Multidimensional Elections*

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Abstract

We propose an approach to two-candidate competition with electoral uncertainty in which elections are both competitive (in the sense that each candidate wins with positive probability) and meaningful (in the sense that the candidates adopt distinct platforms). Candidates may place positive weight on policy and office; these weights may differ across parties; and one party may have an electoral advantage. Existence of equilibria in pure strategies holds in any number of dimensions, even if the extent of electoral uncertainty is arbitrarily small. We characterize equilibria and demonstrate their tractability in applications to distributive politics, to income taxation, and to cultural and economic policy.

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Contents

1 Introduction

Many of the paramount political challenges facing democratic societies are multidimensional, yet the literature on formal political economy does not provide a modeling framework to analyze competitive, meaningful elections in multiple dimensions. Sometimes the multiple dimensions can be collapsed (or "scaled") into a single dimension, but this is not possible if preferences over issues are not highly correlated, and thus one-dimensional models are of limited use for studying inherently multidimensional problems. If we want to improve our theoretical understanding of the distribution of resources across groups, or the conditions under which progressive tax systems arise, or how polities aggregate preferences over economic and cultural policies, to list a several examples, then there are few analytical options available.

This is not without good reason. The classical results of Plott (1967) and Schofield (1983) establish that for general voter preferences, given any initial vector of policies, there is a majority of voters who could replace the status quo with a preferred policy vector. This instability of majority rule translates into equilibrium non-existence in the multidimensional Downsian model in which two office-motivated candidates compete for an elected office. The negative conclusion of Plott and Schofield extends to the model in which candidates place positive weight on policy, as well as office: robust equilibria may exist in two dimensions, but in the model with three or more policy issues, equilibria fail to exist for general voter preferences.[1]

Early work in formal political economy added electoral uncertainty to the multidimensional model. Hinich, Ledyard, and Ordeshook (1972,1973), Hinich (1977), Coughlin and Nitzan (1981), and Lindbeck and Weibull (1987) recovered equilibria in *stochastic bias models,* where voter policy preferences are known to parties, while each voter's choice is subject to an idiosyncratic bias shock. However, equilibria in this model feature vote-maximizing parties that both locate (they "converge") at the unique policy vector that maximizes social welfare. To the extent we think that parties' characteristics (e.g., their electoral motivations or standing vis-à-vis voters) differ, and that these differences are reflected in their holding distinct policy platforms, this approach is not a solution. Wittman (1983, 1990) and Calvert (1985) allow for parties that care about policy as well as office, i.e., they have mixed motives. This leads to equilibria in which the parties choose distinct platforms (they "diverge"), but Wittman assumes each candidate's probability of winning is concave in her own platform. Although he does not provide microfoundations for the concavity assumption, it requires, intuitively, the presence of a substantial amount of uncertainty.[2]

In this paper, we provide a solution to the classical problem of majority rule

[1]See Figure 1 and Theorem 3 of Duggan and Fey (2005).

[2]See pages 154–155 of Wittman (1983) for the existence argument. Note that, in addition, he assumes that the probability of winning is always strictly positive for each candidate; see page 150, below equation (1). Calvert does not address the existence question.

instability by proving equilibrium existence in a general framework of elections in which candidates have mixed motivations, and we provide microfoundations in terms of an aggregate "partisan shock" that is added to candidates' election returns. We then characterize equilibria as we impose increasing structure on the model: we give conditions under which election outcomes are as if there is a single, aggregate voter whose choices are subject to noise; we add the assumption of quadratic utility to obtain a dimensional reduction result; and we then add symmetry to explicitly solve for equilibrium and perform comparative statics. Finally, we demonstrate the tractability of the model in simple applications to distributive politics, income taxation, and cultural and economic policy.

1.1 Discussion of Results

To begin, we consider an abstract model in which the candidates' probability of winning is initially "blackboxed," i.e., it is formalized as an exogenously given function of the candidates' platforms. We prove that a Nash equilibrium in pure strategies exists if three conditions hold: (i) each candidate wins with positive probability whenever both are located at the same platform, (ii) the probability of winning is continuous, and (iii) each candidate's probability of winning is log concave in her own platform. At a technical level, this is an improvement on Wittman's assumption that each candidate's probability of winning is concave in her own platform, and it improves Theorem 1 of Duggan (2025a) by extending it to multiple dimensions.[3] However, because the existence theorem is established in an abstract framework, it is important to provide microfoundations for the assumptions used, log concavity being the key challenge. And to gauge the improvement over the literature, we must understand the purchase gained in fully specified models, in which the probability of winning function is derived from primitives.

We provide such foundations in a model featuring a continuum of policy oriented voters who are partitioned into sets of types, which correspond to voter policy preferences. The model departs from the social choice framework of Plott (1967) and Schofield (1983) by assuming that within each type, an idiosyncratic bias term is distributed across a continuum of voters. This smooths out each candidate's vote share among policy-oriented voters as a function of her platform, but importantly, these bias terms do not introduce aggregate uncertainty: the candidates' platforms generate vote shares in a deterministic way, and thus either one candidate's vote share exceeds one half, or there is a tie. In fact, this is a common reinterpretation of the early model of Hinich, Ledyard, and Ordeshook: because candidates maximize expected vote, a single voter's probability of voting for one candidate can be viewed as the fraction of supportive voters of that type, effectively interpreting integrals as averages rather than expectations. The "continuum interpretation" is adopted, e.g., by Cox and Mc-

[3] See Duggan (2025a) for a discussion of the restrictiveness of concavity (which is assumed by Wittman) relative to log concavity (which is all we need).

Cubbins (1986), Dixit and Londregan (1996), and Roemer (1997,2001), and it is discussed by Banks and Duggan (2005). In the latter articles, candidates are assumed to maximize vote share, and thus in equilibrium, they locate at the same policy platform.

Peress (2010) considers the continuum model in which candidates care about both policy and office, and in contrast to the early literature, payoffs depend on the candidates' probabilities of winning; in particular, the outcome in case of a tie must be specified. The standard approach is to assume that victory is decided by an exogenous randomization device, but it is also possible to allow the tie-breaking rule to be endogenous; then the candidates' probability of winning in case of a tie provides an additional degree of freedom that can be used to support the existence of equilibrium. Peress shows that in case one candidate has an electoral advantage, she must win with probability one, and moreover, if the advantage is not too large, then the candidates have equal vote shares in equilibrium, with the tie being broken in favor of the advantaged candidate.[4] Endogenous tie breaking is required in this case, for if the probability of a victory is exogenously fixed to be strictly between zero and one, then an equilibrium cannot exist. In the remaining case, where neither candidate has an advantage, the candidates must adopt identical platforms. In sum, it is possible to support equilibria in which the candidates adopt distinct platforms, but only if one has an advantage, and then that candidate must win with probability one. Competitive, meaningful elections are impossible in the model with no aggregate uncertainty: there do not exist equilibria in which the candidates adopt distinct platforms (giving voters a meaningful choice) and each wins with positive probability (so both candidates have a stake in the competition).

We propose a framework that introduces electoral uncertainty to the above setting by adding a mass of partisan voters, who vote for one candidate or the other independently of policy, formalized as a random shock that is realized after candidates' platform choices. This *stochastic partisans model* is formally equivalent to Roemer's (2001) "error-distribution model" of elections, and we view it as a natural and convenient form of uncertainty.[5]

Our general existence result applies as long as the distribution of the partisan shock satisfies a weak log concavity condition, allowing candidates to have general mixed motivations, placing arbitrary (and possibly different) weight on policy relative to winning. Importantly, pure strategy equilibria exist *even if the amount of aggregate uncertainty assumed is arbitrarily small.* If the partisan shock is distributed uniformly, for example, then the support of the distribution may be an arbitrarily small interval; or if it is normally distributed, no assumption on the variance (other than being positive) is needed. In contrast,

[4]In fact, in some parameterizations of the model, a continuum of equilibria can be supported with these features.

[5]This type of uncertainty is also used by Bernhardt, Krasa, and Squintani (2024) and Bernhardt, Bouton, Krasa, and Squintani (2025) in their analysis of strategic voting with multiple candidates.

to apply the result of Wittman (1983,1990), a substantial amount of uncertainty is required to satisfy his concavity condition. In our model, as long as both candidates place positive weight on office, each wins with positive probability; and when both place positive weight on policy, we apply a result of Duggan (2025b) to show that for almost all pairs of policy weights, the candidates' equilibrium platforms are distinct. Therefore, equilibria exist generally, and elections are almost always competitive and meaningful.

Our sharpest characterization result provides foundations for an *aggregate voter,* whose utility function acts as a potential function for the candidates' margin among policy-oriented voters: for example, candidate A's margin among policy-oriented voters is

$$\kappa + V(x) - V(y),$$

where V is an aggregate utility function, and κ is a constant measuring A's electoral advantage, which may be positive, negative, or zero. The key assumption here is that idiosyncratic bias is distributed uniformly within each voter type—a strong assumption, but one that is common in applications and that purchases considerable analytic leverage. It is straightforward to show that in equilibrium, a candidate's platform must be Pareto optimal for the aggregate voter and herself, and if a candidate is purely office motivated, then she simply chooses the ideal policy of the aggregate voter. This Pareto optimality result implies that each candidate's platform can be restricted to the contract curve between her ideal point and the aggregate ideal point, essentially reducing her strategy space to a one-dimensional manifold. Assuming politicians and voters have quadratic policy utility, the aggregate utility function is also quadratic, so a candidate simply chooses a platform on the one-dimensional line segment between her ideal point and the aggregate ideal point. That is, candidate strategy sets essentially reduce to one-dimensional intervals—a *dimensional reduction* that greatly simplifies comparative statics and numerical examples.

A strength of our analysis is that we allow for arbitrarily small electoral uncertainty, in the sense that the distribution of the partisan voter shock may be as close to degenerate as one wants. We then perform the asymptotic analysis of characterizing the limit of equilibrium platforms as noise in the model goes to zero. For one example, if one candidate has a large electoral advantage, then her platform simply converges to her ideal point. For another, if the candidates have opposed preferences, in the sense that each prefers the aggregate ideal point to any platform on the contract curve between her opponent and the aggregate voter, and if one candidate has an electoral advantage that is not too great, then the limit of equilibrium platforms has a straightforward characterization: the disadvantaged candidate converges to the aggregate ideal point, while the advantaged candidate converges to the platform that makes the aggregate voter indifferent between electing one or the other; in addition, the advantaged candidate's probability of winning converges to one. Intuitively, the advantaged candidate leverages her relative popularity with voters, and the disadvantaged candidate effectively acts as an anchor that prevents further movement by the advantaged candidate.

As the advantage decreases, the gap between the candidates decreases to zero, with the limiting platform of the advantaged candidate converging to the aggregate ideal point. At the extreme, when neither candidate has an electoral advantage, the equilibrium platforms of the candidates both converge to the aggregate ideal point as uncertainty becomes small. This limit is independent of the weights the candidates place on policy and office.

An interesting case arises when the advantage of one candidate is at an intermediate level. Assuming that candidates and voters have generalized quadratic utility, the characterization depends on the location of the candidates' ideal points vis-à-vis the aggregate voter. If the candidates are polarized, i.e., the lines from the aggregate ideal point to the candidates' ideal points form an obtuse angle, then the disadvantaged candidate again converges to the aggregate voter's ideal point, while the advantaged candidate converges to the platform on her contract curve that makes the aggregate voter indifferent.

However, if the candidates are less polarized, i.e., the lines from the candidates' ideal points form an acute angle, then the latter characterization may break down: letting A be the advantaged candidate, it may be that when uncertainty is small enough, the disadvantaged candidate B would prefer for A to win, rather than win herself at the aggregate ideal point. In this exceptional scenario, the limit of equilibrium platforms is characterized as the solution to a system of two equations in two unknowns: candidate A's platform converges to a policy such that the aggregate voter is indifferent between the candidates, while candidate B's platform converges to a policy such that B is indifferent between winning or losing to A. Interestingly, as noise is removed from the model, there is a unique limit equilibrium that belongs to the class of "irregular" equilibria identified by Peress (2010). Thus, although they are not the focus of his analysis, our results indicate that irregular (and not "regular") equilibria may sometimes be of greater interest, as they are robust to the introduction of electoral uncertainty.

Our last set of results focuses on the symmetric model, where we assume that the candidates and aggregate voter have quadratic utility, with candidate ideal points equidistant from the aggregate ideal point, and that the candidates place identical weight on office. Thus, policy preferences are summarized by the angle formed by the candidate ideal points relative to the voter, with larger angles closer to 180 degrees meaning that the candidate preferences are in opposition to each other, and smaller angles corresponding to alignment of candidate preferences. We show that there is a unique equilibrium that is symmetric, in the sense that the candidates choose platforms on their contract curves that are equidistant from the aggregate voter. In fact, we solve explicitly for the symmetric equilibrium, and we demonstrate expected comparative statics: the candidates moderate, by moving their platforms closer to the aggregate ideal point, when the weight on office increases; when the variance of the partisan shock increases; and when the angle between the candidate increases. The latter result relies critically on multidimensionality, and it arises because as the angle becomes larger, her opponent's platform becomes worse for a candidate, which magnifies the threat of losing and incentives greater moderation.

Comparative statics that introduce asymmetry are more complicated, but important. We show that, beginning from the symmetric equilibrium in the symmetric model, if we increase candidate A's weight on policy slightly, while holding B's objectives fixed, then, as intuition would suggest, A's platform becomes more extreme by shifting toward her ideal point. Interestingly, B's equilibrium response is more nuanced and depends on the angle between the candidates. When the angle is larger, candidate B becomes more moderate; effectively, A's movement increases the threat of losing for B, causing her to trade policy for probability of winning by moving toward the aggregate voter. When the angle is smaller, however, B also shifts toward her ideal point: if candidate preferences are aligned, A's shift can improve her platform from B's perspective, which decreases the threat of losing, and incentivizes B to become more extreme.

Beginning from the symmetric model, if we introduce a small electoral advantage for candidate A, then A's equilibrium platform moves in the direction of her ideal point, becoming more extreme as she leverages her higher popularity. In response, candidate B's equilibrium platform moderates, moving in the direction of the aggregate voter, independently of the angle between the candidates. In contrast to the comparative static with respect to the weight on policy, now there is a direct effect of A's move on B's optimal policy: the marginal policy returns from her platform decrease as A's becomes more likely to win, offsetting the change in the payoff from losing to A, and inducing B to pursue the higher marginal returns to the probability of winning by moderating her platform. Interestingly, this comparative static runs counter to a counterintuitive result of Groseclose (2001), proved in the one-dimensional model with fixed valence and stochastic median, where the effect of giving one candidate a small valence advantage is that she becomes more moderate, while the disadvantaged candidate moves toward her own ideal point.

To demonstrate the usefulness of our approach, we provide three applications. First, we consider a version of the distributive politics model of Lindbeck and Weibull (1987) with aggregate uncertainty and candidates having mixed motivations. Equilibria exist and feature candidates adopting distinct platforms. After adding functional form assumptions and assuming candidates are symmetric relative to the aggregate voter, we solve explicitly for the unique symmetric equilibrium, and we show that as the candidates become more policy motivated, or as the amount of uncertainty about the election outcome increases, the candidates shift resources to their own groups from the mass electorate.

Second, we apply the framework to income taxation and public good provision, initially assuming two types of voters, those with low income and those with high income, with each candidate representing one of the groups. We show that the equilibrium platform of the left-wing party is always more progressive than that of the right-wing party. Moreover, if the distribution of bias among low-income voters has lower variance than the bias among high income voters, then the aggregate voter's ideal point is a progressive tax system. Then the left-wing party's platform is always a progressive tax policy, and if the right-wing party is sufficiently office motivated, then it too adopts a progressive tax policy. Alternatively, if the variance of bias is the same among the two groups, then the right-wing party adopts a regressive tax policy. We show that if the income disparity between rich and poor exceeds a particular level, then the left-wing party's platform also finances a higher level of public good.

Third, we apply our results for symmetric equilibria to a model of elections in which cultural and economic policy are the salient issues. We assume that one party's ideal is liberal policy on both dimensions, and that the other's is conservative policy on both dimensions. Voters have heterogeneous preferences, and to fix ideas, we assume that there are more voters with liberal economic preferences (preferring unmodeled redistribution or government programs) and conservative cultural preferences (perhaps preferring immigration restrictions or lax gun laws). Assuming the parties place positive weight on policy, they will each choose platforms in the direction of their ideal points, relative to the aggregate voter, but the liberal party is more misaligned on culture than on economics, with the reverse for the conservative party, which is more misaligned on economics than on culture.

1.2 Other Models of Electoral Uncertainty

Previous work on one-dimensional models has used electoral uncertainty to generate existence of equilibria that are competitive, in the sense that the candidates each win with positive probability, and meaningful, in the sense that they adopt distinct platforms. Hansson and Stuart (1984) use an abstract framework that captures aspects of the *stochastic median model,* in which the location of the median voter is realized after candidates' platform choices. They assume that each candidate's probability of winning is concave in her own platform, and thus, like Wittman (1983), their analysis is predicated on a substantial amount of uncertainty. Roemer (1997) weakens concavity to log concavity, and so he proves existence of equilibrium in the stochastic median model for arbitrarily small uncertainty, but assuming that the candidates are purely policy motivated; the latter assumption removes problematic discontinuities, thereby simplifying the analysis.[6] Bernhardt, Duggan, and Squintani (2009) analyze the symmetric model and prove existence of symmetric equilibrium, while allowing candidates to place arbitrary (common) weight on office relative to policy. However, when the candidates are asymmetric, equilibrium existence is problematic: Ball

[6]Roemer (2001) replicates the analysis using his error-distribution model, i.e., the stochastic partisans model, to generate uncertainty.

(1999), using asymmetric candidate motivations, and Groseclose (2001), using a fixed valence advantage, provide examples in which pure strategy equilibria fail to exist in the stochastic median model.

In the one-dimensional *stochastic valence model* of elections, Londregan and Romer (1993) assume that the policy preferences of voters are known to the candidates, but following platform choices, an aggregate valence shock is realized and added to the utility from one candidate's platform. Londregan and Romer argue that an equilibrium exists, but they assume candidates are purely policy motivated, and that each candidate's probability of winning is concave in her own platform; thus, like Wittman (1983) and Hansson and Stuart (1984), the analysis of Londregan and Romer is predicated on a substantial amount of uncertainty. Duggan (2025a) considers the general stochastic valence model in one dimension, allowing for mixed motivations and arbitrarily small uncertainty, and he establishes equilibrium existence as a special case of our analysis; in particular, when the density of the valence shock is log concave, the probability of winning functions satisfy our assumptions, and thus equilibria exist. In addition, he provides comparative statics, an asymptotic analysis as uncertainty is removed from the model, and uniqueness for the special case in which utility functions are defined by absolute value loss functions.

The literatures on the stochastic median model and the stochastic valence model take as their starting point the classical social choice framework of Downs (1957) and Plott (1967), in which voter policy preferences are known to the candidates, and each voter casts their ballot for the candidate offering the preferred policy platform (perhaps randomizing when indifferent). Much of the existing literature on probabilistic elections augments the classical model by adding uncertainty in the form of noise—either on the location of the median voter or the net valence of the candidates—and the work of Roemer (1997), Bernhardt et al. (2009), and Duggan (2025a) delivers existence of equilibrium even if the amount of noise is arbitrarily small. As a consequence, the social choice model can be "perturbed," by adding a small amount of noise, in a way that leads to existence of equilibria corresponding to competitive, meaningful elections in which the candidates adopt distinct platforms and each wins with positive probability. However, the latter analyses are predicated on narrow assumptions: all assume a one-dimensional policy space, and in addition, Roemer assumes pure policy motivation while Bernhardt et al. assume the election is symmetric.

Our model adds uncertainty—an amount that may be arbitrarily small—to the continuum model of elections, which departs significantly from the classical framework of Downs and Plott: in this alternative benchmark, voter policy preferences are known, but in addition, within each voter type, an idiosyncratic bias term is distributed across a continuum of voters. We establish a continuity result showing that some such departure from the classical spatial model is necessary: if a pure strategy equilibrium fails to exist in the benchmark model with no uncertainty, then adding a small amount of uncertainty to the model cannot deliver pure strategy equilibria. This result is proved using a flexible notion of "adding a small amount of uncertainty," and it complements a similar result of Duggan and Fey (2005) by considering the case in which candidates place positive (possibly different) weights on office. Since equilibria typically fail to exist in the Downsian model with mixed motivations,[7] it is therefore impossible to prove a general existence result if we start with the multidimensional spatial model and, e.g., add a valence shock with arbitrarily small variance.

Our continuity result demarcates the limits of existence in multidimensional settings and shows that the majority rule instability problem raised by Plott is a fundamental feature of the classical spatial model, one that is robust to perturbations of the model. In light of this negative conclusion, we propose to take the continuum model of Peress (2010) as the chasis on which electoral uncertainty is layered. In the context of large elections, we view the assumption of a continuum of voters as a technical device, and we posit that the bias term, which is distributed across voters within each type and is orthogonal to policy, is a plausible factor in determining real-world elections—in other words, it is a feature, not a bug.

Further from our approach, Roemer (2001) proposes the party unanimity equilibrium model, which uses a relaxed notion of equilibrium in which a deviation by a party is profitable only if it raises the party's probability of winning (satisfying the opportunist faction) and the new platform is itself an improvement for the party (satisfying the militant faction). He studies properties of party unanimity equilibria, but existence becomes moot (it is trivially an equilibrium for the parties to locate at their ideal points), and they are generally indeterminant (a continuum of equilibria exist). In a series of articles and books, Schofield (e.g., Schofield 2003, 2004, 2006, 2009) analyzes a multidimensional model with multiple vote-maximizing parties and a stochastic valence shock that is extreme value distributed and added to the policy utility from each party. He argues for existence by adding a "party activist" term to each party's vote total, and assuming that activist turnout depends only on the party's platform (it is independent of the platforms of other parties) and that the term is sufficient concave.

2 General Model

2.1 Setting

Two candidates, A and B, compete in a Downsian election: the candidates simultaneously commit to platforms, x and y, respectively, belonging to a compact, convex policy space $Z \subseteq \mathbb{R}^d$. Policy preferences of the candidates are given by utility functions $u_A, u_B \colon Z \to \mathbb{R}$, which we assume are concave and continuous. We also assume that u_A has unique ideal point $\hat{x}$, that u_B has

[7] See Theorem 3 and Proposition 6 of Duggan and Fey (2005).

unique ideal point $\hat{y}$, and that these ideal points are distinct: $\hat{x} \neq \hat{y}$.

In this abstract general model, we "black box" the electorate: given $x, y \in Z$, let $P_A(x, y)$ and $P_B(x, y) = 1 - P_A(x, y)$ denote the probability of winning for candidates A and B, respectively. Assume candidates have mixed motives, where A places weight $\lambda_A \in [0, 1]$ on policy and weight $1 - \lambda_A$ on office, while B places weight $\lambda_B \in [0, 1]$ on policy and $1 - \lambda_B$ on office. Then A's expected payoff from platform pair (x, y) is

$$\begin{aligned} & \lambda_A[P_A(x,y)u_A(x) + (1 - P_A(x,y))u_A(y)] + (1 - \lambda_A)P_A(x,y) \\ \propto \quad & \lambda_A P_A(x,y)[u_A(x) - u_A(y)] + (1 - \lambda_A)P_A(x,y) \\ = \quad & P_A(x,y)[\lambda_A(u_A(x) - u_A(y)) + (1 - \lambda_A)], \end{aligned}$$

and similarly, candidate B's expected payoff is an affine transformation of

$$(1 - P_A(x,y))[\lambda_B(u_B(y) - u_B(x)) + (1 - \lambda_B)].$$

Note that we allow the candidates to trade off policy and office differently. Pure policy motivation for A is captured by $\lambda_A = 1$, while pure office motivation is captured by $\lambda_A = 0$, and similarly for B.

Henceforth, we examine pure-strategy Nash equilibria of the strategic game between candidates A and B with strategy space Z for each candidate and payoff functions

$$U_A(x,y) \quad = \quad P_A(x,y)[\lambda_A(u_A(x) - u_A(y)) + (1 - \lambda_A)] \tag{1}$$

and

$$U_B(x,y) \quad = \quad (1 - P_A(x,y))[\lambda_B(u_B(y) - u_B(x)) + (1 - \lambda_B)]. \tag{2}$$

In the central analysis of the paper, we will assume that P_A (and thus P_B) is continuous, so that the electoral game features compact, convex strategy spaces and continuous payoffs. By the Debreu-Fan-Glicksberg theorem, existence of equilibrium then hinges on convexity properties of payoff functions: the critical question is whether $U_A(x, y)$and $U_B(x, y)$ are quasi-concave in, respectively, x and y.[8]

In an interior equilibrium (x^*, y^*) such that P_A is differentiable in x at (x^*, y^*) and such that u_A is differentiable at x^*, the general first order condition for candidate A is

$$\begin{aligned} & D_x P_A(x^*,y^*)[\lambda_A(u_A(x) - u_A(y)) + (1 - \lambda_A)] \\ = \quad & -\lambda_A P_A(x^*,y^*) D u_A(x^*). \end{aligned} \tag{3}$$

[8]Peress (2010) works with more general policy utility that depends on both policies being proposed and a binary parameter $w \in \{0, 1\}$ that indicates the winner of the election, writing the utilities as $u_A(w, x, y)$ and $u_B(w, x, y)$. Our existence result below extends to this setting. We maintain the simpler formulation to stay closer to the previous literature.

Thus, an optimal platform for A balances two forces: the left-hand side of the first order condition reflects the marginal changes in A's expected payoff through her probability of winning as she varies the coordinates of x^*, and the right-hand side reflects the marginal change in her expected payoff through her policy utility. At an optimum, these marginal forces must be exactly balanced. Likewise, the first order condition for B is

$$\begin{aligned} & D_y P_B(x^*, y^*)[\lambda_B(u_B(y) - u_B(x)) + (1-\lambda_B)] \\ = & \quad -\lambda_B P_B(x^*, y^*) D u_B(y^*), \end{aligned}$$

with a similar interpretation. Although the first order condition is generally only necessary, we show in Lemma 8 of Section 6 that it is essentially sufficient when the probability of winning functions satisfy a log concavity condition.

A special case of the abstract model is the *Downsian model* of elections, where we assume a finite, odd number $n \geq 3$ of voters with policy preferences represented by utility functions $u_i \colon Z \to \mathbb{R}$ that are continuous and strictly quasi-concave. Denote the ideal point of voter i by $\hat{z}^i$, and define the dominance relations $\succ$ and $\succeq$ as follows: we write $z \succ z'$ if a majority of voters strictly prefer z to z',

$$z \succ z' \quad \Leftrightarrow \quad \#\{i : u_i(z) > u_i(z')\} > \frac{n}{2},$$

and we write $z \succeq z'$ if a majority of voters weakly prefer z to z',

$$z \succeq z' \quad \Rightarrow \quad \#\{i : u_i(z) \geq u_i(z')\} > \frac{n}{2}.$$

The *majority core* consists of every policy $z \in Z$ such that: for all $z' \in Z$, we have $z \succeq z'$; or equivalently, there does not exist $z' \in Z$ such that $z' \succ z$. In other words, a majority core point is weakly majority preferred to every policy. Since voter utilities are strictly quasi-concave and the number of voters is odd, it follows that if there is a majority core point z, then it is a *Condorcet winner,* in the sense that for all $z' \neq z$, we have $z \succ z'$, i.e., z is strictly majority preferred to every other policy.

In the Downsian model, candidate A's probability of winning satisfies

$$P_A(x, y) \quad = \quad \begin{cases} 1 & \text{if } x \succ y \\ 0 & \text{if } y \succ x, \end{cases} \tag{4}$$

and in case the candidates take the same platform, we assume there exist $\underline{\tau}, \overline{\tau} \in (0,1)$ such that for all $z \in Z$, we have

$$\underline{\tau} \;\leq\; P_A(z, z) \;\leq\; \overline{\tau}. \tag{5}$$

In the remaining case that platforms are distinct and the outcome is decided by indifferent voters, i.e., $x \neq y$ and both $x \succeq y$ and $y \succeq x$, we specify the probability of winning $P_A(x, y) \in (0,1)$ as any strictly positive number less than one. Of course, candidate B's probability of winning is $P_B = 1 - P_A$.

2.2 Existence of Equilibrium

In this section, we establish existence of a Nash equilibrium in the abstract model under general conditions: in addition to our maintained assumptions of continuity and concavity of policy utilities, we only need to require that win probabilities are positive when the candidates locate at the same platform, that they are continuous, and that each candidate's probability of winning is log concave in her own platform. In contrast to Hinich, Ledyard, and Ordeshook's (1972,1973) existence result for the stochastic bias model, we allow for candidates to have policy preferences, as voters do, and we do not rely on a substantial amount of uncertainty. Wittman (1983,1990) allows candidates to have mixed motivations (placing positive weight on policy and office), but he assumes concavity of $P_A(x, y)$ in x and convexity in y. This implicitly relies on the presence of a substantial amount of uncertainty about electoral outcomes, whereas our weak log concavity condition has no such implications. In Section 3, we provide microfoundations for our assumptions in terms of an explicit model of voter preferences and electoral uncertainty.

We first assume that each candidate's probability of winning is strictly positive (but not necessarily equal to one half) when they locate at the same platform:

Condition 1. *For all $z \in Z$, we have $0 < P_A(z, z) < 1$.*

Next, we assume that each candidate's probability of winning is continuous:

Condition 2. *The probability of winning $P_A(x, y)$, and thus $P_B(x, y)$, is jointly continuous in x and y.*

Condition 1 is much weaker than the common assumption that win probabilities are always strictly positive, and Condition 2 is self-evident. It precludes models in which electoral uncertainty is generated by imperfect information about the location of a pivotal voter (the median voter, in one dimension), where counterexamples to equilibrium existence are known (Ball 1999, Groseclose 2001); however, it is satisfied in many environments, including the stochastic partisans model of Section 3. Together, the conditions permit a characterization of optimal policies in the following lemma, which is useful in establishing existence of equilibrium.

Lemma 1. *For all y, if x^* solves*

$$\max_{x \in Z_A(y)} \ln(P_A(x, y)) + \ln(\lambda_A(u_A(x) - u_A(y)) + 1 - \lambda_A), \tag{6}$$

where $Z_A(y) = \{x \in Z : U_A(x, y) > 0\}$, then it solves

$$\max_{x \in Z} U_A(x, y). \tag{7}$$

Under Condition 1, assuming that $\lambda_A < 1$, or that both Condition 2 and $y \neq \hat{x}$ hold, then the converse also holds: if x^* solves (7), then it solves (6). Analogous statements hold for candidate B, with $Z_B(x) = \{y \in Z : U_B(x, y) > 0\}$.

Proofs not contained in the main text appear in the appendix.

Our final assumption for equilibrium existence is that each candidate's probability of winning is log concave in her own platform. For all $x, y \in Z$, we let

$$S_A(y) \quad = \quad \{x \in Z : P_A(x, y) > 0\}$$

denote the *support set* of A at y, and we let

$$S_B(x) \quad = \quad \{y \in Z : P_A(x, y) > 0\}$$

denote the support set of B at x. Note that by Condition 1, these sets are nonempty.

Condition 3. *For all $x, y \in Z$, $P_A(\cdot, y)$ is log concave on the convex set $S_A(y)$, and $P_B(x, \cdot)$ is log concave on the convex set $S_B(x)$.*

The next theorem establishes that the above conditions are sufficient for existence of a Nash equilibrium.

Theorem 1. *Under Conditions 1–3, there is a Nash equilibrium in the general model.*

For the case of candidates who place positive weight on office, the proof of Theorem 1 consists of a straightforward application of Kakutani's fixed point theorem. The case in which at least one candidate, say A, is purely policy motivated is potentially complicated by the fact that Lemma 1 does not directly apply when B locates at $\hat{x}$. When $y = \hat{x}$, it is true that a best response for A is to also locate at $\hat{x}$, but there may be other best responses—A could just as well locate at any platform that loses with probability one—and these could form a non-convex set. This can be finessed by considering a refinement of A's best response correspondence,[9] but the problem is solved straightforwardly by using the fact that the equilibrium correspondence of the game has closed graph.

Remark 1. Theorem 1 can be applied in the one-dimensional stochastic valence model. Duggan (2025a) notes that when the valence density f is log concave, the functions F and $1 - F$ inherit this property. Then the log of candidate A's probability of winning,

$$\ln(F(u_m(x) - u_m(y))),$$

is an increasing, concave transformation $\ln \circ F$ of a concave function $u_m(x) - u_m(y)$ of x, and thus $P_A(x, y)$ is concave in x. Similarly, B's probability of winning

$$\ln(1 - F(u_m(x) - u_m(y))),$$

[9] See Roemer (1997,2001) for an example in the stochastic median model.

is a de creasing, concave transformation $\ln \circ (1-F)$ of a convex function $u_m(x) - u_m(y)$ of y, and thus $P_B(x,y)$ is concave in y. It follows that under Conditions 1–3, and Theorem 1 yields existence of equilibria in the stochastic valence model. Of course, the Downsian and stochastic median models violate continuity, so Theorem 1 does not apply. □

Remark 2. A special case of the stochastic valence model is that of normally distributed valence, i.e., F is the normal distribution with mean μ and standard deviation σ. By the preceding remark, the one-dimensional model with normal shock admits a Nash equilibrium—regardless of the mean and variance of the shock distribution. In particular, fixing μ, we can analyze equilibria as uncertainty is removed from the model, i.e., $\sigma \to 0$. When the distribution of valence is concentrated near μ, the function F approximates a step function, and the probability of winning $F(u_m(x) - u_m(y))$ will fail to be concave in x. Whereas existence results that assume substantial uncertainty (concavity of the probability of winning function) cannot be applied, Theorem 1 ensures that the analysis is not vacuous: equilibria exist for all σ, even as the variance converges to zero, and we can therefore examine the limiting properties of equilibria. Assuming that $\mu > 0$, and that the advantaged candidate B places positive weight on policy, Duggan (2025a) shows that candidate A's equilibrium platform converges to the median voter's ideal point, and B's platform converges to the platform (on B's side of the median) that makes the median voter indifferent between the candidates. That is, candidate B leverages her valence advantage to obtain preferred policy outcomes, and moreover, B's probability of winning converges to one along the sequence. □

2.3 Uniqueness with Office-Motivated Candidates

When candidates are purely office motivated, an equilibrium exists by Theorem 1. We show next that under a strict quasi-concavity condition on the probability of winning functions, the equilibrium is, in fact, unique. Of course, because the electoral game between office-motivated candidates is constant sum, each candidate's platform secures at least her equilibrium probability of winning, regardless of the other's policy position. The equilibrium platforms are what Calvert (1985) refers to as the "estimated medians" of the candidates. When the probability of winning is constant on the diagonal, where both candidates choose the same platform, we furthermore show that the equilibrium features "convergence," i.e., the candidates locate at the same platform. That platform is the unique position securing each candidate at least her equilibrium probability of winning. Each candidate wins with positive probability, but because we do not assume symmetry on the diagonal, it is possible that one wins with higher probability than the other.

Theorem 2. *Under Conditions 1–3, assume that for all $x, y \in Z$, $P_A(x,y)$ is a strictly quasi-concave function of x on $S_A(y)$, and it is strictly quasi-convex in y on $S_B(x)$. If the candidates are purely office motivated, i.e., $\lambda_A, \lambda_B = 0$, then* *there is a unique Nash equilibrium (x^*, y^*) in the general model, and it satisfies: for all $z \in Z$,*

$$P_A(z, y^*) \;\leq\; P_A(x^*, y^*) \;\leq\; P_A(x^*, z).$$

Moreover, if $P_A(z, z)$ is constant in z, then $x^* = y^*$.

Under the assumptions of Theorem 2, we let z^* denote the unique equilibrium platform when both candidates are purely office motivated.

Remark 3. Note that to obtain platform convergence, Calvert (1985) imposes the assumption that the game is symmetric, i.e., $P_A(x, y) = P_B(y, x)$, which implies that $P_A(z, z) = \frac{1}{2}$ for all z. We use the weaker assumption that the probability of winning is constant along the diagonal. $\square$

2.4 Competitiveness of Elections

Characterization of equilibrium at this abstract level is challenging, but very generally, each candidate wins with positive probability in equilibrium; that is, elections are typically competitive.

We say a platform pair (x, y) is *non-standard* if one candidate wins with probability one at the other's ideal point: formally, either (i) $y = \hat{x}$ and $P_A(x, y) = 0$, or (ii) $x = \hat{y}$ and $P_A(x, y) = 1$. Otherwise, the pair (x, y) is *standard.* Obviously, if Condition 1 is strengthened so that each candidate's probability of winning is always positive, then all platform pairs are standard. Moreover, by Lemma 1, there do not exist non-standard Nash equilibria when both candidates place positive weight on office: if (x^*, y^*) is an equilibrium, then the lemma implies that each candidate's payoff is strictly positive, and thus each wins with positive probability. In the stochastic partisans model of Section 3, we give weak conditions under which non-standard equilibria *never* exist in the interior of the policy space, regardless of the supports of the probability of winning functions, and (essentially) irrespective the candidates' policy weights.

