Competitive Experimentation with Private Information*

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Abstract

Two privately-informed firms challenge each other in a R&D race with fixed experimentation intensity and winner-take-all termination. The innovation and resulting prize can arrive independently to each firm still in the race, according to identical Poisson processes of unknown arrival rate. Each firm observes initially a noisy private signal of the invention arrival rate, namely of the "promise" or feasibility of the research project, and subsequently whether the rival is still in the race or not. Due to the common-value nature of the game, the equilibrium displays a "winner's curse", which is more extreme than in standard or all-pay ascending common-value auctions (wars of attrition.) From a normative viewpoint, equilibrium expenditure in R&D may be either too high or too low with respect to the social optimum, depending on the (dis)agreement of private information. Specifically, whenever the private signals are sufficiently different, the more pessimistic firm underestimates the optimism of the opponent and prematurely exits the race, so that R&D activity is inefficiently postponed. Conversely, overinvestment due to duplication costs arises in equilibrium when private signals are sufficiently similar, because players herd and fool each other into believing that the project is quite promising.

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1. Introduction

Investment in Research and Development is widely considered a powerful engine of economic growth. Political platforms often include pledges to subsidize and promote R&D as a key form of investment for the future. There appears to be an implicit but widespread consensus that, left to market forces, the equilibrium amount of investment in R&D would be socially suboptimal. Institutions such as patents arose precisely to address some of the sources of underinvestment, but they still appear inadequate, and further support is often called for. It would be hard to mention one public statement stigmatizing the "excess" investment in R&D and the consequent need to curb it.

The theoretical economics literature has emphasized different sources of distortions in R&D investment, but its conclusions do not seem to uniformly support this widely held belief. Paradoxically, partial equilibrium analysis systematically comes to the conclusion that equilibrium R&D investment is socially excessive, due to duplication costs. This literature has developed around the workhorse Poisson model of inventions originally proposed by Reinganum (1981, 1982). In the strategic Multi-armed bandit literature (Bolton and Harris 1999), equilibrium experimentation is sub-optimal only because players cannot conceal their findings from each other: underexperimentation merely consists in the underprovision of a public good. But this context hardly applies to R&D, whose results are carefully protected for well-known reasons, that we will recall shortly.

In a general equilibrium growth context, Aghion and Howitt (1992) isolate four distortions in R&D investment. A tendency to underinvest is due to two *non-strategic* reasons. First, the "appropriability" effect: the innovator who obtains a patent anticipates to receive *ex post* "only" the monopoly profits from the invention, less than the social surplus. This is an undesirable side-effect of the patent system, which was designed to stimulate investment to begin with. Second, the intertemporal knowledge spillover: each innovator "is a dwarf standing on giants' shoulders", but fails to fully internalize the impact of his contribution on future research (on future dwarfs sitting on his shoulders). The other two forces lead to excess R&D investment. In particular, the "creative destruction" effect is akin to the duplication cost argument of partial equilibrium analysis: a successful innovation displaces incumbent monopolists, making existing technology redundant, a negative externality ignored by the innovator.

In this paper we identify a novel, *purely strategic* force that may generate equilibrium underinvestment in research. This force originates from imperfect aggregation of information about the feasibility or "promise" of research projects, before the investment necessary to verify this promise occurs.[1] The intuition behind this force is simple. It is not uncommon for an innovation to have many authors, who reach independently and simultaneously the same conclusion. Thus, often, an idea "is in the air", and *ex ante* several players conceive the research project as viable and promising. In this case competitive pressures and randomness are likely to lead to duplication costs and overinvestment in research.[2] In many other circumstances, however, one single researcher or research center is much more optimistic than others about a specific project, based on preliminary information/investigation. Negative information tends to be made public, as pessimistic players give up early or even fail to engage the project. *Positive information tends to be concealed, because disclosing it would lure competitors into the race*, and possibly imitators. However, the mere fact of pursuing a research project reveals one's optimism about its feasibility: one cannot explore an idea without suggesting to alert potential competitors the investment opportunity. Hence, when private information is *ex ante* very heterogeneous, the average public expectation of the promise of a project tends to fall short of the expectation under fully shared (complete) information. Due to borrowing or talent constraints, the optimistic player may be limited in the amount of resources she can devote to the project. Thus, the resulting aggregate level of experimentation is too low.

To summarize: suppose that potential competitors for an innovation can observe each other's engagement in the research project and roughly agree *ex ante* about its promise, but do not know that they agree. Then, they tend to fool each other into believing that the project is more promising than they individually thought, and overinvest in the aggregate. Even when the project is quickly successful, they duplicate their efforts. Conversely,

[1] As an example of how large corporations go about scouting promising projects, Pfizer, currently the largest pharmaceutical company in the world after recently absorbing Warner-Lambert/Parke-Davis, invested in 2002 $5.3 billion in R&D through its specialized arm Pfizer Global Research and Development, which employs 12,500 scientists. In addition to this massive structure, quoting from the company official website,"250 partners in academia and industry strengthen our position on the cutting edge of science and biotechnology by providing access to novel R&D tools and to key data on emerging trends."

[2] A famous and thrilling example, illustrated in a recent best-seller (Brenda Maddox 2002, *Rosalind Franklin: The Dark Lady of DNA*, HarperCollins), is the discovery of the helicoidal structure of DNA by Crick and Watson in England. Their Nobel-winning publication, based to a large extent on Rosalind Franklin's pioneering X-ray pictures of "wet" DNA filaments in Cambridge, preceded by a few weeks a likely identical discovery by Linus Pauling, who had been working independently for months on the same project in California. Pauling, an outspoken anti-nuclear advocate, could not travel to England to see Franklin's images because under investigation for anti-American activities (this was the world before the World Wide Web). Arguably, ex post, Pauling's enormous talent (testified by the two Nobel prizes he received) would have been better allocated elsewhere. This phenomenon is quite pervasive. *Technology Review, MIT's Magazine of Innovation*, is a monthly MIT publication devoted exclusively to R&D, a veritable Who's Who of this "industry". Each issue reports on several research projects, either new basic ideas or developments of existing ones, that are still years away from commercial implementation, and offers the readers a glimpse of the technological innovations to expect years later. Each such article contains a table with the (usually between five and ten) companies involved in that specific research enterprise, and makes clear that typically, based on past experience, at most one of them will be successful.

suppose that competitors strongly disagree *ex ante* about the promise of a project, but do not know it. Then, the pessimistic players herd on the efforts of optimistic ones, but eventually quit the race and in doing so give up their assessment of the project. Hence, information flows only from pessimistic to optimistic players, mostly negative information is aggregated, and total investment is too low.

This argument rests on two central assumptions: agents involved in the same research have private information about its promise, that they have no incentives to disclose, but may publicly observe (at least a noisy signal of) the research effort devoted to the project by competitors. From an empirical viewpoint, the first assumption is hardly questionable: R&D results are carefully protected, and industrial espionage actively promoted in response. Moreover, when private corporations sponsor university research, as a norm they require the faculty and graduate students involved to sign non-disclosure and exclusive-licensing agreements. The second assumption is potentially more controversial: the amount of resources invested by a company in R&D, which may signal by revealed preferences the company's private information, is public information in the US only at the aggregate firm level, for tax reasons, and not at the single project level. In Europe, not even firm-level information is typically disclosed in a verifiable manner. We make our second assumption as a first cut to this problem, and discuss at the end the extension to private information about variable and unobservable intensity of R&D investment at the project level.

Based on these premises, we lay out a simple analytic framework to explore the aggregation of private information in R&D races. Two privately-informed firms challenge each other in a research race with winner-take-all termination. The prize can arrive independently to each player still in the race, with identical Poisson arrival processes of underlying unknown parameter, measuring the "promise" of the project. A private signal informs players of the promise of the project. Over time, each firm decides whether to stay in the race, paying a flow cost, or to drop out of it. Once a firm has dropped out, prohibitive sunk costs make re-entry economically unfeasible.

We show that there is a unique symmetric semi-separating Perfect Bayesian equilibrium of this game. For any initial private signal realization, each firm selects a stopping time conditional on the opponent still being in the race or not. As time goes by and the innovation does not arrive, a firm becomes more and more pessimistic about the innovation's arrival rate, and eventually quits the race. The monotonicity of the equilibrium stopping time in the signal yields two implications. First, when both players are still in the race, each firm infers that the opponent's signal is at least as large as the inverse of the equilibrium strategy, and quits when the flow investment cost equals the expected flow benefit conditional on this information. Surprisingly, the possibility that the opponent wins the race in the next instant turns out to have no effect on the marginal value of waiting, and hence on the equilibrium stopping time. Second, quitting the race suddenly reveals all private information to the opponent, who then holds complete information and quits when the expected flow benefit of investment equals the flow cost.

Due to the common-value nature of the game, the equilibrium displays a rather extreme "winner's curse" property. Specifically, if the two private signal realizations are sufficiently similar, when the more pessimistic firm drops out of the race the other firm immediately follows suit, and regrets not having quit sooner. Initially, the latter firm entertains expectations about the relative optimism of the opponent that *ex post* turn out to be too optimistic; when the opponent quits, beliefs over the promise of the project discontinuously "drop". If the two private signals are very similar, then the negative surprise will be particularly unpleasant, hence the regret.

