Strategy-Proofness and Single-Crossing

Alejandro Saporiti

Working Paper No. 55
March 2008

W. ALLEN
WALLIS
Institute of
POLITICAL
ECONOMY

UNIVERSITY OF
ROCHESTER

Strategy-Proofness and Single-Crossing*

Alejandro Saporiti†

March 2008‡

Abstract

This paper analyzes strategy-proof collective choice rules when individuals have single-crossing preferences on a finite and ordered set of social alternatives. It shows that a social choice rule is anonymous, unanimous and strategy-proof on a maximal single-crossing domain if and only if it is an extended median rule with $n-1$ fixed ballots located at the end points of the set of alternatives. As a by-product, the paper also proves that strategy-proofness implies the tops-only property. And it offers a strategic foundation for the so called "single-crossing version" of the Median Voter Theorem, by showing that the median ideal point can be implemented in dominant strategies by a direct mechanism in which every individual reveals his true preferences.

JEL Codes: C72, D71, D78.

Key words: Single-crossing; strategy-proofness; tops-only; positional dictatorships.

1 Introduction

In social choice theory, a collective decision making process is usually represented by a social choice rule. A social choice rule associates a unique alternative from the set of feasible alternatives to every possible list of preferences of the individuals in the society. A social choice

*I am grateful to two anonymous referees and the Associate Editor for their very helpful comments and corrections on an earlier draft of this article. I would also like to thank Nick Baigent, John Duggan, Christian List, Antoine Loeper, Paola Manzini, Jordi Massó, Marco Mariotti, Bernardo Moreno, Hervé Moulin, Alejandro Neme, Fernando Tohmé, Rodrigo Velez, and seminar audiences at LSE, Manchester, Osnabrück, Queen Mary, Rochester, Warwick, and the 2005 World Congress of the Econometric Society for useful suggestions and discussion. Any remaining errors are my own responsibility.

†Economics, University of Manchester, Arthur Lewis Building, Oxford Road, Manchester M13 9PL, United Kingdom. E-mail: alejandro.saporiti@manchester.ac.uk.

‡First Draft: January 2005.

rule is said strategy-proof if no individual can ever benefit from misrepresenting his true preferences. A fundamental result in Social Choice, known as the Gibbard [18]-Satterthwaite [33] Theorem, shows that, if the set of alternatives contains at least three possible outcomes and individual preferences are not restricted in any particular way, then every strategy-proof social choice rule is dictatorial. That is, there is an individual whose preferences always dictate the final choice regardless of other individuals' preferences.

The Gibbard-Satterthwaite Theorem applies whenever every complete and transitive preference relation constitutes an admissible individual preference. In many economic and political applications, however, preferences satisfy additional properties. A case in point is the single-peaked property. A set of preference relations is single-peaked if there is a linear order of the alternatives such that every preference relation has a unique most preferred alternative (or ideal point) over this ordering, and the preference for any other alternative monotonically decreases by moving away from the ideal point. Single-peaked preferences naturally arise in economics by maximizing a strictly quasi-concave utility function on a linear budget set. They were first proposed by Black [6] to assure the existence of a Condorcet winner, (i.e., an alternative that beats every other alternative in a sequence of pair-wise majority contests). And they represent a simple example of a restricted preference domain where the conclusion of the Gibbard-Satterthwaite Theorem does not apply.

To be more specific, consider the family of efficient extended median rules, which are social choice rules that associate to each preference profile the median alternative from a list consisting of the n ideal points of the individuals and $n-1$ other alternatives from the feasible set of alternatives. An important case within this family is the well-known median choice rule, which assigns the median ideal point to every profile of individual preferences. These rules are obviously non-dictatorial. In fact, they are anonymous, because the names of the individuals play no role in taking social choices. They are also unanimous, in the sense that they respect any unanimous consensus in the society about the most preferred alternative. Furthermore, if individual preferences are single-peaked, then Moulin [25] has shown that every member of this family is strategy-proof. Conversely, every anonymous, unanimous and strategy-proof social choice rule on single-peaked preferences is an efficient extended median rule.

Although single-peakedness is an intuitive domain condition, there are interesting problems in political economy and public economics, such as majority voting over distortionary tax rates, where individual preferences do not exhibit the single-peaked property. In some of these cases, however, preferences do satisfy an alternative restriction called the single-crossing property. This property appears for example in models of income taxation and redistribution (Roberts [29], Meltzer and Richard [23]), local public goods and stratification (Westhoof [34], Epple et al. [13], Epple and Platt [14], Epple et al. [15], Calabrese et al.

[8]), coalition formation (Demange [11], Kung [20]) and, more recently, in models to study the selection of policies in the market for higher education (Epple et al. [16]), the citizen candidate under uncertainty (Eguia [12]) and the choice of constitutional and voting rules (Barberà and Jackson [4]).

Unlike single-peakedness, the single-crossing property does not impose *a priori* any restriction on the shape of each individual preference relation. So, for example, it does not exclude preferences which do not monotonically decrease on both sides of the ideal point. That is the reason why it accommodates non-convexities that arise in some applications of majority voting. If preferences are strict orderings, what the single-crossing property requires is the existence of a linear order over the set of individual preferences with the property that, for every pair of alternatives x and y, whenever two preference relations P' and P'' coincide in ranking x above y, so do all preferences *in between*, so that the subset of preferences ranking one alternative above the other all lie to one side of those who have the inverse ranking.[1] Of course, if indifference between alternatives is permitted, then the set of preference relations for which x is indifferent to y must be located between the subsets with a strict ordering of these two alternatives.

As we show in Section 2.4, in several models, notably in models of redistribution financed by income taxation, the single-crossing property is implied by more fundamental assumptions about preferences and technologies. For instance, it holds when individuals' heterogeneity is generated by a one-dimensional parameter θ, (which be interpreted as income, productivity, elasticity of substitution, discount factor, etc.), and the utility over social alternatives exhibits increasing differences in θ (Milgron and Shannon [24]). In addition, under differentiability and some mild conditions on indifferent curves, the single-crossing property is also equivalent to the more familiar Spence-Mirrlees condition of incentive theory and information economics, which requires that the marginal rate of substitution be increasing in θ (Milgron and Shannon [24]).

The single-crossing property has in many cases a substantive interpretation. A working example is the collective choice of an income tax rate. Suppose a moderately rich individual prefers a high tax rate to another relatively smaller tax rate, so that he reveals a preference for a greater redistribution of income. Then, the single-crossing property requires that a relatively poorer individual, who receives a higher benefit from redistribution, also prefers the higher tax rate. Sometimes this is interpreted in the literature by saying that there is a complementary between income and taxation, in the sense that lower incomes increase the

[1]When preferences are strict, it is also possible and convenient to derive a linear order over the set of alternatives from the order of the preference relations, by defining alternative x "smaller than" alternative y if and only if the preference relations for which x is preferred to y lie on the left of those relations who rank y above x (Saporiti and Tohmé [32]).

incremental benefit of greater tax rates. For another example, consider a strong army which prefers a large territorial concession and a small probability of war to a small concession and a high probability of war. Then, under single-crossing, a weaker army, with a lower expected payoff from war, should also prefer the large concession.

Like the single-peaked property, single-crossing also guarantees the existence of a Condorcet winner and provides a simple characterization of it. The Condorcet winner is the ideal point of the median agent, where the latter is the individual whose preference takes up the median position over the ordering of individual preferences for which the single-crossing property is satisfied.[2] This result appeared first in the seminal works of Roberts [29] and Grandmont [19] and, more recently, in Rothstein [31], Gans and Smart [17] and Austen-Smith and Banks [1]. It is referred to by Myerson [27] as the "single-crossing version" of the Median Voter Theorem (MVT). Alternatively, due to the existence of a median individual who is decisive for every nonempty subset of alternatives, it is termed in Rothstein [31] the *Representative Voter Theorem* (RVT).

The problem with Representative Voter Theorem is that, unlike the MVT over single-peaked preferences, whose non-cooperative foundation was provided by Moulin [25], the RVT is based on the assumption that individuals honestly reveal their preferences. A natural question is therefore how legitimate the Representative Voter Theorem is when preferences are private information and individuals can report them insincerely. This question has been recently addressed by Saporiti and Tohmé [32]. They showed that the single-crossing property is sufficient to ensure the existence of social choice rules which are immune to any individual and group misrepresentation of individual preferences. In particular, this is true for the median choice rule.

Building on Saporiti and Tohmé [32], this paper characterizes the family of anonymous, unanimous and strategy-proof social choice rules on a maximal single-crossing domain.[3] This family coincides with the class of *positional dictatorships*, which are extended median rules with $n-1$ fixed ballots located at the end points of the set of feasible alternatives. They include the median choice rule as a particular case.

Although the term "dictatorship" may initially provoke a negative impression about our characterization, it is worth noting that the result is far from a negative one. A positional dictatorship is a social choice rule which only considers the most preferred alternatives announced by the individuals, and always chooses one at a specified rank; e.g., the first ideal

[2]Instead, under single-peakedness, the Condorcet winner is given by the median ideal point over the ordering of the alternatives for which the single-peaked property holds.

[3]A set of preference relations with the single-crossing property is maximal if there does not exist another set of preferences that contains the former set and satisfies the single-crossing property (see Definition 2 in Section 2.3).

point, the second, the median, etc. The preselected *position* is a "dictator". However, in different profiles the ideal points of different individuals can be located at that position. Therefore, there is no a dictator as it is defined in social choice theory.

In our model, positional dictatorships refer to the simple majority rule and other supra-majorities. Hence, the main message coming out from the analysis is that the single-crossing property is another simple domain restriction where majority voting works with "maximal" incentive properties. The article explains the root of this good property of single-crossing domains, and how far we can go in changing the majority rule.

The rest of the paper is organized as follows. Section 2 presents the model, the notation and the definitions. It also offers two applications which provide intuitions about our abstract setup. Section 3 contains the main results of the paper, included the characterization of positional dictatorships and the relationship between strategy-proofness and the tops-only property. As it happens in other cases, in our model every strategy-proof social choice rule ignores all information about preferences except individuals' most preferred alternatives. The proof of this property constitutes a major step in establishing our characterization, and we devote a considerable space to develop the formal argument that proves this result. For expositional convenience, this is done in Appendix A. For the same reason, we relegate the proof of the characterization of positional dictatorships to Appendix B. Section 4 analyzes the robustness of our results to preference reports outside the single-crossing domain. Final remarks appear in Section 5.

2 The model, notation and definitions

2.1 Individuals

Let $N = \{1, \ldots, n\}$ be a finite set of individuals. Except where otherwise noted, $n \geq 2$.

2.2 Alternatives

Let $X = \{x, y, z, \ldots\}$ be a finite set of alternatives, with $|X| > 2$.[4]

2.3 Preferences

Let $\mathcal{P}$ be the set of all complete, transitive and antisymmetric binary relations on X. A preference ordering over the elements of X is represented by an element P of $\mathcal{P}$, with the usual interpretation that for any pair $x, y \in X$, "$x\,P\,y$" denotes a strict preference for x

[4]For every set A, $|A|$ stands for the cardinality of the set, and $\bar{A}$ for the complement of A.

against y. Sometimes we write $P = (x\, y\, z\, \ldots)$ to indicate that $x\,P\,y$, $y\,P\,z$, etc. For any $P \in \mathcal{P}$, and any $Y \subseteq X$, let $\tau|_Y(P) = \arg\max_Y (P)$. For simplicity, we denote $\tau(P) = \tau|_X(P)$.

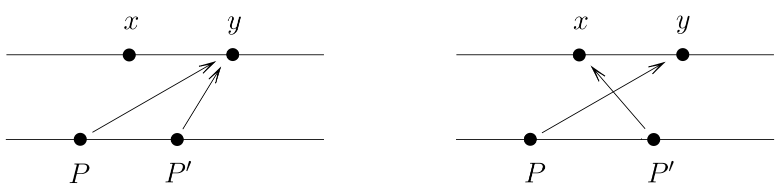
Definition 1 (SC) *A set of preferences $\mathcal{SC} \subset \mathcal{P}$ exhibits the* ***single-crossing property*** *on X if there is a linear order $>$ of X and a linear order $\succ$ of $\mathcal{SC}$ such that $\forall x, y \in X$ and $\forall P, P' \in \mathcal{SC}$,*

$$[y > x,\ P' \succ P \ \&\ y\,P\,x] \ \Rightarrow\ y\,P'\,x, \qquad SC1$$

and

$$[y > x,\ P' \succ P \ \&\ x\,P'\,y] \ \Rightarrow\ x\,P\,y.^5 \qquad SC2$$

To help the reader gain more insight about this property, Figure 1.a offers a graphical illustration of condition $SC1$. On the other hand, Figure 1.b exhibits a case where neither $SC1$ nor $SC2$ are satisfied. In both graphs, arrows denote "preference direction", so that for example an arrow from P to y in the presence of x stands for "$y\,P\,x$".



(a) Condition $SC1$ (b) Violation of $SC1$ & $SC2$

Figure 1: Illustration of Definition 1

In words, a set of preference relations $\mathcal{SC}$ on the set of alternatives X exhibits the single-crossing property (or, for conciseness, $\mathcal{SC}$ is single-crossing) if there is a linear order $>$ of X and a linear order $\succ$ of $\mathcal{SC}$ such that whenever any preference relation $P \in \mathcal{SC}$ ranks any alternative y above (respectively, below) any other alternative x and $y > x$, then so does every other preference relation $P' \in \mathcal{SC}$ for which $P' \succ P$ (respectively, $P' \prec P$).

As we will see in Section 2.4, in the applications where this domain restriction is used, the structure of the models induces a natural order of X and of $\mathcal{SC}$, and unequivocally determines a unique set of single-crossing preferences. For example, in the paper by Barberà and Jackson [4], $X = \{1, \ldots, n\}$ is a set of voting rules, each of them represented by the number of individuals needed to approve a proposal b against the status quo a. The elements

[5] For any $x, y \in X$, we write (1) $x = y$ if and only if $x > y$ and $y > x$; and (2) $x \geq y$ if and only if either $x = y$ or $x > y$. On the other hand, for any two distinct preferences $P, P' \in \mathcal{SC}$, we say that $P \prec P'$ if and only if $\neg[P \succ P']$.

of X are ordered according with the usual order of the natural numbers. On the other hand, the set of preferences with the single-crossing property is $\mathcal{SC} = \{P(\alpha) \in \mathcal{P} : \alpha \in (0,1)\}$, where α denotes the probability that an individual prefers b to a at the time of voting between these alternatives, and for any two preference relations $P(\alpha'), P(\alpha'') \in \mathcal{SC}$, $P(\alpha') \succ P(\alpha'')$ if and only if $\alpha'' > \alpha'$. Thus, the order $\succ$ over $\mathcal{SC}$ is induced by the natural order of the probabilities on the interval $(0,1)$. A more detailed discussion about this model and how the set of preferences $\mathcal{SC}$ is derived is postponed until Section 2.4.2.

The single-crossing property is closely related to other preference restrictions, such as *hierarchical adherence* (Roberts [29]), *intermediateness* (Grandmont [19]), *order-restriction* (Rothstein [30] and [31]), and *unidimensional alignment* (List [21]).[6] In all these preference domains the salient feature is the existence of a linear order of the preference relations with the property that, for each pair of alternatives x and y, the relation x preferred to y (or the reverse) partitions the line over which the preferences are ordered in two disjoint intervals. If indifference between alternatives is permitted, then three of such intervals arise.

When individuals only differ in their preferences, these domain restrictions can also be defined with respect to an ordering of the agents, instead of the preference relations (see, for example, Rothstein [30] and [31], Gans and Smart [17] and Persson and Tabellini [28]). That is, the existence of a linear order over the preference relations with the property described above implies that "we can order individuals in such a way that for any pair of alternatives x and y, the first $j(xy) \geq 0$ individuals in the ordering strictly prefer x to y (respectively, y to x), the final $k(xy) \geq 0$ individuals in the ordering strictly prefer y to x (respectively, x to y), and the middle group of individuals, if any, are indifferent between the two", (Austen-Smith and Banks [1], p. 107).[7]

Scenarios where such strict ordering of individuals exists are quite common in political economy. "For example, in redistributive politics policy makers are concerned with reallocating resources from rich to poor people, subject to the constraint (typically) that such redistributions do not reverse the rank-order of individuals' wealth. So, while there does not exist an obvious ordering of the alternative distributions of wealth, there does exist a natural ordering of individuals and their preferences in terms of individual wealth", (Austen-Smith and Banks [1], p. 107).

From a technical perspective, the importance of single-crossing in political economy and public economics is due to the fact that, like single-peakedness, this domain restriction is sufficient to guarantee the existence of a Condorcet winner, especially in cases where the

[6] See also Barberà and Moreno [5], who recently proposed a weaker condition, called *top-monotonicity*, which encompasses single-crossing, order-restriction and single-peakedness.

[7] As the notation indicates, the "cut-offs" agents $j(\cdot)$ and $k(\cdot)$ can depend on the pair of alternatives under consideration.

single-peaked property does not hold.[8] However, apart from this, it is worth noting that both conditions are totally independent, in the sense that neither property is logically implied by the other. Examples 1 and 2 below illustrate this point.

Example 1 Consider the set of preference relations $\{P^1, P^2, P^3\}$ of Table 1. Recall that, for example, $P^1 = (x\,y\,z)$ stands for $x\,P^1\,y\,P^1\,z$. Simple inspection shows that this set has the single-crossing property on $X = \{x, y, z\}$ with respect to $z > y > x$ and $P^3 \succ P^2 \succ P^1$. On the other hand, for every ordering of the alternatives, $\{P^1, P^2, P^3\}$ violates the single-peaked property, because every alternative is ranked bottom in one preference relation.

Example 2 Consider the set of preferences displayed in Table 2. This set has the single-peaked property on X with respect to $z > y > x > w$. However, $\{P^1, P^2, P^3\}$ violates Definition 1, because for every ordering of the binary relations and for every ordering of the alternatives, there exist a pair of preference relations in $\{P^1, P^2, P^3\}$ and a pair of alternatives in X such that $SC1$ and $SC2$ are both contradicted. (For example, if $z > y > x > w$, then $P^1 \succ P^3$ contradicts $SC1$ and $SC2$ for the pair $\{x, y\}$, while $P^3 \succ P^1$ does so for $\{z, w\}$.)

Table 1: Single-crossing

$P^1 = (x\,y\,z)$
$P^2 = (x\,z\,y)$
$P^3 = (z\,y\,x)$

Table 2: Single-peakedness

$P^1 = (x\,y\,z\,w)$
$P^2 = (z\,y\,x\,w)$
$P^3 = (y\,x\,w\,z)$

Since the main purpose of this article is to characterize the family of strategy-proof social choice rules on single-crossing domains, in what follow we restrict the analysis to the *largest* or *maximal* sets of preference relations with the single-crossing property. These sets contain the largest number of possible deviations. Therefore, they are the appropriate framework to study incentive compatibility.

Definition 2 *A set of preferences $\mathcal{SC}$ with the single-crossing property on X is* ***maximal*** *if there does not exist $\mathcal{SC}' \subset \mathcal{P}$ such that $\mathcal{SC} \subset \mathcal{SC}'$ and $\mathcal{SC}'$ exhibits the single-crossing property on X.*

Example 3 To illustrate Definition 2, consider again Example 1. Notice that the set of preference relations $\{P^1, P^2, P^3\}$ is not the largest set that satisfies Definition 1 on $X =$

[8] A preference relation $P \in \mathcal{P}$ is ***single-peaked*** on X if there is a linear order $>$ of X and an alternative $\tau(P) \in X$ such that $\forall x, y \in X$, (i) $\tau(P) > y > x \;\Rightarrow\; \tau(P)\,P\,y\,P\,x$, and (ii) $x > y > \tau(P) \;\Rightarrow\; \tau(P)\,P\,y\,P\,x$. A set of preference relations $\mathcal{S} \subset \mathcal{P}$ exhibits the ***single-peaked property*** on X if there is a linear order $>$ of X such that every $P \in \mathcal{S}$ is single-peaked on X with respect to $>$.

$\{x, y, z\}$, because there exists a preference $P^4 = (z\,x\,y)$ such that $\{P^1, P^2, P^3, P^4\}$ is single-crossing with respect to $z > y > x$ and $P^3 \succ P^4 \succ P^2 \succ P^1$. On the other hand, $\{P^1, P^2, P^3, P^4\}$ is indeed maximal. However, it is not unique. If we consider the preference relations $P^5 = (y\,x\,z)$ and $P^6 = (y\,z\,x)$, then the set $\{P^1, P^5, P^6, P^3\}$ is also single-crossing with respect to $z > y > x$, for $P^3 \succ P^6 \succ P^5 \succ P^1$. Moreover, the union of $\{P^1, P^5, P^6, P^3\}$ and $\{P^1, P^2, P^3, P^4\}$ covers all preferences on X.