Nevertheless, the following example demonstrates that, in lieu of additional background conditions, non-standard equilibria may exist in the model.

Example 1. Assume that $Z \subseteq \Re$, that A is purely policy motivated, and that B is purely office motivated, so $\lambda_A = 1$ and $\lambda_B = 0$. Assume that for some $x \in Z$, we have $P_A(x, \hat{x}) = 0$; intuitively, x is sufficiently extreme that when B locates at A's ideal point, she wins with probability one. Then $(x, \hat{x})$ is an equilibrium, as each candidate receives her maximum payoff. As discussed above, this example relies on the assumption that A's probability of winning is not always positive, and that it is not the case that both candidates place positive weight on office. $\square$

The following theorem establishes that in every standard equilibrium, each candidate's payoff is positive. Note that the result relies only on Conditions

1 and 2, without the presence of Condition 3, which we would add to ensure existence.

Theorem 3. *Under Conditions 1 and 2, for every standard Nash equilibrium (x^*, y^*) in the general model, we have $U_A(x^*, y^*) > 0$ and $U_B(x^*, y^*) > 0$. Therefore, if $\lambda_A, \lambda_B < 1$, then the candidates' payoffs are strictly positive in every Nash equilibrium.*

In particular, each wins with positive probability, a fact that we use later in our analysis, in Theorem 12, of competitiveness of elections.

Corollary 1. *Under Conditions 1 and 2, for every standard Nash equilibrium (x^*, y^*) in the general model, the candidates each win with positive probability: $P_A(x^*, y^*) > 0$ and $P_B(x^*, y^*) > 0$.*

2.5 Continuity of Equilibrium

This section complements our existence result with a general analysis of equilibrium continuity with respect to parameters of the general model. The conclusion, when the Downsian model of electoral competition is taken as the limiting model, is that the addition of a small amount of noise to the Downsian model—either with purely policy-motivated candidates or with candidates who place positive weight on office—cannot solve the problem of non-existence in the multidimensional setting. We imbed the general model in a space Θ, where each model $\theta \in \Theta$ corresponds to a full specification of parameters:

$$(u_A^\theta, u_B^\theta, \lambda_A^\theta, \lambda_B^\theta, P_A^\theta, P_B^\theta),$$

where u_A^θ and u_B^θ are continuous and concave, $\lambda_A^\theta, \lambda_B^\theta \in [0,1]$, and $P_B^\theta = 1 - P_A^\theta$. We specify candidate payoffs as above, using the notation U_A^θ and U_B^θ to bring out the model explicitly.

Fixing a sequence $\{(\theta^m, x^m, y^m)\}$ of models and platform pairs and a "limiting" triple (θ, x, y), and given $\epsilon > 0$, we say z is an *ϵ-robust better reply to (x, y) at θ for A* if (1) $U_A^\theta(z, y) > U_A^\theta(x, y)$, (2) $P_A^\theta(z, y) > 0$, and (3) there exists a natural number $\overline{m}$ such that for all $m \geq \overline{m}$,

$$\frac{P_A^{\theta^m}(x^m, y^m)}{P_A^{\theta^m}(z, y^m)} \quad < \quad \frac{P_A^\theta(x, y)}{P_A^\theta(z, y)} + \epsilon, \tag{8}$$

where $B_\epsilon(z)$ denotes the open ball of radius epsilon containing z. In words, z is a profitable deviation, and even if the candidates' platforms are perturbed slightly, then z increases A's probability of winning (relative to x^m) by an amount that is not too much less than the increase from z (relative to x). Likewise, z is an *ϵ-robust better reply to (x, y) at θ for B* if (1) $U_B^\theta(x, z) > U_B^\theta(x, y)$, (2) $P_B^\theta(x, z) > 0$, and (3) there exists a natural number $\overline{m}$ such that for all $m \geq \overline{m}$,

$$\frac{P_B^{\theta^m}(x^m, y^m)}{P_B^{\theta^m}(x^m, z)} \quad < \quad \frac{P_B^\theta(x, y)}{P_B^\theta(x, z)} + \epsilon.$$

We use the concept of robust better reply to formally isolate sequences along which the equilibrium correspondence has closed graph.

A sequence $\{(\theta^m, x^m, y^m)\}$ of models and platform pairs satisfies the *closed graph criterion at* (θ, x, y) if we have

(i) $u_A^{\theta^m} \to u_A^\theta$ and $u_B^{\theta^m} \to u_B^\theta$ uniformly, $\lambda_A^{\theta^m} \to \lambda_A^\theta$, and $\lambda_B^{\theta^m} \to \lambda_B^\theta$.

(ii) $(x^m, y^m) \to (x, y)$,

(iii) if (x, y) is not a Nash equilibrium at θ, then for all $\epsilon > 0$, some candidate has an ϵ-robust better reply to (x, y) at θ.

Condition (iii) is a lower semi-continuity condition. It says that if (x, y) is not an equilibrium at θ, then one candidate, say A, has an ϵ-robust better reply. This implies that if z increases candidate A's probability of winning relative to x in the model θ, then even if the initial platforms x and y are allowed to vary slightly, then z achieves close to (or high than) that relative increase in nearby models. This condition is automatically satisfied if the probability of winning is jointly continuous at (θ, x, y) and (θ, z, y). But the condition is much weaker than that: it may be that one candidate and not the other has an ϵ-robust better supply; and it may be that not all better replies have this robustness property. Furthermore, the condition allows for the possibility that candidate A can obtain a higher (by a strictly positive increment) relative increase in her probability of winning by moving to a nearby platform, and thus it does not imply continuity of the probability of winning.

Remark 4. The closed graph criterion above is similar to Reny's (1999) better reply security condition, which he uses to establish existence of equilibria in games with discontinuous payoffs. His condition implies that if a profile of strategies is not an equilibrium, then some player has a better reply that can secure a higher payoff, even if others' strategies are allowed to vary slightly. Unlike his condition, ours considers a sequence of models, rather than a fixed model, and it is not stated in terms of payoffs directly, but only on the underlying probability of winning. □

Next, we state a general continuity result for the general model. Although our notion of approximation is not topological, the theorem is essentially a closed graph result for the equilibrium correspondence. An implication of the result (using compactness of Z) is that if there is no equilibrium in the limiting model θ, then for high enough m, there is no equilibrium in model θ^m.

Theorem 4. *Assume that* $\{(\theta^m, x^m, y^m)\}$ *satisfies the closed graph criterion at* (θ, x, y)*. If* (x^m, y^m) *is a Nash equilibrium of* θ^m *for all* m*, then* (x, y) *is a Nash equilibrium of* θ*.*

Importantly, because it permits discontinuities in the probability of winning, Theorem 4 can be applied to the Downsian model as the limiting case. Fix candidate utilities u_A and u_B, weights $\lambda_A, \lambda_B \in [0, 1)$, and let θ be the Downsian model with continuous, strictly quasi-concave voter utilities u_i, so that A's probability of winning P_A^θ satisfies equations (4) and (5). Consider a sequence of models $\{\theta^m\}$ such that for all $x, y \in Z$, we have:

(a) if $x \succ y$, then there exist open neighborhoods G of x and H of y such that $P_A^{\theta^m}(\cdot) \to 1$ uniformly on $G \times H$,

(b) if $y \succ x$, then there exist open neighborhoods G of x and H of y such that $P_A^{\theta^m}(\cdot) \to 0$ uniformly on $G \times H$,

where $\succ$ refers to the relation of majority dominance in model θ.[10] Thus, if one candidate's platform dominates the other's in majority voting, then along the sequence of models, her probability of winning converges to one, even if the platforms of the candidates are allowed to vary slightly. Our analysis uses the following lemma, which is well-known and straightforward to prove using strict quasi-concavity of voter utilities.

Lemma 2. *For all distinct $x, y \in Z$ and all $\alpha \in (0, 1)$, if $x \succeq y$, then $\alpha x + (1 - \alpha)y \succ y$.*

In the next result, which assumes the candidates place strictly positive weight on office, we consider a sequence of models satisfying conditions (a) and (b) above, and any corresponding sequence of equilibria. We show that there is a subsequece of equilibria that satisfies the closed graph criterion, and thus Theorem 4 applies. Of course, conditions (i) and (ii) in the definition of the closed graph criterion are satisfied by construction, so the key is the lower semi-continuity condition (iii).

Theorem 5. *Fix continuous, concave utilities u_A and u_B, and payoff parameters $\lambda_A, \lambda_B < 1$. Let θ be a Downsian model of elections, and consider a sequence $\{\theta^m\}$ of models such that for all $x, y \in Z$, conditions (a) and (b) hold. If (x^m, y^m) is a Nash equilibrium of θ^m for all m with $(x^m, y^m) \to (x, y)$, then (x, y) is a Nash equilibrium of θ.*

Remark 5. Theorem 5 of Duggan and Fey (2005) proves the closed graph result for the case of pure policy motivation in the limit model, i.e., $\lambda_A^\theta = \lambda_B^\theta = 0$, with the additional assumption that win probabilities in each model θ^m are positive for both candidates. That paper imposes the background assumption of differentiable policy utility, but the differentiability assumption is not used in the proof of the theorem. Our Theorem 5 above covers the case of mixed motivations with positive weight on office for both candidates. $\square$

[10] Propositions 4 and 5 of Duggan and Fey (2005) provide two sets of conditions under which (a) and (b) hold. The first is the stochastic bias model with shocks independently distributed across voters such that the shock distribution converges weakly to the unit mass on zero. The other is a multidimensional version of the stochastic median model, in which preference shocks converge weakly to a degenerate distribution.

We end this section with two interesting implications of Theorem 5. First, in the one-dimensional model, if both candidates place positive weight on office, then a median voter result holds: the unique equilibrium in the limiting model is that both candidates locate at the median voter's ideal point, and by our continuity result, if we add a small amount of uncertainty to the model, in the sense of (a) and (b), then equilibria (if they exist) in models close to θ must be close to the median.

Corollary 2. *Assume $Z \subseteq \mathbb{R}$, and fix continuous, concave utilities u_A and u_B and payoff parameters $\lambda_A, \lambda_B < 1$. Let θ be a one-dimensional Downsian model of elections, and consider a sequence $\{\theta^m\}$ of models such that for all $x, y \in Z$, conditions (a) and (b) hold. If (x^m, y^m) is a Nash equilibrium of θ^m for all m with $(x^m, y^m) \to (x, y)$, then the candidates' platforms converge to the ideal point of the median voter: $x = y = \hat{z}_M$.*

The above implication is relevant when the policy space is one-dimensional, but in the multidimensional Downsian model, it is generically the case that the majority core is empty: for all y, there is some z such that $z \succ y$. In this case, assuming the candidates place positive weight on office, equilibria fail to exist. To see why, note that in any equilibrium (x, y), each candidate must win with positive probability: if, say, candidate A's probability of winning were zero, then she could increase her payoff by locating at y.

Moreover, the candidates platforms must be located at the same policy: if $x \neq y$, then using Lemma 2, candidate A could locate at $z_\alpha = \alpha x + (1-\alpha)y$, and for $\alpha \in (0,1)$, she would win with probability one; since $u_A(x) \geq u_A(y)$, this would increase her payoff. Finally, the policy at which the candidate locate must belong to the majority core: if $x = y$ and $z \succ y$, then $z_\alpha = \alpha x + (1-\alpha)z$ would be a profitable deviation for $\alpha \in (0,1)$ close enough to one. This line of argument establishes the second implication of Theorem 5.

Corollary 3. *Fix continuous, concave utilities u_A and u_B and payoff parameters $\lambda_A, \lambda_B < 1$. Let θ be a multidimensional Downsian model of elections such that the majority core is empty, and consider a sequence $\{\theta^m\}$ of models such that for all $x, y \in Z$, conditions (a) and (b) hold. Then for high enough m, there are no Nash equilibria in the model θ^m.*

Corollary 3 demonstrates that, assuming candidates place positive weight on office, if there is no equilibrium in the multidimensional Downsian model, then adding a small amount of uncertainty to the model *cannot* solve the existence problem. If we want a theory of elections that can be applied in environments with low uncertainty, which is permitted under the conditions of Theorem 1, then we must look to alternatives to the Downsian model for microfoundations. We turn to this in the following section.

Remark 6. Banks and Duggan (2005) prove similar results for the stochastic bias model. Theorem 10 of that article is a closed graph result for the special case in which the limiting model is Downsian. It establishes that if a sequence of models approaches θ, and if a corresponding sequence of pure strategy equilibria converges to (x, y), then the limiting platform pair must be such that x and y belong to the core of the Downsian model; therefore, if the core is empty in the Downsian model, then adding a small amount of noise cannot create equilibria in pure strategies. In contrast to our results, Banks and Duggan (2005) restrict attention to vote-maximizing candidates, and their closed graph result allows for sequences of mixed strategy equilibria, rather than focusing on pure strategies, as we do here. $\square$

3 Stochastic Partisans Model

3.1 Microfoundations

In this section, we open the black box of the probability of winning function to provide a microfounded model satisfying Conditions 1–3 for equilibrium existence, while allowing for multiple policy dimensions and low uncertainty. By Corollary 3, the Downsian model does not provide such a foundation: if equilibria do not exist in the Downsian model, the typical the case in multiple dimensions, then adding a small amount of uncertainty does not create equilibria. The critical condition here is log concavity in Condition 3. It is possible to add a small amount of uncertainty to smooth out the probability of winning and to satisfy the other two conditions, but non-convexities present in the Downsian model are inherited by nearby probabilistic election models. We therefore propose an alternative foundation, in which a continuum of policy-oriented voters is partitioned into a set of types, each corresponding to a policy utility function; within each type, there is a distribution of bias; and aggregate uncertainty is introduced via a stochastic mass of partisan voters who turnout to support the candidates, regardless of their policy platforms.

Formally, a unit mass N of *policy oriented* voters is partitioned into a probability space $(T, \mathscr{T}, \tau)$ of voter types t, where the fraction of policy-oriented voters with types belonging to a measurable set $S \in \mathscr{T}$ is $\tau(S)$. We assume that there is a continuum of voters of each type t, and that the policy utilities of the type t voters are given by $u_t\colon Z \to \mathbb{R}$, where $u_t(z)$ is concave and continuous in z, with ideal point $\hat{z}^t$, and it is also measurable in t with the Borel sigma-algebra on $\mathbb{R}$.

In addition to policy, voters' evaluations of the candidates incorporate idiosyncratic bias terms β_i, which are distributed according to F_t among the type t voters. We maintain the assumption that $F_t(\beta_i)$ is continuous in β_i, and although it is not important for equilibrium existence, we assume that the restriction of F_t to its support set, denoted $S_{F_t} = \{\beta_i : 0 < F_t(\beta_i) < 1\}$, is differentiable, and that the restriction of the density f_t to S_{F_t} is continuous.

Moreover, we assume that the support set S_{F_t} contains all possible utility differences for the type t voters, i.e., for all $x, y \in Z$,

$$f_t(u_t(x) - u_t(y)) \quad > \quad 0. \tag{9}$$

Finally, we assume that $F_t(\beta_i)$ is measurable in t, and that $f_t(\beta_i)$ is uniformly τ-integrable as a function of t, which implies that $f_t(\beta_i)$ is also measurable in t (Aliprantis and Burkinshaw, 1990, Theorem 20.3).[11]

Remark 7. This model of voter types is general, and it admits the case of a finite set of voter types as a special case, where we index the type set as $T = \{t_1, \ldots, t_n\}$ and give T the discrete sigma-algebra. Then we can define τ_j as the share of type t_j voters among the policy-oriented electorate. Integrability is then moot, and integrals are replaced by weighted sums. $\square$

We interpret β_i as a net bias for candidate B, so a voter i who is type t supports A if and only if $u_t(x) \geq u_t(y) + \beta_i$. Then the fraction of type t voters supporting A is $F_t(u_t(x) - u_t(y))$, and thus the mass of policy oriented voters who vote for A is

$$\int F_t(u_t(x) - u_t(y)) d\tau,$$

and the mass of policy oriented voters who vote for B is

$$1 - \int F_t(u_t(x) - u_t(y)) d\tau.$$

Thus, the net mass of votes in favor of candidate A among policy oriented voters is

$$2\left(\int F_t(u_t(x) - u_t(y)) d\tau\right) - 1,$$

and the net vote in favor of B is negative one times this. Since $f_t(u_t(x) - u_t(y)) > 0$, it follows that regardless of the platform choices of the candidates, some type t voters will support A, and some will support B.

Since we assume a continuum of voters of each type, the idiosyncratic biases do not introduce aggregate uncertainty into the election. To do so, we assume a set P of *partisan voters* who stochastically vote for one candidate or the other independent of policy after the candidates adopt their platforms. The relevant quantity is π, the net mass of partisan voters who vote in favor of candidate B, which we model as a random variable distributed according to the distribution function G, where we assume that G is continuous, that the restriction of G to its support set, denoted $S_G = \{\pi : 0 < G(\pi) < 1\}$, is differentiable, and that the restriction of the density g to S_G is differentiable. This allows us to capture the family of uniform densities as particular cases.

[11] We say $f_t(\beta_i)$ is *uniformly τ-integrable* if there is a τ-integrable function $h\colon T \to \mathbb{R}$ such that for all $\beta_i \in \mathbb{R}$ and all $t \in T$, we have $h(t) \geq |f_t(\beta_i)|$. Note that uniform τ-integrability is implied if $f_t(\beta_t)$ is bounded, i.e., there exists $b > 0$ such that for all for all $\beta_i \in \mathbb{R}$ and all $t \in T$, we have $f_t(\beta_i) \leq b$.

Since G is nonatomic, we can ignore the possibility of a tied election, and therefore candidate A wins if and only if

$$\pi \leq 2\left(\int F_t(u_t(x) - u_t(y))d\tau\right) - 1,$$

which occurs with probability

$$P_A(x,y) = G\left(2\left(\int F_t(u_t(x) - u_t(y))d\tau\right) - 1\right). \tag{10}$$

We assume that the support set S_G is convex, but it may be small; we assume only that

$$0 < G\left(2\left(\int F_t(0)d\tau\right) - 1\right) < 1, \tag{11}$$

so that when the candidates locate at the same platform, each wins with positive probability. By (11), Condition 1 is automatically satisfied, and continuity of the distributions F_t and G implies that the probability of winning is continuous for each candidate, delivering Condition 2.

Remark 8. Although we assume that each F_t has support containing all possible utility differences of the type t voters, we allow the support of G to be small, subject to satisfying (11). Thus, for example, we allow G to be uniform with arbitrarily small support, as long as each candidate's probability of winning is strictly positive when the candidates adopt the same platform. □

Finally, we impose the following condition on the curvature of F_t and G to fulfill Condition 3.[12]

Condition 4. *The density g is log concave on its convex support, and for all t, $F_t(u_t(x) - u_t(y))$ is concave in x and convex in y.*

By Theorems 1 and 3 of Bagnoli and Bergstrom (2005), Condition 4 implies that G and $1 - G$ are log concave on the support of G. Then, as a function of x, $\ln(P_A(x,y))$ is the composition of an increasing, concave function $\ln \circ G$, with a concave function, the integral of $F_t(u_t(x) - u_t(y))$ across types, making it concave. Similarly, as a function of y, $\ln(P_B(x,y))$ is the composition of a decreasing, concave function $\ln \circ (1 - G)$, with a convex function, making it again concave. We conclude that the functions P_A and P_B satisfy Conditions 1–3. It would obviously be sufficient for to assume G and $1-G$, rather than the density g, are log concave; we maintain the stronger assumption in Condition 4, for simplicity.

Remark 9. Persson and Tabellini (2000, Section 3.4) consider a one-dimensional model with a similar decomposition of shocks. They assume three voter types,

[12] Condition 4 implies that the closure of the support of each F_t contains all possible utility differences, $u_t(x) - u_t(y)$, as x and y vary over Z. This is implied by inequality (9).

a continuum of voters of each type, and two candidates who are purely office motivated. Each voter has an idiosyncratic bias in favor of one candidate, and following the choice of platforms, a net valence for that candidate is drawn. Persson and Tabellini assume that the idiosyncratic bias and the stochastic valence shock are both uniformly distributed over a sufficiently large interval. This determines a probability of winning function similar to (10), but which is linear in the voters' utility differences. □

Remark 10. Desai (2025) analyzes a two-dimensional policy space, and a continuum of voters with four voter types. He also uses a similar decomposition of shocks, assuming an idiosyncratic bias term and an aggregate shock, both uniformly distributed with large support. In his model, parties are policy motivated and complete on two dimensions, a linear tax and an identity-based social cleavage dimension on which they incur a cost of campaigning. He proves existence and uniqueness of equilibrium in this model. □

3.2 Existence of Equilibrium

Having verified Conditions 1–3, Theorem 1 immediately delivers existence of equilibrium in the general stochastic partisans model.

Theorem 6. *Under Condition 4, there is a Nash equilibrium in the stochastic partisans model.*

Theorem 6 allows for any number of policy dimensions. Condition 4 does assume the idiosyncratic bias is sufficiently dispersed, a condition familiar from Hinich, Ledyard, and Ordeshook (1972,1973), Lindbeck and Weibull (1987,1993), Peress (2010), and others. However, the log concavity of g is quite weak, and it permits many familiar density functions from probability theory. In particular, equilibria exist even if the variance of the aggregate shock is low, and the distribution of π is close to degenerate.

All of our results hold with a modification of the model in which each type t voter also receives idiosyncratic, expressive cost of voting, say, $\gamma_i \geq 0$. A type t voter i with idiosyncratic bias ϵ_i supports candidate A if and only if

$$\gamma_i \quad \leq \quad u_t(x) - u_t(y) - \epsilon_i,$$

Now, let F_t^A denote the distribution function for the sum $\gamma_i + \epsilon_i$, so the fraction of type t voters who support A is $F_t^A(u_t(x) - u_t(y))$. Voter i supports B if and only if

$$\gamma_i \quad \leq \quad u_t(y) - u_t(x) + \epsilon_i,$$

and letting F_t^B denote the distribution for $\gamma_i - \epsilon_i$, the fraction who support B is $F_t^B(u_t(y) - u_t(x))$. We again assume stochastic partisans, and we extend

Condition 4 so that $F_t^B(u_t(y) - u_t(x))$ is convex in x and concave in y. Then the net mass of policy oriented voters in favor of A is

$$\int (F_t^A(u_t(x) - u_t(y)) - F_t^B(u_t(y) - u_t(x)))d\tau,$$

which is concave in x and convex in y, and our existence result goes through.

3.3 Pareto Optimality of Equilibrium

As a first step toward characterizing equilibria, we next make the simple observation that in equilibrium, each candidate's platform is such that there is no policy change that helps the candidate without hurting at least one voter. Informally, the candidates do not "waste" policy or probability of winning.

Formally, we say x is *Pareto optimal for* $T \cup \{A\}$ if there does not exist $x' \in Z$ such that (i) $u_A(x') \geq u_A(x)$ and for all $t \in T$, $u_t(x') \geq u_t(x)$, and (ii) either $u_A(x') > u_A(x)$ or there exists measurable $S \subseteq T$ with $\tau(S) > 0$ such that for all $t \in S$, we have $u_t(x') > u_t(x)$. We define y to be *Pareto optimal for* $T \cup \{B\}$ if an analogous condition holds, with B substituted for A. Note that we restrict attention to standard equilibria; later, in Subsection 3.5, we provide weak conditions under which all interior equilibria are standard.

Theorem 7. *Under Condition 4, assume $\lambda_A, \lambda_B > 0$. For every standard Nash equilibrium (x^*, y^*) in the stochastic partisans model, (i) x^* is Pareto optimal for $T \cup \{A\}$, and (ii) y^* is Pareto optimal for $T \cup \{B\}$.*

We can sharpen the Pareto optimality result by adding differentiability of policy utilities. Note that Condition 5, below, implies that the derivative $Du_t(z)$ is measurable in t, and that for all $x, y \in Z$, $F_t(u_t(x) - u_t(y))$ is measurable as a function of t.[13] Moreover, since $D_x u_t(x)$ is uniformly bounded, the derivative,

$$D_x F_t(u_t(x) - u_t(y)) \quad = \quad f_t(u_t(x) - u_t(y)) D_x u_t(x),$$

is uniformly τ-integrable. It follows that $\int F_t(u_t(x) - u_t(y))d\tau$ is also differentiable, and that we can take the derivative with respect to x through the integral sign,

$$D_x \int F_t(u_t(x) - u_t(y))d\tau \quad = \quad \int f_t(u_t(x) - u_t(y)) D_x u_t(x) d\tau,$$

[13] Note that $h_0(t, \beta) \equiv F_t(\beta)$ is measurable in t and continuous in β, so it is a Caratheodory function, and it is therefore measurable with respect to the product sigma-algebra $\mathscr{T} \otimes \mathscr{B}_{\mathbb{R}}$ on $T \times \mathbb{R}$ and the Borel sigma-algebra $\mathscr{B}_{\mathbb{R}}$ on $\mathbb{R}$ (Aliprantis and Border, 2006, Lemma 4.51). Similarly, the mapping $h_2(t, x) \equiv u_t(x) - u_t(y)$ is measurable with $\mathscr{T} \otimes \mathscr{B}_Z$ on $T \times Z$, where $\mathscr{B}_Z$ is the Borel sigma-algebra on Z, and with $\mathscr{B}_{\mathbb{R}}$ on the real line. Of course, the projection mapping $h_1(t, x) \equiv t$ is $(\mathscr{T} \otimes \mathscr{B}_Z, \mathscr{T})$-measurable. Then the Cartesian product of mappings $(h_1, h_2)(t, x) \equiv (t, u_t(x) - u_t(y))$ is measurable with $\mathscr{T} \otimes \mathscr{B}_Z$ on $T \times Z$ and $\mathscr{T} \otimes \mathscr{B}_{\mathbb{R}}$ on $T \times \mathbb{R}$. Finally, we can view $F_t(u_t(x) - u_t(y))$ as the composition of h_0 and (h_1, h_2), which is $(\mathscr{T} \otimes \mathscr{B}_Z, \mathscr{B}_{\mathbb{R}})$-measurable.

with analogous remarks for the derivative with respect to y (Aliprantis and Burkinshaw, 1990, Theorem 20.3).

Condition 5. *Candidate utility functions u_A and u_B are differentiable, that for each t, u_t is differentiable, and that $Du_t(z)$ is uniformly bounded, i.e., there exists $c > 0$ such that for all $z \in \mathsf{int}\, Z$ and all $t \in T$, $\|Du_t(z)\| \leq c$.*

In an interior equilibrium (x^*, y^*), candidate A's platform satisfies the necessary first order condition in (3), which becomes:

$$2g[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \int f_t Du_t(x^*) d\tau + \lambda_A G Du_A(x^*) \quad = \quad 0,$$

where g and G are evaluated at $2\int F_t(u_t(x^*) - u_t(y^*))d\tau - 1$, and f_t is evaluated at $u_t(x^*) - u_t(y^*)$. This is also the first order condition for a welfare maximization problem with concave objective function, and thus, A's equilibrium policy maximizes a weighted sum of candidate and voter utility, as we establish in the next theorem.

Theorem 8. *Under Condition 5, for every interior, standard Nash equilibrium (x^*, y^*) in the stochastic partisans model, x^* solves*

$$\max_{x \in Z} \alpha u_A(x) + (1 - \alpha) \int F_t(u_t(x) - u_t(y^*)) d\tau,$$

where

$$\alpha \quad = \quad \frac{G\lambda_A}{G\lambda_A + 2g[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A]},$$

with a corresponding result holding for candidate B.

In Figure 1, we depict candidate A's optimal platform choice, given the platform y^* of candidate B. Here, we assume A has quadratic utility; we show her indifference curve (a perfect circle) through x^*; and we illustrate sample contour sets of the function

$$\int F_t(u_t(x) - u_t(y^*)) d\tau, \tag{12}$$

which is parameterized by y^*. Consistent with Theorem 8, the optimal platform for A is such that the gradients of u_A and (12) point in opposite directions. Adding the assumption that F_t is uniform, we will see in Section 4 that B's policy enters only through a constant term, and thus does not affect the level sets of this function. This allows us to provide foundations for an aggregate voter result, in which the electoral game is "as if" there is a single policy oriented voter.

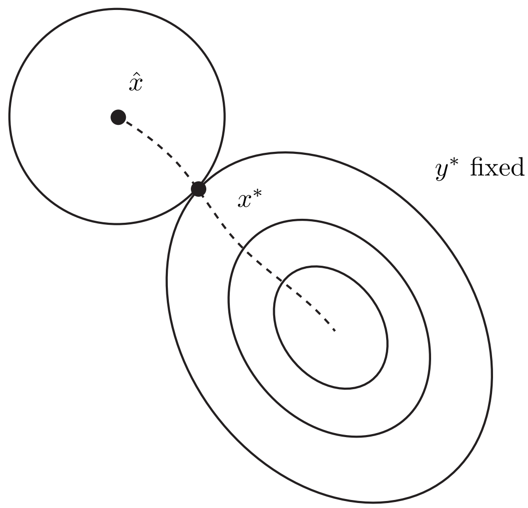

Figure 1: Optimality of x^*

3.4 Office-Motivated Candidates and Welfare Optimality

We now consider the classical case of purely office-motivated candidates, in which the electoral game is constant sum. In the general model, Theorem 2 provides a strict quasi-concavity condition under which the equilibrium is unique, with both candidates adopting the platform z^*. Here, we use the structure of the stochastic partisans model to sharpen that result, and to provide an approximate welfare optimality result as the candidates place high weight on office. We verify the strict quasi-concavity condition of Theorem 2 to obtain equilibrium uniqueness, and since A's probability of winning is constant and equal to

$$P_A(z,z) \quad = \quad G\left(2\left(\int F_t(0)d\tau\right)-1\right)$$

when the candidates choose the same platform, it follows that the equilibrium is "convergent," with $x^* = y^* = z^*$.

Moreover, if it is interior, then the equilibrium platform maximizes a social welfare function that is a weighted sum of voter utility, where the weight on the measurable set S of voters is $\int_S f_t(0)d\tau$, representing the marginal impact on the support from voters in S from a small change in a candidate's platform.

Theorem 9. *Under Condition 4, assume that the candidates are purely office motivated, i.e., $\lambda_A = \lambda_B = 0$, and that candidate A's share of policy-oriented voters, in (12), is strictly quasi-concave in x and strictly quasi-concave in y. There is a unique Nash equilibrium (x^*, y^*) in the stochastic partisans model, and this satisfies $x^* = y^* = z^*$. If the equilibrium is interior, and if each u_t is* *differentiable, then z^* is the unique solution to the welfare maximization problem*

$$\max_{z\in Z} \int f_t(0)u_t(z)d\tau. \tag{13}$$

This characterization is closely related to findings by Hinich (1977), Lindbeck and Weibull (1987,1993), and Banks and Duggan (2005), who assume that candidates are vote maximizing, rather than seeking to maximize probability of winning. The latter article establishes that, under the conditions of Theorem 9, there is a unique pure strategy equilibrium in the game between vote-maximizing candidates. Because we assume there is no abstention on the part of voters, we can define the payoff of candidate A in that game as her plurality among policy-oriented voters,

$$V_A(x,y) \quad = \quad 2\int F_t(u_t(x) - u_t(y))d\tau - 1,$$

and we define candidate B's payoffs by $V_B = -V_A$. Let (w^*, w^*) denote the unique equilibrium of this game.

The next result highlights the close connection: the equilibrium in the stochastic partisans model with purely office-motivated candidates is also the unique equilibrium in the game between vote-maximizing candidates, in which there is no partisan shock. Indeed, if $z^* \neq w^*$, then we can assume that candidate A has a profitable deviation from (z^*, z^*) in the game between vote-maximizing candidates, so there exists $x' \in Z$ such that

$$2\int F_t(u_t(x') - u_t(z^*))d\tau - 1 \;=\; V_A(x', z^*) \;>\; V_A(z^*, z^*) \;=\; 2\int F_t(0)d\tau - 1,$$

but by (11), it follows that $U_A(x', z^*) > U_A(z^*, z^*)$, contradicting the fact that (z^*, z^*) is an equilibrium in the stochastic partisans model with purely office-motivated candidates. This yields the following theorem.

Theorem 10. *Under Condition 4, assume that the candidates are purely office motivated, i.e., $\lambda_A = \lambda_B = 0$, and that candidate A's share of policy-oriented voters, in (12), is strictly quasi-concave in x and strictly quasi-concave in y. Then*

$$\max_{x\in Z}\int F_t(u_t(x) - u_t(z^*)d\tau \quad = \quad \min_{y\in Z}\int F_t(u_t(z^*) - u_t(y))d\tau,$$

and thus $z^* = w^*$.

While Theorem 9 applies directly to environments in which candidates are purely office motivated, it has implications for the model when the candidates are close to purely office motivated: if we introduce a small amount of policy motivation into the model, then the candidates will typically diverge slightly, but their equilibrium platforms will be close to welfare maximizing policies. Thus, analysis of electoral politics simply as the solution of the welfare maximization problem in (13) are robust to the objectives of the candidates. Given Theorem 9, the proof of our *approximate welfare optimality theorem,* stated next, follows directly from continuity of candidate payoffs, compactness of the policy space, and upper hemicontinuity of the equilibrium correspondence with respect to policy weights.

Theorem 11. *Under Condition 4, assume that candidate A's share of policy-oriented voters, in (12), is strictly quasi-concave in x and strictly quasi-concave in y. Let $\{\lambda_A^m, \lambda_B^m)\}$ be a sequence of policy weights such that the candidates become arbitrarily office motivated, i.e.,*

$$\lim_{m\to\infty} \lambda_A^m \;=\; \lim_{m\to\infty} \lambda_B^m \;=\; 0.$$

Then there is a corresponding sequence $\{(x^m, y^m)\}$ of Nash equilibria in the stochastic partisans model, and for every such sequence, the limit points of all convergent subsequences of $\{x^m\}$ and of $\{y^m\}$ maximize voter welfare. Moreover, if each u_t is strictly concave, then equilibrium platforms approximate the social welfare optimum:

$$\lim_{m\to\infty} x^m \;=\; \lim_{m\to\infty} y^m \;=\; z^*.$$

Note that our framework differs from that of Hinich, Lindbeck-Weibull, and Banks-Duggan, in that candidate objectives are flexibly parameterized by the policy weights λ_A and λ_B. Our robustness result cannot be stated in the vote maximization approach, where it is challenging to introduce policy concerns on behalf of candidates, and where equilibrium existence is problematic in the absence of the vote-maximizing assumption.

3.5 Competitive and Meaningful Elections

The literature on multidimensional elections has lacked an approach that guarantees existence of equilibria in which elections are both competitive (in the sense that each candidate wins with positive probability) and meaningful (in the sense that the candidates adopt distinct platforms). Corollary 1 shows, in the general model, that each candidate wins with positive probability in every standard Nash equilibrium, i.e., every equilibrium for which it is not the case that one candidate wins with probability one at the other's ideal point. In connection with that result, we noted that non-standard equilibria typically fail to exist: for example, this is true when the probability of winning functions are always positive, or when both candidates place positive weight on office.

Next, we assume that it is not the case that one candidate is purely policy motivated and the other is purely office motivated, and we give a simple differentiability condition that precludes interior, non-standard equilibria: it is enough that the distribution G of the partisan shock be differentiable everywhere.[14]

Lemma 3. *Under Condition 5, assume that $\lambda_A(1-\lambda_B) < 1$ and $(1-\lambda_A)\lambda_B < 1$, and that G is differentiable on $\mathbb{R}$. Then every interior Nash equilibrium in the stochastic partisans model is standard.*

Remark 11. The differentiability assumption of Lemma 3 is not satisfied by distributions that are non-differentiable at the boundary points of their support, such as the uniform distribution. However, such a distribution can be uniformly approximated by a smooth distribution function to any desired degree. □

Lemma 3 omits the cases $\lambda_A = 1 - \lambda_B \in \{0, 1\}$, where the candidates' objectives are diametrically opposed, i.e., one is purely office motivated, and the other is purely policy motivated. Recall that, in the context of the general model, Example 1 demonstrated a non-standard equilibrium when candidate A is purely policy motivated, and B is purely office motivated, so that $\lambda_A(1-\lambda_B) = 1$. The only assumption there on the candidates' probability of winning was that $P_A(x, \hat{x}) = 0$ for some x, and it is straightforward to specify the stochastic partisans model to satisfy that condition.

