Majority Rule Dynamics with Endogenous Status Quo

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Abstract

We analyze a stochastic bargaining game in which a new dollar is divided among committee members in each of an infinity of periods. In each period, a committee member is recognized and offers a proposal for the division of the dollar. The proposal is implemented if it is approved by a majority. If the proposal is rejected, then last period's allocation is implemented. We show existence of equilibrium in Markovian strategies. It is such that irrespective of the initial status quo, the discount factor, or the probabilities of recognition, the proposer extracts the entire dollar in all periods but the initial two. We also derive a fully strategic version of McKelvey's (1976), (1979) dictatorial agenda setting, so that a player with exclusive access to the formulation of proposals can extract the entire dollar in all periods except the first. The equilibrium collapses when within period payoffs are sufficiently concave. Winning coalitions may comprise players with high instead of low recognition probabilities, *ceteris paribus*.

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1 Introduction

Social choice theory has had a profound impact on our thinking about political interaction due to the counter-intuitive nature of its conclusions. When it comes to majority rule, prominent in the

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parade of paradoxical results of this literature are those by Plott (1967), and McKelvey (1976, 1979). The former convincingly demonstrates that stable policies (that beat every other by majority vote) in a continuous space of social alternatives are generically non-existent in more than one policy dimensions. The latter, assuming the non-existence of stable policies, establishes that the entire set of alternatives are entangled in a majority preference cycle. These results are suggestive but not conclusive about the stability and/or predictability (or lack thereof) of collective decision making under majority rule. In particular, neither precludes the existence of non-cooperative equilibrium a la Nash. Building on that observation and the sequential bargaining model of Rubinstein (1982), a non-cooperative literature on collective decision making has flourished in the last two decades.[1]

While this literature fills one lacuna in the theory of collective choice by providing a palpable solution concept, it is ill-suited to address the dynamic implications of social preference intransitivity engendered by individual preference aggregation via majority rule. For that purpose we need to specify game forms that trace collective choice over time, yet most of the existing literature assumes that interaction ceases once the committee has reached a decision. The goal of this paper is to further our understanding of equilibrium dynamics in a committee setting in which policies drawn from a multidimensional policy space can be revised *ad infinitum*. Specifically we study a stochastic game with an odd number of five[2] or more players who decide on the division of a fixed budget (a dollar) under majority rule. In each period, one of the committee members is recognized with some fixed probability and makes a proposal. If the proposal is approved by a majority then the dollar is divided in that period accordingly. Otherwise, the status quo allocation (which is defined as last period's division) is implemented. The above setup constitutes a natural framework for the study of dynamic political interaction. Independently of variation in the process of agenda formation, all existing constitutional democracies require that new legislation must be pitted against the status quo in a final vote before its promulgation. Thus, as in the current analysis, the status quo policy remains in effect and accrues payoffs to players, until it is beaten via majority vote by the policy that replaces it.

A first issue we have to confront in this environment is that of existence of equilibrium. We focus our analysis on simple equilibria such that players' behavior in each period is only conditioned on payoff relevant information (Tirole and Maskin (2001)), but our assumptions specify a stochastic

[1] *E.g.*, Baron and Ferejohn (1989), Merlo and Wilson (1995), Banks and Duggan (2000), (2006), etc.

[2] The case of three-member committees is special, as we discuss shortly, and is covered under more restrictive assumptions in Kalandrakis (2004).

game with continuous action and state spaces, for which such equilibria need not exist.[3] We are able to overcome this difficulty and establish and characterize a (refined) Markov Perfect Nash equilibrium via a combination of direct methods and abstract arguments. The existence proof is based on two steps. First (Proposition 1), we obtain a closed form solution for equilibirum strategies in a subset of the state space that is absorbing given these strategies. In the second step (Lemma 1), we apply a fixed point theorem in order to show that, given the above strategies, there exist extensions of these strategies to the entire state space that move the game (with probability one) in the absorbing set characterized in the first step. In Proposition 2 we show that these extended strategies constitute an equilibrium.

With existence of equilibrium established, the important questions revolve around the nature of equilibrium policy dynamics. Intuition offers an abundance of forces to induce policy moderation in our framework. This follows from reasoning about the incentives of players both in their role as voters as well as as proposers. By adopting an extreme (but desirable) division in the current period, players face the risk that future proposers and coalitions will achieve passage of undesirable divisions because they will legislate with a status quo division that disadvantages many committee members. Thus, optimal divisions of the dollar should balance a trade-off between immediate gains and a potentially averse stream of future decisions. In sharp contrast to the above intuition, the equilibrium we characterize is such that the proposer extracts the entire dollar in every period except (possibly) the initial two. This is true for all initial status quo divisions of the dollar and independent of the discount factor. In effect, despite the fact that players are strategic and farsighted, the long-run behavior of the system is identical to the one that would prevail in the same model with myopic behavior (or impatient players).

A similar equilibrium is obtained by Kalandrakis (2004) in the special case of a committee with three players. Although equilibrium behavior is identical in the long-run (the proposer extracts the entire dollar in each period), with $n = 3$ players there exist initial status quo such that, with positive probability, absorption to the set of long-run equilibrium divisions may not occur for any finite period t. This is because there is positive probability that the excluded member from last period's winning coalition is recognized, and this player is unable to extract the entire dollar when the other two players have a positive status quo allocation. As we demonstrate in section 3, when $n \geq 5$ there always exist a bare majority of members with zero status quo allocation for any

[3] For a detailed discussion of the equilibrium existence problem in these settings, see Duggan and Kalandrakis (2007).

proposer recognized in period $t = 3$. Thus absorption to the long-run equilibrium set of policies occurs with probability one in period $t = 3$ in the present study. Besides the substantive difference in equilibrium dynamics, this discrepancy implies that the case with $n = 3$ players cannot be subsumed in the present study using the same line of proof (or vice versa). Kalandrakis (2004) obtains an equilibrium using direct methods and exploiting the symmetry of the game stemming from the assumption that players have equal recognition probabilities. Also, Kalandrakis (2004) requires linear stage payoffs.

In the present study, we consider the case stage payoffs exhibit diminishing returns, and relax the assumption of equal recognition probabilities. The fact that we consider general recognition probabilities imposes a significant additional burden on the analysis, as the extra heterogeneity across committee members makes it hard to determine analytically the composition of optimal coalitions for any given proposer and status quo allocation. This is because, in any given period, possible coalition partners are now characterized by two features: their status quo allocation *and* their recognition probability. It is, of course, much easier to sort out the least expensive players for inclusion in the proposer's optimal coalition when players differ in one as opposed to two dimensions. Thus, unlike Kalandrakis (2004), our approach does not involve the analytical derivation of equilibrium proposal strategies but we are able to establish an equilibrium with general probabilities of recognition. An immediate payoff from this generality is that we tackle an important question that was originally posed by McKelvey in his seminal papers (1976), (1979). In particular, McKelvey discusses how a dictatorial agenda setter (a person that formulates the proposal with probability one in each period) can eventually pass her ideal point via an appropriate sequence of binary votes between the status quo and a new alternative. McKelvey's construction is a direct consequence of his intransitivity result but relies on the unrealistic assumption (as McKelvey explicitly points out) that players vote on each pair of alternatives myopically, without anticipating the eventual perils from their immediate gratification. Can McKelvey's dictatorial agenda setting result be obtained when voters are farsighted? Our analysis gives a conclusive answer for the case the space of agreements is the division of a dollar (Proposition 3). If a committee member is recognized with probability one in every period, she can extract the entire dollar in all but the very first period, for all initial status quo and every discount factor.

In addition, we show that the characterized equilibrium collapses when utility from the share of the dollar displays significant diminishing returns, for fixed committee size, or – for fixed level of diminishing returns – if the legislature is small (Proposition 5). Thus, concavity seems to play a different role in this model compared to the models where the legislature adjourns once a decision is reached, since in such settings risk aversion allows the proposer to extract more of the surplus (Harrington (1990)). Finally, also contrary to the comparative statics in the Baron and Ferejohn model established by Eraslan (2002), we find that players that have high probability of becoming the proposer are less expensive coalition partners, *ceteris paribus* (Proposition 4). In particular, we show that there exist equilibria such that for certain status quo players with higher recognition probabilities (but equal status quo allocation) are included in the winning coalition with higher probability.

Before we move to the detailed presentation and analysis of the model, we further discuss related contributions. Closely related to the present model is that analyzed by Epple and Riordan (1987). They study subgame perfect equilibria of a divide-the-dollar game in which three players alternate making proposals and establish that at least two radically different sequences of divisions of the dollar can be supported in equilibrium. This result can be interpreted as a justification for the focus on Markovian equilibria, as it suggests that a folk-theorem may obtain for these games. The first study of Markov Perfect equilibria with the game form we consider in the present study is by Baron (1996), who analyzes the case of a one-dimensional policy space and shows that policies converge to the median in the long run. Baron and Herron (2003) numerically analyze a finitely repeated version of the same game with two policy dimensions and three legislators. They find that equilibrium decisions tend to be more centrally located with a higher discount factor and a longer time horizon. Thus, both Baron (1996) and Baron and Herron (2003) obtain qualitatively different long-run equilibrium outcomes compared to the present analysis, suggesting that the discrepancy may originate from the different policy spaces. While the above studies are concerned with applications in special policy spaces, Duggan and Kalandrakis (2007) study a general model with only smoothness conditions imposed on players' preferences and minimal restrictions on the policy space. Among other results, they establish existence and continuity properties of pure strategy Markovian equilibria, show that all such equilibria are essentially pure, and obtain sufficient conditions for the policy process to have a unique invariant distribution. Despite their generality, these results do not apply in the model considered in this study, because Duggan and Kalandrakis assume stochastic shocks on preferences and the status quo which are not captured in the present model.

Related to the setup of Baron (1996), Kalandrakis (2004), and the present study is the model with a one-dimensional state space of Cho (2005a) who studies a multi-party parliamentary democracy with both bargaining and elections. Cho (2005b) and Fong (2005) both study two-dimensional models with transferable utility but different bargaining protocols from those considered presently, the former focusing on cabinet dissolution and the vote of confidence institution. Transferable utility is also assumed by Gomez and Jehiel (2005) who study efficiency properties of equilibrium in a dynamic coalitional game with a finite state space. Battaglini and Coate (2007) characterize stationary equilibria in a model of public good provision, particularistic spending, and taxation when interaction within periods takes the form of a finite Baron and Ferejohn (1989) protocol, and the dynamic link across periods is determined by the stock of the public good. Bernheim, Rangel, and Rayo (2006) analyze interaction for the determination of a policy in a single legislative period assuming a sequence of votes on proposals such that each victorious proposal moves to the next voting round (without being implemented), with the winning proposal in the last voting round being the implemented policy, and derive conditions so that this implemented policy coincides with the ideal policy of the last proposer. In section 6 of their study they discuss an extension to a dynamic model such that implemented agreements can be revised a finite number of periods. Due to the special institution assumed by the authors for legislative interaction within periods this multiperiod model does not generate strategic links between implemented agreements across periods (as a consequence of their Corollary 1, page 1167). In a general setting applying social choice theoretic equilibrium notions, Roger Lagunoff (2005a), (2005b) studies the dynamics of institutional stability and reform. Penn (2005) analyzes a dynamic model in which proposals arise exogenously and voting by the committee on these proposals is probabilistic. Random proposals and myopic voting are analyzed by Ferejohn, McKelvey, and Packel (1984).

In what follows we present the model in detail and define the equilibrium solution concept. We establish existence of equilibrium in sections 3 and 4. In section 5 we discuss properties of equilibrium and extensions. We conclude in section 6.

2 Model & Preliminaries

Consider a set $N = \{1, ..., n\}$ of $n = 2\kappa + 1$ committee members, $\kappa \geq 2$. They convene in each period $t = 1, 2, ...$ to reach an agreement $\mathbf{x}^t$ drawn from a set X. Our ultimate goal is to analyze the case when X represents all possible divisions of a fixed budget (a dollar) among the n players. In this case we set $X = \Delta$, where $\Delta = \{\mathbf{x} \in \mathbf{R}^n_+ : \sum_{i=1}^n x_i = 1\}$. In section 3, it will prove convenient to solve an auxiliary game in which the space of possible agreements, X, is restricted to a proper subset of Δ. The game proceeds as follows. At the beginning of each period $t = 1, 2, ...$ player $i \in N$ is recognized with probability $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$, to make a proposal $\mathbf{z} \in X$. Having observed the proposal players vote *yes* or *no*. If a majority of $\kappa + 1$ or more players vote *yes* then the proposed agreement is implemented, *i.e.*, $\mathbf{x}^t = \mathbf{z}$. Otherwise, if $\mathbf{z}$ does not receive the approval of a majority, then period t's status quo policy $\mathbf{s}^t \in X$ is implemented (*i.e.*, $\mathbf{x}^t = \mathbf{s}^t$). The game then moves to the next period $t + 1$, with a status quo now being period t's agreement, $\mathbf{s}^{t+1} = \mathbf{x}^t$, and a new round of proposal making and voting.

Players derive stage utility $u_i : X \to \mathbf{R}$, $i \in N$, from the implemented agreement, $\mathbf{x}^t$. We assume players' utility depends only on their share of the dollar, so that $u_i(\mathbf{x}) = u(x_i)$ for all $i \in N$, for a utility function $u : [0, 1] \to \mathbf{R}$ with $u' > 0$. We normalize payoffs so that $u(0) = 0$ and $u(1) = 1$. For analytical tractability, we state some of the more general results under the restriction that $u'' = 0$ and payoffs are linear with $u(x_i) = x_i$. As we explicitly discuss in section 5, our conclusions are qualified in some important respects in the presence of diminishing returns, so we admit $u'' \leq 0$ unless otherwise stated. Players discount the future with a common factor $\delta \in (0, 1)$, and their payoff in the game is given by the discounted sum of stage payoffs.

