{\mathbf{x}}_i$, and (b) are perfectly consistent with the m voting decisions of each of the n legislators as required by condition (2).

Hence, we show:

Theorem 3 *For all roll call voting records and all possible ideal points for the n legislators, there exist voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$ that render the roll call voting record rationalizable with these ideal points for all legislators $i \in N$.*

It is important to emphasize that theorem 3 states that all possible roll call records and all possible ideal points for the n legislators can *all* be rationalized by appropriately choosing the location of the voting alternatives. That is, one choice of the location of the voting alternatives will work for all legislators at the same time, *i.e.*, we need not choose different voting alternatives for each legislator separately. Furthermore, theorem 3 is valid *a fortiori* if we impose weaker requirements on legislators' utility functions, for instance, if we relax concavity to quasi-concavity. Quasi-concave allows for the shape of the utility function used (in symmetric parametric form) by Poole and Rosenthal, 1985, 1991.

Theorem 3 is shown by construction, which is illustrated in figure 7. Figure 7(a), represents an example of this construction for the one-dimensional case. The essence of the result stems from the fact that there exists a way to arrange the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$, such that every roll call voting record is necessarily rationalizable for all legislators. In the one dimensional case, this arrangement amounts to locating one of the two voting alternatives in each voting item in some arbitrary order, then locating the remaining voting alternatives in a non-overlapping interval with the reverse order of voting items than the one previously used. Locating the voting alternatives in this manner guarantees condition (4) is not violated no matter what the roll call voting record is. It is then a simple additional step to translate the above arrangement in the space of alternatives where the given ideal points have already been located in order to ensure that the added restrictions of condition (6) are not violated.

[insert figure 7 about here]

We emphasize that the construction used in proving theorem 3 is not 'knife edge.' Typically, the choice of location for the voting alternatives $\mathbf{z}^j$, $\mathbf{y}^j$, can be perturbed in an open set containing these alternatives. Furthermore, in the generic case when legislators' ideal points are distinct, the proof in theorem 3 ensures that at least one of alternatives $\mathbf{z}^j, \mathbf{y}^j$, lies in the Pareto set for each voting item j.[11] In the next section, we discuss the implications of our results for the estimation of legislators' ideal points from roll call voting data, and for the interpretation of the output of existing techniques using such data.

[11] The requirement for Pareto optimality of the voting alternatives is not imposed by any existing ideal point estimation techniques.

4. DISCUSSION

In order to convey the full power of theorem 3, we discuss its implications for the connection between roll call data and the ideological location of legislators in the familiar context of the US Senate. In figure 8, we represent a one dimensional policy space, construed as a typical left-right dimension. In figure 8(a) we depict the ideal points of six Senators. These ideal points are obtained from the analysis of the actual roll call record (with a total of $m = 675$ items) of all senators in the 108th Congress, using the W-NOMINATE procedure. As is obvious from the reported relative ranking of these Senators, we have chosen Senators that lie at both the extreme, and moderate positions within each of the two parties, according to these estimates.

[insert figure 8 about here]

In figures 8(b) and 8(c), we display different hypothetical ideal points for these six senators. These ideal point locations are intentionally in stark conflict with those represented in figure 8(a). For example, in figure 8(c), the two most extreme Senators from the left and the right according to NOMINATE, are both placed at the extreme left, while the two most moderate senators are both placed at the extreme right. Yet, theorem 3 ensures that we can locate all the voting alternatives for the 675 roll call items in this one dimensional space, and we can find strictly concave utility functions for these six senators with the ideal points in figure 8(c) (or 8(b)), in order to perfectly account for all voting decisions of these senators, for all 675 items in the roll call record.

In other words, the roll call voting record imposes no constraints whatsoever on legislators' ideal points. In fact, all possible ideal points for all 100 senators are consistent with the hypothesis that the voting decisions of these legislators arise from strictly concave utility functions with the postulated ideal points. This is always true even if we limit the space of voting alternatives to be one-dimensional. The mere voting decisions recorded in the roll call data simply cannot reject the hypothesis that legislators' ideal points are configured in some arbitrary manner.

At this point the reader may object that prevalent ideal point estimators produce 'reasonable' estimates of ideal points, despite the indeterminacy established in theorem 3. For example, the configuration of ideal points in figure 8(a) is a very plausible rendition (at least in relative terms) of the perceived actual ideological position of these senators. But the fact that these estimates conform with our prior beliefs cannot constitute a standard on the basis of which we can proclaim the validity of the implied inferences. Unless we are willing to assert that these senators have symmetric (Gaussian, quadratic, or other) preferences, we have not learned, upon observing the roll call data, that senator Nickles (R-OK) is to the right of senator Dole (R-NC), as one interpretation of figure 8(a) might suggest, simply because the voting decisions of these two Senators and their 98 colleagues in the Senate are also consistent with them having the ideal policies depicted in figure 8(b) or (c), or any other.

Of course, 'reasonable' estimates need not be taken to be those that conform with prior beliefs. In evaluating the estimates from roll call analyses we may also ask whether these estimates are an efficient, low-dimensional summary of the patterns of voting and coalition formation in the legislature. There are numerous criteria via which the roll call record, viewed as a high-dimensional matrix of zeros and ones, can be 'condensed' into a lower dimensional picture, and ideal point estimation techniques constitute one successful method for achieving such a summary. Theorem 3 has no ramifications for our ability to achieve this type of successful data reduction of roll call records. Nevertheless, the reader should be cautioned to the fact that very similar summaries can be obtained using estimators that are not directly motivated on the spatial voting model.[12]

Under this lens, the scores assigned to the six senators in figure 8(a) can be interpreted to mean that, e.g., senator Sarbanes' (D-MD) voting behavior is 'closer' to that of Senator Schumer (D-NY) than to the voting behavior of Senator Nelson (D-FL), where this proximity is evaluated according to some appropriate metric criterion. Independently of the particular metric used in order to evaluate the proximity of observed voting behavior between different legislators, the important distinction here is that it is not necessarily the case that proximity as measured by these data reduction methods arises from the ideological proximity of these legislators. To return to the example, senators Sarbanes (D-MD) and Schumer (D-NY) seem to behave similarly, compared to senator Nelson (D-FL), but not necessarily, as theorem 3 cautions, because they are more liberal than senator Nelson (D-FL).

[12] In the US Congress setting, De Leeuw, 2006, p. 32, attributes this to the "extreme polarization of US politics."

In short, some interpretations of the output of ideal point estimation techniques must be qualified in view of our results. The use of roll call data in order to locate legislators' ideal policies on some underlying ideological scale rests critically on symmetry or stronger parametric restrictions imposed on legislators' utility functions. In the absence of such restrictions, our results establish that in order to use the roll call data for inferences regarding the ideological location of legislators, we must incorporate additional information on the location of voting alternatives in our estimation methods.