Example 2. Assume that $Z = [-2, 2]$, that A's ideal point is $\hat{x} = -1$, that there is a single type $t = 1$ of policy-oriented voter, with ideal point at zero and quadratic utility, i.e., $u_1(z) = -z^2$. Let F_1 be uniform on $[-4, 4]$, and let G is uniform on $[-\frac{1}{2}, \frac{1}{2}]$. Then for all $x, y \in Z$ with $-2 \leq -x^2 + y^2 \leq 2$, we have

$$F_1(u_1(x) - u_1(y)) \quad = \quad \frac{1}{2} + \frac{-x^2 + y^2}{8},$$

and A's probability of winning satisfies

$$P_A(x, y) \quad = \quad G(2F_1(u_1(x) - u_1(y)) - 1) \quad = \quad \frac{1}{2} + \frac{-x^2 + y^2}{4}.$$

We have $P_A(x, y) = 0$ when $-x^2+y^2 < -2$, and $P_A(x, y) = 1$ when $-x^2+y^2 > 2$. Setting $x = 2$, we then have $P_A(x, \hat{x}) = 0$, satisfying the assumptions of Example 1. Thus, if A is purely policy motivated, and B is purely office motivated, then there is a non-standard equilibrium. □

The preceding example shows that Lemma 3 is tight with respect to the assumption $\lambda_A(1 - \lambda_B) < 1$, and by a symmetric example, the assumption $(1 - \lambda_A)\lambda_B < 1$ also cannot be dropped. To see the role of differentiability, we elaborate on Examples 1 and 2.

Example 3. Assume that $Z = [-2, 2]$, that A is purely policy motivated with quadratic utility, and that B places small but positive weight on policy, also

[14] We maintain the assumption that G is differentiable on its support $S_G = \{\pi : 0 < G(\pi) < 1\}$, but this assumption allows for points of non-differentiability at the boundary.

with quadratic utility. Assume the ideal points of the candidates are both to the left of the voter, with $\hat{x} = -1$ and $\hat{y} = -2$. Then $\lambda_A = 1$, and $\lambda_B > 0$, and the candidates' objective functions satisfy the assumptions of Lemma 3. However, we drop the assumption that G is differentiable on the real line, and as in Example 2, we let G be uniform on $[-\frac{1}{2}, \frac{1}{2}]$. Set $\tilde{x} = \sqrt{3}$ and $\tilde{y} = \hat{x} = -1$. Then $-\tilde{x}^2 + \tilde{y}^2 = -2$, so $P_A(\tilde{x}, \tilde{y}) = 0$, and A loses to B at her own ideal point. Since A obtains her maximum payoff, the candidate cannot deviate profitably. It is clear that B cannot profitably deviate by moving rightward, as her probability of winning is already one, so such a move could only generate worse policy outcomes and/or lower probability of winning. To see that she cannot profitably deviate by a leftward move, note that

$$U_B(\sqrt{3}, y) \;\; = \;\; \left(\frac{1}{2} + \frac{-y^2+3}{4}\right)[\lambda_B(-(-2-y)^2 + (-2-\sqrt{3})^2) + 1 - \lambda_B].$$

When $\lambda_B = 0$, this function is concave in $y \in [-2, -1]$ and maximized at $y = -1$. Moreover, the derivative,

$$\frac{\partial}{\partial y} U_B(\sqrt{3}, \tilde{y}) \;\; = \;\; -\frac{\tilde{y}}{2},$$

is strictly positive on the interval $[-2, -1]$. By continuity, the derivative is strictly positive on $[-2, -1]$ with $\lambda_B > 0$ is sufficiently small, and thus $\tilde{y} = -1$ is a best response for candidate B. We have shown that $(\tilde{x}, \tilde{y})$ is a non-standard equilibrium, exploiting the non-differentiability of the uniform density. □

From Corollary 1 and Lemma 3, we have the obvious implication that if it is not the case that candidate objectives are diametrically opposed, and if the partisan shock distribution is differentiable, then in every interior equilibrium, each candidate wins with positive probability.

Theorem 12. *Under Condition 5, assume that $\lambda_A(1 - \lambda_B) < 1$ and $(1 - \lambda_A)\lambda_B < 1$, and that G is differentiable on $\mathbb{R}$. Then for every interior Nash equilibrium (x^*, y^*) in the stochastic partisans model, we have $P_A(x^*, y^*) > 0$ and $P_B(x^*, y^*) > 0$.*

Meaningfulness of elections, in the minimal sense that voters have a choice between distinct platforms, is clearly possible in the stochastic partisans framework; in fact, we will show that it is a ubiquitous feature of elections in our model. When candidates are purely policy motivated, Calvert's (1985) Theorem 5 provides extremely weak conditions under which the equilibrium platforms of the candidates are distinct, i.e., equilibria are "divergent." However, when the candidates place positive weight on policy, examples of convergent equilibria can be constructed. This is true for the extreme case of pure office motivation, by Theorem 9, but Duggan (2025b) provides an example, of a convergent equilibrium when candidates have mixed motivations, placing positive weight on both policy and office. That example requires a precise specification of policy weights for the candidates, and in fact, equilibrium platform convergence is impossible for typical parameter values.

Duggan (2025b) provides conditions for generic divergence of interior equilibria. In addition to differentiability, the assumptions needed to apply Theorem 5 of the latter paper to the stochastic partisans model are that: (i) $P_A(z, z)$ is constant in z, (ii) when both candidates locate at one's ideal point, the partial derivative of P_A with respect to the latter's platform is non-zero, and (iii) the derivatives of $D_x P_A(z, z)$ and $D_y P_B(z, z)$ with respect to z are negative semi-definite. In the stochastic partisans model, we have

$$P_A(z, z) \quad = \quad G\left(2\left(\int F_t(0) d\tau\right) - 1\right),$$

which is obviously constant in z. The partial derivative of P_A with respect to x at $(\hat{x}, \hat{x})$ is non-zero if and only if

$$g\left(2\left(\int F_t(0) d\tau\right) - 1\right) 2 \int f_t(0) D u_t(\hat{x}) d\tau \quad \neq \quad 0, \tag{14}$$

which would only be violated for precise specifications of the model. Similarly,

$$D_y P_B(\hat{y}, \hat{y}) \quad \neq \quad 0, \tag{15}$$

would only be violated in very special conditions. Finally, note that

$$\begin{aligned} D_z[D_x P_A(z, z)] &= D_z\left[g\left(2\left(\int F_t(0) d\tau\right) - 1\right) 2 \int f_t(0) D u_t(z) d\tau\right] \\ &= g\left(2\left(\int F_t(0) d\tau\right) - 1\right) 2 \int f_t(0) D^2 u_t(z) d\tau, \end{aligned}$$

which inherits negative semi-definiteness from the Hessians of voter utility functions, as does $D_z[D_y P_B(z, z)]$. Thus, we have the following result establishing generic platform divergence for equilibria of the stochastic partisans model.

Theorem 13. *Under Condition 5, assume:*

(i) u_A, u_B, *that each* u_t *are twice continuously differentiable,*

(ii) $D^2 u_t(z)$ *is uniformly bounded with respect to the max norm* $\|D^2 u_t(z)\| = \max_{i,j} |D^2_{i,j} u_t(z)|$,

(iii) inequalities (14) and (15) hold,

(iv) for all $z \in \mathsf{int}\, Z$, *the Hessian matrices* $D^2 u_A(z)$ *and* $D^2 u_B(z)$ *are negative definite, and for all* t, $D^2 u_t(z)$ *is negative semi-definite.*

For almost all $(\lambda_A, \lambda_B) \in [0, 1]^2$, *if* (x^*, y^*) *is an interior Nash equilibrium of the stochastic partisans model, then* $x^* \neq y^*$.

4 Aggregate Voter Results

4.1 Uniform Idiosyncratic Biases

In this section, we impose greater structure on the stochastic partisans model to deduce a substantial simplification of the candidates' probability of winning functions, one that will facilitate later comparative statics analysis and will have practical usefulness in applications. An implication of Theorem 8 is that in any equilibrium, candidate A's platform x^* lies on the locus of tangencies generated by A's policy utility and the contour sets of the function (12). As noted above, however, these contour sets are parameterized by B's platform y^*, and thus the usefulness of the characterization is limited, at that level of generality. We investigate conditions under which A's probability of winning has a representation in terms of an aggregate voter, who is pinned down by the primitives of the model. This in turn implies that the contour sets of $P_A(\cdot, y)$ are independent of B's platform, and when utility functions are quadratic, it leads to a dimensional reduction, in which the equilibrium platforms of each candidate belong to a pre-specified "interval" of policies between her ideal point and that of the aggregate voter. Even if the policy space is highly multidimensional, the space spanned by the candidates' possible platforms will be two-dimensional, and as the preferences of the candidates become very aligned or very opposed, their platforms are better explained by a simple one-dimensional model.

The key property underlying our aggregate voter theorem is that idiosyncratic bias is uniformly distributed for each type. While this assumption is strong, as is any parametric restriction on the distribution of a random shock, it yields a substantial return in the form of tractability. The next condition also strengthens the maintained assumption of concavity to strict concavity of voter utility functions.

Condition 6. *For all t, u_t is strictly concave, and F_t is uniform on an interval $[\mu_t - \frac{1}{2\rho_t}, \mu_t + \frac{1}{2\rho_t}]$, where the mean μ_t and precision ρ_t are measurable functions of voter types t, and the product $\rho_t \mu_t$ is τ-integrable.*

The second part of Condition 6 means that F_t has the linear functional form

$$F_t(\epsilon_i) \quad = \quad \kappa_t + \rho_t \epsilon_i$$

over its support, where $\kappa_t = \frac{1}{2} - \rho_t \mu_t$. Note that by the maintained assumption of (9), we require that for all $x, y \in Z$ and all $t \in T$, the inequalities

$$\mu_t - \frac{1}{2\rho_t} \;\leq\; u_t(x) - u_t(y) \;\leq\; \mu_t + \frac{1}{2\rho_t}$$

hold, so the support is sufficiently large, and the precision is correspondingly small, relative to the range of possible utility differences.[15]

[15]This large support condition is common in probabilistic models of elections, although it is sometimes not made explicit.

We now define an artificial actor, denoted V, with the interpretation being that of an aggregate voter whose utility determines the probability of winning for each candidate. Abusing notation slightly, define the policy utility function of the aggregate voter as

$$V(z) \quad = \quad 2\int \rho_t u_t(z) d\tau$$

which is a positive affine transformation of the social welfare function in (13). Also, define the constant term

$$\kappa \quad = \quad \left(2\int \kappa_t d\tau\right) - 1,$$

which is A's electoral advantage among policy-oriented voters. Then it is straightforward to see that

$$\begin{aligned} \kappa + V(x) - V(y) \quad &= \quad \left(2\int (\kappa_t + \rho_t(u_t(x) - u_t(y))d\tau\right) - 1 \\ &= \quad 2\left(\int F_t(u_t(x) - u_t(y))d\tau\right) - 1, \end{aligned}$$

which implies

$$P_A(x,y) \quad = \quad G(\kappa + V(x) - V(y)). \tag{16}$$

Observe that the contour sets of $\kappa + V(x) - V(y)$, as a function of x, do not depend on y; and since $P_A(x,y)$ is a monotonic transformation of $\kappa + V(x) - V(y)$, the same holds for A's probability of winning. Of course,

$$P_B(x,y) \quad = \quad 1 - G(\kappa + V(x) - V(y)), \tag{17}$$

and the same observations hold for candidate B. Note that V is continuous and, by Condition 6, strictly concave, and we denote the unique maximizer of V by $\hat{z}$, the ideal point of the aggregate voter. We maintain the assumption that $\hat{z} \notin \{\hat{x}, \hat{y}\}$, so neither candidate can maximize her probability of winning by simply choosing her ideal point.

4.2 Aggregate Voter Theorem

The main result of this section is the *aggregate voter theorem,* which sharpens Theorems 7–9 in the presence of the uniform bias assumption. Clearly, if candidate A is purely office motivated, then she chooses x to maximize $\kappa + V(x) - V(y)$, i.e., the candidate chooses $x = \hat{z}$. Otherwise, if she places positive weight on policy, then in equilibrium, A's platform will not typically coincide with $\hat{z}$, but she maximizes a weighted sum of her own policy utility, u_A, and that of the aggregate voter, V. In general, following the proof of Theorem 7, A's equilibrium platform will be Pareto optimal for the candidate and the aggregate voter. Parelleling Theorem 8, if utilities are differentiable and an equilibrium is interior, then each candidate's platform maximizes a weighted sum of her utility and the aggregate voters. Finally, if both candidates are purely office motivated, then the unique equilibrium is $(\hat{z}, \hat{z})$, which implies that the unique equilibrium platform z^* from Theorem 9 is simply the aggregate ideal point, $\hat{z}$.

Theorem 14. *Under Condition 6, let (x^*, y^*) be an equilibrium in the stochastic partisan model with uniform bias. Then:*

(i) P_A and P_B have the forms in (16) and (17), respectively,

(ii) x^* is Pareto optimal for $\{A, V\}$, and y^* is Pareto optimal for $\{B, V\}$,

(iii) under Condition 5, if (x^*, y^*) is interior, then x^* solves

$$\max_{z \in Z} \alpha u_A(x) + (1 - \alpha)V(x),$$

where α is defined in Theorem 8, with a corresponding result for B,

(iv) if candidate A is purely office motivated, i.e., $\lambda_A = 0$, then $x = \hat{z}$ is a dominant strategy, so $x^* = \hat{z}$, and likewise for B.

(v) and thus if both candidates are purely office motivated, then the unique equilibrium is $(\hat{z}, \hat{z})$, which implies $z^* = \hat{z}$.

Remark 12. Persson and Tabellini (2000, Section 3.4), assuming a one-dimensional policy space, three voter types, and purely office-motivated candidates, deduce that in equilibrium, both candidates maximize a weighted sum of utilities and choose the aggregate ideal point $\hat{z}$, as implied by part (iii) of Theorem 14. Our result generalizes their insight to multiple dimensions, to an arbitrary set of voter types, and to candidates who have general mixed motivations, with any weights (possibly different across candidates) on policy and office. □

An immediate implication of the aggregate voter theorem, assuming the differentiability condition of part (ii), is that if candidate A places positive weight on policy, i.e., $\lambda_A > 0$, then she does not locate at the aggregate voter's ideal point. Moreover, if A places some weight on office, then she does not locate at her own ideal point.

Corollary 4. *Under Conditions 5 and 6, assume that $\hat{x}, \hat{z} \in \mathsf{int}\, Z$. In any equilibrium (x^*, y^*) of the stochastic partisans model with uniform bias, if $\lambda_A \in (0, 1]$, then $x^* \neq \hat{z}$, and if $\lambda_A \in [0, 1)$, then $x^* \neq \hat{x}$. An analogous result holds for candidate B.*

Another simple implication is that if one candidate is purely office motivated, then there is a unique equilibrium: the office-motivated candidate locates at the aggregate ideal point $\hat{z}$, and the other best responds. Note that Condition 4 is only used in the following corollary to ensure that candidate B, who may place positive weight on policy, has a unique best response to A.

Corollary 5. *Under Conditions 4 and 6, if $\lambda_A = 0$, then the stochastic partisans model with uniform bias has a unique equilibrium (x^*, y^*), and $x^* = \hat{z}$. An analogous statement holds for candidate B.*

A final important implication of Theorem 14 is that the contract curve for candidate A and the aggregate voter, denoted $\overline{AV}$, is determined by the primitives of the model, as is the contract curve for B and the aggregate voter, $\overline{BV}$. In terms of Figure 1, the contour sets of A's probability of winning are now independent of y, and similarly for B. This essentially reduces the strategy sets of the players to one-dimensional manifolds, thereby simplifying the analysis of the model. In Subsection 4.4, we impose further structure on voter utility and trace its implications for the aggregate voter and candidate platforms. Before that, we study the implications of the aggregate voter theorem in a model of redistributive politics.

4.3 Application: Balanced-Budget Redistribution

We apply the above analysis to elections in an environment of balanced-budget redistribution, in the spirit of Lindbeck and Weibull (1987), but allowing for candidate with mixed motivations. We show below that policy preferences of candidates introduce some divergence between the candidates; this converges to zero as the candidates' weight on office increases, and their equilibrium platforms converge to the social optimum, consistent with the approximate welfare optimality theorem, Theorem 11, and the result of Lindbeck and Weibull that both parties locate at the platform that maximizes voter welfare in the case of pure office motivation. For the general case of mixed motivations, we provide a closed form solution that permits intuitive comparative statics with respect to the candidates' policy weight, the size of different voter types in the population of policy-oriented voters, and the distribution of the partisan shock.

We specialize the stochastic partisans model by assuming there are three types of policy-oriented voter, $T = \{1, 2, 3\}$, where type 1 represents the group of liberal elites, type 2 represents the conservative elites, and type 3 represents the remaining mass electorate. The policy space is the unit simplex in $\mathbb{R}^3$,

$$Z \quad = \quad \{z \in \mathbb{R}^3_+ : z_1 + z_2 + z_3 = 1\},$$

where a policy $z = (z_1, z_2, z_3)$ represents an allocation of resources to the groups, with the share going to group t being z_t. Here, we interpret z_t as being a group-specific public good, which is enjoyed by all voters of type t. We assume that voter utility from consumption is

$$u_t(z) \quad = \quad v(z_t),$$

where $v\colon \mathbb{R}_+ \to \mathbb{R}$ is a concave, differentiable function such that $v'(r) > 0$. This makes u_t continuous, concave, with idea point equal to e^t, the tth unit coordinate vector, i.e., the vertex of the unit simplex that allocates all resources to the type t voters. In addition to the policy oriented voters, a mass of partisan voters is distributed according to G, which satisfies the log concavity property of Condition 4.

Candidates A and B compete in an election, where A is associated with the type 1 voters, and candidate B is associated with the type 2 voters, so both candidates are elite, with each representing a different faction of elite voters. We assume that candidate utility is linear,

$$u_A(z) \;=\; z_1 \quad \text{and} \quad u_B(z) \;=\; z_2,$$

and that A and B have mixed motivations. Theorem 6 immediately delivers existence of an equilibrium (x^*, y^*), but we will focus on equilibria that are *symmetric,* in the sense that $x_3^* = y_3^*$, $x_1^* = y_2^*$, and $x_2^* = y_1^*$; that is, the candidates allocate the same amount of resources to the masses, the same to themselves, and the same to the competing elite group. To this end, we further assume that $\lambda_A = \lambda_B = \lambda > 0$, so candidates place the same positive weight on policy, that $\kappa = 0$, so neither candidate has an electoral advantage, and that g is symmetric around zero.

We assume Condition 6, so that the aggregate voter theorem holds, and we denote the shares of voter types among policy-oriented voters by τ_1, τ_2, and τ_3. Then aggregate voter utility is

$$V(z) \quad = \quad \omega_1 u_1(z) + \omega_2 u_2(z) + \omega_3 u_3(z),$$

where the weights used in the sum are

$$\omega_t \quad = \quad 2\rho_t \tau_t.$$

To impose symmetry on the candidates, we assume that the weights of type 1 and type 2 voters each equal ω, which is less than one third of the policy-oriented voters:

$$\omega_1 \;=\; \omega_2 \;=\; \omega \;<\; \frac{1}{3} \quad \text{and} \quad \omega_3 \;=\; 1 - 2\omega \;>\; \frac{1}{3}.$$

Then aggregate voter utility simplifies to

$$V(z) \quad = \quad \omega v(z_1) + \omega v(z_2) + (1 - 2\omega) v(z_3),$$

We consider two cases of interest for voter utility.

The first, and simplest case, is linear voter utility, i.e., $v(r) = r$. Then aggregate utility is also linear, and the highest coefficient on utility is $1 - 2\omega$, which means that the aggregate ideal point is the third unit coordinate vector $\hat{z} = e^3$, so that all resources are allocated to the masses. Our interest will be in equilibria such that neither candidate allocates to resources to the other's group, i.e., $x_2^* = y_1^* = 0$, but we first consider the general problem of a candidate in a symmetric equilibrium; without loss of generality, we focus on candidate A. At a symmetric equilibrium (x^*, y^*), candidate A solves the following constrained maximization problem:

$$\begin{aligned} \max_{x \in Z} \; G(V(x) - V(y^*))[\lambda(u_A(x) - u_A(y^*)) + 1 - \lambda] \\ \text{s.t. } x_1 + x_2 + x_3 = 1 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

At such an equilibrium, A's platform satisfies the Kuhn-Tucker first order condition,

$$\begin{aligned} g(0)DV(x^*)\Delta + \tfrac{\lambda}{2}Du_A(x^*) + (1,1,1)\mu + (\nu_1, \nu_2, \nu_3) \;&=\; 0 \\ \nu_1 x_1^* = 0, \;\; \nu_2 x_2^* = 0, \;\; \nu_3 x_3^* = 0& \\ \nu_1 \geq 0, \;\; \nu_2 \geq 0, \;\; \nu_3 \geq 0,& \end{aligned}$$

where μ is the multiplier on the equality constraint, and ν_t is the multiplier on the non-negativity constraint of the consumption of type t voters, and where we use the fact that $V(x^*) - V(y^*) = 0$, and

$$\Delta \;=\; \lambda(u_A(x^*) - u_A(y^*)) + 1 - \lambda$$

is A's gain from winning. Analysis of the first order condition yields an explicit solution for the unique symmetric equilibrium.

Proposition 1. *Under Condition 6, in the symmetric model of balanced-budget redistribution, assume linear voter utility, i.e., $v(r) = r$, and*

$$\frac{2g(0)(1-3\omega)}{2g(0)(1-3\omega)+1} \;<\; \lambda \;<\; 2g(0)(1-3\omega).$$

There is a unique symmetric equilibrium: candidate A's platform is

$$\begin{aligned} x_1^* &= \frac{1}{2g(0)(1-3\omega)} - \frac{1-\lambda}{\lambda} \\ x_2^* &= 0 \\ x_3^* &= 1 - x_1^*, \end{aligned}$$

and candidate B's platform is $y^* = (0, x_1^*, x_3^*)$.

Several intuitive comparative statics arise immediately for symmetric equilibria when candidates allocate resources to themselves and to the masses. First, as the candidates' policy weight increases, each allocates more resources to her elite group, reflecting higher marginal returns from consumption. Second, when the masses grow in size, i.e., ω decreases, the candidates allocate more resources to them, reflecting the increased electoral returns. Third, when λ is close enough to one, the candidates actually consume all of the resource, while if λ is close enough to zero, they allocate all resources to the masses, thereby maximizing aggregate voter welfare. This is consistent with the approximate welfare optimality theorem, Theorem 11, which holds for the general stochastic partisans model. Because of the linear utility specification here, we see that optimality is actually attained, not just approximated, when candidates are sufficiently office motivated.

With linear voter utility, candidates can face a meaningful trade off, but that it limited, given that one group of voters is always excluded. An economically more interesting situation is that in which the marginal utility for voters from consuming all resources is zero, which is assumed by Lindbeck and Weibull (1987) and implies that the aggregate voter ideal point is located in the interior of the simplex. For tractability, we maintain linear utility for the candidates; since Lindbeck and Weibull assume candidates simply maximize votes, they do not formulate any notion of policy utility for the candidates. For voters, we assume utility from consumption has a quadratic form:[16]

$$v(r) \quad = \quad -(1-r)^2.$$

Then the aggregate ideal point is found by solving the following constrained maximization problem,

$$\max_{z \in Z} \ \omega v(z_1) + \omega v(z_2) + (1-2\omega) v(z_3)$$
$$\text{s.t. } z_1 + z_2 + z_3 = 1,$$

which has first order condition

$$\begin{aligned} \omega v'(z_1) + \mu &= 0 \\ \omega v'(z_2) + \mu &= 0 \\ (1-2\omega) v'(z_3) + \mu &= 0. \end{aligned}$$

By the first two equations, we have $\hat{z}_1 = \hat{z}_2$, and then, using quadratic utility and substituting for μ, the third simplifies to

$$(1-2\omega)(2\hat{z}_1) - \omega(1-\hat{z}_1) \quad = \quad 0.$$

Solving this, we obtain the aggregate ideal point as

$$\hat{z}_1 = \frac{\omega}{2-3\omega}, \qquad \hat{z}_2 = \frac{\omega}{2-3\omega}, \qquad \hat{z}_3 = \frac{2-5\omega}{2-3\omega}.$$

Clearly, as ω increases, the aggregate ideal point shifts away from the mass electorate, and it moves toward the allocation that gives all resources to the masses as ω becomes small.

[16] Lindbeck and Weibull (1987) also assume that marginal voter utility increases to infinity as consumption goes to zero. Our quadratic specification does not satisfy this condition, but it generates the same qualitative properties of equilibria.

We are now interested in equilibria such that all groups receive positive resources, so we consider the best response problem of candidate A in such a symmetric equilibrium,

$$\max_{x \in Z} G(V(x) - V(y^*))[\lambda(u_A(x) - u_A(y^*)) + 1 - \lambda]$$
$$\text{s.t. } x_1 + x_2 + x_3 = 1,$$

with first order condition

$$g(0)DV(x^*)\Delta + \frac{\lambda}{2} Du_A(x^*) + (\mu, \mu, \mu) \quad = \quad 0.$$

Analysis of the first order condition, though somewhat more complicated, again leads to an explicit solution for the unique symmetric equilibrium. For example, see the pair (x^*, y^*) in Figure 2, which is computed for parameter values $\lambda = .1$, $\omega = .28$, and $g(0) = .25$, and which depicts contours of the aggregate voter utility function and the policy utility function of candidate A.

Proposition 2. *Under Condition 6, in the symmetric model of balanced-budget redistribution, assume quadratic voter utility, i.e.,* $v(r) = -(1-r)^2$*. There is a unique symmetric equilibrium: in the range of parameters for which all three voters types receive a positive level of resources, candidate A's platform is*

$$\begin{aligned} x_1^* &= \frac{\omega}{2-3\omega} + \left(\frac{1-\omega}{2-3\omega}\right)\delta \\ x_2^* &= \frac{\omega}{2-3\omega} - \left(\frac{1-2\omega}{2-3\omega}\right)\delta \\ x_3^* &= \frac{2-5\omega}{2-3\omega} - \left(\frac{\omega}{2-3\omega}\right)\delta, \end{aligned}$$

where

$$\delta \quad = \quad \frac{1}{2}\left[-\frac{1-\lambda}{\lambda} + \sqrt{\left(\frac{1-\lambda}{\lambda}\right)^2 + \frac{1}{g(0)\omega}}\,\right],$$

and candidate B's platform is $y^* = (0, x_1^*, x_3^*)$.

Several observations are in order. First, if we fix the weights ω and vary other parameters, then the effect on equilibrium platforms is only through the term δ. Thus, as those parameters vary, candidate A's equilibrium policy platforms vary along a line segment in the simplex, with one endpoint at the aggregate ideal point—demonstrating the implication of the aggregate voter theorem, Theorem 14, that the candidate's strategy space is essentially a one-dimensional manifold, in this case a linear one. In Figure 2, this is the orange line emanating from $\hat{z}$. Because candidate A's policy preferences only place positive weight on the

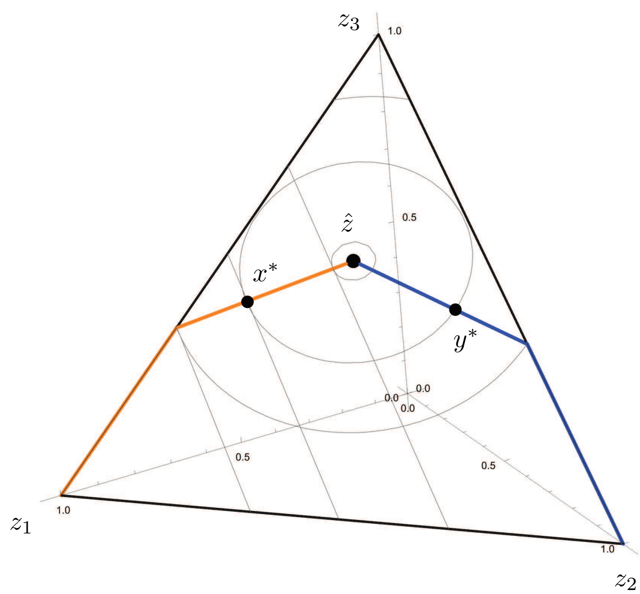

Figure 2: Balanced budget equilibria

liberal elite (type 1) voters, this line hits the edge of the simplex when δ is high enough, and at that point, it follows the edge of the simplex to the candidate's ideal point; thus, the set of possible equilibrium platforms for A is the piecewise linear path in red, while the set for B is the blue path. Any values of λ and $g(0)$ will determine symmetric equilibrium platforms for the candidates, where A chooses a platform on the red path, and B chooses the corresponding one on the blue path.

Second, as we vary the parameters λ and $g(0)$, we obtain the same intuitive comparative statics as in the linear case: as $g(0)$ increases from a low value, δ decreases; and the candidates shift resources from their own elite groups to the masses. When $g(0)$ becomes sufficient high, a candidate's platform moves toward the aggregate voter along the edge of the simplex, which also increases the amount of resources going to the other voter types. As the candidates become more office motivated, λ decreases, and now δ increases, which has the opposite effect: as λ increases from a low value, the candidates move away from the aggregate voter, initially reallocating resources from both other groups to their own, and eventually taking resources from the masses and moving along the edge of the simplex to her own ideal point. Moreover, we have $\lambda \to 0$, and thus $\frac{1-\lambda}{\lambda} \to \infty$. This entails $\delta \to 0$, and we see from the solution in Proposition 2 that $x^* \to \hat{z}$, as called for by the approximate welfare optimality theorem, Theorem 11.

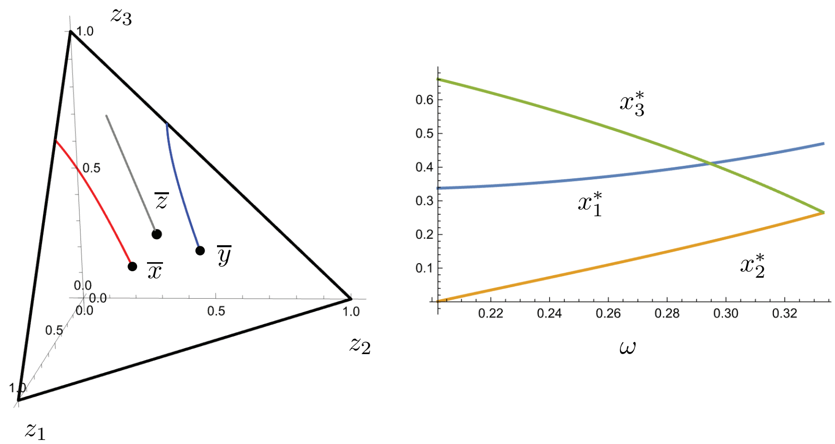

Figure 3: Comparative statics with respect to ω

Finally, the parameter ω affects equilibrium platforms through δ but also directly. As ω increases, our solution for δ shows that the difference between the resources allocated by a candidate to her own type compared to the other elite group becomes smaller, i.e., δ decreases. At the same time, the aggregate ideal point moves away from $\hat{z}_3$, closer to the ideal points of the elite types, and the indifference contours of the aggregate voter change shape. In the left-hand panel of Figure 3, we fix $\lambda = .065$ and $g = .25$, and we depict the paths of equilibrium policies for the candidates as ω increases from .202, at which the candidates choose policies on the edge of the simplex, to .333, where the candidates locate in the interior of the simplex. As $\omega \to 1/3$, the aggregate ideal point converges to $\overline{z} = (1/3, 1/3, 1/3)$, and we have

$$x^* \;\; \to \;\; \left(\frac{1}{3} + \frac{2\delta}{3}, \frac{1}{3} - \frac{\delta}{3}, \frac{1}{3} - \frac{\delta}{3}\right),$$

where this limit it denoted by $\overline{x}$ in Figure 3. The right-hand panel of the figure shows the resource allocation to each voter type as a function of ω and illustrates the non-linear effect of group size on equilibrium platforms.

4.4 Generalized Quadratic Utility and Dimension Reduction

Building on the aggregate voter theorem, we next show that if utility for voters and candidates is generalized quadratic, with the same matrix of coefficients for the candidates and aggregate voter, then the strategy sets of the candidates effectively become one-dimensional line segments, regardless of the dimensionality of the policy space. We say a function $u\colon Z \to \mathbb{R}$ is *generalized quadratic* if there exist a maximizer $z \in Z$, a constant $\zeta \in \mathbb{R}$, and a symmetric, positive definite, $d \times d$ matrix M such that for all $w \in Z$, we have

$$u(w) \quad = \quad -(w-z)M(w-z)+\zeta.$$

The function is *weighted quadratic* if, in addition, the coefficient matrix M is diagonal, with entries along the diagonal representing the weights placed on each dimension; this produces indifference curves that are elliptical in shape and oriented along the axes. It is simply *quadratic* if M is equal to the identity matrix.

Condition 7. *For all t, u_t is generalized quadratic with ideal point $\hat{z}^t$, constant ζ_t, and coefficient matrix M_t, where these are bounded, measurable functions of voter types t.*

The next lemma establishes that when individual voter utility is generalized quadratic, then so is aggregate voter utility. The aggregate voter ideal point is a linear function of voter ideal points, and the matrix of coefficients for the aggregate voter is an integral over voter matrices. In particular, if voter utility is weighted quadratic, then so is aggregate voter utility; and if voter utility is quadratic, then this is again inherited by the aggregate voter utility. In the latter special case, note that the aggregate ideal point reduces to a simple weighted average of voter ideal points,

$$\hat{z} \quad = \quad \int \left(\frac{\rho_t \eta_t}{\int \rho_s \eta_s \tau(ds)}\right) \hat{z}^t \tau(dt).$$

In the statement of the lemma, we use the fact that the integral of positive definite matrices is positive definite, and thus is also invertible.

Lemma 4. *Under Conditions 6 and 7, the aggregate utility V in the stochastic partisans model with uniform bias and generalized quadratic voter utility is generalized quadratic with coefficient matrix*

$$M \quad = \quad 2\int \rho_t M_t d\tau,$$

and aggregate ideal point

$$\hat{z} \quad = \quad M^{-1}\int \rho_t M_t \hat{z}^t d\tau.$$

It can be fruitful to combine Condition 7, using Lemma 4, with the assumption that the candidates also have generalized quadratic policy preferences with coefficient matrix M given in the lemma. The following condition presupposes quadratic utility on the part of policy-oriented voters.

Condition 8. *Candidate utility functions u_A and u_B are generalized quadratic with coefficient matrices $M_A = M_B = M$.*

When the candidates and aggregate voter have generalized quadratic utility with the same matrix of coefficients, the contract curves $\overline{AV}$ and $\overline{BV}$ are simply line segments connecting the ideal point of a candidate with the aggregate ideal point. Indeed, let M be the coefficient matrix for the aggregate voter, and let the coefficient matrix for candidate A be σM, where $\sigma > 0$. Note that a policy z^* is Pareto efficient for $\{A, V\}$ if and only if z^* solves

$$\max_{z \in Z} \alpha u_A(z) + (1-\alpha)V(z)$$

for some $\alpha \in [0,1]$. The first order condition for the above problem is

$$2\alpha\sigma M(\hat{x} - z^*) + 2(1-\alpha)M(\hat{z} - z^*) \quad = \quad 0,$$

or equivalently,

$$M\left(\frac{\alpha\sigma}{\alpha\sigma + 1 - \alpha}\hat{x} + \frac{1-\alpha}{\alpha\sigma + 1 - \alpha}\hat{z}\right) \quad = \quad Mz^*.$$

Therefore, applying the inverse M^{-1} to both sides, z^* lies on $\overline{AV}$ if and only if

$$z^* \quad = \quad \frac{\alpha\sigma}{\alpha\sigma + 1 - \alpha}\hat{x} + \frac{1-\alpha}{\alpha\sigma + 1 - \alpha}\hat{z}$$

for some $\alpha \in [0,1]$. It follows that the contract curve $\overline{AV}$ is just the line segment between the ideal points of candidate A and the aggregate voter, and similar analysis holds for candidate B.

As stated in the following *dimensional reduction theorem,* we conclude that the strategy sets of the candidates reduce to one-dimensional intervals, even if the underlying policy space is multidimensional. See Figure 4 for an illustration.

Theorem 15. *Under Conditions 6–8, for all Nash equilibria (x^*, y^*) in the stochastic partisans model with uniform bias and generalized quadratic utility, there exist $\alpha, \beta \in [0,1]$ such that*

$$x^* \;=\; \alpha\hat{x} + (1-\alpha)\hat{z} \quad \textit{and} \quad y^* \;=\; \beta\hat{y} + (1-\beta)\hat{z}.$$

Theorem 13 provides generic conditions on policy weights under which the candidates adopt distinct platforms in equilibrium. In this spirit, the dimensional reduction theorem leads to a simple set of sufficient conditions for divergence for general policy weights: the equilibrium platforms of the candidates are divergent if

(a) at least one candidate places positive weight on policy,

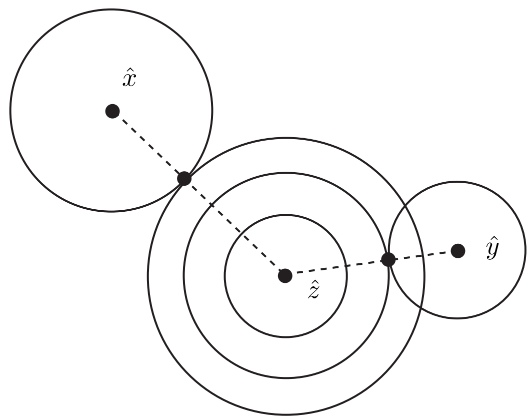

Figure 4: Dimensionality reduction with generalized quadratic utility

(b) candidates and the aggregate voter have generalized quadratic utility with common coefficient matrix, and

(c) the vectors $\hat{x} - \hat{z}$ and $\hat{y} - \hat{z}$ are not positively dependent, i.e., there do not exist $\alpha, \beta > 0$ such that $\alpha(\hat{x} - \hat{z}) = \beta(\hat{y} - \hat{z})$.