We focus the analysis on Markov Perfect equilibria.[4] Existence of such equilibria requires mixing at the proposal stage of the game, so we represent a (mixed) Markov *proposal strategy* for player i as a function $\pi_i : X \to \mathcal{P}[X]$, where $\mathcal{P}[X]$ is the space of Borel probability measures over X.[5] We use the somewhat abusive notation $\pi_i[\cdot \mid \mathbf{s}] \in \mathcal{P}[X]$ to denote player i's randomization over proposals when recognized with status quo $\mathbf{s}$. A Markov *voting strategy* is a function $\alpha_i : X \times X \to \{yes, no\}$, so that $\alpha_i(\mathbf{s}, \mathbf{z}) = yes$ indicates player i votes *yes* on proposal $\mathbf{z}$ when the status quo is $\mathbf{s}$. In the sequel, we opt to work with the equivalent representation of voting

[4]There are well developed arguments in the literature (*e.g.*, Maskin and Tirole, 2001, and the references therein) that justify this focus on Markov strategies.

[5]In general, additional measurability conditions on proposal strategies are necessary in order for players' expected payoffs to be well defined. For the sake of simplicity, we omit such explicit restrictions and secure measurability of continuation payoffs in the relevant subset of the state space by solving analytically for continuation value functions v_i (as in (5)) in an absorbing subset of the policy space X. Alternatively, we could start with the restriction that proposal strategies π_i are Markov transitions (Aliprantis and Border (1999), definition 18.8, page 594) so that the functional equation (5) maps the space of bounded measurable functions into itself (Aliprantis & Border (1999), theorem 18.7, page 593), and use additional arguments, which are available upon request, to show that this restriction can be met by proposal strategies in the equilibrium characterized in the present study.

strategy α_i by a correspondence $A_i : X \rightrightarrows X$, that maps each status quo $\mathbf{s}$ to an *acceptance set* $A_i(\mathbf{s}) = \{\mathbf{z} \in X : \alpha_i(\mathbf{s}, \mathbf{z}) = yes\}$. Given Markov voting strategies A_i, $i \in N$, we can compute the *win set* of $\mathbf{s}$:

$$W(\mathbf{s}) = \left\{ \mathbf{y} \in X \mid \sum_{i=1}^{n} I_{A_i(\mathbf{s})}(\mathbf{y}) \geq \kappa + 1 \right\}, \tag{1}$$

where $I_A(\mathbf{y})$ is the indicator function. The win set $W(\mathbf{s}) \subseteq X$ contains the agreements that defeat status quo $\mathbf{s}$ by majority rule.

Our equilibrium notion is that of Markov Perfect Nash (Maskin and Tirole (2001)) with a standard refinement on voting strategies:

Definition 1 *An equilibrium is a set of proposal and voting strategies π_i^*, A_i^*, such that for all $i \in N$, and for all status quo $\mathbf{s} \in X$:*

$$\mathbf{y} \in A_i^*(\mathbf{s}) \Leftrightarrow U_i(\mathbf{y}) \geq U_i(\mathbf{s}), \text{ and} \tag{2}$$

$$\pi_i^*[\arg\max\{U_i(\mathbf{x}) \mid \mathbf{x} \in W(\mathbf{s})\} \mid \mathbf{s}] = 1, \tag{3}$$

where $U_i : X \to \mathbf{R}$ represents i's expected payoff from implemented agreements

$$U_i(\mathbf{x}^t) = (1-\delta)u_i(\mathbf{x}^t) + \delta v_i^*(\mathbf{x}^t), \tag{4}$$

and the continuation value $v_i^* : X \to \mathbf{R}$ in equation (4) satisfies

$$v_i^*(\mathbf{s}) = \sum_{j=1}^{n} p_j \int_X [(1-\delta)u_i(\mathbf{x}) + \delta v_i^*(\mathbf{x})] \, \pi_j[d\mathbf{x} \mid \mathbf{s}]. \tag{5}$$

Equilibrium condition (2) amounts to the requirement that players vote *yes* to proposals if and only if they weakly prefer them over the status quo $\mathbf{s}$. Thus, we eliminate a – rather large – class of uninteresting equilibria that involve majorities approving proposals not preferred over the status quo (or vice versa) solely because individual players are not pivotal and, hence, are indifferent between their voting actions. Equilibrium condition (3) requires that committee members choose proposals optimally when recognized. Observe that proposers are restricted to choose among the set of alternatives that defeat the status quo, $W(\mathbf{s})$. Since proposals $\mathbf{y} \notin W(\mathbf{s})$ effectively preserve the status quo policy, and since the status quo $\mathbf{s} \in W(\mathbf{s})$ for all $\mathbf{s} \in X$ (by equilibrium condition (2)), this restriction does not impair the optimality of players' proposal strategies.

We can now proceed to the analysis of the game. Our first goal is to establish existence of equilibrium. We accomplish this in sections 3 and 4. At the same time, we will obtain a characterization of equilibrium outcomes. To pave the way for this analysis, we introduce necessary notation. Partition the space of possible divisions of the budget into subsets $\Delta_\theta \subset \Delta$, where θ, $0 \leq \theta \leq n-1$, indicates the number of players receiving zero share of the dollar, *i.e.*, $\Delta_\theta = \left\{\mathbf{x} \in \Delta \mid \sum_{i=1}^{n} I_{\{0\}}(x_i) = \theta\right\}$. Further, for $\beta > \alpha$, $\alpha, \beta \in \{0, 1, .., n-1\}$, define

$$\Delta_\alpha^\beta = \bigcup_{\theta=\alpha}^{\beta} \Delta_\theta.$$

Δ_α^β is the set of all allocations of the dollar with α, or $\alpha+1, ...,$ or β players receiving zero.

The solution of the game we characterize is built from the intuition that equilibrium proposals involve 'minimum winning coalitions' (Riker (1962)), such that at most $\kappa+1$ players receive a positive fraction of the dollar in each period. As a result, we conjecture that Δ_κ^{n-1} is an absorbing set, one that is reached in at most one period from any initial status quo allocation. Capitalizing on the above conjecture, we execute our proof strategy in two steps. First, we derive equilibrium strategies in closed form for an auxiliary game in which the space of possible agreements $X = \Delta_\kappa^{n-1}$. In the second step, we extend the specified equilibrium strategies to the entire space of agreements $X = \Delta$. We execute the first step in section 3, the second in section 4. In section 5 we discuss properties of the equilibrium and additional results.

3 Equilibrium, $X = \Delta_\kappa^{n-1}$

Throughout this section, we will assume a status quo $\mathbf{s} \in \Delta_\kappa^{n-1}$ such that $s_{i+1} \geq s_i$, $i = 1, ..., n-1$. This is without loss of generality. Our goal is to derive an equilibrium when the space of possible agreements is restricted to $X = \Delta_\kappa^{n-1}$. This equilibrium is obtained via a 'conjecture and verify' approach. The conjecture is that players with zero status quo allocation accept proposals in $\Delta_{\kappa+1}^{n-1}$ that allocate them zero.[6] If this is the case, then any proposer i is able to obtain the approval of κ other players in order to extract the whole dollar when the status quo $\mathbf{s} \in \Delta_{\kappa+1}^{n-1}$, or when the

[6]Indeed, we will show that players with zero status quo allocation may even strictly prefer such proposals in equilibrium.

status quo $\mathbf{s} \in \Delta_\kappa$ and $s_i > 0$. Observe that, with these proposal strategies, players' continuation value for any status quo $\mathbf{s} \in \Delta_{\kappa+1}^{n-1}$ is given by

$$v_i(\mathbf{s}) = p_i,\ i \in N,\ \mathbf{s} \in \Delta_{\kappa+1}^{n-1}. \tag{6}$$

With proposals specified as above, it remains to determine proposals when the status quo $\mathbf{s} \in \Delta_\kappa$ and the proposer's allocation is zero (*i.e.*, proposer is $i \in \{1, ..., \kappa\}$). In these cases, the proposer must allocate a positive amount to one among players $j \in \{\kappa + 1, ..., n\}$ with $s_j > 0$. Of course, the proposer wishes to coalesce with the least expensive player which, intuitively, is the player with the lowest positive status quo allocation, *i.e.*, player $j = \kappa + 1$. We shall now demonstrate that, depending on the exact value of the status quo, $\mathbf{s} \in \Delta_\kappa$, it is not an equilibrium strategy for the proposer $i \in \{1, ..., \kappa\}$ to allocate a positive amount to $j = \kappa + 1$ with probability one. To see this is true, suppose that player $\kappa + 1$ is allocated an amount z whenever player $i \in \{1, ..., \kappa\}$ is the proposer, with i retaining the rest of the dollar. The corresponding allocation $\mathbf{z}$ is an element of Δ_{n-2}, so that by equation (6) player $\kappa + 1$'s expected utility from the proposal is $U_{\kappa+1}(\mathbf{z}) = (1 - \delta)u(z) + \delta p_{\kappa+1}$. On the other hand, the expected utility from maintaining the status quo $\mathbf{s} \in \Delta_\kappa$ is (given assumed proposal strategies)

$$\begin{aligned} U_{\kappa+1}(\mathbf{s}) &= (1-\delta)u(s_{\kappa+1}) + \delta((1-\delta)(\sum_{i=1}^{\kappa} p_i u(z) + p_{\kappa+1}u(1) + \sum_{i=\kappa+2}^{n} p_i u(0)) + \delta p_{\kappa+1}) \\ &= (1-\delta)u(s_{\kappa+1}) + \delta(\sum_{i=1}^{\kappa} p_i(1-\delta)u(z) + p_{\kappa+1}). \end{aligned}$$

Thus, the optimal allocation z can be obtained by solving $U_{\kappa+1}(\mathbf{z}) = U_{\kappa+1}(\mathbf{s})$ to get

$$u(z) = \frac{u(s_{\kappa+1})}{1 - \delta \sum_{i=1}^{\kappa} p_i}.$$

But, with these proposal strategies, players $h = \kappa + 2, ..., n$, have expected payoff
$U_h(\mathbf{s}) = (1 - \delta)\, u(s_h) + \delta p_h$. As a consequence, proposer $i \in \{1, ..., \kappa\}$ can allocate an amount s_h to player $h = \kappa + 2, ..., n$, in order to obtain h's vote, and retain the rest of the dollar. Thus, for some status quo $\mathbf{s} \in \Delta_\kappa$ such that

$$\frac{u(s_{\kappa+1})}{1 - \delta \sum_{i=1}^{\kappa} p_i} > u(s_{\kappa+2}), \tag{7}$$

the assumed proposal strategies are not part of an equilibrium. Player $\kappa+1$ becomes too expensive because she expects a positive allocation with probability $\sum_{i=1}^{\kappa} p_i$ while players $h = \kappa+2, ..., n$ expect zero from $i \in \{1, ..., \kappa\}$, instead.

To reconcile these incentives with the underlying equilibrium conditions, we consider mixed proposal strategies by proposers $i \in \{1, ..., \kappa\}$ for status quo $\mathbf{s} \in \Delta_\kappa$. Specifically, players $i \in \{1, ..., \kappa\}$ mix by allocating an amount we denote by $z_b(\mathbf{s})$ to one among b players $j \in \{\kappa+1, ..., \kappa+b\}$. By a similar method to that used above, we show this amount $z_b(\mathbf{s})$ must be such that

$$u(z_b(\mathbf{s})) = \frac{\sum_{j=\kappa+1}^{\kappa+b} u(s_j)}{b - \delta \sum_{i=1}^{\kappa} p_i}. \tag{8}$$

Furthermore, the integer $b \in \{1, ..., \kappa+1\}$ is uniquely[7] determined by two equilibrium conditions:

$$u(z_b(\mathbf{s})) < u(s_{\kappa+b+1}), \text{ if } b = 1, ..., \kappa, \text{ and} \tag{9}$$

$$u(z_b(\mathbf{s})) \geq u(s_j), j = \kappa+1, ..., \kappa+b. \tag{10}$$

Condition (9) is a generalization of condition (7) and requires that the utility received by each of the b players $\kappa+1, ..., \kappa+b$ is smaller than that demanded by player $\kappa+b+1$. Thus, (9) ensures proposers do not have an incentive to coalesce with any of players $\kappa+b+1, ..., n$, instead of choosing one among players $\kappa+1, ..., \kappa+b$. Condition (10) implies that players $\kappa+1, ..., \kappa+b$ receive (and demand) a larger amount than their status quo allocation s_j, $j \in \{\kappa+1, ..., \kappa+b\}$, in order to approve a proposal. On the one hand, these players' utility stream in the event they become the proposer in future periods is identical under the two alternatives, that is, these players can extract the whole dollar in the future whether they accept the equilibrium proposal or retain the status quo. On the other hand, upon accepting an equilibrium proposal, players $j \in \{\kappa+1, ..., \kappa+b\}$ receive zero by all proposers $h \neq j$ in future periods, whereas, by maintaining the status quo, these players expect to receive a positive amount as coalition partners with positive probability. Thus, the proposed allocation must be larger than the status quo allocation in order for these players to vote against the status quo.