Ideal Point Restrictions & Additional Data

In the last part of this section, we briefly discuss the possible sources of additional information that can be combined with the conditions derived in our analysis in order to derive information on legislators' ideal points from roll call voting records. A number of scholars have already moved in this direction, by explicitly modelling or accounting for the structure of the legislative agenda. Londregan, 2000, suggests use of the information from the identity of the sponsors of bills. The strategic calculus of the proposer may imply a particular configuration of the status quo, the proposal, and the ideal point of the sponsor of the bill. Likewise, Clinton and Meirowitz, 2001, impose the restriction that successful bills become the status quo voting alternative in ensuing votes,[13] while Clinton and Meirowitz, 2003, model additional structure of the agenda.

Several alternatives can complement these attempts for a detailed account of the legislative structure and/or the agenda formation process. One obvious possibility is to elicit the location of the voting alternatives on pre-specified scales via expert surveys. These expert survey responses can then produce estimates of the location of voting alternatives, in order to test the rationalizability of the data via condition (4), and/or obtain restrictions on legislators' ideal points via condition (6). In fact, order restrictions on the location of voting alternatives are sufficient for that purpose, at least in one policy dimension. In other words, these surveys need not pin down the exact location of these alternatives, as long as survey respondents are able to identify the relative location of voting

[13] The identifying role of this additional restriction is strongest in one dimension, but not in higher dimensions.

alternatives from different voting items in the roll call.

Alternatively, since survey responses will typically reflect *ex post* uncertainty about the location of the voting alternatives, such survey data can be used to formulate informative priors on the location of the voting alternatives. With such priors, as well as proper priors on the location of legislators' ideal points, one can in principle obtain a posterior on the location of these legislators' ideal points via application of condition (6) and Bayes' rule. Note that a common (*i.e.* identical) non-degenerate prior on the location of all these quantities (voting alternatives[14] & ideal points) will necessarily lead to different posteriors for the ideal points, unless the roll call data are badly conditioned.

In some (admittedly rare) instances, the location of policy alternatives is readily available from the content of the legislation. For example, Krehbiel and Rivers, 1988, analyze roll call votes on the minimum wage, the level of which generates a natural one dimensional scale. Even if the content of legislation does not readily translate in Euclidean space, in some legislative or other political contexts it may be possible to locate the voting alternatives through the content analysis of the text of these alternatives in a factor analytic fashion, such as is used by the Manifesto project (*e.g.,* Budge et al., 2001).

5. CONCLUSIONS

We have derived necessary and sufficient conditions in order for roll call voting data to be consistent with the choices of legislators with strictly concave utility representations. These conditions imply simple testable restrictions on the location of legislators' ideal points when the location of the alternatives being voted upon by the legislature is known. These testable restrictions arise solely through the assumption of concavity of legislators' utility function, without any parametric assumptions or symmetry requirements on legislators' preferences.

If the location of voting alternatives is unrestricted (as is assumed in prevalent political methodology techniques for the estimation of legislators' ideal points) then the derived conditions

[14] The question of whether an exchangeability assumption on the location of voting alternatives is appropriate, is beyond the scope of the present study.

are vacuously satisfied for arbitrary ideal points for the legislators. If we fix arbitrary ideal points for these legislators, we can always locate the voting alternatives *and* find strictly concave utility functions for these legislators that both (a) have the postulated ideal points, and (b) perfectly account for the observed voting decisions. This is true no matter how large the roll-call voting record is, even if we restrict the space of alternatives in one dimension.

If we maintain an epistimological perspective that requires parsimonious interpretation of the output of our empirical investigations, then our results place a caveat against certain interpretations of the output of various ideal point estimation techniques. In particular, we cannot conclude using the mere 'yea' or 'nay' decisions in a roll call record that particular legislators are more or less conservative, liberal, or other, unless, of course, we assert that legislators have utility functions that are symmetric or assume standard parametric functional forms. In the absence of such restrictions on utility functions, and if our goal is to estimate legislators' ideological position, then we have to incorporate more information in our estimators. In particular, the necessary and sufficient rationalizability conditions developed in this study imply that information on the location of voting alternatives, combined with roll call voting records, can bear estimates of the ideological location of individual legislators.

APPENDIX

In this appendix we provide proofs of the theorems and corollaries stated in section 3. For the reader's ease of reference, we restate the results to be proven. We start by showing theorem 1:

Theorem 1 *Fix the location of the voting alternatives. The roll call voting record of legislator i is rationalizable if and only if*

$$N_i(M) \setminus C(Y_i(M)) \neq \emptyset, \text{ for all } M \subseteq \{1, ..., m\}. \tag{4}$$

Proof. **Necessity:** Suppose there exists strictly concave $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ that rationalizes i's record, but (4) fails for some $\widetilde{M} \subseteq \{1, ..., m\}$. By lemmas 5.114, and 5.113 in Aliprantis and Border (1999), u_i is minimized at an extreme point $\widetilde{\mathbf{x}}$ of $C(Y_i(\widetilde{M}))$ so that $u_i(\widetilde{\mathbf{x}}) = \min\left\{u_i(\mathbf{x}) : \mathbf{x} \in C(Y_i(\widetilde{M}))\right\}$, and by lemma 5.123 in the same reference, $\widetilde{\mathbf{x}} \in Y_i(\widetilde{M})$. Without loss of generality, let $\widetilde{\mathbf{x}} = \mathbf{z}^j$, $j \in \widetilde{M}$. Since u_i rationalizes i's record and (4) fails for $\widetilde{M}$, we have $u_i(\mathbf{z}^j) > u_i(\mathbf{y}^j)$ and $\mathbf{y}^j \in C(Y_i(\widetilde{M}))$, a contradiction. This establishes the necessity of (4).

Sufficiency: We now assume that (4) holds. We wish to show that there exists $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ that rationalizes i's roll call record. We shall use the following lemma:

Lemma 1 *Consider a finite set $K \subset \mathbb{R}^d$ and strictly concave $u_i : C(K) \longrightarrow \mathbb{R}$ that represents i's preferences over K. If $\mathbf{x} \in \mathbb{R}^d$ is such that $\mathbf{x} \notin C(K)$ and $\mathbf{x}' \succ_i \mathbf{x}$ for all $\mathbf{x}' \in K$, then there exists another strictly concave $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ that represents i's preferences over $K \cup \{\mathbf{x}\}$.*

Proof. Condition (G′) of Richter and Wong, 2004, page 345, is satisfied for $K \cup \{\mathbf{x}\}$, since $\mathbf{x}' \succ_i \mathbf{x}$ for all $\mathbf{x}' \in K$. ■

Intuitively, lemma 1 is obvious since we can assign a sufficiently small value on $u_i(\mathbf{x})$ in order to ensure that (3) holds in $C(K \cup \{\mathbf{x}\})$. We shall now pursue a proof of the sufficiency of (4) by induction on the number of votes m included on the roll-call voting record. In particular, the sufficiency of (4) of theorem 1 is trivially true for $m = 1$.