Condition (c) is satisfied in the typical case that the ideal points of the candidates and aggregate voter are not collinear, and it is satisfied if they line up with $\hat{x}$ and $\hat{y}$ on opposite sides of $\hat{z}$. It implies that the intersection of contract curves is just the aggregate ideal point, i.e., $\overline{AV} \cap \overline{BV} = \{\hat{z}\}$.

Theorem 16. *Under Conditions 6–8, assume that $\hat{z} \in \mathsf{int}\, Z$, and that (a)–(c) above hold. If either $\lambda_A > 0$ or $\lambda_B > 0$, then in every Nash equilibrium (x^*, y^*) of the stochastic partisans model with uniform bias and generalized quadratic utility, we have $x^* \neq y^*$.*

4.5 Application: Income Taxation

In the absence of results establishing the existence and characterization of equilibrium in voting models with multidimensional policy spaces, the formal literature on income taxation has made limited progress in explaining the political determinants of progressive or regressive tax schedules. Romer (1975), Roberts (1977), and Meltzer and Richard (1981) analyze redistributive taxation in settings with a linear tax rate and exogenously given revenue requirement; because of their linearity restriction, these articles are silent on the question of progressivity vs. regressivity of taxes. In this section, we apply our general results to provide conditions under which one or both parties propose progressive tax policies in a pure strategy equilibrium of the electoral game, to characterize those equilibrium policies, and we examine the effect of parameters on progressiveness of tax policy.

Current work on progressivity of income tax schedules includes Snyder and Kramer (1988) and Brett and Weymark (2016), who study the problem by restricting the set of tax policies that the parties can propose to the set of policies that are the ideal point for some voter type, while Berliant and Gouveia (2025) use ex post feasibility to reduce the size of the policy space. Roemer (1999,2001) uses his party unanimity equilibrium approach, in which existence is not an issue, but rather multiplicity of equilibria is a challenge. He restricts tax schedules to a two-parameter family and assumes a balanced-budget constraint, and he gives conditions such that in all equilibria (of which there is a continuum), both parties adopt progressive tax policies. Other contributions include Marhuenda and Ortuno-Ortin (1995) and De Donder and Hindriks (2004), who provide partial equilibrium stories for progressive tax outcomes. Carbonell-Nicolau and Klor (2003) and Carbonell-Nicolau and Ok (2007) study a very general setting, with the latter paper showing existence of equilibria in mixed strategies, and the former extending the model to allow for candidate entry.

Instead, we apply the stochastic partisans model to analyze the political determinants of tax policy, without restricting ex ante the set of tax policies. Rather than take a revenue constraint as exogenously given, we assume that tax revenue finances the provision of a public good, and we simplify the analysis by assuming that voters supply labor inelastically, so each voter is characterized by an income level. Given this context, we can without loss of generality assume that a tax schedule specifies an amount paid by, or equivalently a tax rate on the income of, each voter type. In a simple, two-type model, in which each party represents one type, we allow arbitrary mixed motivations for parties (formerly "candidates," in our earlier terminology); we show that equilibria always exist, and that the low-income party always proposes a tax schedule that is more progressive that of the high-income party. Furthermore, we provide conditions under which both parties adopt progressive tax policies, and we extend the analysis to the general model with an arbitrary, finite set of voter types.

Formally, we assume a unit mass of policy-oriented voters, and we initially assume each voter is one of two types, labeled L and H, with the share of type L being $\tau_L = \omega \in [1/2, 1)$ and type H being $\tau_H = 1 - \omega \in (0, 1/2]$. Type L has income $w_L > 0$ and type H has income $w_H > w_L$, and thus we refer to L as the "low-income type" and to H as the "high-income type." A tax policy is a schedule $z = (z_L, z_H) \in [0, 1]^2$, which specifies tax rate z_L on the income of type L voters, and a tax rate z_H on the income of type H voters. Taxes are used to finance a public good produced by the following technology:

$$h(z) \quad = \quad \gamma\left[\left(z_L - \frac{(z_L)^2}{2K}\right)\omega w_L + \left(z_H - \frac{(z_H)^2}{2K}\right)(1-\omega)w_H\right],$$

where γ and K are parameters. The utility to a voter of type t from tax policy

$z = (z_L, z_H)$ is

$$u_t(z) \;\; = \;\; (1 - z_t)w_t + h(z),$$

so that γ can be interpreted as an implicit weight on the public good, and the quadratic terms in $h(z)$ capture the deadweight loss from taxation. Within each voter type t, a net bias in favor of party B is distributed according to F_t, which satisfies the dispersion property in Condition 4. In addition, a mass of partisan voters turn out to vote stochastically, with the net mass in favor of party B distributed according to G, which satisfies the log concavity property of Condition 4.[17]

Following standard definitions in the public finance literature, we say a tax policy $z = (z_L, z_H)$ is *progressive* if higher income voters are taxed at a higher rate than lower income voters, i.e., $z_L < z_H$. A tax policy z is *regressive* if $z_L > z_H$, and it is a *flat tax* if $z_L = z_H$. A policy $\tilde{z}$ is *more progressive* than policy z if taxation rises more steeply with type, i.e., $z_H - z_L < \tilde{z}_H - \tilde{z}_L$, in which case we write $\tilde{z} \succ_p z$. Finally, a policy $\tilde{z}$ is *more expansionist* than policy z if it provides a greater amount of the public good, i.e., $h(\tilde{z}) > h(z)$, which we write as $\tilde{z} \succ_e z$.

We maintain the assumption that γ and K satisfy

$$0 \;<\; K \;<\; 1 \;<\; \gamma \min_t \omega_t, \tag{A1}$$

which guarantees that the ideal tax rates of each voter type on each dimension lie strictly between zero and one.[18] In particular, low- and high-income voters have ideal points

$$\hat{z}^L = \left(\big(1 - \frac{1}{\gamma\omega}\big)K, K \right) \quad \text{and} \quad \hat{z}^H = \left(K, \big(1 - \frac{1}{\gamma(1-\omega)}\big)K \right),$$

respectively, so that each type prefers to tax the other at rate K and to tax their own type at a lower (but still positive) rate. Immediately, we see that low-income voters prefer a progressive policy, while high-income voters prefer a regressive policy. Figure 5 depicts these ideal policies, $\hat{z}_L$ and $\hat{z}_H$, as lying in the progressive and regressive regions, respectively: the set of progressive tax policies is the set of points above the dashed 45 degree line; the set of regressive policies is the set of points below the line; and the dashed line is the set of flat tax policies. It is a routine calculation to verify that $h(\hat{z}^L) > h(\hat{z}^H)$ if and only if $w_L/\omega < w_H/(1-\omega)$, which indeed holds by our assumptions that $\omega \in [1/2, 1)$ and $w_H > w_L > 0$; therefore, left-wing party A's ideal policy is more expansionist than right-wing party B's.

[17] We can assume that the set of partisan voters (including those who vote and who do not vote) is partitioned into low and high types, and thus contribute to the public good, by including those masses in the calculation of ω. The critical property is that these citizens vote for the parties independently of their tax policies.

[18] By our assumption that $\tau_L = \omega \geq 1/2 \geq 1 - \omega = \tau_H$, this means that $\min_t \tau_t = 1 - \omega$.

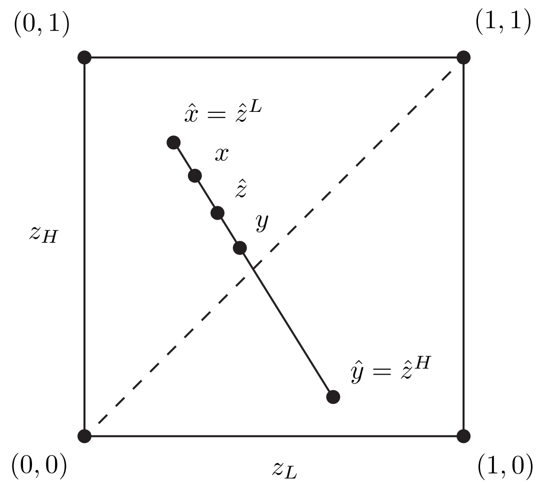

Figure 5: Configuration of tax policies

The following comparative statics of $\hat{z}^L$ and $\hat{z}^H$ follow by inspection. As their population share decreases (ω declines), or as the marginal returns to production of the public good become smaller (γ declines), the ideal policy of low-income voters becomes more progressive, but less expansionist. In contrast, as their population share rises ($1-\omega$ increases) or as the returns to production of the public good improve (γ increases), the ideal policy of high-income voters becomes more progressive and more expansionist. Both voter types prefer greater expansion when the deadweight loss to taxation declines (K rises).

Two parties, labeled A and B, compete on policy by adopting tax policies $x, y \in [0,1]^2$, the components of which represent tax rates on income of each of the two types. We assume that party A is a "left-wing party" that represents the interests of low-income types L, i.e., $u_A = u_L$, while party B is a "right-wing party" that represents the interests of high-income types H, i.e., $u_B = u_H$. Thus, the ideal points of the parties are the same as those of the income types they represent: party A has an ideal policy of $\hat{x} = \hat{z}^L$ while B has an ideal policy of $\hat{y} = \hat{z}^H$. We assume that each party has mixed motivations, with weights λ_A and λ_B on policy for A and B, respectively, and Theorem 6 immediately delivers existence of an equilibrium. To characterize equilibria and exploit the general analysis, we impose further structure on the model.

We assume the conditions needed for the aggregate voter theorem, so each type t voter has a net bias for party B that satisfies Condition 6, so that by Theorem 14, the parties' probabilities of winning satisfy (16) and (17), with the aggregate voter having utility

$$V(z) \quad = \quad 2\left[\omega\rho_L u_L(z) + (1-\omega)\rho_H u_H(z)\right].$$

The aggregate ideal point is thus

$$\hat{z} \;=\; \left((1 - \frac{\rho_L}{\gamma[\rho_L \omega + \rho_H(1-\omega)]})K, (1 - \frac{\rho_H}{\gamma[\rho_L \omega + \rho_H(1-\omega)]})K \right).$$

Note that $\hat{z}^t > 0$ for both $t \in \{L, H\}$, by the parametric assumptions in (A1) made above. In fact, $\hat{z}$ always lies on the line segment connecting $\hat{z}^L$ with $\hat{z}^H$, which is always negatively sloped, as in Figure 5, with slope equal to $-\omega/(1-\omega)$. Thus, the aggregate ideal point $\hat{z}$ taxes type L voter at a higher rate, and the type H voter at a lower rate, than $\hat{z}^L$; and it taxes type L at a lower rate, and type H at a higher rate, than $\hat{z}^H$. In particular, $\hat{z}^L \succ_p \hat{z} \succ_p \hat{z}^H$. While Figure 5 depicts the aggregate ideal point as progressive, this does not hold in general; below, we specify the conditions under which aggregate progressivity holds.

As Conditions 6–8 are satisfied in the model of income taxation, the dimensional reduction theorem, Theorem 15, holds. Thus, the left-wing party A always sets its equilibrium policy on the line segment between $\hat{x} = \hat{z}^L$ and $\hat{z}$, while the right-wing party B always sets its equilibrium policy on the segment connecting $\hat{y} = \hat{z}^H$ with $\hat{z}$. This immediately implies that party A's equilibrium policy is weakly more progressive than party B's. They are equally progressive if and only if both parties set their policies to $\hat{z}$, which in turn occurs if and only if they are both fully office-motivated, with $\lambda_A = \lambda_B = 0$.

The aggregate ideal point $\hat{z}$ is progressive if and only if $\rho_L > \rho_H$, i.e., low-income voters are more sensitive to tax policy than high-income voters are. When this is the case, the aggregate ideal point must lie above the 45 degree line, as depicted in Figure 5, and the equilibrium platform of party A is always progressive. By part (iv) of Theorem 14, the equilibrium platform of party B will then be progressive if she is purely office motivated, and by upper hemicontinuity of the equilibrium correspondence, party B's equilibrium platform is progressive if she is sufficiently office motivated. Figure 5 depicts this case, with x and y both lying in the progressive region of tax policies.

The model does not imply the progressive taxes are the only possibility. The situation is reversed when $\rho_L < \rho_H$, as then party B's equilibrium policy is always regressive, and party A's tax policy will be regressive if she is sufficiently office motivated. It is also possible that the parties take opposing positions on tax policy: for example, if $\rho_L = \rho_H$, and if the parties place positive weight on policy, i.e., $\lambda_A, \lambda_B < 1$, then the aggregate ideal point lies on the 45 degree line, A's equilibrium policy is progressive, and B's is regressive. The next proposition summarizes the preceding observations.

Proposition 3. *Under Conditions 6–8, in the two-type model of income taxation, assume (A1). There is a Nash equilibrium, and in every equilibrium* (x^*, y^*)*, we have:*

(i) If $\lambda_A, \lambda_B > 0$*, then* $\hat{x} \succ_p x^* \succ_p \hat{z} \succ_p y^* \succ_p \hat{y}$.

(ii) If $\rho_L > \rho_H$, then $\hat{z}$ is progressive, and thus x^* is progressive. Moreover, there exists $\overline{\lambda} > 0$ such that if $\lambda_B < \overline{\lambda}$ (holding other parameters fixed), then y^* is progressive.

(iii) If $\rho_L < \rho_H$, then $\hat{z}$ is regressive, and thus y^* is regressive. Moreover, there exists $\overline{\lambda} > 0$ such that if $\lambda_A < \overline{\lambda}$ (holding other parameters fixed), then x^* is regressive.

(iv) If $\rho_L = \rho_H$ and $\lambda_A, \lambda_B > 0$, then x^* is progressive, and y^* is regressive.

To gain insight into expansionism of the parties, it is easily calculated that $h(\hat{z}) > h(\hat{z}^H)$ holds if and only if

$$\frac{w_L}{w_H} \quad < \quad \frac{\omega}{1-\omega} + 2\frac{\rho_H}{\rho_L} \tag{18}$$

which always holds because $w_L/w_H < 1$ while $1 \leq \omega/(1-\omega)$. Thus, the aggregate ideal tax policy is more expansionist than the right-wing party B's ideal policy. Because the equilibrium policy of party B lies between $\hat{z}$ and $\hat{z}^H$, strict concavity of h and (18) imply that B's platform is more expansionist than its ideal policy. In contrast, $h(\hat{z}^L) > h(\hat{z})$ holds if and only if

$$\frac{1-\omega}{\omega} + 2\frac{\rho_L}{\rho_H} \quad < \quad \frac{w_H}{w_L}. \tag{19}$$

Unlike (18) above, inequality (19) is not implied by our parametric assumptions, because ρ_L/ρ_H may be arbitrarily large. When it does hold, however, we obtain conditions under which party A is more expansionist than party B. Informally, under these conditions, B is the party of "small government," while party A is the party of "big government."

Proposition 4. *Under Conditions 6–8, in the two-type model of income taxation, assume (A1). Given any Nash equilibrium (x^*, y^*), we have $y^* \succ_e \hat{z}^H$. Moreover:*

(i) if (19) holds, then we have $x^* \succ_e \hat{z} \succ_e y^* \succ_e \hat{z}^H$,

(ii) and if it fails, then $x^* \succ_e \hat{x}$.

Finally, we discuss how the model generalizes to the case of $|T| \geq 3$ voter types. We now index types as $t = 1, \ldots, n$, with incomes ordered as $0 < w_1 < w_2 < ... < w_n$. The population shares of these types are $\omega_t > 0$, with $\sum_t \omega_t = 1$. Tax policy is now an n-tuple $z = (z_t)_{t=1}^n \in [0,1]^n$, and the public good is produced by the following technology:

$$h(z) \quad = \quad \gamma \sum_{t=1}^{n} \left(z_L - \frac{(z_L)^2}{2K} \right) \omega_t w_t.$$

We can continue to state the main parametric restriction as (A1), with the understanding that $\min_t \omega_t$ is now taken over all of the n types. Extending our earlier terminology, a tax policy z is *weakly progressive* if $z_1 \leq z_2 \leq ... \leq z_n$ with at least one inequality strict, and it is *progressive* if all inequalities are strict. A policy that is not weakly progressive is *regressive*. A tax policy $\tilde{z}$ is *more progressive* than another policy z if $z_{t+1} - z_t \leq \tilde{z}_{t+1} - \tilde{z}_t$ for $t = 1, ..., n-1$, with at least one of the inequalities being strict, and it is *more expansionist* if $h(\tilde{z}) > h(z)$.

Party A is a left-wing party that represents the lowest income type's interests, so that $u_A = u_1$, while party B is a right-wing party that looks after the highest income type, so $u_B = u_T$. All other features of the model are the same, including that within each type t, voters have a net bias for party B distributed according to F_t, and that that there is a stochastic mass of partisan voters, with the net mass in favor of B distributed by G. We assume F_t and G satisfy Condition 4, so that Theorem 14 continues to hold.

Analogous to the two-type case, the ideal policy for a type t voter is now to tax her own type t at rate $\left(1 - \frac{1}{\gamma\omega_t}\right)K$, and to tax all other types at the rate K. The ideal tax policies of parties A and B are thus, respectively,

$$\hat{x} \;=\; \hat{z}^L \;=\; \left(\left(1 - \frac{1}{\gamma\omega_1}\right)K, K, \ldots, K\right)$$

and

$$\hat{y} \;=\; \hat{z}^H \;=\; \left(K, \ldots, K, \left(1 - \frac{1}{\gamma\omega_n}\right)K\right),$$

and party A's ideal policy is weakly progressive, while party B's is regressive. For the aggregate voter, we have $V(z) = 2\sum_t \omega_t \rho_t u_t(z)$, the aggregate ideal policy is then

$$\hat{z} \;=\; \left(\left(1 - \frac{\rho_1}{\gamma\sum_t \rho_t\omega_t}\right)K, ..., \left(1 - \frac{\rho_n}{\gamma\sum_t \rho_t\omega_t}\right)K\right).$$

Thus, the aggregate voter's ideal policy is weakly progressive if and only if $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$, with at least one inequality strict, and it is progressive if and only if all these inequalities are strict.

With at least three types, the aggregate ideal point $\hat{z}$ no longer lies on the line segment connecting $\hat{z}^L$ with $\hat{z}^H$. Indeed, the aggregate ideal $\hat{z}$ is to tax all types $t = 2, ..., n-1$ at a rate strictly lower than K (since $\rho_t > 0$ for all t) while every point on the segment between $\hat{z}^L$ and $\hat{z}^H$ taxes these types at the rate K. Whether the segment connecting $\hat{z}$ and $\hat{z}^L$ and the one connecting $\hat{z}$ and $\hat{z}^H$ together form an acute or an obtuse angle depends on the values of $\rho_1, ..., \rho_n$. If ρ_1 and ρ_n are very small, for example, while $\rho_2, ..., \rho_{n-1}$ are very large, then the angle is acute. If the reverse is true, then it is obtuse. Because $\hat{z}$ no longer lies on the line segment connecting $\hat{z}^L$ with $\hat{z}^H$ we can no longer say that the equilibrium policies of party A are always weakly more progressive than the equilibrium policies of party B.

However, we can give sufficient conditions under which party A adopts a more progressive policy in equilibrium than does party B. The following proposition reports such a condition.

Proposition 5. *Under Conditions 6–8, in the n-type model of income taxation, assume (A1) and $\rho_1 > \rho_2 > ... > \rho_n$. There exist $\underline{\lambda} > 0$ and $\overline{\lambda} < 1$ such that if $\lambda_A \in [0, \underline{\lambda}]$ and $\lambda_B \in [\overline{\lambda}, 1]$, then for every Nash equilibrium (x^*, y^*), we have $x^* \succ_p y^*$.*

To see why the proposition holds, note that under our specification of the model, Conditions 6–8 continue to be satisfied, so the dimensional reduction theorem holds. Therefore, party A's and party B's equilibrium platforms satisfy

$$x^* \;=\; \alpha \hat{z}^L + (1-\alpha)\hat{z} \quad \text{and} \quad y^* \;=\; \beta \hat{z}^H + (1-\beta)\hat{z},$$

respectively, for some $\alpha, \beta \in [0,1]$. We claim that if $\rho_1 > \rho_2 > ... > \rho_n$, so that the aggregate ideal point $\hat{z}$ is progressive, then party A's equilibrium policy is more progressive than party B's equilibrium policy if and only if $\alpha < \beta$. This claim, which is proven in the appendix, immediately implies the sufficient conditions stated in the proposition, because when $\lambda_A = 0$ and $\lambda_B = 1$, part (iv) of Theorem 14 implies that $x^* = \hat{z}$, and part (iii) of the theorem implies that $y^* \neq \hat{z}$. By upper hemicontinuity of the equilibrium correspondence, it follows that if A is sufficiently office motivated and B is sufficiently policy motivated, then $\alpha < \beta$.

5 Taking Uncertainty to Zero

Our equilibrium existence result, Theorem 6, only relies on log concavity of the density of the partisan vote, and thus it is consistent with arbitrarily low variance of the partisan shock π. For example, we can let g be the normal density with standard deviation σ, and equilibria will exist as $\sigma \to 0$. In this section, we characterize the limit of equilibria as we remove uncertainty from the model. We find that, very generally, if one candidate has an electoral advantage, in the sense that she receives support from a majority of policy-oriented voters whenever the candidates adopt the same platform, then her probability of winning approaches one when uncertainty is small. Thus, even a small advantage measured in voter support translates to an extreme advantage in terms of election outcomes.

We then characterize the limit of candidate platforms in the presence of an aggregate voter. We first consider the case in which the candidates have opposing preferences, in the sense that each prefers the aggregate ideal point to any platform on the contract curve between her opponent and the voter. If the electoral advantage κ is large, then the advantaged candidate converges to her ideal point; otherwise, if κ is positive and not too large, then the platform of the disadvantaged candidate converges to the aggregate ideal point, and the advantaged candidate converges to the platform that makes the aggregate voter indifferent—essentially leveraging her advantage to win with probability one at the closest possible platform to her ideal point, while the disadvantaged candidate acts as an anchor, constraining her opponent from adopting even more extreme platforms. If neither candidate is advantaged, i.e., $\kappa = 0$, then competitive forces lead the candidates to take platforms approaching the aggregate ideal, regardless of their underlying policy preferences, as uncertainty goes to zero.

In case the candidates' preferences are more aligned, so that one candidate may prefer her opponent's platform to the aggregate ideal point, the analysis is more involved, and it is more challenging to characterize the limit of equilibrium platforms. We then add the structure of quadratic utility, and we give a full characterization of the limiting platforms as a function of parameters of the model, including preferences and the electoral advantage of candidate A. Interestingly, the limit is independent of the particular sequence of distributions used, and it gives us a selection from equilibria of the limiting model with no aggregate uncertainty, analyzed by Peress (2010). Thus, the selected equilibrium can be used as approximation of equilibria in more complex models with aggregate uncertainty, when the level of uncertainty is low.

5.1 Characterization with General Utility

We now maintain the general assumptions of the stochastic partisans model to analyze the electoral prospects of the candidates, when one has an advantage vis-á-vis the preferences of policy-oriented voters. Formally, we say candidate A is *advantaged* if she receives the support of a majority of policy-oriented voters when the candidates adopt the same platform, i.e.,

$$\int F_t(0)d\tau \quad > \quad \frac{1}{2}.$$

Of course, we say candidate B is *advantaged* if the above strict inequality is reversed. We consider a sequence $\{G^n\}$ of distributions that converges weak* to the unit mass on zero, and a sequence $\{(x^n, y^n)\}$ of corresponding standard equilibria. We use the convention that the candidates' probability of winning functions in the game with distribution G^n are denoted P^n_A and P^n_B, and their payoff functions are U^n_A and U^n_B.

We also use the fact that if $(x^n, y^n) \to (x^*, y^*)$ and

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad > \quad \frac{1}{2},$$

then candidate A's probability of winning converges to one. Indeed, we can choose π' such that for sufficiently high n, we have

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \;\; > \;\; \pi' \;\; > \;\; \frac{1}{2}.$$

Then for such n,

$$\begin{aligned} P_A^n(x^n, y^n) &= G^n\left(\int F_t(u_t(x^n) - u_t(y^n)) - \frac{1}{2}\right) \\ &\geq G^n\left(\pi' - \frac{1}{2}\right) \\ &\to 1, \end{aligned}$$

as claimed. We establish, quite generally, that if a candidate has an electoral advantage, then her probability of winning converges to one.

Furthermore, we give two partial restrictions on limiting platforms under additional conditions. First, if the equilibrium platforms of the advantaged candidate, say A, do not approach her ideal point, and if her ideal point does not garner a majority of support from the policy-oriented voters, then in the limit, the platforms of the candidates must result in a tie. Second, in the case of such a tie, if the platforms of the disadvantaged candidate, B, do not approach a platform that maximizes her support from policy-oriented voters against the limit of A's platforms, then B must be indifferent between losing to A at platform x^*, on the one hand, and winning herself with platform B, on the other.

The following result augments the baseline conditions for the stochastic partisans model only by adding strict quasi-concavity in x (respectively, strict quasi-convexity in y) to Condition 4, thereby precluding probability of winning functions that take constant values on open regions of the policy space.

Theorem 17. *Under Condition 4, assume that candidate A is advantaged, and that candidate A's share of policy-oriented voters, in (12), is strictly quasi-concave in x and strictly quasi-concave in y. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11), each has density g^n that is continuously differentiable on its support S_{G^n}, and $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \to (x^*, y^*)$, such that for all n, (x^n, y^n) is a standard Nash equilibrium of the stochastic partisans model with partisan shock distribution G^n. Then:*

(i) Candidate A's probability of winning converges to one, i.e.,

$$\lim_{n\to\infty} P_A^n(x^n, y^n) \;\; = \;\; 1.$$

(ii) If $x^* \neq \hat{x}$, if $\lambda_A > 0$, and if A's ideal point does not gain a majority of support among policy-oriented voters against y^*, i.e.,

$$\int F_t(u_t(\hat{x}) - u_t(y^*))d\tau \quad \leq \quad \frac{1}{2}, \tag{20}$$

then the election is tied among policy-oriented voters, i.e.,

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad = \quad \frac{1}{2}. \tag{21}$$

(iii) If x^* and y^* are tied, as in (21), and if y^* does not maximize B's support among policy-oriented voters against x^*, i.e.,

$$y^* \notin \operatorname*{argmin}_{z \in Z} \int F_t(u_t(x^*) - u_t(z))d\tau, \tag{22}$$

then candidate B is indifferent between losing to A at x^* or winning herself at y^*, i.e.,

$$\lambda_B(u_B(y^*) - u_B(x^*)) + 1 - \lambda_B \quad = \quad 0. \tag{23}$$

To analyze the limiting probabilities of winning for the model in which neither candidate is advantaged, we assume that the candidates place positive weight on office, and that the partisan shock becomes degenerate in a way that is not too asymmetric; formally, $G^n(0)$ does not go to zero or one. Then we establish that the equilibrium probability of each candidate has a positive limit. To characterize the limits of equilibrium platforms when neither candidate has an advantage, we show that if a candidate places positive weight on policy, and if her equilibrium platforms do not approach her ideal point, then her limiting platform does not garner majority support among policy-oriented voters against the limiting platform of her opponent. An implication is that if each candidate places positive weight on policy, and if we have $x^* \neq \hat{x}$ and $y^* \neq \hat{y}$, then the limiting platforms are tied. We further show that in the case of a tie, the limiting platforms must actually coincide. Finally, in the case of a tie, adding the assumption that the candidates each place positive weight on office, we prove that the equilibrium platforms approach the unique equilibrium (z^*, z^*) of the model with purely office-motivated candidates.[19]

Theorem 18. *Under Condition 4, assume that neither candidate is advantaged, and that candidate A's share of policy-oriented voters, in (12), is strictly quasi-concave in x and strictly quasi-concave in y. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11), each has density g^n that is continuously differentiable on its support S_{G^n}, and $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \rightarrow (x^*, y^*)$, such that for all n, (x^n, y^n) is a standard Nash equilibrium of the stochastic partisans model with partisan shock distribution G^n. Then:*

(i) If candidate A places positive weight on office, i.e., $\lambda_A < 1$, and if

$$\liminf_{n \rightarrow \infty} G^n(0) \quad > \quad 0,$$

[19]Note that by Theorem 10, the equilibrium platform z^* in the model with purely office-motivated candidates is defined independently of the distribution G^n.

then A's equilibrium probability of winning has a positive lower bound, i.e.,

$$\lim_{n\to\infty} P_A^n(x^n, y^n) \quad > \quad 0,$$

with an analogous result for B.

(ii) If $x^* \neq \hat{x}$ and $\lambda_A > 0$, then

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad \leq \quad \frac{1}{2},$$

with an analogous result for B.

(iii) If the limiting platforms are tied among policy-oriented voters, i.e.,

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad = \quad \frac{1}{2}, \tag{24}$$

then the limiting platforms of the candidates coincide, i.e., $x^* = y^*$.

(iv) If (24) holds and $\lambda_A, \lambda_B < 1$, then $x^* = y^* = z^*$.

5.2 Limits of Equilibria with Aggregate Voter

Theorems 17 and 18 inform us of the limits of equilibrium platforms as uncertainty becomes small under very general conditions; in this subsection, to gain further analytical traction, we impose more structure: we assume that idiosyncratic bias is uniformly distributed, so that the aggregate voter theorem applies, and the unique equilibrium platform in the model with purely office-motivated candidates is just the aggregate ideal point: $z^* = \hat{z}$. Our analysis in this subsection, and the next on quadratic utility, relies on two lemmas, which are easily extracted from the preceding two theorems.

The first lemma gives conditions under which the aggregate voter must be indifferent between electing candidates A or B, when given their limiting platforms. Essentially, if A's advantage is not too large, and if A's platforms do not approach her ideal point, then the aggregate voter must be indifferent: if she strictly preferred to elect A, then the candidate could increase her payoffs by moving closer to her ideal point and still win with probability approaching one; and if she strictly preferred to elect B, then A could shift in the direction of the aggregate ideal point to win and increase her payoffs. The result is a simple translation of part (ii) of Theorem 17 to the model with aggregate voter: inequality (20) becomes $V(\hat{x})+\kappa \leq V(y^*)$, and the conclusion (21) of the theorem becomes (25), below.

Lemma 5. *Under Conditions 4 and 6, assume that V is strictly quasi-concave; that $\lambda_A, \lambda_B > 0$; that candidate A is advantaged, i.e., $\kappa > 0$; and that the candidates' contract curves with the aggregate voter intersect only at the aggregate* *ideal point, i.e.,* $\overline{AV} \cap \overline{BV} = \{\hat{z}\}$. *Let* $\{G^n\}$ *be a sequence of distributions such that each* G^n *satisfies (11), each has density* g^n *that is continuously differentiable on its support* S_{G^n}*, and* $\{G^n\}$ *converges weak* to the unit mass on zero. Let* $\{(x^n, y^n)\}$ *be a convergent sequence, with* $(x^n, y^n) \to (x^*, y^*)$*, such that for all n,* (x^n, y^n) *is a standard Nash equilibrium of the stochastic partisans model with partisan shock distribution* G^n*. If* $V(\hat{x}) + \kappa \leq V(y^*)$ *and* $x^* \neq \hat{x}$*, then the aggregate voter is indifferent between candidate A winning with* x^* *and B winning with* y^**, i.e.,*

$$V(x^*) + \kappa \quad = \quad V(y^*). \tag{25}$$

The second lemma provides conditions for another form of indifference, now for the disadvantaged candidate, B. If the aggregate voter is indifferent between electing one candidate or the other, and if B's platforms do not approach the aggregate ideal point, then she must be indifferent between losing to A at the limiting platform x^* or winning herself with y^*. If candidate B strictly preferred to lose to A at x^*, then her equilibrium payoff would be negative when uncertainty is sufficiently small, contradicting Theorem 3, which establishes that candidate payoffs are strictly positive at all standard equilibria; and if B strictly preferred to win with y^*, then she could shift toward the aggregate voter to win with probability close to one, thereby increasing her payoffs. The result is a simple translation of part (iii) of Theorem 17 to the model with aggregate voter: inequality (20) becomes $V(\hat{x}) + \kappa \leq V(y^*)$, and the assumption (22) becomes $y^* \neq \hat{z}$, giving us (26).

Lemma 6. *Under Conditions 4 and 6, assume that* V *is strictly quasi-concave, that* $\lambda_A, \lambda_B > 0$ *and that candidate A is advantaged, i.e.,* $\kappa > 0$*. Let* $\{G^n\}$ *be a sequence of distributions such that each* G^n *satisfies (11), each has density* g^n *that is continuously differentiable on its support* S_{G^n}*, and* $\{G^n\}$ *converges weak* to the unit mass on zero. Let* $\{(x^n, y^n)\}$ *be a convergent sequence, with* $(x^n, y^n) \to (x^*, y^*)$*, such that for all n,* (x^n, y^n) *is a standard Nash equilibrium of the stochastic partisans model with partisan shock distribution* G^n*. If* $V(x^*) + \kappa = V(y^*)$ *and* $y^* \neq \hat{z}$*, then candidate B is indifferent between losing to A at* x^* *or winning herself at* y^**, i.e.,*

$$\lambda_B(u_B(y^*) - u_B(x^*)) + 1 - \lambda_B \quad = \quad 0. \tag{26}$$

We can now characterize the limits of equilibrium platforms, when one candidate has an electoral advantage, candidates and voters have general utility functions, the candidates' preferences are opposed, in the sense that each prefers the aggregate ideal point to any platform on the contract curve between her opponent and the voter. If the electoral advantage κ is large, then the advantaged candidate, A, converges to her ideal point; otherwise, if κ is not too large, then B's platform converges to the aggregate ideal point, and A's converges to the platform that makes the aggregate voter indifferent. Intuitively, candidate A leverages her advantage to win with probability one at the closest possible platform to her ideal point, while candidate B acts as an anchor, constraining A

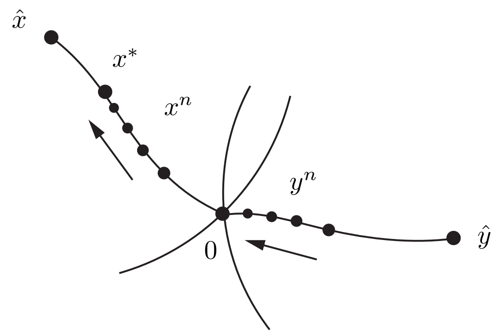

Figure 6: Limiting platforms with opposed candidate preferences

from adopting even more extreme platforms. See Figure 6 for a depiction in which the candidates and aggregate voter have quadratic utility.

Theorem 19. *Under Conditions 4 and 6, assume that V is strictly quasi-concave; that $\lambda_A, \lambda_B > 0$; that candidate A is advantaged, i.e., $\kappa > 0$; and that candidate B prefers $\hat{z}$ to all other platforms on A's contract curve with the aggregate voter, i.e., for all $x \in \overline{AV}$, we have $u_B(0) > u_B(x)$. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11), each has density g^n that is continuously differentiable on its support S_{G^n}, and $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \to (x^*, y^*)$, such that for all n, (x^n, y^n) is a standard Nash equilibrium of the stochastic partisans model with partisan shock distribution G^n.*

(i) If $\kappa \geq V(0) - V(\hat{x})$, then $x^* = \hat{x}$, and otherwise,

(ii) if $\kappa < V(0) - V(\hat{x})$, then $y^* = 0$ and $V(x^*) + \kappa = V(0)$.

To characterize the limits of equilibrium platforms when the electoral playing field is level, i.e., $\kappa = 0$, we continue to assume some disagreement in the candidates' preferences, now in the sense that the contract curves of the candidates with the aggregate voter intersect only at the aggregate ideal point. We show that competitive forces lead the candidates to take platforms approaching the aggregate ideal, regardless of their underlying policy preferences, as uncertainty goes to zero. Interestingly, this results holds even if the candidates care solely about policy, rather than office.