Our welfare analysis compares the aggregate discounted amounts of experimentation in equilibrium and in the efficient team solution. The winner-take-all assumption makes the two firms unwilling to share their private information, unless they are joined in a single team. If future payoffs are discounted, we show the optimal team policy is to gather information on the arrival rate of the project as quickly as possible: the team runs both firms' facilities in parallel until the expected joint marginal benefit of waiting is larger than the joint marginal cost, and then stops them simultaneously.

In equilibrium, instead, a firm may prematurely drop out of the race because her private information is very negative relative to joint aggregate information. Leaving the race early means not testing the idea and pursuing the project extensively. Therefore, the remaining player learns the opponent's private information, but does not observe the same amount of negative public information (no prize arrival) as in the team solution. Thus, this player is more optimistic than the team, and "overshoots" the team solution by experimenting longer. In the Poisson linear structure of our model, in equilibrium the premature exit of the pessimistic firm exactly offsets the belated exit of the remaining firm, and the joint experimentation durations are the same as in the team solution. But some experimentation is postponed to the future, inefficiently slowing down the joint rate of innovation arrival. Due to discounting, if the firms' signals are very disperse, then the equilibrium discounted expenditure in R&D will be too low with respect to the social optimum due to this postponement. If instead, the firms' signal are very close, then the "winner's curse" property of the equilibrium implies that both firms will suboptimally delay their exit from the race, and equilibrium overexperimentation takes place.

From an institutional viewpoint, R&D cooperatives have become increasingly popular arrangements, in the US between universities and corporations , in Japan among competing firms. Although their primary rationale appears to be pooling resources to overcome borrowing constraints and to share fixed costs, they also involve transfers of technology (see e.g. Adams *et alii* (2000)). Thus, in our perspective they might also be greatly beneficial to pool information about the preliminary promise of a project.

In Section 7 we discuss various extensions of the model, and show the robustness of our key normative insight linking the dispersion of private information to equilibrium (in)efficiency. We propose two tractable ways to accommodate gradual accumulation of private information over time, concerning intermediate outcomes of research; we relax the assumption of common arrival rate, to capture different approaches to the same research question; and we consider time-variable but unobservable experimentation intensity.

Section 2 reviews the related theoretical literature, Section 3 lays out the model, Section 4 characterizes the team solution, Section 5 the unique symmetric monotonic equilibrium, Section 6 compares the two solutions from a normative viewpoint, Section 7 discusses robustness, an Appendix contains the proofs.

2. Related Literature

Our work is related to several strands of literature, but presents important conceptual differences with respect to each of them.

First and foremost, the key benchmark are R&D races modeled either as differential or stopping games. The differential game approach is put forth in Reinganum (1981, 1982). At each moment in time t, each firm i selects an experimentation intensity $u_i(t)$, paying a quadratic cost. The intensity affects linearly the Poisson rate of arrival of the invention, which is $u_i(t)\lambda$. Innovation arrivals are independent across firms, and the first firm to achieve the innovation wins the race. A simplified version of this differential game, where each firm experiments with fixed intensity until it drops out of the race, can be understood as a stopping game. Choi (1985) takes this simplified route to extend the analysis to the case of uncertain λ with commonly known prior. This work is further extended by Malueg and Tsutsui (1999) in a full-fledged Poisson differential game à la Reinganum. In these models with symmetric information, equilibrium R&D investment is socially excessive, due to duplication costs.[3] This occurs essentially because the patent system allows the

[3] Before the full-fledged dynamic approach of Reinganum (1981), the duplication costs effect is identified in the "static" models of Dasgupta and Stiglitz (1980), Loury (1979) and Lee and Wilde (1980). Reinganum (1981) also identifies a source of under-investment in the following technological externality. Say that in the team's problem, each project's success rate increases with the expenditure in both facilities, because of so called "knowledge sharing." As a result, fractioning expenditure across facilities increases the joint hazard rate of the innovation. In order to separate the welfare effect of information aggregation from this purely technological and well-understood inefficiency, our model assumes constant returns to scale in R&D: the total arrival rate of an invention rises linearly in the number of active players.

first innovator to fully appropriate the entire associated stream of profits. In its strategic decision, each firm trades off the expected benefit for winning the race against her own experimentation costs in the race, and does not internalize the opponents' experimentation costs. We adopt Choi's simple model to address, for the first time in this literature, the effects of private information. We identify both instances of overinvestment, due to the "winner curse" effect, and of underinvestment, due to the postponement of research activity that follows inefficient information aggregation.[4]

An alternative approach to modeling R&D competition is the "tug-of-war": firms take turns in making costly steps towards a "finish line." In the absence of uncertainty, when solved by backward induction these games predict a rather dramatic preemption effect: once a firm is known to be ahead in the race, the opponents drop out of the race, and the winner acts as if she faced no competition in the race (Fudenberg, Gilbert, Stiglitz and Tirole (1983), Harris and Vickers (1985)). As a result, the equilibrium displays no duplication costs, and it is indeed socially efficient. However, once we reintroduce uncertainty in the duration of each step (Lippman and McCardle (1985), Harris and Vickers (1987)) the preemption effect vanishes, and equilibrium R&D investment is again socially excessive, due to duplication costs.

While our game shares elements of an all-pay ascending auction, or equivalently of a war of attrition[5] with common values (see Krishna and Morgan (1997) for a general treatment), the payoff specifications are different, and this induces a radically different equilibrium behavior. To see this, consider as a benchmark the symmetric equilibrium strategy of the standard common-value ascending second-price auctions (Milgrom (1981)): a player with private signal x quits the game just at the first time $\tau(x)$ when the flow cost of waiting equals the expected flow benefit conditioned on *both* players holding signal x. In the monotonic symmetric equilibrium of a common-value war of attrition, each player leaves the race much *earlier* than this $\tau(x)$. If in fact the opponent adopts strategy τ, as the conjectured optimal best-response own stopping time $\tau(x)$ is approaching, the player increasingly believes that the opponent's signal is likely to be larger than x, and hence that

[4]Beyond the problem of patent races, some have investigated the welfare implications of the patents institution determined by its innovation diffusion function. Scherer (1967) and Horstmann, MacDonald and Slivinski (1985) point out that firms may choose not to patent, because patents allow for imitation. At an institutional level, Scotchmer and Green (1990) compare First-to-File with First-to-Invent rule, and Choi (1998) focuses on litigations. Llobet, Hopenhayn and Mitchell (2001) introduce a mechanism-design approach to reward cumulative innovations.

[5]Fudenberg and Tirole (1985, 1986) model a duopoly war with changing demand as a war of attrition with private information of private values (marginal costs). Similarly to our results, they show that the weakest firm may exit too early or too late with respect to social optimum, but this is purely coincidental. While in our R&D race problem the welfare analysis compares the efficient team of firms' solution with the equilibrium durations, in their dupoly problem welfare is assessed by comparing equilibrium consumers' surplus before and after the weakest firm exits.

the race is lost. As the expected benefit of staying in the race vanishes, the firm anticipates exit to avoid paying the cost of attrition. In our game, prize arrivals are i.i.d. across firms, with same but unknown arrival rate. Hence the information that the opponent's signal is larger than x does not imply that the firm will lose the race, but rather conveys good news on the firm's prize promise (arrival rate), so that such a firm holding x is induced to postpone exit *after* $\tau(x)$. In a sense, the informational spillover that derives from common value and independent arrivals places our game on the opposite side of common-value wars of attritions, with common-value standard auctions in between.[6]

Chamley and Gale (1994) [CG] study a discrete-time timing game of common interest, where firms are privately informed on the "state of the economy" and may have investment opportunity, that can be irreversibly exercised at any period. As in our model, an irreversible timing decision is the only instrument to communicate private information, and this results in imperfect information aggregation. Two key differences distinguish our analysis from CG. First, in CG firms face a coordination problem, where either all should invest or none; if it were feasible for them to communicate, they would have no reason to conceal their private information. We analyze a game of conflicting interests, with payoff congestion, so firms have every incentive to conceal their private information, and would always downplay the promise of the project, if they could persuade outsiders. But, they are forced to slowly and imperfectly reveal their precious private information if they want to act upon it, a tension that is crucial to our results and that is absent in CG. Second, CG focus on prior-information aggregation, and allow for no joint information accumulation over time. So, their game ends almost immediately if time periods become very short. Inefficiency then results because firms may quickly coordinate on a wrong decision (either invest in a good state, or underinvest in bad state). Herding delays investment, as each firm would like to wait and see how many opponents choose to invest. In our R&D game, instead, *public* learning over time plays a central role, as the negative public information that the prize has not arrived works against the good news that the opponent is still in the race. Also, the rate of accumulation of public information is proportional to the number of players still in the race. Herding extends experimentation durations and works in favor of overinvestment.[7]

[6]Inefficiency of the equilibrium of our R&D game originates from imperfect information aggregation, partly arising from an informational cascade in the sense of Smith and Sorensen (2000). Our firms invest too much when their private signals are in sufficient agreement: mutual observation of actions leads to place too much weight on the public information that the opponent is still in the game, inefficiently correlating the equilibrium behaviors of the two players. Unlike herding models, we consider two long-run and forward-looking players, who act at endogenous times.