Observe that we have so far proceeded to characterize proposal strategies under the intuitive assumption that players wish to maximize their own allocation when proposing. Although this is

[7] As we establish in Lemma 2 in the Appendix, for every status quo $\mathbf{s} \in \Delta_\kappa$, conditions (9) and (10) jointly determine a unique number of players, b, that are potential recipients of positive allocations in equilibrium.

indeed the case when stage payoffs are linear in individual allocations, it turns out not to be true, in general, when stage utilities exhibit diminishing returns, *i.e.*, when $u'' < 0$. In particular, the equilibrium we have characterized with $X = \Delta_\kappa^{n-1}$ requires certain restrictions on the concavity of players' stage preferences. Thus we defer a detailed discussion of the effect of strict concavity until section 5, and state the equilibrium under the restriction that stage payoffs are linear in individual allocations:

Proposition 1 *Assume $X = \Delta_\kappa^{n-1}$ and $u(x) = x$. Consider allocations $\mathbf{s} \in \Delta_\kappa^{n-1}$ such that $s_{i+1} \geq s_i$, $i = 1, ..., n-1$. There exists an equilibrium such that:*

1. *The proposer i extracts the whole dollar for all status quo $\mathbf{s} \in \Delta_{\kappa+1}^{n-1}$, $i \in N$, or for all status quo $\mathbf{s} \in \Delta_\kappa$ if $i = \kappa + 1, ..., n$,*

2. *For all status quo $\mathbf{s} \in \Delta_\kappa$, and all $i \in \{1, ..., \kappa\}$, i proposes $\mathbf{z}^{ij} \in \Delta_{n-2}$, with probability $\mu_i^j = \dfrac{u(z_b(\mathbf{s})) - u(s_j)}{\delta u(z_b(\mathbf{s})) \sum_{i=1}^{\kappa} p_i}$, $j = \kappa + 1, ..., \kappa + b$. Proposal $\mathbf{z}^{ij}$ is such that $z_j^{ij} = z_b(\mathbf{s})$ and $z_h^{ij} = 0, h \neq i, j$, and b satisfies (9) and (10).*

3. *The equilibrium expected utility, $U_i(\mathbf{s}), \mathbf{s} \in \Delta_\kappa^{n-1}$, is continuous and given by:*

$$U_i(\mathbf{s}) = \begin{cases} (1-\delta)u(s_i) + \delta p_i & \text{if} \quad i = \kappa + b + 1, ..., n, \\ (1-\delta)u(z_b(\mathbf{s})) + \delta p_i & \text{if} \quad i = \kappa + 1, ..., \kappa + b, \\ \delta(p_i(1-\delta)u(1 - z_b(\mathbf{s})) + \delta p_i) & \text{if} \quad i = 1, ..., \kappa. \end{cases} \tag{11}$$

Proof. See the appendix. ■

The proof of Proposition 1 is straightforward except, perhaps, in ascertaining the optimality of players' proposals, as the awkward shape of the objective function of the proposer implied by (11) makes it hard to verify equilibrium condition (3). Nevertheless, when stage payoffs are linear ($u(x) = x$) we can show (Lemma 3 in the Appendix) that the preferences represented by (11) have similar properties with conventional preferences over a divide-the-dollar space, which ensures that the proposals in Proposition 1 are optimal. We put these results to use in the following section, where we focus the analysis on the linear case, $u(x) = x$, and we use Proposition 1 in order to establish the existence of an equilibrium for the game with unrestricted agreement space $X = \Delta$.

4 Equilibrium, $X = \Delta$

Throughout this section, assume linear payoffs, *i.e.*, $u(x) = x$. Suppose the game is played in the manner we have characterized in Proposition 1 for all status quo $\mathbf{s} \in \Delta_\kappa^{n-1}$. Since these strategies render $\Delta_\kappa^{n-1} \subset \Delta$ an absorbing set, in order to establish an equilibrium for the entire game we must extend the proposal strategies of Proposition 1 to status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$ and ensure that the resultant strategies defined over all $\mathbf{s} \in \Delta$ are mutual best responses. It turns out that this is possible by, at the same time, imposing considerable structure on these strategies. We develop these arguments in detail in what follows.

First, we require that for each status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$, the support of player i's randomization over proposals is contained in a set $\Delta(i) \subset \Delta_\kappa^{n-1}$. Denote this randomization by $\widehat{\pi}_i \in \mathcal{P}[\Delta(i)]$. Set $\Delta(i) = \cup_{C \subset N \setminus \{i\} : |C| = \kappa} \{\mathbf{x} \in \Delta : x_i = 0, i \in C\}$, which is compact as the union of compact sets. In words, $\Delta(i)$ contains allocations such that a bare majority of players *including* player i receive a strictly positive amount, or allocations such that the set of players that receive a positive amount is a minority. We can afford this restriction[8] on i's proposals since (by (11)) if player i can implement a proposal in $\Delta(i)$, then no allocation in $\Delta_\kappa^{n-1} \setminus \Delta(i)$ can improve on i's utility. Assume that $\widehat{\pi}_i$ is such that all proposals in its support are approved and implemented. Then, if we denote the vector of such randomizations by all players as $\widehat{\pi} \in \times_{i \in N} \mathcal{P}[\Delta(i)]$, players' expected utility when the status quo is $\mathbf{s} \in \Delta_0^{\kappa-1}$ can be computed as:

$$\widehat{U}_i(\widehat{\pi}, \mathbf{s}) = (1 - \delta)s_i + \delta \sum_{h=1}^{n} p_h \int U_i(\mathbf{z}) \widehat{\pi}_h[d\mathbf{z}], i \in N, \tag{12}$$

where $U_i(\mathbf{z})$ is given in equation (11) of Proposition 1. We emphasize that (12) is derived under the assumption that players play according to the strategies in Proposition 1 for $\mathbf{s} \in \Delta_\kappa^{n-1}$.

Using this expected utility, $\widehat{U}_i(\widehat{\pi}, \mathbf{s})$, we obtain the proposals in Δ_κ^{n-1} that are accepted by player $i \in N$ when the status quo is $\mathbf{s} \in \Delta_0^{\kappa-1}$ and players use randomizations $\widehat{\pi} \in \times_{h \in N} \mathcal{P}[\Delta(h)]$ as:

$$\widehat{A}_i(\widehat{\pi}, \mathbf{s}) = \{\mathbf{x} \in \Delta_\kappa^{n-1} \mid U_i(\mathbf{x}) \geq \widehat{U}_i(\widehat{\pi}, \mathbf{s})\}.$$

Define for each player $i \in N$, status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$, and proposal lotteries $\widehat{\pi} \in \times_{h \in N} \mathcal{P}[\Delta(h)]$, the

[8] Indeed, we need it in order to obtain Lemma 4 in the Appendix, since non-emptiness of $\widehat{W}_i(\widehat{\pi}, \mathbf{s})$ does not generally obtain if we allow proposers to (provisionally) randomize among all alternatives in Δ_κ^{n-1}.

set

$$\widehat{W}_i(\widehat{\pi}, \mathbf{s}) = \{\mathbf{y} \in \Delta(i) \mid \sum_{h \neq i} I_{\widehat{A}_h(\widehat{\pi}, \mathbf{s})}(\mathbf{y}) \geq \kappa\}. \tag{13}$$

Given our construction, $\widehat{W}_i(\widehat{\pi}, \mathbf{s})$ contains those among proposals available to player i that are approved by at least κ *other* players when the status quo is $\mathbf{s} \in \Delta_0^{\kappa-1}$, players use lotteries over proposals given by $\widehat{\pi}$, and the game is played according to Proposition 1 for status quo $\mathbf{s} \in \Delta_\kappa^{n-1}$. In Lemma 4 in the Appendix, we show that $\widehat{W}_i$ is a non-empty, upper-hemicontinuous correspondence of $\widehat{\pi}$, and that player i can always find a proposal in $\widehat{W}_i(\widehat{\pi}, \mathbf{s})$ such that i's allocation is strictly positive.

For each player i, we now define the correspondence of best response proposals

$$M_i(\widehat{\pi}, \mathbf{s}) = \arg\max\{U_i(\mathbf{x}) \mid \mathbf{x} \in \widehat{W}_i(\widehat{\pi}, \mathbf{s})\}. \tag{14}$$

Suppose that for any initial or provisional randomizations $\widehat{\pi}$, we pick new randomizations $\widehat{\pi}'$ by restricting players to choose optimal proposals (*i.e.*, those in $M_i(\widehat{\pi}, \mathbf{s})$). Thus, we define the correspondence $B_i(\widehat{\pi}, \mathbf{s}) = \mathcal{P}[M_i(\widehat{\pi}, \mathbf{s})]$, and require $\widehat{\pi}'_i \in B_i(\widehat{\pi}, \mathbf{s})$. A significant step in proving existence of equilibrium is to establish the following Lemma:

Lemma 1 *Assume $u(x) = x$ and consider any allocation $\mathbf{s} \in \Delta_0^{\kappa-1}$. The correspondence $B : \times_{i \in N}\mathcal{P}[\Delta(i)] \rightrightarrows \times_{i \in N}\mathcal{P}[\Delta(i)]$ defined by $B(\widehat{\pi}, \mathbf{s}) = \times_{h=1}^{n} B_h(\widehat{\pi}, \mathbf{s})$ has a fixed point $\widehat{\pi}^* \in B(\widehat{\pi}^*, \mathbf{s})$.*

Proof. See the Appendix. ■

Lemma 1 states that for any status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$ we can restrict players to propose in Δ_κ^{n-1} in a consistent manner. In particular, players' expectation about lotteries over proposals are correct. Given these expectations, the proposals in the support of these lotteries are both acceptable by κ players other than the proposer and maximize the proposer's expected utility. Define the correspondence $B^* : \Delta_0^{\kappa-1} \rightrightarrows \times_{i \in N}\mathcal{P}[\Delta(i)]$ that maps status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$ to the fixed points $\widehat{\pi}^* \in \times_{i \in N}\mathcal{P}[\Delta(i)]$ of $B(\widehat{\pi}, \mathbf{s})$. Combining a selector from B^* with the proposal strategies of Proposition 1, we obtain proposal strategies $\pi_i^* : \Delta \to \mathcal{P}[\Delta]$, for each player $i \in N$. In Proposition 2 we show that these proposal strategies form part of an equilibrium.

Proposition 2 *Suppose $X = \Delta$, and $u(x) = x$. Combine any selector from B^* with the proposal strategies from Proposition 1. The resultant proposal strategies $\pi_i^* : \Delta \to P[\Delta]$, $i \in N$, form part* *of an equilibrium. Thus, at least one equilibrium exists.*

Proof. See the Appendix. ■

We have established a very counter-intuitive equilibrium such that proposers are able to eventually extract the whole dollar in the long-run. In order to understand the forces behind this counter-intuitive result, it is useful to decompose players' incentives generated by a given allocation into two effects. The first effect arises from players' utility stream as coalition partners to other proposers: a small allocation reduces a player's immediate utility, as well as the demands of that player as a voter in future periods. The second effect stems from players' expected utility stream from their role as proposers. In this regard, holding a player's own allocation fixed, there is an incentive to favor allocations that are least equitable: the less equitable the status quo, the easier it is for the proposer to extract more of the dollar (*e.g.*, Romer and Rosenthal, 1978). In equilibrium, the second effect dominates. Because of majority rule, some players in the minority are excluded from equilibrium allocations. Once a player receives an allocation equal to zero, there is no future loss or gain to be had for this player in her capacity as a coalition partner. Thus, the only active incentive for that player is that of a proposer, and this incentive sustains the inequitable allocations. Proposition 2 guarantees the existence of an equilibrium, but equilibria with the stated properties need not be unique, although all such equilibria are essentially identical in that they involve the same expected payoffs for allocations in the absorbing set Δ_κ^{n-1}. Note that Proposition 2 does not rule out the existence of other equilibria that are not payoff equivalent.

5 Equilibrium Properties & Discussion

In this section we discuss properties of the equilibrium established in Proposition 2 and derive certain implications and additional results. We start with a discussion of equilibrium dynamics.

5.1 Equilibrium Dynamics

In combination with Proposition 1, Proposition 2 provides a sharp description of the equilibrium. The Markov process over policy outcomes induced by this equilibrium is depicted graphically in Figure 1. Note that if a decision $\mathbf{x} \in \Delta_\kappa$ prevails in period $t = 1$, then there is probability $\widetilde{p} = \sum_{i=1}^{n} I_{\{0\}}(x_i)p_i$ that a decision in Δ_{n-2} is reached in period $t = 2$. Thus, within a maximum

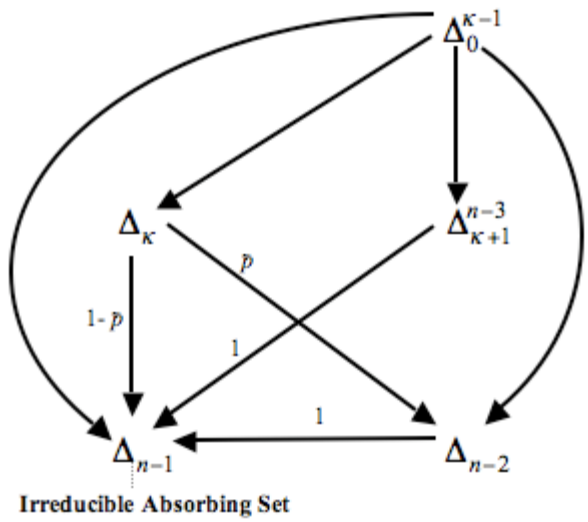

Figure 1: From any initial status quo allocation, equilibrium decisions are absorbed in Δ_{n-1} in at most three periods. $\tilde{p}$ is the sum of recognition probabilities of players with zero status quo allocation when the status quo $\mathbf{s} \in \Delta_\kappa$.

of three periods all proposers extract the entire dollar, *i.e.*, all equilibrium allocations are drawn from Δ_{n-1}. Note the difference with the corresponding distribution in the version of this game with $n = 3$ players analyzed by Kalandrakis (2004). With three players, it is possible that decisions are drawn outside the absorbing set Δ_{n-1} with positive probability for any finite period t. This is because for certain status quo such that a single player, say i, receives zero in period t, i cannot extract the whole dollar. Hence, if i is recognized in period t, the status quo in period $t+1$ must also involve a single player, say j, receiving zero. The same is possible in $t+2$, if j is recognized in period $t+1$, etc. On the contrary, when $n \geq 5$, there always exists a bare minority of κ players *other* than the proposer that have zero status quo allocation in period $t = 3$.