At this point, it may be useful to compare the size of the set of all single-peaked preferences and the size of the maximal sets with the single-crossing property, for a given ordering of X.[9] For the former, it is well-known to be $2^{|X|-1}$. For single-crossing, the largest size is $|X| \cdot \frac{|X|-1}{2} + 1$, therefore much smaller. To see this, draw a line for each pair of distinct alternatives in X. Observe that, under single-crossing, for each pair $a, b \in X$, the relation a preferred to b (or the reverse), partitions the line associated with $\{a, b\}$ in two disjoint intervals: one interval where the preference relations for which a is preferred to b are ordered; and the other where the relations with the opposite ranking of a and b are ordered (see Figure 2 for the case where $X = \{x, y, z\}$). There are $|X| \cdot \frac{|X|-1}{2}$ such partitions. And the projection of these partitions into a line forms at most $|X| \cdot \frac{|X|-1}{2} + 1$ different subintervals. In each subinterval, the preference relation is entirely determined. Hence, the given number $|X| \cdot \frac{|X|-1}{2} + 1$ is an upper bound for the cardinality of the maximal sets of preferences with the single-crossing property.[10]

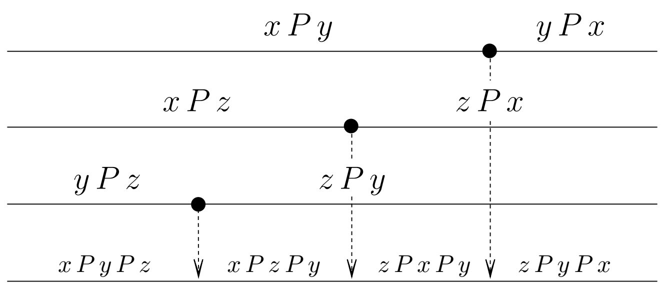


Figure 2: Maximal sets of single-crossing preferences

Fix now for the rest of the analysis a maximal set $\mathcal{SC} \subset \mathcal{P}$ of preference relations with the single-crossing property on X with respect to $>$ and $\succ$. Suppose each individual $i \in N$ is endowed with a preference $P_i \in \mathcal{SC}$. Let P_i be agent i's private information. Assume everybody knows the set $\mathcal{SC}$; everybody knows that every agent has preferences on X out of $\mathcal{SC}$; and so

[9] As we noted in Example 3, there may be several maximal sets of single-crossing preferences for a given ordering of X. Instead, the set of all single-peaked preferences is unique once alternatives are ordered.

[10] I am grateful to Professor Moulin who has made this observation in personal correspondence.

on. Denote by the n-fold Cartesian product $\mathcal{SC}^n$ the set of all single-crossing preference profiles. As usual, for any profile $\mathbf{P} = (P_1, \ldots, P_n) \in \mathcal{SC}^n$, let $\mathbf{P}_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$; for each $\hat{P}_i \in \mathcal{SC}$, denote $(\hat{P}_i, \mathbf{P}_{-i}) = (P_1, \ldots, P_{i-1}, \hat{P}_i, P_{i+1}, \ldots, P_n)$; and, for every set $S \subseteq N$, let $\mathbf{P}_S = (P_i)_{i \in S}$.

In the next section, we provide two applications which illustrate how our collective choice model with single-crossing preferences can naturally arise in political economy. These applications have been selected to show that the single-crossing property can be easily derived from more fundamental assumptions about preferences and technologies. Other examples of this domain restriction can be found in Ashworth and Bueno de Mesquita [3], Barberà and Moreno [5] and Persson and Tabellini [28], among others.

2.4 Applications

2.4.1 Income taxation

Consider a simplified version of the well-known model of redistribution financed by a linear income tax scheme, formulated by Roberts [29] and Meltzer and Richard [23]. In this version, individual i's preferences are represented by a utility function $U_i(c_i, l_i) = c_i + V(l_i)$, where c_i denotes private consumption, l_i leisure time and $V(\cdot)$ a continuous and concave function. Let $(1-t)h_i + T \geq c_i$ be individual i's budget constraint, where $t \in (0,1)$ is an income tax rate, h_i the individual labor supply, and $T = (\sum_{i \in N} t\, h_i)/n$ a lump-sum transfer. The real wage is exogenous and normalized at 1. Assume each individual i is endowed with a productivity $\theta_i \in \Theta \subseteq \mathbb{R}$, and let $1 - \theta_i \geq l_i + h_i$ be agent i's effective time constraint.

If we solve the maximization problem of each individual $i \in N$ of type θ_i for a given tax rate $t \in (0,1)$ and substitute the optimal consumption, $c_i^*(t, \theta_i)$, and the optimal leisure time, $l_i^*(t, \theta_i)$, into the utility function $U_i(c_i, l_i)$, then the indirect utility associated with t and θ_i is

$$\begin{aligned} W(t, \theta_i) &= U_i(c_i^*(t, \theta_i), l_i^*(t, \theta_i)) \\ &= H(t) + V[1 - H(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta}), \end{aligned}$$

where $H(t) = 1 - \bar{\theta} - V_l^{-1}(1-t)$ is the average labor supply, V_l the first derivative of $V(\cdot)$, and $\bar{\theta}$ the mean productivity. Note that, for any two policies t', $t'' \in (0,1)$, with $t' > t''$, the difference $W(t', \theta) - W(t'', \theta) = \{H(t') + V[1 - H(t') - \bar{\theta}] - (1-t')(\theta - \bar{\theta})\} - \{H(t'') + V[1 - H(t'') - \bar{\theta}] - (1-t'')(\theta - \bar{\theta})\}$ is strictly increasing in θ, because

$$\frac{\partial (W(t', \theta) - W(t'', \theta))}{\partial \theta} = t' - t'' > 0.$$

Now, fix a set with three different tax rates $X = \{t^1, t^2, t^3\}$, $t^1 > t^2 > t^3$, and consider the set of induced preferences over X, $\{P(\theta) \in \mathcal{P} : \theta \in \Theta\}$, where for all $x, y \in X$,

$x\,P(\theta)\,y \;\Leftrightarrow\; W(x,\theta) > W(y,\theta)$. Define a linear order $\succ$ over $\{P(\theta) \in \mathcal{P} : \theta \in \Theta\}$ in such a way that for all $\theta', \theta'' \in \Theta$, $P(\theta') \succ P(\theta'') \;\Leftrightarrow\; \theta' > \theta''$. We claim that the set $\{P(\theta) \in \mathcal{P} : \theta \in \Theta\}$ exhibits the single-crossing property with respect to $\succ$ and the order of X. On the contrary, suppose that there exist $x, y \in X$ and $\theta', \theta'' \in \Theta$, $\theta' \neq \theta''$, such that $y > x$, $P(\theta') \succ P(\theta'')$, $y\,P(\theta'')\,x$ and $x\,P(\theta')\,y$. Note that $y\,P(\theta'')\,x \;\Rightarrow\; W(y,\theta'') - W(x,\theta'') > 0$; and $P(\theta') \succ P(\theta'') \;\Rightarrow\; \theta' > \theta''$. Hence, since $W(y,\theta) - W(x,\theta)$ is strictly increasing in θ, we have that $W(y,\theta') - W(x,\theta') > 0$, contradicting that by hypothesis $x\,P(\theta')\,y$. Therefore, $\{P(\theta) \in \mathcal{P} : \theta \in \Theta\}$ is single-crossing on X.

2.4.2 Choosing how to choose

Consider next Barberà and Jackson's [4] model on self-stable constitutions, where individuals have induced preferences over different voting rules (constitutions). In this model, a society $N = \{1, \ldots, n\}$ chooses in period 1 the voting rule $s \in \{1, \ldots, n\}$, which is used in period 2 to make a social choice between alternatives a and b. Suppose b is chosen if at least s votes say b, and a is chosen otherwise. In period 1, individuals do not yet know their preferences over a and b. Each agent i is characterized by a probability $\alpha_i \in (0,1)$ that he will prefer b to a at the time of the vote (i.e., in period 2). Each individual receives a payoff of 1 if his preferred alternative is chosen, and 0 otherwise.

Given the likelihood of different patterns of support for a and b, agent i's expected utility $W(s, \alpha_i)$ at period 1 under voting rule s is as follows. For any $k \in \{0, \ldots, n-1\}$, let $B_i(k)$ denote the probability that exactly k of the individuals in $N \backslash \{i\}$ support b:

$$B_i(k) = \sum_{C \subset N\backslash\{i\} : |C| = k} \times_{j \in C}\ \alpha_j\ \times_{\ell \notin C}\ (1 - \alpha_\ell).$$

The indirect utility associated with each voting rule $s \in \{1, \ldots, n\}$ and each probability type $\alpha_i \in (0,1)$ is

$$W(s, \alpha_i) = \alpha_i \sum_{k=s-1}^{n-1} B_i(k) + (1 - \alpha_i) \sum_{k=0}^{s-1} B_i(k). \tag{1}$$

Lemma 2 in Barberà and Jackson [4] shows that, for every $s' > s''$, the difference $W(s', \alpha) - W(s'', \alpha)$ is decreasing in $\alpha \in (0,1)$, capturing in this way the intuition that the incremental benefit of a lower quota raises as the probability of preferring b to a increases.

Like in the previous application, let $\{P(\alpha) \in \mathcal{P} : \alpha \in (0,1)\}$ be the set of induced preferences over the voting rules $\{1, \ldots, n\}$, where for every $s', s'' \in \{1, \ldots, n\}$, $s'\,P(\alpha)\,s''$ if and only if $W(s', \alpha) > W(s'', \alpha)$. Define a linear order $\succ$ over $\{P(\alpha) \in \mathcal{P} : \alpha \in (0,1)\}$ with the property that, for every $\alpha', \alpha'' \in (0,1)$, $P(\alpha') \succ P(\alpha'')$ if and only if $\alpha'' > \alpha'$. Next we prove that the set of preference relations $\{P(\alpha) \in \mathcal{P} : \alpha \in (0,1)\}$ has the single-crossing property on the set of voting rules with respect to the linear order $\succ$ and the natural order of $\{1, \ldots, n\}$. On the contrary, suppose that there exist $x, y \in \{1, \ldots, n\}$ and $\alpha', \alpha'' \in (0,1)$, $\alpha' \neq \alpha''$, such that $y > x$, $P(\alpha') \succ P(\alpha'')$, $y\,P(\alpha'')\,x$ and $x\,P(\alpha')\,y$. Note that $y\,P(\alpha'')\,x \;\Rightarrow\; W(y,\alpha'') - W(x,\alpha'') > 0$; and $P(\alpha') \succ P(\alpha'') \;\Rightarrow\; \alpha'' > \alpha'$. Hence, since $W(y,\alpha) - W(x,\alpha)$ is decreasing in α, we have that $W(y,\alpha') - W(x,\alpha') > 0$, contradicting that by hypothesis $x\,P(\alpha')\,y$. Thus, $\{P(\alpha) \in \mathcal{P} : \alpha \in (0,1)\}$ is single-crossing on $\{1, \ldots, n\}$.

2.5 Aggregation process

The problem of the society described in Sections 2.1, 2.2 and 2.3 is to make a social choice from the set of alternatives X. Each individual is entitled to report a preference relation on X from the set of admissible preferences $\mathcal{SC}$, which is assumed to be commonly known. These reports are intended to provide information about the profile of true preferences, although agents' sincerity cannot be ensured.

A ***social choice rule*** is a single-valued mapping $f : \mathcal{SC}^n \to X$ that associates to each preference profile $\mathbf{P} \in \mathcal{SC}^n$ a unique outcome $f(\mathbf{P}) \in X$. Denote by $r_f = \{x \in X : \exists\, \mathbf{P} \in \mathcal{SC}^n$ such that $f(\mathbf{P}) = x\}$ the range of f. Given a social choice rule $f : \mathcal{SC}^n \to X$, a nonempty set $S \subseteq N$ and a profile $\mathbf{P}_{\bar{S}} \in \mathcal{SC}^{|\bar{S}|}$, let $O_S^f(\mathbf{P}_{\bar{S}}) = \{x \in X : \exists\, \mathbf{P}_S \in \mathcal{SC}^{|S|}$ such that $f(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) = x\}$ be the ***option set*** of S, given that the remaining individuals in $\bar{S} = N \backslash S$ have reported $\mathbf{P}_{\bar{S}}$. If $S = N$, it is assumed that $O_N^f(\cdot) = r_f$.

We are interested in social choice rules that satisfy the following properties on $\mathcal{SC}^n$. The main one is that individuals never have incentives to misrepresent their preferences.

Definition 3 (SP) *A social choice rule $f : \mathcal{SC}^n \to X$ is **strategy-proof** if $\forall\, i \in N$ and $\forall\,(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$, there is no $\hat{P}_i \in \mathcal{SC}$ such that $f(\hat{P}_i, \mathbf{P}_{-i})\,P_i\,f(P_i, \mathbf{P}_{-i})$.*

There are two interpretations of Definition 3. According with the first interpretation, a social choice rule f on $\mathcal{SC}^n$ is strategy-proof if for any individual $i \in N$, any possible preference $P_i \in \mathcal{SC}$ for i and *any* collection of preferences $\mathbf{P}_{-i} \in \mathcal{SC}^{n-1}$ that the other individuals could report, individual i is not better off, according to P_i, by reporting a preference $\hat{P}_i \in \mathcal{SC}$ different from P_i.

Alternatively, following Austen-Smith and Banks [2], p. 21., we could also say that a social choice rule f on $\mathcal{SC}^n$ is strategy-proof if for every $i \in N$, regardless of the true preferences $\mathbf{P}_{-i} \in \mathcal{SC}^{n-1}$ of all individuals other than i, agent i can do no better than report his true preferences $P_i \in \mathcal{SC}$, no matter which P_i individual i is endowed with. In this second view, a strategy-proof social choice rule provides no opportunities for any individual to profitably change the social outcome by misrepresenting the true preferences, given that all others report the truth.

Independently of the interpretation given to Definition 3, if a social choice rule f is not strategy-proof, then there must exist one agent, say $i \in N$, who can be strictly better off in at least one case, say at $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$, by announcing a preference $\hat{P}_i \in \mathcal{SC}$ different from his true ordering P_i. In that case, we say f is ***manipulable*** by $i \in N$ at $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ via $\hat{P}_i \in \mathcal{SC}$.

To study the possibility of group deviations, it is also possible to define the concept of *group strategy-proofness*, which can be obviously interpreted in a similar way than strategy-proofness, except for the fact that it is a coalition of individuals who deviate from the true preferences.

Definition 4 (GSP) *A social choice rule $f : \mathcal{SC}^n \to X$ is **group strategy-proof** if $\forall\, S \subseteq N$ and $\forall\, (\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathcal{SC}^n$, there is no $\hat{\mathbf{P}}_S \in \mathcal{SC}^{|S|}$ such that $\forall i \in S,\ f(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})\, P_i\, f(\mathbf{P}_S, \mathbf{P}_{\bar{S}})$.*

Another property that we may seek in a social choice rule is *unanimity*. This property ensures that, if all agents have the same most preferred alternative, then that alternative is socially selected.

Definition 5 (UN) *A social choice rule $f : \mathcal{SC}^n \to X$ is **unanimous** if $\forall\, x \in X$ and $\forall\, \mathbf{P} \in \mathcal{SC}^n$ such that $\tau(P_i) = x\ \forall i \in N,\ f(\mathbf{P}) = x$.*

A profile $\mathbf{P} \in \mathcal{SC}^n$ is a permutation of another profile $\hat{\mathbf{P}} \in \mathcal{SC}^n$ if there is a one-to-one function $\sigma : N \to N$ such that for every individual $i \in N$, P_i is identical to $\hat{P}_{\sigma(i)}$. That is, $\mathbf{P}$ is a permutation of $\hat{\mathbf{P}}$ if the lists of preferences under $\mathbf{P}$ and $\hat{\mathbf{P}}$ are identical up to a renaming of agents.

Definition 6 (AN) *A social choice rule $f : \mathcal{SC}^n \to X$ is **anonymous** if $f(\mathbf{P}) = f(\hat{\mathbf{P}})$ for every permutation $\mathbf{P}$ of $\hat{\mathbf{P}} \in \mathcal{SC}^n$.*

In words, a social choice rule is anonymous if the names of the individuals holding particular preferences are immaterial in deriving social choices.

One last property that a social choice rule may satisfy is the *tops-only* property. We say that f is tops-only if for any admissible preference profile, the social choice is exclusively determined by individuals' most preferred alternatives on the range of the social choice rule.

Definition 7 (TO) *A social choice rule $f : \mathcal{SC}^n \to X$ is **tops-only** if $\forall\, \mathbf{P}, \hat{\mathbf{P}} \in \mathcal{SC}^n$ such that $\tau|_{r_f}(P_i) = \tau|_{r_f}(\hat{P}_i)\ \forall\, i \in N,\ f(\mathbf{P}) = f(\hat{\mathbf{P}})$.*

The tops-only property severely constrains the scope for manipulation. No agent can expect to be able to affect the social outcome without modifying the peak on the range of his reported preference. Perhaps not surprisingly, we show later in Proposition 2 that this condition is closely related to strategy-proofness, in the sense that every strategy-proof social choice rule on a maximal single-crossing domain is tops-only.[11] The next remark, which follows immediately from Definition 7, will be useful in the proof of Proposition 2.

Remark 1 *A social choice rule $f : \mathcal{SC}^n \to X$ is tops-only if and only if $\forall i \in N$, $\forall (P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ and $\forall \hat{P}_i \in \mathcal{SC}$ such that $\tau|_{r_f}(\hat{P}_i) = \tau|_{r_f}(P_i)$, $f(P_i, \mathbf{P}_{-i}) = f(\hat{P}_i, \mathbf{P}_{-i})$.*

Now we define a class of social choice rules that plays a crucial role in Section 3. To do that we introduce the following notation. For any odd positive integer k, we say that $m^k : X^k \to X$ is the *k-median* function on X^k if for each $\mathbf{x} = (x_1, \ldots, x_k) \in X^k$, $|\{x_i : m^k(\mathbf{x}) \geq x_i\}| \geq \frac{(k+1)}{2}$ and $|\{x_j : x_j \geq m^k(\mathbf{x})\}| \geq \frac{(k+1)}{2}$. Since k is odd, $m^k(\mathbf{x})$ is always well defined.

Definition 8 (EMR) *A social choice rule $f : \mathcal{SC}^n \to X$ is an **extended median rule** if there are $n+1$ fixed ballots $\alpha_1, \ldots, \alpha_{n+1} \in X$ such that for every preference profile $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = m^{2n+1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n+1})$.*

We denote by f^e a social choice rule that satisfies Definition 8, and by EMR the family of all such rules. A particular case of interest within this family is the well-known ***median choice rule***, denoted f^m. This rule is obtained from f^e by assigning $(n+1)/2$ fixed ballots at $\underline{X} = \min X$ and the rest at $\overline{X} = \max X$, if n is odd; and $n/2$ at $\underline{X}$ and $n/2+1$ at $\overline{X}$ if n is even and f^m breaks the ties in favor of the largest median peak. Alternatively, when n is even and f^m breaks the ties in favor of the smallest median peak, then $n/2+1$ fixed ballots are placed at $\underline{X}$ and the remaining $n/2$ at $\overline{X}$.

Proceeding in a similar way, we can derive other rules from EMR, by restricting each α_i to a particular value of X. For example, if $\alpha_i = \alpha \in X$ for all $i = 1, \ldots, n+1$, then f^e is completely insensitive to the preferences reported by the individuals. We might want to exclude such undesirable rules and, in particular, require Pareto efficiency. A social choice rule $f : \mathcal{SC}^n \to X$ is ***Pareto efficient*** if $\forall \mathbf{P} \in \mathcal{SC}^n$ there is no $y \in X$ such that $\forall i \in N$, $y \, P_i \, f(\mathbf{P})$. Hence, to eliminate the possibility of inefficiency, we set $\alpha_n = \underline{X}$ and $\alpha_{n+1} = \overline{X}$. By doing so, we derive a social choice rule f^* with the property that for all $\mathbf{P} \in \mathcal{SC}^n$, $f^*(\mathbf{P}) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n-1})$. This rule is called an ***efficient extended median rule***, and it is characterized by $n-1$ fixed ballots located on X. The set of all such rules is denoted by EMR^*.

Finally, we can also restrict each α_i to take its value at either $\underline{X}$ or $\overline{X}$, so that each fixed ballot is either a *leftist* or a *rightist* ballot. The family of social choice rules derived in that way was first introduced by Moulin [26], and is known as ***positional dictatorships***.