Theorem 20. *Under Conditions 4 and 6, assume that V is strictly quasi-concave; that $\lambda_A, \lambda_B > 0$; that neither candidate is advantaged, i.e., $\kappa = 0$; and that the candidates' contract curves with the aggregate voter intersect only at the aggregate ideal point, i.e., $\overline{AV} \cap \overline{BV} = \{\hat{z}\}$. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11), each has density g^n that is continuously* *differentiable on its support S_{G^n}, and $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \to (x^*, y^*)$, such that for all n, (x^n, y^n) is a standard Nash equilibrium of the stochastic partisans model with partisan shock distribution G^n. Then $x^* = y^* = \hat{z}$.*

Remark 13. Duggan and Ma (2023) analyze a multidimensional model of bargaining with a fixed agenda setter, an arbitrary number of voters, and an arbitrary voting rule. Translating the aggregate voter in our model into the agenda setter, and the two candidates into two voters, a platform at which the candidates' contract curves cross, i.e., $z \in \overline{AV} \cap \overline{BV}$, is referred to as a "constrained core" alternative by Duggan and Ma. That paper shows formally that in three or more dimensions, the constrained core is generically empty, i.e., for "almost all" specifications of utility functions, the unique intersection of the contract curves is the aggregate ideal point. Thus, when $d \geq 3$, the assumption $\overline{AV} \cap \overline{BV} = \{\hat{z}\}$ is generically satisfied. □

Theorem 19 does not pin down the limit of equilibrium platforms when candidate preferences are not opposed, but Lemmas 5 and 6 do have implications for the general case. If (x^*, y^*) is the limit of standard equilibria in the model with an aggregate voter and with A having an electoral advantage, then we can say that one of the following holds:

- $x^* = \hat{x}$,
- $y^* = \hat{z}$ and $V(x^*) + \kappa = V(\hat{z})$,
- (x^*, y^*) solve the system of equations

$$\begin{aligned} V(x^*) + \kappa &= V(y^*) \\ \lambda_B u_B(y^*) + 1 - \lambda_B &= \lambda_B u_B(x^*). \end{aligned}$$

However, this observation does not imply that there is a unique limit of equilibrium platforms; in case there is a unique limit, it does not identify the relevant case in terms of model parameters; and it does not shed light on the nature of the solution to the above system of equations.

5.3 Full Characterization with Quadratic Utility

The characterization result of Theorem 19 addresses the case in which the candidates have opposing preferences, but the analysis is incomplete, as it does not account for equilibria in which one candidate prefers her opponent's platform to the aggregate ideal point. In this subsection, we complete the analysis by adding the structure of generalized quadratic utility: aggregate voter utility is generalized quadratic with coefficient matrix M; candidates also have generalized quadratic utility each with coefficient matrix M; and ideal points $\hat{x}$, $\hat{y}$, and

$\hat{z} = 0$ are distinct and such that $\hat{x}$ and $\hat{y}$ are not positive scalar multiples of each other.

Theorem 19 immediately applies to the quadratic model, where candidates' preferences are suitably opposed as long as the angle formed by their ideal points with the aggregate ideal point is obtuse, in the sense that $\hat{x}M\hat{y} < 0$; see Figure 6 for an illustration. In fact, because all equilibria are standard when the candidates' ideal points do not point in the same direction, we can drop that explicit restriction from the following corollary.

Corollary 6. *Under Conditions 6–8, assume that $\lambda_A, \lambda_B > 0$; that $\hat{z} = 0$ with $\hat{x}$ and $\hat{y}$ forming an obtuse angle, i.e., $\hat{x}M\hat{y} < 0$; and that candidate A is advantaged, i.e., $\kappa > 0$. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11) and each has density g^n that is continuously differentiable on its support S_{G^n}, and assume that $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \to (x^*, y^*)$, such that for all n, (x^n, y^n) is a Nash equilibrium of the stochastic partisans model with uniform bias, generalized quadratic utility, and partisan shock distribution G^n. Then candidate A's probability of winning converges to one, $y^* = 0$, and conditions (i) and (ii) of Theorem 19 hold.*

Theorem 20, which considers the case in which neither candidate is advantaged, also applies. The key assumption there was that the candidates' contract curves with the aggregate voter intersect only at the aggregate ideal point; in the quadratic environment, this condition is automatically satisfied as long as the candidate ideal points do not point in the same direction from $\hat{z} = 0$. Thus, equally matched candidates are induced to choose platforms close to the aggregate ideal point as electoral uncertainty becomes small.

Corollary 7. *Under Conditions 6–8, assume that $\lambda_A, \lambda_B > 0$; that $\hat{z} = 0$ with $\hat{x}$ and $\hat{y}$ not pointing in the same direction, i.e., $\hat{x}\hat{y} < \|\hat{x}\| \, \|\hat{y}\|$; and that neither candidate is advantaged, i.e., $\kappa = 0$. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11) and each has density g^n that is continuously differentiable on its support S_{G^n}, and assume that $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \to (x^*, y^*)$, such that for all n, (x^n, y^n) is a Nash equilibrium of the stochastic partisans model with uniform bias, generalized quadratic utility, and partisan shock distribution G^n. Then $x^* = y^* = 0$.*

When one candidate is advantaged and the candidates' preferences are not opposed, the characterization of limiting platforms is complex. It is still the case that when κ is large, the advantaged candidate A locates at her ideal point, and when κ is lower, it may be that the disadvantaged candidate B acts as an anchor at the aggregate ideal point, which A's platform makes the aggregate voter indifferent between electing either candidate. However, we also find an "exceptional case" in which the limit of platforms is not as in Corollary 6. The advantaged candidate A still pulls policy as far as possible toward (but not to) her ideal point while winning with probability close to one, and the disadvantaged candidate B still acts as an anchor, but not at $\hat{z} = 0$. Rather, B's platform is positioned away from $\hat{z} = 0$, giving A greater latitude in choosing her platform.

The exceptional case is when (i) A's electoral advantage is not too great, (ii) B does not place too much weight on office, and (iii) polarization among the candidates is low enough, in the sense that the angle formed by their ideal points relative to the aggregate voter is small enough. Under these conditions, the situation described in part (ii) of Corollary 6, where B positions at $\hat{z} = 0$ and A makes the aggregate voter indifferent, *cannot* be obtained as the limit of equilibria as uncertainty is removed from the model. The issue is that when uncertainty is small, candidate B would prefer that candidate A wins, rather than winning herself near the voter ideal point, and she thus becomes unwilling to act as an anchor for A. See the left-hand panel of Figure 7, where B has an incentive to deviate to platforms near x^* to increase her payoff. In this exceptional case, the limit of equilibrium platforms requires a different characterization, which we provide in closed form.

Lemmas 5 and 6 provide the key insight needed for the analysis. If the advantaged candidate A's equilibrium platforms do not converge to her ideal point, i.e., $x^* \neq \hat{x}$, then the indifference condition (25) for the aggregate voter must hold; and if the disadvantaged candidate B does not position at the aggregate ideal point, then the indifference condition (26) for B must hold. This gives us two equations specified at the end of the preceding subsection,

$$\begin{aligned} V(\alpha\hat{x}) + \kappa &= V(\beta\hat{y}) && (27) \\ \lambda_B u_B(\beta\hat{y}) + 1 - \lambda_B &= \lambda_B u_B(\alpha\hat{x}), && (28) \end{aligned}$$

in two unknowns, α and β, where we use the dimensionality reduction theorem for quadratic utility. The right-hand panel of Figure 7 depicts a solution (α, β) to the system of equations for which $\beta > 0$ and $\alpha < 1$, where B's platform is positioned to make the aggregate voter indifferent between electing one candidate or the other, and A's platform is positioned to make B indifferent between winning and losing.

The remainder of this subsection establishes the limiting characterization of equilibria, with special attention on the exceptional case. Toward this end, we solve (27) and (28) using the following shorthand:

$$\begin{aligned} X &= \lambda_B \hat{x} M \hat{x} & \qquad \Gamma &= \tfrac{\lambda_B (\hat{x} M \hat{y})}{Y} \\ Y &= \lambda_B \hat{y} M \hat{y} & \qquad \Delta &= \tfrac{1 - \lambda_B + \kappa\lambda_B}{2Y}, \end{aligned}$$

where $X, Y > 0$ follows from positive definiteness of M, and $\Delta > 0$, since $\lambda_B \in (0, 1]$. Then equation (27) simplifies to

$$-\alpha^2 X + \beta^2 Y + \kappa\lambda_B = 0. \qquad (29)$$

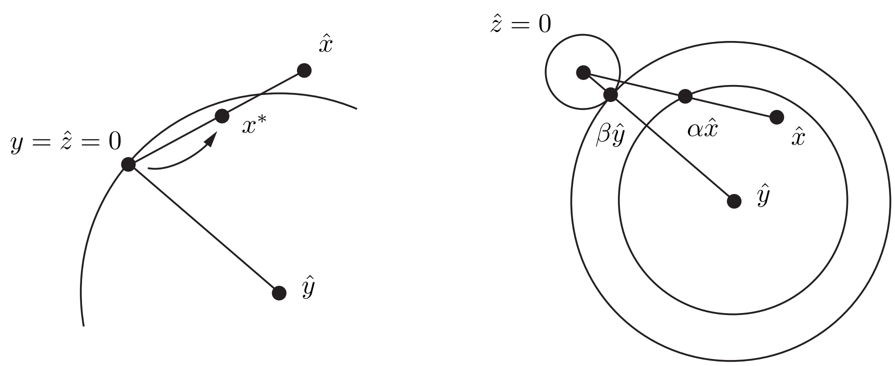

Figure 7: Limit of equilibria in the exceptional case

Equation (28) becomes

$$2\beta Y - \beta^2 Y + 1 - \lambda_B - 2\alpha\lambda_B(\hat{x}M\hat{y}) + \alpha^2 X \quad = \quad 0, \tag{30}$$

and substituting in (29), we obtain the restriction

$$2\beta Y - 2\alpha\lambda_B(\hat{x}M\hat{y}) + 1 - \lambda_B + \kappa\lambda_B \quad = \quad 0.$$

This allows us to solve for β in terms of α, as follows:

$$\beta \quad = \quad \left(\frac{\lambda_B(\hat{x}M\hat{y})}{Y}\right)\alpha - \frac{1 - \lambda_B + \kappa\lambda_B}{2Y} \quad = \quad \Gamma\alpha - \Delta.$$

Substituting the latter expression into (30) gives us a quadratic equation in α, which we can solve for: in case the angle between $\hat{x}$ and $\hat{y}$ is obtuse, i.e., $\hat{x}M\hat{y} \le 0$, set

$$\alpha^0 \quad = \quad \frac{-\Gamma\Delta Y - \sqrt{(\Gamma\Delta Y)^2 + (X - \Gamma^2 Y)(\Delta^2 Y + \kappa)}}{X - \Gamma^2 Y},$$

and in case the angle is acute, i.e., $\hat{x}M\hat{y} > 0$, set

$$\alpha^0 \quad = \quad \frac{-\Gamma\Delta Y + \sqrt{(\Gamma\Delta Y)^2 + (X - \Gamma^2 Y)(\Delta^2 Y + \kappa)}}{X - \Gamma^2 Y}.$$

In either case, we have

$$\beta^0 \quad = \quad \Gamma\alpha^0 - \Delta.$$

In the expression for α^0, note that since M is symmetric and positive definite, $XY \ge (xMy)^2$ follows from the Cauchy-Schwartz inequality, and the inequality in fact holds strictly, since the ideal points $\hat{x}$ and $\hat{y}$ are not collinear with, and on the same side of, $\hat{z} = 0$. Thus, when $\hat{x}M\hat{y} \le 0$, we have $\alpha^0 \le 0$, and when $\hat{x}M\hat{y} > 0$, we have $\alpha^0 > 0$.

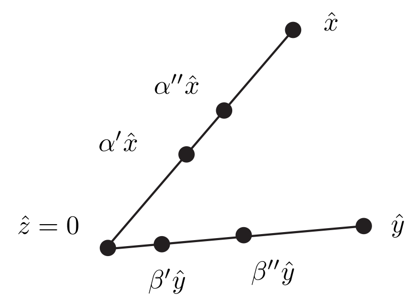

Figure 8: Sliding lemma

The analysis is facilitated by the following *sliding lemma,* which allows us to leverage the indifference conditions of Lemmas 5 and 6. To convey the idea, imagine pairs (α', β') and (α'', β'') with $0 < \alpha'' < \alpha'$, $0 < \beta'' < \beta'$, and $\alpha'\|\hat{x}\| > \beta'\|\hat{y}\|$. See Figure 8 for an illustration of the situation. Then the aggregate voter utility at platforms $\alpha'\hat{x}$ and $\beta'\hat{y}$, is greater, respectively, than the utility at $\alpha''\hat{x}$ and $\beta''\hat{y}$. Suppose that the aggregate utility difference is the same, i.e., $V(\alpha''\hat{x}) - V(\alpha'\hat{x}) = V(\beta''\hat{y}) - V(\beta'\hat{y})$. Since $\alpha'\|\hat{x}\| > \beta'\|\hat{y}\|$, and since V is concave, it follows that the distance between $\alpha'\hat{x}$ and $\alpha''\hat{x}$ is smaller than the distance between $\beta'\hat{y}$ and $\beta''\hat{y}$. Moreover, assume that candidate B is indifferent between losing to A with $\alpha'\hat{x}$ or winning herself with $\beta'\hat{y}$.

Now, we "slide up" the contract curves to $\alpha''\hat{x}$ and $\beta''\hat{y}$. Recall that the move from $\beta'\hat{y}$ to $\beta''\hat{y}$ is larger than the move from $\alpha'\hat{x}$ to $\alpha''\hat{x}$. In addition, because the former move is in the direction of B's ideal point $\hat{y}$, and because $\beta'\hat{y}$ is further from her ideal point, the gradient of her utility function is higher along this move. Combined, this means that the utility difference for candidate B between $\beta''\hat{y}$ and $\beta'\hat{y}$ *exceeds* the utility difference between $\alpha''\hat{x}$ and $\alpha'\hat{x}$. Since she was initially indifferent between winning and losing, candidate B now strictly prefers to win with $\beta''\hat{y}$, rather than lose to A at $\alpha''\hat{x}$. Alternatively, if we begin by assuming that candidate B is indifferent between losing to $\alpha''\hat{x}$ or winning with $\beta''\hat{y}$, then we "slide down" the contract curves to conclude that she strictly prefers to lose to A with $\alpha''\hat{x}$, rather than win with $\beta'\hat{y}$. Here, we assumed $\alpha'\|\hat{x}\| > \beta'\|\hat{y}\|$, but the logic also holds when $\alpha''\|\hat{x}\| > \beta''\|\hat{y}\|$.

Lemma 7. *Under Conditions 6–8, let* $0 < \alpha'' < \alpha'$ *and* $\beta'' < \beta'$, *with* $\beta' + \beta'' > 0$. *If*

$$V(\alpha'\hat{x}) - V(\alpha''\hat{x}) \quad = \quad V(\beta'\hat{y}) - V(\beta''\hat{y}) \tag{31}$$

and either $\alpha'\|\hat{x}\| > \beta'\|\hat{y}\|$ *or* $\alpha''\|\hat{x}\| > \beta''\|y\|$, *then*

$$u_B(\alpha'\hat{x}) - u_B(\alpha''\hat{x}) \quad < \quad u_B(\beta'\hat{y}) - u_B(\beta''\hat{y}).$$

The next theorem characterizes the limit of equilibria as we remove uncertainty from the model. When the candidates' ideal points form an obtuse angle with the aggregate ideal point, i.e., $\hat{x}M\hat{y} \leq 0$, Corollary 6 applies, so we now focus on the acute angle case, $\hat{x}M\hat{y} > 0$. The theorem identifies a cutoff level of electoral advantage such that for κ exceeding that level, candidate A's equilibrium platforms approach her ideal point as uncertainty becomes small. For a lower level of advantage, it is possible that candidate B's platforms converge to the aggregate ideal point, anchoring the platform of candidate A, who leverages her advantage by choosing platforms that converge to a point making the aggregate voter indifferent. The remaining case is the exceptional case discussed above, in which A's advantage is not too large, B does not place too much weight on office, and the candidates' preferences are sufficiently aligned. In that case, the candidates' equilibrium platforms converge to the unique solution (α^0, β^0) to equations (27) and (28).

Theorem 21. *Under Conditions 6–8, assume that $\lambda_A, \lambda_B > 0$; that $\hat{z} = 0$ with $\hat{x}$ and $\hat{y}$ forming an acute angle, i.e., $0 < \hat{x}M\hat{y}$; that $\hat{x}$ and $\hat{y}$ do not point in the same direction, i.e., $\hat{x}\hat{y} < \|\hat{x}\|\,\|\hat{y}\|$; and that candidate A is advantaged, i.e., $\kappa > 0$. Let $\{G^n\}$ be a sequence of distributions such that each G^n satisfies (11) and each has density g^n that is continuously differentiable on its support S_{G^n}, and assume that $\{G^n\}$ converges weak* to the unit mass on zero. Let $\{(x^n, y^n)\}$ be a convergent sequence, with $(x^n, y^n) \to (x^*, y^*)$, such that for all n, (x^n, y^n) is a Nash equilibrium of the stochastic partisans model with uniform bias, generalized quadratic utility, and partisan shock distribution G^n. Then candidate A's equilibrium probability of winning converges to one, and*

(i) if $\kappa \geq \overline{\kappa}$, then $x^* = \hat{x}$,

(ii) otherwise, if $\kappa < \overline{\kappa}$ and $\lambda_B(u_B(0) - u_B(\overline{\alpha}\hat{x})) + 1 - \lambda_B \geq 0$, then $(x^*, y^*) = (\overline{\alpha}\hat{x}, 0)$,

(iii) and if $\kappa < \overline{\kappa}$ and $\lambda_B(u_B(0) - u_B(\overline{\alpha}\hat{x})) + 1 - \lambda_B < 0$, then $(x^*, y^*) = (\alpha^0\hat{x}, \beta^0\hat{y})$,

where $\overline{\alpha}$ and $\overline{\kappa}$ are defined by

$$V(\overline{\alpha}\hat{x}) + \kappa \;=\; V(0) \quad \text{and} \quad \overline{\kappa} \;=\; V(\overline{\beta}\hat{y}) - V(\hat{x}),$$

and $\overline{\beta}$ is the lower solution to

$$\lambda_B(u_B(\overline{\beta}\hat{y}) - u_B(\hat{x})) + 1 - \lambda_B \;=\; 0.$$

Example 4. In Figure 9, we depict equilibrium platforms and candidate A's probability of winning as we remove uncertainty from the model, when the angle formed by the candidates' ideal points is obtuse. In both panels, we assume quadratic utility, $\|\hat{x}\| = \|\hat{y}\| = 1$, $\hat{x} \cdot \hat{y} = -.8$, $\kappa = .5$, and $\lambda_A = \lambda_B = 1$. Since A's advantage is not too large, part (ii) of Theorem 21 applies, so we know

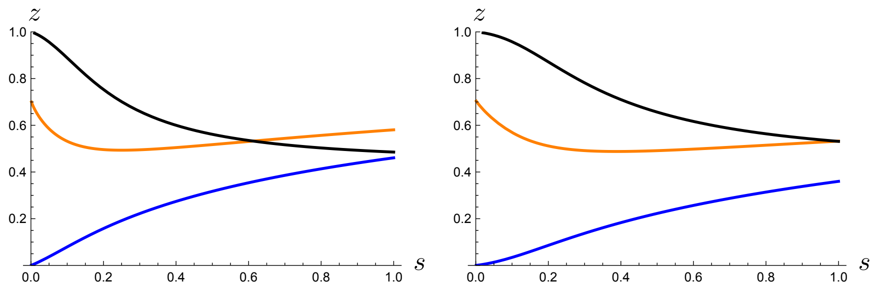

Figure 9: Removing uncertainty: obtuse angle

that the advantaged candidate A's probability of winning (black) converges to one and that A's platform (orange) converges to $\tilde{\alpha} = .7071$, while B's platform (blue) converges to the aggregate voter's ideal point at zero. In the left-hand panel, we compute equilibria assuming G is the logistic distribution, with scale parameter s, and in the right-hand panel, we assume G is normal with standard deviation s. Notably, we see that convergence of A's platform is non-monotonic: when the variance of the partisan shock decreases, A's probability of winning initially increases in a convex way, and she moves toward the aggregate voter to take advantage of the voter's responsiveness to her platform. As A's probability of winning becomes concave, the candidate pulls her platform toward her ideal point, leveraging her electoral advantage, at negligible cost to her probability of winning, which converge to one. $\square$

Example 5. In Figure 10, we maintain the assumptions of Figure 9, but we now examine the exceptional case by setting the angle between the candidates' platforms to be acute: $\hat{x} \cdot \hat{y} = .8$. By Theorem 21, we know that candidate A's probability of winning (black) converges to one, that A's platform (orange) converges to $\alpha^0 = .8123$, and that B's platform (blue) converges to $\beta^0 = .3999$. In the left-hand panel, we compute equilibria assuming G is the logistic distribution, with scale parameter s, and in the right-hand panel, we assume G is normal with standard deviation s. As in the previous example, we see that convergence to the limit is non-monotonic for the advantaged candidate, as A does not exert her leverage until the variance of the partisan shock is relatively small. $\square$

5.4 Limiting Model of Peress (2010)

The limit of our model, once uncertainty is completely removed, is analyzed by Peress (2010). That paper characterizes equilibria in which one candidate, say A, has an electoral advantage: assuming the electoral advantage is not too large, the disadvantaged candidate locates at the "vote-maximizing point," and the

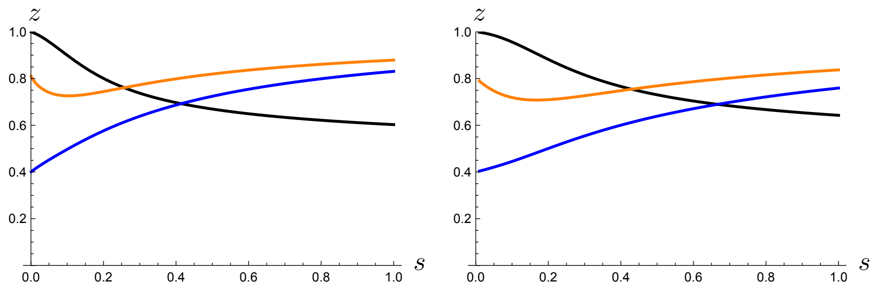

Figure 10: Removing uncertainty: exceptional case

advantaged candidate locates at the best element of the "pseudo-core." Peress states his result in the general model, without the aggregate voter structure of the previous subsection. With that structure, the description of equilibria in his model simplifies: candidate B locates at the aggregate ideal point $y = \hat{z} = 0$, and candidate A locates on the contract curve with the aggregate voter to make the voter indifferent, i.e., A's platform x solves

$$V(0) \quad = \quad V(x) + \kappa, \tag{32}$$

where a solution exists as long as $\kappa \leq V(0) - V(\hat{x})$. This creates a tie, as half of all voters support A, while the other half support B, and Peress assumes that the tie is broken in favor of the advantaged candidate: A wins with probability one. The remainder of the discussion assumes $\kappa < V(0) - V(\hat{x})$, so that A cannot locate at her ideal point and win with certainty, irrespective of B's location. In this case, let $\overline{x}$ denote the unique solution to (32). We also assume for simplicity that aggregate voter utility and candidate utilities are quadratic, so $\overline{x} = \overline{\alpha}\hat{x}$ lies on the line segment between A's ideal point and $\hat{z} = 0$.

The platforms described above form an equilibrium of the model of Peress (2010) with no partisan shock, by a suitable choice of winning probability for the disadvantaged candidate B when her platform creates a tie. In the stochastic partisans model, when $\hat{x}\hat{y} \leq 0$, the platforms also characterize the limit of equilibria as uncertainty goes to zero. This characterization continues to hold when $\hat{x}M\hat{y} > 0$ and B's office motivation is high enough, as long as the angle formed by the candidates' ideal points is not too small. The next example discusses the exceptional case, where the equilibrium described by Peress (2010) fails to capture the limit of equilibria in our model.

Example 6. In the stochastic partisans model, assume that the aggregate voter result holds, that A is advantaged (so $\kappa > 0$), that utilities are quadratic, that B is purely policy motivated, and that $\hat{x}$ and $\hat{y}$ each have norm one, with acute angle $\hat{x} \cdot \hat{y} > 0$ formed between them. Let $\{(x^n, y^n)\}$ be a sequence of equilibria as $\{G^n\}$ converges weak* to the unit mass on zero, and suppose toward a contradiction that: $y^n \to \hat{z} = 0$, that $x^n \to \overline{x}$, where $\overline{x}$ is the solution to (32), and that candidate A's probability of winning converges to one. We specify the location of candidate ideal points as in the left-hand panel of Figure 7, and we specify $\kappa > 0$ such that $\overline{x}$ is located as in the figure; in particular, candidate B strictly prefers $\overline{x}$ to $\hat{z}$. Then for high enough n, candidate B's equilibrium expected payoff satisfies

$$U_B(x^n, y^n) \;=\; P_B(x^n, y^n)[u_B(y^n) - u_B(x^n)] < 0,$$

whereas B could replicate A's platform to obtain $U_B(x^n, x^n) = 0$. Thus, for high enough n, (x^n, y^n) is not an equilibrium, a contradiction. In the model of Peress (2010) with no partisan shock, the configuration in the left-hand panel of Figure 7 can be supported as an equilibrium by specifying that B loses with probability one whenever her platform creates a tie with A. $\square$

Theorem 21 demonstrates that the characterization of limits of equilibria in the stochastic partisans model is more nuanced in the exceptional case, with limits x^* and y^* being the unique solution to the equations (27) and (28). Interestingly, the pair (x^*, y^*) gives us an additional equilibrium in the limiting model of Peress with no uncertainty.

Example 7. In the model of Peress (2010) with no partisan shock, we continue to assume that utilities are quadratic, $\|\hat{x}\| = \|\hat{y}\| = 1$, and that $\hat{x} \cdot \hat{y} > 0$. The right-hand panel of Figure 7 depicts a solution to equations (27) and (28), where we assume $\lambda_B < 1$. Here, candidate B locates at the platform $\beta\hat{y}$ that makes her indifferent between winning at $\beta^0\hat{y}$ or losing to A at $\alpha^0\hat{x}$. Since $1 - \lambda_B > 0$, this implies that on policy grounds, $\alpha^0\hat{x}$ is preferred to $\beta^0\hat{y}$. The aggregate voter is indifferent between electing A and B, so the election is tied, and we break the tie in favor of candidate A. Note that B has the option of stealing victory from A by choosing a platform even closer (in the small circle around the origin), but by equation (28), this deviation is not profitable for B. Meanwhile, if A deviates to a preferred policy, then victory shifts to B, so this is not profitable. Thus, $(\alpha^0\hat{x}, \beta^0\hat{y})$ comprises an equilibrium of the limiting model. For example, specifying $\hat{x} = \hat{y} = 1$ with $\hat{x} \cdot \hat{y} = \frac{\pi}{6}$, and setting $\lambda_B = .9$ and $\kappa = .15$, the unique solution to the two equations is $\alpha^0 = .082$ and $\beta^0 = .396$. $\square$

Peress (2010) defines an equilibrium to be "regular" if the advantaged candidate chooses a platform in the pseudo-core. In the context of this subsection, that means candidate A chooses a platform x such that $V(0) \leq V(x) + \kappa$, which ensures that she garners at least half of the vote. Proposition 4 of that paper shows that every regular equilibrium has the form described above in part (ii) of Theorem 21: candidate B locates at $y = \hat{z} = 0$, and candidate A locates at $\overline{\alpha}\hat{x}$, where $\overline{\alpha}$ solves (32). Proposition 5 of Peress (2010) states that under quasi-concavity conditions satisfied here, there is no irregular equilibrium in which each candidate chooses a platform distinct from her ideal point. However, the preceding example demonstrates that such an equilibrium does indeed exist.[20]

[20] Peress checked for irregular equilibria in the one-dimensional model and only found examples in which both candidate ideal points were on the same side of the vote-maximizing point. This is the extreme case of $\hat{x} \cdot \hat{y} = 1$ in our example, but equilibria of the sort we illustrate will exist for an open set of parameters.

Moreover, part (iii) of Theorem 21 shows that for some parameter values, these irregular equilibria are the ones selected by the limit of equilibria in the model with the partisan shock, as uncertainty goes to zero.

6 Symmetric Model

In the symmetric version of the stochastic partisans model, with uniform bias, we assume that the politicians' and voters' utility function are quadratic, that candidates' ideal points are equidistant from the aggregate ideal point, that preference parameters are symmetric between the candidates, that neither candidate has a bias advantage, and that the distribution of the net mass of partisan voters is symmetric around zero.

Thus, in the *symmetric model* with uniform bias, we assume: (i) u_A, u_B, and V are quadratic with $\|\hat{x}\| = \|\hat{y}\| > 0 = \hat{z}$, (ii) the vectors $\hat{x} - \hat{z}$ and $\hat{y} - \hat{z}$ are not positively dependent, so that conditions (a)–(c) of Theorem 16 hold, (iii) there exists λ such that $\lambda_A = \lambda_B = \lambda$, (iv) $\kappa = 0$, and (v) for all π, $g(\pi) = g(-\pi)$. The placement of the aggregate voter at the origin is, of course, a normalization that is without loss of generality, and we let $r = \|\hat{x}\| = \|\hat{y}\|$ denote the extremity of the candidates, relative to the aggregate voter. Let $\theta = (\hat{x} \cdot \hat{y})/r^2$ denote the angle between the candidates' ideal points. By Theorem 15, in any equilibrium (x^*, y^*), there exist $\alpha, \beta \in [0,1]$ such that

$$x^* \;=\; \alpha\hat{x} \quad \text{and} \quad y^* \;=\; \beta\hat{y}.$$

Thus, we can view the strategy set of each candidate as the unit interval $[0,1]$, which corresponds to a platform between the candidate and aggregate voter. For simplicity, we maintain the assumption that $\overline{AV}$ and $\overline{BV}$ are contained in the interior of Z, so that boundary issues are moot.

6.1 Uniqueness of Symmetric Equilibrium

In this symmetric setting, we are especially interested in symmetric equilibria, which we can represent by a single scalar $\gamma \in [0,1]$ such that $\alpha = \beta = \gamma$. See Figure 11. Recall that at an equilibrium (x^*, y^*), candidate A's platform satisfies the necessary first order condition:

$$g(0)DV(x^*)[\lambda(u_A(x^*) - u_A(y^*)) + 1 - \lambda] + \frac{\lambda}{2}Du_A(x^*) \;=\; 0,$$

where we use the facts that $\kappa + V(x^*) - V(y^*) = 0$ and that $G(0) = \frac{1}{2}$. Using $DV(z) = -2z$ and $Du_A(z) = 2(\hat{x} - z)$, $x^* = \gamma\hat{x}$, and $y^* = \gamma\hat{y}$, the first order condition becomes

$$\lambda(1-\gamma)\hat{x} \;=\; g(0)2\gamma\hat{x}[\lambda(u_A(\gamma\hat{x}) - u_A(\gamma\hat{y})) + 1 - \lambda].$$

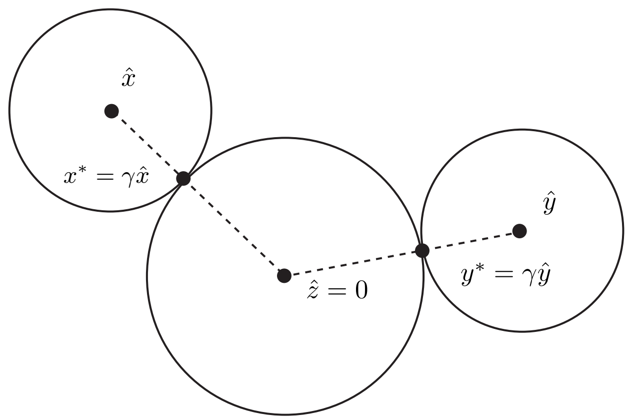

Figure 11: Symmetric equilibrium in the symmetric model

Because optimal platforms for A always lie on the contract curve $\overline{AV}$, which is the line segment from 0 to $\hat{x}$, we can restrict attention to derivatives in the direction $\frac{1}{\|\hat{x}\|}\hat{x}$. Taking the dot product with this unit vector, and dividing by $\|\hat{x}\|$, the first order condition simplifies further to

$$\lambda(1-\gamma) \quad = \quad g(0)2\gamma[\lambda(u_A(\gamma\hat{x}) - u_A(\gamma\hat{y})) + 1 - \lambda].$$

Moreover, using $\hat{x}\hat{x} = \hat{y}\hat{y}$, the utility difference is

$$u_A(\gamma\hat{x}) - u_A(\gamma\hat{y}) \quad = \quad 2\gamma(\hat{x}\hat{x} - \hat{x}\hat{y}),$$

where, as $\hat{x}$ and $\hat{y}$ are not positively dependent, we have $\hat{x}\hat{x} - \hat{x}\hat{y} > 0$. Since $r = \|\hat{x}\|$, this can be written in terms of the angle between the candidates as

$$u_A(\gamma\hat{x}) - u_A(\gamma\hat{y}) \quad = \quad 2\|\hat{x}\|^2(1-\theta)\gamma.$$

Finally, the first order condition for a symmetric equilibrium becomes

$$0 \quad = \quad 2g(0)\left[2\|\hat{x}\|^2(1-\theta)\gamma + \frac{1-\lambda}{\lambda}\right]\gamma + \gamma - 1. \tag{33}$$

The right-hand side of this equation is increasing in γ, since it is negative at $\gamma = 0$, and since it is positive at $\gamma = 1$, the intermediate value theorem implies that a solution exists; and since the right-hand side is strictly increasing in γ, it therefore has exactly one solution.

Since (33) is quadratic, we can solve for the unique symmetric equilibrium explicitly and perform several simple comparative statics. For the comparative statics analysis, note that the right-hand side in (33) is strictly increasing in $g(0)$, and it is strictly decreasing in λ and θ. It follows that when the election outcome becomes more sensitive to platforms, in the sense that $g(0)$ increases, the equilibrium coefficient γ decreases, i.e., the candidates shift toward the aggregate voter, becoming more moderate. In contrast, when the candidates become more policy motivated, or when their interests become more aligned, in the sense that θ increases, they shift toward their ideal points and become more extreme relative to the aggregate voter. While the first of these two observations are intuitive and can be found in the one-dimensional version of the model (where they are analogous to results of Bernhardt, Duggan, and Squintani (2009)), the third comparative static on alignment θ is essentially multidimensional. Our result suggests, for example, that if two parties are controlled by elites with similar preferences on one dimension, then the equilibrium platforms of the parties will be pulled in the direction of their common preference—even if this leaves some probability of winning on the table.

The next theorem summarizes the analysis of symmetric equilibria in the symmetric model.

Theorem 22. *Under Conditions 4 and 6, the symmetric stochastic partisans model with uniform bias and quadratic utility has a unique symmetric Nash equilibrium γ^*, which is the unique solution to equation (33). Moreover, $\gamma^* = 0$ if and only if the candidates are purely office motivated, i.e., $\lambda = 0$, and otherwise, the candidates choose divergent platforms. Assuming $\lambda > 0$, the symmetric equilibrium is given by*

$$\gamma^* \quad = \quad \frac{-\left(1+\frac{2g(0)(1-\lambda)}{\lambda}\right)+\sqrt{\left(1+\frac{2g(0)(1-\lambda)}{\lambda}\right)^2+16g(0)\|\hat{x}\|^2(1-\theta)}}{8g(0)\|\hat{x}\|^2(1-\theta)}.$$

Finally, the candidates become more moderate as the outcome of the election becomes more sensitive to policy; they become more extreme as they place greater weight on policy or their preferences become more aligned:

$$D_{g(0)}\gamma^* \ < \ 0, \quad D_\lambda\gamma^* \ > \ 0, \quad \textit{and} \ \ D_\theta\gamma^* \ > \ 0.$$

6.2 Comparative Statics

So far, we have restricted attention to symmetric equilibria, but if we seek to understand the equilibrium response to parameters that enter asymmetrically, then the analysis must account for equilibria in the asymmetric model, recognizing that such equilibria will likely be asymmetric. Before proceeding to the comparative statics, we address some calculus preliminaries in the general framework of Section 2. First, to represent the candidates' first order conditions, we define the function $\phi = (\phi_A, \phi_B)\colon (\mathsf{int}\, Z)^2 \to \mathbb{R}^2$ as follows: given any $x, y \in \mathsf{int}\, Z$, assuming P_A, P_B, u_A, and u_B are continuously differentiable,

$$\phi(x,y) \quad = \quad \begin{bmatrix} (D_xP_A)\Delta_A + \lambda_A P_A Du_A \\ (D_yP_B)\Delta_B + \lambda_B P_B Du_B \end{bmatrix},$$

where P_A, P_B, D_xP_A, and D_yP_B are evaluated at (x, y), Du_A is evaluated at x, Du_B is evaluated at y, and

$$\begin{aligned}\Delta_A &= \lambda_A(u_A(x) - u_A(y)) + 1 - \lambda_A \\ \Delta_B &= \lambda_B(u_B(y) - u_B(x)) + 1 - \lambda_B.\end{aligned}$$

Of course, an interior platform pair (x, y) satisfies the first order conditions for the candidates if and only if $\phi(x, y) = 0$.

The following lemma establishes that at any interior platform pair generating positive payoffs for the candidates, the necessary first order condition is, in fact, sufficient for equilibrium, and that the second order condition holds strictly at any solution to $\phi = 0$. The result assumes that the candidates place strictly positive weight on policy, and it hinges on log concavity of the probability of winning functions. The lemma does not depend on quadratic utility or symmetry, and it is stated for the general model under general differentiability conditions.