We shall show that if (4) is sufficient for $m = m'$ voting items, then it is also sufficient for $m = m' + 1$. So suppose that (4) is satisfied for $m = m' + 1$ voting items. Since (4) holds for $\widetilde{M} = \{1, ..., m', m' + 1\}$, the set $\widetilde{K} = N_i(\widetilde{M}) \setminus C(Y_i(\widetilde{M}))$ is non-empty. Among the finite number of alternatives $\mathbf{x} \in \widetilde{K}$, select one that is a vertex of $C(\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\})$. Such a vertex point must exist since $\widetilde{K}$ is finite, $\widetilde{K} \cap C(Y_i(\widetilde{M})) = \emptyset$, and $C(Y_i(\widetilde{M})) \subset C(\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\})$. Without loss of generality, assume this alternative is $\mathbf{z}^{m'+1} \in N_i(\widetilde{M})$.

By the inductive hypothesis that condition (4) is sufficient for the rationalizability of $m = m'$ votes, there exist strictly concave $u_i : C(\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\}) \longrightarrow \mathbb{R}$ that rationalizes i's voting record for all votes $j \in \{1, ..., m'\}$. If necessary,[15] take the restriction of this function u_i to $C(\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \setminus \{\mathbf{z}^{m'+1}\})$. We now distinguish two cases:

Case 1, $\mathbf{y}^{m'+1} \in C(\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \setminus \{\mathbf{z}^{m'+1}\})$: We set $K = \cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\} \setminus \{\mathbf{z}^{m'+1}\}$ and apply lemma 1 to this K and $\mathbf{z}^{m'+1}$, assuming that $\mathbf{x} \succ_i \mathbf{z}^{m'+1}$ for all $\mathbf{x} \in K$. By that lemma, i's entire voting record is rationalizable.

Case 2, $\mathbf{y}^{m'+1} \notin C(\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \setminus \{\mathbf{z}^{m'+1}\})$: In this case we first apply lemma 1 in the same manner as above to $\mathbf{y}^{m'+1}$ with $K = \cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \setminus \{\mathbf{z}^{m'+1}\}$. We then proceed to apply the same lemma a second time to $\mathbf{z}^{m'+1}$ assuming $K = \cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\} \setminus \{\mathbf{z}^{m'+1}\}$.

This completes the proof of sufficiency. ■

We now proceed to show corollary 1:

Corollary 1 *Suppose one issue dimension ($d = 1$) and fix the location of the voting alternatives. A roll call voting record is rationalizable for legislator i if and only if*

$$N_i(\{h, j\}) \setminus C(Y_i(\{h, j\})) \neq \emptyset, \textit{ for all distinct } h, j \in \{1, ..., m\}. \tag{5}$$

Proof. We only need to show sufficiency. If $d = 1$, for any M with $|M| \geq 2$, $C(Y_i(M)) = C(Y_i(\{h, j\}))$ for some $h, j \in M$, where without loss of generality we can assume $C(Y_i(\{h, j\})) = [z^j, z^h]$. Thus, if condition (4) fails for M, we must have $y^j, y^h \in [z^j, z^h]$, *i.e.*, it must be that condition (5) fails. ■

The proof of theorem 2 essentially duplicates arguments used in proving theorem 1. We state the proof separately in order to avoid notation clutter.

Theorem 2 *Fix the location of the voting alternatives. The roll-call voting record of legislator i is*

[15] The exclusion of $z^{m'+1}$ is necessary (or of consequence) only if there exist multiple $x \in \widetilde{K}$ with $x = z^{m'+1}$.

rationalizable with ideal point $\widehat{\mathbf{x}}_i$ *if and only if*

$$N_i(M) \setminus C(Y_i(M) \cup \{\widehat{\mathbf{x}}_i\}) \neq \emptyset, \textit{ for all } M \subseteq \{1, ..., m\}\,. \tag{6}$$

Proof. Necessity: see proof of theorem 1.

Sufficiency: We now assume that (6) holds. We wish to show that there exists strictly concave $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ with ideal point $\widehat{\mathbf{x}}_i$ that rationalizes the voting record of legislator i. We first show that there exists u_i that rationalizes i's preferences over $\cup_{j=1}^{m}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}$. We proceed by induction on the number of votes m included on the roll-call voting record. In particular, condition (6) is sufficient for $m = 1$, since we can write a strictly concave utility function for the alternatives $\{\mathbf{z}^j, \mathbf{y}^j, \widehat{\mathbf{x}}_i\}$.

We shall now show that if condition (6) is sufficient for $m = m'$ voting items, then it is also sufficient for $m = m'+1$. So suppose that (6) is satisfied for $m = m'+1$ voting items. Then, for $\widetilde{M} = \{1, ..., m', m'+1\}$, the set $\widetilde{K} = N_i(\widetilde{M}) \setminus C(Y_i(\widetilde{M}) \cup \{\widehat{\mathbf{x}}_i\})$ is non-empty. Among the finite number of alternatives $\mathbf{x} \in \widetilde{K}$, select one that is a vertex of $C(\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\})$. Such a vertex point must exist since $\widetilde{K}$ is finite, $\widetilde{K} \cap C(Y_i(\widetilde{M}) \cup \{\widehat{\mathbf{x}}_i\}) = \emptyset$, and $C(Y_i(\widetilde{M}) \cup \{\widehat{\mathbf{x}}_i\}) \subset C(\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\})$. Without loss of generality, assume this alternative is $\mathbf{z}^{m'+1} \in N_i(\widetilde{M})$.

By the inductive hypothesis that (6) is sufficient for $m = m'$ votes, there exist strictly concave $u_i : C(\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \longrightarrow \mathbb{R}$, that rationalizes legislator i's voting record for voting items $j \in \{1, ..., m'\}$. If necessary, take the restriction of this function u_i to $C((\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \setminus \{\mathbf{z}^{m'+1}\})$. We now distinguish two cases:

Case 1, $\mathbf{y}^{m'+1} \in C((\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \setminus \{\mathbf{z}^{m'+1}\})$: We set $K = (\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \setminus \{\mathbf{z}^{m'+1}\}$ and apply lemma 1 to this K and $\mathbf{z}^{m'+1}$, requiring that $\mathbf{x} \succ_i \mathbf{z}^{m'+1}$ for all $\mathbf{x} \in K$.

Case 2, $\mathbf{y}^{m'+1} \notin C((\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \setminus \{\mathbf{z}^{m'+1}\})$: In this case we first apply lemma 1 in the same manner as above to $\mathbf{y}^{m'+1}$ with $K = (\cup_{j=1}^{m'}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \setminus \{\mathbf{z}^{m'+1}\}$. We then proceed to apply the same lemma a second time to $\mathbf{z}^{m'+1}$ assuming $K = (\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\} \cup \{\widehat{\mathbf{x}}_i\}) \setminus \{\mathbf{z}^{m'+1}\}$.