[11] A similar result holds when preferences are single-peaked, since every strategy-proof social choice rule whose range is an interval satisfies tops-only (see, for instance, Weymark [35] and Ching [10]).

Definition 9 (PD) *A social choice rule* $f : \mathcal{SC}^n \to X$ *is a* ***positional dictatorship*** *if there are* $n-1$ *fixed ballots* $\alpha_1, \ldots, \alpha_{n-1} \in \{\underline{X}, \overline{X}\}$ *such that for all* $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n),\, \alpha_1, \ldots, \alpha_{n-1})$.

These rules select the j-th peak among the tops of the reported preference orderings, for some $j \in \{1, \ldots, n\}$. For example, if $j = 1$, we have the *leftist rule*, which always chooses the smallest reported peak. The median choice rule f^m is also a particular case. We denote by f^j the positional dictatorship that selects, for all $\mathbf{P} \in \mathcal{SC}^n$, the alternative of the sequence $\tau(P_1), \ldots, \tau(P_n)$ placed at the j-th position according with the order $>$ of X. This rule is obtained from f^* by locating $n-j$ fixed ballots at $\underline{X}$ and $j-1$ at $\overline{X}$. The family of all such rules is denoted by $PD = \{f^j\}_{j \in N}$.

3 Characterization

In this section, we prove that the set of positional dictatorships is the only family of social choice rules that satisfies unanimity, anonymity and strategy-proofness on a maximal single-crossing domain. At the end, we also show that this is a tight characterization, in the sense that relaxing any of the previous conditions enlarges indeed the family of social choice rules.

We start by proving that every positional dictatorship is group strategy-proof.

Proposition 1 *Every positional dictatorship* f^j *is group strategy-proof on* $\mathcal{SC}^n$.

Proof. Fix $f^j \in PD$. Suppose, by contradiction, there exist a coalition $S \subseteq N$, a profile $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathcal{SC}^n$, and a joint deviation $\hat{\mathbf{P}}_S \in \mathcal{SC}^{|S|}$ for S such that $f^j(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})\, P_i\, f^j(\mathbf{P}_S, \mathbf{P}_{\bar{S}})$ for all $i \in S$. To simplify, denote $\tau = f^j(\mathbf{P}_S, \mathbf{P}_{\bar{S}})$ and $\hat{\tau} = f^j(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})$, and let $\hat{\tau} > \tau$.

By definition, $f^j \in PD$ implies $\alpha_i \in \{\underline{X}, \overline{X}\}$ for all $i = 1, \ldots, n-1$. Hence, τ and $\hat{\tau}$ must coincide with the tops reported by two individuals. Denote these agents by k and k', and their preferences by P_k and $P_{k'}$, respectively. We show next that, for all $i \in S$, $\tau(P_i) > \tau$. Suppose not. That is, assume $\tau \geq \tau(P_i)$ for some agent $i \in S$. If $\tau(P_i) = \tau$, then $\tau\, P_i\, \hat{\tau}$, which contradicts our initial hypothesis. Instead, suppose $\tau > \tau(P_i)$. Since $\hat{\tau}\, P_i\, \tau$ and $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathcal{SC}^n$, by $SC1$ we have that $\hat{\tau}\, P\, \tau$ for all $P \succ P_i$. Hence, $P_i \succ P_k$. Otherwise, $\hat{\tau} > \tau$, $P_k \succ P_i$ and $\hat{\tau}\, P_i\, \tau$ would imply $\hat{\tau}\, P_k\, \tau$, contradicting that $\tau = \tau(P_k)$. And again, by $SC1$, it follows that $\tau\, P_k\, \tau(P_i) \;\Rightarrow\; \tau\, P_i\, \tau(P_i)$, a contradiction. Hence, $\tau(P_i) > \tau\ \forall i \in S$.

By definition,

$$\tau = m^{2n-1}(\{\tau(P_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$$

and

$$\hat{\tau} = m^{2n-1}(\{\tau(\hat{P}_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \ldots, \alpha_{n-1}).$$

Hence, there must exist an individual $i \in S$ for whom $\tau > \tau(\hat{P}_i)$. Otherwise, if $\tau(\hat{P}_i) \geq \tau$ for all $i \in S$, we would have that $\hat{\tau} = \tau$, because we already saw that $\tau(P_i) > \tau$ for all $i \in S$. Thus, if we rename $(\{\tau(\hat{P}_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$ as $(y_1, \ldots, y_{2n-1})$, it follows that $|\{j \in \{1, \ldots, (2n-1)\} : \tau \geq y_j\}| \geq n$. But then $\tau \geq m^{2n-1}(y_1, \ldots, y_{2n-1})$. That is, $f^j(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \geq f^j(\hat{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})$, contradicting that $\hat{\tau} > \tau$. Therefore, f^j is GSP on $\mathcal{SC}^n$. ■

Falling short of Moulin's [25] results, who proved that every extended median rule is strategy-proof on single-peaked preferences, Proposition 1 shows that every positional dictatorship is group strategy-proof (and, consequently, strategy-proof) on any single-crossing domain. Instead, we claim that other extended median rules, which allow the social choice to be a fixed ballot, are not guaranteed to be strategy-proof on single-crossing preferences. Example 4 illustrates this claim.

Example 4 Consider a society with three agents, $N = \{1, 2, 3\}$, and three alternatives, $X = \{x, y, z\}$. Let individual preferences on X be as follows: $P_1 = (x\,y\,z)$, $P_2 = (x\,z\,y)$ and $P_3 = (z\,y\,x)$. As we said in Example 1, these preferences are single-crossing with respect to $z > y > x$ and $P_3 \succ P_2 \succ P_1$. Fix a social choice rule $f \in EMR^*$, and assume that $\alpha_1 = y$ and $\alpha_2 = z$. Note that α_1 coincides with neither individuals' most preferred alternatives nor with the end points of X. Moreover, $f(\mathbf{P}) = m^5(x, x, z, \alpha_1, \alpha_2) = y$. Thus, individual 2, who prefers that the group's choice be either x or z instead of alternative y, can manipulate f by declaring the insincere preference $\hat{P}_2 = (z\,y\,x)$. This causes the outcome to become $f(\hat{P}_2, \mathbf{P}_{-2}) = m^5(x, z, z, \alpha_1, \alpha_2) = z$. Therefore, agent 2's deviation is profitable and individual manipulation cannot be excluded.

The previous example shows that strategy-proofness is not assured for *every* efficient extended median rule because, with the exception of the subclass of positional dictatorships, all other extended median rules do not guarantee that the chosen alternative is always the most preferred alternative declared by an individual. However, as the proof of Proposition 1 illustrates, this information is used in a fundamental way to rule out preferences that may create incentives for manipulation. The reason for that lies in the fact that the single-crossing property is a restriction on the distribution of preferences across individuals, but it does not exclude a priori any preference relation.

Thus, to get rid of the undesirable orderings, i.e. those which provide incentives to misrepresent the true preferences, the argument cannot rely on the shape of individual preferences, as it happens under single-peakedness. On the contrary, the proof of Proposition 1 shows that the argument exploits (i) that the social choice is the most preferred alternative declared by an individual, (ii) the preference ordering of that agent; and (iii) the correlation between preferences in a set with the single-crossing property. Remarkably, no information about the shape of the preference relations is necessary to guarantee strategy-proofness.

Of course, the conjecture that only positional dictatorships are not manipulable on a maximal single-crossing domain stands in sharp contrast with the main result under the single-peakedness restriction, where every extended median rule (not just positional dictatorships) has been shown to be strategy-proof. In the next theorem we formalize this conjecture and we show that positional dictatorships can be characterized by strategy-proofness, anonymity and unanimity. The proof of this result will occupy the remainder of the paper.

Theorem 1 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. A social choice rule $f : \mathcal{SC}^n \to X$ is unanimous, anonymous and strategy-proof if and only if f is a positional dictatorship.*

Proof. See Appendix B. ■

The proof of Theorem 1, which is carried out in Appendix B for expositional convenience, rests on three main results, each of them important in it own right. The first one, summarized in Proposition 2, shows that on a maximal set of single-crossing preferences the tops-only property is implied by strategy-proofness. This result is a major step in doing the proof of Theorem 1, and is consistent with other results in the literature on strategy-proofness. In short, it captures the intuitive idea that social choice rules that use too much information from individuals' preferences are easier to manipulate.

Proposition 2 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. A social choice rule $f : \mathcal{SC}^n \to X$ is strategy-proof only if f is tops-only.*

Proof. See Appendix A. ■

Apart from Proposition 2, the proof of Theorem 1 also involves two additional results, which are summarized in Lemma 1 and 2, respectively. The first of these lemmas points out that, if a social choice rule is unanimous and strategy-proof (and therefore tops-only), then no individual must be able to profit by reporting extreme preference relations, unless such extreme preferences constitute the individual's true ordering. This "median property" at the individual level must simultaneously hold for *every* agent.

To present this more formally, in the sequel we use $\underline{P}$ (respectively, $\overline{P}$) to denote the most leftist (respectively, rightist) preference relation on X according with the linear order $>$, so that for all $x, y \in X$, $x \, \underline{P} \, y$ (respectively, $y \, \overline{P} \, x$) if and only if $y > x$. Clearly, $\tau(\underline{P}) = \underline{X}$ and $\tau(\overline{P}) = \overline{X}$. Moreover, it is easy to check that these rankings always belong to $\mathcal{SC}$.

Lemma 1 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. A social choice rule $f : \mathcal{SC}^n \to X$ is unanimous and strategy-proof only if for all $i \in N$ and all $\mathbf{P} \in \mathcal{SC}^n$,*

$$f(P_i, \mathbf{P}_{-i}) = m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})).$$

Proof. Let f be UN and SP on $\mathcal{SC}^n$. By Proposition 2, f is TO on $\mathcal{SC}^n$. Fix a profile $\mathbf{P} \in \mathcal{SC}^n$ and an individual $i \in N$. If $f(\underline{P}_i, \mathbf{P}_{-i}) > f(\overline{P}_i, \mathbf{P}_{-i})$, then $f(\underline{P}_i, \mathbf{P}_{-i}) \, \overline{P}_i \, f(\overline{P}_i, \mathbf{P}_{-i})$. Thus, agent i would like to manipulate f at $(\overline{P}_i, \mathbf{P}_{-i})$ via $\underline{P}_i$, a contradiction. Hence, $f(\overline{P}_i, \mathbf{P}_{-i}) \geq f(\underline{P}_i, \mathbf{P}_{-i})$.

Two cases are possible.

Case 1: $f(\underline{P}_i, \mathbf{P}_{-i}) \geq \tau(P_i)$. Then, $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = f(\underline{P}_i, \mathbf{P}_{-i})$. Assume, by contradiction, $f(\mathbf{P}) \neq f(\underline{P}_i, \mathbf{P}_{-i})$. First, suppose $f(\underline{P}_i, \mathbf{P}_{-i}) > f(\mathbf{P})$. If $P_i \succ \underline{P}_i$, $SC1$ would imply that $f(\underline{P}_i, \mathbf{P}_{-i}) \, P_i \, f(P_i, \mathbf{P}_{-i})$, which contradicts SP. Thus, since $\underline{P}_i$ is agent i's most leftist preference relation, P_i is identical to $\underline{P}_i$ and $f(P_i, \mathbf{P}_{-i}) = f(\underline{P}_i, \mathbf{P}_{-i})$, which contradicts that by hypothesis $f(\mathbf{P}) \neq f(\underline{P}_i, \mathbf{P}_{-i})$. So, $f(\mathbf{P}) > f(\underline{P}_i, \mathbf{P}_{-i})$, implying that $f(P_i, \mathbf{P}_{-i}) > \tau(P_i)$. By SP, $f(P_i, \mathbf{P}_{-i}) \, P_i \, f(\underline{P}_i, \mathbf{P}_{-i})$. Hence, $\tau(P_i) \neq f(\underline{P}_i, \mathbf{P}_{-i})$. Furthermore, $f(\underline{P}_i, \mathbf{P}_{-i}) \neq \tau(\underline{P}_i)$, because $f(\underline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) \geq \tau(\underline{P}_i) = \underline{X}$. In fact, as can be inferred from Figure 3, $f(\underline{P}_i, \mathbf{P}_{-i}) \neq \tau(P_j)$ for all $j \neq i$. Otherwise, if $f(\underline{P}_i, \mathbf{P}_{-i}) = \tau(P_j)$ for some $j \in N \backslash \{i\}$, then $P_j \succ P_i$, because $f(\underline{P}_i, \mathbf{P}_{-i}) > \tau(P_i)$. However, by $SC2$, $P_j \succ P_i$, $f(P_i, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$ and $f(\underline{P}_i, \mathbf{P}_{-i}) \, P_j \, f(P_i, \mathbf{P}_{-i})$ would imply $f(\underline{P}_i, \mathbf{P}_{-i}) \, P_i \, f(P_i, \mathbf{P}_{-i})$, a contradiction.

Step 1. Assume there is a preference $P_i^\alpha \in \mathcal{P}$, between $\underline{P}_i$ and P_i, such that

(i) $\tau(P_i^\alpha) = \tau(P_i)$, and

(ii) $f(\underline{P}_i, \mathbf{P}_{-i}) \, P_i^\alpha \, f(P_i, \mathbf{P}_{-i})$, (see Figure 3).

If $P_i^\alpha \in \mathcal{SC}$, we are done. By TO, $f(P_i^\alpha, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$.[12] By definition of P_i^α, $f(\underline{P}_i, \mathbf{P}_{-i}) \, P_i^\alpha \, f(P_i^\alpha, \mathbf{P}_{-i})$, which contradicts strategy-proofness.

Step 2. On the other hand, if $P_i^\alpha \notin \mathcal{SC}$, then there must exist a preference $P_i^\beta \in \mathcal{SC}$ which stops P_i^α to be part of $\mathcal{SC}$. That is, there must be a $P_i^\beta \in \mathcal{SC}$ such that $P_i^\beta \prec P_i$ and

(i) $\tau(P_i) > \tau(P_i^\beta)$, and

(ii) $f(P_i, \mathbf{P}_{-i}) \, P_i^\beta \, f(\underline{P}_i, \mathbf{P}_{-i})$.

Figure 3 represents this ordering. Clearly, P_i^α must be above P_i^β, because the ideal point $\tau(P_i^\alpha)$ is greater than $\tau(P_i^\beta)$. At the same time, P_i^α must be below P_i^β, because $f(\underline{P}_i, \mathbf{P}_{-i}) \, P_i^\alpha \, f(P_i, \mathbf{P}_{-i})$ and $f(P_i, \mathbf{P}_{-i}) \, P_i^\beta \, f(\underline{P}_i, \mathbf{P}_{-i})$, being $f(P_i, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$. Of course, these two requirements cannot be simultaneously satisfied. Therefore $P_i^\alpha \notin \mathcal{SC}$.

Step 3. If $f(\underline{P}_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i})$, then i can manipulate f at $(\underline{P}_i, \mathbf{P}_{-i})$ via P_i^β because, by definition of $\underline{P}_i$, a lower alternative is always preferred. On the other hand, if

[12] Note that unanimity implies that, for all $i \in N$, and all $P_i \in \mathcal{SC}$, $\tau(P_i) \in r_f$. Hence, $\tau(P_i) = \tau|_{r_f}(P_i)$.

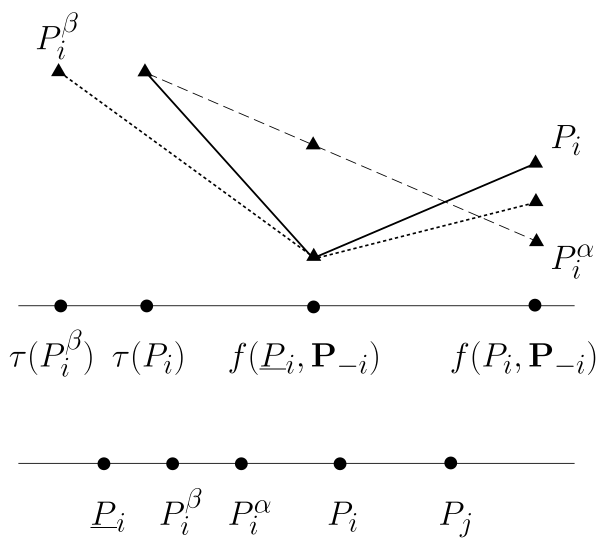


Figure 3: What's going on in Case 1?

$f(P_i^{\beta}, \mathbf{P}_{-i}) = f(\underline{P}_i, \mathbf{P}_{-i})$, then i can manipulate f at $(P_i^{\beta}, \mathbf{P}_{-i})$ via P_i because, by definition of P_i^{β}, $f(P_i, \mathbf{P}_{-i})\, P_i^{\beta}\, f(\underline{P}_i, \mathbf{P}_{-i})$. Hence, $f(P_i^{\beta}, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$. Furthermore, $f(P_i, \mathbf{P}_{-i}) \geq f(P_i^{\beta}, \mathbf{P}_{-i})$. To see this, recall that $P_i \succ P_i^{\beta}$ and, by SP, $f(P_i^{\beta}, \mathbf{P}_{-i})\, P_i^{\beta}\, f(P_i, \mathbf{P}_{-i})$, Thus, if $f(P_i^{\beta}, \mathbf{P}_{-i}) > f(P_i, \mathbf{P}_{-i})$, then $SC1$ would imply $f(P_i^{\beta}, \mathbf{P}_{-i})\, P_i\, f(P_i, \mathbf{P}_{-i})$, a contradiction. To summarize, $f(P_i, \mathbf{P}_{-i}) \geq f(P_i^{\beta}, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$.

Step 4. Repeating Step 1, suppose there is a preference $P_i^{\alpha+1} \in \mathcal{SC}$, between $\underline{P}_i$ and P_i^{β}, such that (i) $\tau(P_i^{\alpha+1}) = \tau(P_i^{\beta})$, and (ii) $f(\underline{P}_i, \mathbf{P}_{-i})\, P_i^{\alpha+1}\, f(P_i^{\beta}, \mathbf{P}_{-i})$. By TO, $f(P_i^{\alpha+1}, \mathbf{P}_{-i}) = f(P_i^{\beta}, \mathbf{P}_{-i})$. Therefore, $f(\underline{P}_i, \mathbf{P}_{-i})\, P_i^{\alpha+1}\, f(P_i^{\alpha+1}, \mathbf{P}_{-i})$, contradicting that f is SP.

On the contrary, if $P_i^{\alpha+1} \notin \mathcal{SC}$, then, repeating the argument behind Step 2, there must exist a preference $P_i^{\beta+1} \in \mathcal{SC}$ such that $P_i^{\beta+1} \prec P_i^{\beta}$ and (i) $\tau(P_i^{\beta}) > \tau(P_i^{\beta+1})$, and (ii) $f(P_i^{\beta}, \mathbf{P}_{-i})\, P_i^{\beta+1}\, f(\underline{P}_i, \mathbf{P}_{-i})$. By Step 3, $f(P_i^{\beta}, \mathbf{P}_{-i}) \geq f(P_i^{\beta+1}, \mathbf{P}_{-i}) > f(\underline{P}_i, \mathbf{P}_{-i})$.

If we go back to Step 1 and we continue applying Steps 1 to 3 repeatedly, then either we eventually get the desired contradiction or, after say ℓ interactions, we get a preference $P_i^{\beta+\ell} \in \mathcal{SC}$, between $\underline{P}_i$ and $P_i^{\beta+\ell-1}$, such that $\tau(P_i^{\beta+\ell}) = \tau(\underline{P}_i)$ and $f(P_i^{\beta+\ell-1}, \mathbf{P}_{-i})\, P_i^{\beta+\ell}\, f(\underline{P}_i, \mathbf{P}_{-i})$. By the tops-only property, $f(P_i^{\beta+\ell}, \mathbf{P}_{-i}) = f(\underline{P}_i, \mathbf{P}_{-i})$. Therefore, i can manipulate f at $(P_i^{\beta+\ell}, \mathbf{P}_{-i})$ via $P_i^{\beta+\ell-1}$.

Case 2: $f(\overline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) > f(\underline{P}_i, \mathbf{P}_{-i})$.[13] Then, $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = \tau(P_i)$. Assume, by contradiction, $f(\mathbf{P}) \neq \tau(P_i)$. Without loss of generality, suppose $\tau(P_i) > f(\mathbf{P})$, so that $f(\overline{P}_i, \mathbf{P}_{-i}) > f(\mathbf{P})$. By SP, $f(P_i, \mathbf{P}_{-i})\, P_i\, f(\overline{P}_i, \mathbf{P}_{-i})$ and $f(\overline{P}_i, \mathbf{P}_{-i})\, \overline{P}_i\, f(P_i, \mathbf{P}_{-i})$.