[7]Gul and Lundholm (1995) study a two-player *continuous*-time coordination timing game that shares many similarities with CG. Again, players would like to exchange information if they were allowed to, and the herding effect goes in the direction of delaying an irreversible decision with random consequences (i.e.

3. The Game

Two players, A and B, play the following optimalstopping game. A prize $b > 0$ arrives to player $i = A, B$ at a random time $t_i \geq 0$, according to a Poisson process of constant hazard rate λ, c.d.f. $F(t_i|\lambda) = 1 - e^{-\lambda t_i}$ and density $f(t_i|\lambda) = \lambda e^{-\lambda t_i}$. Conditional on the common λ, the two arrivals are independent. In order to know t_i and receive the prize, player i must keep paying a flow cost $c > 0$. Stopping payments of such costs implies that the prize is abandoned irreversibly and t_i will never be learned. We make a *winner-take-all* assumption: the first player to receive the prize ends the game. Costs and prizes are discounted at rate r.

The common hazard rate of arrival of the prize, $\lambda \geq 0$, is drawn by Nature, unobserved by the players, from a Gamma distribution:

$$\pi(\lambda) = \frac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha\lambda} \lambda^{\beta-1}, \text{ for } \alpha > 0, \beta > 0.$$

Before starting to pay costs, each player i observes a private signal $z_i \leq 0$ distributed according to a negative exponential distribution: for every $Z \leq 0$ and $z = z_i$,

$$H(Z|\lambda) = \Pr(z \leq Z|\lambda) = e^{\lambda Z} \text{ with density } H'(Z|\lambda) = h(Z|\lambda) = \lambda e^{\lambda Z}.$$

The two private signals z_A, z_B are conditionally (on λ) independent.

We will refer to a "project" as the possibility of paying c to activate the arrival of a prize. In our game each player has a "project"; based on the realization of the private signal, he decides whether to pursue it or not and, if so, when to stop it irreversibly conditional on the other player being still in the game or not. The canonical application of the model is as follows. Each of two firms might start an R&D project of the same nature. Before starting the project, each firm observes a private signal on its "promise." This key parameter is common to both projects, because they revolve around the same question, device etc.; but, conditional on the promise, the actual winner is determined by luck and by willingness to continue investing resources in research.

Belief Updating. Two posterior beliefs play a key role in our analysis. We consider player $A's$ updating, player $B's$ being symmetric. First, suppose that player A is fully informed about the signal realizations $z_A = x$, $z_B = y$, and that project A has not delivered a prize by time t and project B by time t'. Conditional on this information, the posterior

delaying investment in CG's terminology). As in our game, the first players who takes her timing decision reveals all her private information, leaving the opponent to act fully informed.

belief density on λ is a Gamma with parameters $\alpha - x - y + t + t'$ and $\beta + 2$:

$$\begin{aligned}
\pi(\lambda | z_A = x, z_B = y, t_A \geq t, t_B \geq t') &= \frac{\pi(\lambda) h(x|\lambda) h(y|\lambda) [1 - F(t|\lambda)] [1 - F(t'|\lambda)]}{\int_\Lambda \pi(\lambda') h(x|\lambda') h(y|\lambda') [1 - F(t|\lambda')] [1 - F(t'|\lambda')] d\lambda'} \\
&= \frac{e^{-\alpha\lambda} \lambda^{\beta-1} \lambda e^{\lambda x} \lambda e^{\lambda y} e^{-\lambda t} e^{-\lambda t'}}{\int_\Lambda e^{-\alpha\lambda'} \lambda'^{\beta-1} \lambda' e^{\lambda' x} \lambda' e^{\lambda' y} e^{-\lambda' t} e^{-\lambda' t'} d\lambda'} \\
&= \frac{e^{-\lambda(\alpha - x - y + t + t')} \lambda^{\beta+1}}{\Gamma(\beta+2) (\alpha - x - y + t + t')^{-\beta-2}}.
\end{aligned}$$

for ease of notation, we shall henceforth denote $\pi(\lambda | z_A = x, z_B = y, t_A \geq t, t_B \geq t')$ by $\pi_{t,t'}(\lambda|x, y)$, with c.d.f. $\Pi_{t,t'}(\lambda|x, y)$.

Second, suppose that player A is fully informed about her own private signal realization $z_A = x$, that own prize has not arrived yet by t, the other prize has not arrived by t', and that the opponent has a signal z_B *not smaller than* y. Conditional on this information, the posterior belief density on λ is:

$$\begin{aligned}
\pi(\lambda | z_A = x, z_B \geq y, t_A \geq t, t_B \geq t') &= \frac{\pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)]}{\int_\Lambda \pi(\lambda') h(x|\lambda') [1 - H(y|\lambda')] [1 - F(t|\lambda')] [1 - F(t'|\lambda')] d\lambda'} \\
&= \frac{e^{-\alpha\lambda} \lambda^{\beta-1} \lambda e^{\lambda x} \left(1 - e^{\lambda y}\right) e^{-\lambda t} e^{-\lambda t'}}{\int_\Lambda e^{-\alpha\lambda'} \lambda'^{\beta-1} \lambda' e^{\lambda' x} \left(1 - e^{\lambda' y}\right) e^{-\lambda' t} e^{-\lambda' t'} d\lambda'} \\
&= \frac{\lambda^\beta \left[e^{-\lambda(\alpha + t + t' - x)} - e^{-\lambda(\alpha + t + t' - x - y)}\right]}{\Gamma(\beta+1) \left[(\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1}\right]}
\end{aligned}$$

For ease of notation, we shall henceforth denote $\pi(\lambda | z_A = x, z_B \geq y, t_A \geq t, t_B \geq t')$ by $\pi_{t,t'}(\lambda|x, y+)$, with c.d.f. $\Pi_{t,t'}(\lambda|x, y+)$.

The key statistic to determine optimal stopping in the model is the expected hazard rate of prize arrival. Conditional on "complete" information (x, y, t, t') this is:

$$\begin{aligned}
\mathbb{E}_{t,t'}[\lambda| x, y] &= \int_\Lambda \lambda \pi_{t,t'}(\lambda|x, y) d\lambda \\
&= \frac{\int_\Lambda e^{-\lambda(\alpha - x - y + t + t')} \lambda^{\beta+2} d\lambda}{\Gamma(\beta+2) (\alpha - x - y + t + t')^{-\beta-2}} \\
&= \frac{\beta + 2}{\alpha - x - y + t + t'};
\end{aligned}$$

and conditional on the information that the opponent's signal is above y:

$$\begin{aligned}
\mathbb{E}_{t,t'}[\lambda| x, y+] &= \int_\Lambda \lambda \pi_{t,t'}(\lambda|x, y+) d\lambda \\
&= \int_\Lambda \frac{\lambda^{\beta+1} \left[e^{-\lambda(\alpha + t + t' - x)} - e^{-\lambda(\alpha + t + t' - x - y)}\right]}{\Gamma(\beta+1) \left[(\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1}\right]} d\lambda \\
&= (\beta + 1) \frac{(\alpha + t + t' - x)^{-\beta-2} - (\alpha + t + t' - x - y)^{-\beta-2}}{(\alpha + t + t' - x)^{-\beta-1} - (\alpha + t + t' - x - y)^{-\beta-1}}.
\end{aligned}$$

It is immediate to see that $\mathbb{E}_{t,t'}[\lambda| x,y]$ is increasing in x and y and decreasing in t,t'; the next Lemma shows that this is the case also for $\mathbb{E}_{t,t'}[\lambda| x,y+]$.

Lemma 1. *The posterior expected hazard rates of arrival* $\mathbb{E}_{t,t'}[\lambda| x,y]$, $\mathbb{E}_{t,t'}[\lambda| x,y+]$, *conditional on: no prize* A *arrival by time* t, *no prize* B *arrival by time* t', *the realization of the signal* $z_A = x$, *and the realization of signal* z_B *being (respectively) equal to or larger than* y, *are both strictly decreasing in* t *and in* t', *and strictly increasing in* x *and in* y. *Furthermore,*

$$\lim_{t\to\infty} \mathbb{E}_{t,t'}[\lambda| x,y] = 0, \quad \lim_{t\to\infty} \mathbb{E}_{t,t'}[\lambda| x,y+] = 0.$$

Next, knowing that the opponent's private signal realization equals y for sure is bad news compared to knowing only that it is larger than y.