Figure 1 allows for the possibility that the equilibrium Markov process can be absorbed in Δ_{n-1} even from status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$. In fact, this is possible even if *all* players receive a positive status quo allocation. We illustrate this in the following example:

Example 1 *Assume $u(x) = x$, $\kappa = 2$ ($n = 5$), $p_i = \frac{1}{5}$, $i \in N$, and initial status quo $\mathbf{s} = (\varepsilon, \varepsilon, \frac{1}{6}, \frac{5}{12} - \varepsilon, \frac{5}{12} - \varepsilon) \in \Delta_0$. For small $\varepsilon > 0$, equilibrium proposals are identical to those that prevail for status quo $\mathbf{s}' = (0, 0, \frac{1}{6}, \frac{5}{12}, \frac{5}{12}) \in \Delta_2$ in Proposition 1. Specifically, players $j = 3, 4, 5$ extract the whole dollar, while players $i = 1, 2$ optimize by allocating $z_3 = \frac{\frac{1}{6}}{1-\delta\frac{2}{5}} = \frac{5}{6(5-2\delta)}$ to player*

3 and retaining the rest of the dollar. Indeed, player i's, $i = 1, 2$, expected utility from the status quo with the above proposal strategies is given by

$$U_i(\mathbf{s}) = (1-\delta)\varepsilon + \delta(\tfrac{(1-\delta)}{5}(1 - \tfrac{5}{6(5-2\delta)}) + \delta p_i) = (1-\delta)\varepsilon - \tfrac{\delta(1-\delta)}{6(5-2\delta)} + \delta p_i,$$

so that $U_i(\mathbf{s}) < \delta p_i$ for sufficiently small ε.

In example 1 all successful proposals have at most two players receiving a positive fraction of the dollar, even though all status quo allocations are positive, *i.e.*, $s_i > 0$ for all $i \in N$. Furthermore, players 1 and 2 *strictly prefer* a proposal that allocates the whole dollar to $j = 3, 4, 5$ over the status quo $\mathbf{s}$. This is because replacing the status quo $\mathbf{s} \in \Delta_0$ produces the externality of reducing the future coalition building costs for players 1 and 2.

5.2 Proposer Power & McKelvey's Dictatorial Agenda Setter

Proposition 2 holds for all possible values of recognition probabilities, p_i, $i \in N$. Thus, with linear payoffs $u(x) = x$, players' long-run equilibrium expected payoff can be any fraction of the available 'pie', depending on recognition probabilities p_i, $i \in N$. If we take the perspective that a player's expected payoff represents her *power* in this setting, then Proposition 2 yields a partial extension of the result of Kalandrakis (2006) on the relation between recognition probabilities and political power: for any level of power $\mathbf{x} \in \Delta$ and any discount factor, there exists an assignment of proposal probabilities, so that players' equilibrium level of power in the long-run coincides with $\mathbf{x}$.[9]

The case when $p_i = 1$ for some player i is theoretically significant as it yields an equilibrium derivation of dictatorial agenda setting under the institution assumed by McKelvey (1976), (1979). McKelvey's dictator uses a sequence of binary votes between the status quo and appropriate proposals. Each proposal is implemented and becomes the status quo in the next round of proposal making until the proposer eventually implements her ideal point. In his analysis, voters approve these proposals to their eventual detriment, because they are assumed to be myopic ($\delta = 0$). In the present setup, this type of dictatorial agenda setting is obtained as part of a Nash equilibrium, in fact a Markov Perfect equilibrium, under the assumption that voters are farsighted, and for every value of the discount factor $\delta < 1$. Remarkably, it only takes two periods for player i to extract the

[9]This is a partial extension because Kalandrakis (2006) considers all possible voting rules and all assymmetric discount factors. Furthermore, his analysis is obtained under a restriction to stationary equilibria in pure strategies.

whole dollar, while in general it may take three periods for absorption in Δ_{n-1} in the equilibrium of Proposition 2. In particular, we obtain the following result:

Proposition 3 (Smooth Dictator) *Assume* $X = \Delta$, $p_i = 1$, $u(x) = x$ *and any initial status quo* $\mathbf{s} \in X$. *There exists an equilibrium such that* i *extracts the whole dollar in every period* $t \geq 2$.

Proof. Consider any equilibrium among those shown to exist in Proposition 2. By construction, the proposer i with $p_i = 1$ implements some $\mathbf{x} \in \Delta(i)$ in period $t = 1$. Now in period $t = 2$, we have $\mathbf{s}^t = \mathbf{x}$, and at least κ players other than i with zero status quo allocation. Thus, according to the proposal strategies in Proposition 1, the proposer i can extract the whole dollar with probability one in period $t = 2$. ■

In order to illustrate how the agenda setting established in Proposition 3 can be achieved, consider the following five-player example.

Example 2 *Assume* $u(x) = x$, $\kappa = 2$, $p_1 = 1$, $p_h = 0$, $h = 2, ..., 5$, *and an initial status quo* $\mathbf{s} = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. *Player* 1 *needs two votes in addition to her own in order to have a proposal approved. Consider a proposal strategy such that player 1 allocates an amount* $z_i = z_j = z = \frac{1}{2(2-\delta)}$ *to two randomly chosen players* $i, j \in \{2, ..., 5\}$ *and retains* $z_1 = \frac{1-\delta}{2-\delta}$. *From (11) and the fact that* $p_h = 0$, $h \in \{2, ..., 5\}$ *we calculate expected payoffs from such proposals* $\mathbf{z} \in \Delta_2$ *as* $U_h(\mathbf{z}) = (1-\delta)z_h$, $h \in \{2, ..., 5\}$. *On the other hand, the expected payoff of players* $h \in \{2, ..., 5\}$ *from the status quo,* $\mathbf{s}$, *is given by* $U_h(\mathbf{s}) = (1-\delta)s_h + \delta v_h(\mathbf{s})$. *With the above proposal strategy for player 1, and since each player* $h \in \{2, ..., 5\}$ *has probability* $\frac{1}{2}$ *of receiving an allocation* z, *we get* $v_h(\mathbf{s}) = \frac{1}{2}z(1-\delta)$, *so* $U_h(\mathbf{s}) = \frac{(1-\delta)}{2(2-\delta)}$, $h \in \{2, ..., 5\}$. *Thus, in period* $t = 1$ *the specified proposals* $\mathbf{z} \in \Delta_2$ *are optimal and receive majority approval. Henceforth, player 1 can extract the whole dollar in all periods* $t = 2, \ldots$.

Note that the proposer in example 2 receives $\frac{1-\delta}{2-\delta}$ in the first period, an amount that tends to zero when players are patient ($\delta \to 1$). Since players 2 to 5 are recognized with probability zero in future periods, they require a higher compensation to overturn the status quo. On the other hand, player 1 is content with a low allocation in period 1 since that allocation will allow player 1 to extract the dollar in all future periods. In fact, player 1 strictly prefers any allocation $\mathbf{z}$ that excludes two other players over the status quo $\mathbf{s}$ (even if $z_1 = 0$).

5.3 Composition of Equilibrium Coalitions

This discussion points to a more general pattern concerning the effect of probabilities of recognition. Players with high probability of being recognized are more willing to accept a 'bad' proposal in the current period, since it allows them to extract more of the dollar in the following period, *ceteris paribus*. In fact, it is possible that such players are included in the winning coalition with higher probability for certain status quo, as we show in example 3. Before we state this example we define a proxy of the 'cost' or *demand* of a player:

Definition 2 *Consider an equilibrium from Proposition 2 with equilibrium expected payoffs* $U_h^*(\mathbf{s})$*,* $h \in N$*. The* demand *of player* $h \in N$ *is defined as* $d_h(\mathbf{s}) = (1-\delta)^{-1} \max\{0, U_h^*(\mathbf{s}) - \delta p_h)\}$.

In effect, the demand of player h is the allocation necessary for h to accept a proposal in $\Delta_{\kappa+1}^{n-1}$. Now consider the following:

Example 3 *Assume* $u(x) = x$*,* $\kappa = 3$*,* $\delta = \frac{9}{10}$*,* $p_l = \frac{4}{15}$*,* $l = 1, 2$*,* $p_3 = \frac{1}{5}$*,* $p_l = \frac{1}{15}$*,* $l = 4, ..., 7$*, and an initial status quo* $\mathbf{s} = \left(0, 0, \frac{1}{14}, \frac{1}{14}, \frac{2}{7}, \frac{2}{7}, \frac{2}{7}\right)$ *. Note that* $s_3 = s_4$ *but* $p_3 > p_4$*. Consider proposal strategies such that players* $l = 4, ..., 7$*, allocate* $z_3 = d_3(\mathbf{s})$*, and retain* $z_l = 1 - d_3(\mathbf{s})$*, player* 3 *allocates* $z_4 = d_4(\mathbf{s})$ *and retains* $z_3 = 1 - d_4(\mathbf{s})$*, while players* $l = 1, 2$ *allocate* $z_h = d_h(\mathbf{s}), h = 3, 4$*, and retain* $z_l = 1 - d_3(\mathbf{s}) - d_4(\mathbf{s})$*. With these proposal strategies, demands* $d_h(\mathbf{s}), h \in N$*, are obtained using (11) as solutions to:*

$$
\begin{aligned}
d_3(\mathbf{s}) &= \tfrac{1}{14} + \delta\left(\tfrac{1}{5}(1 - d_4(\mathbf{s})) + \tfrac{4}{5} d_3(\mathbf{s})\right) - \tfrac{\delta}{5}, \\
d_4(\mathbf{s}) &= \tfrac{1}{14} + \delta\left(\tfrac{1}{15}(1 - d_3(\mathbf{s})) + \tfrac{11}{15} d_4(\mathbf{s})\right) - \tfrac{\delta}{15}, \\
d_l(\mathbf{s}) &= \tfrac{2}{7} + \delta\left(\tfrac{1}{15}(1 - d_3(\mathbf{s}))\right) - \tfrac{\delta}{15}, l = 5, ..., 7, \\
d_l(\mathbf{s}) &= \max\left\{\delta\left(\tfrac{4}{15}(1 - d_3(\mathbf{s}) - d_4(\mathbf{s}))\right) - \tfrac{4\delta}{15}, 0\right\}, l = 1, 2.
\end{aligned}
$$

It is straightforward to verify that $d_1(\mathbf{s}) = d_1(\mathbf{s}) = 0 < d_3(\mathbf{s}) = \frac{200}{1477} < d_4(\mathbf{s}) = \frac{275}{1477} < d_l(\mathbf{s}) = \frac{410}{1477}$*,* $l = 5, ..., 7$*. Thus, the above described proposals are optimal and player 3 is included in the winning coalition with higher probability than player 4 even while* $p_3 = \frac{1}{5} > p_4 = \frac{1}{15}$ *and* $s_3 = s_4$*.*

In the model of Baron and Ferejohn (1989), a higher recognition probability implies both a higher cost of inclusion in the coalition (higher demand), and a smaller probability of inclusion in the winning coalition (see Eraslan (2002)). Let $\mu_h^l(\mathbf{s}) \leq p_h$ be the probability that player h makes proposals $\mathbf{z}$ such that $U_l^*(\mathbf{z}) \geq U_l^*(\mathbf{s})$, *i.e.*, such that player $l \neq h$ is included in the winning coalition. Contrary to the effect of probabilities of recognition on a players' likelihood of inclusion in the winning coalition in the Baron and Ferejohn model, we have just shown:

Proposition 4 *There exist probabilities of recognition satisfying $p_i > p_j$ for two players $i, j \in N$, and an equilibrium (as in Proposition 2) for the associated game, such that for some status quo $\mathbf{s} \in \Delta$ with $s_i = s_j$ we have $d_j(\mathbf{s}) > d_i(\mathbf{s})$ and $\sum_{h \neq i,j} \mu_h^j(\mathbf{s}) < \sum_{h \neq i,j} \mu_h^i(\mathbf{s})$.*

In fact, when it comes to equilibrium demands we can show a stronger result, *i.e.*, that $d_j(\mathbf{s}) \geq d_i(\mathbf{s})$ for all players $i \neq j$ with $p_i > p_j$ and all status quo $\mathbf{s} \in \Delta$ such that $s_i = s_j$.[10] We emphasize that both in the present study and in the Baron and Ferejohn model players are better off with higher recognition probabilities. Put otherwise, in our analysis recognition probabilities have (a) a positive effect on players' long-run expected payoff (Proposition 1, Proposition 2); and, (b) under certain conditions, a negative effect on players demand and a positive effect on players' probability of being included in the winning coalition. Thus, it is the relation between recognition probabilities and the *composition* of equilibrium coalitions (not equilibrium expected payoffs) that is different between Baron and Ferejohn type of bargaining and the fully dynamic model we analyze in this study.

5.4 Diminishing Returns

The equilibrium established in section 4 requires linear payoffs, $(u(x) = x)$. Thus, the equilibrium is Pareto optimal, so that equilibrium allocations, and any plan of division of the dollar for that matter, are efficient from an economic perspective even though they may imply a politically disturbing inequality of payoffs. This conclusion is no longer valid if players' stage utility, u, is strictly concave. Under this assumption, the equilibrium is obviously inefficient. Since $u(p_i) > p_i$, $p_i \in (0,1)$, every player $i \in N$ strictly prefers a constant share p_i of the dollar after period $t = 3$ rather than receiving the whole dollar with probability $p_i \in (0,1)$. Thus, strict concavity generates incentives for more equitable allocations, and one may question whether the equilibrium in Proposition 2 survives in

[10] The proof is lengthy and tedious because of the complications that arise by the non-standard form of players' expected utility (11) when players propose allocations $\mathbf{z} \in \Delta_\kappa$ (see Kalandrakis, 2003). For the same reason, it is not necessarily true that a player with lower demand is included in the winning coalition with higher probability, exactly due to the shape of equilibrium expected utility for allocations $\mathbf{z} \in \Delta_\kappa$.

the presence of these incentives. Consistent with this intuition, we will show that the proposal strategies prescribed in Proposition 1 are not optimal if players' stage preferences are sufficiently concave.