[13] The remaining case where $\tau(P_i) \geq f(\overline{P}_i, \mathbf{P}_{-i})$ is similar to Case 1.

Step 1. Suppose there is a preference $P_i^\alpha \in \mathcal{SC}$, between P_i and $\overline{P}_i$, such that

(i) $\tau(P_i^\alpha) = \tau(P_i)$, and

(ii) $f(\overline{P}_i, \mathbf{P}_{-i})\, P_i^\alpha\, f(P_i, \mathbf{P}_{-i})$, (see Figure 4).

By TO, $f(P_i^\alpha, \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$. Thus, $f(\overline{P}_i, \mathbf{P}_{-i})\, P_i^\alpha\, f(P_i^\alpha, \mathbf{P}_{-i})$, a contradiction.

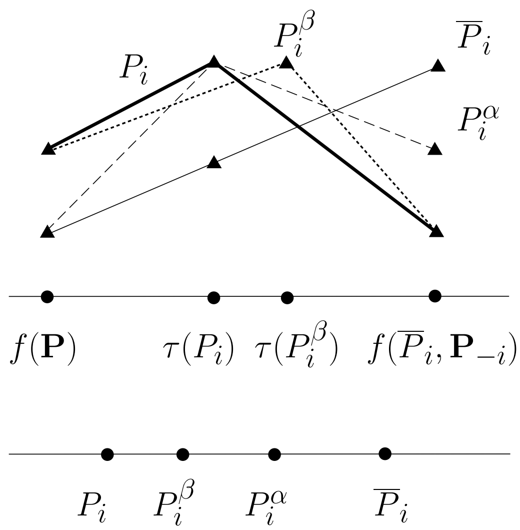


Figure 4: What's going on in Case 2?

Step 2. If $P_i^\alpha \notin \mathcal{SC}$, then there is a preference $P_i^\beta \in \mathcal{SC}$ such that $P_i^\beta \succ P_i$ and

(i) $\tau(P_i^\beta) > \tau(P_i)$, and

(ii) $f(P_i, \mathbf{P}_{-i})\, P_i^\beta\, f(\overline{P}_i, \mathbf{P}_{-i})$, (see Figure 4).

Step 3. If $f(P_i^\beta, \mathbf{P}_{-i}) > f(\overline{P}_i, \mathbf{P}_{-i})$, then i can manipulate f at $(\overline{P}_i, \mathbf{P}_{-i})$ via P_i^β because, by definition of $\overline{P}_i$, a greater alternative is always preferred. On the other hand, if $f(P_i^\beta, \mathbf{P}_{-i}) = f(\overline{P}_i, \mathbf{P}_{-i})$, then i can manipulate f at $(P_i^\beta, \mathbf{P}_{-i})$ via P_i because, by definition of P_i^β, $f(P_i, \mathbf{P}_{-i})\, P_i^\beta\, f(\overline{P}_i, \mathbf{P}_{-i})$. Hence, $f(\overline{P}_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i})$. Furthermore, $f(P_i^\beta, \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$. To see this, recall that $P_i \prec P_i^\beta$ and, by SP, $f(P_i^\beta, \mathbf{P}_{-i})\, P_i^\beta\, f(P_i, \mathbf{P}_{-i})$, Thus, if $f(P_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i})$, then $SC2$ would imply $f(P_i^\beta, \mathbf{P}_{-i})\, P_i\, f(P_i, \mathbf{P}_{-i})$, a contradiction. To summarize, $f(\overline{P}_i, \mathbf{P}_{-i}) > f(P_i^\beta, \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$.

Step 4. Repeating Step 1, suppose there is a preference $P_i^{\alpha+1} \in \mathcal{SC}$, between $\overline{P}_i$ and P_i^β, such that (i) $\tau(P_i^{\alpha+1}) = \tau(P_i^\beta)$, and (ii) $f(\overline{P}_i, \mathbf{P}_{-i})\, P_i^{\alpha+1}\, f(P_i^\beta, \mathbf{P}_{-i})$. By TO, $f(P_i^{\alpha+1}, \mathbf{P}_{-i}) = f(P_i^\beta, \mathbf{P}_{-i})$. Therefore, $f(\overline{P}_i, \mathbf{P}_{-i})\, P_i^{\alpha+1}\, f(P_i^{\alpha+1}, \mathbf{P}_{-i})$, a contradiction.

On the contrary, if $P_i^{\alpha+1} \notin \mathcal{SC}$, then, repeating the argument behind Step 2, there must exist a preference $P_i^{\beta+1} \in \mathcal{SC}$ such that $P_i^{\beta+1} \succ P_i^{\beta}$ and (i) $\tau(P_i^{\beta+1}) > \tau(P_i^{\beta})$, and (ii) $f(P_i^{\beta}, \mathbf{P}_{-i}) \, P_i^{\beta+1} \, f(\overline{P}_i, \mathbf{P}_{-i})$. By Step 3, $f(\overline{P}_i, \mathbf{P}_{-i}) > f(P_i^{\beta+1}, \mathbf{P}_{-i}) \geq f(P_i^{\beta}, \mathbf{P}_{-i})$.

If we go back to Step 1 and we continue applying Steps 1 to 3 over and over again, then in the end either we get the desired contradiction or, after say ℓ interactions, we find a $P_i^{\beta+\ell} \in \mathcal{SC}$, between $\overline{P}_i$ and $P_i^{\beta+\ell-1}$, such that $\tau(P_i^{\beta+\ell}) = \tau(\overline{P}_i)$ and $f(P_i^{\beta+\ell-1}, \mathbf{P}_{-i}) \, P_i^{\beta+\ell} \, f(\overline{P}_i, \mathbf{P}_{-i})$. By TO, $f(P_i^{\beta+\ell}, \mathbf{P}_{-i}) = f(\overline{P}_i, \mathbf{P}_{-i})$. Therefore, i can manipulate f at $(P_i^{\beta+\ell}, \mathbf{P}_{-i})$ via $P_i^{\beta+\ell-1}$, contradicting that f is strategy-proof. ■

Finally, the proof of Theorem 1 also benefits from Lemma 2, according to which a strategy-proof and unanimous social choice rule must satisfy a property called *top-monotonicity.* Roughly speaking, this property ensures that collective choices do not respond perversely to changes in individuals' ideal points.

Definition 10 (TM) *A social choice rule $f : \mathcal{SC}^n \to X$ is* ***top-monotonic*** *if $\forall i \in N$, $\forall (P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ and $\forall \hat{P}_i \in \mathcal{SC}$ such that $\tau(\hat{P}_i) \geq \tau(P_i)$, $f(\hat{P}_i, \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$.*

Like before, assume $\underline{P}$ (respectively, $\overline{P}$) denote the most leftist (respectively, rightist) preference relation on X according with the linear order $>$.

Lemma 2 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathcal{SC}^n \to X$ is unanimous and strategy-proof, then f is top-monotonic.*

Proof. Let f be UN and SP on $\mathcal{SC}^n$. Consider any individual $i \in N$, any profile $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ and any admissible deviation $P_i' \in \mathcal{SC}$, such that $\tau(P_i') \geq \tau(P_i)$. We want to show that $f(P_i', \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$. Three cases are possible.

Case 1. If we have that $\tau(P_i) \geq f(\overline{P}_i, \mathbf{P}_{-i})$, then $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = m^3(\tau(P_i'), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i}))$, because SP implies that $f(\overline{P}_i, \mathbf{P}_{-i}) \geq f(\underline{P}_i, \mathbf{P}_{-i})$, and $\tau(P_i') \geq \tau(P_i)$ by hypothesis. Therefore, by Lemma 1, $f(P_i', \mathbf{P}_{-i}) = f(P_i, \mathbf{P}_{-i})$.

Case 2. If $f(\overline{P}_i, \mathbf{P}_{-i}) > \tau(P_i) > f(\underline{P}_i, \mathbf{P}_{-i})$, then $m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = \tau(P_i)$; and, given that $\tau(P_i') \geq \tau(P_i)$, $m^3(\tau(P_i'), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) \geq \tau(P_i)$. Therefore, by Lemma 1, $f(P_i', \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$.

Case 3. Finally, if $f(\underline{P}_i, \mathbf{P}_{-i}) \geq \tau(P_i)$, then $m^3(\tau(P_i'), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) \geq m^3(\tau(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i})) = f(\underline{P}_i, \mathbf{P}_{-i})$. Hence, by Lemma 1, $f(P_i', \mathbf{P}_{-i}) \geq f(P_i, \mathbf{P}_{-i})$. ■

Under the hypotheses of Theorem 1, the social choice always coincides with an individual's most preferred alternative. Thus, a corollary that can be immediately derived from it is that, on a maximal set of single-crossing preferences, every unanimous, anonymous and strategy-proof social choice rule satisfies Pareto efficiency.

Corollary 1 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathcal{SC}^n \to X$ is unanimous, anonymous and strategy-proof, then f is Pareto efficient.*

Proof. Suppose, by contradiction, there exists a social choice rule f that satisfies the hypotheses of Corollary 1, but f is not Pareto efficient. Then, there must exist $\mathbf{P} \in \mathcal{SC}^n$, and a pair x, $y \in X$, $x \neq y$, such that $f(\mathbf{P}) = x$, while $y \, P_i \, x$ for all $i \in N$. Hence, for all $i = 1, \ldots n$, $f(\mathbf{P}) \neq \tau(P_i)$, contradicting that, by Theorem 1, $f \in PD$. ■

In addition to the previous corollary, under the hypotheses of Theorem 1, it is also possible to show that the set of admissible preferences has the single-peaked property over the range of the social choice rule. More formally, for any set $Y \subset X$ and any preference $P \in \mathcal{SC}$, let $P|_Y$ be the restriction of the binary relation P over the elements of Y. Denote by $\mathcal{SC}|_Y$ the set containing the restriction of each preference $P \in \mathcal{SC}$ over Y. We refer to $\mathcal{SC}|_Y$ as the restriction of $\mathcal{SC}$ on the set $Y \subset X$.

Lemma 3 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathcal{SC}^n \to X$ is unanimous, anonymous and strategy-proof, then the restriction of $\mathcal{SC}$ on the range r_f has the single-peaked property.*

Proof. The proof is based on Saporiti and Tohmé [32]. Fix a maximal set $\mathcal{SC} \subset \mathcal{P}$ with the single-crossing property with respect to $>$ and $\succ$. Take a UN, AN and SP social choice rule $f : \mathcal{SC}^n \to X$. Assume, by contradiction, there exists a preference $P|_{r_f} \in \mathcal{SC}|_{r_f}$ which is not single-peaked on r_f with respect to the order linear $>$ of X. Then, there must be a triple $x, y, z \in r_f$ such that $x > y > z$ and $x \, P \, y$ and $z \, P \, y$. By Theorem 1, $y = \tau|_{r_f}(P')$ for some $P' \in \mathcal{SC}$. If $P' \succ P$, then, by $SC1$, $x > y$ and $x \, P \, y$ imply $x \, P' \, y$, contradicting that $y = \tau|_{r_f}(P')$. Hence, $P \succ P'$. However, since $y > z$ and $z \, P \, y$, $SC2$ implies $z \, P' \, y$, a contradiction. Therefore, the set $\mathcal{SC}|_{r_f}$ has the single-peaked property on r_f. ■

An immediate corollary of Lemma 3 is therefore that $\mathcal{SC}|_{r_f}$ is a *regular domain.* A social choice rule $f : \mathcal{SC} \to X$ has a regular domain if for every $\alpha \in r_f$ there is a preference $P^\alpha \in \mathcal{SC}$ such that $\tau|_{r_f}(P^\alpha) = \alpha$ (Weymark [35]).

Corollary 2 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathcal{SC}^n \to X$ is unanimous, anonymous and strategy-proof, then the restriction of $\mathcal{SC}$ on the range r_f is a regular domain.*

Proof. Immediate from Lemma 3. ■

Finally, we close this section discussing the independence of the axioms used in Theorem 1, as well as the role of the maximal domain condition specified in Definition 2. First, consider the consequence of relaxing strategy-proofness. As we explained before, any efficient extended median rule that it is not a positional dictatorship may be subject to individual manipulation on a single-crossing domain (see Example 4). However, all of them are anonymous and unanimous. Thus, the family that satisfies these two axioms on $\mathcal{SC}^n$ is larger than the set of positional dictatorships.

Second, consider the consequence of relaxing unanimity. Define a social choice rule f in such a way that, for each $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = a \in X$. It is clear that f is anonymous and strategy-proof. However, f violates unanimity, since $r_f = \{a\}$. Hence, $f \notin PD$.

Third, relax anonymity, by fixing an agent $j \in N$ and defining a social choice rule f in such a way that, for all $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = \tau(P_j)$. It is immediate to see that f is unanimous and strategy-proof. However, it violates anonymity, because f is dictatorial.

Lastly, to illustrate why the maximal domain condition is needed to derive the main results of this paper, let $N = \{1, 2\}$, $X = \{x, y, z\}$, with $z > y > x$, and $\mathcal{SC} = \{\underline{P}, \overline{P}\}$, where $\underline{P} = (x\,y\,z)$ and $\overline{P} = (z\,y\,x)$. As is clear from Example 3, the set of preferences $\mathcal{SC}$ is not a maximal set with the single-crossing property. Define f by setting $f(\underline{P}, \underline{P}) = x$, $f(\overline{P}, \overline{P}) = z$ and $f(\underline{P}, \overline{P}) = f(\overline{P}, \underline{P}) = y$. This function satisfies unanimity, anonymity and strategy-proofness. However, f is not a positional dictatorship, because y is chosen at $(\underline{P}, \overline{P})$ and at $(\overline{P}, \underline{P})$, but y is the most preferred alternative of nobody at those profiles.[14]

4 Robustness

So far, we have assumed that every individual $i \in N$ is endowed with a preference P_i draw from the restricted domain $\mathcal{SC}$, and is entitled to report a preference relation (not necessarily the true one) from the same admissible set. That is, we have restricted *both* the true preferences of all individuals and their strategies, i.e., the orderings they are permitted to announce, to the *same* maximal set of single-crossing preferences. The main result obtained from that assumption is that a social choice rule is anonymous, unanimous and strategy-proof if and only if it is a positional dictatorship.

There are two main concerns regarding this result. First, there is a question of how easy to describe is the set of admissible preferences. We saw in Section 2.4 that in the applications the set of individual preferences with the single-crossing property is well defined and can be derived from standard assumptions of economics. However, to be able to describe this set, the mechanism designer would probably need to already possess some information about individuals' preferences, though not about any particular individual's ordering. Given that the goal of this paper is to study social choice problems where individual preferences are

[14]I am grateful to one of the referees for suggesting this example.

privately observed, the information require by the planner to specify the set of admissible reports weakens the contribution of Theorem 1.

Nevertheless, as Campbell and Kelly [9], p. 567, say, "there is a sense in which results based on a domain of single-peaked preferences have the same drawback: Although single-peaked domains can be defined as product sets, single-peakedness is characterized by means of a particular linear ordering, and an individual would have to know the linear ordering to which the reported preference is admissible, before being convinced that his own reported preference is admissible".

Furthermore, while in some cases this ordering is *natural* and, therefore, the assumption that it is commonly known (included by the planner) is not too demanding, in others it is not necessarily obvious. Suppose, for example, that alternatives are political candidates. Then, the way in which individuals agree to locate these candidates on a one-dimensional political scale is not immediate. Moreover, that ordering not only determines which preferences can be declared, but it also provides information about other individuals' preferences. For instance, if $X = \{x, y, z\}$ and $(P_i)_{i \in N}$ is single-peaked with respect to $x > y > z$, then the order of the alternatives reveals that nobody holds a preference which ranks y bottom (such as the relations $P = (x\,z\,y)$ and $P' = (z\,x\,y)$).

Apart from the difficulty to specify the set of possible reports, a second concern is that, even if the mechanism designer would have had the information to do so, it would still be unclear how to deal with declarations which are not in the admissible set. Can we tell an individual that, despite the fact that preferences are not directly observed, on the basis of our beliefs about "how they should be", he cannot submit a certain preference relation because we consider it somehow "unreasonable" and, therefore, it has been removed from the set of possible declarations?

Once again, this affects not only the analysis on single-crossing preferences, but also on other domain restrictions. Consider, for instance, the case where preferences satisfy the single-peaked property over the real line. For the planner, it would not be difficult to describe the set of admissible preferences, because alternatives are ordered according with the usual order of the real numbers. However, suppose individual i reports a preference that is not single-peaked on that order. What can we do in such situation? Can we say to individual i that he is not entitled to have such preference relation? In a democracy, every individual is free to order the alternatives in the way he wishes to do so, independently of how sensible we think these orderings are. Thus, assuming that preference relations which do not verify the domain restriction will not be permitted seems neither realistic nor democratic.

To deal with this problem, in this section we analyze the possibility of strengthening the result of Theorem 1, by eliminating the requirement that individual reports be restricted to be in the set $\mathcal{SC}$. The analysis is inspired by Blin and Satterthwaite [7], who have done a similar exercise to assess the robustness of the strategy-proof result of majority rule with Borda completion on single-peaked preferences when individual reports are allowed to be outside the single-peaked domain.

Our findings are positive: If the true preferences of the society satisfy the single-crossing property, then no individual can ever profitably manipulate a positional dictatorship by reporting a preference which is not his true preference relation, independently of whether the insincere preference belongs to the single-crossing domain or not. Conversely, if we allow deviations outside the single-crossing domain, every anonymous, unanimous and strategy-proof social choice rule must be a positional dictatorship on the set of preferences with the single-crossing property.

To show this more formally, let us now redefine a social choice rule so that it associates a feasible alternative to every profile of complete, transitive and antisymmetric preferences; i.e., let $f : \mathcal{P}^n \to X$. Following Blin and Satterthwaite [7], a social choice rule f is said manipulable on $\mathcal{SC}^n$ if there exists an individual $i \in N$, a profile $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ and a deviation $\hat{P}_i \in \mathcal{P}$ such that $f(\hat{P}_i, \mathbf{P}_{-i})\, P_i\, f(P_i, \mathbf{P}_{-i})$. A social choice rule is strategy-proof on $\mathcal{SC}^n$ if and only if it is not manipulable on $\mathcal{SC}^n$. Notice that in Definition 3 we have omitted the qualification "on $\mathcal{SC}^n$" when we defined strategy-proofness, because that was also the domain of the social choice rule. Instead, here the social choice rule is defined on a larger domain, actually on the set of all strict preferences; but it is required to satisfy strategy-proofness only on the domain of individuals' true preferences.

Proceeding in a similar way, we can redefine anonymity and unanimity. A social choice rule $f : \mathcal{P}^n \to X$ is unanimous on $\mathcal{SC}^n$ if $\forall\, x \in X$ and $\forall\, \mathbf{P} \in \mathcal{SC}^n$ such that $\tau(P_i) = x\; \forall i \in N$, $f(\mathbf{P}) = x$. Similarly, $f : \mathcal{P}^n \to X$ is anonymous on $\mathcal{SC}^n$ if $f(\mathbf{P}) = f(\hat{\mathbf{P}})$ for every permutation $\mathbf{P}$ of $\hat{\mathbf{P}} \in \mathcal{SC}^n$. Finally, we can extend the definition of positional dictatorships to the set of all complete, transitive and antisymmetric preference profiles, and to any nonempty subset of it. Specifically, a social choice rule $f : \mathcal{P}^n \to X$ is a positional dictatorship on $\mathcal{D} \subseteq \mathcal{P}^n$ if there are $n-1$ fixed ballots $\alpha_1, \ldots, \alpha_{n-1} \in \{\underline{X}, \overline{X}\}$ such that $\forall \mathbf{P} \in \mathcal{D}$, $f(\mathbf{P}) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n),\, \alpha_1, \ldots, \alpha_{n-1})$. For simplicity, when $\mathcal{D}$ coincides with $\mathcal{P}^n$, we simply say that f is a positional dictatorship.