To complete the proof, note that the application of lemma 1 above gives us strictly concave utility functions that rationalize legislator i's voting decisions over $\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\}$ and ensures that $\widehat{\mathbf{x}}_i$ is strictly preffered to all altnernatives in $\cup_{j=1}^{m'+1}\{\mathbf{z}^j, \mathbf{y}^j\}$. Yet, this leaves open the possibility that $\widehat{\mathbf{x}}_i$ is not a global maximum of this utility function. The following lemma show this is not an issue:

Lemma 2 *Consider a finite set* $K \subset \mathbb{R}^d$ *and strictly concave* $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ *that represents i's preferences over* K*. If* $\widehat{\mathbf{x}} \in K$ *is such that* $\widehat{\mathbf{x}} \succ_i \mathbf{x}$ *for all* $\mathbf{x} \in K$*,* $\mathbf{x} \neq \widehat{\mathbf{x}}$*, then there exists another strictly concave* $u_i : \mathbb{R}^d \longrightarrow \mathbb{R}$ *that represents i's preferences over* K *such that* $u_i(\widehat{\mathbf{x}}) > u_i(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^d$*,* $\mathbf{x} \neq \widehat{\mathbf{x}}$*.*

Proof. For unknowns $u_i^{\mathbf{z}} \in \mathbb{R}$, and $\mathbf{d}^{\mathbf{z}} \in \mathbb{R}^d$, one for each $\mathbf{z} \in K$, consider the following set of equalities and inequalities:

$$\begin{aligned} u_i^{\mathbf{z}} - u_i^{\mathbf{y}} &> 0, \text{ for all } \mathbf{z}, \mathbf{y} \in K \text{ with } \mathbf{z} \succ_i \mathbf{y} \\ u_i^{\mathbf{z}} - u_i^{\mathbf{y}} &= 0, \text{ for all } \mathbf{z}, \mathbf{y} \in K \text{ with } \mathbf{z} \sim_i \mathbf{y} \\ u_i^{\mathbf{z}} - u_i^{\mathbf{y}} - (\mathbf{d}^{\mathbf{z}})^T (\mathbf{z} - \mathbf{y}) &> 0, \text{ for all } \mathbf{z} \in K, \text{ all } \mathbf{y} \in K, \mathbf{y} \neq \mathbf{z}. \end{aligned}$$

Since preferences over K can be represented by strictly concave u_i, there exists a solution to this system (by setting $u_i^{\mathbf{z}} = u_i(\mathbf{z})$, and $\mathbf{d}^{\mathbf{z}}$ equal to certain derivative[16] of u_i at $\mathbf{z}$). Obviously, by setting $\mathbf{d}^{\widehat{\mathbf{x}}} = \mathbf{0}$, and maintaining the remaining values, we still have a solution. This new solution produces a strictly concave utility function $\widetilde{u}_i : \mathbb{R}^d \longrightarrow \mathbb{R}$, as follows:

$$\widetilde{u}_i(\mathbf{x}) \equiv \min_{\mathbf{z} \in K} \left\{ u_i^{\mathbf{z}} + (\mathbf{d}^{\mathbf{z}})^T (\mathbf{x} - \mathbf{z}) - \varepsilon (\mathbf{x} - \mathbf{z})^T (\mathbf{x} - \mathbf{z}) \right\}$$

for small enough $\varepsilon > 0$ (see Richter and Wong, 2004, page 356), which is obviously maximized at $\widehat{\mathbf{x}}$. ■

This completes the proof of sufficiency. ■

Now we show corollary 2:

Corollary 2 *Suppose one issue dimension* ($d = 1$) *and fix the location of the voting alternatives. The roll call voting record of legislator i is rationalizable with ideal point* $\widehat{x}_i$ *if and only if condition (5) holds and*

$$N_i(\{j\}) \cap C(Y_i(\{j\}) \cup \{\widehat{x}_i\}) = \emptyset, \text{ for all } j \in \{1, ..., m\}. \tag{7}$$

[16] Geometrically, $d^{\mathbf{z}}$ is the 'slope' of a hyperplane that is tangent to u_i at z, and lies above $u_i(\mathbf{x})$ for all $x \neq z$.

Proof. We only need show sufficiency. In particular, we shall show that if condition (6) fails for some M with $|M| \geq 2$ when $d = 1$, then either condition (5) or condition (7) must fail. So suppose condition (6) fails for some M with $|M| \geq 2$. Then, either $\widehat{x}_i$ is in the interior of $C(Y_i(M) \cup \{\widehat{x}_i\})$, in which case condition (5) must fail as well, or (without loss of generality) $C(Y_i(M) \cup \{\widehat{x}_i\}) = \left[z^j, \widehat{x}_i\right]$, for some $j \in M$. Then $y^j \in \left[z^j, \widehat{x}_i\right]$, *i.e.* condition (7) fails, and the proof is complete. ■

We conclude this appendix with a proof of theorem 3.

Theorem 3 *For all roll call voting records and all possible ideal points for the* n *legislators, there exist voting alternatives* $\mathbf{z}^j$, $\mathbf{y}^j$ *that render the roll call voting record rationalizable with these ideal points for all legislators* $i \in N$.

Proof. We shall first show the truth of the theorem assuming the ideal points $\widehat{\mathbf{x}}_1$,...,$\widehat{\mathbf{x}}_n$ are distinct from each other. The proof is by construction, and we shall establish it first for the case $d = 1$, then generalize to $d > 1$. Without loss of generality, let $\widehat{x}_i < \widehat{x}_{i+1}$, $i = 1, ..., n-1$. Further assume without loss of generality that $Y_1(\{1, ..., m\}) = \left\{z^1, ..., z^m\right\}$ and that $Y_n(\{1, ..., m\}) = \left\{z^1, ..., z^k, y^{k+1}, ..., y^m\right\}$. Then position alternatives $z^1, ..., z^m$ in the interval $(\widehat{x}_1, \widehat{x}_2)$, so that $\widehat{x}_1 < z^1 < ... < z^m < \widehat{x}_2$. Position alternatives $y^{k+1}, ..., y^m$ in $(\widehat{x}_{n-1}, \widehat{x}_n)$, so that $\widehat{x}_{n-1} < y^m < ... < y^{k+1} < \widehat{x}_n$. Finally, position alternatives $y^1, ..., y^k$ in $(\widehat{x}_n, +\infty)$ so that $\widehat{x}_n < y^k < ... < y^1$.

By construction, for any $j, h = 1, ..., m$ which (again without loss of generality) satisfy $j > h$, we have $z^h < z^j < y^j < y^h$. Thus, for any $i \in N$, $C(Y_i(\{h, j\}))$ can be either equal to $\left[y^j, y^h\right]$, or $\left[z^j, y^h\right]$, or $\left[z^h, y^j\right]$, or $\left[z^h, z^j\right]$. In all four cases, condition (5) is satisfied for any $j, h = 1, ..., m$ and $i \in N$, so that the roll call record is rationalizable. Thus, to complete the proof when $d = 1$, we need to verify condition (7), *i.e.*, we need to show that the roll call record is rationalizable with the specified ideal points. But condition (7) is trivially satisfied for legislators $i = 2, ..., n-1$, and is true by construction for legislators 1 and n. Now the proof is complete by corollary 2.