Lemma 8. *Under Conditions 1–3, in the general model, assume $\lambda_A, \lambda_B > 0$ and:*

(i) u_A and u_B are twice continuously differentiable,

(ii) the Hessian matrices $D^2u_A(z)$ and $D^2u_B(z)$ are negative definite on $\mathsf{int}\, Z$,

(iii) for all $y \in Z$, $P_A(\cdot, y)$ is twice continuously differentiable on $\{x \in \mathsf{int}\, Z : P_A(x, y) > 0\}$, and

(iv) for all $x \in Z$, $P_B(x, \cdot)$ is twice continuously differentiable on $\{y \in \mathsf{int}\, Z : P_B(x, y) > 0\}$.

Then for all $(x^*, y^*) \in (\mathsf{int}\, Z)^2$ such that $U_A(x^*, y^*) > 0$ and $U_B(x^*, y^*) > 0$, if $\phi(x^*, y^*) = 0$, then the pair (x^*, y^*) is a Nash equilibrium, and the second partial derivatives $D_x^2U_A(x^*, y^*)$ and $D_y^2U_B(x^*, y^*)$ are negative definite.

In the context of the model with quadratic utility and uniformly distributed bias, the function $\phi \colon (0, 1)^2 \to \mathbb{R}^2$ takes the simpler form

$$\phi(\alpha, \beta) \quad = \quad \left[\begin{array}{c} g(V(\alpha\hat{x}) + \kappa - V(\beta\hat{y}))[\lambda_A(u_A(\alpha\hat{x}) - u_A(\beta\hat{y})) + 1 - \lambda_A]D_\alpha V(\alpha\hat{x}) \\ +\lambda_A G(V(\alpha\hat{x}) + \kappa - V(\beta\hat{y}))D_\alpha u_A(\alpha\hat{x}) \\ \\ g(V(\alpha\hat{x}) + \kappa - V(\beta\hat{y}))[\lambda_B(u_B(\beta\hat{y}) - u_B(\alpha\hat{x})) + 1 - \lambda_B]D_\beta V(\beta\hat{y}) \\ +\lambda_B(1 - G(V(\alpha\hat{x}) + \kappa - V(\beta\hat{y})))D_\beta u_B(\beta\hat{y}) \end{array}\right],$$

where we suppress dependence on parameters of potential interest, including λ_A, λ_B, and κ. The next lemma, which is key to signing comparative statics on parameters such as λ_A and κ, provides the sign of the determinant of the system of first order conditions: beginning from a symmetric solution to the candidates' first order conditions in the symmetric model with quadratic utility and uniform bias, the determinant has positive sign.

Lemma 9. *Under Conditions 4 and 6, in the symmetric stochastic partisans model with uniform bias and quadratic utility, for all $\gamma \in [0,1]$ such that $\phi(\gamma,\gamma) = 0$, the Jacobian of the system of first order conditions has the form*

$$D\phi(\gamma,\gamma) \quad = \quad \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \tag{34}$$

where $|b| < -a$, and b has the sign

$$\mathsf{sign}\, \lambda\left(\frac{1+\theta}{2} - \gamma\right).$$

In particular, the determinant of the Jacobian is positive at every symmetric solution (γ,γ) to the first order conditions:

$$\mathsf{det}\, D\phi(\gamma,\gamma) \quad > \quad 0.$$

We can now conduct comparative statics analysis, beginning from the symmetric equilibrium of the symmetric model, that allows asymmetric perturbations of the electoral game. Let $\sigma = (\alpha,\beta)$ denote a platform pair, and let θ be any finite-dimensional parameter of interest. We then write $\phi(\sigma,\theta)$ for the parameterized first order conditions of the candidates, $D_\sigma\phi(\sigma,\theta)$ for the Jacobian of the system, i.e., the derivative with respect to the endogenous variables α and β, and $D_\theta\phi(\sigma,\theta)$ for the derivative with respect to the coordinates of θ. Given a symmetric solution $\tilde{\sigma} = (\tilde{\gamma},\tilde{\gamma})$ to the first order conditions for a given value $\tilde{\theta}$, Lemma 9 shows that $D_\sigma\phi(\tilde{\sigma},\tilde{\theta})$ has non-zero determinant and is thus non-singular. By the implicit function theorem, for an open set around $\tilde{\theta}$, there is a unique equilibrium $\sigma^*(\theta)$ near $\tilde{\sigma}$, and moreover, $\sigma^*(\cdot)$ is a C^1 function with derivative

$$D_\theta\sigma^*(\tilde{\theta}) \quad = \quad -[D_\sigma\phi(\tilde{\sigma},\tilde{\theta})]^{-1} D_\theta\phi(\tilde{\sigma},\tilde{\theta})$$

at $\tilde{\theta}$. Letting $D_\sigma\phi(\tilde{\sigma},\tilde{\theta})$ have the form in (34), it follows that

$$[D_\sigma\phi(\tilde{\sigma},\tilde{\theta})]^{-1} \quad = \quad \frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix},$$

so we can obtain the comparative static of $\tilde{\sigma}$ at $\tilde{\theta}$ by simple matrix multiplication.

For example, let $\tilde{\sigma} = (\tilde{\gamma},\tilde{\gamma})$ be the unique symmetric equilibrium in the symmetric model with policy weights equal to λ, and consider the effect of λ_A increasing above $\tilde{\lambda}_A = \lambda > 0$. Note that

$$D_{\lambda_A}\phi(\tilde{\sigma},\tilde{\lambda}_A) \quad = \quad \begin{bmatrix} g[u_A(\tilde{\gamma}\hat{x}) - u_A(\tilde{\gamma}\hat{y})) - 1]D_\gamma V(\tilde{\gamma}\hat{x}) + G D_\gamma u_A(\tilde{\gamma}\hat{x}) \\ 0 \end{bmatrix}.$$

Since $\phi_A(\tilde{\gamma},\tilde{\gamma}) = 0$, it is straightforward to see that the upper entry in the above matrix is strictly positive. Then the change in equilibrium platforms is given by

$$D_{\lambda_A}\sigma^*(\tilde{\lambda}_A) \quad = \quad -\frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}\begin{bmatrix} c \\ 0 \end{bmatrix},$$

where $|b| < -a$ and $c > 0$.

By Lemma 9 and the above discussion, we conclude that when candidate A's weight on policy increases, she becomes more extreme, as her equilibrium platform shifts in the direction of her ideal point, away from the aggregate voter:

$$D_{\lambda_A}\alpha^*(\tilde{\lambda}_A) \quad = \quad -\frac{ac}{a^2-b^2} \quad > \quad 0,$$

In contrast, the effect on candidate B's platform, namely,

$$D_{\lambda_A}\beta^*(\tilde{\lambda}_A) \quad = \quad \frac{bc}{a^2-b^2},$$

has the same sign as b, i.e.,

$$\mathsf{sign}\, D_{\lambda_A}\beta^*(\tilde{\lambda}_A) \quad = \quad \mathsf{sign}\left(\frac{1+\theta}{2} - \gamma\right).$$

For example, when $\theta = -1$, so the candidates are diametrically opposed, the sign of the comparative static is negative, so that candidate B moderates by shifting toward the aggregate voter. When the candidates are more aligned, this comparative static can be reversed: if $\theta \in (-1, 1)$, then for $\lambda > 0$ close enough to zero, γ will be small, and $D_{\lambda_A}\beta^*$ will be positive. At work is the fact that when the candidates are highly opposed, A's shift toward her ideal point increases the threat to B, incentivizing the latter candidate to moderate. When the candidates are more aligned and the symmetric equilibrium is not too extreme (so γ is not too close to one), A's shift can be beneficial to B, decreasing the threat and allowing her to also become more extreme.

Theorem 23. *Under Conditions 4 and 6, in the symmetric stochastic partisans model with uniform bias and quadratic utility, assume $\lambda > 0$. At the symmetric Nash equilibrium γ^*, the equilibrium platform of candidate A becomes more extreme as she places greater weight on policy; candidate B's equilibrium platform becomes more extreme if the degree of equilibrium moderation is not too great relative to the degree of alignment between the candidates. Formally,*

$$D_{\lambda_A}\alpha^*(\tilde{\lambda}_A) \;>\; 0 \quad \text{and} \quad \mathsf{sign}\, D_{\lambda_A}\beta^*(\tilde{\lambda}_A) \;=\; \mathsf{sign}\left(\frac{1+\theta}{2} - \gamma^*\right),$$

where $\tilde{\lambda}_A = \lambda$.

To examine the effect of increasing candidate A's electoral advantage to small $\kappa > 0$, note that if $\tilde{\sigma}$ is the symmetric equilibrium at $\kappa = 0$, then

$$D_\kappa \phi(\tilde{\sigma}, 0) \quad = \quad \begin{bmatrix} \lambda g(0) D_\gamma u_A(\gamma\hat{x}) \\ -\lambda g(0) D_\gamma u_B(\gamma\hat{x}) \end{bmatrix},$$

where we use $g'(0) = 0$. Assuming $\lambda > 0$, the change in equilibrium platforms is then given by

$$D_\kappa \sigma^*(0) \quad = \quad -\frac{1}{a^2-b^2}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix}\begin{bmatrix} d \\ -d \end{bmatrix},$$

where $|b| < -a$ and $d > 0$. We conclude that when A's advantage increases, the candidate leverages her advantage to pull policy in the direction of her ideal point:

$$D_\kappa \alpha^*(0) \quad = \quad -\frac{ad + bd}{a^2 - b^2} \quad > \quad 0.$$

In contrast, B is forced to moderate her platform by moving toward the aggregate voter :

$$D_\kappa \beta^*(0) \quad = \quad \frac{bd + ad}{a^2 - b^2} \quad < \quad 0.$$

This intuitive comparative static runs counter to a result found by Groseclose (2001) in a different model; whereas Groseclose assumes a single policy dimension and uncertainty about the location of the median voter, we allow multiple dimensions and assume that voter policy preferences are known to the candidates.[21]

Theorem 24. *Under Conditions 4 and 6, in the symmetric stochastic partisans model with uniform bias and quadratic utility, assume $\lambda > 0$. At the symmetric Nash equilibrium, as the electoral advantage of candidate A increases, her equilibrium platform becomes more extreme, and the platform of candidate B becomes more moderate:*

$$D_\kappa \alpha^*(0) \ > \ 0 \quad \textit{and} \quad D_\kappa \beta^*(0) \ < \ 0.$$

6.3 Application: Cultural and Economic Polices

Following the contributions of Romer (1975), Roberts (1977), and Meltzer and Richard (1981), which treat economic policy as the main (and in their models, the only) dimension of political conflict, there has been growing recognition that redistributive policies may also be affected by competition along a second, cultural and social dimension. Roemer (1998) and Roemer, Lee, and Van Der Straeten (2007) consider models in which, in addition to tax policy, parties compete on issues like religion, affirmative action, and immigration policy, employing the party unanimity equilibrium concept to establish the existence of an equilibrium. Other papers, such as Krasa and Polborn (2012) and Besley and Persson (2023), allow for a cultural dimension, but they assume that competition is on only one of the two dimensions. Enke, Polborn, and Wu (2025) allow for a second dimension that they refer to as moral values, but treat the parties' positions as exogenous; in an extension where they endogenize these positions, they assume that the parties maximize a weighted average of the utilities of voters who support them. Buisseret and van Weelden (2025) allow for party

[21] The intuitive comparative static result is also found by Duggan (2025a) in the stochastic valence model; see the latter paper for a discussion about the difference in candidate incentives that leads to the contrary findings.

competition on both taxes and culture, but cultural policy is binary and they establish existence of pure strategy equilibrium under the assumption that the policy motivation of the parties is endogenously determined by the sets of voters who support them.

In this section, we show how our results for the stochastic partisans model can be applied to study party competition in two dimensions of policy, an economic dimension and a cultural dimension, which stands in for issues such as LGBTQ rights, abortion, immigration, and gun control. We assume the policy space is $Z = [0, 1]^2$, and a generic policy is denoted $z = (z_e, z_c)$, with z_e being economic policy and z_c being cultural policy. We do not specify a model of tax and redistribution for these preferences in this application, but our previous application on progressive taxation and the voluminous political economy literature, beginning with seminal contributions cited above, suggest foundations for how such preferences could be induced.

We assume that a fraction $\nu < 1/2$ of voters are left-leaning on both economic and cultural policy, and thus have ideal point $(0, 0)$, while another fraction ν are right-leaning on both issues and have ideal point $(1, 1)$. A fraction $(1-2\nu)\mu$ are culturally progressive but economically conservative, with ideal point $(1, 0)$, while the remaining fraction $(1-2\nu)(1-\mu)$ are culturally conservative but economically left-leaning, with ideal point $(0, 1)$. We maintain the assumption that $\mu < 1/2$, which means that more voters are economically left leaning than are right leaning, and that more are culturally right leaning than are left leaning.[22] We identify each of four voter types with one of these ideal points, where voter preferences are quadratic. For voter type t with ideal point $\hat{z}^t = (\tilde{z}_e^t, \tilde{z}_c^t)$, the utility from policy (z_e, z_c) is then

$$u_{\tilde{z}_e^t, \tilde{z}_c^t}(z_e, z_c) = -\frac{1}{2}(z_e - \tilde{z}_e^t)^2 - \frac{1}{2}(z_c - \tilde{z}_c^t)^2. \tag{35}$$

The two parties A and B share the policy preferences of the type $(0, 0)$ and $(1, 1)$ voters, respectively, so that $u_A = u_{0,0}$ and $u_B = u_{1,1}$, and the parties' ideal points are $\hat{x} = (0, 0)$ and $\hat{y} = (1, 1)$ respectively. Thus, A is the party of the left, representing voters who are left-leaning on both policy dimensions, and B is the party of the right, representing those who are consistently right-leaning. See Figure 12, where the corners of the policy space are the ideal points of each voter type, with A and B located on the 45 degree line.

In addition to these policy oriented voters, there is a mass of partisan voters distributed by a function G satisfying Condition 4. We also assume that each type of t voter has a net bias for party B that satisfies Condition 6, so that the aggregate voter result of Theorem 14 applies, and the parties' probabilities of winning satisfy (16) and (17). Finally, we assume that $\rho_t = 1/2$ for all types t,

[22] These assumptions are consistent with survey results for 982 voters reported by the 2025 June Omnibus Political Quadrant Analysis: `https://public.tableau.com/shared/KFCNKGP28?:display_count=n&:origin=viz_share_link`

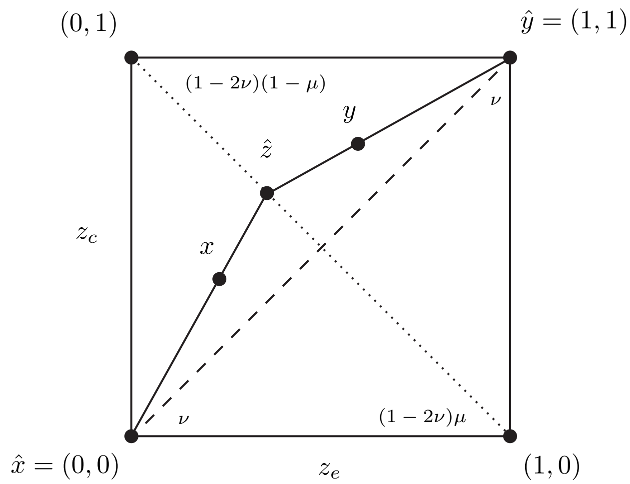

Figure 12: Cultural and economic policies

to ensure that the support of the bias term is large enough to satisfy (9).[23] In the present context, aggregate utility is then

$$V(z) \quad = \quad \nu u_{0,0}(z) + (1-2\nu)(1-\mu)u_{0,1}(z) + (1-2\nu)\mu u_{1,0}(z) + \nu u_{1,1}(z),$$

and thus the aggregate ideal point is

$$\hat{z} \quad = \quad (\hat{z}_e, \hat{z}_c) \quad = \quad \big(\nu + (1-2\nu)\mu, \nu + (1-2\nu)(1-\mu)\big).$$

Here, the first coordinate is simply the share of voters who are conservative on the economic dimension, and the second coordinate is the share who are liberal on economic policy. Equivalently, by the symmetry built into the model, the first and second coordinates of the aggregate ideal point are the shares of voters who are liberal and conservative, respectively, on cultural policy. By our assumption that $\mu < 1/2$, the aggregate ideal point $\hat{z}$ lies due northwest of the "neutral" policy $(1/2, 1/2)$ on the dotted line depicted in Figure 12. Note that the angle θ formed by the party ideal points (relative to the aggregate ideal point) is necessarily obtuse, because $\overline{AV}$, the line segment between $\hat{x}$ and $\hat{z}$, is steeper than 45 degrees, and $\overline{BV}$, the line between $\hat{y}$ and $\hat{z}$, is flatter.

Because Conditions 7 and 8 also hold, Theorem 15 applies, so party A chooses a platform on the contract curve $\overline{AV}$ in Figure 12, while party B chooses a platform on the contract curve $\overline{BV}$. Note that the lengths of these line segments are the same by the fact that $\hat{z}_e + \hat{z}_c = 1$. Formally, in equilibrium, party A

[23]Technically, the results of Subsections 6.1 and 6.2 implicitly assume that the coefficient matrix M for the aggregate voter utility function is the identity matrix. Here, it is $M = \frac{1}{4}I$, which has the same effect as scaling the value $g(0)$ by $1/4$ in the earlier subsections. Thus, our comparative statics results for the symmetric model carry over unchanged.

chooses a policy $\alpha\hat{x} + (1-\alpha)\hat{z}$ for some $\alpha \in [0,1]$, and party B chooses a policy $\beta\hat{y}+(1-\beta)\hat{z}$ for some $\beta \in [0,1]$. We furthermore assume that the parties have the same level of policy motivation, i.e., $\lambda_A = \lambda_B = \lambda$ for some $\lambda \in [0,1]$, and that neither party has an electoral advantage, i.e., $\kappa = 0$. Therefore, the assumptions of Section 6 are satisfied by the symmetric model of cultural and economic policy—up to a translation of the ideal points—and so the main conclusions of Theorems 22–24 apply.

In particular, Theorem 22 implies that there is a unique symmetric equilibrium, with $\alpha^* = \beta^* = \gamma^*$, where γ^* is determined by the parameters of the model as follows:

$$\gamma^* = \frac{-\left(1 + \frac{g(0)(1-\lambda)}{2\lambda}\right) + \sqrt{\left(1 + \frac{g(0)(1-\lambda)}{2\lambda}\right)^2 + 4g(0)}}{2g(0)}. \tag{36}$$

The expression above is actually simpler than the formula for γ^* in Theorem 22, as the terms $\|\hat{x}\|^2(1-\theta)$ are not present. Letting r denote the magnitude $\|\hat{z}\|$, note that if we translate ideal points so that the aggregate voter is located at the origin, then r is just the norm of party A's ideal point. Using the fact that $\hat{z}$ lies on the dotted line in Figure 12, it can be shown that $r^2(1-\gamma) = 1$, which gives us the simpler expression in (36).

Substantively, the solution for γ^* in (36) means that, unsurprisingly, if $\lambda > 0$, then in equilibrium party B will be to the right of the aggregate voter on both economic policy and culture, while party A will be to the left of the aggregate voter on both dimension. More interestingly, however, because $\overline{AV}$ is steeper, and $\overline{BV}$ is flatter, than the 45 degree line, the right-leaning party B is further to the right of the aggregate voter on economic policy than it is on culture, and the left-leaning party A is more out of step with the voter on culture than on economics. Comparing across parties, A is more misaligned (relative to the aggregate voter) than B on culture, and the situation is reversed on economic policy, where B is more misaligned that A.

If μ decreases, so that the economically left-wing, and socially right-wing, position $(0,1)$ becomes more popular relative to the position $(1,0)$, then this causes the aggregate ideal point to shift upward along the dotted line, further to the northwest in Figure 12. This leaves γ^* unaffected, as discussed above, but it causes $\overline{AV}$ to become steeper, and $\overline{BV}$ flatter. In turn, this exaggerates the imbalances across dimensions within each party: party A becomes more misaligned on culture and less misaligned on economic policy, while party B becomes more misaligned on economic policy and less on culture. For example, party A's degree of *misalignment* on the culture dimension relative to the aggregate ideal point is

$$\nu + (1-2\nu)(1-\mu) - \gamma(\nu + (1-2\nu)(1-\mu)),$$

and the derivative of this with respect to μ is $-(1-\gamma)(1-2\nu) < 0$. That is, as μ decreases, cultural misalignment increases at the rate $(1-\gamma)(1-2\nu)$, as does the rate of economic misalignment for party B. Similarly, party A's degree of misalignment on the economic dimension is

$$\nu + (1 - 2\nu)\mu - \gamma(\nu + (1 - 2\nu)\mu),$$

and this decreases at the rate $(1 - \gamma)(1 - 2\nu)$ as μ decreases, as does the rate of B's misalignment on the cultural dimension.

Proposition 6. *Under Conditions 4 and 6, in the symmetric model of cultural and economic policy with uniform bias and quadratic utility, assume $\lambda > 0$. There is a unique symmetric Nash equilibrium γ^*, and assuming $\lambda > 0$, this is given by (36). Then:*

(i) party A's misalignment relative to the aggregate voter is greater on the culture dimension than on the economic dimension, i.e.,

$$(1 - \gamma)(\nu + (1 - 2\nu)(1 - \mu)) \quad > \quad (1 - \gamma)(\nu + (1 - 2\nu)\mu),$$

(ii) party B's misalignment relative to the aggregate voter is greater on the economic dimension than on the culture dimension,

(iii) as the mass of culturally conservative and economically liberal voters increases, i.e., μ decreases, A's misalignment on the culture dimension increases, and its misalignment on the economic dimension decreases,

(iv) and B's misalignment on the culture dimension decreases, and its misalignment on the economic dimension increases.

As the parties become more policy motivated, i.e., λ increases, or as voting becomes less sensitive to platforms, i.e., $g(0)$ decreases, Theorem 22 tells us that the symmetric equilibrium platforms shift toward the parties' ideal points, so parties become more polarized. Although the parties diverge from the aggregate voter by the same amount, λ and $g(0)$ have differential effects across dimensions: because the slope of the line segment $\overline{AV}$ is greater than one, party A's shift away from the aggregate ideal point is greater on the culture dimension than on the economic dimension, and analogously, party B's moves away more on the economic dimension than on the culture dimension. If μ decreases, so that the segment $\overline{AV}$ becomes steeper, then these imbalances are magnified, and in the limit, when $\mu = 0$, the aggregate ideal point is at $(0, 1)$, party A is aligned with the voter on economic policy, while B is aligned on cultural policy; in this case, A's shift is entirely on the culture dimension, while B's is entirely on the economic dimension.

Proposition 7. *Under Conditions 4 and 6, in the symmetric model of cultural and economic policy with uniform bias and quadratic utility, assume $\lambda > 0$. As $g(0)$ decreases or λ increases:*

(i) party A's misalignment relative to the aggregate voter increases on both dimensions, and the rate of increase is greater on the culture dimension than on the economic dimension, i.e.,

$$D_\lambda x_c^* \; < \; D_\lambda x_e^* \; < \; 0 \quad \text{and} \quad D_{g(0)} x_c^* \; > \; D_{g(0)} x_e^* \; > \; 0,$$

where $x^* = \gamma^* \hat{x} + (1 - \gamma^*)\hat{z}$ is A's symmetric equilibrium platform,

(ii) party B's misalignment relative to the aggregate voter increases on both dimensions, and the rate of increase is greater on the economic dimension than on the culture dimension,

(iii) the increase in misalignment of party A on the culture dimension is greater than the increase in party B's misalignment on the culture dimension, i.e.,

$$-D_\lambda x_c^* \; > \; D_\lambda y_c^* \quad \text{and} \quad D_{g(0)} x_c^* \; > \; -D_{g(0)} y_c^*,$$

where $y^* = \gamma^* \hat{y} + (1 - \gamma^*)\hat{z}$ is B's symmetric equilibrium platform,

(iv) the increase in A's misalignment on the economic dimension is less than the increase in misalignment of B on the economic dimension.

We can also apply Theorems 23 and 24 to examine the effects of asymmetric changes in parameters, such as an increase in party A 's electoral advantage or its weight on policy (holding B's fixed). Theorem 24 implies that in the first case, as κ becomes slightly positive, party A's platform shifts down the segment $\overline{AV}$ toward A's ideal point and away from the aggregate voter, while party B's shifts down $\overline{BV}$ away from its ideal point and toward the aggregate voter. Theorem 23 tells us that if A's policy motivation λ_A increases slightly, then the party again shifts toward its ideal point, whereas B's response is ambiguous. We do not re-state those results here, but we end by noting that if μ is close to one half, then the aggregate ideal point approaches $(1/2, 1/2)$, and the angle becomes close to 180 degrees, i.e., θ is close to -1. Since $\gamma^* > 0$ is unaffected by μ, Theorem 23 implies that for μ close enough to one half, we have $D_{\lambda_A}\beta^* < 0$. In this case, as A becomes more policy motivated, B responds by unambiguously shifting toward the aggregate voter.

7 Conclusion

The previous literature has not offered a framework for modeling multidimensional candidate competition that generates competitive and meaningful elections. We provide such a framework: equilibria exist generally in the stochastic partisans model in any number of dimensions, while candidates have mixed motivations that typically lead to equilibria in which they adopt distinct platforms and each win with positive probability. At the same time, uncertainty may be arbitrarily small, and we obtain the social optimality result from the literature with vote-maximizing candidates as the limiting case when uncertainty is removed from our model. We give conditions under which election outcomes are as if there is a single, aggregate voter whose choices are subject to noise; we add the assumption of quadratic utility to obtain a dimensional reduction result; and we then add symmetry to explicitly solve for equilibrium and perform comparative statics. Simple applications to distributive politics, income taxation, and cultural and economic policy confirm the tractability of the framework, and they demonstrate the promise for future applications to understand how democratic elections aggregate voter preferences to address inherently multidimensional problems.

A Proofs Omitted from Text

A.1 Proof of Lemma 1

Fix $y \in Z$. First, assume x^* solves (6). Since the exponential function is a monotonic transformation, it then solves

$$\max_{x \in Z_A(y)} U_A(x,y),$$

and thus it solves (7). Conversely, assume Condition 1 holds. In case $\lambda_A < 1$, candidate A can set $x = y$, and by Condition 1, we have $P_A(y,y) > 0$, and thus $U_A(y,y) = P_A(y,y)(1-\lambda_A) > 0$. This implies that $x \in Z_A(y)$. Consider the remaining case $\lambda_A = 1$, Condition 2 holds, and $y \neq \hat{x}$. Since $P_A(y,y) > 0$ and P_A is continuous, we can choose $\alpha \in (0,1)$ small enough that

$$P_A(\alpha\hat{x} + (1-\alpha)y) \;>\; 0 \quad \text{and} \quad u_A(\alpha\hat{x} + (1-\alpha)y) \;>\; u_A(y),$$

which imply $\alpha\hat{x} + (1-\alpha)y \in Z_A(y)$. In either case, we conclude that $Z_A(y) \neq \emptyset$, and thus all best response platforms for A must belong to $Z_A(y)$. In particular, we can restrict A's maximization problem to $Z_A(y)$, and as the log function is a monotonic transformation, we conclude that if x^* solves (7), then it solves (6).

A.2 Proof of Theorem 1

The electoral game features compact, convex strategy sets, and candidate payoffs are continuous. It follows immediately that the candidates' best response correspondences have nonempty values and closed graph. First, assume $\lambda_A, \lambda_B < 1$. To invoke Kakutani's theorem, we need only show that the best response correspondences of the candidates are convex valued. Focusing on candidate A, we will argue that for all $y \in Z$,

$$BR_A(y) \;\equiv\; \arg\max_{x \in Z} U_A(x,y)$$

is convex. By Lemma 1, candidate A's best response platforms are just the solutions to (6), and by Condition 3, the objective function in (6) is concave in x. Thus, $BR_A(y)$ is convex. Let $BR_B(x)$ denote the set of best response platforms for B. By Kakutani's theorem, the correspondence $\psi\colon Z \times Z \rightrightarrows Z \times Z$ defined by $\psi(x, y) = BR_A(y) \times BR_B(x)$ has a fixed point, (x^*, y^*), which is a Nash equilibrium. Finally, given any $\lambda_A, \lambda_B \in [0, 1]$, we can let $\{(\lambda_A^m, \lambda_B^m)\}$ be any sequence satisfying $\lambda_A^m, \lambda_B^m < 1$ for all m and $(\lambda_A^m, \lambda_B^m) \to (\lambda_A, \lambda_B)$. For each m, we have shown there exists an equilibrium (x^m, y^m). Since $Z \times Z$ is compact, the sequence $\{(x^m, y^m)\}$ has a convergent subsequence, and by closed graph of the equilibrium correspondence, the limit of this subsequence is an equilibrium of the model with parameters λ_A and λ_B.

A.3 Proof of Theorem 2

Since the game with office-motivated candidates is constant sum, the claimed inequalities follow by definition of equilibrium. Next, suppose toward a contradiction that (x^*, y^*) and $(\tilde{x}, \tilde{y})$ are distinct equilibria, and assume without loss of generality that $x^* \neq \tilde{x}$. Equilibria of constant-sum games are interchangeable, so $(\tilde{x}, y^*)$ is also an equilibrium. By Condition 1, candidate A's equilibrium payoff is positive, and thus $x^*, \tilde{x} \in S_A(y^*)$. Letting $x' = \frac{1}{2}x^* + \frac{1}{2}y^*$, strict quasi-concavity yields

$$P_A(x', y^*) \;>\; \min\{P_A(x^*, y^*), P_A(\tilde{x}, y^*)\},$$

contradicting the fact that both platform pairs are equilibria. Thus, there is a unique equilibrium, say, (x^*, y^*). Finally, assume $P_A(z, z)$ is constant in z, and suppose toward a contradiction that $x^* \neq y^*$. Then

$$P_A(x^*, x^*) \;\geq\; P_A(x^*, y^*) \;\geq\; P_A(y^*, y^*) \;=\; P_A(x^*, x^*),$$

where the first inequality follows since y^* is a best response to x^* for B, and the second follows since x^* is a best response to y^* for A. We conclude that $P_A(x^*, y^*) = P_A(y^*, y^*)$. Again letting $x' = \frac{1}{2}x^* + \frac{1}{2}y^*$, strict quasi-concavity gives us

$$P_A(x', y^*) \;>\; \min\{P_A(x^*, y^*), P_A(y^*, y^*)\} \;=\; P_A(x^*, y^*),$$

a contradiction, from which we conclude that $x^* = y^*$.

A.4 Proof of Theorem 3

Under Condition 1, Lemma 1 directly implies that if a candidate places positive weight on office, then her equilibrium payoff is always positive. Now, let $\lambda_A = 1$, and consider any equilibrium (x^*, y^*). Assume that $U_A(x^*, y^*) = 0$. We claim that $y^* = \hat{x}$. Indeed, otherwise, since $P_A(y^*, y^*) > 0$ and P_A is continuous, we can choose $\alpha \in (0,1)$ such that $\alpha\hat{x} + (1-\alpha)y^*$ gives candidate A a positive payoff. This contradicts the fact that x^* is a best response for A to y^*, and it follows that $y^* = \hat{x}$, as claimed. We conclude that if $U_A(x^*, y^*) = 0$, then (x^*, y^*) is non-standard, and by an analogous argument, if $U_B(x^*, y^*) = 0$, then the equilibrium is non-standard. Equivalently, if (x^*, y^*) is standard, then $U_A(x^*, y^*) > 0$ and $U_B(x^*, y^*) > 0$.

A.5 Proof of Theorem 4

Let $\{(\theta^m, x^m, y^m)\}$ be a sequence of models and corresponding equilibria that satisfies the closed graph criterion at (θ, x, y). Suppose toward a contradiction that (x, y) is not an equilibrium in model θ. To simplify notation, define

$$\begin{aligned} \Delta_x &= \lambda_A^\theta(u_A^\theta(x) - u_A^\theta(y)) + 1 - \lambda_A^\theta \\ \Delta_x^m &= \lambda_A^{\theta^m}(u_A^{\theta^m}(x^m) - u_A^{\theta^m}(y^m)) + 1 - \lambda_A^{\theta^m}. \end{aligned}$$

By condition (iii), for all $\epsilon > 0$, some candidate has an ϵ-robust better reply. Let E_A denote the set of $\epsilon > 0$ such that A has an ϵ-robust better reply, and let E_B be the analogous set for B. Without loss of generality, assume E_A contains arbitrarily small $\epsilon > 0$, i.e., 0 is an accumulation point of E_A. Given any $\epsilon \in E_A$, let $z \in Z$ be an ϵ-robust better reply, which implies $U_A^\theta(z, y) > U_A^\theta(x, y)$, or equivalently,

$$\begin{aligned} & P_A^\theta(z, y)[\lambda_A^\theta(u_A^\theta(z) - u_A^\theta(y)) + 1 - \lambda_A^\theta] \\ & \quad > \ P_A^\theta(x, y)[\lambda_A^\theta(u_A^\theta(x) - u_A^\theta(y)) + 1 - \lambda_A^\theta]. \end{aligned}$$

Defining

$$\Delta_z = \lambda_A^\theta(u_A^\theta(z) - u_A^\theta(y)) + 1 - \lambda_A^\theta,$$

this becomes

$$\frac{\Delta_z}{\Delta_x} - \frac{P_A^\theta(x, y)}{P_A^\theta(z, y)} > 0.$$

For each integer k, choose $\epsilon_k \in E_A$ such that

$$\epsilon_k < \min\left\{\frac{1}{k}, \frac{\Delta_z}{\Delta_x} - \frac{P_A^\theta(x, y)}{P_A^\theta(z, y)}\right\}.$$

By the definition of ϵ-robust better reply, there exists $\overline{m}_k$ such that for all $m \geq \overline{m}_k$, inequality (8) holds. For each k, choose m_k high enough that $m_k \geq \overline{m}_k$ and $m_k > m_{k-1}$, making $\{x^{m_k}\}$ a subsequence of $\{x^m\}$, and $\{y^{m_k}\}$ a subsequence of $\{y^m\}$.

For each k, we then have

$$\frac{P_A^{\theta^{m_k}}(x^{m_k}, y^{m_k})}{P_A^{\theta^{m_k}}(z, y^{m_k})} < \frac{P_A^\theta(x, y)}{P_A^\theta(z, y)} + \epsilon_k. \tag{37}$$

Define

$$\Delta_z^k = \lambda_A^{\theta^{m_k}}(u_A^{\theta^{m_k}}(z) - u_A^{\theta^{m_k}}(y^{m_k})) + 1 - \lambda_A^{\theta^{m_k}},$$

and note that $\Delta_z^k \to \Delta_z$ as $k \to \infty$. Then

$$\frac{\Delta_z^k}{\Delta_x^{m_k}} \to \frac{\Delta_z}{\Delta_x}$$

as $k \to \infty$, and we can choose k high enough that

$$\max\left\{\epsilon_k, \frac{\Delta_z}{\Delta_x} - \frac{\Delta_z^k}{\Delta_x^{m_k}}\right\} < \frac{1}{2}\left(\frac{\Delta_z}{\Delta_x} - \frac{P_A^\theta(x,y)}{P_A^\theta(z,y)}\right).$$

In particular, for such k, we have

$$\epsilon_k + \frac{\Delta_z}{\Delta_x} - \frac{\Delta_z^k}{\Delta_x^{m_k}} < \frac{\Delta_z}{\Delta_x} - \frac{P_A^\theta(x,y)}{P_A^\theta(z,y)},$$

which implies

$$\epsilon_k < \frac{\Delta_z^k}{\Delta_x^{m_k}} - \frac{P_A^\theta(x,y)}{P_A^\theta(z,y)}.$$

From (37), it follows that for such k,

$$\begin{aligned}\frac{P_A^{\theta^{m_k}}(x^{m_k}, y^{m_k})}{P_A^{\theta^{m_k}}(z^k, y^{m_k})} &< \frac{P_A^\theta(x,y)}{P_A^\theta(z,y)} + \frac{\Delta_z^k}{\Delta_x^{m_k}} - \frac{P_A^\theta(x,y)}{P_A^\theta(z,y)} \\ &= \frac{\Delta_z^k}{\Delta_x^{m_k}}.\end{aligned}$$

Then, however, we have

$$\frac{P_A^{\theta^{m_k}}(x^{m_k}, y^{m_k})}{P_A^{\theta^{m_k}}(z, y^{m_k})} < \frac{\Delta_z^k}{\Delta_x^{m_k}},$$

or equivalently,

$$P_A^{\theta^{m_k}}(z, y^{m_k})\Delta_z^k > P_A^{\theta^{m_k}}(x^{m_k}, y^{m_k})\Delta_x^{m_k}.$$

This implies that $U_A^{\theta^{m_k}}(z, y^{m_k}) > U_A^{\theta^{m_k}}(x^{m_k}, y^{m_k})$ for high enough k, contradicting the fact that (x^{m_k}, y^{m_k}) is an equilibrium in θ^{m_k}.