We will provide two examples in order to substantiate this claim. In the first of the two examples, the proposer prefers to deviate from proposals prescribed in Proposition 1 (assuming players subsequently adhere to the strategies prescribed in that equilibrium) in the following way: instead of buying the vote of a single player among those with a positive allocation by exclusively giving a positive amount to that player, the proposer is better off allocating an equal amount to $\kappa + 1$ players including herself.

Example 4 *Assume $p_i = \frac{1}{n}$ for all $i \in N$ and an initial status quo $\mathbf{s} = (0, ..., 0, \frac{1}{\kappa+1}, ..., \frac{1}{\kappa+1}) \in \Delta_\kappa$. Note that $b = \kappa+1$ satisfies conditions (9) and (10) for $\mathbf{s}$. Thus, according to the proposal strategies in Proposition 1, player $i \in \{1, ..., \kappa\}$ allocates $z_b(\mathbf{s}) = u^{-1}(\frac{(\kappa+1)u(\frac{1}{\kappa+1})}{\kappa+1-\delta\frac{\kappa}{n}})$ to one of players $\kappa+1, ..., n$, and retains $1 - z_b(\mathbf{s})$. Player i's utility from this proposal is given by $(1-\delta)u(1-z_b(\mathbf{s})) + \frac{\delta}{n}$. Now suppose stage utility function u satisfies*

$$\frac{(\kappa + 1)u(\frac{1}{\kappa+1})}{\kappa + 1 - \delta\frac{\kappa}{n}} > u(\tfrac{1}{2}).$$

Then we deduce $\frac{(\kappa+1)u(\frac{1}{\kappa+1})}{\kappa+1-\delta\frac{\kappa}{n}} > u(\frac{1}{2}) \Leftrightarrow u(z_b(\mathbf{s})) > u(\frac{1}{2}) \Leftrightarrow u(z_b(\mathbf{s})) > u(1 - z_b(\mathbf{s}))$, the first step obtained by substituting from (8). Thus, player i can improve her utility by proposing $\mathbf{x} \in \Delta_\kappa$ with $x_i = \frac{1}{\kappa+1}$ and $x_h = \frac{1}{\kappa+1}$ to κ more players. From (11), the expected payoff from this proposal is $U_i(\mathbf{x}) = \frac{(1-\delta)(\kappa+1)u(\frac{1}{\kappa+1})}{\kappa+1-\delta\frac{\kappa}{n}} + \frac{\delta}{n}$, if the game is subsequently played according to the proposal strategies specified in Proposition 1, which is larger than $(1-\delta)u(1-z_b(\mathbf{s})) + \frac{\delta}{n}$.

Thus, for the proposal strategies in Proposition 1 to form an equilibrium when $u'' < 0$ and recognition probabilities are equal, we must have:

$$\frac{(\kappa + 1)u(\frac{1}{\kappa+1})}{\kappa + 1 - \delta\frac{\kappa}{n}} \leq u(\frac{1}{2}). \tag{15}$$

Figure 2 depicts a situation where condition (15) is violated when $\kappa = 2$.

Now consider a different example:

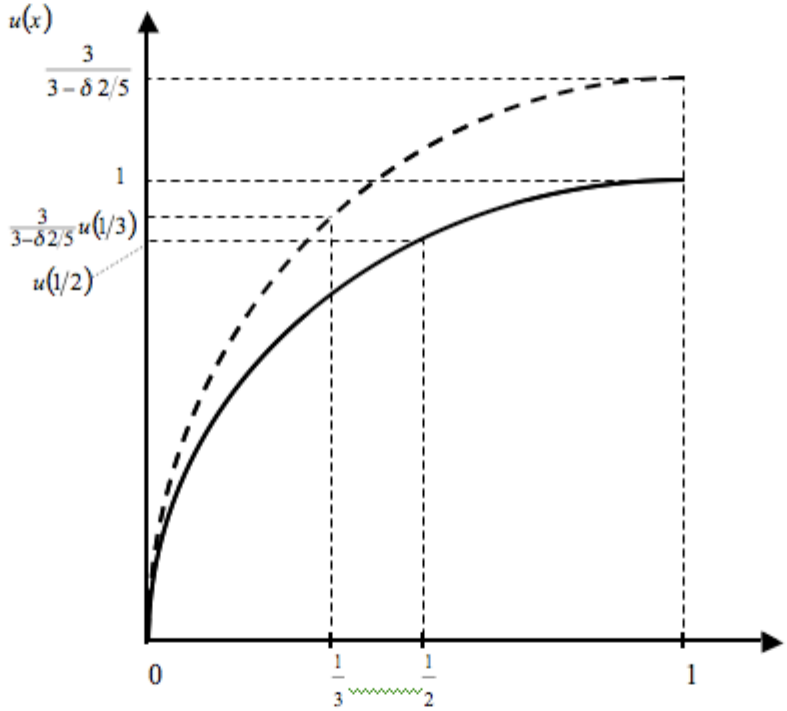

Figure 2: Stage utility function, u, violates condition (15) when $\kappa = 2$, since $\frac{3u(\frac{1}{3})}{3-\delta\frac{2}{5}} > u(\frac{1}{2})$.

Example 5 *Assume* $p_i = \frac{1}{n}$*,* $i \in N$*, and an initial status quo* $\mathbf{s} = (0, ..., 0, s, ..., s, 1 - \kappa s) \in \Delta_\kappa$*, where* s *is such that* $\frac{\kappa u(s)}{\kappa - \delta\frac{\kappa}{n}} < u(1-\kappa s)$*. Now* $b = \kappa$ *satisfies conditions (9) and (10). Thus, according to the proposal strategies in Proposition 1, player* $i \in \{1, ..., \kappa\}$ *allocates* $z_b(\mathbf{s}) = u^{-1}(\frac{\kappa}{\kappa - \delta\frac{\kappa}{n}} u(s))$ *to one of players* $\kappa + 1, ..., n - 1$*, and retains* $1 - z_b(\mathbf{s})$*. Now suppose that* u *is such that*

$$u(\kappa s) < \frac{\kappa u(s)}{\kappa - \delta\frac{\kappa}{n}}.$$

Then we have $u(\kappa s) < \frac{\kappa u(s)}{\kappa - \delta\frac{\kappa}{n}} \Leftrightarrow \kappa s < z_b(\mathbf{s}) \Leftrightarrow u(1 - \kappa s) > u(1 - z_b(\mathbf{s}))$*. Thus, if the game is subsequently played according to the strategies in Proposition 1, the prescribed proposal is not optimal since player* $i \in \{1, ..., \kappa\}$ *can propose* $\mathbf{x} \in \Delta_\kappa$ *with* $x_i = 1 - \kappa s$ *and* $x_h = s$ *to* κ *more players and get higher utility.*

In a manner similar to example 4, in example 5 the proposer can improve her expected utility by allocating an equal amount to κ other players rather than allocating a positive amount to a single player. Thus, we obtain another necessary condition for the proposal strategies in Proposition 1 to be part of an equilibrium in the presence of diminishing returns, namely:

$$\frac{\kappa u(s)}{\kappa - \delta\frac{\kappa}{n}} \leq u(\kappa s), \text{ if } s \text{ is such that } \frac{\kappa u(s)}{\kappa - \delta\frac{\kappa}{n}} < u(1-\kappa s). \tag{16}$$

Note that both conditions (15) and (16) effectively impose a bound on the steepness of stage utility $u(x)$ at small fractions of the dollar. When either condition is violated, the immediate gain from extracting larger shares of the dollar diminishes so much, so that such a gain is not preferred over the option of receiving a smaller amount in the current period with a prospect of also receiving a positive fraction in the future as a coalition partner. Note that for fixed utility function, u, conditions (15) and (16) are easier met when the committee is larger (larger κ). Both conditions (15) and (16) are always satisfied in the case of risk neutrality, $u(x) = x$. In fact, we can show that these conditions are sufficient for the strategies described in Proposition 1 to form an equilibrium, assuming recognition probabilities are equal.

Proposition 5 *Assume $X = \Delta_\kappa^{n-1}$, $p_i = \frac{1}{n}$ for all $i \in N$, and $u'' < 0$. The proposal strategies in Proposition 1 form part of an equilibrium under these assumptions if and only if u satisfies conditions (15) and (16).*

Proof. See the appendix. ■

Proposition 5 points to another instance of the discrepancy between models without recurring policy decisions and the model we analyze. In particular, in a model with the same institutions as in Baron and Ferejohn (1989), Harrington (1990), shows that higher degrees of risk aversion increase the power of the proposer. In contrast, we have shown that an equilibrium that is quite favorable for the proposer collapses when risk aversion is high.

6 Conclusions

We have analyzed a dynamic majority rule bargaining game over a distributive policy space with an endogenous status quo policy. Although such an equilibrium may fail to exist for games in the class we analyze, we established existence of a (refined) Markov Perfect Nash equilibrium. The equilibrium produces a number of novel and in many instances counter-intuitive findings. First, we have shown that dynamic bargaining over a distributive policy space does not guarantee sufficient strategic incentives for players to converge to equitable allocations both within and across periods. In fact, the long-term dynamics induced in the equilibrium we characterized are identical to the dynamics that would prevail in a world were players are myopic or non-strategic. Second, in Proposition 3 we obtained the grim consequences of McKelvey's (1976), (1979), dictatorial agenda setting construction despite the fact that voters are farsighted.

In some regards, the results of our analysis leave a more upbeat note than many interpretations of the conclusions of social choice theory. Instead of encompassing the entire space of alternatives, the long-run absorbing set of possible divisions in the characterized equilibrium is a finite set. Thus, instead of chaos, there are at most n possible policy outcomes after period 3. In addition, there is *ex ante* positive probability equal to $\sum_{i=1}^{n} p_i^2$ that the same decision prevails between consecutive periods, so that we don't observe perpetual instability of decisions in equilibrium. We emphasize that these results do not depend on the restriction to Markov equilibria. The equilibrium we characterize, and many more, exists *a fortiori* if we consider weaker equilibrium notions such as subgame perfect or simple Nash equilibrium. Also, these results do not depend on the way we resolve voter indifference at the critical voting period when alternatives that allocate zero to more than a majority of players prevail for the first time. As illustrated in example 1, players that vote *yes* in these cases and receive zero may strictly prefer the proposal over the status quo, despite the fact that they receive a smaller allocation.

Under certain restrictions, the inequality of long-run equilibrium allocations persists even in the presence of diminishing returns on players' stage preferences, in which case the equilibrium is inefficient. On the other hand, we also show players' incentives for sharing the budget do manifest themselves in the presence of sufficient concavity in stage preferences, since the characterized inefficient equilibrium does not survive when such concavity is marked. Besides the counter-intuitive nature of the equilibrium, the significance of our findings also stems from the fact that in many respects they differ from results in other studies that impose similar equilibrium restrictions. For example, if we consider similar institutions and ideological policy spaces (*e.g.*, Baron (1996), Baron and Herron (2003)), we obtain long-run policy dynamics concentrated at the center of the policy space. If, instead, we maintain the distributive space of our analysis but substitute the institutional arrangements with the Baron and Ferejohn (1989) closed rule, we get a different composition of winning coalitions in regards to probabilities of recognition (Proposition 4 vs. Eraslan, (2002)) or the effect of risk aversion (Proposition 5 vs. Harrington (1990)). The first comparison suggests that at least some aspects of our theories of legislative politics cannot be independent of the underlying policy space. The second comparison highlights the significance (at least theoretically) of modeling choices that trade between analytical tractability and realism by assuming legislative interaction that ceases after a decision is reached or incorporate dynamic interaction with endogenous status quo.

APPENDIX

In this appendix we first state and prove two Lemmas (Lemmas 2 and 3), which are used in the proofs of Proposition 1, Lemma 1, and Proposition 2. We start with Lemma 2:

Lemma 2 *Consider any $\mathbf{s} \in \Delta_{\kappa}^{n-1}$, any stage utility function u with $u'' \leq 0$, and assume without loss of generality that $s_{i+1} \geq s_i$, $i = 1, ..., n-1$.*

(i) There exists unique b, $1 \leq b \leq \kappa+1$, that satisfies (9) and (10).

(ii) If b satisfies (9) and (10), then

$$u(z_b(\mathbf{s})) \leq \frac{(\kappa+1)u(\frac{1}{\kappa+1})}{\kappa+1-\delta\sum_{i=1}^{\kappa} p_i}.$$

Proof. We start by showing the following equivalence:

$$[u(z_b(\mathbf{s})) < u(s_{\kappa+b+1}) \Leftrightarrow u(z_{b+1}(\mathbf{s})) < u(s_{\kappa+b+1})], b = 1, ..., \kappa. \tag{17}$$

Indeed, making use of (8) we write $u(z_b(\mathbf{s})) < u(s_{\kappa+b+1}) \Leftrightarrow \sum_{j=\kappa+1}^{\kappa+b} u(s_j) < (b-\delta\sum_{i=1}^{\kappa} p_i)u(s_{\kappa+b+1})$ $\Leftrightarrow \sum_{j=\kappa+1}^{\kappa+b+1} u(s_j) < (b+1-\delta\sum_{i=1}^{\kappa} p_i)u(s_{\kappa+b+1}) \Leftrightarrow u(z_{b+1}(\mathbf{s})) < u(s_{\kappa+b+1})$.
The contra-positive of (17) also gives us

$$[u(z_b(\mathbf{s})) \geq u(s_{\kappa+b+1}) \Leftrightarrow u(z_{b+1}(\mathbf{s})) \geq u(s_{\kappa+b+1})], b = 1, ..., \kappa. \tag{18}$$

Now, to show existence of b satisfying (9) and (10) consider the algorithm:
1. Start with $b = 1$; if $u(z_1(\mathbf{s})) < u(s_{\kappa+2})$ then $b = 1$.
2. If $u(z_b(\mathbf{s})) \geq u(s_{\kappa+b+1})$, consider $b' = b+1$. (18) ensures that b' satisfies (10). If b' also satisfies (9) then stop.