Lemma 2. *For any* t,t',x, *and* $y<0$, $\Pi_{t,t'}(\lambda|x,y) \prec_{FSD} \Pi_{t,t'}(\lambda|x,y+) \prec_{FSD} \Pi_{t,t'}(\lambda|x,0)$. *Hence*

$$\mathbb{E}_{t,t'}[\lambda| x,y] < \mathbb{E}_{t,t'}[\lambda| x,y+] < \mathbb{E}_{t,t'}[\lambda| x,0].$$

4. The Team Solution

We begin our analysis by studying the first-best solution, in which players join forces in a single team, sharing four pieces of information: the two signal realizations x,y, that project A has not delivered a prize by time t and project B by time t'. In principle the team may choose to stop projects in sequence. We first consider the case in which one of the two projects (project "1") has been irreversibly stopped at time $T_1 \geq 0$, and the other (project "2") is still ongoing at time $T_2 \geq T_1$. This will allow us later, by backward induction, to solve the problem where both projects are still ongoing. Project 1 may be indifferently either project A or B, as they have identical statistical properties.

4.1. Optimal Stopping of the Last Project by the Team

Value Equation. At time t, conditional on a true value of the prize hazard rate λ, unknown to the players, and on no prize arrival to date, the expected value of planning to stop a single project at some future date $T_2 \geq t$ equals

$$U_{2,t}(T_2|\lambda) = \int_t^{T_2} \frac{f(s|\lambda)}{1-F(t|\lambda)} \left[\int_t^s (-c)\, e^{-r(v-t)} dv + e^{-r(s-t)} b\right] ds + \frac{1-F(T_2|\lambda)}{1-F(t|\lambda)} \int_t^{T_2} (-c)\, e^{-r(v-t)} dv$$

where the subscript "2" denotes the relevance of this value for the second project, the first project being already off line. The first term is the expected discounted return in case the prize arrives before the project is stopped. Here $f(s|\lambda)/[1-F(t|\lambda)] = \lambda e^{-\lambda(s-t)}$ is the density of the prize arrival time, conditioned on no prize having arrived so far. The second term is the expected discounted return in case the prize does not arrive by the planned quitting time T_2, premultiplied by the chance that this happens $[1 - F(T_2|\lambda)] / [1 - F(t|\lambda)] = e^{-\lambda(T_2 - t)}$. The team, having stopped the first project at calendar time T_1, plans at time $t \geq T_1$ to stop the second project at time $T_2 \geq t$ to maximize the expectation of this value given posterior beliefs, namely

$$\max_{T_2 \geq t} W_{2,t}(T_2|x, y, T_1) = \max_{T_2 \geq t} \int_\Lambda U_{2,t}(T_2|\lambda)\, \pi_{t,T_1}(\lambda|x, y) d\lambda, \tag{4.1}$$

where recall that $\pi_{t,T_1}(\lambda|x, y)$ denotes the density of posterior beliefs conditional on the signal realizations x, y being known exactly, and on the facts that neither prize had arrived by time T_1, when the first project was stopped, and that the second project running alone did not arrive in $[T_1, t)$ either. The optimal stopping time of the second project, after stopping the first at T_1, and as planned at time $t \geq T_1$, is thus:

$$T^*_{2,t}(x, y, T_1) = \arg\max_{T_2 \geq t} W_{2,t}(T_2|x, y, T_1).$$

In order to determine this optimal choice, we may differentiate the value $W_{2,t}(T_2|x, y, T_1)$ with respect to current time t and obtain a differential equation for the value, which is the Bellman equation for this problem. However, we choose to work directly on the integral form of the value, as written above, for two reasons. First, the integral form allows to solve for the value function without having to guess its functional form, as is commonly done in dynamic programming or in solving differential equations. Second, this approach is constructive, thus rigorous and transparent, and needs no indirect arguments based on the applicability of recursive methods, which often lack sufficient conditions for an optimum.

First-Order Condition. Notice that the team's expected value $W_{2,t}(T_2|x, y, T_1) = \mathbb{E}_{t,T_1}[U_{2,t}(T_2|\lambda)|\, x, y]$ of stopping the second project at time T_2, after stopping the first at time T_1, depends on T_2 only through the integrand value $U_{2,t}(T_2|\lambda)$ conditional on λ. Therefore, $W_{2,t}(T_2|x, y, T_1)$, just like $U_{2,t}(T_2|\lambda)$, is $\mathbb{C}^2$ in T_2 for every $T_2 \geq t$ and every x, y, t, T_1. To find the team's optimal stopping time of the remaining running project, we take a derivative of the expected value function (4.1). Since this type of manipulations will be used repeatedly in later omitted proofs, it is instructive to go through them at least once:

$$\frac{dW_{2,t}(T_2|x, y, T_1)}{dT_2} = \int_\Lambda \frac{d}{dT_2} U_{2,t}(T_2|\lambda)\, \pi_{t,T_1}(\lambda|x, y)\, d\lambda$$

$$= \int_\Lambda \left[\frac{f(T_2|\lambda)}{1 - F(t|\lambda)} \left(\int_t^{T_2} (-c)\, e^{-r(v-t)} dv + e^{-r(T_2 - t)} b \right) - \frac{f(T_2|\lambda)}{1 - F(t|\lambda)} \int_t^{T_2} (-c)\, e^{-r(v-t)} dv \right.$$

$$+\frac{1-F(T_2|\lambda)}{1-F(t|\lambda)}(-c)\,e^{-r(T_2-t)}\Bigg]\frac{\pi(\lambda)\,h(x|\lambda)\,h(y|\lambda)\,[1-F(t|\lambda)]\,[1-F(T_1|\lambda)]}{\int_\Lambda \pi(\lambda')\,h(x|\lambda')\,h(y|\lambda')\,[1-F(t|\lambda')]\,[1-F(T_1|\lambda')]\,d\lambda'}d\lambda$$

$$\begin{aligned}
&= \Sigma_{T_2-t}(T_2,T_1,t,T_1,x,y)\int_\Lambda\left[\frac{f(T_2|\lambda)}{1-F(T_2|\lambda)}b-c\right]\cdot \\
&\quad \cdot\frac{\pi(\lambda)\,h(x|\lambda)\,h(y|\lambda)\,[1-F(T_2|\lambda)]\,[1-F(T_1|\lambda)]}{\int_\Lambda \pi(\lambda')\,h(x|\lambda')\,h(y|\lambda')\,[1-F(T_2|\lambda')]\,[1-F(T_1|\lambda')]\,d\lambda'}d\lambda \\
&= \Sigma_{T_2-t}(T_2,T_1,t,T_1,x,y)\,(b\mathbb{E}_{T_2,T_1}[\lambda|\,x,y]-c) \\
&\propto b\mathbb{E}_{T_2,T_1}[\lambda|\,x,y]-c
\end{aligned}$$

where recall $\mathbb{E}_{T_2,T_1}[\lambda|\,x,y]$ is the expected value of λ conditional on the posterior beliefs $\pi_{T_2,T_1}(\lambda|x,y)$, and we introduce the normalizing factor:

$$\Sigma_{T_2-t}(T_2,T_1,t,T_1,x,y)\equiv e^{-r(T_2-t)}\frac{\int_\Lambda \pi(\lambda)\,h(x|\lambda)\,h(y|\lambda)\,[1-F(T_2|\lambda)]\,[1-F(T_1|\lambda)]\,d\lambda}{\int_\Lambda \pi(\lambda')\,h(x|\lambda')\,h(y|\lambda')\,[1-F(t|\lambda')]\,[1-F(T_1|\lambda')]\,d\lambda'}>0.$$

Therefore, the FOC simply equates the posterior expected hazard rate of the remaining prize to the cost/benefit ratio:

$$b\mathbb{E}_{T_2,T_1}[\lambda|\,x,y]=c. \tag{4.2}$$

Intuitively, the marginal cost c of proceeding an extra instant must equal the marginal benefit, which consists of the prize b multiplied by its expected hazard rate conditional on all available information. Due to exponential discounting, this condition is independent of the planning time t; therefore an optimal stopping time $T_2\in[T_1,\infty)$, if it exists, is time-consistent.

Optimal Stopping Time. We now determine the optimal stopping time of the second project, $T^*_{2,t}(x,y,T_1)$ for any signal pair x,y, current calendar time t, and time T_1 when the first project was stopped. Since

$$\mathbb{E}_{T_2,T_1}[\lambda|\,x,y]=\frac{\beta+2}{\alpha-x-y+T_1+T_2},$$

the FOC (4.2) yields the unique solution:

$$T_2(x,y,T_1)=\frac{b}{c}(\beta+2)+x+y-\alpha-T_1. \tag{4.3}$$

Lemma 3. *For every pair of signals x,y, if the team has stopped the first project at time T_1, the optimal stopping time of the second project is*

$$T^*_2(x,y,T_1)=\max\{T_1,\frac{b}{c}(\beta+2)+x+y-\alpha-T_1\}.$$

Proof. We are left to verify a sufficient condition. For every x, y, t, T_1,

$$\frac{dW_{2,t}(T_2|x,y,T_1)}{dT_2} \propto b\mathbb{E}_{T_2,T_1}[\lambda|\, x, y] - c = b\frac{\beta+2}{\alpha - x - y + T_1 + T_2} - c,$$

this immediately implies:

$$T_2 < (>) \frac{b}{c}(\beta+2) + x + y - \alpha - T_1 \Longleftrightarrow \frac{dW_{2,t}(T_2|x,y,T_1)}{dT_2} > (<) 0.$$

Therefore $T_2(x, y, T_1)$ if positive, is the unique local and thus global maximum of the value function $W_{2,t}(T_2|x, y, T_1)$, independently of t. It follows that if $t < T_2(x, y, T_1)$, then the second project is kept open at time t, whereas if $t \geq T_2(x, y, T_1)$, it is optimal to turn the second project off. This in turn implies that the optimal stopping time is $T_2^*(x, y, T_1) = \max\{T_1, T_2(x, y, T_1)\}$. ■

The above Lemma implies that if the team carries on both projects for a long enough time, before the stopping one of them, then it must stop both of them at the same time.