A.6 Proof of Theorem 5

Assume that the sequence $\{\theta^m\}$ of models satisfies (a) and (b), and that (x^m, y^m) is an equilibrium of each θ^m, with $(x^m, y^m) \to (x, y)$. We show that a subsequence of $\{(\theta^m, x^m, y^m)\}$ satisfies the closed graph criterion at (θ, x, y). Since conditions (i) and (ii) are satisfied by construction, we focus on condition (iii). To this end, assume that (x, y) is not an equilibrium at θ, and let $\epsilon > 0$. We show that some candidate has an ϵ-robust better reply.

We claim that either

$$\liminf_{m\to\infty} P_A^{\theta^m}(x^m, y^m) \quad \leq \quad P_A^{\theta}(x, y)$$

or

$$\liminf_{m\to\infty} P_B^{\theta^m}(x^m, y^m) \quad \leq \quad P_B^{\theta}(x, y).$$

Indeed, if the first inequality does not hold, then

$$\liminf_{m\to\infty} P_A^{\theta^m}(x^m, y^m) \quad > \quad P_A^{\theta}(x, y),$$

which implies

$$\limsup_{m\to\infty} (1 - P_A^{\theta^m}(x^m, y^m)) \quad < \quad 1 - P_A^{\theta}(x, y),$$

which implies the second inequality. Therefore, we can assume without loss of generality that the first inequality holds. We then move to a convergent subsequence of $\{\theta^m\}$ (still indexed by m, for simplicity) such that

$$\lim_{m\to\infty} P_A^{\theta^m}(x^m, y^m) \quad \leq \quad P_A^{\theta}(x, y). \tag{38}$$

Note that for all m, candidate A has the option of choosing platform y^m, so since (x^m, y^m) is an equilibrium, we have

$$U_A^{\theta^m}(x^m, y^m) \ \geq \ U_A^{\theta^m}(y^m, y^m) \ \geq \ \underline{\tau}(1 - \lambda_A),$$

where the inequality follows from (5). Borrowing notation from the proof of Theorem 4, this implies

$$P_A^{\theta^m}(x^m, y^m) \quad \geq \quad \frac{\underline{\tau}(1 - \lambda_A)}{\Delta_x^m}.$$

Letting $\overline{\Delta}_A$ be an upper bound for Δ_x^m, it follows that

$$P_A^{\theta^m}(x^m, y^m) \ \geq \ \frac{\underline{\tau}(1 - \lambda_A)}{\overline{\Delta}_A^m} \ > \ 0,$$

so that $P_A^{\theta^m}(x^m, y^m)$ has a positive lower bound. By inequality (38), we conclude that $P_A^{\theta}(x, y) > 0$, which implies $x \succeq y$.

We claim that $y \succeq x$. Indeed, suppose toward a contradiction that $x \succ y$. By condition (a), there exist open sets G containing x and H containing y such that

$P_A^{\theta^m}(\cdot) \to 1$ uniformly on $G \times H$. In particular, observe that $P_A^{\theta^m}(x^m, y^m) \to 1$, i.e., $P_B^{\theta^m}(x^m, y^m) \to 0$. However, an argument analogous to the above

$$P_B^{\theta^m}(x^m, y^m) \;\geq\; \frac{(1-\overline{\tau})(1-\lambda_B)}{\overline{\Delta}_B} \;>\; 0,$$

where $\overline{\Delta}_B$ is an upper bound for $\lambda_B(u_B(y) - u_B(x)) + 1 - \lambda_B$. This contradicts the observation that B's probability of winning converges to zero. We conclude that $y \succ x$, as claimed, which means that in the limiting model, the platforms x and y create a majority tie, and thus $P_A^{\theta}(x, y) < 1$.

We break the remainder of the proof into two cases. The first case is $x \neq y$. For $\alpha \in (0,1)$, we can define $z_\alpha = \alpha x + (1-\alpha)y$, and since $x \succeq y$, Lemma 2 implies $z_\alpha \succ y$. This gives us $P_A^{\theta}(z_\alpha, y) = 1 > 0$. Moreover, by condition (a), we have $P_A^{\theta^m}(z_\alpha, y^m) \to 1$. We claim that for $\alpha \in (0,1)$ close enough to one, z_α is a profitable deviation for A at (x, y) in θ, or equivalently,

$$\frac{P_A^{\theta}(x,y)}{P_A^{\theta}(z_\alpha, y)} \;<\; \frac{\Delta_{z_\alpha}}{\Delta_x}.$$

Indeed, note that the left-hand side above is equal to $P_A^{\theta}(x, y) < 1$, while the right-hand side converges to one as $\alpha \uparrow 1$, as claimed. Thus, we can choose $\alpha \in (0,1)$ close enough to one that $U_A^{\theta}(z_\alpha, y) > U_A^{\theta}(x, y)$. Now, inequality (8) becomes

$$\frac{P_A^{\theta^m}(x^m, y^m)}{P_A^{\theta^m}(z, y^m)} \;<\; P_A^{\theta}(x,y) + \epsilon.$$

Taking the limit as $m \to \infty$, and using (38) and $P_A^{\theta^m}(z_\alpha, y^m) \to 1$, we see that the above inequality holds for high enough m, and we conclude that z_α is an ϵ-robust better reply.

The second case is $x = y$. Since (x, y) is not an equilibrium, there is a policy $z \neq y$ such that $z \succeq y$. For $\alpha \in (0,1)$, define z_α as above, so that $z_\alpha \succ y$ and $P_A^{\theta}(z_\alpha, y) = 1 > 0$. Again, we have $U_A^{\theta}(z_\alpha, y) > U_A^{\theta}(x, y)$ for α close enough to one, and inequality (8) holds for m sufficiently high. We conclude that z_α is an ϵ-robust better reply.

A.7 Proof of Theorem 7

Let (x^*, y^*) be a standard equilibrium, and consider candidate A. Suppose toward a contradiction that x^* is not Pareto optimal for $T \cup \{A\}$, and let x' be a platform that makes A and all policy-oriented voters weakly better off, and either gives A strictly higher utility than does x, or it gives a measurable set

$S \subseteq T$ of voters a strictly higher utility than x. By (10), it follows that

$$\begin{aligned} P_A(x', y^*) &= G\Big(2\Big(\int F_t(u_t(x') - u_t(y^*))d\tau\Big) - 1\Big) \\ &\geq G\Big(2\Big(\int F_t(u_t(x^*) - u_t(y^*))d\tau\Big) - 1\Big) \\ &= P_A(x^*, y^*). \end{aligned}$$

By Theorem 3, we have $U_A(x^*, y^*) > 0$, and therefore

$$P_A(x', y^*) \ \geq \ P_A(x^*, y^*) \ > \ 0,$$

and

$$\lambda_A(u_A(x') - u_A(y^*)) + 1 - \lambda_A \ \geq \ \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A \ > \ 0.$$

Now, in case $u_A(x') > u_A(x^*)$, we have

$$\begin{aligned} U_A(x', y^*) &= P_A(x', y^*)[\lambda_A(u_A(x') - u_A(y^*)) + 1 - \lambda_A] \\ &\geq P_A(x^*, y^*)[\lambda_A(u_A(x') - u_A(y^*)) + 1 - \lambda_A] \\ &> P_A(x^*, y^*)[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \\ &= U_A(x^*, y^*), \end{aligned}$$

which contradicts the fact that x^* is a best response to y^*. In the remaining case that $u_t(x') > u_t(x^*)$ for all $t \in S$ with $\tau(S) > 0$, we observe that since $0 < P_A(x^*, y^*) < 1$, we have

$$0 \ < \ G\Big(2\Big(\int F_t(u_t(x^*) - u_t(y^*))d\tau\Big) - 1\Big) \ < \ 1,$$

so $2(\int F_t(u_t(x^*) - u_t(y^*))d\tau - 1$ belongs to the support set S_G of G. Since

$$2\Big(\int F_t(u_t(x') - u_t(y^*))d\tau\Big) - 1 \quad > \quad 2\Big(\int F_t(u_t(x^*) - u_t(y^*))d\tau\Big) - 1,$$

it follows that $P_A(x', y^*) > P_A(x^*, y^*)$. Finally, we have

$$\begin{aligned} U_A(x', y^*) &= P_A(x', y^*)[\lambda_A(u_A(x') - u_A(y^*)) + 1 - \lambda_A] \\ &> P_A(x^*, y^*)[\lambda_A(u_A(x') - u_A(y^*)) + 1 - \lambda_A] \\ &\geq P_A(x^*, y^*)[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \\ &= U_A(x^*, y^*), \end{aligned}$$

again a contradiction. We conclude that (x^*, y^*) is Pareto optimal for $T \cup \{A\}$, and an analogous argument proves the Pareto optimality result for $T \cup \{B\}$.

A.8 Proof of Theorem 9

By Theorem 6, an equilibrium exists. To show uniqueness, it suffices by Theorem 2 to argue that $P_A(x,y)$ is strictly quasi-concave in $x \in S_A(y)$, with an analogous argument applying to candidate B. To this end, consider any distinct $x, x' \in S_A(y)$, and assume without loss of generality that $P_A(x,y) \leq P_A(x',y)$. Note that the support set S_G of G contains both

$$2\left(\int F_t(u_t(x) - u_t(y))d\tau\right) - 1 \quad \text{and} \quad 2\left(\int F_t(u_t(x') - u_t(y))d\tau\right) - 1,$$

where the first term above is less than or equal to the second. Setting $x'' = \frac{1}{2}x + \frac{1}{2}x'$, our strict quasi-concavity assumption implies

$$2\left(\int F_t(u_t(x'') - u_t(y))d\tau\right) - 1 \quad > \quad 2\left(\int F_t(u_t(x) - u_t(y))d\tau\right) - 1.$$

Then we have

$$\begin{aligned} P_A(x'',y) &= G\left(2\left(\int F_t(u_t(x'') - u_t(y))d\tau\right) - 1\right) \\ &> G\left(2\left(\int F_t(u_t(x) - u_t(y))d\tau\right) - 1\right) \\ &= P_A(x,y), \end{aligned}$$

and we conclude that $P_A(x,y)$ is strictly quasi-concave in $x \in S_A(y)$. Theorem 2 then implies that there is a unique equilibrium (x^*, y^*), and since A's probability of winning $P_A(z,z)$ when both candidates choose the same platform is constant , we have $x^* = y^*$. Next, assume that x^* is interior, and that voter utilities are differentiable. Since x^* solves

$$\max_{z \in Z} \sum_{t \in T} \omega_t F_t(u_t(z) - u_t(y^*)),$$

it satisfies the first order condition

$$\sum_{t \in T} \omega_t f_t(u_t(x^*) - u_t(y^*)) D u_t(x^*) \quad = \quad 0.$$

Using $x^* = y^*$, this coincides with the first order condition of the problem (13), and since the objective function in the latter problem is concave, we conclude that x^* solves (13), as required.

A.9 Proof of Lemma 3

Consider any interior equilibrium (x^*, y^*). In case one candidate is purely office motivated, then under the assumptions of the theorem, the other must place positive weight on office. It follows from Lemma 1 that each wins with positive probability, so the equilibrium is standard. In the remaining case, each candidate places positive weight on policy, i.e., $\lambda_A, \lambda_B > 0$. Since the equilibrium is interior, the first order condition for each candidate must hold; for example, candidate A's first order condition is

$$2g[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \int f_t Du_t(x^*) d\tau + \lambda_A G Du_A(x^*) \quad = \quad 0.$$

If (x^*, y^*) is non-standard, then one candidate, say A, wins with probability one, so $G = 1$ in the above first order condition. Then, since G is differentiable, we must have

$$g\left(2\left(\int F_t(u_t(x^*) - u_t(y^*)\right) d\tau - 1\right) \quad = \quad 0,$$

so the first order condition implies that $Du_A(x^*) = 0$, i.e., $x^* = \hat{x} \neq \hat{y}$. Then for $\alpha \in (0,1)$ small enough, we have $u_B(\alpha\hat{y} + (1-\alpha)x^*) > u_B(x^*)$ and $P_B(x^*, \alpha\hat{y} + (1-\alpha)x^*) > 0$, but this implies that

$$U_B(x^*, \alpha\hat{y} + (1-\alpha)x^*) \;\; > \;\; 0 \;\; = \;\; U_B(x^*, y^*),$$

contradicting the fact that y^* is a best response to x^*. We conclude that (x^*, y^*) is standard.

A.10 Proof of Theorem 14

The argument for part (i) proceeds as in the proof of Theorem 7 and is left to the reader. Now, assume that u_A and each u_t are differentiable and x^* is interior. Then A's optimization problem is

$$\max_{x \in Z} G(\kappa + V(x) - V(y))[\lambda_A(u_A(x) - u_A(y)) + 1 - \lambda_A],$$

with first order condition

$$gV'[\lambda_A(u_A(x) - u_A(y)) + 1 - \lambda_A] + G\lambda_A Du_A \quad = \quad 0$$

holding at x^*. Thus, x^* solves the first order condition for maximizing $\alpha u_A + (1-\alpha)V$, and since the latter function is concave, part (ii) follows. It is clear that if candidate A is office motivated, then her unique optimal policy is $x = \hat{z}$, regardless of B's platform, which gives us part (iii).

A.11 Proof of Corollary 4

First, assume that $\lambda_A > 0$, so that $\alpha > 0$. If candidate A located at $x^* = \hat{z}$ in equilibrium, then by part (ii) of Theorem 14, the first order condition

$$\alpha Du_A(\hat{z}) + (1-\alpha)DV(\hat{z}) \quad = \quad 0$$

would need to hold. Of course, $DV(\hat{z}) = 0$, but since $\hat{x} \neq \hat{z}$, and since u_A is concave, we have $\alpha Du_A(\hat{z}) \neq 0 = (1-\alpha)DV(\hat{z})$. From this contradiction, we conclude $x^* \neq \hat{z}$. Next, assume $\lambda_A \in [0,1)$, which implies $\alpha \in [0,1)$. If candidate A located at $x^* = \hat{x}$ in equilibrium, then since $Du_A(\hat{x}) = 0 \neq DV(\hat{x})$, the first order order condition for the problem in part (ii) of Theorem 14 again leads to a contradiction, and we conclude that $x^* \neq \hat{x}$.

A.12 Proof of Corollary 5

The first part of the corollary follows because, by Condition 6, it is a strictly dominant strategy for the office-motivated candidate to choose $\hat{z}$. For the second part, we show that the other candidate has a unique optimal policy; we consider candidate A's best response to $\hat{z}$. Clearly, A's unique optimal policy is $\hat{x}$ if she is purely office motivated, so assume $\lambda_A > 0$. Since $y = \hat{z} \neq \hat{x}$, Lemma 1 implies that candidate A's optimal platforms are just those solving (6), and since $\lambda_A > 0$, it follows that $\ln(\lambda_A(u_A(x) - u_A(y)) + 1 - \lambda_A)$ is strictly concave of x on $Z_A(y)$. The term $\ln(P_A(x,y))$ is concave in x, by Condition 4, and thus the objective function in (6) is strictly concave and has a unique maximizer, say x^*, and we conclude that $(x^*, \hat{z})$ is the unique equilibrium.

A.13 Proof of Proposition 1

Recall the maximization problem

$$\max_{x \in Z} G(V(x) - V(y^*))[\lambda(u_A(x) - u_A(y^*)) + 1 - \lambda]$$
$$\text{s.t. } x_1 + x_2 + x_3 = 1$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0,$$

with Kuhn-Tucker first order condition,

$$\begin{aligned} g(0)DV(x^*)\Delta + \tfrac{\lambda}{2}Du_A(x^*) + (1,1,1)\mu + (\nu_1,\nu_2,\nu_3) &= 0 \\ \nu_1 x_1^* = 0,\ \nu_2 x_2^* = 0,\ \nu_3 x_3^* = 0 \\ \nu_1 \geq 0,\ \nu_2 \geq 0,\ \nu_3 \geq 0, \end{aligned}$$

where μ is the multiplier on the equality constraint, and ν_t is the multiplier on the non-negativity constraint of the consumption of type t voters, and where

$$\Delta = \lambda(u_A(x^*) - u_A(y^*)) + 1 - \lambda.$$

is A's gain from winning.

Using $DV(x) = (\omega, \omega, 1-2\omega)$ and $Du_A(x) = (1,0,0)$, the first order condition implies

$$\begin{aligned}
g(0)\Delta\omega + \frac{\lambda}{2} + \mu + \nu_1 &= 0 \\
g(0)\Delta\omega + \mu + \nu_2 &= 0 \\
g(0)\Delta(1-2\omega) + \mu + \nu_3 &= 0.
\end{aligned}$$

Note that

$$g(0)\Delta\omega + \nu_2 \;=\; -\mu \;=\; g(0)\Delta(1-2\omega) + \nu_3,$$

and since $1-2\omega > \omega$, this implies that $\nu_2 > \nu_3 \geq 0$. By the complementary slackness condition $\nu_2 x_2 = 0$, we conclude that $x_2 = 0$; that is, candidate A allocates no resources to the conservative elites. Since $x_1 + x_3 = 1$, we have $\nu_1 = 0$ or $\nu_2 = 0$ or both. We focus on equilibria for which $x_1 > 0$ and $x_3 > 0$, which will exist for a range of policy weights. Then the first order conditions imply

$$\begin{aligned}
g(0)\Delta\omega + \frac{\lambda}{2} + \mu &= 0 \\
-g(0)\Delta(1-2\omega) &= \mu.
\end{aligned}$$

Substituting the second equation into the first, we have

$$g(0)\Delta(3\omega - 1) + \frac{\lambda}{2} \;=\; 0$$

Since $\Delta = \lambda x_1^* + 1 - \lambda$, we can solve the latter equation to obtain the only possible symmetric equilibrium strategy for A as

$$\begin{aligned}
x_1^* &= \frac{1}{2g(0)(1-3\omega)} - \frac{1-\lambda}{\lambda} \\
x_2^* &= 0 \\
x_3^* &= 1 - x_1^*,
\end{aligned}$$

where B's platform is $y^* = (0, x_1^*, x_3^*)$. From the above solution, we see that such an equilibrium exists when

$$\frac{2g(0)(1-3\omega)}{2g(0)(1-3\omega)+1} \;<\; \lambda \;<\; 2g(0)(1-3\omega),$$

whereas outside that range, the candidates allocate all resources to their own elites when λ is too high, and they allocate all resources to the masses when λ is too low.

A.14 Proof of Proposition 2

We are interested in equilibria such that all groups receive positive resources, so we consider the best response problem of candidate A in such a symmetric equilibrium,

$$\max_{x \in Z} G(V(x) - V(y^*))[\lambda(u_A(x) - u_A(y^*)) + 1 - \lambda]$$
$$\text{s.t. } x_1 + x_2 + x_3 = 1,$$

with first order condition

$$g(0)DV(x^*)\Delta + \frac{\lambda}{2} Du_A(x^*) + (\mu, \mu, \mu) \quad = \quad 0.$$

Using our assumptions that voter utility from consumption is quadratic and candidate utility is linear, this becomes

$$g(0)\omega(1 - x_1^*)\Delta + \frac{\lambda}{2} + \mu \quad = \quad 0 \tag{39}$$
$$g(0)\omega(1 - x_2^*)\Delta + \mu \quad = \quad 0 \tag{40}$$
$$g(0)(1 - 2\omega)(1 - x_3^*)\Delta + \mu \quad = \quad 0. \tag{41}$$

Combining the first two equations, (39) and (40), we have

$$g(0)\omega(x_2^* - x_1^*)\Delta + \frac{\lambda}{2} \quad = \quad 0 \tag{42}$$

By symmetry, we have $y_1^* = x_2^*$, and thus

$$\Delta \;=\; \lambda(x_1^* - y_1^*) + 1 - \lambda \;=\; \lambda(x_1^* - x_2^*) + 1 - \lambda.$$

Substituting this expression into (42), and writing $\delta = x_1^* - x_2^*$, we have a quadratic equation

$$g(0)\omega\delta\left[\delta + \frac{1-\lambda}{\lambda}\right] - \frac{1}{2} \quad = \quad 0,$$

which gives us an explicit solution

$$\delta \quad = \quad \frac{1}{2}\left[-\frac{1-\lambda}{\lambda} + \sqrt{\left(\frac{1-\lambda}{\lambda}\right)^2 + \frac{1}{g(0)\omega}}\right]$$

for the difference in a candidate's consumption and her opponents consumption in a symmetric equilibrium platform pair.

To calculate the amount allocated to the other elite group, x_2^*, in terms of δ, we use (40) and (41) to obtain

$$g(0)\omega(1 - x_2^*)\Delta \quad = \quad g(0)(1 - 2\omega)(1 - x_3^*)\Delta,$$

which gives us

$$x_3^* \quad = \quad \frac{1 - 3\omega}{1 - 2\omega} + \left(\frac{\omega}{1 - 2\omega}\right)x_2^*. \tag{43}$$

Since $x_1^* = 1 - x_2^* - x_3^*$, we can then write x_1^* in terms of x_2^* as

$$x_1^* = \frac{\omega}{1-2\omega} - \left(\frac{1-\omega}{1-2\omega}\right)x_2^*. \tag{44}$$

Then we have

$$\delta = x_1^* - x_2^* = \frac{\omega}{1-2\omega} - \left(\frac{2-3\omega}{1-2\omega}\right)x_2^*,$$

from which we deduce

$$x_2^* = \frac{\omega}{2-3\omega} - \left(\frac{1-2\omega}{2-3\omega}\right)\delta.$$

Using (44) and (43), we obtain the consumption of types 1 and 3 as

$$\begin{aligned} x_1^* &= \frac{\omega}{2-3\omega} + \left(\frac{1-\omega}{2-3\omega}\right)\delta \\ x_3^* &= \frac{2-5\omega}{2-3\omega} - \left(\frac{\omega}{2-3\omega}\right)\delta, \end{aligned}$$

while, by symmetry, candidate B's platform is $y^* = (x_2^*, x_1^*, x_3^*)$.

A.15 Proof of Lemma 4

Letting $\tilde{M}^t = 2\rho_t M^t$ and $\tilde{\zeta}_t = 2\rho_t\zeta_t$ for each voter type t, set M and $\hat{z}$ as in the statement of the lemma, and set

$$\zeta = \hat{z}M\hat{z} - \int \hat{z}^t\tilde{M}_t\hat{z}^t d\tau + \int \tilde{\zeta}_t d\tau.$$

Note that

$$\begin{aligned} V(z) &= 2\int \rho_t u_t(z)d\tau \\ &= 2\int \rho_t[-(z-\hat{z}^t)M_t(z-\hat{z}^t) + \zeta_t]d\tau \\ &= \int [-z\tilde{M}_t z + 2z\tilde{M}_t\hat{z}^t - \hat{z}^t\tilde{M}_t\hat{z}^t + 2\rho_t\zeta_t]d\tau \\ &= -z\left(\int \tilde{M}_t d\tau\right)z + 2z\int \tilde{M}_t\hat{z}^t d\tau - \int \hat{z}^t\tilde{M}_t\hat{z}^t d\tau + \int \tilde{\zeta}_t d\tau \\ &= -zMz + 2zM\hat{z} - \hat{z}M\hat{z} + \zeta \\ &= -(z-\hat{z})M(z-\hat{z}) + \zeta, \end{aligned}$$

and thus V is generalized quadratic with ideal point $\hat{z}$.

A.16 Proof of Theorem 16

Assume $\lambda_A > 0$, and let (x^*, y^*) be an equilibrium. Then x^* lies on $\overline{AV}$ and by Corollary 4, we have $x^* \neq \hat{z}$ (for this conclusion, we use the assumption that $\hat{z}$ is interior, but we do not require that $\hat{x}$ is also interior). As well, y^* lies on $\overline{BV}$. Since the ideal points are not positively dependent, we have $(\overline{AV} \setminus \{\hat{z}\}) \cap \overline{BV} = \emptyset$, and thus $x^* \neq y^*$.

A.17 Proof of Proposition 5

By the argument in the paragraph following the statement of the proposition, it is enough to prove our claim that when $\rho_1 > \rho_2 > ... > \rho_n$, party A's equilibrium policy is more progressive than party B's if and only if $\alpha < \beta$. It is a routine calculation to verify that:

$$\begin{aligned}
(x_2 - x_1) - (y_2 - y_1) &= \left[\frac{\alpha}{\gamma \omega_1} + \frac{\beta - \alpha}{\gamma \sum_{\tilde{t}} \rho_{\tilde{t}} \omega_{\tilde{t}}} (\rho_1 - \rho_2) \right] K \\
(x_{t+1} - x_t) - (y_{t+1} - y_t) &= \frac{\beta - \alpha}{\gamma \sum_{\tilde{t}} \rho_{\tilde{t}} \omega_{\tilde{t}}} (\rho_t - \rho_{t+1}) K, \quad t = 2, ..., n-1 \\
(x_n - x_{n-1}) - (y_n - y_{n-1}) &= \left[\frac{\beta}{\gamma \omega_n} + \frac{\beta - \alpha}{\gamma \sum_{\tilde{t}} \rho_{\tilde{t}} \omega_{\tilde{t}}} (\rho_{n-1} - \rho_n) \right] K.
\end{aligned}$$

Now, if A's policy is more progressive than party B's policy, i.e., $x_{t+1} - x_t > y_{t+1} - y_t$ for all $t = 1, ..., n-1$, then from the expressions above for any $t = 2, ..., n-1$, using $\rho_t > \rho_{t+1}$, we obtain $\beta - \alpha > 0$. Next, assuming that $\beta > \alpha \geq 0$, this means that all of the expressions on the right hand sides of the above displayed equations are positive whenever $\rho_1 > \rho_2 > ... > \rho_n$. Therefore, party A's policy is more progressive than party B's.

A.18 Proof of Theorem 17

For part (i), first assume $y^* = \hat{x}$. Since A is advantaged and $y^n \to \hat{x}$, we have

$$\int F_t(u_t(\hat{x}) - u_t(y^n)) d\tau \;\; \to \;\; \int F_t(0) d\tau \;\; > \;\; \frac{1}{2},$$

and therefore $P_A^n(\hat{x}, y^n) \to 1$. Then

$$\begin{aligned}
&\left(\lim_{n\to\infty} P_A^n(x^n, y^n) \right) [1 - \lambda_A] \\
&\geq \left(\lim_{n\to\infty} P_A^n(x^n, y^n) \right) [\lambda_A (u_A(x^*) - u_A(\hat{x})) + 1 - \lambda_A] \\
&= \lim_{n\to\infty} U_A(x^n, y^n)
\end{aligned}$$

$$
\begin{aligned}
&\geq \lim_{n\to\infty} U_A(\hat{x}, y^n) \\
&= 1 - \lambda_A.
\end{aligned}
$$

In case $\lambda_A < 1$, this limit implies $P_A^n(x^n, y^n) \to 1$. In the remaining case that $\lambda_A = 1$, suppose toward a contradiction that there exists $\overline{p} \in [0, 1)$ such that for arbitrarily high n, we have $P_A^n(x^n, y^n) \leq \overline{p}$. By Theorem 3, the candidate's equilibrium payoff is positive, and so $u_A(x^n) > u_A(y^n)$ for all n, and thus we have

$$
U_A(x^n, y^n) \quad \leq \quad \overline{p}(u_A(x^n) - u_A(y^n))
$$

for arbitrarily high n. Setting $p' \in (\overline{p}, 1)$, since $P_A^n(\hat{x}, y^n) \to 1$, we have $P^n(\hat{x}, y^n) \geq p'$ for sufficiently high n, and for arbitrarily high n, it follows that

$$
U_A(\hat{x}, y^n) \;\geq\; p'(u_A(\hat{x}) - u_A(y^n)) \;>\; \overline{p}(u_A(\hat{x}) - u_A(y^n)) \;\geq\; U_A(x^n, y^n),
$$

where we use $u_A(\hat{x}) \geq u_A(x^n) > u_A(y^n)$. Therefore, there exists n such that $\hat{x}$ is a profitable deviation for candidate A, a contradiction.

Next, assume $y^* \neq \hat{x}$. Using the assumption that A is advantaged, choose $\alpha \in (0, 1)$ small enough that

$$
\int F_t(u_t((1-\alpha)y^* + \alpha\hat{x}) - u_t(y^*))d\tau \quad > \quad \frac{1}{2},
$$

which implies

$$
P_A^n((1-\alpha)y^* + \alpha\hat{x}, y^n) \quad \to \quad 1.
$$

If $\lambda_A = 0$, then this yields the result, as A's equilibrium probability of winning from x^n is no less than that from $\alpha\hat{x} + (1-\alpha)y^*$. Now, assume $\lambda_A > 0$. Since $u_A((1-\alpha)y^* + \alpha\hat{x}) > u_A(y^*)$, we have

$$
\begin{aligned}
U_A^n((1-\alpha)y^* + \alpha\hat{x}, y^n) \quad &\to \quad \lambda_A(u_A((1-\alpha)y^* + \alpha\hat{x}) - u_A(y^*)) + 1 - \lambda_A \\
&> \quad 1 - \lambda_A,
\end{aligned}
$$

and thus

$$
\lim_{n\to\infty} U_A^n(x^n, y^n) \;\geq\; \lim_{n\to\infty} U_A^n((1-\alpha)y^* + \alpha\hat{x}, y^n) \;>\; 1 - \lambda_A.
$$

Since the above limit of equilibrium payoffs is positive and Z is compact, this implies that A's equilibrium probability of winning has a positive lower bound. This in turn implies that

$$
\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad \geq \quad \frac{1}{2}.
$$

In addition, since that limit is strictly higher than $1-\lambda_A$, we also have $u_A(x^*) > u_A(y^*)$. Now, suppose toward a contradiction that there exists $\overline{p} \in [0,1)$ such that for arbitrarily high n, we have $P_A^n(x^n, y^n) \leq \overline{p}$. Then for such n,

$$\begin{aligned} U_A^n(x^n, y^n) &\leq \overline{p}(\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A) \\ &< \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A. \end{aligned}$$

However, for $\gamma \in (0,1)$, strict quasi-concavity implies

$$\int F_t(u_t((1-\gamma)x^* + \gamma y^*) - u_t(y^*))d\tau \;>\; \int F_t(0)d\tau \;>\; \frac{1}{2},$$

and for γ small enough, we have

$$u_A((1-\gamma)x^* + \gamma y^*) - u_A(y^*) \;>\; \overline{p}(u_A(x^*) - u_A(y^*)).$$

Then we have $P_A^n((1-\gamma)x^* + \gamma y^*, y^n) \to 1$ and

$$\begin{aligned} \lim_{n\to\infty} U_A^n((1-\gamma)x^* + \gamma y^*, y^*) &= \lambda_A(u_A((1-\gamma)x^* + \gamma y^*) - u_A(y^*)) + 1 - \lambda_A \\ &> \overline{p}(\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A) \\ &\geq U_A^n(x^n, y^n), \end{aligned}$$

but then there exists n such that $(1-\gamma)x^* + \gamma y^*$ is a profitable deviation for A, a contradiction.

For part (ii), assume $x^* \neq \hat{x}$. There are two cases to preclude. The first is that

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \;<\; \frac{1}{2}.$$

In this case, we have $P_A^n(x^n, y^n) \to 0$, contradicting part (i). The remaining case is that the reverse strict inequality holds. Then there exists $\alpha \in (0,1)$ small enough that

$$\int F_t(u_t((1-\alpha)x^* + \alpha\hat{x}) - u_t(y^*))d\tau \;>\; \frac{1}{2}.$$

Then we have

$$P_A^n(x^n, y^n) \to 1 \quad \text{and} \quad P_A^n((1-\alpha)x^* + \alpha\hat{x}, y^n) \to 1,$$

and

$$\begin{aligned} &\lim_{n\to\infty} U_A^n((1-\alpha)x^* + \alpha\hat{x}, y^n) \\ &= \lambda_A(u_A((1-\alpha)x^* + \alpha\hat{x}) - u_A(y^*)) + 1 - \lambda_A \\ &> \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A \\ &= \lim_{n\to\infty} U_A^n(x^n, y^n), \end{aligned}$$

where the inequality uses concavity of u_A and $\lambda_A > 0$. This implies that $(1-\alpha)x^* + \alpha\hat{x}$ is a profitable deviation for A when n is large, a contradiction.

For part (iii), assume there exists $z' \in Z$ such that

$$\int F_t(u_t(x^*) - u_t(z'))d\tau \quad < \quad \int F_t(u_t(x^*) - u_t(y^*))d\tau,$$

and suppose toward a contradiction the strict inequality

$$\lambda_B(u_B(y^*) - u_B(x^*)) + 1 - \lambda_B \quad < \quad 0,$$

which then holds at (x^n, y^n) for sufficiently high n, by continuity. For each n, since (x^*, y^*) is a standard equilibrium, Theorem 3 implies that candidate B's payoff is positive, i.e., $U_B^n(x^n, y^n) > 0$, and in particular, $P_B^n(x^n, y^n) > 0$. But the above inequality implies that for high enough n,

$$U_B^n(x^n, y^n) \;=\; P_B^n(x^n, y^n)[\lambda_B(u_B(y^n) - u_B(x^n)) + 1 - \lambda_B] \;<\; 0,$$

a contradiction. Next, suppose the strict inequality holds in the reverse direction. For $\beta \in (0,1)$, our strict quasi-concavity assumption implies

$$\int F_t(u_t(x^*) - u_t((1-\beta)y^* + \beta z'))d\tau \;<\; \int F_t(u_t(x^*) - u_t(y^*))d\tau \;=\; \frac{1}{2},$$

where the equality holds by (21). We then have

$$P_B^n(x^n, (1-\beta)y^* + \beta z') \quad \to \quad 1.$$

Moreover, for $\beta > 0$ small enough, we have

$$\lambda_B(u_B((1-\beta)y^* + \beta z') - u_B(x^*)) + 1 - \lambda_B \quad > \quad 0,$$

and thus,

$$\begin{aligned}
&\lim_{n\to\infty} U_B^n(x^n, (1-\beta)y^* + \beta z') \\
&\quad= \lambda_B(u_B((1-\beta)y^* + \beta z') - u_B(x^*)) + 1 - \lambda_B \\
&\quad> 0 \\
&\quad= \lim_{n\to\infty} U_B^n(x^n, y^n),
\end{aligned}$$

and again, we obtain a profitable deviation for B when n is high enough, a contradiction.

A.19 Proof of Theorem 18

For part (i), assume $\lambda_A < 1$ and $\liminf_{n\to\infty} G^n(0) > 0$. Note that

$$\begin{aligned}
U_A^n(x^n, y^n) &\geq U_A^n(y^n, y^n) \\
&= P_A^n(y^n, y^n)[1 - \lambda_A] \\
&= G^n\left(2\int F_t(0)d\tau - 1\right)[1 - \lambda_A] \\
&= G^n(0)[1 - \lambda_A],
\end{aligned}$$

where we use the assumption that neither candidate is advantaged. Thus, we have

$$\liminf_{n\to\infty} U_A^n(x^n, y^n) \quad \geq \quad [1-\lambda_A] \liminf_{n\to\infty} G^n(0).$$

Then the limit infimum of A's equilibrium payoffs is strictly positive, which implies $\lim_{n\to\infty} P_A^n(x^n, y^n) \neq 0$. Of course, an analogous argument holds for candidate B.

For part (ii), assume $x^* \neq \hat{x}$, and suppose toward a contradiction that

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad > \quad \frac{1}{2}.$$

Then there exists $\alpha \in (0,1)$ small enough that

$$\int F_t(u_t((1-\alpha)x^* + \alpha\hat{x}) - u_t(y^*)) \quad > \quad \frac{1}{2},$$

which implies

$$P_A^n((1-\alpha)x^* + \alpha\hat{x}, y^n) \quad \to \quad 1,$$

but then, since u_A is concave and $\lambda_A > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} U_A^n((1-\alpha)x^* + \alpha\hat{x}, y^*) \quad &= \quad \lambda_A(u_A((1-\alpha)x^* + \alpha\hat{x}) - u_A(y^*)) + 1 - \lambda_A \\
&> \quad \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A \\
&= \quad \lim_{n\to\infty} U_A^n(x^n, y^n).
\end{aligned}$$

This implies that $(1-\alpha)x^* + \alpha\hat{x}$ is a profitable deviation for A when n is sufficiently high, a contradiction.

For part (iii), assume that

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \quad = \quad \frac{1}{2},$$

and suppose toward a contradiction that $x^* \neq y^*$. Letting

$$\overline{p} \quad = \quad \lim_{n\to\infty} P_A^n(x^n, y^n),$$

since at least one candidate has positive probability of winning in the limit, we can assume without loss of generality that $\overline{p} < 1$. We first consider the case $\lambda_A < 1$ and $u_A(x^*) \geq u_A(y^*)$. Define $\alpha_m = \frac{1}{m+1}$. Note that since neither candidate is advantaged, we have

$$\int F_t(u_t(y^*) - u_t(y^*)d\tau \quad = \quad \int F_t(0)d\tau \quad = \quad \frac{1}{2},$$

so our strict quasi-concavity assumption implies

$$\int F_t(u_t((1-\alpha_m)x^* + \alpha_m y^*) - u_t(y^*))d\tau \quad > \quad \frac{1}{2}$$

for all m, and in particular,

$$\lim_{n\to\infty} P_A^n((1-\alpha_m)x^* + \alpha_m y^*, y^*) \quad = \quad 1.$$

It follows that for each m, we can choose $n_m > n_{m-1}$ sufficiently high that

$$P_A^{n_m}((1-\alpha_m)x^* + \alpha_m y^*, y^{n_m}) \quad > \quad \frac{m-1}{m},$$

and thus

$$\begin{aligned}
&\lim_{m\to\infty} U_A^{n_m}((1-\alpha_m)x^* + \alpha_m y^*, y^{n_m}) \\
&\quad = \quad \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A \\
&\quad > \quad \overline{p}[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \\
&\quad = \quad \lim_{m\to\infty} U_A^{n_m}(x^{n_m}, y^{n_m}).
\end{aligned}$$

Thus, for high enough m, candidate A can profitably deviate to $(1-\alpha_{n_m})x^* + \alpha_{n_m} y^*$ in the model with shock distribution G^{n_m}, a contradiction.