Theorem 2 *Every positional dictatorship $f^j : \mathcal{P}^n \to X$ is strategy-proof on $\mathcal{SC}^n$. Conversely, if $\mathcal{SC}$ is a maximal set of single-crossing preferences, then every social choice rule $f : \mathcal{P}^n \to X$ that is unanimous, anonymous and strategy-proof on $\mathcal{SC}^n$ is a positional dictatorship on $\mathcal{SC}^n$.*

Proof. To prove the first part of Theorem 2, fix any positional dictatorship $f^j : \mathcal{P}^n \to X$ and suppose f^j is manipulable by $i \in N$ at a profile $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ via a preference relation $\hat{P}_i \in \mathcal{P}$, which is not necessarily in $\mathcal{SC}$. Without loss of generality, suppose that

$f^j(\hat{P}_i, \mathbf{P}_{-i}) > f^j(P_i, \mathbf{P}_{-i})$. Since f^j always chooses an individual's most preferred alternative, let $f^j(P_i, \mathbf{P}_{-i})$ coincide with individual k's ($k \neq i$) ideal point $\tau(P_k)$ according with P_k. By $SC1$, $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ and $f^j(\hat{P}_i, \mathbf{P}_{-i})\, P_i\, f^j(P_i, \mathbf{P}_{-i})$ imply $P_i \succ P_k$. Therefore, $\tau(P_i) > f^j(P_i, \mathbf{P}_{-i})$. Moreover, $f^j(P_i, \mathbf{P}_{-i}) > \tau(\hat{P}_i)$. Otherwise, we would have $f^j(\hat{P}_i, \mathbf{P}_{-i}) = f^j(P_i, \mathbf{P}_{-i})$. Hence, by definition of f^j, $f^j(P_i, \mathbf{P}_{-i}) \geq f^j(\hat{P}_i, \mathbf{P}_{-i})$, which contradicts the initial hypothesis that $f^j(\hat{P}_i, \mathbf{P}_{-i}) > f^j(P_i, \mathbf{P}_{-i})$. Therefore, f^j is SP on $\mathcal{SC}^n$.

The proof of the second part of Theorem 2 is immediate. Consider a social choice rule $f : \mathcal{P}^n \to X$ that is unanimous, anonymous and strategy-proof on $\mathcal{SC}^n$. Define the social choice rule $g : \mathcal{SC}^n \to X$ such that, for all $\mathbf{P} \in \mathcal{SC}^n$, $g(\mathbf{P}) = f(\mathbf{P})$. Simple inspection shows that g is AN, UN and SP. Thus, by Theorem 1, $g \in PD$; and, by definition, f is a positional dictatorship on $\mathcal{SC}^n$. ■

Notice that the first part of Theorem 2 is proved applying the same argument used in the proof to Proposition 1. This is because the proof of Proposition 1 does not use any particular structure of the deviation profile $(\hat{P}_i, \mathbf{P}_{-i})$. What really matters is that $(P_i, \mathbf{P}_{-i})$ is in $\mathcal{SC}^n$. On the other hand, the second part of Theorem 2 holds because every social choice rule which is not manipulable when individuals can report *any* strict preference relation must be so when they are allowed to declare only preferences from a strictly smaller subset. However, we have already shown in Theorem 1 that, when reports are restricted to $\mathcal{SC}$, every unanimous, anonymous and strategy-proof social choice rule is a positional dictatorship. Hence, if we dispense of the assumption that declarations are restricted to the set with the single-crossing property, we must still obtain the same family of rules on the restricted domain.

Finally, note that Theorem 2 does not provide a full characterization, because we haven't determined the form of a unanimous, anonymous and strategy-proof social choice rule outside the domain of preferences with the single-crossing property. However, it does show that the rules obtained in any of such characterizations coincide over a maximal single-crossing domain with the rules characterized in Theorem 1. This, together with the fact that every positional dictatorship is strategy-proof on single-crossing preferences, allow us to conclude that the result stated in Theorem 1 is robust to the kind of perturbations introduced in this section.

5 Final remarks

This paper analyzes strategy-proof collective choice rules when individuals have single-crossing preferences on a finite and ordered set of social alternatives. While the single-crossing property has been shown to be sufficient to ensure the existence of a Condorcet winner, this result has been derived assuming that individuals sincerely declare their preferences. This naturally raises the issue of potential individual and group manipulation, motivating the current research.

The main contributions of this article are the following. First of all, it shows that, in addition to single-peakedness, single-crossing is another meaningful domain that guarantees the existence of strategy-proof social choice rules. Specifically, it proves that every positional dictatorship is group strategy-proof on any set of preferences with the single-crossing property. Conversely, every social choice rule that satisfies anonymity, unanimity and strategy-proofness on a maximal single-crossing domain is shown to be a member of this family. These results are robust to deviations outside the single-crossing domain, provided that individuals' true preferences belong to that set.

A natural consequence of the previous characterization is that anonymity, unanimity and strategy-proofness imply Pareto efficiency. Furthermore, although in our framework individual preferences need not be convex over the set of alternatives, anonymity, unanimity and strategy-proofness also imply that preferences must satisfy single-peakedness over the range of the social choice rule.[15] So, our results indicate that to rule out incentives to misrepresent individual preferences some convexity and regularity of the domain is indeed necessary.

Another important conclusion coming out from this research is that, on a maximal single-crossing domain, strategy-proofness implies the tops-only property. The proof of this claim does not completely follow the proof strategy recently proposed by Weymark [35], because the non-convexities of single-crossing preferences make quite difficult to directly prove that the Le Breton–Weymark's [22] regularity condition is satisfied. To avoid this, first we prove the claim in a two-person case, where unanimity over the range can be used without further complications. And then we obtain a partial characterization of the social choice rule for the case with only two individuals, which allows to prove the claim for more than two agents by reducing the analysis to a situation where only two tops are different.

Finally, this papers also shows that the Representative Voter Theorem has a well defined strategic foundation, in the sense that the median voter's ideal point can be implemented in dominant strategies by a direct mechanism. However, this conclusion holds on a subdomain of single-crossing preferences, the Cartesian product domain. Therefore, relaxing the assumption that individuals sincerely reveal their preferences is not free.

Moreover, given that the domain of single-crossing preferences is somehow less natural than the single-peaked domain, we also find that the single-crossing version of the Median Voter Theorem under incomplete information of individuals' preferences would not probably have the same appeal as its counterpart on single-peaked domains.

[15]Given a set X and a linear order $>$ of X, a preference $P \in \mathcal{P}$ on X is convex with respect to $>$ if for every three distinct alternatives $x, y, z \in X$, $x\,P\,y \;\Rightarrow\; y\,P\,z$ whenever y is between x and z.

6 Appendix A: Proof of Proposition 2

In order to prove Proposition 2, the following lemma will be extremely useful.

Lemma 4 *Suppose $f : \mathcal{SC}^n \to X$ is a strategy-proof social choice rule with $n \geq 1$. For any nonempty set $S \subseteq N$, any $x \in r_f$ and every profile $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathcal{SC}^n$ such that $\tau|_{O_S^f(\mathbf{P}_{\bar{S}})}(P_i) = x$ for all $i \in S$, $f(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) = x$.*

Proof. The proof is domain independent and is based on Proposition 2 in Le Breton and Weymark [22]. Assume f is SP on $\mathcal{SC}^n$, and consider any coalition $S \subseteq N$, any alternative $x \in r_f$ and any profile $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathcal{SC}^n$ such that $\forall i \in S$, $\tau|_{O_S^f(\mathbf{P}_{\bar{S}})}(P_i) = x$. Suppose, by contradiction, $f(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) = y \neq x$ for some $y \in X$. Define the social choice rule $g : \mathcal{SC}^{|S|} \to X$, where for all $\mathbf{P}'_S \in \mathcal{SC}^{|S|}$, $g(\mathbf{P}'_S) = f(\mathbf{P}'_S, \mathbf{P}_{\bar{S}})$. It is easy to show that g is SP and $r_g = O_S^f(\mathbf{P}_{\bar{S}})$. Since $x \in O_S^f(\mathbf{P}_{\bar{S}})$, there exists $\widetilde{\mathbf{P}}_S \in \mathcal{SC}^{|S|}$ such that $g(\widetilde{\mathbf{P}}_S) = f(\widetilde{\mathbf{P}}_S, \mathbf{P}_{\bar{S}}) = x$. Let $S = \{i_1, \ldots, i_{|S|}\}$ and consider the sequence of profiles

$$\begin{aligned}
\mathbf{P}_S^0 &= (P_{i_1}, \ldots, P_{i_{|S|}}), \\
\mathbf{P}_S^1 &= (\widetilde{P}_{i_1}, P_{i_2}, \ldots, P_{i_{|S|}}), \\
&\vdots \\
\mathbf{P}_S^{|S|-1} &= (\widetilde{P}_{i_1}, \ldots, \widetilde{P}_{i_{|S|-1}}, P_{i_{|S|}}), \\
\mathbf{P}_S^{|S|} &= (\widetilde{P}_{i_1}, \ldots, \widetilde{P}_{i_{|S|}}).
\end{aligned}$$

For all $k = 0, 1, \ldots, |S|$, denote $z^k = g(\mathbf{P}_S^k) = f(\mathbf{P}_S^k, \mathbf{P}_{\bar{S}})$. Let $j = \inf\{1, \ldots, |S|\}$ such that $g(\mathbf{P}_S^j) = f(\mathbf{P}_S^j, \mathbf{P}_{\bar{S}}) = x$. Such a j exists because $g(\mathbf{P}_S^{|S|}) = f(\widetilde{\mathbf{P}}_S, \mathbf{P}_{\bar{S}}) = x$. Moreover, $j \neq 0$, because by hypothesis $g(\mathbf{P}_S^0) = f(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) = y \neq x$. But, then agent $i_j \in S$ can manipulate g at $\mathbf{P}_S^{j-1}$ via $\widetilde{P}_{i_j}$, a contradiction. ■

From Lemma 4, we can derive Corollaries 3 and 4, whose proofs follow immediately by setting $S = \{i\}$ and $S = N$, respectively.

Corollary 3 *If $f : \mathcal{SC}^n \to X$ is a strategy-proof social choice rule, then for all $i \in N$, every $x \in r_f$ and all $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ such that $\tau|_{O_i^f(\mathbf{P}_{-i})}(P_i) = x$, $f(P_i, \mathbf{P}_{-i}) = x$.*

Corollary 4 *If $f : \mathcal{SC}^n \to X$ is a strategy-proof social choice rule, then for all $x \in r_f$ and all $(P_i, \mathbf{P}_{-i}) \in \mathcal{SC}^n$ such that $\tau|_{r_f}(P_i) = x$ for all $i \in N$, $f(P_i, \mathbf{P}_{-i}) = x$.*

In words, Corollary 4 points out that a strategy-proof social choice rule must respect unanimity over the range, in the sense that if everyone has the same most preferred alternative on the range of the social choice rule, then that alternative must be the social choice.

Finally, one last result that can be derived from Lemma 4 is the following:

Corollary 5 *If $f : \mathcal{SC}^n \to X$ is a strategy-proof social choice rule, then for any nonempty set $S \subset N$, every $x \in r_f$ and all $(\mathbf{P}_S, \mathbf{P}_{\bar{S}}) \in \mathcal{SC}^n$ such that $\tau|_{r_f}(P_i) = x$ for all $i \in S$, $x \in O^f_{\bar{S}}(\mathbf{P}_S)$.*

The proof of Corollary 5 is immediate from Corollary 4 (just take a preference relation for each individual in the set $\bar{S}$ with the most preferred alternative over the range equal to x). Roughly speaking, it says that if a social choice rule is strategy-proof and all agents in a certain coalition agree on the most preferred alternative over the range of the rule, then that alternative must be available in the option set of the remaining agents.

Our next lemma shows that, when there are only two individuals in the society, a social choice rule is strategy-proof only if it satisfies the tops-only property. This, in turns, implies that every two preferences in the admissible domain with the same most preferred alternative over the range of the social choice rule must necessarily have the same top on any option set generated by the preference of the other individual. This implication is an immediate consequence of Remark 1 and Corollary 3.

Lemma 5 *Let $|N| = 2$ and suppose $\mathcal{SC}$ is a maximal set of single-crossing preferences. A social choice rule $f : \mathcal{SC}^2 \to X$ is strategy-proof only if f is tops-only.*

Proof. Assume, by contradiction, there exists a strategy-proof social choice rule $f : \mathcal{SC}^2 \to X$ which is not tops-only. By Remark 1, there must exist a profile $(P_1, P_2) \in \mathcal{SC}^2$ and a preference $\hat{P}_1 \in \mathcal{SC}$ such that $\tau|_{r_f}(\hat{P}_1) = \tau|_{r_f}(P_1)$ and $f(\hat{P}_1, P_2) = y \neq x = f(P_1, P_2)$. By Corollary 3, $\tau|_{O^f_1(P_2)}(P_1) = x$ and $\tau|_{O^f_1(P_2)}(\hat{P}_1) = y$. Thus, if the tops-only property is contradicted, there must be two preferences P_1 and $\hat{P}_1$ in $\mathcal{SC}$, with the same most preferred alternative on r_f, but with different tops on an option set $O^f_1(P_2)$ generated by the preference of the other agent. The rest of the proof consists in showing that this supposition leads to a contradiction with the fact that f is a strategy-proof social choice rule. A similar argument will be used along the proof of Lemma 7.

Without loss of generality, assume $\tau|_{O^f_1(P_2)}(\hat{P}_1) = y > x = \tau|_{O^f_1(P_2)}(P_1)$. Hence, $\hat{P}_1 \succ P_1$. Obviously, $x\, P_1\, y$, $y\, \hat{P}_1\, x$, $\tau|_{r_f}(P_1) \neq x$ and $\tau|_{r_f}(\hat{P}_1) \neq y$. Furthermore, note that $x\, P_2\, \tau|_{r_f}(P_1)$. Otherwise, by Corollary 4, agent 2 can manipulate f at (P_1, P_2) via a $\tilde{P}_2$ equal to P_1, (which renders $\tau|_{r_f}(P_1)$). Using a similar argument, $y\, P_2\, \tau|_{r_f}(P_1)$.

Two cases are possible, depending on the location of $\tau|_{r_f}(P_1)$: (1) $y > \tau|_{r_f}(P_1) > x$; and (2) $y > x > \tau|_{r_f}(P_1)$.[16] Notice that the first case contradicts that $O^f_1(P_2)$ is an interval of r_f, because by hypothesis $\tau|_{r_f}(P_1) \notin O^f_1(P_2)$. On the other hand, the second case goes against the regularity of $\mathcal{SC}$, since x is between y and $\tau|_{r_f}(P_1)$ and $y\, \hat{P}_1\, x$ and $\tau|_{r_f}(P_1)\, \hat{P}_1\, x$, implying

[16]The remaining situation, where $\tau|_{r_f}(P_1) > y > x$, is similar to the second case.

that no preference in the domain can rank x best. As we explain below, however, both cases are ruled out by strategy-proofness. The reason for this is as follows.

Case 1: $y > \tau|_{r_f}(P_1) > x$. If $P_2 \succ P_1$, we have that $\tau|_{r_f}(P_1) > x$ and $x\, P_2\, \tau|_{r_f}(P_1)$ imply, by $SC2$, $x\, P_1\, \tau|_{r_f}(P_1)$, a contradiction. Thus, $P_1 \succ P_2$. But, then $y > \tau|_{r_f}(P_1)$ and $y\, P_2\, \tau|_{r_f}(P_1)$ imply, by $SC1$, that $y\, P_1\, \tau|_{r_f}(P_1)$, a contradiction.

Case 2: $y > x > \tau|_{r_f}(P_1)$. Firstly, suppose $\tau|_{r_f}(P_2) = x$. Then, $P_2 \succ P_1$. Otherwise, $P_1 \succ P_2$, $x > \tau|_{r_f}(P_1)$ and $x\, P_2\, \tau|_{r_f}(P_1)$ would imply, by $SC1$, $x\, P_1\, \tau|_{r_f}(P_1)$. Similarly, $\hat{P}_1 \succ P_2$, since $x\, P_2\, y$ implies $x\, P\, y\ \forall P \prec P_2$, and $y\, \hat{P}_1\, x$ by hypothesis. But, $\tau|_{r_f}(P_1)\, \hat{P}_1\, x$ implies $\tau|_{r_f}(P_1)\, P\, x\ \forall P \prec \hat{P}_1$, contradicting that $\tau|_{r_f}(P_2) = x$. Hence, $\forall j = 1, 2$, $x \neq \tau|_{r_f}(P_j)$.

Secondly, if $P_1 \succ P_2$, then $\tau|_{r_f}(P_1)\, P_1\, x$ implies $\tau|_{r_f}(P_1)\, P_2\, x$, a contradiction. Thus, $P_2 \succ P_1$ and, therefore, $\tau|_{r_f}(P_2) > \tau|_{r_f}(P_1)$. Similarly, if $\hat{P}_1 \succ P_2$, then $\tau|_{r_f}(P_1)\, \hat{P}_1\, \tau|_{r_f}(P_2)$ implies $\tau|_{r_f}(P_1)\, P_2\, \tau|_{r_f}(P_2)$. So, $P_2 \succ \hat{P}_1$. Moreover, $y\, P_2\, x$, since $x\, P_2\, y$ would imply $x\, \hat{P}_1\, y$. Finally, if $y > \tau|_{r_f}(P_2)$, then $\tau|_{r_f}(P_2)\, P_2\, y$ would imply $\tau|_{r_f}(P_2)\, \hat{P}_1\, y$, contradicting SP, because, by Corollary 4, agent 1 would profitably manipulate f at $(\hat{P}_1, P_2)$ via a $\tilde{P}_1$ equal to P_2. Hence, $\tau|_{r_f}(P_2) \geq y$, and we face a situation as in Figure 5.

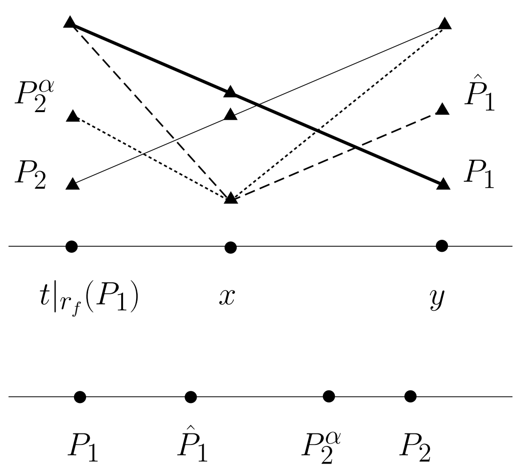


Figure 5: What's going on in Case 2?

Note that $y \notin O_2^f(P_1)$. Otherwise, there must be a $P_2' \in \mathcal{SC}$ such that $f(P_1, P_2') = y$. And, because $y\, P_2\, x$, it would follow that individual 2 can manipulate f at (P_1, P_2) via P_2'.

Step 1. Take a preference $P_2^\alpha \in \mathcal{P}$, between P_2 and $\hat{P}_1$, such that

(i) $\tau|_{O_2^f(\hat{P}_1)}(P_2^\alpha) = y = \tau|_{O_2^f(\hat{P}_1)}(P_2)$, and

(ii) $\tau|_{O_2^f(P_1)}(P_2^\alpha) = \tau|_{r_f}(P_1)$, (see Figure 5).

If $P_2^\alpha \in \mathcal{SC}$, we are done. By Corollary 3, $f(\hat{P}_1, P_2^\alpha) = y$ and $f(P_1, P_2^\alpha) = \tau|_{r_f}(P_1)$. Hence, agent 1 can manipulate f at $(\hat{P}_1, P_2^\alpha)$ via P_1, contradicting that f is SP.

Step 2. On the contrary, if $P_2^\alpha \notin \mathcal{SC}$, then there must exist a preference $P_2^\beta \in \mathcal{SC}$, $P_2^\beta \prec P_2$, and an alternative $z^\beta \in O_2^f(P_1)$ such that

(i) $y > \tau|_{O_2^f(\hat{P}_1)}(P_2^\beta)$, and

(ii) $\tau|_{O_2^f(P_1)}(P_2^\beta) = z^\beta > \tau|_{r_f}(P_1)$, (see Figure 6).

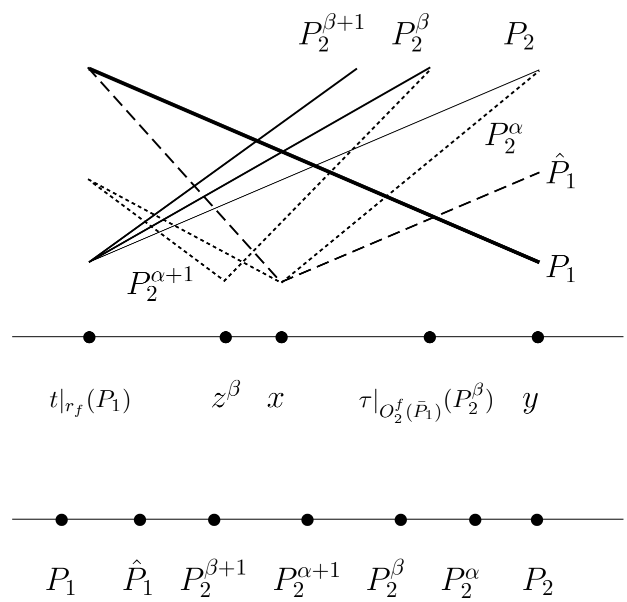


Figure 6: How do P_2^β, $P_2^{\alpha+1}$ and $P_2^{\beta+1}$ look like?