Corollary 1. *If the team stops one project at a time T_1 such that*

$$T_1 \geq \max\left\{0, \frac{1}{2}\left[(\beta+2)\frac{b}{c} + x + y - \alpha\right]\right\} \equiv T^*(x, y), \tag{4.4}$$

then it optimally stops also the second project simultaneously at $T_2^*(x, y, T_1) = T_1$.

The expression for $T^*(x, y)$ (when positive) is intuitive. The stopping time is longer the larger the benefit/cost ratio b/c, the higher the signal realizations x, y, the higher the prior mean β/α and the prior variance β^2/α of beliefs about the hazard rate λ. The variance effect stems from a standard option value of information: stopping later means (in the language of Moscarini and Smith 2001) experimenting more, because one sacrifices payoffs today in the hope of a random return to new knowledge in the future.

4.2. Optimal Stopping of the First Project by the Team

We may now calculate by backward induction the optimal stopping time of the first project $T_1^*(x, y)$. Specifically, we will show that the team's optimal policy always prescribes to stop both projects simultaneously, so that $T_1^*(x, y) = T^*(x, y) = T_2^*(x, y, T_1^*(x, y))$.

Let $\hat{F}(s|\lambda)$ be the c.d.f. associated with the arrival of the prize to either one of the two projects at time s, thus

$$\hat{F}(s|\lambda) = 1 - (1 - F(s|\lambda))^2, \text{ with density } \hat{F}'(s|\lambda) = \hat{f}(s|\lambda) = 2f(s|\lambda)(1 - F(s|\lambda)).$$

For any $t \geq 0$, conditional on a known value of λ, the value of stopping the first project at time T_1 equals:

$$\begin{aligned} U_{1,t}(T_1|\lambda) &= \int_t^{T_1} \frac{2f(s|\lambda)(1-F(s|\lambda))}{(1-F(t|\lambda))^2}\left[\int_t^s (-2c)\, e^{-r(v-t)}dv + e^{-r(s-t)}b\right] ds \\ &\quad + \frac{(1-F(T_1|\lambda))^2}{(1-F(t|\lambda))^2}\left[\int_t^{T_1} (-2c)\, e^{-r(v-t)}dv + e^{-r(T_1-t)}U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)\right] \end{aligned}$$

where the first line collects payoffs in case the prize arrives while both projects run together (before T_1), and the second line the cost of running two projects fruitlessly until T_1 and then collecting the payoff of continuing optimally with one project from that moment forward. The team plans at time t a stopping time T_1 to maximize the expectation of this value given current posterior beliefs:

$$W_{1,t}(T_1|x,y) = \int_\Lambda U_{1,t}(T_1|\lambda)\, \pi_{t,t}(\lambda|x,y)\, d\lambda.$$

The optimal stopping time as planned at t is thus:

$$T_{1,t}^*(x,y) = \arg\max_{T_1 \geq t} W_{1,t}(T_1|x,y).$$

To find the optimal stopping time, again, we take a derivative of this value function $W_{1,t}$ with respect to T_1, and present the final result of substantial manipulations in the following:

Lemma 4. *For every pair of signals x, y, and any time $t \geq 0$, the marginal value of waiting to stop the first project is proportional to*

$$\frac{dW_{1,t}(T_1|x,y)}{dT_1} \propto -c + \mathbb{E}_{T_1,T_1}\left[\lambda\left(b - U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)\right)|x,y\right].$$

Intuitively, by delaying the stopping time T_1 of the first project, the team pays the flow cost c and receives the expected marginal benefit $\mathbb{E}_{T_1,T_1}\left[\lambda\left(b - U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)\right)|x,y\right]$. At hazard rate λ, the prize arrives and the benefit b is incurred, but on the other hand the continuation value $U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)$ of proceeding with only one project is lost. Notice that $U_{2,T_1}(T_2^*(x,y,T_1)|\lambda) < b$, as in the continuation the team can earn at most b, and not immediately a.s. Furthermore, the hazard rate λ of the prize and the continuation value conditional on λ, namely $U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)$, are multiplied *within* the posterior expectation: the team cares about their covariance induced by the common dependence on λ. If both projects are stopped together, namely if T_1 is such that $T_2^*(x,y,T_1) = T_1$, then clearly the continuation value of the second project alone is zero: $U_{2,T_1}(T_2^*(x,y,T_1)|\lambda) = 0$. In this case the familiar expression $dW_{1,t}(T_1|x,y)/dT_1 \propto (b\mathbb{E}_{T_1,T_1}[\lambda|\, x,y] - c)$ obtains. Lemma 4 and Corollary 1 together immediately imply that the optimal stopping time

$T_1^*(x,y)$ of the first project cannot exceed the magnitude $T^*(x,y)$ defined in (4.4). In fact, by Corollary 1, if $T_1 > T^*(x,y)$, then $T_2^*(x,y,T_1) = T_1$, and thus by Lemma 4:

$$\begin{aligned}\frac{dW_{1,t}(T_1|x,y)}{dT_1} &\propto -c + \mathbb{E}_{T_1,T_1}\left[\lambda\left(b - U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)\right)|x,y\right] \\ &= -c + b\mathbb{E}_{T_1,T_1}\left[\lambda|x,y\right] = -c + b\frac{\beta+2}{\alpha - x - y + T_1 + T_1} \\ &< -c + b\frac{\beta+2}{\alpha - x - y + 2T^*(x,y)} = 0.\end{aligned}$$

If parameters are such that $T^*(x,y) = 0$, then $T_2^*(x,y,T_1^*(x,y)) = T_1^*(x,y) = 0$ and no project is ever started. But, in the case that $T^*(x,y) > 0$, the key question is still whether it is best for the team to stop the two projects simultaneously at time $T^*(x,y)$, or to stop them sequentially, so that $T_1^*(x,y) < T_2^*(x,y,T_1^*(x,y))$. By Corollary 1 sequential stopping requires that $T_1^*(x,y) < T^*(x,y)$; the following Lemma shows that this inequality is in fact impossible.

Lemma 5. *For every pair of signals x,y such that $T^*(x,y) > 0$, the optimal stopping time $T_1^*(x,y)$ of the first project cannot be smaller than $T^*(x,y)$.*

The results of this section are summarized in the following Proposition.

Proposition 1. ***(The Team Solution)*** *For every pair of signals x,y on the unobserved promise λ of the two projects, the team optimally stops both projects simultaneously at time*

$$T_1^*(x,y) = T_2^*(x,y,T_1^*(x,y)) = \max\{0, T^*(x,y)\},$$

where

$$T^*(x,y) \equiv \frac{1}{2}[(\beta+2)\,\frac{b}{c} + x + y - \alpha].$$

5. Equilibrium in Semi-Separating Strategies

5.1. Definition

In the game, each player observes only her own private signal and whether her rival is still in the race. Owing to the winner-take-all assumption, no player would reveal her private signal truthfully to the opponent. Hence, private information may only be revealed through quitting decisions. A player draws information from elapsing time in two ways. First, she verifies that no prize has arrived (when one prize arrives, the game is over); second, she can see whether the opponent is still in the game or not. The first piece of information is always bad news, because posterior beliefs deteriorate in a FSD sense by Lemma 2;

the second is either bad or good news, depending on the monotonicity of the opponent's quitting strategy in her private signal.

For each player $i = A, B$, a pure strategy in this game is a pair of functions (τ_1^i, τ_2^i). We focus on symmetric equilibria, hence we omit the superscript i from the strategies.[8] The stopping time function as "quitter no. 1", given own private signal *and* that the opponent is still in the race, is denoted by $\tau_1 : \mathbb{R}_- \to \overline{\mathbb{R}}_+$. Here, for any x, the stopping time $\tau_1(x)$ prescribes that the player stays in the race until time $\tau_1(x)$ unless observing that the opponent has left the race at any time $\hat{\tau} < \tau_1(x)$. The choice of the extended positive real numbers $\overline{\mathbb{R}}_+$ as the range of τ_1 is made to allow for the possibility that a player may decide to stay in the game and wait for the prize forever, given some signal realization x, and given that the opponent is not leaving the game either. Note that the stopping time $\tau_1(x) = 0$ prescribes that the player should not enter the race at all.