In case $\lambda_A < 1$ and $u_A(x^*) < u_A(y^*)$, setting $\alpha_1 = \frac{1}{2}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} U_A^n((1-\alpha_1)x^* + \alpha_1 y^*, y^n) \\
&\quad = \quad \lambda_A(u_A((1-\alpha_1)x^* + \alpha_1 y^*) - u_A(y^*)) + 1 - \lambda_A \\
&\quad \geq \quad \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A \\
&\quad > \quad \overline{p}[\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \\
&\quad = \quad \lim_{n\to\infty} U_A^n(x^n, y^n),
\end{aligned}$$

and A has a profitable deviation to $(1-\alpha_1)x^* + \alpha y^*$ for high enough n, again a contradiction.

In the remaining case that $\lambda_A = 1$, since (x^n, y^n) is a standard equilibrium, we have $u_A(x^n) \geq u_A(y^n)$ for all n, and by continuity, $u_A(x^*) \geq u_A(y^*)$. Fixing $\gamma \in (\overline{p}, 1)$ and setting $x' = \gamma x^* + (1-\gamma)y^*$, our strict quasi-concavity assumption implies

$$\int F_t(u_t(x') - u_t(y^*))d\tau \quad > \quad \frac{1}{2},$$

and since u_A is concave with $u_A(x^*) \geq u_A(y^*)$, we also have

$$u_A(x') \quad \geq \quad \overline{p}u_A(x^*) + (1-\overline{p})u_A(y^*).$$

For small enough $\alpha \in (0,1)$, the inequality

$$\int F_t(u_t((1-\alpha)x' + \alpha\hat{x}) - u_t(y^*))d\tau \quad > \quad \frac{1}{2}$$

is maintained, and now

$$u_A((1-\alpha)x' + \alpha\hat{x}) \quad > \quad \overline{p}u_A(x^*) + (1-\overline{p})u_A(y^*).$$

Then we have $P_A^n((1-\alpha)x' + \alpha\hat{x}, y^n) \to 1$, and

$$\begin{aligned}
& \lim_{n\to\infty} U_A^n((1-\alpha)x' + \alpha\hat{x}, y^n) \\
& \quad = \quad \lambda_A(u_A((1-\alpha)x' + \alpha\hat{x}) - u_B(y^*)) + 1 - \lambda_A \\
& \quad > \quad \lambda_A(\overline{p}u_A(x^*) + (1-\overline{p})u_A(y^*) - u_A(y^*)) + 1 - \lambda_A \\
& \quad = \quad \overline{p}[\lambda_A((u_A(x^*) - u_A(y^*)) + 1 - \lambda_A] \\
& \quad = \quad \lim_{n\to\infty} U_A^n(x^n, y^n),
\end{aligned}$$

but then for high enough n, candidate A can profitably deviate to $(1-\alpha)x' + \alpha\hat{x}$, a contradiction. We conclude that $x^* = y^*$.

For part (iv), assuming (24), part (iii) delivers $x^* = y^*$. Suppose toward a contradiction that $x^* = y^* \neq z^*$. Then in the model with vote-maximizing candidates, (x^*, y^*) is not an equilibrium. Thus, one candidate, say A, has a profitable deviation, i.e., there exists $x' \in Z$ such that

$$\begin{aligned}
\int F_t(u_t(x') - u_t(y^*))d\tau \quad & > \quad \int F_t(u_t(x^*) - u_t(y^*))d\tau \\
& = \quad \int F_t(0)d\tau \\
& = \quad \frac{1}{2},
\end{aligned}$$

where the last equality uses the assumption that neither candidate is advantaged. Again, let $\alpha_m = \frac{1}{m+1}$, so that

$$\int F_t(u_t((1-\alpha_m)x^* + \alpha_m x') - u_t(y^*))d\tau \quad > \quad \frac{1}{2}$$

for all m. As in the proof of part (ii), we can choose a subsequence $\{n_m\}$ such that

$$\lim_{m\to\infty} P_A^{n_m}((1-\alpha_m)x^* + \alpha_m x', y^{n_m}) \quad = \quad 1.$$

This implies that

$$\begin{aligned}
\lim_{m\to\infty} U_A^{n_m}((1-\alpha_m)x^* + \alpha_m x', y^{n_m}) \quad & = \quad \lambda_A(u_A(y^*) - u_A(y^*)) + 1 - \lambda_A \\
& > \quad \overline{p}[1 - \lambda_A] \\
& = \quad \lim_{n\to\infty} U_A^n(x^n, y^n),
\end{aligned}$$

which again yields a profitable deviation, a contradiction.

A.20 Proof of Theorem 19

For part (i), assume $\kappa \geq V(0) - V(\hat{x})$. Then for all $\alpha \in (0,1)$, we have

$$V(\alpha\hat{x}) + \kappa \;>\; V(0) \;\geq\; V(y^*),$$

and thus $P_A^n(\alpha\hat{x}, y^n) \to 1$. Then for all $\alpha \in (0,1)$, we have

$$\begin{aligned} \lim_{n\to\infty} U_A^n(x^n, y^n) &\geq \lim_{n\to\infty} U_A^n(\alpha\hat{x}, y^n) \\ &= \lambda_A(u_A(\alpha\hat{x}) - u_A(y^*)) + 1 - \lambda_A. \end{aligned}$$

Therefore, taking the supremum over $\alpha \in (0,1)$, we have

$$\begin{aligned} \lambda(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A &\geq \lim_{n\to\infty} U_A^n(x^n, y^n) \\ &\geq \lambda_A(u_A(\hat{x}) - u_A(y^*)) + 1 - \lambda_A. \end{aligned}$$

Since $\lambda_A > 0$, this implies that $x^* = \hat{x}$.

Next, assume $\kappa < V(0) - V(\hat{x})$. We claim that $V(\hat{x}) + \kappa \leq V(y^*)$, for suppose otherwise. Then we have

$$V(x^*) + \kappa \;\geq\; V(\hat{x}) + \kappa \;>\; V(y^*)$$

and $P_A^n(x^n, y^n) \to 1$. If $x^* \neq \hat{x}$, then there exists $\alpha \in (0,1)$ small enough that $V((1-\alpha)x^* + \alpha\hat{x}) + \kappa > V(y^*)$, which implies

$$P_A^n((1-\alpha)x^* + \alpha\hat{x}, y^n) \;\to\; 1,$$

but then

$$\begin{aligned} \lim_{n\to\infty} U_A^n((1-\alpha)x^* + \alpha\hat{x}, y^*) &= \lambda_A(u_A((1-\alpha)x^* + \alpha\hat{x}) - u_A(y^*)) + 1 - \lambda_A \\ &> \lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A \\ &= \lim_{n\to\infty} U_A^n(x^n, y^n), \end{aligned}$$

which implies that $(1-\alpha)x^* + \alpha\hat{x}$ is a profitable deviation for A when n is sufficiently high. Thus, we have $x^* = \hat{x}$, but then the assumption of the claim implies

$$V(0) \;>\; V(\hat{x}) + \kappa \;=\; V(x^*) + \kappa,$$

which implies $P_B^n(x^n, 0) \to 1$. Since $u_B(0) > u_B(\hat{x})$, we then have

$$\lim_{n\to\infty} U_B^n(x^n, 0) \;=\; \lambda_B(u_B(0) - u_B(\hat{x})) + 1 - \lambda_B \;>\; 0 \;=\; \lim_{n\to\infty} U_B^n(x^n, y^n),$$

which implies that $y = 0$ is a profitable deviation for B when n is high enough. This establishes the claim.

To finish the proof of part (ii), consider the case $x^* \neq \hat{x}$. Using the preceding claim, Lemma 5 implies that $V(x^*) + \kappa = V(y^*)$, and in particular, $x^* \in \overline{AV} \setminus \{0\}$. If $y^* \neq 0$, then Lemma 6 implies that

$$\lambda_B(u_B(y^*) - u_B(x^*)) + 1 - \lambda_B \;=\; 0,$$

but we have $u_B(y^*) \geq u_B(0) > u_B(x^*)$, which contradicts the above equality. Thus, $y^* = 0$, so Lemma 5 gives us $V(x^*) + \kappa = V(0)$. In the remaining case that $x^* = \hat{x}$, note that candidate B can adopt $y = 0$, and then $P_B^n(x^n, 0) \to 1$, and once again, B can profitably deviate to $y = 0$, a contradiction.

A.21 Proof of Theorem 20

First, consider the case that $x^* \neq 0$ and $y^* \neq 0$. Since $x^* \in \overline{AV} \setminus \{0\}$, it follows that $x^* \notin \overline{BV}$, so there exists y' such that $u_B(y') > u_B(x^*)$ and $V(y') > V(x^*)$. Then $P_B^n(x^n, y') \to 1$, and we have

$$\begin{aligned} \lim_{n\to\infty} U_B^n(x^n, y^n) &\geq \lim_{n\to\infty} U_B^n(y', x^n) \\ &= \lambda_B(u_B(y') - u_B(x^*)) + 1 - \lambda_B \\ &> 0. \end{aligned} \tag{45}$$

In particular, candidate B's equilibrium probability of winning has a positive lower bound, i.e.,

$$\liminf_{n\to\infty} P_B^n(x^n, y^n) > 0,$$

which in turn implies $V(x^*) \leq V(y^*)$, or equivalently,

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau \leq \frac{1}{2}.$$

A symmetric argument for candidate B, using $y^* \in \overline{BV} \setminus \{0\}$, yields the opposite weak inequality, giving us (24). By part (iii) of Theorem 18, it follows that $x^* = y^*$, but since $\overline{AV} \cap \overline{BV} = \{0\}$, this is a contradiction. Next, consider the case that $x^* = 0$ and $y^* \neq 0$. Then $V(x^*) > V(y^*)$, or equivalently,

$$\int F_t(u_t(x^*) - u_t(y^*))d\tau > \frac{1}{2},$$

so part (ii) of Theorem 18 implies that $x^* = \hat{x}$, a contradiction. Similarly, we cannot have $x^* \neq 0$ and $y^* = 0$. The only possibility is the remaining case that $x^* = y^* = 0$.

A.22 Proof of Lemma 7

Because M is symmetric and positive definition, the operation $(x, y) \mapsto xMy$ is an inner product. To simplify notation, we then define the inner product norm $\|\cdot\|_M$ by

$$\|z\|_M = \sqrt{zMz},$$

so that aggregate voter utility is $V(z) = -\|z\|_M^2$, and candidate utility functions are $u_A(z) = -\|\hat{x} - z\|_M^2$ and $u_B(z) = -\|\hat{y} - z\|_M^2$. Since

$$\frac{d}{d\alpha}V(\alpha\hat{x}) = -2\alpha\|\hat{x}\|_M^2 \quad \text{and} \quad \frac{d}{d\alpha}V(\beta\hat{y}) = -2\beta\|\hat{y}\|_M^2,$$

we have

$$V(\alpha'\hat{x}) - V(\alpha''\hat{x}) = \int_{\alpha''}^{\alpha'} -2\alpha\|\hat{x}\|_M^2 d\alpha \tag{46}$$

$$V(\beta'\hat{y}) - V(\beta''\hat{y}) = \int_{\beta''}^{\beta'} -2\beta\|\hat{y}\|_M^2 d\beta. \tag{47}$$

Furthermore, since

$$\frac{d}{d\alpha}u_B(\alpha\hat{x}) = -\frac{d}{d\alpha}\|\hat{y} - \alpha\hat{x}\|_M^2 = 2\hat{x}M(\hat{y} - \alpha\hat{x})$$

and

$$\frac{d}{d\beta}u_B(\beta\hat{y}) = -\frac{d}{d\beta}\|\hat{y} - \beta\hat{y}\|_M^2 = 2(1-\beta)\|\hat{y}\|_M^2,$$

we also have

$$u_B(\alpha'\hat{x}) - u_B(\alpha''\hat{x}) = \int_{\alpha''}^{\alpha'} 2\hat{x}M(\hat{y} - \alpha\hat{x})d\alpha$$

$$u_B(\beta'\hat{x}) - u_B(\beta''\hat{x}) = \int_{\beta''}^{\beta'} 2(1-\beta)\|\hat{y}\|_M^2 d\beta.$$

Then

$$u_B(\alpha'\hat{x}) - u_B(\alpha''\hat{x}) = \int_{\alpha''}^{\alpha'} 2\hat{x}M\hat{y}d\alpha - \int_{\alpha''}^{\alpha'} 2\alpha\|\hat{x}\|_M^2 d\alpha$$

and

$$u_B(\beta'\hat{x}) - u_B(\beta''\hat{x}) = \int_{\beta''}^{\beta'} 2\|\hat{y}\|_M^2 d\beta - \int_{\beta''}^{\beta'} 2\beta\|\hat{y}\|_M^2 d\beta.$$

By (31)–(47), the desired inequality holds if and only if

$$\int_{\beta''}^{\beta'} 2\|\hat{y}\|_M^2 d\beta > \int_{\alpha''}^{\alpha'} 2\hat{x}M\hat{y}d\alpha,$$

which is equivalent to

$$\|\hat{y}\|_M^2(\beta' - \beta'') > \hat{x}M\hat{y}(\alpha' - \alpha'').$$

We can write the latter as

$$\|\hat{y}\|_M(\beta' - \beta'') > \|\hat{x}\|_M(\alpha' - \alpha'')\left(\frac{\hat{x}M\hat{y}}{\|\hat{x}\|_M\,\|\hat{y}\|_M}\right), \tag{48}$$

where $\hat{x}M\hat{y} \leq \|\hat{x}\|_M \, \|\hat{y}\|_M$, by the Cauchy-Schwartz inequality.

To show (48), we claim that

$$(\beta' - \beta'')\|\hat{y}\|_M \quad > \quad (\alpha' - \alpha'')\|\hat{x}\|_M. \tag{49}$$

Indeed, note that (31) can be written as

$$(\alpha' - \alpha'')(\alpha' + \alpha'')\|\hat{x}\|_M^2 \quad = \quad (\beta' - \beta'')(\beta' + \beta'')\|\hat{y}\|_M^2,$$

so the claimed inequality (49) holds if $\beta' + \beta'' > 0$ and

$$(\beta' + \beta'')\|\hat{y}\|_M \quad < \quad (\alpha' + \alpha'')\|\hat{x}\|_M.$$

In case $\beta''\|\hat{y}\|_M < \alpha''\|\hat{x}\|_M$, note that the reverse inequality leads to:

$$\begin{aligned}(\beta' + \beta'')\|\hat{y}\|_M \;&\geq\; (\alpha' + \alpha'')\|\hat{x}\|_M \\ \Rightarrow \quad \beta'\|\hat{y}\|_M \;&>\; \alpha'\|\hat{x}\|_M \\ \Rightarrow \quad (\beta' - \beta'')\|\hat{y}\|_M \;&>\; (\alpha' - \alpha'')\|\hat{x}\|_M.\end{aligned}$$

And in case $\beta'\|\hat{y}\|_M < \alpha'\|\hat{x}\|_M$, the reverse inequality $(\beta' + \beta'')\|\hat{y}\|_M \geq (\alpha' + \alpha'')\|\hat{x}\|_M$ implies $\beta''\|\hat{y}\|_M > \alpha''\|\hat{x}\|_M > 0$. But then

$$(\beta')^2\|\hat{y}\|_M^2 \;<\; (\alpha')^2\|\hat{x}\|_M^2 \quad \text{and} \quad (\beta'')^2\|\hat{y}\|_M^2 \;>\; (\alpha'')^2\|\hat{x}\|_M^2,$$

which implies

$$(-(\alpha')^2 + (\alpha'')^2)\|\hat{x}\|_M^2 \quad < \quad (-(\beta')^2 + (\beta'')^2)\|\hat{y}\|_M^2,$$

contradicting (31). We conclude that inequality (49) holds. Finally, using the Cauchy-Schwartz inequality, (49) implies (48), as required.

A.23 Proof of Theorem 21

Because each candidate chooses a platform on the contract curve between herself and the aggregate voter, and these line segments intersect only at the aggregate ideal point, every equilibrium (x^n, y^n) is standard. Therefore, by Corollary 1, each candidate wins with positive probability in equilibrium. By Theorem 17, candidate A's probability of winning converges to one, so it suffices to show (i)–(iii) in the remainder of the proof.

For part (i), assume $\kappa \geq \overline{\kappa}$, which implies $V(\hat{x}) + \kappa \geq V(\overline{\beta}\hat{y})$. Suppose toward a contradiction that $x^* \neq \hat{x}$. Then we cannot have $V(\hat{x}) + \kappa > V(y^*)$, else candidate A could deviate to platforms closer to her ideal point and still win with probability converging to one. Thus, $V(\hat{x}) + \kappa \leq V(y^*)$, and by Lemma 5, the indifference condition (25) for the aggregate voter holds at (x^*, y^*). Let $\tilde{\beta} \geq 0$ satisfy $V(\hat{x}) + \kappa = V(\tilde{\beta}\hat{y})$, so, letting $\tilde{y} = \tilde{\beta}\hat{y}$, voter indifference also holds at $(\hat{x}, \tilde{y})$. Note that

$$V(\tilde{y}) - V(\hat{x}) \;\;=\;\; \kappa \;\;\geq\;\; V(\overline{\beta}\hat{y}) - V(\hat{x}),$$

which implies $\tilde{\beta} \leq \overline{\beta}$. Then

$$\lambda_B(u_B(\tilde{\beta}\hat{y}) - u_B(\hat{x})) + 1 - \lambda_B \;\leq\; \lambda_B(u_B(\overline{\beta}\hat{y}) - u_B(\hat{x})) + 1 - \lambda_B \;=\; 0$$

Given voter indifference at (x^*, y^*) and at $(\hat{x}, \tilde{\beta}\hat{y})$, we can slide down from the second to the first to conclude that candidate B would rather lose to A at x^* than win herself at y^*. Technically, we have

$$V(x^*) + \kappa \;=\; V(y^*) \quad \text{and} \quad V(\hat{x}) + \kappa \;=\; V(\tilde{\beta}\hat{y}),$$

which implies

$$V(x^*) - V(y^*) \;=\; V(\hat{x}) - V(\tilde{\beta}\hat{y}).$$

Then Lemma 7 implies that

$$u_B(\hat{x}) - u_B(x^*) \;<\; u_B(\tilde{\beta}\hat{y}) - u_B(y^*),$$

which implies

$$\begin{aligned}
\lambda_B(u_B(y^*) - u_B(x^*)) + 1 - \lambda_B &< \lambda_B(u_B(\tilde{\beta}\hat{y}) - u_B(\hat{x})) + 1 - \lambda_B \\
&\leq \lambda_B(u_B(\overline{\beta}\hat{y}) - u_B(\hat{x})) + 1 - \lambda_B \\
&= 0,
\end{aligned}$$

where the equality holds by definition of $\overline{\beta}$. But then, for sufficiently high n, this implies

$$U_B^n(x^n, y^n) \;=\; P_B^n(x^n, y^n)[\lambda_B(u_B(y^n) - u_B(x^n)) + 1 - \lambda_B] \;\leq\; 0,$$

contradicting Theorem 3.

Henceforth, assume $\kappa < \overline{\kappa}$. Note that since $P_A^n(x^n, y^n) \to 1$, candidate B's equilibrium payoff converges to zero: $U_B^n(x^n, y^n) \to 0$. Furthermore, we claim that $x^* \neq \hat{x}$. Indeed, suppose toward a contradiction that $x^* = \hat{x}$. Since $V(\hat{x}) + \kappa < V(\overline{\beta}\hat{y})$, it follows that there exists $\beta' \in (\overline{\beta}, 1]$ close enough to $\overline{\beta}$ such that $V(\hat{x}) + \kappa < V(\beta'\hat{y})$. Letting $y' = \beta'\hat{y}$, we have $P_B^n(x^n, y') \to 1$, and since $u_B(y') > u_B(\overline{\beta}\hat{y})$, we also have

$$\begin{aligned}
\lim_{n\to\infty} U_B^n(x^n, y') &= \lambda_B(u_B(y') - u_B(x^*)) + 1 - \lambda_B \\
&> \lambda_B(u_B(\overline{\beta}\hat{y}) - u_B(\hat{x})) + 1 - \lambda_B \\
&= 0 \\
&= \lim_{n\to\infty} U_B^n(x^n, y^n).
\end{aligned}$$

Then for high enough n, y' is a profitable deviation for candidate B, a contradiction.

For part (ii), add the assumption that $\lambda_B(u_B(0) - u_B(\overline{\alpha}\hat{x})) + 1 - \lambda_B \geq 0$. Suppose toward a contradiction that $x^* \neq \overline{\alpha}\hat{x}$. Letting $x^* = \alpha^*\hat{x}$, we cannot have $\alpha^* < \overline{\alpha}$, else candidate A could deviate to platforms closer to her ideal point and still win with probability converging to one. Then $\alpha^* > \overline{\alpha}$, and thus, by voter indifference, we have $y^* \neq 0$. Letting $y^* = \beta^* \hat{y}$, the above claim and Lemma 6 together imply that (α^*, β^*) solves (27) and (28), and therefore $x^* = \alpha^0 \hat{x}$ and $y^* = \beta^0 \hat{y}$. Since voter indifference holds at (x^*, y^*) and $(\overline{\alpha}\hat{x}, 0)$, we slide from $(\alpha^0 \hat{x}, \beta^0 \hat{y})$ down to $(\overline{\alpha}\hat{x}, 0)$, and we conclude that

$$u_B(x^*) - u_B(\overline{\alpha}\hat{x}) \quad < \quad u_B(y^*) - u_B(0),$$

which implies

$$\lambda_B(u_B(0) - u_B(\overline{\alpha}\hat{x})) + 1 - \lambda_B \;\; < \;\; \lambda_B(u_B(y^*) - u_B(x^*)) + 1 - \lambda_B \;\; = \;\; 0,$$

which contradicts the assumption of part (ii).

For part (iii), instead add the assumption that $\lambda_B(u_B(0) - u_B(\overline{\alpha}\hat{x})) + 1 - \lambda_B < 0$. We have established that $x^* \neq \hat{x}$, so $V(x^*) + \kappa = V(y^*)$. Then we cannot have $y^* = 0$, for that would imply $x^* = \overline{\alpha}\hat{x}$, but we have assumed that B would rather lose to A at $\overline{\alpha}\hat{x}$ than win herself at 0. This means that for n high enough, B's equilibrium payoff is negative, which is impossible. Thus, we have $y^* \neq 0$, and as argued in the proof of part (ii), this implies $x^* = \alpha^0 \hat{x}$ and $y^* = \beta^0 \hat{y}$.

A.24 Proof of Lemma 8

Consider $(x^*, y^*) \in (\mathsf{int}\, Z)^2$ such that $U_A(x^*, y^*) > 0$ and $U_B(x^*, y^*) > 0$, and assume that $\phi(x^*, y^*) = 0$. In particular, we have

$$D_x U_A(x^*, y^*) \;\; = \;\; (D_x P_A)\Delta_A + \lambda_A P_A D u_A \;\; = \;\; 0.$$

Focusing on candidate A, note that $x^* \in Z_A(y^*)$ by assumption, and that

$$D_x \ln(U_A(x^*, y^*)) \;\; = \;\; \frac{1}{U_A(x^*, y^*)} D_x U_A(x^*, y^*) \;\; = \;\; 0.$$

Since

$$\ln(U_A(x^*, y^*)) \quad = \quad \ln(P_A(x^*, y^*)) + \ln(\lambda_A(u_A(x^*) - u_A(y^*)) + 1 - \lambda_A)$$

is concave in x, it follows that x^* maximizes $\ln(U_A(x, y^*))$ over $x \in Z_A(y^*)$. Then Lemma 1 implies that x^* is a best response to y^*. A symmetric argument for B implies that y^* is a best response to x^*, i.e., (x^*, y^*) is an equilibrium.

For the second part of the lemma, note that the second partial derivative of U_A with respect to x is

$$[(D_x^2 P_A)\Delta_A + \lambda_A (D_x P_A)^T D u_A] + \lambda_A (D_x P_A)^T D u_A + \lambda_A P_A D^2 u_A. \qquad (50)$$

The third term above is negative definite by $\lambda_A > 0$, by negative definiteness of $D^2 u_A$, and by the assumption that $U_A(x^*, y^*) > 0$, which implies $P_A > 0$. For the second term, note that the first order condition for A implies

$$\lambda_A D u_A = -\frac{\Delta_A}{P_A} D_x P_A, \tag{51}$$

and thus the second term in (50) equals

$$-\frac{\Delta_A}{P_A} (D_x P_A)^T D_x P_A,$$

which, since $U_A(x^*, y^*) > 0$ implies $\Delta_A > 0$, is negative semi-definite.

For the first term in (50), note that by log concavity of $P_A(\cdot, y)$, the second derivative

$$D_x^2 \ln(P_A(x^*, y^*)) = -\frac{1}{P_A^2} (D_x P_A)^T D_x P_A + \frac{1}{P_A} D_x^2 P_A$$

is negative semi-definite, and thus so is

$$-\frac{1}{P_A} (D_x P_A)^T D_x P_A + D_x^2 P_A.$$

Substituting the first order condition from (51) into the expression in brackets, we obtain

$$(D_x^2 P_A)\Delta_A - \frac{\Delta_A}{P_A} (D_x P_A)^T D_x P_A = (D_x^2 P_A) - \frac{1}{P_A} (D_x P_A)^T D_x P_A,$$

which is negative semi-definite, by log concavity. This shows that $D_x^2 U_A(x^*, y^*)$ is negative definite, and a symmetric argument delivers the result for B.

A.25 Proof of Lemma 9

In the general symmetric model, the partial derivative of ϕ_A with respect to α is

$$\begin{aligned} D_\alpha \phi_A(\alpha, \beta) &= g' \Delta_A D_\alpha V(\alpha \hat{x})^2 + g \lambda_A D_\alpha u_A(\alpha \hat{x}) D_\alpha V(\alpha \hat{x}) + g \Delta_A D_\alpha^2 V(\alpha \hat{x}) \\ &\quad + \lambda_A g D_\alpha V(\alpha \hat{x}) D_\alpha u_A(\alpha \hat{x}) + \lambda_A G D_\alpha^2 u_A(\alpha \hat{x}), \end{aligned}$$

and the partial derivative of ϕ_A with respect to β is

$$\begin{aligned} D_\beta \phi_A(\alpha, \beta) &= -g' D_\beta V(\beta \hat{y}) \Delta_A D_\alpha V(\alpha \hat{x}) - g \lambda_A D_\beta u_A(\beta \hat{y}) D_\alpha V(\alpha \hat{x}) \\ &\quad - \lambda_A g D_\beta V(\beta \hat{y}) D_\alpha u_A(\alpha \hat{x}), \end{aligned}$$

where $\|\hat{x}\| = \|\hat{y}\|$, $\kappa = 0$, and $\lambda = \lambda_A = \lambda_B$. Given a symmetric platform pair (γ, γ), the above expressions simplify: $\alpha = \beta = \gamma$, $\Delta = \Delta_A = \Delta_B$, and we also have $V(\alpha\hat{x}) = V(\beta\hat{y})$, which implies that $g'(V(\alpha\hat{x}) + \kappa - V(\beta\hat{y})) = g'(0) = 0$. Similarly, $G(V(\alpha\hat{x})+\kappa-V(\beta\hat{y})) = G(0) = \frac{1}{2}$, which implies $g(0) > 0$. Moreover, we have $D_\alpha V(\alpha\hat{x}) = D_\beta V(\beta\hat{y})$ as well.

The above derivatives then simplify to

$$D_\alpha\phi_A \quad = \quad g\Delta D_\alpha^2 V + 2\lambda g D_\alpha V D_\alpha u_A + \frac{\lambda D_\alpha^2 u_A}{2} \tag{52}$$

and

$$D_\beta\phi_A \quad = \quad -g\lambda D_\beta u_A D_\alpha V - \lambda g D_\alpha V D_\alpha u_A, \tag{53}$$

where g is evaluated at zero, $D_\alpha V$, $D_\alpha u_A$, and $D_\alpha^2 u_A$ are evaluated at $\alpha\hat{x} = \gamma\hat{x}$, and $D_\beta u_A$ is evaluated at $\beta\hat{y} = \gamma\hat{y}$. We now assume that (γ, γ) satisfies the candidates' first order conditions, and in particular, $\phi_A(\gamma,\gamma) = 0$, and we will establish that the second partial derivative dominates the cross partial derivative: $-D_\alpha\phi_A > |D_\beta\phi_A|$.

We separate out the case that $\Delta = 0$, which is only possible if $\lambda = 1$ and the candidates locate at the same platform, i.e., $\gamma\hat{x} = \gamma\hat{y}$. Since

$$\phi_A(\gamma,\gamma) \quad = \quad g\Delta D_\alpha V + \frac{\lambda D_\alpha u_A}{2} \quad = \quad 0, \tag{54}$$

this implies that $D_\alpha u_A = 0$, which implies that $\gamma = 1 = \theta$. In this case, we have $D_\alpha\phi_A = D_\alpha^2 u_A/2 < 0$ and $D_\beta\phi_A = 0$, so the desired inequality holds. In the remainder of the analysis, we assume $\Delta > 0$. Under the latter assumption, if $\lambda = 0$, then $-D_\alpha\phi_A > |D_\beta\phi_A|$ reduces to $-g\Delta D_\alpha^2 V > 0$, which clearly holds. Henceforth, we also assume that $\lambda > 0$. Returning to (52) and (53), note that each of the four terms in the expression for $D_\alpha\phi_A$ is negative, the first and third strictly so, and thus $D_\alpha\phi_A < 0$. We also have $-gD_\alpha V D_\alpha u_A \geq 0$, but the sign of $-gD_\beta u_A D_\alpha V$ depends on the angle θ and the coefficient γ.

To show $-D_\alpha\phi_A > D_\beta\phi_A$, note that the inequality holds if

$$-\lambda D_\alpha V D_\alpha u_A \quad \geq \quad -\lambda D_\alpha V D_\beta u_A. \tag{55}$$

This holds trivially if $D_\alpha V = 0$. Otherwise, we have $D_\alpha V < 0$, and since utility is quadratic, the inequality (55) is equivalent to

$$(\hat{x} - \gamma\hat{x})\hat{x} \quad \geq \quad (\hat{x} - \gamma\hat{y})\hat{y}.$$

Using $\|\hat{x}\| = \|\hat{y}\|$, this holds if and only if $\hat{x}\hat{x} \geq \hat{x}\hat{y}$, which indeed holds by the Cauchy-Schwartz inequality.

The remaining inequality to be verified is $-D_\alpha\phi_A > -D_\beta\phi_A$, which is equivalent to

$$g\Delta D_\alpha^2 V + 3\lambda g D_\alpha V D_\alpha u_A + \frac{\lambda D_\alpha^2 u_A}{2} \quad < \quad -g\lambda D_\beta u_A D_\alpha V.$$

Since $\Delta, \lambda > 0$, candidate A's first order condition (54) implies that $D_\alpha u_A > 0$. Substituting

$$\frac{\lambda}{2} \quad = \quad -\frac{g\Delta D_\alpha V}{D_\alpha u_A}$$

from the first order condition, the inequality in question becomes

$$\Delta D_\alpha^2 V + 3\lambda D_\alpha V D_\alpha u_A - \frac{\Delta D_\alpha V D_\alpha^2 u_A}{D_\alpha u_A} \quad < \quad -\lambda D_\beta u_A D_\alpha V. \tag{56}$$

The three terms on the left-hand side above are negative, the first and third strictly so, and thus (56) holds if

$$-\lambda D_\beta u_A D_\alpha V - \lambda D_\alpha V D_\alpha u_A \quad \geq \quad 0.$$

Next, we focus on the case in which the opposite inequality holds.

Note that the left-hand side of the desired inequality (56) is independent of θ, so we can choose $\theta = -1$ as the worst case; indeed, $D_\beta u_A(\beta\hat{y}) = 2(\hat{x} - \beta\hat{y})\hat{y} = 2(\hat{x}\hat{y} - \beta\hat{y}\hat{y})$ is decreasing in $\hat{x}\hat{y}$, so the right-hand side is decreasing in $\hat{x}\hat{y}$ as well. The inequality therefore holds if

$$\frac{D_\alpha V D_\alpha^2 u_A}{D_\alpha u_A} \quad \geq \quad \frac{\lambda D_\beta u_A D_\alpha V + \lambda D_\alpha V D_\alpha u_A}{\Delta}.$$

The latter inequality holds trivially if $D_\alpha V = 0$, and otherwise, we have $D_\alpha V < 0$. Dividing both sides by $D_\alpha V$, the inequality holds if

$$\frac{D_\alpha^2 u_A}{D_\alpha u_A} \quad \leq \quad \frac{\lambda D_\beta u_A + \lambda D_\alpha u_A}{\Delta} \quad = \quad \frac{D_\beta u_A + D_\alpha u_A}{u_A(\gamma\hat{x}) - u_A(\gamma\hat{y}) + \frac{1-\lambda}{\lambda}}.$$

Since the numerator on the right-hand side is non-positive, $\lambda = 1$ is the worst case, so the inequality holds if

$$\frac{D_\alpha^2 u_A}{D_\alpha u_A} \quad \leq \quad \frac{D_\beta u_A + D_\alpha u_A}{u_A(\gamma\hat{x}) - u_A(\gamma\hat{y})}.$$

Since utility is quadratic, we have $D_\alpha^2 u_A(\alpha\hat{x}) = -2\hat{x}\hat{x}$, so the inequality holds if

$$\frac{-2\hat{x}\hat{x}}{2(\hat{x} - \gamma\hat{x})\hat{x})} \quad \leq \quad \frac{2(\hat{x} - \gamma\hat{y})\hat{y} + 2(\hat{x} - \gamma\hat{x})\hat{x}}{(\hat{x} - \gamma\hat{y})(\hat{x} - \gamma\hat{y}) - (\hat{x} - \gamma\hat{x})(\hat{x} - \gamma\hat{x})},$$

or equivalently,

$$\frac{1}{1-\gamma} \quad \geq \quad -\frac{2(\hat{x}\hat{y} - \gamma\hat{y}\hat{y} + \hat{x}\hat{x} - \gamma\hat{x}\hat{x})}{\hat{x}\hat{x} - 2\gamma\hat{x}\hat{y} + \gamma^2\hat{y}\hat{y} - \hat{x}\hat{x} + 2\gamma\hat{x}\hat{x} - \gamma^2\hat{x}\hat{x}}.$$

Using $\|\hat{x}\| = \|\hat{y}\|$, the latter inequality simplifies to

$$\frac{1}{1-\gamma} \quad \geq \quad \frac{-\hat{x}\hat{y} + 2\gamma\hat{x}\hat{x} - \hat{x}\hat{x}}{-\gamma\hat{x}\hat{y} + \gamma\hat{x}\hat{x}}.$$

Dividing the numerator and denominator of the right-hand side by $\hat{x}\hat{x}$, and assuming the worst case scenario $\theta = -1$, this becomes

$$\frac{1}{1-\gamma} \quad \geq \quad \frac{2\gamma}{2\gamma},$$

which indeed holds.

At a platform pair (γ, γ) such that $\phi(\gamma, \gamma) = 0$, symmetry implies that the Jacobian of the system of first order conditions has the form in (34), namely,

$$D\phi(\gamma, \gamma) \quad = \quad \begin{bmatrix} a & b \\ b & a \end{bmatrix},$$

and we have shown that $|b| < -a$. From (53), it is apparent that if $\lambda = 0$, then $b = 0$. If $\lambda > 0$, then the first order condition (54) implies that $D_\alpha V < 0$, so the sign of $D_\beta \phi_A$ is

$$\begin{aligned} \mathsf{sign}\,(D_\beta u_A + D_\alpha u_A) \quad &= \quad \mathsf{sign}\,((\hat{x} - \gamma\hat{y})\hat{y} + (\hat{x} - \gamma\hat{x})\hat{x}) \\ &= \quad \mathsf{sign}\,(\hat{x}\hat{x} + \hat{x}\hat{y} - 2\gamma\hat{x}\hat{x}) \\ &= \quad \mathsf{sign}\left(\frac{1+\theta}{2} - \gamma\right), \end{aligned}$$

giving us the desired sign of b. Finally, we conclude that

$$\mathsf{det}\, D\phi(\gamma, \gamma) \quad = \quad a^2 - b^2 \quad > \quad 0,$$

as required.

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