The above construction generalizes for the case $d > 1$. In particular, (as illustrated in figure 7(b) for the case $d = 2$) we can place alternatives $\{\mathbf{z}^j, \mathbf{y}^j\}$ in the same fashion as above on the line formed by ideal points $\widehat{\mathbf{x}}_1$ and $\widehat{\mathbf{x}}_n$. Without loss of generality, we can assume the line segment defined by these two ideal points $\widehat{\mathbf{x}}_1$ and $\widehat{\mathbf{x}}_n$, contains all other ideal points, say $\widehat{\mathbf{x}}_h$, which are collinear with $\widehat{\mathbf{x}}_1$ and $\widehat{\mathbf{x}}_n$. Clearly, condition (4) is satisfied for all $i \in N$, because (5) is satisfied. Furthermore, for all legislators i with ideal points on the line formed by points $\widehat{\mathbf{x}}_1$ and $\widehat{\mathbf{x}}_n$ condition (6) is satisfied because (7) is. Finally, since (4) is satisfied for all $i \in N$, then condition (6) is trivially satisfied for all $i \in N$ with ideal points outside the line formed by points $\widehat{\mathbf{x}}_1$ and $\widehat{\mathbf{x}}_n$.

The above completes the proof for all cases when two ideal points with the properties of $\widehat{\mathbf{x}}_1$, $\widehat{\mathbf{x}}_n$ above are distinct from all remaining ideal points. If there is no pair of distinct ideal points with these properties, there is an obvious amendment to the above construction, for instance by placing all $\mathbf{z}^j$, $j = 1, ..., m$ to the left of $\widehat{\mathbf{x}}_1$, and all $\mathbf{y}^j$, $j = 1, ..., m$ to the right of $\widehat{\mathbf{x}}_n$. ■

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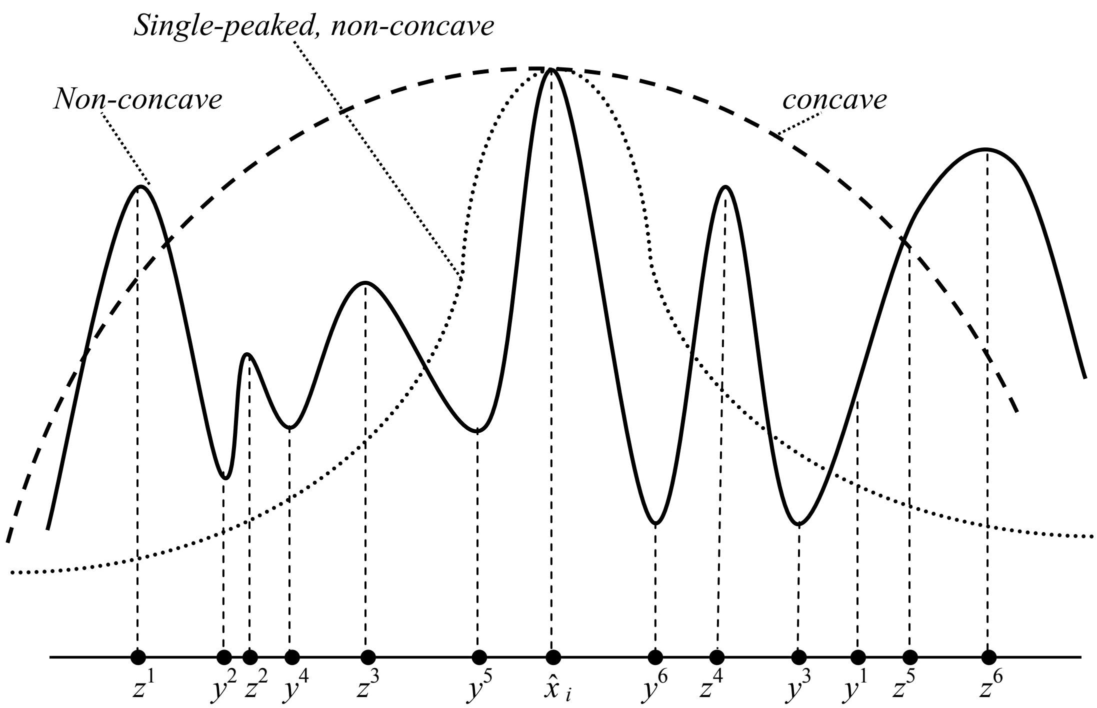
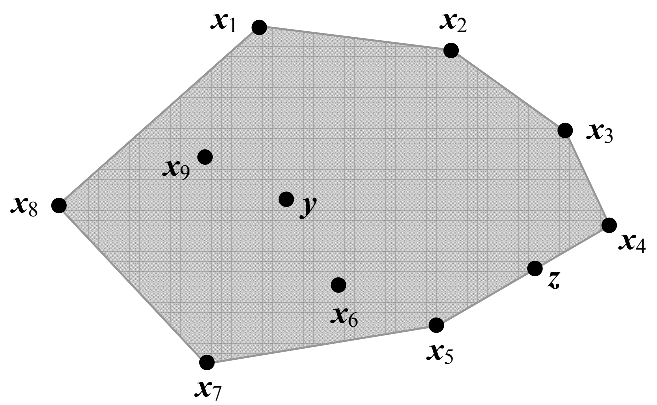
Figure 1: Roll Call Records and Non-concave Preferences