The stopping time function as "quitter no. 2", given own private signal and that the opponent has already left the game, is denoted by $\tau_2 : \mathbb{R}_- \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$. Here $\tau_2(x, \hat{\tau}) \geq \hat{\tau}$ describes the player's stopping time when holding signal x and after the opponent has quit at time $\hat{\tau}$. We restrict attention to strategies where τ_1 satisfies the following monotonicity requirement: there exist $\underline{x}$, $\bar{x} \leq 0$ such that

$$\tau_1(x) = \begin{cases} 0 & \text{if } x \leq \underline{x} \\ \text{positive and strictly increasing} & \text{if } \underline{x} < x \leq \bar{x} \\ \infty & \text{if } x > \bar{x}. \end{cases}$$

In particular, if $\bar{x} = 0$, then the stopping time $\tau_1(x)$ is finite for any x. We denote by g the inverse function of τ_1 on the domain $[\underline{x}, \bar{x}]$. We look for a *Symmetric Monotone Perfect Bayesian Equilibrium* (SMPBE): both players adopt the same equilibrium strategy (τ_1^*, τ_2^*), which is a best response to itself, and τ_1^* also satisfies the above monotonicity requirement with thresholds denoted by $\underline{x}^*$, $\bar{x}^*$. The equilibrium is semi-separating, because not entering the game reveals to the opponent only an upper bound $\underline{x}^*$ to the observed private signal.

5.2. Equilibrium Play After the Opponent Quits

In a SMPBE, when a player enters the game at time 0, and then quits first at time $\hat{\tau} > 0$ according to the equilibrium strategy τ_1^*, the remaining player perfectly infers

[8] Without loss in generality, we can restrict attention to time-consistent stopping times chosen by the players, function of the private signal and of how many players are left in the game, but not of calendar time. In fact, consider (for the sake of illustration) player A; even if he revises his stopping decision as time goes by, all that matters to player B is when player A does quit, because player B observes only player $A's$ actions, not his intentions. So any previous plans made by A and later revised are immaterial: player B cares about $A's$ decision to stop at time τ as planned at time τ.

her opponent's private information $g^*(\hat{\tau})$. From that moment on, the remaining player updates beliefs and solves a single-project decision problem equivalent to that of the team. It follows immediately that the optimal stopping time of the last player is given by the team's stopping rule, conditional on the first player having left at $\hat{\tau} > 0$, namely $\tau_2^*(y, \hat{\tau}) = T_2^*(y, g(\hat{\tau}), \hat{\tau})$.

Proposition 2. ***(Equilibrium Play after the Opponent Has Quit)*** *In any SMPBE, for any $\hat{\tau} > 0$, the optimal stopping time of a player with signal x, after the opponent quits at time $\hat{\tau}$ and reveals her private information $g^*(\hat{\tau})$, equals the team's optimal stopping time of the second project conditional on the same information:*

$$\tau_2^*(x, \hat{\tau}) = \max\{\hat{\tau}, \frac{b}{c}(\beta + 2) + x + g^*(\hat{\tau}) - \alpha - \hat{\tau}\}.$$

As mentioned, if one player even fails to join the game, then her private signal cannot be perfectly inferred by the opponent, because the equilibrium strategy τ_1^* is not invertible for $x \leq \underline{x}^*$, or $\hat{\tau} = 0$. Assuming that player B does not join the game, for the sake of illustration, the expected continuation value of player A at time t for quitting at time τ^A may be expressed as:

$$W_{2,t}(\tau^A | x, g^*(0) -, 0) \equiv \int_\Lambda U_{2,t}\left(\tau^A | \lambda\right) \pi(\lambda | z_A = x, z_B \leq g^*(0), t_A \geq t, t_B \geq 0) d\lambda,$$

where by definition, $g^*(0) = \underline{x}^*$. This allows to conclude the following:

Proposition 3. ***(Equilibrium Play after the Opponent Has Not Joined the Game)*** *In any SMPBE, the optimal stopping time of a player with signal x, conditional on the opponent not having joined the game, equals*

$$\tau_2^*(x, 0) = \max\{0, \frac{b}{c}(\beta + 1) + x + \underline{x}^* - \alpha\}.$$

Since entering the game for an arbitrarily small length of time, and then quitting, perfectly reveals own private information x, while not joining the game at all only reveals an upper bound $\underline{x}^*$ to x, there is a natural discontinuity in the equilibrium strategy $\tau_2^*(x, \hat{\tau})$ at $\hat{\tau} = 0$. In fact, for any $x > \alpha - (\beta + 2) b/c - \underline{x}^*$, so that entering the game for some time is optimal,

$$\lim_{\hat{\tau} \downarrow 0} \tau_2^*(x, \hat{\tau}) = \frac{b}{c}(\beta + 2) + x + \underline{x}^* - \alpha > \max\left\{0, \frac{b}{c}(\beta + 1) + x + \underline{x}^* - \alpha\right\} = \tau_2^*(x, 0).$$

5.3. Equilibrium Play Before the Opponent Quits

The most complex part of the equilibrium characterization concerns the earlier phase of the game, when both players are still in the game. Each player must plan an optimal stopping time based on the hypothesis that the opponent will quit later, and on the resulting information about the opponent's private information.

The Value Function. We first determine the value function of a player at any time $t > 0$ for quitting at time $\tau \geq t$, conditional on the facts that opponent has not quit yet at time τ and is adopting a monotonic strategy (τ_1, τ_2), with associated inverse $g = \tau_1^{-1}$. We need to distinguish four events that can take place at any time s between the current time t and any future date τ at which the player plans to quit first (i.e. provided the opponent has not quit by then). We consider the problem of player A who contemplates stopping first at time τ^A, the other player's calculations being symmetric.

1. Player A'a prize arrives at $t_A \in [t, \tau^A)$, before (the prize arrives to the rival at time) t_B and before (the opponent quits first at time) τ^B. In this case, A wins and takes all, the game is over at time t_A. Conditional on the true arrival rate λ, the c.d.f. of this event is

$$\begin{aligned}\Pr(t_A &\leq s, t_B > t_A, \tau^B > t_A | t_A > t, t_B > t, \tau^B > t, \lambda) = \\ \Pr(t_A &\leq s | t_A > t, \lambda) \Pr(t_B > t_A | t_B > t, \lambda) \Pr(\tau^B > t_A | \tau^B > t, \lambda) \\ &= \frac{F(s|\lambda)}{1 - F(t|\lambda)} \frac{1 - F(t_A|\lambda)}{1 - F(t|\lambda)} \frac{1 - H(g(t_A)|\lambda)}{1 - H(g(t)|\lambda)}\end{aligned}$$

   The density of this event for $t_A = s \in [t, \tau^A]$ is

$$\frac{d}{ds}\left[\frac{F(s|\lambda)}{1 - F(t|\lambda)} \frac{1 - F(t_A|\lambda)}{1 - F(t|\lambda)} \frac{1 - H(g(t_A)|\lambda)}{1 - H(g(t)|\lambda)}\right]_{s=t_A} = \frac{f(s|\lambda)}{1 - F(t|\lambda)} \frac{1 - F(s|\lambda)}{1 - F(t|\lambda)} \frac{1 - H(g(s)|\lambda)}{1 - H(g(t)|\lambda)}$$

2. Player B's prize arrives at $t_B \in [t, \tau^A)$, before A's prize arrives at t_A and before A quits at τ^A. As a result, B wins and takes all, the game is over at time t_B. Reasoning as above, the density of this event for $t_A = s \in [t, \tau^A]$ is

$$\frac{f(s|\lambda)}{1 - F(t|\lambda)} \frac{1 - F(s|\lambda)}{1 - F(t|\lambda)} \frac{1 - H(g(s)|\lambda)}{1 - H(g(t)|\lambda)}.$$

3. Player B quits at $\tau^B \in [t, \tau^A)$ first, i.e. before either prize arrives. Then the signal $z_B = y$ is revealed to A by inverting $y = g(\tau^B)$. The density of this event for $t_A = s \in [t, \tau^A]$ is

$$\frac{h(g(s)|\lambda) g'(s)}{1 - H(g(t)|\lambda)} \left(\frac{1 - F(s|\lambda)}{1 - F(t|\lambda)}\right)^2.$$

4. Nothing happens in the time interval $[t, \tau^A)$: no one quits and no prize arrives. In such a case, player A quits at τ^A. The probability of this event is

$$\begin{aligned} & \Pr\left(t_A > \tau^A | t_A > t, \lambda\right) \Pr\left(t_B > \tau^A | t_B > t, \lambda\right) \Pr\left(\tau^B > \tau^A | \tau^B > t, \lambda\right) \\ = & \left(\frac{1 - F\left(\tau^A | \lambda\right)}{1 - F\left(t | \lambda\right)}\right)^2 \frac{1 - H(g\left(\tau^A\right) | \lambda)}{1 - H(g\left(t\right) | \lambda)} \end{aligned}$$