3. Otherwise, if $u(z_{b'}(\mathbf{s})) \geq u(s_{\kappa+b'+1})$ proceed as in 2 until $u(z_b(\mathbf{s})) < u(s_{\kappa+b+1})$ for some $b \leq \kappa$.
4. If condition (9), $u(z_b(\mathbf{s})) < u(s_{\kappa+b+1})$ fails for all $b \leq \kappa$, then $u(z_\kappa(\mathbf{s})) \geq u(s_n)$, and $b = \kappa + 1$. Thus b exists.

To show uniqueness, suppose there exist distinct, b, b' with $b < b'$ that satisfy (9) and (10) to get a contradiction. Then, we have $u(z_b(\mathbf{s})) < u(s_{\kappa+b+1})$ from (9) and certainly $u(z_{b'}(\mathbf{s})) \geq u(s_{\kappa+b+\lambda})$, $\lambda = 1, ..., (b'-b)$ from (10). From the last $(b'-b)$ inequalities we deduce

$$(b'-b)u(z_{b'}(\mathbf{s})) \geq \sum_{\lambda=1}^{(b'-b)} u(s_{\kappa+b+\lambda}) \Leftrightarrow$$

$$(b'-b)\frac{\sum_{h=\kappa+1}^{\kappa+b} u(s_h) + \sum_{\lambda=1}^{(b'-b)} u(s_{\kappa+b+\lambda})}{b + (b'-b) - \delta \sum_{i=1}^{\kappa} p_i} \geq \sum_{\lambda=1}^{(b'-b)} u(s_{\kappa+b+\lambda}) \Leftrightarrow$$

$$\frac{\sum_{h=\kappa+1}^{\kappa+b} u(s_h)}{b - \delta \sum_{i=1}^{\kappa} p_i} \geq \frac{\sum_{\lambda=1}^{(b'-b)} u(s_{\kappa+b+\lambda})}{b'-b} \Leftrightarrow$$

$$u(z_b(\mathbf{s})) \geq \frac{\sum_{\lambda=1}^{(b'-b)} u(s_{\kappa+b+\lambda})}{b'-b} \geq u(s_{\kappa+b+1}),$$

which contradicts condition (9) for b. This concludes the proof of part (i).

To show part (ii), we will first show that $u(z_b(\mathbf{s})) \leq u(z_{\kappa+1}(\mathbf{s}))$. This is trivial if $b = \kappa + 1$, so consider the case $b \leq \kappa$. Then, by condition (9) and the fact that $s_{i+1} \geq s_i$, $i = 1, ..., n-1$ we have $u(z_b(\mathbf{s})) < u(s_{\kappa+b+1})$, which implies that

$$\frac{\sum_{i=\kappa+1}^{\kappa+b} u(s_i)}{b - \delta \sum_{h=1}^{\kappa} p_h} < \frac{\sum_{i=\kappa+b+1}^{n} u(s_i)}{\kappa + 1 - b} \Leftrightarrow$$

$$(\kappa + 1 - b) \sum_{i=\kappa+1}^{\kappa+b} u(s_i) < (b - \delta \sum_{h=1}^{\kappa} p_h) \sum_{i=\kappa+b+1}^{n} u(s_i) \Leftrightarrow$$

$$(\kappa + 1) \sum_{i=\kappa+1}^{\kappa+b} u(s_i) < b \sum_{i=\kappa+1}^{n} u(s_i) - \delta \sum_{h=1}^{\kappa} p_h \sum_{i=\kappa+b+1}^{n} u(s_i) \Leftrightarrow$$

$$(\kappa + 1 - \delta \sum_{h=1}^{\kappa} p_h) \sum_{i=\kappa+1}^{\kappa+b} u(s_i) < (b - \delta \sum_{h=1}^{\kappa} p_h) \sum_{i=\kappa+1}^{n} u(s_i) \Leftrightarrow$$

$$u(z_b(\mathbf{s})) = \frac{\sum_{i=\kappa+1}^{\kappa+b} u(s_i)}{b - \delta \sum_{h=1}^{\kappa} p_h} < \frac{\sum_{i=\kappa+1}^{n} u(s_i)}{\kappa + 1 - \delta \sum_{h=1}^{\kappa} p_h} = u(z_{\kappa+1}(\mathbf{s})).$$

Thus, we conclude that $u(z_b(\mathbf{s})) \leq u(z_{\kappa+1}(\mathbf{s}))$ when $b = 1, ..., \kappa+1$ satisfies (9) and (10), as we wished to show. Now, concavity of u implies that

$$u(z_{\kappa+1}(\mathbf{s})) \leq \frac{(\kappa+1)u(\frac{1}{\kappa+1})}{\kappa+1-\delta\sum_{i=1}^{\kappa} p_i},$$

which completes the proof of part (ii) and the Lemma. ■

We continue by showing Lemma 3 which ensures that, starting with any allocation in Δ_κ^{n-1}, the preferences represented by (11) are such so that we can incrementally reduce the allocation of one player in order to increase the payoff of a bare minority that contains all players receiving a positive amount in the original allocation:

Lemma 3 *Assume $u(x) = x$ and preferences over Δ_κ^{n-1} given by (11). Consider allocation $\mathbf{x} \in \Delta_\kappa^{n-1}$ and let $C = \{h \in N : x_h > 0\}$. For every $\varepsilon > 0$, every coalition $K \subset N$ with $|K| = \kappa+1$ and $C \subseteq K$, and every $i \in C$, there exists $\mathbf{y} \in \Delta_\kappa^{n-1}$ such that $U_j(\mathbf{y}) > U_j(\mathbf{x})$, $j \in K \setminus \{i\}$, and $|U_i(\mathbf{y}) - U_i(\mathbf{x})| < \varepsilon$.*

Proof. If $\mathbf{x} \in \Delta_{\kappa+1}^{n-1}$, then $U_h(\mathbf{x}) = (1-\delta)x_h + \delta p_h$ for all $h \in N$, and the proof is straightforward: set $y_i = x_i - \kappa\eta$, where $\kappa\eta > 0$, and η is as small as is necessary for $|U_i(\mathbf{y}) - U_i(\mathbf{x})| < \varepsilon$ to hold, and set $y_j = x_j + \eta$ for all $j \in K \setminus \{i\}$, and set $y_h = x_h = 0$ for all $h \notin K$.

Thus, it remains to consider $\mathbf{x} \in \Delta_\kappa$. Assume without loss of generality that $x_{h+1} \geq x_h$, $h = 1, ..., n-1$, so that $C = K = \{\kappa+1, ..., n\}$, and assume b satisfies (9) and (10) for $\mathbf{x}$. There are two cases:

Case 1, $i > \kappa + b$: Set $y_i = x_i - \eta(\kappa+1-b) > 0$, and $\eta > 0$. Set $y_j = \frac{(\sum_{j=\kappa+1}^{\kappa+b} x_j)+\eta}{b}$, $j = \kappa+1, ..., \kappa+b$, and $y_j = x_j + \eta$, for $j \in \{\kappa+b+1, ..., n\} \setminus \{i\}$. For sufficiently small η, $|U_i(\mathbf{y}) - U_i(\mathbf{x})| < \varepsilon$ and we have

$$U_j(\mathbf{y}) = \frac{(1-\delta)(\sum_{j=\kappa+1}^{\kappa+b} x_j + \eta)}{b - \delta\sum_{h\notin K} p_h} + \delta p_j > \frac{(1-\delta)\sum_{j=\kappa+1}^{\kappa+b} x_j}{b - \delta\sum_{h\notin K} p_h} + \delta p_j = U_j(\mathbf{x}), j = \kappa+1, ..., \kappa+b,$$

while $U_j(\mathbf{y}) = U_j(\mathbf{x}) + \eta$, for $j \in \{\kappa+b+1, ..., n\} \setminus \{i\}$.

Case 2, $i \leq \kappa + b$: We have $z_b(\mathbf{x}) = \frac{\sum_{j=\kappa+1}^{\kappa+b} x_j}{b-\delta\sum_{h\notin K} p_h}$. Note that $\gamma = (\sum_{j=\kappa+1}^{\kappa+b} x_j) - (b-1)z_b(\mathbf{x}) = \frac{(1-\delta\sum_{h\notin K} p_h)(\sum_{j=\kappa+1}^{\kappa+b} x_j)}{b-\delta\sum_{h\notin K} p_h} > 0$. Thus it is feasible to set $y_i = \gamma - \kappa\eta$, with $\eta > 0$ and sufficiently small, $y_j = z_b(\mathbf{x}) + \eta$, $j \in \{\kappa+1, ..., \kappa+b\} \setminus \{i\}$, and $y_j = x_j + \eta$, for $j \in \{\kappa+b+1, ..., n\}$. Now we have

$U_i(\mathbf{y}) = \frac{(1-\delta)y_i}{b-\delta\sum_{h\notin K} p_h} + \delta p_i = (1-\delta)(z_b(\mathbf{x}) - \frac{\kappa\eta}{1-\delta\sum_{h\notin K} p_h}) + \delta p_i$, versus $U_i(\mathbf{x}) = (1-\delta)z_b(\mathbf{x}) + \delta p_i$, while $U_j(\mathbf{y}) = U_j(\mathbf{x}) + \eta$, $j \in K - \{i\}$. ■

Next, we prove Propositions 1 and 5. Many arguments in the two proofs are identical, so we economize on space by presenting both proofs at the same time. We explicitly identify cases when either of the two propositions requires special arguments.

Proof of Propositions 1 and 5. To ensure that the associated proposals are well defined, note that by part (i) of Lemma 2, b exists and is unique. Also, part (ii) of Lemma 2 ensures that $z_b(\mathbf{s}) < \frac{1}{2}$. In particular, in the case of Proposition 1 with $u(x) = x$ we have from part (ii) of Lemma 2 that $z_b(\mathbf{s}) \leq \frac{1}{\kappa+1-\delta} < \frac{1}{2}$, while in the case of Proposition 5 condition (15) similarly ensures $z_{\kappa+1}(\mathbf{s}) \leq \frac{(\kappa+1)u(\frac{1}{\kappa+1})}{\kappa+1-\delta\frac{\kappa}{n}} \leq \frac{1}{2}$ when $u'' < 0$ and $p_i = \frac{1}{n}$ for all $i \in N$. Thus it is feasible to construct proposals $\mathbf{z}^{ij} \in \Delta_{n-2}$ with $z_j^{ij} = z_b(\mathbf{s})$. Lastly, to show that mixing probabilities lie between zero and one and sum up to one, it suffices to show that $\sum_{j=\kappa+1}^{\kappa+b} \mu_i^j = 1$ and that $\mu_i^j \geq 0$ for all j. The latter is equivalent to $u(z_b(\mathbf{s})) \geq u(s_j)$, $j = \kappa+1, ..., \kappa+b$ which is true by (10). We also have $\sum_{j=\kappa+1}^{\kappa+b} \mu_i^j = \frac{bu(z_b(\mathbf{s})) - \sum_{j=\kappa+1}^{\kappa+b} u(s_j)}{\delta u(z_b(\mathbf{s}))\sum_{i=1}^{\kappa} p_i} = 1$, after substitution from (8).

Continuity of $U_i(\mathbf{s})$ follows easily either by direct arguments or by the fact that proposal probabilities and proposals are continuous functions of $\mathbf{s}$. Next we show that the expected utilities in (11) are derived from the described proposal strategies and are such that all proposals are accepted when players play stage-undominated voting strategies. As already argued in stating (6), we have $v_i(\mathbf{s}) = p_i$ for all $\mathbf{s} \in \Delta_{\kappa+b}$, $b = 1, ..., \kappa+1$. Further note that for such status quo, b satisfies (9) and (10), $z_b(\mathbf{s}) = s_{\kappa+b} = 0$ and the expected payoff in (11) reduces to $U_i(\mathbf{s}) = (1-\delta)u(s_i) + \delta p_i$ as required by (4). Now consider $\mathbf{s} \in \Delta_\kappa$. Since proposals offered for such status quo, $\mathbf{z}^{ij}$, belong in $\Delta_{\kappa+1}^{n-1}$, we can write players' continuation value as $v_i(\mathbf{s}) = \sum_{h=1}^{n} p_h \sum_{j=\kappa+1}^{\kappa+b} \mu_h^j((1-\delta)u(z_i^{hj}) + \delta p_i)$. After substitution for μ_h^j and a bit of algebra we obtain for each $\mathbf{s} \in \Delta_\kappa^{n-1}$:

$$v_i(\mathbf{s}) = \begin{cases} p_i & \text{if} \quad i = \kappa+b+1, ..., n, \\ (1-\delta)\delta^{-1}(u(z_b(\mathbf{s})) - u(s_i)) + p_i & \text{if} \quad i = \kappa+1, ..., \kappa+b, \\ (1-\delta)p_i u(1-z_b(\mathbf{s})) + \delta p_i & \text{if} \quad i = 1, ..., \kappa. \end{cases}$$

Direct application of the definition in equation (4) using the above yields equation (11) as desired. Note that with the expected payoffs in (11) all players $i \in \{1, ..., \kappa\}$ accept any proposal $\mathbf{z} \in \Delta_{\kappa+1}^{n-1}$ with $z_i = 0$ for every $\mathbf{s} \in \Delta_\kappa^{n-1}$, while the same is true for players $j \in \{\kappa+1, ..., \kappa+b\}$ as long as

$z_j = z_b(\mathbf{s})$. Thus proposals are approved by majorities.