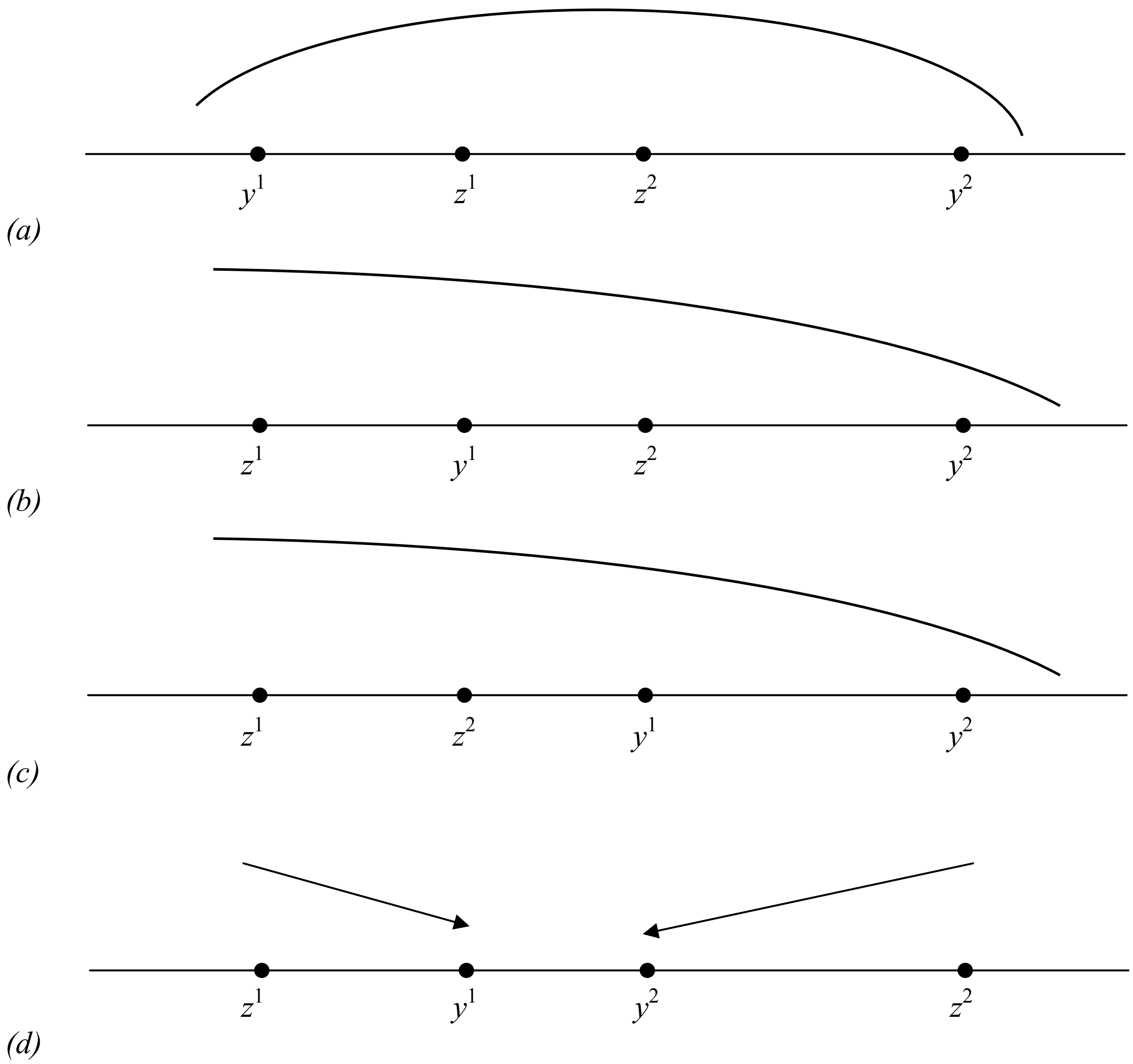
Key: *The utility function represented with a solid line is constructed so that it is consistent with the voting decisions of legislator i, who votes 'yea' in all roll call voting items. It is also consistent with this legislator having an ideal point at* $\hat{x}_i$*. This function is not concave. The utility function represented with a dashed line is concave. This function maintains the same ideal point but is inconsistent with the voting decisions of legislator i, e.g., i's vote on the sixth voting item. The utility function represented with a dotted line is single-peaked but not concave.*

Figure 2: The Convex Hull of a finite Set



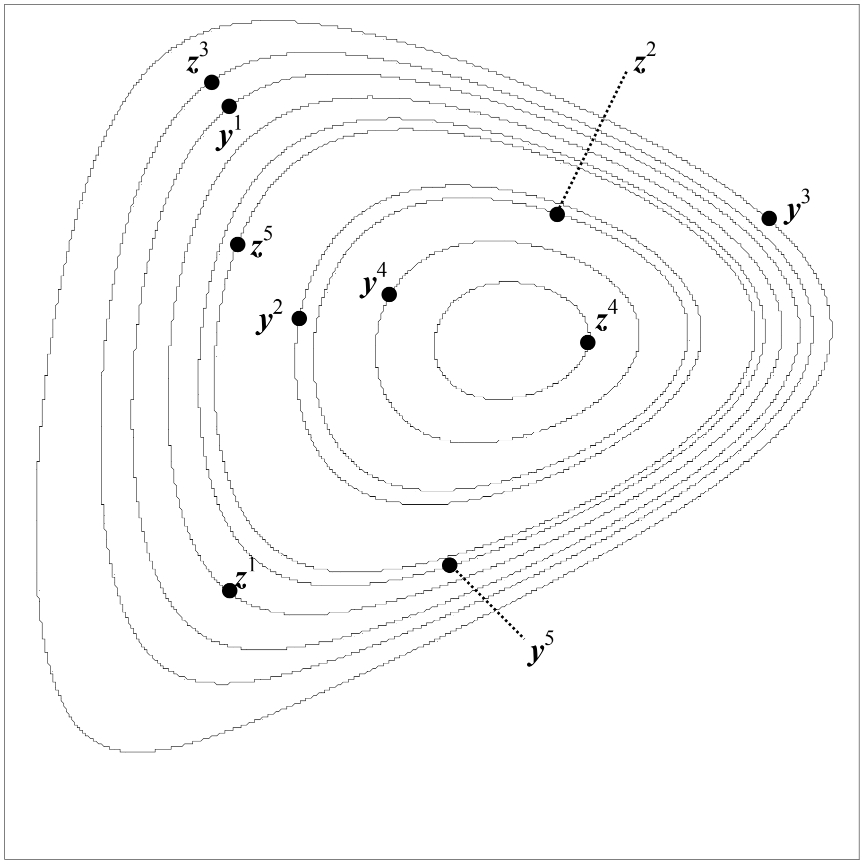
Key: *The gray area marks the convex hull of the set K =* $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5, \boldsymbol{x}_6, \boldsymbol{x}_7, \boldsymbol{x}_8, \boldsymbol{x}_9\}$. *Every point in this area, such as* $\boldsymbol{y}$ *or* $\boldsymbol{z}$, *can be written as a convex combination of elements of K.*

Figure 3: Illustration of Condition (4) in One Dimension ($d = 1$)



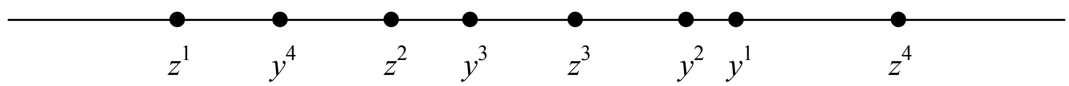
Key: $Y_i(\{1, 2\}) = \{z^1, z^2\}$, $d = 1$. *Cases (a), (b), and (c) satisfy condition (4) of theorem* 1*, so there exists a strictly concave utility function that represents i's preferences. In case (d) that fails condition (4), any utility function that rationalizes i's preferences must reverse monotonicity, initially strictly decreasing, then strictly increasing. Such a function cannot be concave.*

Figure 4: A Rationalizable Voting Record in Two Dimensions (d = 2)



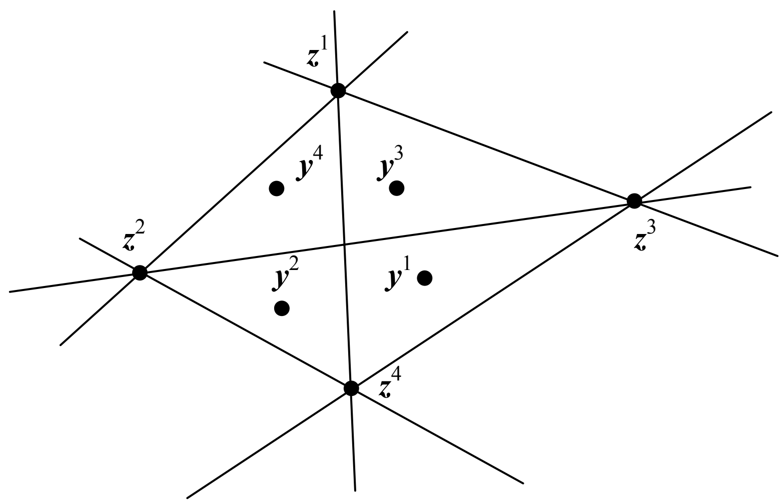
Key: *The roll call voting record of legislator i, $Y_i(\{1, 2, 3, 4, 5\}) = \{\boldsymbol{z}^1, \boldsymbol{z}^2, \boldsymbol{z}^3, \boldsymbol{z}^4, \boldsymbol{z}^5\}$, is rationalizable. As a result, there exists a concave utility function that is consistent with legislator i's voting decisions. The figure depicts the indifference contours for one such possible utility function.*