Each of the four events has associated a corresponding PDV of payoffs. We can write the expected value at time t for planning at time t to stop at some time $\tau > t$, conditional on λ. Following the order of the four events, and using the subscript "1" to denote the stopping time as quitter no. 1:

$$\begin{aligned} Q_{1,t}\left(\tau | \lambda\right) = & \int_t^\tau \frac{f\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)} \frac{1 - F\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)} \frac{1 - H(g\left(s\right) | \lambda)}{1 - H(g\left(t\right) | \lambda)} \left[\int_t^s \left(-c\right) e^{-r(v-t)} dv + e^{-r(s-t)} b\right] ds \\ & + \int_t^\tau \frac{f\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)} \frac{1 - F\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)} \frac{1 - H(g\left(s\right) | \lambda)}{1 - H(g\left(t\right) | \lambda)} \left[\int_t^s \left(-c\right) e^{-r(v-t)} dv\right] ds \\ & + \int_t^\tau \left(\frac{1 - F\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)}\right)^2 \frac{h(g\left(s\right) | \lambda) g'\left(s\right)}{1 - H(g\left(t\right) | \lambda)} \left[-\int_t^s c e^{-r(v-t)} dv + e^{-r(s-t)} W_{s,s}(\tau_2^*(x,s) | x, g(s))\right] ds \\ & + \left(\frac{1 - F\left(\tau | \lambda\right)}{1 - F\left(t | \lambda\right)}\right)^2 \frac{1 - H(g\left(s\right) | \lambda)}{1 - H(g\left(t\right) | \lambda)} \left[-\int_t^\tau c e^{-r(v-t)} dv\right] \end{aligned}$$

which can be compacted as follows:

$$\begin{aligned} = & \int_t^\tau \frac{f\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)} \frac{1 - F\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)} \frac{1 - H(g\left(s\right) | \lambda)}{1 - H(g\left(t\right) | \lambda)} \left[\int_t^s \left(-2c\right) e^{-r(v-t)} dv + e^{-r(s-t)} b\right] ds \\ & + \int_t^\tau \left(\frac{1 - F\left(s | \lambda\right)}{1 - F\left(t | \lambda\right)}\right)^2 \frac{h(g\left(s\right) | \lambda) g'\left(s\right)}{1 - H(g\left(t\right) | \lambda)} \left[\int_t^s \left(-c\right) e^{-r(v-t)} dv + e^{-r(s-t)} W_{2,s}(\tau_2^*(x,s) | x, g(s), s)\right] ds \\ & + \left(\frac{1 - F\left(\tau | \lambda\right)}{1 - F\left(t | \lambda\right)}\right)^2 \frac{1 - H(g\left(\tau\right) | \lambda)}{1 - H(g\left(t\right) | \lambda)} \left[\int_t^\tau \left(-c\right) e^{-r(v-t)} dv.\right] \end{aligned}$$

Here $W_{2,s}(\tau_2^*(x,s) | x, g(s), s)$ is the expected continuation value if the opponent leaves at s and the player optimally stops at $\tau_2^*(x,s) \geq s$, given that no prize has arrived by s. At any time $t > 0$, the expected value of this strategy given t-current posterior beliefs is therefore

$$V_{1,t}\left(\tau | x\right) = \int_\Lambda Q_{1,t}\left(\tau | \lambda\right) \pi_{t,t}\left(\lambda | x, g\left(t\right) +\right) d\lambda.$$

Given that the opponent adopts a monotonic strategy τ_1^*, with inverse g^*, quitting first at time $\tau_1 = \tau_1^*(x)$ after observing private signal $x = g^*(\tau_1)$, each player chooses the optimal stopping time as quitter no. 1 by solving

$$\tau_{1,t}^*(x) = \arg\max_{\tau \geq t} V_{1,t}\left(\tau | x\right).$$

The First-Order Condition: Necessity and Sufficiency. In order to find the optimal stopping time of a player before the opponent quits, we differentiate the expected value $V_{1,t}(\tau|x)$ with respect to the stopping time τ:

$$
\begin{aligned}
V'_{1,t}(\tau|x) = &\int_\Lambda \Bigg[\frac{f(\tau|\lambda)}{1-F(t|\lambda)}\frac{1-F(\tau|\lambda)}{1-F(t|\lambda)}\frac{1-H(g(\tau)|\lambda)}{1-H(g(t)|\lambda)}\left(\int_t^\tau(-2c)\,e^{-r(v-t)}dv+e^{-r(\tau-t)}b\right)\\
&+\left(\frac{1-F(\tau|\lambda)}{1-F(t|\lambda)}\right)^2\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(t)|\lambda)}\left[\int_t^\tau(-c)\,e^{-r(v-t)}dv+e^{-r(\tau-t)}W_{2,\tau}(\tau_2^*(x,\tau)|x,g(\tau),\tau)\right]\\
&-\frac{2(1-F(\tau|\lambda))\,f(\tau|\lambda)\,[1-H(g(\tau)|\lambda)]+[1-F(\tau|\lambda)]^2\,h(g(\tau)|\lambda)g'(\tau)}{[1-F(t|\lambda)]^2\,[1-H(g(t)|\lambda)]}\int_t^\tau(-c)\,e^{-r(v-t)}dv\\
&+\left(\frac{1-F(\tau|\lambda)}{1-F(t|\lambda)}\right)^2\frac{[1-H(g(\tau)|\lambda)]}{[1-H(g(t)|\lambda)]}(-c)\,e^{-r(\tau-t)}\Bigg]\pi_{t,t}(\lambda|x,g(t)+)\,d\lambda
\end{aligned}
$$

$$
\begin{aligned}
= &\int_\Lambda \Bigg[\frac{f(\tau|\lambda)}{1-F(t|\lambda)}\frac{1-F(\tau|\lambda)}{1-F(t|\lambda)}\frac{1-H(g(\tau)|\lambda)}{1-H(g(t)|\lambda)}e^{-r(\tau-t)}b\\
&+\left(\frac{1-F(\tau|\lambda)}{1-F(t|\lambda)}\right)^2\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(t)|\lambda)}e^{-r(\tau-t)}W_{2,\tau}(\tau_2^*(x,\tau)|x,g(\tau),\tau)\\
&+\left(\frac{1-F(\tau|\lambda)}{1-F(t|\lambda)}\right)^2\frac{[1-H(g(\tau)|\lambda)]}{[1-H(g(t)|\lambda)]}(-c)\,e^{-r(\tau-t)}\Bigg]\pi_{t,t}(\lambda|x,g(t)+)\,d\lambda
\end{aligned}
$$

$$
= \Sigma_{t,t}(\tau,x,g(\tau)+)\int_\Lambda\left\{\frac{f(\tau|\lambda)}{1-F(\tau|\lambda)}b+\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)}W_{2,\tau}(\tau_2^*(x,\tau)|x,g(\tau),\tau))-c\right\}\pi_{\tau,\tau}(\lambda|x,g(\tau)+)d\lambda
$$

$$
\propto b\mathbb{E}_{\tau,\tau}[\lambda|\,x,g(\tau)+]+W_{2,\tau}(\tau_2^*(x,\tau)|x,g(\tau),\tau)\mathbb{E}_{\tau,\tau}\left[\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)}\middle|\,x,g(\tau)+\right]-c \quad (5.1)
$$

where we introduce a new normalizing factor

$$
\Sigma_{t,t}(\tau,x,g(\tau)+)\equiv e^{-r(\tau-t)}\frac{\int_\Lambda\pi(\lambda)\,h(x|\lambda)\,[1-H(g(\tau)|\lambda)]\,[1-F(\tau|\lambda)]^2\,d\lambda}{\int_\Lambda\pi(\lambda)\,h(x|\lambda)\,[1-H(g(t)|\lambda)]\,[1-F(t|\lambda)]^2\,d\lambda'}>0.
$$

The last line of Equation (5.1) describes the marginal value at τ of waiting before dropping out of the race. It depends on the flow cost $-c$ and on two flow benefit terms, the first one is the prize value b times the expected hazard rate $\mathbb{E}_{\tau,\tau}[\lambda|\,x,g(\tau)+]$ of prize arrival, and the second one is the product of the expected hazard rate $\mathbb{E}_{\tau,\tau}\left[\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)}\middle|\,x,g(\tau)+\right]$ of the opponent leaving the game and of the returns from this event, expressed by the continuation value $W_{2,\tau}(\tau_2^*(x,\tau)|x,g(\tau),\tau)$.

Remarkably, *the possibility that the opponent receives the prize does not enter the marginal value of waiting to quit the game first.* The derivative $V'_{1,t}(\tau|x)$ captures the difference in value at τ between, on the one hand, staying in the race for an extra $\Delta\tau$ and then leaving at time $\tau+\Delta\tau$, and on the other hand of leaving immediately at τ, for $\Delta\tau$

small. In this period of time $\Delta\tau$, the cost $c\Delta\tau$ is paid upfront and sunk, and either one of the two prizes of size b or $W_{2,\tau}(\tau_2^*(x,\tau)|x, g(\tau), \tau)$ may arrive. But nothing may arrive, in which case one quits, or the opponent may win in the meantime: either way, the payoff is zero. It is immaterial to this cost-benefit analysis that the prize could also arrive to the opponent: the only effect of this is to end to game before $\tau + \Delta\tau$ at no further cost, nor benefit to the player. Since the player is anyway ending the game at time $\tau + \Delta\tau$, the arrival of the prize to the opponent at any given $t \in (\tau, \tau + \Delta\tau)$ bears no change in the player's marginal value for waiting as $\Delta\tau$ vanishes. If the loser of the race were either to pay a fine or to receive a runner-up prize, then this remarkable fact would not be true.