In words, if $P_2^\alpha \notin \mathcal{SC}$, then there must be a P_2^β in $\mathcal{SC}$ such that (i) P_2^β is more leftist than P_2^α regarding the top on $O_2^f(\hat{P}_1)$; and (ii) P_2^β is more rightist than P_2^α with respect to the top on $O_2^f(P_1)$.[17]

Step 3. By Corollary 3, $f(\hat{P}_1, P_2^\beta) = \tau|_{O_2^f(\hat{P}_1)}(P_2^\beta)$ and $f(P_1, P_2^\beta) = z^\beta > \tau|_{r_f}(P_1)$. Hence, $\tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) \neq \tau|_{r_f}(P_1)$. Otherwise, if $\tau|_{O_2^f(\hat{P}_1)}(P_2^\beta) = \tau|_{r_f}(P_1)$, then agent 1 can manipulate f at (P_1, P_2^β) via $\hat{P}_1$. Furthermore, if $z^\beta > x$, then $P_2 \succ P_2^\beta$ and $z^\beta \, P_2^\beta \, x$ imply $z^\beta \, P_2 \, x$, contradicting that $x = \tau|_{O_2^f(P_1)}(P_2)$. Therefore, $x \geq z^\beta > \tau|_{r_f}(P_1)$.

Step 4. Proceeding like in Step 1, take a preference $P_2^{\alpha+1} \in \mathcal{P}$ such that

[17]The other possibility that can stop P_2^α to be in $\mathcal{SC}$ is a preference with a top lower than $\tau|_{r_f}(P_1)$ on $O_2^f(P_1)$ and a top greater than y on $O_2^f(\hat{P}_1)$. However, such order is ruled out by the existence of P_1 in $\mathcal{SC}$.

(i) $\tau|_{O_2^f(\hat{P}_1)}(P_2^{\alpha+1}) = \tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta})$, and

(ii) $\tau|_{O_2^f(P_1)}(P_2^{\alpha+1}) = \tau|_{r_f}(P_1)$, (see Figure 6).

If $P_2^{\alpha+1} \in \mathcal{SC}$, we are done. By Corollary 3, we have that $f(P_1, P_2^{\alpha+1}) = \tau|_{r_f}(P_1)$ and $f(\hat{P}_1, P_2^{\alpha+1}) = \tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta}) \neq \tau|_{r_f}(P_1)$. Hence, agent 1 can manipulate f at $(\hat{P}_1, P_2^{\alpha+1})$ via P_1, contradicting that f is SP.

On the contrary, if $P_2^{\alpha+1} \notin \mathcal{SC}$, then repeating the reasoning in Step 2, there must exist a preference $P_2^{\beta+1} \in \mathcal{SC}$, $P_2^{\beta+1} \prec P_2^{\beta}$, and an alternative $z^{\beta+1} \in O_2^f(P_1)$ such that

(i) $\tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta}) > \tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta+1})$, and

(ii) $\tau|_{O_2^f(P_1)}(P_2^{\beta+1}) = z^{\beta+1} > \tau|_{r_f}(P_1)$, (see Figure 6).

Using the argument of Step 3, $z^{\beta} \geq z^{\beta+1} > \tau|_{r_f}(P_1)$. Going back to Step 1 and repeating the analysis over and over again, then in the end either, we get the desired contradiction at some point in the process, or after a number of repetitions, say ℓ, we eventually arrive at a preference $P_2^{\beta+\ell} \in \mathcal{SC}$, between $P_2^{\beta+\ell-1}$ and $\hat{P}_1$, such that (i) $\tau|_{O_2^f(\hat{P}_1)}(P_2^{\beta+\ell}) = \tau|_{r_f}(P_1)$; and (ii) $\tau|_{O_2^f(P_1)}(P_2^{\beta+\ell}) = z^{\beta+\ell} > \tau|_{r_f}(P_1)$. By Corollary 3, $f(\hat{P}_1, P_2^{\beta+\ell}) = \tau|_{r_f}(P_1)$ and $f(P_1, P_2^{\beta+\ell}) = z^{\beta+\ell} \neq \tau|_{r_f}(P_1)$. Hence, agent 1 can manipulate f at $(P_1, P_2^{\beta+\ell})$ via $\hat{P}_1$, a contradiction. Therefore, f is TO on $\mathcal{SC}^2$. ■

Corollary 6 *Let $|N| = 2$ and suppose $\mathcal{SC}$ is a maximal set of single-crossing preferences. A social choice rule $f : \mathcal{SC}^2 \to X$ is strategy-proof only if for all $i \in N$ and all $\mathbf{P} \in \mathcal{SC}^2$, $f(P_i, \mathbf{P}_{-i}) = m^3(\tau|_{r_f}(P_i), f(\underline{P}_i, \mathbf{P}_{-i}), f(\overline{P}_i, \mathbf{P}_{-i}))$.*

Proof. Immediate from Lemmas 1 and 5. ■

Now, before generalizing Lemma 5 to the case where $|N| > 2$, we first extend the tops-only property to the option sets generated by a strategy-proof social choice rule. We do this in two steps. First, we prove in Lemma 6 that the option set of any single individual $i \in N$ satisfies a tops-only property when there is agreement among the individuals in $N\backslash\{i\}$ as to which alternative is best on the range. Then, in Lemma 7, we generalize this result to the option set of any nonempty coalition of individuals and when the remaining agents do not necessarily agree on the most preferred alternative over the range.

Lemma 6 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathcal{SC}^n \to X$ is strategy-proof, then for every individual $i \in N$ and every two profiles $\mathbf{P}'_{-i}, \mathbf{P}''_{-i} \in \mathcal{SC}^{n-1}$ for which $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_k)$ for all $j, k \in N\backslash\{i\}$, $O_i^f(\mathbf{P}'_{-i}) = O_i^f(\mathbf{P}''_{-i})$.*

Proof. Consider any $i \in N$ and any two profiles $\mathbf{P}'_{-i}, \mathbf{P}''_{-i} \in \mathcal{SC}^{n-1}$,

$$\begin{aligned} \mathbf{P}'_{-i} &= (P'_1, \ldots, P'_{i-1}, P'_{i+1}, \ldots, P'_n), \text{ and} \\ \mathbf{P}''_{-i} &= (P''_1, \ldots, P''_{i-1}, P''_{i+1}, \ldots, P''_n), \end{aligned}$$

such that, for all $j, k \in N \backslash \{i\}$, $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_k) = z$ for some $z \in X$. To simplify the notation, assume $P'_j = P'_k$ and $P''_j = P''_k$ for all $j, k \in N \backslash \{i\}$, so that we can write

$$\begin{aligned} \mathbf{P}'_{-i} &= (\underbrace{P', \ldots, P'}_{n-1 \ times}), \text{ and} \\ \mathbf{P}''_{-i} &= (\underbrace{P'', \ldots, P''}_{n-1 \ times}). \end{aligned}$$

We want to show that $O_i^f(\mathbf{P}'_{-i}) = O_i^f(\mathbf{P}''_{-i})$. To do that, define the sequence

$$\begin{aligned} \mathbf{P}^0_{-i} &= (P', \ldots, P'), \\ \mathbf{P}^1_{-i} &= (P'', P', \ldots, P'), \\ \mathbf{P}^2_{-i} &= (P'', P'', P', \ldots, P'), \\ &\vdots \qquad\qquad \vdots \\ \mathbf{P}^{n-1}_{-i} &= (P'', \ldots, P''). \end{aligned}$$

To establish the result, it is enough to prove that, for all $j = 1, \ldots, n-1$, $O_i^f(\mathbf{P}^j_{-i}) = O_i^f(\mathbf{P}^{j-1}_{-i})$. Assume, by contradiction, there exists $x \in X$ such that for some $1 \leq j^* \leq n-1$, $x \in O_i^f(\mathbf{P}^{j^*-1}_{-i})$ and $x \notin O_i^f(\mathbf{P}^{j^*}_{-i})$. By Corollary 5, $z \in O_i^f(\mathbf{P}^{j^*-1}_{-i}) \cap O_i^f(\mathbf{P}^{j^*}_{-i})$. Therefore $z \neq x$. Moreover, since $x \in O_i^f(\mathbf{P}^{j^*-1}_{-i})$, there exists $\widetilde{P}_i \in \mathcal{SC}$ such that $f(\widetilde{P}_i, \mathbf{P}^{j^*-1}_{-i}) = x$.

Notice that the preference profiles $\mathbf{P}^{j^*-1}_{-i}$ and $\mathbf{P}^{j^*}_{-i}$ differ only in one preference relation. Without loss of generality, suppose it is the preference of agent $\ell \in N \backslash \{i\}$:

$$\begin{aligned} \mathbf{P}^{j^*-1}_{-i} &= \big(\underbrace{P'', \ldots, P''}_{j^*-1}, P'_\ell, \underbrace{P', \ldots, P'}_{n-j^*-1}\big), \\ \mathbf{P}^{j^*}_{-i} &= \big(\underbrace{P'', \ldots, P''}_{j^*-1}, P''_\ell, \underbrace{P', \ldots, P'}_{n-j^*-1}\big). \end{aligned}$$

Fix $\mathbf{P}^{j^*-1}_{-\{i,\ell\}} = \big(\underbrace{P'', \ldots, P''}_{j^*-1}, \underbrace{P', \ldots, P'}_{n-j^*-1}\big) = \mathbf{P}^{j^*}_{-\{i,\ell\}}$. Define the two-person social choice rule $g : \mathcal{SC}^2 \to X$ in such a way that, for all $(P_i, P_\ell) \in \mathcal{SC}^2$, $g(P_i, P_\ell) = f(P_i, P_\ell, \mathbf{P}^{j^*-1}_{-\{i,\ell\}})$. It is easy to show that g is strategy-proof and $r_g = O^f_{\{i,\ell\}}(\mathbf{P}^{j^*-1}_{-\{i,\ell\}})$. By Corollary 5, $z \in r_g$.

Hence, $\tau|_{r_g}(P''_\ell) = \tau|_{r_g}(P'_\ell) = z$, because by hypothesis $\tau|_{r_f}(P''_\ell) = \tau|_{r_f}(P'_\ell) = z$ and $r_g \subseteq r_f$. By Lemma 5, g is tops-only. Therefore, $g(\widetilde{P}_i, P''_\ell) = g(\widetilde{P}_i, P'_\ell)$. By definition of g, $g(\widetilde{P}_i, P'_\ell) = f(\widetilde{P}_i, \mathbf{P}^{j^*-1}_{-i}) = x$. Thus, $g(\widetilde{P}_i, P''_\ell) = x$. That is, $f(\widetilde{P}_i, \mathbf{P}^{j^*}_{-i}) = x$. Therefore, $x \in O^f_i(\mathbf{P}^{j^*}_{-i})$, a contradiction. Hence, $O^f_i(\mathbf{P}'_{-i}) = O^f_i(\mathbf{P}''_{-i})$. ■

Lemma 7 *Let $\mathcal{SC}$ be a maximal set of single-crossing preferences. If a social choice rule $f : \mathcal{SC}^n \to X$ is strategy-proof, then for every coalition $S \subset N$ and every two profiles $\mathbf{P}', \mathbf{P}'' \in \mathcal{SC}^n$ such that $\tau|_{r_f}(P'_i) = \tau|_{r_f}(P''_i)$ for all $i \in S$, $O^f_{\bar{S}}(\mathbf{P}'_S) = O^f_{\bar{S}}(\mathbf{P}''_S)$.*

Proof. We make the proof in four steps, though the crucial one is Step 1:

Step 1. Consider any individual $i \in N$ and any two profiles $\hat{\mathbf{P}}_{-i}, \check{\mathbf{P}}_{-i} \in \mathcal{SC}^{n-1}$, such that for all $j, k \in N\backslash\{i\}$, $\hat{P}_j = \hat{P}_k$ and $\check{P}_j = \check{P}_k$, and for each $j \in N\backslash\{i\}$, $\tau|_{r_f}(\hat{P}_j) = \tau|_{r_f}(\check{P}_j) = z$ for some $z \in X$. By Lemma 6, $O^f_i(\hat{\mathbf{P}}_{-i}) = O^f_i(\check{\mathbf{P}}_{-i})$.

Fix any individual $j \neq i$ and any two preferences $P'_j, P''_j \in \mathcal{SC}$, such that $\tau|_{r_f}(P'_j) = w = \tau|_{r_f}(P''_j)$ for some $w \in X$. We want to show that $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \hat{\mathbf{P}}_{-\{i,j\}})$. Define the 2-person social choice rule $g : \mathcal{SC}^2 \to X$ in such a way that for all $(P_i, P_j) \in \mathcal{SC}^2$, $g(P_i, P_j) = f(P_i, P_j, \hat{\mathbf{P}}_{-\{i,j\}})$. Since f is SP on $\mathcal{SC}^n$, g is SP on $\mathcal{SC}^2$, with range $r_g = O^f_{\{i,j\}}(\hat{\mathbf{P}}_{-\{i,j\}})$. By Lemma 5, g is TO over r_g. If $\tau|_{r_g}(P'_j) = \tau|_{r_g}(P''_j)$, then by applying Lemma 6 to g we get $O^g_i(P'_j) = O^g_i(P''_j)$. Hence, by definition, $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \hat{\mathbf{P}}_{-\{i,j\}})$.

Instead, if $\tau|_{r_g}(P'_j) = a \neq b = \tau|_{r_g}(P''_j)$, then Lemma 6 cannot be used, because it rests on the existence of a common peak on the range of the social choice rule. So, we proceed as follows. Without loss of generality, let $b > a$, implying that $P''_j \succ P'_j$. Assume, by contradiction, $O^g_i(P'_j) \neq O^g_i(P''_j)$. That is, suppose there is $\alpha \in r_g$ such that $\alpha \in O^g_i(P'_j)$ and $\alpha \notin O^g_i(P''_j)$. Hence, there must be a $\widetilde{P}_i \in \mathcal{SC}$ such that $g(\widetilde{P}_i, P'_j) = \alpha$. Since $\alpha \notin O^g_i(P''_j)$, let $g(\widetilde{P}_i, P''_j) = \beta \neq \alpha$. By SP, $\alpha\, P'_j\, \beta$ and $\beta\, P''_j\, \alpha$. By single-crossing, $P''_j \succ P'_j$ implies $\beta > \alpha$.

We would like to find two preferences $P^\alpha_i, P^\beta_i \in \mathcal{SC}$, not necessarily different, such that: (i) $\tau|_{r_f}(P^\alpha_i) = w$; (ii) $g(P^\alpha_i, P'_j) = \alpha$; (iii) $\tau|_{r_f}(P^\beta_i) = w$; and (iv) $g(P^\beta_i, P''_j) = \beta$. We show below that such preferences exist in $\mathcal{SC}$. First, note that if $\widetilde{P}_i$ is between P'_j and P''_j, then we already have the desired preferences, because in that case $\tau|_{r_f}(\widetilde{P}_i) = w$. So, without loss of generality, suppose $\widetilde{P}_i \succ P''_j$, implying that $\tau|_{r_g}(\widetilde{P}_i) = c \geq b$. Clearly, $\beta\, \widetilde{P}_i\, \alpha$, because $\beta\, P''_j\, \alpha$. Therefore, $\tau|_{r_g}(\widetilde{P}_i) = c \neq \alpha$. By Corollary 6,

$$g(\widetilde{P}_i, P'_j) = m^3(\tau|_{r_g}(\widetilde{P}_i), g(\underline{P}_i, P'_j), g(\overline{P}_i, P'_j)), \tag{2}$$

where $\overline{P}_i$ (respectively, $\underline{P}_i$) represents the most rightist (respectively, leftist) ranking on X. Applying Corollary 6 once again to $g(\underline{P}_i, P'_j)$ and to $g(\overline{P}_i, P'_j)$, we get

$$g(\underline{P}_i, P'_j) = m^3(\tau|_{r_g}(P'_j), g(\underline{P}_i, \underline{P}_j), g(\underline{P}_i, \overline{P}_j)), \quad \text{and} \tag{3}$$

$$g(\overline{P}_i, P'_j) = m^3(\tau|_{r_g}(P'_j), g(\overline{P}_i, \underline{P}_j), g(\overline{P}_i, \overline{P}_j)). \tag{4}$$

By Corollary 4, $g(\underline{P}_i, \underline{P}_j) = \underline{X}_{r_g}$ and $g(\overline{P}_i, \overline{P}_j) = \overline{X}_{r_g}$, where $\underline{X}_{r_g} = \min(r_g)$ and $\overline{X}_{r_g} = \max(r_g)$. Therefore, (3) can be rewritten as $g(\underline{P}_i, P'_j) = m^3(a, \underline{X}_{r_g}, g(\underline{P}_i, \overline{P}_j))$, while (4) becomes $g(\overline{P}_i, P'_j) = m^3(a, g(\overline{P}_i, \underline{P}_j), \overline{X}_{r_g})$. It is immediate to see that $g(\overline{P}_i, P'_j) \geq g(\underline{P}_i, P'_j)$, because (4) is at least a, while (3) is at most a. Hence, $c > g(\underline{P}_i, P'_j)$, because $c \geq b > a$. Moreover, $\tau|_{r_g}(\widetilde{P}_i) = c$ cannot be between $g(\overline{P}_i, P'_j)$ and $g(\underline{P}_i, P'_j)$. Otherwise, (2) would imply that $g(\widetilde{P}_i, P'_j) = c$, contradicting the initial hypothesis that $g(\widetilde{P}_i, P'_j) = \alpha$ (recall that $c \neq \alpha$, because $\beta\, \widetilde{P}_i\, \alpha$). Thus, $c > g(\overline{P}_i, P'_j) \geq g(\underline{P}_i, P'_j)$ and, therefore, $\alpha = g(\overline{P}_i, P'_j)$.

Take a preference P_i^α equal to P''_j. By Corollary 6, $g(P_i^\alpha, P'_j) = m^3(b, g(\underline{P}_i, P'_j), g(\overline{P}_i, P'_j))$. Thus, if $b \geq g(\overline{P}_i, P'_j)$, we are done, since $g(P_i^\alpha, P'_j) = \alpha$ and, by definition, $\tau|_{r_f}(P_i^\alpha) = w$. Instead, if $g(\overline{P}_i, P'_j) > b$, then $\alpha > b$. Moreover, $\alpha > a$, because $b > a$. And, by (4), $\alpha = m^3(a, g(\overline{P}_i, \underline{P}_j), \overline{X}_{r_g}) = g(\overline{P}_i, \underline{P}_j)$. Consider $g(\widetilde{P}_i, P''_j)$. By Corollary 6, $g(\widetilde{P}_i, P''_j) = m^3(b, g(\widetilde{P}_i, \underline{P}_j), g(\widetilde{P}_i, \overline{P}_j))$, where

$$g(\widetilde{P}_i, \underline{P}_j) = m^3(c, \underline{X}_{r_g}, g(\overline{P}_i, \underline{P}_j)), \quad \text{and} \tag{5}$$

$$g(\widetilde{P}_i, \overline{P}_j) = m^3(c, g(\underline{P}_i, \overline{P}_j), \overline{X}_{r_g}). \tag{6}$$

Note that (5) can be rewritten as $g(\widetilde{P}_i, \underline{P}_j) = m^3(c, \underline{X}_{r_g}, \alpha) = \alpha$, because $c > \alpha$ (recall that $c > g(\overline{P}_i, P'_j) = \alpha$). Moreover, since (6) is at least equal to c and we have assumed above that $\alpha > b$, it follows that $g(\widetilde{P}_i, P''_j) = m^3(b, \alpha, g(\widetilde{P}_i, \overline{P}_j)) = \alpha$, contradicting that, by hypothesis, $g(\widetilde{P}_i, P''_j) = \beta \neq \alpha$.

Therefore, the previous argument shows that a preference P_i^α with the properties specified above exists in $\mathcal{SC}$. In fact, it says that P_i^α can be set equal to P''_j. Following a similar reasoning, we prove next that P_i^β also exists in $\mathcal{SC}$ and is equal to P''_j. To do that, note that by Corollary 6,

$$g(\widetilde{P}_i, P''_j) = m^3(c, g(\underline{P}_i, P''_j), g(\overline{P}_i, P''_j)), \tag{7}$$

where $g(\underline{P}_i, P''_j) = m^3(b, \underline{X}_{r_g}, g(\underline{P}_i, \overline{P}_j)$ is at most equal to b, and $g(\overline{P}_i, P''_j) = m^3(b, \overline{X}_{r_g}, g(\overline{P}_i, \underline{P}_j))$ is at least equal to b. Therefore, $g(\overline{P}_i, P''_j) \geq b \geq g(\underline{P}_i, P''_j)$.