Figure 5: Non-rationalizable voting records in $d = 1$ and $d > 1$ dimensions



$Y_i(\{1, 2, 3, 4\}) = \{z^1, z^2, z^3, z^4\}$

(a)

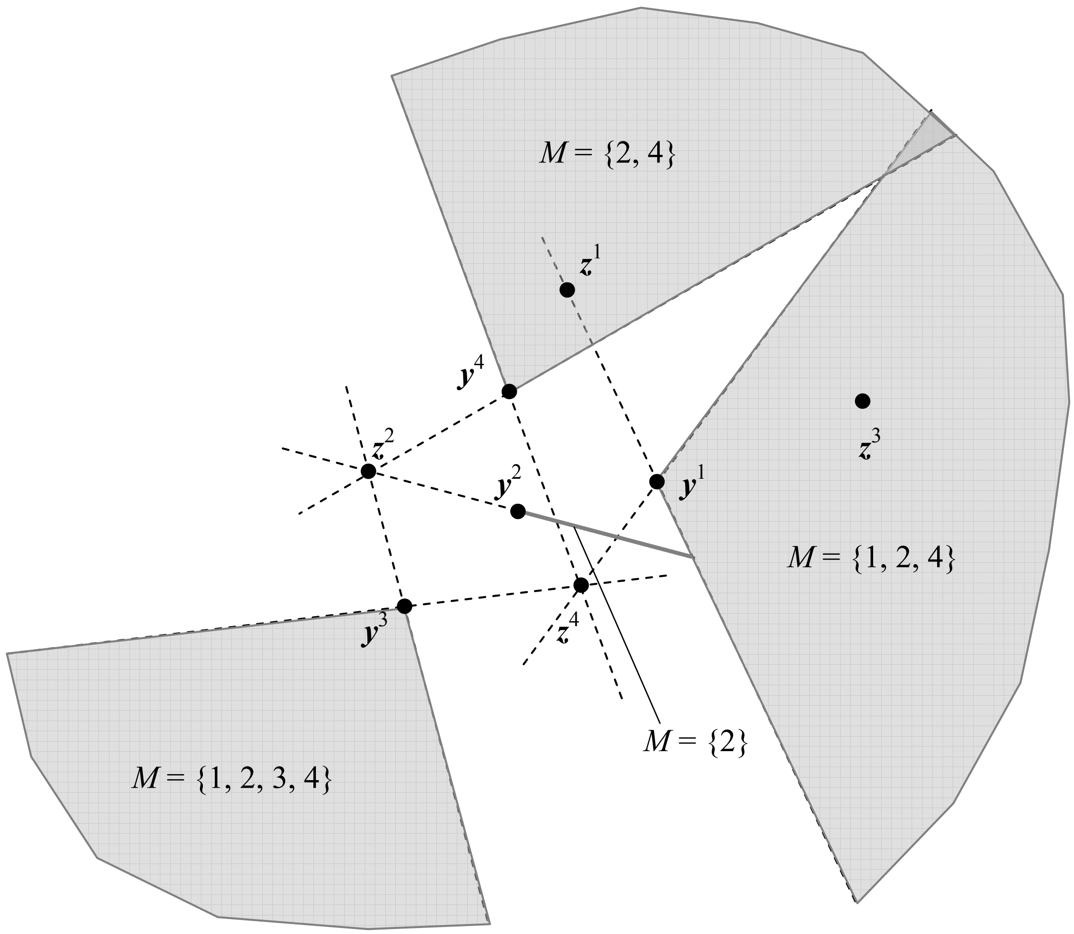


$Y_i(\{1, 2, 3, 4\}) = \{z^1, z^2, z^3, z^4\}$

(b)

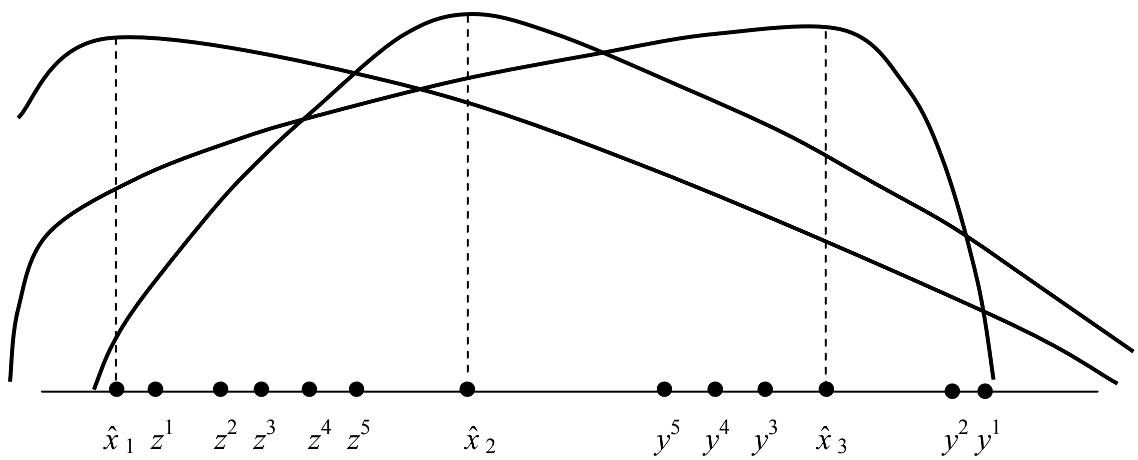
Key: *When d =* 1*, if condition (*4*) fails for subsets M of k >* 2 *voting items, then it must fail for some pair of votes M = {h, j}. E.g. in case (a) the condition fails for M = {*1*,* 2*,* 3*,* 4*} but also for M = {*1*,* 4*}. This is not true in d >* 1 *dimensions. In case (b) with d =* 2*, condition (*4*) is satisfied for all subsets of* 3 *or less voting items, but the condition fails for four voting items, M = {*1*,* 2*,* 3*,* 4*}.*