To complete the proof of the proposition, it remains to show optimality of proposals. We pursue a proof that covers both cases (Proposition 1 and 5). Note that $(1-\delta)u(1)+\delta p_i = \max\{U_i(\mathbf{x}) : \mathbf{x} \in \Delta_\kappa^{n-1}\}$. Thus, since extracting the entire dollar is a global maximum, we only need consider cases when the proposer does not extract the entire dollar. This occurs for status quo $\mathbf{s} \in \Delta_\kappa$ and proposer i with $s_i = 0$. So assume:

- a status quo $\mathbf{s} \in \Delta_\kappa$,
- a proposer i with $s_i = 0$, and
- an integer b that uniquely satisfies (9) and (10) for $\mathbf{s}$.

Prescribed equilibrium proposals $\mathbf{z}^{ih} \in \mathbf{\Delta}_{n-2}$, $h = \kappa+1, ..., \kappa+b$ for proposer i are optima among feasible alternatives in $\Delta_{\kappa+1}^{n-1}$, because maximization for the proposer among alternatives in $\Delta_{\kappa+1}^{n-1}$ clearly amounts to maximizing her stage allocation. Thus, we need to show that there exists no $\mathbf{y} \in \arg\max\{U_i(\mathbf{z}) : \mathbf{z} \in W(\mathbf{s}) \bigcap \Delta_\kappa\}$, with $U_i(\mathbf{y}) > U_i(\mathbf{z}^{ih})$. So further assume, in order to get a contradiction, that there exists $\mathbf{y} \in W(\mathbf{s}) \cap \Delta_\kappa$ such that $U_i(\mathbf{y}) > U_i(\mathbf{z}^{ih})$. From the assumption that $U_i(\mathbf{y}) > U_i(\mathbf{z}^{ih})$ and the fact that $z_b(\mathbf{s}) \leq \frac{1}{2}$ by part (ii) of Lemma 2, we have

$$y_i > z_i^{ih} \geq \tfrac{1}{2} \geq z_h^{ih} = z_b(\mathbf{s}). \tag{19}$$

Without loss of generality relabel players so that $y_{l+1} \geq y_l$, $l = 1, ..., n-1$, and let integer b' uniquely satisfy (9) and (10) for $\mathbf{y}$. After relabeling, we have $i = n$ from (19) and the fact that $z_{b'}(\mathbf{y}) \leq \frac{1}{2}$, the latter again from part (ii) of Lemma 2 and (15). We must also have $U_j(\mathbf{y}) \geq U_j(\mathbf{s}) > \delta p_j$ for at least one player j with $s_j > 0$ else $\mathbf{y} \notin W(\mathbf{s})$. Since $s_j > 0$, (11) and (9) imply that

$$(1-\delta)^{-1}(U_j(\mathbf{y}) - \delta p_j) \geq u(z_b(\mathbf{s})) > 0, \tag{20}$$

so $y_j > 0$ and, again after relabeling players, $j \in \{\kappa+1, ..., n-1\}$. We now have three cases all of which lead to a contradiction emanating from the assumption that $U_i(\mathbf{y}) > U_i(\mathbf{z}^{ih})$.

Case 1 ($j > \kappa + b'$): Then from (11) we have $U_j(\mathbf{y}) = (1-\delta)u(y_j) + \delta p_j \geq U_j(\mathbf{s})$. Thus, from (19) and (20) we deduce $y_j \geq z_b(\mathbf{s}) = z_h^{ih}$. But then from (19) we have $1 - z_h^{ih} = z_i^{ih} < y_i < 1 - y_j$, which yields $z_h^{ih} > y_j$, a contradiction.

Case 2 ($j \leq \kappa + b', b' = \kappa$): Then from (11) and (8) we have $U_j(\mathbf{y}) = \frac{(1-\delta)\sum_{l=\kappa+1}^{n-1} u(y_l)}{\kappa - \delta \sum_{l=1}^{\kappa} p_l} + \delta p_j$. From (20) we have $\frac{\sum_{l=\kappa+1}^{n-1} u(y_l)}{\kappa - \delta \sum_{l=1}^{\kappa} p_l} \geq u(z_b(\mathbf{s}))$. This last inequality implies that if u is linear then $\sum_{l=\kappa+1}^{n-1} y_l > z_b(\mathbf{s})$, while if u is strictly concave with $p_l = \frac{1}{n}$ for all $l \in N$, we have from concavity and (16) that $u(z_b(\mathbf{s})) \leq \frac{\sum_{l=\kappa+1}^{n-1} u(y_l)}{\kappa - \delta \frac{\kappa}{n}} < \frac{\kappa u(\overline{y})}{\kappa - \delta \frac{\kappa}{n}} \leq u(\kappa \overline{y})$, where $\overline{y} = \frac{\sum_{l=\kappa+1}^{n-1} y_l}{\kappa}$. From (19) we have $y_i > z_i^{ih} \geq z_h^{ih} = 1 - z_i^{ih} > 1 - y_i = \sum_{l=\kappa+1}^{n-1} y_l$. But then we have $u(z_b(\mathbf{s})) = u(z_h^{ih}) > u(\sum_{h=\kappa+1}^{n-1} y_h) > u(z_b(\mathbf{s}))$, a contradiction.

Case 3 ($j \leq \kappa + b', b' < \kappa$): Then $U_j(\mathbf{y}) = \frac{(1-\delta)\sum_{l=\kappa+1}^{\kappa+b'} u(y_l)}{b' - \delta \sum_{l=1}^{\kappa} p_l} + \delta p_j$. From (9), (19), and (20) we have $u(y_{\kappa+b'+1}) > \frac{\sum_{l=\kappa+1}^{\kappa+b'} u(y_l)}{b' - \delta \sum_{l=1}^{\kappa} p_l} \geq u(z_b(\mathbf{s})) = u(z_h^{ih})$, which implies $z_i^{ih} = 1 - z_h^{ih} > 1 - y_{\kappa+b'+1} > y_i \Rightarrow U_i(\mathbf{y}) < U_i(\mathbf{z}^{ih})$, which is the final contradiction. ■

We continue with Lemma 4 that is used in the proof of Lemma 1. The two Lemmas in combination yield the proof of Proposition 2.

Lemma 4 *Assume $u(x) = x$. Consider any player $i \in N$ and any allocation $\mathbf{s} \in \Delta_0^{\kappa-1}$. $\widehat{W}_i(\widehat{\pi}, \mathbf{s})$ defined in (13) is non-empty and upper-hemicontinuous as a correspondence of $\widehat{\pi}$. Furthermore, for all $\widehat{\pi} \in \times_{i \in N} \mathcal{P}[\Delta(i)]$, there exists $\mathbf{x} \in \widehat{W}_i(\widehat{\pi}, \mathbf{s})$ such that $x_i > 0$.*

Proof. The proof consists of four steps. First, we establish a lower bound on $\widehat{U}_i(\widehat{\pi}, \mathbf{s})$. Then, we use this bound to show that the sum of the demands of an appropriate set of players is less than unity. In steps 3 and 4 we use these results to prove the Lemma.

Claim 1: *For all $i \in N$,*

$$\widehat{U}_i(\widehat{\pi}, \mathbf{s}) > \frac{-\delta^2(1-\delta)p_i(1-p_i)}{\kappa} + \delta^2 p_i. \tag{21}$$

From (11) we have

$$\min\{U_i(\mathbf{x}) : \mathbf{x} \in \Delta(i)\} = \delta p_i. \tag{22}$$

Also from (11) we determine the minimum possible payoff that can be received by player i when other players propose, i.e.,

$$\min\left\{U_i(\mathbf{x}) : \mathbf{x} \in \Delta_\kappa^{n-1}\right\} = \delta(p_i(1-\delta)(1 - z_b(\mathbf{x})) + \delta p_i) = -\delta p_i(1-\delta) z_b(\mathbf{x}) + \delta p_i,$$

for some $\mathbf{x} \in\Delta_\kappa \setminus \Delta(i)$ that results in $z_b(\mathbf{x})$ (b satisfying (9) and (10)) that is as large as possible. By part (ii) of Lemma 2 we must have $z_b(\mathbf{x}) < \frac{1}{\kappa}$ so that

$$\min\left\{U_i(\mathbf{x}) : \mathbf{x} \in\Delta_\kappa^{n-1}\right\} > -\delta p_i(1-\delta)\frac{1}{\kappa} + \delta p_i. \tag{23}$$

We can now obtain the desired bound on $\widehat{U}_i(\widehat{\pi}, \mathbf{s})$ by combining (22) and (23). In particular, since i is the proposer with probability p_i we have from (12):

$$\widehat{U}_i(\widehat{\pi}, \mathbf{s}) > (1-\delta)s_i + \delta\left(p_i(\delta p_i) + (1-p_i)\left(-\delta p_i(1-\delta)\frac{1}{\kappa} + \delta p_i\right)\right).$$

But the right hand side is larger or equal to $\delta\left(\frac{-\delta p_i(1-\delta)(1-p_i)}{\kappa} + \delta p_i\right)$. Hence we have

$$\widehat{U}_i(\widehat{\pi}, \mathbf{s}) > \frac{-\delta^2(1-\delta)p_i(1-p_i)}{\kappa} + \delta^2 p_i, \text{ all } i \in N,$$

as we wished to show.

For the next step we define $\widehat{d}_j = (1-\delta)^{-1}\max\left\{0, \widehat{U}_i(\widehat{\pi}, \mathbf{s}) - \delta p_i\right\}$. We show:

Claim 2: *Assume without loss of generality that* $\widehat{d}_{h+1} \geq \widehat{d}_h$, $h = 1, ..., n-1$. *Then*

$$\sum_{h=2}^{\kappa+1} \widehat{d}_h < 1.$$

Let $l = \min\left\{i \in N : \widehat{d}_i > 0\right\}$. Obviously, if $l > \kappa + 1$ then $\sum_{h=2}^{\kappa+1} \widehat{d}_h = 0$, so we only need consider cases with $l \leq \kappa + 1$. Note that $\sum_{h=l}^{n} \widehat{d}_h = (1-\delta)^{-1}\sum_{h=l}^{n}\left(\widehat{U}_h(\widehat{\pi}, \mathbf{s}) - \delta p_i\right)$ and that $(1-\delta)^{-1}\sum_{h=1}^{n}\left(\widehat{U}_h(\widehat{\pi}, \mathbf{s}) - \delta p_i\right) = 1$. Thus, $\sum_{h=2}^{\kappa+1} \widehat{d}_h < 1$ follows trivially if $l = 1$. It remains to consider the case $\kappa + 1 \geq l > 1$, whence

$$\sum_{h=l}^{n} \widehat{d}_h = 1 - (1-\delta)^{-1}\sum_{h=1}^{l-1}\left(\widehat{U}_h(\widehat{\pi}, \mathbf{s}) - \delta p_h\right).$$

We now invoke (21) to deduce that

$$\sum_{h=l}^{n} \widehat{d}_h < 1 + \frac{\delta^2}{\kappa}\sum_{h=1}^{l-1} p_h(1-p_h) + \delta\sum_{h=1}^{l-1} p_h.$$

We also have $\frac{\sum_{h=l}^{\kappa+1}\widehat{d}_h}{\kappa+2-l} \leq \frac{\sum_{h=l}^{n}\widehat{d}_h}{2\kappa+2-l}$, because $\widehat{d}_{h+1} \geq \widehat{d}_h$. Combining the two inequalities we obtain:

$$\sum_{h=l}^{\kappa+1}\widehat{d}_h < \frac{\kappa+2-l}{2\kappa+2-l}\left(1+\frac{\delta^2}{\kappa}\sum_{h=1}^{l-1}p_h\left(1-p_h\right)+\delta\sum_{h=1}^{l-1}p_h\right), l=2,...,\kappa+1.$$

Further note that $\sum_{h=1}^{l-1}p_h\left(1-p_h\right) \leq (l-1)(\frac{1}{l-1})(1-\frac{1}{l-1}) = \frac{l-2}{l-1}$. Thus, since $\delta < 1$, the following inequality holds:

$$\sum_{h=l}^{\kappa+1}\widehat{d}_h < \frac{\kappa+2-l}{2\kappa+2-l}\left(2+\frac{1}{\kappa}\left(\frac{l-2}{l-1}\right)\right), l=2,...,\kappa+1.$$

The right hand side of this inequality is equal to 1 when $l=2$, and decreases with l. Since $\sum_{h=l}^{\kappa+1}\widehat{d}_h = \sum_{h=2}^{\kappa+1}\widehat{d}_h$ (because $\widehat{d}_h = 0, h < l$, and $l > 1$) we have shown that $\sum_{h=2}^{\kappa+1}\widehat{d}_h < 1$, as desired.

Claim 3: $\widehat{W}_i\left(\widehat{\pi},\mathbf{s}\right)$ *is non-empty and contains* $\mathbf{x} \in\Delta\left(i\right)$ *such that* $x_i > 0$. By Claim 2, (still assuming $\widehat{d}_{h+1} \geq \widehat{d}_h$) we can construct proposal $\mathbf{x} \in\Delta\left(i\right)$ with $x_h = \widehat{d}_h$, $h \in \{1,...,\kappa+1\} \setminus \{i\}$ if $i \in \{1,...,\kappa\}$, or $x_h = \widehat{d}_h$, $h \in \{1,...,\kappa\}$ if $i \in \{\kappa+1,...,n\}$. Since $\sum_{h=2}^{\kappa+1}\widehat{d}_h < 1$, this is possible, and i can retain $x_i > 0$. By (11) and the definition of $\widehat{d}_h$ we easily infer that $U_h(\mathbf{x}) \geq \widehat{U}_h\left(\widehat{\pi},\mathbf{s}\right)$ so that $\mathbf{x} \in \widehat{W}_i\left(\widehat{\pi},\mathbf{s}\right)$.