Since $c \geq b$, we have two possibilities. First, suppose $g(\overline{P}_i, P''_j) > c \geq b \geq g(\underline{P}_i, P''_j)$. Hence, $g(\overline{P}_i, P''_j) = g(\overline{P}_i, \underline{P}_j)$. Consider $g(\widetilde{P}_i, P'_j) = m^3(c, g(\underline{P}_i, P'_j), g(\overline{P}_i, P'_j))$. Note that $g(\underline{P}_i, P''_j) \geq g(\underline{P}_i, P'_j)$, because $g(\underline{P}_i, P''_j) = m^3(b, \underline{X}_{r_g}, g(\underline{P}_i, \overline{P}_j))$ and $g(\underline{P}_i, P'_j) = m^3(a, \underline{X}_{r_g}, g(\underline{P}_i, \overline{P}_j))$. On the other hand, $g(\overline{P}_i, P'_j) = m^3(a, \overline{X}_{r_g}, g(\overline{P}_i, \underline{P}_j))$. However, we have shown before that $g(\overline{P}_i, \underline{P}_j) > b$, because $g(\overline{P}_i, \underline{P}_j) = g(\overline{P}_i, P''_j) > c \geq b$. Therefore, $g(\overline{P}_i, P'_j) = g(\overline{P}_i, \underline{P}_j)$. To summarize, we have that $g(\overline{P}_i, P'_j) = g(\overline{P}_i, P''_j) > c \geq g(\underline{P}_i, P''_j) \geq g(\underline{P}_i, P'_j)$. Therefore, by (2) and (7), $g(\widetilde{P}_i, P'_j) = c = g(\widetilde{P}_i, P''_j)$, a contradiction.

Thus, $c \geq g(\overline{P}_i, P''_j) \geq g(\underline{P}_i, P''_j)$ and, by (7), $g(\widetilde{P}_i, P''_j) = g(\overline{P}_i, P''_j)$. If $g(\overline{P}_i, P''_j) = g(\underline{P}_i, P''_j)$, then the desired result is obtained by setting P_i^β equal to P''_j because, by definition of P_i^β, $\tau|_{r_f}(P_i^\beta) = w$ and, by Corollary 6, $g(P_i^\beta, P''_j) = m^3(b, g(\underline{P}_i, P''_j), g(\overline{P}_i, P''_j)) = g(\overline{P}_i, P''_j)$. So, suppose $g(\overline{P}_i, P''_j) > g(\underline{P}_i, P''_j)$.

If $b > g(\overline{P}_i, P''_j)$, then by choosing P_i^β equal to P''_j we have that $g(P_i^\beta, P''_j) = m^3(b, g(\underline{P}_i, P''_j), g(\overline{P}_i, P''_j)) = g(\overline{P}_i, P''_j) \neq b$, contradicting Corollary 4. Hence, $g(\overline{P}_i, P''_j) = m^3(b, g(\overline{P}_i, \underline{P}_j), \overline{X}_{r_g}) \geq b$. Assume, by contradiction, $g(\overline{P}_i, P''_j) > b$. Then, $g(\overline{P}_i, P''_j) = g(\overline{P}_i, \underline{P}_j)$. Moreover, $g(\overline{P}_i, \underline{P}_j) = \beta$, because $g(\overline{P}_i, P''_j) = g(\widetilde{P}_i, P''_j)$ and $g(\widetilde{P}_i, P''_j) = \beta$. Hence, $\beta > b$ and $g(\overline{P}_i, P'_j) = m^3(a, g(\overline{P}_i, \underline{P}_j), \overline{X}_{r_g}) = m^3(a, \beta, \overline{X}_{r_g}) = \beta$. However, since $g(\underline{P}_i, P'_j) = m^3(a, g(\underline{P}_i, \overline{P}_j), \underline{X}_{r_g})$ is at most a, it follows that $g(\widetilde{P}_i, P'_j) = m^3(c, \beta, g(\underline{P}_i, P'_j))$ is either c or β, contradicting that by hypothesis $g(\widetilde{P}_i, P'_j) = \alpha$ and $\alpha \neq \beta$ and $c \neq \alpha$.

Therefore, it must be that $g(\overline{P}_i, P''_j) = b$, implying that $g(\widetilde{P}_i, P''_j) = m^3(c, b, g(\underline{P}_i, P''_j)) = b$. But, then the desired result is obtained by setting P_i^β equal to P''_j.

Now, to complete the analysis, we proceed as in the proof of Lemma 5. First, recall that $w \neq \alpha$ and $w \neq \beta$, because $w \notin r_g$. Otherwise, we would have had $\tau|_{r_g}(P'_j) = \tau|_{r_g}(P''_j)$. Moreover, by Corollary 5, $w \neq z$, because $z \in r_g$.

Case 1. Suppose $\beta > w > \alpha$. If $\hat{P} \succ P'_j$, then $w \, P'_j \, \alpha$ implies $w \, \hat{P} \, \alpha$ (recall $\hat{P}$ is the common preference of the profile $\hat{\mathbf{P}}_{-\{i,j\}}$). Define the sequence of profiles $\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \ldots, \hat{P})$, $\mathbf{P}^1_{-\{i,j\}} = (P^\alpha, \hat{P}, \ldots, \hat{P}), \ldots, \mathbf{P}^{n-2}_{-\{i,j\}} = (P^\alpha, \ldots, P^\alpha)$. For each $k = 0, \ldots, n-2$, let $x^k = f(P_i^\alpha, P'_j, \mathbf{P}^k_{-\{i,j\}})$. By strategy-proofness, $\alpha = x^0 \, \hat{P} \, x^1 \, \hat{P} \, \ldots, \hat{P} \, x^{n-2} = w$. Therefore, by transitivity, $\alpha \, \hat{P} \, w$, a contradiction.

In a similar way, if $P'_j \succ \hat{P}$, then $w \, P'_j \, \beta$ implies $w \, \hat{P} \, \beta$. Define the sequence of profiles $\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \ldots, \hat{P})$, $\mathbf{P}^1_{-\{i,j\}} = (P^\beta, \hat{P}, \ldots, \hat{P}), \ldots, \mathbf{P}^{n-2}_{-\{i,j\}} = (P^\beta, \ldots, P^\beta)$. For each $k = 0, \ldots, n-2$, let $y^k = f(P_i^\beta, P''_j, \mathbf{P}^k_{-\{i,j\}})$. By strategy-proofness, $\beta = y^0 \, \hat{P} \, y^1 \, \hat{P} \, \ldots, \hat{P} \, y^{n-2} = w$. Therefore, $\beta \, \hat{P} \, w$, a contradiction.

Case 2. Finally, suppose $\beta > \alpha > w$. The remaining case, where $w > \beta > \alpha$, is similar. If $P''_j \succ \hat{P}$, then $w \, \hat{P} \, \alpha$ and we can use the same type of argument than in Case 1. Hence, assume $\hat{P} \succ P''_j$. By SP, $\beta \, P''_j \, \alpha$. By $SC1$, $\beta \, \hat{P} \, \alpha$. If $\beta \in O^f_{-\{i,j\}}(P_i^\alpha, P'_j)$, then there must be a $\bar{\bar{\mathbf{P}}}_{-\{i,j\}} \in \mathcal{SC}^{n-2}$ such that $f(P_i^\alpha, P'_j, \bar{\bar{\mathbf{P}}}_{-\{i,j\}}) = \beta$. Define the sequence of profiles $\mathbf{P}^0_{-\{i,j\}} = (\hat{P}, \ldots, \hat{P})$, $\mathbf{P}^1_{-\{i,j\}} = (\bar{\bar{P}}_{\ell_1}, \hat{P}, \ldots, \hat{P}), \ldots, \mathbf{P}^{n-2}_{-\{i,j\}} = (\bar{\bar{P}}_{\ell_1}, \ldots, \bar{\bar{P}}_{\ell_{n-2}})$. For each $k = 0, \ldots, n-2$, let $x^k = f(P_i^\alpha, P'_j, \mathbf{P}^k_{-\{i,j\}})$. By strategy-proofness, $\alpha = x^0 \, \hat{P} \, x^1 \, \hat{P} \, \ldots \, \hat{P} \, x^{n-2} = \beta$. By transitivity, $\alpha \, \hat{P} \, \beta$, a contradiction. Therefore, $\beta \notin O^f_{-\{i,j\}}(P_i^\alpha, P'_j)$.

By Corollary 5, $w \in O^f_{-\{i,j\}}(P^\alpha_i, P'_j) \cap O^f_{-\{i,j\}}(P^\alpha_i, P''_j)$. Moreover, $\beta \in O^f_{-\{i,j\}}(P^\alpha_i, P''_j)$, because $f(P^\alpha_i, P''_j, \hat{\mathbf{P}}_{-\{i,j\}}) = \beta$. Consider a preference $P^\epsilon \in \mathcal{P}$, between P''_j and $\hat{P}$, such that

(i) $\tau|_{O^f_{-\{i,j\}}(P^\alpha_i, P''_j)}(P^\epsilon) = \beta = \tau|_{O^f_{-\{i,j\}}(P^\alpha_i, P''_j)}(\hat{P})$, and

(ii) $\tau|_{O^f_{-\{i,j\}}(P^\alpha_i, P'_j)}(P^\epsilon) = w = \tau|_{O^f_{-\{i,j\}}(P^\alpha_i, P'_j)}(P''_j)$.

If $P^\epsilon \in \mathcal{SC}$, we are done. By Lemma 4, $f(P^\alpha_i, P''_j, \mathbf{P}^\epsilon_{-\{i,j\}}) = \beta$ and $f(P^\alpha_i, P'_j, \mathbf{P}^\epsilon_{-\{i,j\}}) = w$. Therefore, agent j manipulates f at $(P^\alpha_i, P''_j, \mathbf{P}^\epsilon_{-\{i,j\}})$ via P'_j. On the contrary, if $P^\epsilon \notin \mathcal{SC}$, then the desired contradiction is found following the same argument applied in Steps 2 to 4 in the second case of the proof of Lemma 5.

Hence, by Case 1 and 2, we conclude that $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \hat{\mathbf{P}}_{-\{i,j\}})$. Applying a similar reasoning, we also have that $O^f_i(P'_j, \check{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \check{\mathbf{P}}_{-\{i,j\}})$.

Step 2. Next we prove that $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \check{\mathbf{P}}_{-\{i,j\}})$. From Step 1, we know that $O^f_i(P''_j, \check{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P'_j, \check{\mathbf{P}}_{-\{i,j\}})$. Hence, it is enough to show that $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P'_j, \check{\mathbf{P}}_{-\{i,j\}})$. Define the sequence of profiles

$$\begin{array}{rcl} \mathbf{P}^0_{-\{i,j\}} & = & (\hat{P}, \ldots, \hat{P}), \\ \mathbf{P}^1_{-\{i,j\}} & = & (\check{P}, \hat{P}, \ldots, \hat{P}), \\ \vdots & & \vdots \\ \mathbf{P}^{n-2}_{-\{i,j\}} & = & (\check{P}, \ldots, \check{P}). \end{array}$$

To show that $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P'_j, \check{\mathbf{P}}_{-\{i,j\}})$, it is enough to prove that for all $k = 1, \ldots, n-2$, $O^f_i(P'_j, \mathbf{P}^{k-1}_{-\{i,j\}}) = O^f_i(P'_j, \mathbf{P}^{k}_{-\{i,j\}})$. Suppose, by contradiction, there exists $1 \leq k^* \leq n-2$ such that

$$O^f_i(P'_j, \mathbf{P}^{k^*-1}_{-\{i,j\}}) \neq O^f_i(P'_j, \mathbf{P}^{k^*}_{-\{i,j\}}). \tag{8}$$

Recall that

$$\begin{array}{rcl} \mathbf{P}^{k^*-1}_{-\{i,j\}} & = & \big(\underbrace{\check{P}, \ldots, \check{P}}_{k^*-1}, \underbrace{\hat{P}, \ldots, \hat{P}}_{n-k^*-1}\big), \quad \text{and} \\ \mathbf{P}^{k^*}_{-\{i,j\}} & = & \big(\underbrace{\check{P}, \ldots, \check{P}, \check{P}}_{k^*}, \underbrace{\hat{P}, \ldots, \hat{P}}_{n-k^*-2}\big). \end{array}$$

That is, profiles $\mathbf{P}^{k^*-1}_{-\{i,j\}}$ and $\mathbf{P}^{k^*}_{-\{i,j\}}$ differ only in one preference relation. Abusing a bit the notation, assume this ordering corresponds to agent k^*. Then, (8) can be rewritten as

$$O^f_i(P'_j, \hat{P}_{k^*}, \mathbf{P}^{k^*-1}_{-\{i,j,k^*\}}) \neq O^f_i(P'_j, \check{P}_{k^*}, \mathbf{P}^{k^*}_{-\{i,j,k^*\}}). \tag{9}$$

However, (9) contradicts Step 1, because $(P'_j, \mathbf{P}^{k^*-1}_{-\{i,j,k^*\}})$ is equal to $(P'_j, \mathbf{P}^{k^*}_{-\{i,j,k^*\}})$, and $\tau|_{r_f}(\check{P}_{k^*}) = \tau|_{r_f}(\hat{P}_{k^*}) = z$. Therefore, $O^f_i(P'_j, \hat{\mathbf{P}}_{-\{i,j\}}) = O^f_i(P''_j, \check{\mathbf{P}}_{-\{i,j\}})$.

Step 3. Suppose that for some $K \subset N\backslash\{i\}$ and some $\mathbf{P}'_K$, $\mathbf{P}''_K \in \mathcal{SC}^K$ with the property that $\forall j \in K$, $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_j)$,

$$O^f_i(\mathbf{P}'_K, \hat{\mathbf{P}}_{\bar{K}\backslash\{i\}}) = O^f_i(\mathbf{P}''_K, \check{\mathbf{P}}_{\bar{K}\backslash\{i\}}). \tag{10}$$

Notice that Step 2 deals with the particular case where $K = \{j\}$. Fix any $k \in \bar{K}\backslash\{i\}$ and two preferences P'_k, $P''_k \in \mathcal{SC}$ for which $\tau|_{r_f}(P'_k) = \tau|_{r_f}(P''_k)$. We want to show that

$$O^f_i(\mathbf{P}'_{K\cup\{k\}}, \hat{\mathbf{P}}_{\bar{K}\backslash\{i,k\}}) = O^f_i(\mathbf{P}''_{K\cup\{k\}}, \check{\mathbf{P}}_{\bar{K}\backslash\{i,k\}}). \tag{11}$$

This is equivalent to prove that $O^f_i(\mathbf{P}'_{K\cup\{k\}}, \hat{\mathbf{P}}_{\bar{K}\backslash\{i,k\}}) = O^f_i(\mathbf{P}''_{K\cup\{k\}}, \hat{\mathbf{P}}_{\bar{K}\backslash\{i,k\}})$. Define the $(|K|+2)$-person social choice rule $g : \mathcal{SC}^{|K|+2} \to X$, such that for all $(P_i, \mathbf{P}_{K\cup\{k\}}) \in \mathcal{SC}^{|K|+2}$, $g(P_i, \mathbf{P}_{K\cup\{k\}}) = f(P_i, \mathbf{P}_{K\cup\{k\}}, \hat{\mathbf{P}}_{\bar{K}\backslash\{i,k\}})$. By the argument in Step 2, $O^g_i(\mathbf{P}'_K, P'_k) = O^g_i(\mathbf{P}''_K, P''_k)$. Hence, $O^f_i(\mathbf{P}'_{K\cup\{k\}}, \hat{\mathbf{P}}_{\bar{K}\backslash\{i,k\}}) = O^f_i(\mathbf{P}''_{K\cup\{k\}}, \hat{\mathbf{P}}_{\bar{K}\backslash\{i,k\}})$. In particular, since this is true for any $K \subset N\backslash\{i\}$, we have that $O^f_i(P'_1, \ldots, P'_{i-1}, P'_{i+1}, \ldots, P'_n) = O^f_i(P''_1, \ldots, P''_{i-1}, P''_{i+1}, \ldots, P''_n)$ or, more compactly, $O^f_i(\mathbf{P}'_{-i}) = O^f_i(\mathbf{P}''_{-i})$.

Step 4. Finally, suppose that for some $S \subset N$ and some $\mathbf{P}'_{\bar{S}}, \mathbf{P}''_{\bar{S}} \in \mathcal{SC}^{|\bar{S}|}$ with the property that $\forall j \in \bar{S}$, $\tau|_{r_f}(P'_j) = \tau|_{r_f}(P''_j)$, we have $O^f_S(\mathbf{P}'_{\bar{S}}) = O^f_S(\mathbf{P}''_{\bar{S}})$.[18] Fix any $h \in \bar{S}$. We want to show that $O^f_{S\cup\{h\}}(\mathbf{P}'_{\bar{S}\backslash\{h\}}) = O^f_{S\cup\{h\}}(\mathbf{P}''_{\bar{S}\backslash\{h\}})$. Suppose not. Without loss of generality, assume there exists $x \in X$ such that $x \in O^f_{S\cup\{h\}}(\mathbf{P}'_{\bar{S}\backslash\{h\}})$ and $x \notin O^f_{S\cup\{h\}}(\mathbf{P}''_{\bar{S}\backslash\{h\}})$. Then, there must be a $\widetilde{\mathbf{P}}_{S\cup\{h\}} \in \mathcal{SC}^{|S|+1}$ such that $f(\widetilde{\mathbf{P}}_{S\cup\{h\}}, \mathbf{P}'_{\bar{S}\backslash\{h\}}) = x$. Fix $\widetilde{\mathbf{P}}_S \in \mathcal{SC}^{|S|}$ and define the $|\bar{S}|$-person social choice rule $g : \mathcal{SC}^{|\bar{S}|} \to X$ such that for all $\mathbf{P}_{\bar{S}} \in \mathcal{SC}^{|\bar{S}|}$, $g(\mathbf{P}_{\bar{S}}) = f(\widetilde{\mathbf{P}}_S, \mathbf{P}_{\bar{S}})$. Since g is SP, the argument in Step 3 implies that $O^g_h(\mathbf{P}'_{\bar{S}\backslash\{h\}}) = O^g_h(\mathbf{P}''_{\bar{S}\backslash\{h\}})$. Hence, by definition, $O^f_h(\widetilde{\mathbf{P}}_S, \mathbf{P}'_{\bar{S}\backslash\{h\}}) = O^f_h(\widetilde{\mathbf{P}}_S, \mathbf{P}''_{\bar{S}\backslash\{h\}})$, implying that $x \in O^f_h(\widetilde{\mathbf{P}}_S, \mathbf{P}''_{\bar{S}\backslash\{h\}})$. That is, there is a $\widehat{P}_h \in \mathcal{SC}$ such that $f(\widehat{P}_h, \widetilde{\mathbf{P}}_S, \mathbf{P}''_{\bar{S}\backslash\{h\}}) = x$. So, $x \in O^f_{S\cup\{h\}}(\mathbf{P}''_{\bar{S}\backslash\{h\}})$, a contradiction. Thus, for all $h \in \bar{S}$, $O^f_{S\cup\{h\}}(\mathbf{P}'_{\bar{S}\backslash\{h\}}) = O^f_{S\cup\{h\}}(\mathbf{P}''_{\bar{S}\backslash\{h\}})$. And, since $S \subset N$ and $P', P'' \in \mathcal{SC}$ were arbitrarily chosen, this completes the proof. ■

Now, we are ready to prove Proposition 2:

Proof of Proposition 2. Suppose, by contradiction, there exists $(P'_i, \mathbf{P}'_{-i}) \in \mathcal{SC}^n$ and $P''_i \in \mathcal{SC}$ such that $\tau|_{r_f}(P'_i) = \tau|_{r_f}(P''_i)$, and $f(P'_i, \mathbf{P}'_{-i}) = x \neq y = f(P''_i, \mathbf{P}'_{-i})$. Fix $j \neq i$. Since preferences are strict, $x \neq y$ implies that either $x\, P'_j\, y$ or $y\, P'_j\, x$. Without loss of generality, assume that $y\, P'_j\, x$. By Lemma 7, $O^f_j(P'_i, \mathbf{P}'_{-\{i,j\}}) = O^f_j(P''_i, \mathbf{P}'_{-\{i,j\}})$. Thus,

[18]Note that, if $S = \{i\}$, then we get the previous result, i.e. $O^f_i(\mathbf{P}'_{-i}) = O^f_i(\mathbf{P}''_{-i})$.