Figure 6: A Roll Call Voting Record and Ideal Point Restrictions

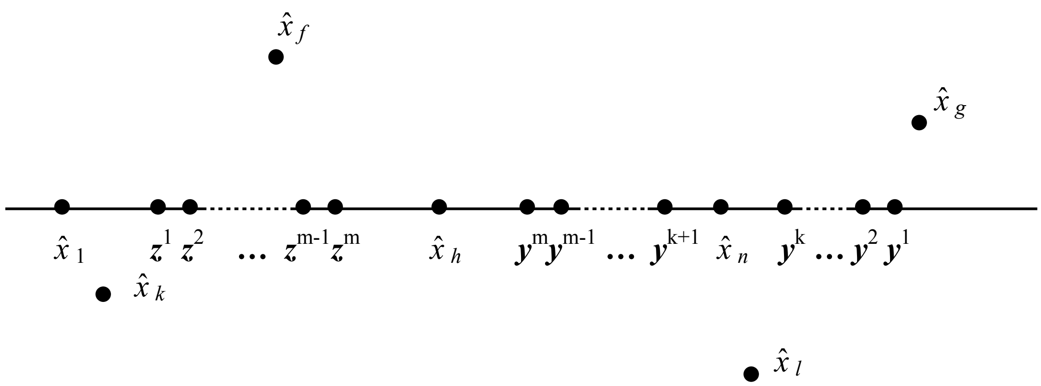


Key: *If legislator i has the rationalizable voting record $Y_i(\{1, 2, 3, 4\}) = \{z^1, z^2, z^3, z^4\}$, then application of condition (6) restricts legislator i's ideal point to lie outside the areas marked gray. The set M attached to each of these restricted areas indicates the set of roll call votes on which condition (6) is applied to obtain the corresponding restriction.*

Figure 7: Illustration of Construction in Theorem 3



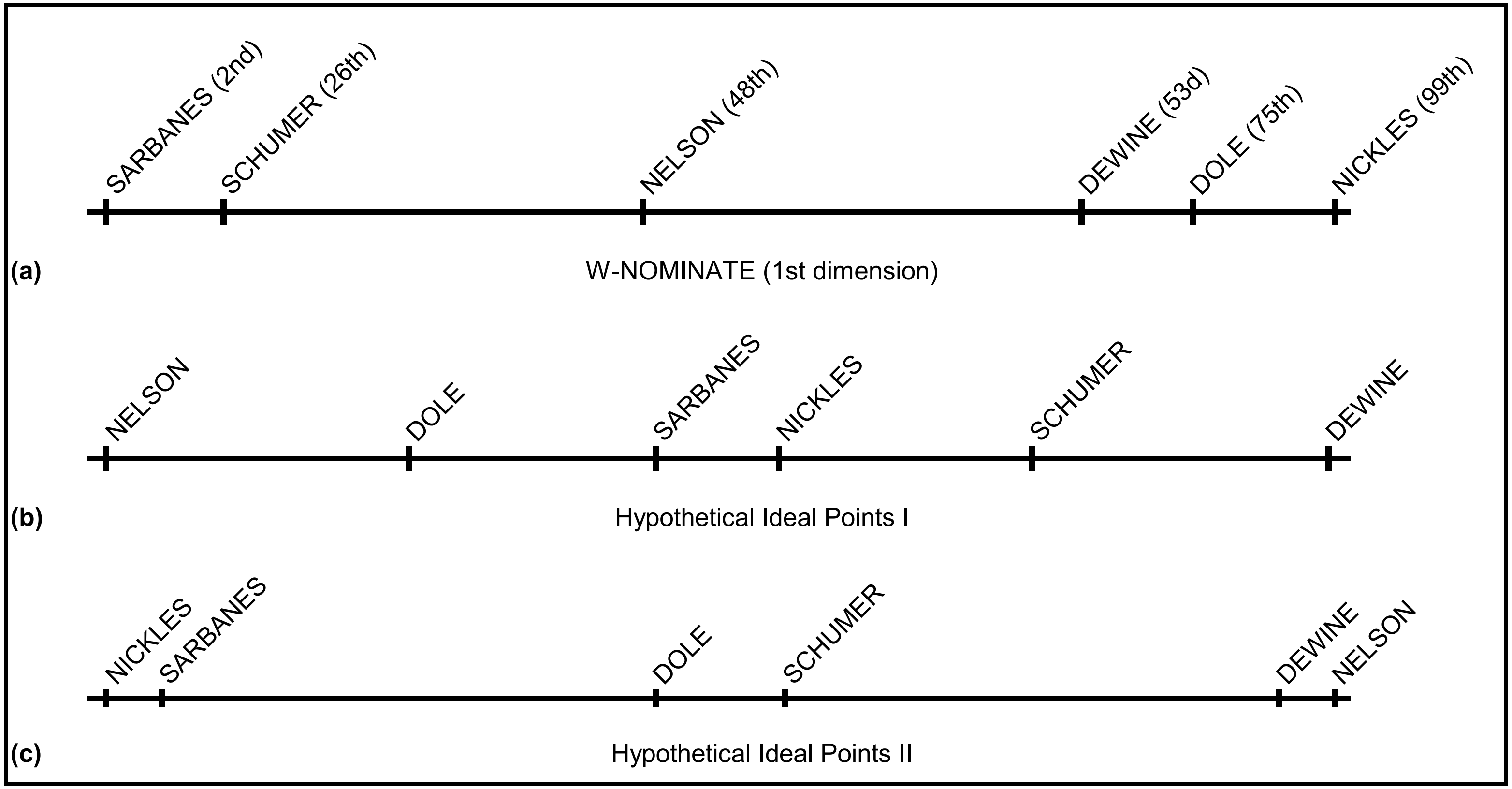
(a)



(b)

Key: *Case (a) represents an example of the construction detailed in the proof of theorem 3 for the case* $d = 1$, $n = 3$, $m = 5$, *and ideal points that satisfy* $\hat{x}_1 < \hat{x}_2 < \hat{x}_3$. *The roll call record in this case is summarized by* $Y_1(\{1,\dots,5\})=\{z^1, z^2, z^3, z^4, z^5\}$, $Y_2(\{1,\dots,5\})=\{y^1, z^2, y^3, y^4, z^5\}$, *and* $Y_3(\{1,\dots,5\})=\{z^1, z^2, y^3, y^4, y^5\}$.
Case (b) depicts a generalization of this construction for $d = 2$ *dimensions. As in the one-dimensional case, condition (6) is satisfied for legislators* 1, *h, and n, with collinear ideal points* $\hat{x}_1$, $\hat{x}_h$, $\hat{x}_n$. *Furthermore, this condition is satisfied for every legislator with ideal point not on the line defined by points* $\hat{x}_1$, $\hat{x}_n$ *(such as* $\hat{x}_f$, $\hat{x}_g$, $\hat{x}_k$, $\hat{x}_l$*).*

Figure 8: Ideal Points & Theorem 3



Key: Figure (a) depicts the first dimension of W-Nominate (Poole and Rosenthal, 1991) scores for the 108-th Congress (source: http://voteview.com/) for six senators (three from each party) at 'extreme' and 'moderate' positions within each party. Their relative ranking based on these estimates is reported in parenthesis. We may, instead, assume that these legislators have the hypothetical ideal points in figures (b) or (c). Theorem 3 ensures that we can locate the voting alternatives for all 675 roll call items recorded during that Congress and we can find strictly concave utility functions for these six senators with any of these hypothetical ideal points, so that these utility functions perfectly account for all the actual voting decisions of these senators, for all 675 items in the roll call record.