We complete the proof with a last step.

Claim 4: $\widehat{W}_i\left(\widehat{\pi},\mathbf{s}\right)$ *is upper-hemicontinuous as a correspondence of* $\widehat{\pi}$. To establish upper-hemicontinuity, notice that $U_i\left(\mathbf{x}\right)$, $\widehat{U}_i\left(\widehat{\pi},\mathbf{s}\right)$ are continuous in $\mathbf{x}$, $\widehat{\pi}$ respectively, thus $\widehat{A}_i\left(\widehat{\pi},\mathbf{s}\right)$ has closed graph for all $i \in N$. By extension, $\widehat{W}_i\left(\widehat{\pi},\mathbf{s}\right)$ has closed graph, since finite unions and intersections of closed sets are closed. Thus, since it also has compact Hausdorff range, $\widehat{W}_i\left(\widehat{\pi},\mathbf{s}\right)$ is upper-hemicontinuous, by the Closed Graph Theorem (Aliprantis and Border (1999), 16.12, p. 529). ■

Armed with Lemma 4, we prove Lemma 1.

Proof of Lemma 1. The proof is by application of Glicksberg's (1952) fixed point theorem. To ensure the conditions of the theorem are met, it suffices to show that $M_i(\widehat{\pi},\mathbf{s})$ is a non-empty, upper-hemicontinous correspondence with respect to $\widehat{\pi}$. If this is true, then, since $B_i(\widehat{\pi},\mathbf{s}) = \mathcal{P}[M_i(\widehat{\pi},\mathbf{s})]$, B_i and B are non-empty, upper-hemicontinuous, and convex valued by theorem 16.14 of Aliprantis and Border (1999), page 530, so that a fixed point exists.

Since $\widehat{W}_i(\widehat{\pi}, \mathbf{s})$ is non-empty and compact, $M_i(\widehat{\pi}, \mathbf{s})$ is non-empty. Thus, it remains to show that $M_i(\widehat{\pi}, \mathbf{s})$ is upper-hemicontinuous. Because $\widehat{W}_i(\widehat{\pi}, \mathbf{s})$ need not be lower-hemicontinuous,[11] we prove upper-hemicontinuity directly instead of following the typical line of proof that invokes the Theorem of the Maximum. In particular, since M_i has compact Hausdorff range it suffices for our purposes to show that $M_i(\widehat{\pi}, \mathbf{s})$ has closed graph (by the Closed Graph Theorem, Aliprantis and Border (1999), 16.12, p. 529). Suppose $M_i(\widehat{\pi}, \mathbf{s})$ does not have closed graph to get a contradiction. Then there exists a sequence

$$(\widehat{\pi}^k, \mathbf{x}^k) \in GrM_i = \{(\widehat{\pi}, \mathbf{x}) \in \times_{i \in N}\mathcal{P}[\Delta(i)] \times \Delta_\kappa^{n-1} : \mathbf{x} \in M_i(\widehat{\pi}, \mathbf{s})\},$$

such that $(\widehat{\pi}^k, \mathbf{x}^k) \to (\widehat{\pi}, \mathbf{x}) \notin GrM_i$. By Lemma 4, $\mathbf{x} \in \widehat{W}_i(\widehat{\pi}, \mathbf{s})$, *i.e.*, $\mathbf{x}$ is feasible. Thus, since $(\widehat{\pi}, \mathbf{x}) \notin GrM_i$, there exists $\mathbf{y} \in \arg\max\{U_i(\mathbf{z}) \mid \mathbf{z} \in \widehat{W}_i(\widehat{\pi}, \mathbf{s})\}$ such that $U_i(\mathbf{y}) > U_i(\mathbf{x})$. Note that by Lemma 4 we must have $y_i > 0$. Otherwise, Lemma 4 guarantees the existence of $\mathbf{z} \in \widehat{W}_i(\widehat{\pi}, \mathbf{s})$ such that $z_i > 0$, hence $U_i(\mathbf{z}) > \delta p_i \geq U_i(\mathbf{y})$ if $y_i = 0$, a contradiction. Thus, $y_i > 0$. Then by the continuity of $\widehat{U}_h, U_h$, all $h \in N$, and by Lemma 3, there is appropriate $\mathbf{y}' \in \widehat{W}_i(\widehat{\pi}^k, \mathbf{s})$ such that $\widehat{U}_i(\mathbf{y}') > \widehat{U}_i(\mathbf{x}^k)$, for large enough k. But this contradicts $(\widehat{\pi}^k, \mathbf{x}^k) \in GrM_i$. Thus, we have arrived at a contradiction that emanates from the working hypothesis that GrM_i is not closed. Hence, M_i is upper-hemicontinuous and the proof of the Lemma is complete. ■

We are now ready to prove Proposition 2.

Proof of Proposition 2. Consider a selector $\widehat{\pi}^* : \Delta_0^{\kappa-1} \to \times_{i \in N}\mathcal{P}[\Delta(i)]$ from B^*. The restriction of $\widehat{\pi}^*$ to $\widehat{\pi}_i^* : \Delta_0^{\kappa-1} \to \mathcal{P}[\Delta(i)]$, determines randomizations over proposals for player i and status quo $\mathbf{s} \in \Delta_0^{\kappa-1}$. Thus, by combining the above selector with the proposal strategies from Proposition 1, we obtain proposal strategies $\pi_i^* : \Delta \to \mathcal{P}[\Delta]$ for each $i \in N$. From these proposal strategies we calculate expected payoffs $U_i^*(\mathbf{x})$, $\mathbf{x} \in \Delta$, in accordance with equations (4) and (5) for each $i \in N$. In particular $U_i^*(\mathbf{x})$ coincides with $U_i(\mathbf{x})$ defined in (11) for all $\mathbf{x} \in \Delta_\kappa^{n-1}$, so that we can trivially compute $v_i^*(\mathbf{x})$ and $U_i^*(\mathbf{x})$ for $\mathbf{x} \in \Delta_0^{\kappa-1}$. Using these expected payoffs U_i^* we obtain voting strategies A_i^*, $i \in N$, that satisfy condition (2). To show that these strategies form

[11] Failure of lower-hemicontinuity occurs at certain suboptimal but feasible proposals that allocate zero to the proposer. Because of the last part of Lemma 4, this does not influence the continuity properties of M_i.

an equilibrium it suffices to show the following:

$$\begin{aligned}&\textit{For all } \mathbf{x} \in \Delta_0^{\kappa-1}\textit{, and all coalitions } C \subset N \textit{ with } |C| = \kappa + 1,\\ &\textit{there exists } \mathbf{y} \in \Delta_\kappa^{n-1} \textit{ such that } U_j^*(\mathbf{y}) \geq U_j^*(\mathbf{x}) \textit{ for all } j \in C.\end{aligned} \tag{*}$$

Indeed, by construction, proposal strategies $\pi_i^* : \Delta \to \mathcal{P}[\Delta]$ are such that any proposer $i \in N$, optimizes over acceptable proposals in Δ_κ^{n-1}. In addition (*) ensures that for any proposal $\mathbf{x} \in \Delta_0^{\kappa-1}$ available to proposer i, there exists an acceptable proposal $\mathbf{y} \in \Delta_\kappa^{n-1}$ that is at least as good. Thus, $\pi_i^* : \Delta \to \mathcal{P}[\Delta]$ satisfy equilibrium condition (3), and we have an equilibrium as desired.

Thus, to prove the Proposition we need show (*) is true. Consider any $\mathbf{x} \in \Delta_0^{\kappa-1}$ and define $d_j^*(\mathbf{x}) = (1-\delta)^{-1} \max\{U_j^*(\mathbf{x}) - \delta p_j, 0\}$, $j \in N$. Without loss of generality, assume players are ranked so that $j > i \Rightarrow d_j(\mathbf{x}) \geq d_i(\mathbf{x})$. Now choose any majority coalition $C \subset N$ with $|C| = \kappa+1$, and let $h = \min\{i \in C\}$. To prove (*) we shall show that $\sum_{i \in C} d_i^*(\mathbf{x}) - \delta(\sum_{i \notin C} p_i) d_h^*(\mathbf{x}) \leq 1$. Then (*) follows since for $\mathbf{y} \in \Delta_\kappa$ with

$$y_j = \begin{cases} d_j^*(\mathbf{x}) & \text{if} \quad j \in C \setminus \{h\}, \\ 1 - \sum_{i \in C \setminus \{h\}} d_i^*(\mathbf{x}) & \text{if} \quad j = h, \\ 0 & \text{otherwise,} \end{cases}$$

we have $y_h = 1 - \sum_{i \in C \setminus \{h\}} d_i^*(\mathbf{x}) \geq (1 - \delta \sum_{i \notin C} p_i) d_h^*(\mathbf{x})$ and it is immediate from (11), (8), and the definition of d_j^* that $U_j^*(\mathbf{y}) \geq U_j^*(\mathbf{x})$ for all $j \in C$.

We thus need to prove that $\sum_{i \in C} d_i^*(\mathbf{x}) - \delta(\sum_{i \notin C} p_i) d_h^*(\mathbf{x}) \leq 1$. Let $l \in N$ be such that $l = \min\{i \in N : d_i(\mathbf{x}) > 0\}$. We start by constructing a lower bound on $U_i^*(\mathbf{x})$ for players with $d_i^*(\mathbf{x}) = 0$:

$$U_i^*(\mathbf{x}) \geq (1-\delta)x_i + \delta(p_i(1-\delta)(1-D) + \delta p_i),\ i = 1, ..., l-1, \tag{24}$$

where $D = \sum_{i=1}^{\kappa+1} d_i(\mathbf{x}) < 1$ by Claim 2 of Lemma 4. To see why (24) holds, first note that all players other than i propose alternatives in $\Delta_{\kappa+1}^{n-1}$. This is because Lemma 3 ensures that a proposer other than i who contemplates a proposal such that $\kappa + 1$ players receive a positive allocation, can profitably reduce the allocation of one player (possibly i) to zero still obtaining majority support. By implication, i receives zero in all proposals by other players, so that i obtains utility δp_i with probability $(1 - p_i)$. Also, i can secure utility of at least $(1-\delta)(1-D) + \delta p_i$ when proposing with probability p_i, simply by allocating $d_j(\mathbf{x})$ to all $j \in \{1, ..., \kappa+1\} \backslash \{i\}$.

We have $\sum_{i=1}^{n}(U_i^*(\mathbf{x}) - \delta p_i) = 1 - \delta$, thus, if $l = 1$ we must have $\sum_{i \in C} d_i^*(\mathbf{x}) \leq 1$. Hence, to show $\sum_{i \in C} d_i^*(\mathbf{x}) - \delta \sum_{i \notin C} p_i d_h^*(\mathbf{x}) < 1$ (and (*)) it remains to consider $l > 1$. By summing both sides of (24) for $i = 1, ..., l-1$ and rearranging terms we get

$$\sum_{i=1}^{l-1}(U_i^*(\mathbf{x}) - \delta p_i) \geq (1-\delta)\sum_{i=1}^{l-1} x_i - \delta(1-\delta)D\sum_{i=1}^{l-1} p_i. \tag{25}$$

Since $j > i \Rightarrow d_j(\mathbf{x}) \geq d_i(\mathbf{x})$, we also have for coalition C with $|C| = \kappa + 1$ that

$$d_h^*(\mathbf{x}) + \sum_{i \notin C} d_i^*(\mathbf{x}) \geq D = \sum_{i=1}^{\kappa+1} d_i(\mathbf{x}). \tag{26}$$

If $h = \min\{i \in C\} \geq l$, we have $\sum_{i=1}^{l-1} p_i \leq \sum_{i \notin C} p_i$, while if $h < l$, $d_h^*(\mathbf{x}) = 0$. In either case we deduce from (26) that

$$\sum_{i \notin C} p_i d_h^*(\mathbf{x}) + \sum_{i \notin C} d_i^*(\mathbf{x}) \geq \sum_{i=1}^{l-1} p_i D.$$

Since $0 < \delta < 1$, and $\sum_{i=1}^{l-1} x_i \geq 0$ the above implies

$$\sum_{i=1}^{l-1} x_i + \delta \sum_{i \notin C} p_i d_h^*(\mathbf{x}) + \sum_{i \notin C} d_i^*(\mathbf{x}) \geq \delta D \sum_{i=1}^{l-1} p_i.$$

Adding $\sum_{i \in C} d_i^*(\mathbf{x})$ on both sides and re-arranging terms this is equivalent to

$$\sum_{i=1}^{l-1} x_i - \delta D \sum_{i=1}^{l-1} p_i + \sum_{i=1}^{n} d_i^*(\mathbf{x}) \geq \sum_{i \in C} d_i^*(\mathbf{x}) - \delta \sum_{i \notin C} p_i d_h^*(\mathbf{x}).$$

Since we have that $\sum_{i=1}^{l-1} d_i^*(\mathbf{x}) = 0$, and from (25), we deduce

$$(1-\delta)^{-1}\sum_{i=1}^{l-1}(U_i^*(\mathbf{x}) - \delta p_i) + \sum_{i=l}^{n} d_i^*(\mathbf{x}) \geq \sum_{i \in C} d_i^*(\mathbf{x}) - \delta \sum_{i \notin C} p_i d_h^*(\mathbf{x}).$$

But the left-hand side is equal to 1, by the fact that $d_i^*(\mathbf{x}) = (1-\delta)^{-1}(U_i^*(\mathbf{x}) - \delta p_i)$ for $i = l, ..., n$, and since $(1-\delta)^{-1}\sum_{i=1}^{n}(U_i^*(\mathbf{x}) - \delta p_i) = 1$. Thus, $\sum_{i \in C} d_i^*(\mathbf{x}) - \delta(\sum_{i \notin C} p_i) d_h^*(\mathbf{x}) \leq 1$ and the proof of (*) and of the Proposition is complete. ∎

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