$y \in O_j^f(P_i', \mathbf{P'}_{-\{i,j\}})$. That is, there exists a $\widehat{P}_j \in \mathcal{SC}$ such that $f(\widehat{P}_j, P_i', \mathbf{P'}_{-\{i,j\}}) = y$. However, since $y\, P_j'\, x$, this means that j would like to manipulate f at $(P_i', \mathbf{P'}_{-i})$ via $\widehat{P}_j$, a contradiction. Therefore, f is TO. ■

7 Appendix B: Proof of Theorem 1

Proof of Theorem 1. (**Sufficiency**) Immediate from Proposition 1 and the definition of positional dictatorships.

(Necessity) Suppose f is UN, AN and SP on $\mathcal{SC}^n$. We want to show that $f \in PD$. By Proposition 2, f is TO on $\mathcal{SC}^n$.

Consider first the case where $|N| = 2$. Fix a profile $\mathbf{P} \in \mathcal{SC}^n$. Without loss of generality, assume $\tau(P_2) \geq \tau(P_1)$. By Lemma 1, $f(P_1, P_2) = m^3(\tau(P_1), f(\underline{P}_1, P_2), f(\overline{P}_1, P_2))$, where the lower (respectively, the upper) bar is used to denote the most leftist (respectively, rightist) preference relation on X. Applying Lemma 1 once again, $f(\underline{P}_1, P_2) = m^3(\tau(P_2), f(\underline{P}_1, \underline{P}_2), f(\underline{P}_1, \overline{P}_2))$, and $f(\overline{P}_1, P_2) = m^3(\tau(P_2), f(\overline{P}_1, \underline{P}_2), f(\overline{P}_1, \overline{P}_2))$. By unanimity, $f(\underline{P}_1, \underline{P}_2) = \underline{X}$ and $f(\overline{P}_1, \overline{P}_2) = \overline{X}$. By anonymity, $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2)$.

We show next that $f(\underline{P}_1, \overline{P}_2), f(\overline{P}_1, \underline{P}_2) \in \{\underline{X}, \overline{X}\}$. Suppose not. Without loss of generality, assume that $f(\underline{P}_1, \overline{P}_2) = z \in X \backslash \{\underline{X}, \overline{X}\}$. Consider a preference $P_1^\alpha \in \mathcal{P}$ such that (i) $\tau(P_1^\alpha) = \tau(\underline{P}_1)$, and (ii) $\overline{X}\, P_1^\alpha\, z$ (see Figure 7). If $P_1^\alpha \in \mathcal{SC}$, we are done. By TO, $f(P_1^\alpha, \overline{P}_2) = f(\underline{P}_1, \overline{P}_2) = z$. Thus, by definition of P_1^α, individual 1 would like to manipulate f at $(P_1^\alpha, \overline{P}_2)$ via $\overline{P}_1$.

On the contrary, if $P_1^\alpha \notin \mathcal{SC}$, then using the order of the preferences, there must exist a preference $P_1^\beta \in \mathcal{SC}$ which blocks P_1^α. That is, there must be a $P_1^\beta \in \mathcal{SC}$ such that (i) $\tau(P_1^\beta) > \tau(\underline{P}_1)$, and (ii) $z\, P_1^\beta\, \overline{X}$ (see Figure 7). Denote $f(P_1^\beta, \overline{P}_2) = z^\beta$. If $z > z^\beta$, then individual 1 would manipulate f at $(\underline{P}_1, \overline{P}_2)$ via P_1^β. Similarly, if $z^\beta = \overline{X}$, then 1 would manipulate f at $(P_1^\beta, \overline{P}_2)$ via $\underline{P}_1$. Therefore, $\overline{X} > z^\beta \geq z$. Consider a preference $P_1^{\alpha+1} \in \mathcal{P}$ such that (i) $\tau(P_1^{\alpha+1}) = \tau(P_1^\beta)$, and (ii) $\overline{X}\, P_1^{\alpha+1}\, z^\beta$ (see Figure 7). If $P_1^{\alpha+1} \in \mathcal{SC}$, we are done. By TO, $f(P_1^{\alpha+1}, \overline{P}_2) = z^\beta$. Thus, by definition of $P_1^{\alpha+1}$, individual 1 would like to manipulate f at $(P_1^{\alpha+1}, \overline{P}_2)$ via $\overline{P}_1$.

On the other hand, if $P_1^{\alpha+1} \notin \mathcal{SC}$, then we can repeat the previous argument and find a preference $P_1^{\beta+1} \in \mathcal{SC}$ such that (i) $\tau(P_1^{\beta+1}) > \tau(P_1^\beta)$, and (ii) $z^\beta\, P_1^{\beta+1}\, \overline{X}$. Since in each step the top of the blocking ordering becomes larger and larger, the sequence of preferences $P_1^\beta, P_1^{\beta+1}, \ldots$ approaches $\overline{P}_1$. Therefore, at same point we will find a preference $P_1^{\beta+\ell} \in \mathcal{SC}$ such that (i) $\tau(P_1^{\beta+\ell}) = \tau(\overline{P}_1)$, and (ii) $z^{\beta+\ell-1}\, P_1^{\beta+\ell}\, \overline{X}$, which provides the desired contradiction (recall that $\tau(\overline{P}_1) = \overline{X}$). Hence, $f(\underline{P}_1, \overline{P}_2), f(\overline{P}_1, \underline{P}_2) \in \{\underline{X}, \overline{X}\}$.

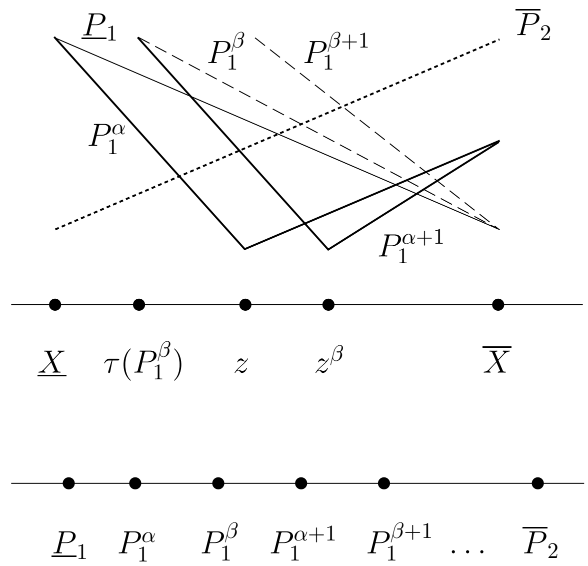


Figure 7: Fixed ballots over X

If $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2) = \underline{X}$, then $f(\underline{P}_1, P_2) = m^3(\tau(P_2), \underline{X}, \underline{X}) = \underline{X}$ and $f(\overline{P}_1, P_2) = m^3(\tau(P_2), \underline{X}, \overline{X}) = \tau(P_2)$. Thus, $f(P_1, P_2) = m^3(\tau(P_1), \underline{X}, \tau(P_2)) = \tau(P_1)$, (i.e., f chooses the smallest peak). Instead, if $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2) = \overline{X}$, then a similar argument shows that $f(P_1, P_2) = m^3(\tau(P_1), \tau(P_2), \overline{X}) = \tau(P_2)$, (i.e., f chooses the largest peak).

Thus, if $|N| = 2$ and f satisfies the hypotheses of Theorem 1, i.e. f is UN, AN and SP, the previous paragraphs show that there exists a parameter (or fixed ballot) $\alpha \in \{\underline{X}, \overline{X}\}$ such that, for all $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = m^3(\tau(P_1), \tau(P_2), \alpha)$. Hence, $f \in PD$.

Now, suppose $|N| = 3$. Take any profile $\mathbf{P} \in \mathcal{SC}^n$. Without loss of generality, relabel N if necessary so that $\tau(P_3) \geq \tau(P_2) \geq \tau(P_1)$. Using Lemma 1, it is easy to see that,

$$\begin{aligned} f(\mathbf{P}) = m^3 \big[&\tau(P_1),\, m^3 \left(\tau(P_2),\, m^3(\tau(P_3), a_3, a_2),\, m^3(\tau(P_3), a_2, a_1)\right), \\ &m^3 \left(\tau(P_2),\, m^3(\tau(P_3), a_2, a_1),\, m^3(\tau(P_3), a_1, a_0)\right)\big], \end{aligned} \tag{12}$$

where $a_3 = f(\underline{P}_1, \underline{P}_2, \underline{P}_3)$, $a_0 = f(\overline{P}_1, \overline{P}_2, \overline{P}_3)$, and

$$a_2 = f(\underline{P}_1, \underline{P}_2, \overline{P}_3) = f(\underline{P}_1, \overline{P}_2, \underline{P}_3) = f(\overline{P}_1, \underline{P}_2, \underline{P}_3), \tag{13}$$

and

$$a_1 = f(\underline{P}_1, \overline{P}_2, \overline{P}_3) = f(\overline{P}_1, \overline{P}_2, \underline{P}_3) = f(\overline{P}_1, \underline{P}_2, \overline{P}_3), \tag{14}$$

where the equalities in (13) and in (14), respectively, follow from the fact that f is AN on $\mathcal{SC}^n$. By UN and TM, we have that $\overline{X} = a_0 \geq a_1 \geq a_2 \geq a_3 = \underline{X}$. By SP, a_1, $a_2 \in \{\underline{X}, \overline{X}\}$.

Otherwise, if for example $f(\underline{P}_1, \overline{P}_2, \overline{P}_3) \notin \{\underline{X}, \overline{X}\}$, we can find a preference $P_1^\alpha \in \mathcal{SC}$ such that, for some $P_1^\beta \in \mathcal{SC}$, $\tau(P_1^\alpha) = \tau(P_1^\beta)$ and $\overline{X}\, P_1^\alpha\, f(P_1^\beta, \overline{P}_2, \overline{P}_3)$. By TO, $f(P_1^\alpha, \overline{P}_2, \overline{P}_3) = f(P_1^\beta, \overline{P}_2, \overline{P}_3)$. Thus, agent 1 can manipulate f at $(P_1^\alpha, \overline{P}_2, \overline{P}_3)$ via $\overline{P}_1$.

There are three cases to consider:

(i) If $a_1 = \underline{X}$, then $a_2 = \underline{X}$ because $a_1 \geq a_2$. Therefore, (12) can be rewritten as $f(\mathbf{P}) = m^3(\tau(P_1), \underline{X}, \tau(P_2)) = \tau(P_1)$, (i.e., f chooses the smallest ideal point);

(ii) Similarly, if $a_2 = \overline{X}$, then $a_1 = \overline{X}$, and $f(\mathbf{P}) = m^3(\tau(P_1), \tau(P_3), \overline{X}) = \tau(P_3)$, (i.e., f chooses the largest ideal point);

(iii) Finally, if $a_1 = \overline{X}$ and $a_2 = \underline{X}$, then (12) can be rewritten as $f(\mathbf{P}) = m^3(\tau(P_1), \tau(P_2), \tau(P_3)) = \tau(P_2)$, (i.e., f selects the median ideal point).

Thus, since $\mathbf{P}$ was arbitrarily chosen, (i)-(iii) imply that, if $|N| = 3$ and f is AN, UN and SP, then there exist $\alpha_1, \alpha_2 \in \{\underline{X}, \overline{X}\}$ such that for all $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = m^5(\tau(P_1), \tau(P_2), \tau(P_3), \alpha_1, \alpha_2)$. Therefore, $f \in PD$.

Now let us extend the proof to $|N| = n > 3$. For all $K \subseteq N$, let $a_{|K|} = f(\underline{\mathbf{P}}_K, \overline{\mathbf{P}}_{\bar{K}})$, where $\bar{K} = N \backslash K$. By unanimity, $K = \emptyset$ implies $a_0 = f(\overline{P}_1, \ldots, \overline{P}_n) = \overline{X}$. Similarly, if $K = N$, then $a_n = f(\underline{P}_1, \ldots, \underline{P}_n) = \underline{X}$. By anonymity,

$$\begin{aligned} a_1 &= f(\underline{P}_i, \overline{\mathbf{P}}_{-i}),\ \forall \{i\} \subset N, \\ a_2 &= f(\underline{\mathbf{P}}_{\{i,j\}}, \overline{\mathbf{P}}_{-\{i,j\}}),\ \forall \{i,j\} \subseteq N, \\ &\vdots \qquad \vdots \\ a_{n-1} &= f(\underline{\mathbf{P}}_{-j}, \overline{P}_j),\ \forall \{j\} \subset N. \end{aligned}$$

Thus, by top-monotonicity, $a_0 \geq a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq a_n$. Moreover, for all $k = 0, 1, \ldots, n$, $a_k \in \{\underline{X}, \overline{X}\}$. In effect, if either $k = 0$ or $k = n$, then the result follows immediately from UN. So, assume that $a_k \in \{\underline{X}, \overline{X}\}$ for some $k = 0, 1, \ldots, n-2$, and let us prove the claim for a_{k+1}. On the contrary, suppose $a_{k+1} = f(\underline{P}_1, \ldots, \underline{P}_{k+1}, \overline{P}_{k+2}, \ldots, \overline{P}_n) \notin \{\underline{X}, \overline{X}\}$. Without loss of generality, let $a_k = f(\underline{P}_1, \ldots, \underline{P}_k, \overline{P}_{k+1}, \ldots, \overline{P}_n) = \overline{X}$. Following the argument illustrated in Figure 7 for $|N| = 2$, we can find preferences $P_{k+1}^\alpha \in \mathcal{SC}$ and $P_{k+1}^\beta \in \mathcal{SC}$ such that $\tau(P_{k+1}^\alpha) = \tau(P_{k+1}^\beta)$ and $\overline{X}\, P_{k+1}^\alpha\, f(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^\beta, \overline{P}_{k+2}, \ldots, \overline{P}_n)$. By TO, $f(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^\alpha, \overline{P}_{k+2}, \ldots, \overline{P}_n) = f(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^\beta, \overline{P}_{k+2}, \ldots, \overline{P}_n)$. Hence, agent $k+1$ can manipulate f at $(\underline{P}_1, \ldots, \underline{P}_k, P_{k+1}^\alpha, \overline{P}_{k+2}, \ldots, \overline{P}_n)$ via $\overline{P}_{k+1}$, a contradiction.

Now, fix any profile $\mathbf{P} \in \mathcal{SC}^n$, and relabel N if necessary, so that $\tau(P_n) \geq \tau(P_{n-1}) \geq \ldots \geq \tau(P_1)$. By repeated application of Lemma 1, it follows that:

(i) If $a_k = \underline{X}$ for all $k = 1, \ldots, n-1$, then $f(\mathbf{P}) = m^3(\tau(P_1), \underline{X}, \tau(P_2)) = \tau(P_1)$, (i.e., f chooses the smallest peak);

(ii) If $a_k = \overline{X}$ for all $k = 1, \ldots, n-1$, then $f(\mathbf{P}) = m^3(\tau(P_1), \tau(P_n), \overline{X}) = \tau(P_n)$, (i.e., f chooses the largest peak);

(iii) Finally, if for some $k = 1, \ldots, n-2$, $a_1 = \cdots = a_k = \overline{X}$ and $a_{k+1} = \cdots = a_{n-1} = \underline{X}$, then $f(\mathbf{P}) = m^3(\tau(P_1), \tau(P_{k+1}), \tau(P_{k+2})) = \tau(P_{k+1})$, (i.e., if k parameters are placed at $\overline{X}$ and the rest at $\underline{X}$, then f chooses the ideal point ranked at the $(k+1)$-th position).

Therefore, since $\mathbf{P} \in \mathcal{SC}^n$ was arbitrarily chosen and, for every $k = 0, 1 \ldots, n$, a_k is independent of $\mathbf{P}$, if f is AN, UN and SP, then items (i)-(iii) imply that there exist $n-1$ fixed ballots $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ on $\{\underline{X}, \overline{X}\}$ such that, for all $\mathbf{P} \in \mathcal{SC}^n$, $f(\mathbf{P}) = m^{2n-1}(\tau(P_1), \tau(P_2), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n-1})$. Hence, $f \in PD$. ■

References

[1] Austen-Smith, D., and J. Banks, (1999), *Positive Political Theory I: Collective Preference*, Ann Arbor: The University of Michigan Press.

[2] Austen-Smith, D., and J. Banks, (2005), *Positive Political Theory II: Strategy and Structure*, Ann Arbor: The University of Michigan Press.

[3] Ashworth, S., and E. Bueno de Mesquita, (2006), Monotone comparative statics for models of politics, *American Journal of Political Science* 50, 214-231.

[4] Barberà, S., and M. Jackson, (2004), Choosing how to choose: Self-stable majority rules and constitutions, *Quarterley Journal of Economics* 119, 1011-1048.

[5] Barberà, S., and B. Moreno, (2007), Top monotonicity: A weak domain restriction encompassing single peakedness, single crossing and order restriction, Working Paper No 297, Centre de Referència en Economia Analítica, Barcelona.

[6] Black, D., (1948), On the rationale of group decision making, *Journal of Political Economy* 56, 23-34.

[7] Blin, J., and M. Satterthwaite, (1976), Strategy-proofness and single-peakedness, *Public Choice* 26, 51-58.

[8] Calabrese, S., Epple, D., Romer, T., and H. Sieg, (2006), Local public good provision: voting, peer effects, and mobility, *Journal of Public Economics* 90, 959-981.

[9] Campbell, D., and J. Kelly, (2003), A strategy-proofness characterization of majority rule, *Economic Theory* 22, 557-568.

[10] Ching, S., (1997), Strategy-proofness and "median voters", *International Journal of Game Theory* 26, 473-490.

[11] Demange, G., (1994), Intermediate preferences and stable coalition structures, *Journal of Mathematical Economics* 23, 45-58.

[12] Eguia, J., (2007), Citizen candidates under uncertainty, *Social Choice and Welfare* 29, 317-331.

[13] Epple, D., Filimon, R., and T. Romer, (1993), Existence of voting and housing equilibrium in a system of communities with property taxes, *Regional Science and Urban Economics* 23, 585-610.

[14] Epple, D., and G. Platt, (1998), Equilibrium among jurisdictions when households differ by preferences and income, *Journal of Urban Economics* 43, 23-51.

[15] Epple, D., Romer, T., and H. Sieg, (2001), Interjurisdictional sorting and majority rule: an empirical analysis, *Econometrica* 69, 1437-1465.

[16] Epple, D., Romano, R., and H. Sieg, (2006), Admission, tuition and financial aid policies in the market for higher education, *Econometrica* 74, 885-928.

[17] Gans, J., and M. Smart, (1996), Majority voting with single-crossing preferences, *Journal of Public Economics* 59, 219-237.

[18] Gibbard, A., (1973), Manipulation of voting schemes: a general result, *Econometrica* 41, 587-601.

[19] Grandmont, J., (1978), Intermediate preferences and the majority rule, *Econometrica* 46, 317-330.

[20] Kung, F., (2006), An algorithm for stable and equitable coalition structures with public goods, *Journal of Public Economic Theory* 8, 345-355.

[21] List, C., (2001), A possibility theorem on aggregation over multiple interconnected propositions, *Mathematical Social Sciences* 45, 1-13.

[22] Le Breton, M., and J. Weymark, (1999), Strategy-proof social choice with continuous separable preferences, *Journal of Mathematical Economics* 32, 47-85.

[23] Meltzer, A., and S. Richard, (1981), A rational theory of the size of government, *Journal of Political Economy* 89, 914-927.

[24] Milgrom, P., and C. Shannon, (1994), Monotone comparative statics, *Econometrica* 62, 157-180.

[25] Moulin, H., (1980), On strategy-proofness and single-peakedness, *Public Choice* 35, 437-455.

[26] Moulin, H., (1988), *Axioms of Cooperative Decision Making*, Cambridge: Cambridge University Press.

[27] Myerson, R., (1996), Fundamentals of social choice theory, Discussion Paper No 1162, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

[28] Persson, T., and G. Tabellini, (2000), *Political Economics: Explaining Economic Policy*, Cambridge, MA: MIT Press.

[29] Roberts, K., (1977), Voting over income tax schedules, *Journal of Public Economics* 8, 329-340.

[30] Rothstein, P., (1990), Order-restricted preferences and majority rule, *Social Choice and Welfare* 7, 331-342.

[31] Rothstein, P., (1991), Representative voter theorems, *Public Choice* 72, 193-212.

[32] Saporiti, A., and F. Tohmé, (2006), Single-crossing, strategic voting and the median choice rule, *Social Choice and Welfare* 26, 363-383.

[33] Satterthwaite, M., (1975), Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions, *Journal of Economic Theory* 10, 187-217.

[34] Westhoff, F., (1977), Existence of equilibria in economies with a local public good, *Journal of Economic Theory* 14, 84-112.

[35] Weymark, J., (2008), Strategy-proofness and the tops-only property, *Journal of Public Economic Theory* 10, 7-26.