At $\Delta\tau = 0$, or $t = \tau$, we know that the opponent prize has not arrived yet by assumption. We now show that a player cannot optimally stay in the game forever, given any signal realization x, and given that the opponent is never going to leave the game with positive probability. This allows us to conclude that there are no SMPBE where the players herd on each other's experimentation so much that they remain in the race forever. Furthermore, we show that it cannot be the case that a player decides to quit the race too early if her signal is good enough.

Lemma 6. *Suppose that player B plays a monotonic strategy τ_1, of quitting first at time $\tau_1(y)$ after privately observing y, with inverse g. For any signal x observed by player A, there exists $\bar{\tau} > 0$ large enough that player A's marginal value of waiting $V'_{1,t}(\tau|x)$ is negative for any $\tau \geq \bar{\tau}$. For any x such that $b\mathbb{E}_{0,0}[\lambda|\, x, g(0)+] > c$, there exists $\underline{\tau} > 0$ small enough that $V'_{1,t}(\tau|x) > 0$ for any $t \leq \tau \leq \underline{\tau}$.*

Lemma 6 shows that if the opponent plays a monotonic strategy with inverse g, then a player that happens to be in the race at time $t > 0$, and that is endowed with a signal x such that $b\mathbb{E}_{0,0}[\lambda|\, x, g(0)+] > c$, optimally chooses to stop at a finite time $\tau_{1,t}(x) = \max\{t, \tau_1(x)\}$, where the stopping time $\tau_1(x)$ must satisfy the first-order condition $V'_{1,t}(\tau|x) = 0$. This is because it is optimal to enter the game, while staying in the game too long eventually has strictly negative marginal value. Since the last line of (5.1) is independent of current time t, this shows that for any observed private signal realization x and any current time t, the sign of the derivative $V'_{1,t}(\tau|x)$ must be independent of t. As a result, for any x and any t, any stopping time $\tau_1(x)$ satisfying the FOC $V'_{1,t}(\tau|x) = 0$ must be independent of t, and hence must be time consistent. All this implies the following: if the opponent plays a monotonic strategy with inverse g, a player who chooses to enter the race and holds a signal x such that $b\mathbb{E}_{0,0}[\lambda|\, x, g(0)+] > c$, must optimally plan to quit the game at a time τ that satisfies the First-Order Condition $V'_{1,t}(\tau|x) = 0$.

Equilibrium Characterization. The key result of this subsection is a new type of winner's curse, that we identify in any monotonic equilibrium of this class of optimal-stopping games of conflicting interests. Suppose that (say) player B adopts a monotonic strategy g, and that player A endowed with signal x plans, in the event that B remains in the race, to quit first at a time τ which satisfies the FOC $V'_{1,t}(\tau|x) = 0$. Then, if B quits first at any time $\hat{\tau}$ earlier than but close enough to τ, then A must also immediately leave the race, *regretting not having left earlier:* formally, $T_2^*(x, g(\hat{\tau}), \hat{\tau}) = \hat{\tau}$. The intuition behind this result is simple. When player A plans to remain in the race at any time $t < \tau$, and to leave exactly at τ, she conditions on player B still being in the race and hence on the expectation $\mathbb{E}[z_B|z_B \geq g(t)]$ with respect to B's signal. If in fact B quits first at a time $\hat{\tau}$ when A is about to do so at τ, namely $\hat{\tau}$ is close to but smaller than τ, then A at $\hat{\tau}$ suddenly realizes that B had observed signal $z_B = g(\hat{\tau})$, which is much smaller than $\mathbb{E}[z_B|z_B \geq g(\hat{\tau})]$ and hence smaller also than $\mathbb{E}[z_B|z_B \geq g(\tau)]$. This wake-up call induces a sudden pessimistic revision of $A's$ beliefs; accordingly, A quits immediately after B, regretting her previous over-optimistic expectation of the rival's assessment of the project's feasibility.

Lemma 7. *Suppose that player B plays a monotonic strategy (τ_1, τ_2) with inverse $g = \tau_1^{-1}$. For any signal x, if player A is planning to quit first at time $\tau > 0$ such that $V'_{1,t}(\tau|x) = 0$, then for any $\hat{\tau} < \tau$ but close enough to τ player A's optimal stopping strategy after B quits at $\hat{\tau}$ is to follow suit:*

$$\tau_2^*(x, \hat{\tau}) = \hat{\tau} > T_2(x, g(\hat{\tau}), \hat{\tau}) = \frac{b}{c}(\beta + 2) + x + g(\hat{\tau}) - \hat{\tau} - \alpha$$

and hence A's continuation value is $W_{2,\hat{\tau}}(\tau_2^*(x, \hat{\tau})|x, g(\hat{\tau}), \hat{\tau}) = 0$.

Proof. By definition, $\tau_2^*(x, \hat{\tau}) = \max\{\hat{\tau}, T_2(x, g(\hat{\tau}), \hat{\tau})\}$, where $T_2(x, g(\hat{\tau}), \hat{\tau})$ satisfies:

$$c = b\mathbb{E}_{T,\hat{\tau}}[\lambda|\, x, g(\hat{\tau})] = b\frac{\beta + 2}{\alpha - \hat{x} - g(\hat{\tau}) + T + \hat{\tau}}.$$

Since the RHS is strictly decreasing in T, the claim follows from:

$$c > b\mathbb{E}_{\hat{\tau},\hat{\tau}}[\lambda|\, x, g(\hat{\tau})] = b\frac{\beta + 2}{\alpha - \hat{x} - g(\hat{\tau}) + \hat{\tau} + \hat{\tau}},$$

and since the RHS is continuous in $\hat{\tau}$, it is enough to show that:

$$c > b\mathbb{E}_{\tau,\tau}[\lambda|\, x, g(\tau)].$$

In fact, using the hypothesis and Equation (5.1), we write:

$$\begin{aligned} 0 &= V'_{1,t}(\tau|x) \\ &\propto b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)+\right] + W_{2,\tau}(\tau_2^*(x,\tau)|x, g(\tau), \tau)\mathbb{E}_{\tau,\tau}\left[\left.\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)}\right| x, g(\tau)+\right] - c \\ &\geq b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)+\right] - c \\ &> b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)\right] - c \end{aligned}$$

the first inequality follows because $\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)} > 0$ and $W_{2,\tau}(\tau_2^*(x,\tau)|x, g(\tau), \tau) > 0$, whereas the second inequality follows from Lemma 2. ■

In light of the Lemma 7, for any x, the First-Order Condition $V'_{1,t}(\tau|x) = 0$ can be rewritten very simply as:

$$c = bE_{\tau,\tau}\left[\lambda|x, g(\tau)+\right] = b(\beta+1)\frac{(\alpha+2\tau-x)^{-\beta-2} - (\alpha+2\tau-x-g(\tau))^{-\beta-2}}{(\alpha+2\tau-x)^{-\beta-1} - (\alpha+2\tau-x-g(\tau))^{-\beta-1}}, \quad (5.2)$$

which says that a player quits first when the flow cost equals the expected flow benefit consisting of the value of the prize b times expected hazard rate λ of arrival of the prize, conditional on own private information x, on the opponent's private signal y being larger than $g(\tau)$, and on neither prize having arrived by time τ. In any SMPBE (τ_1^*, τ_2^*), equilibrium symmetry implies that $g^*(\tau_1^*(x)) = x$ (and in particular $g^*(0) = \underline{x}^*$), so that Equation (5.2) is further simplified as:

$$c = bE_{\tau,\tau}\left[\lambda|\, x, x+\right] = b(\beta+1)\frac{\left[(\alpha+2\tau-x)^{-\beta-2} - (\alpha+2\tau-2x)^{-\beta-2}\right]}{\left[(\alpha+2\tau-x)^{-\beta-1} - (\alpha+2\tau-2x)^{-\beta-1}\right]} \quad (5.3)$$

By Lemma 1, the RHS of this equation is strictly decreasing in τ and strictly increasing in x. Therefore, this equation has a unique positive solution $\tau_1(x)$ for any x such that:

$$b\mathbb{E}_{0,0}\left[\lambda|\, x, g^*(0)+\right] = bE_{\tau,\tau}\left[\lambda|\, x, \underline{x}^*+\right] \geq c,$$

and no solutions otherwise. In particular, $\underline{x}^* = g^*(0)$ is uniquely pinned down by

$$c = bE_{0,0}\left[\lambda|\, \underline{x}^*, \underline{x}^*+\right].$$

Also, notice that by $\tau_1(x)$ is increasing in x for any $x \geq \underline{x}^*$, as required by our monotonicity restriction.

So far, our analysis has singled out as the unique candidate SMPBE stopping strategy, the monotone function τ_1^* such that: $\tau_1^*(x)$ is the unique solution of the FOC (5.3) if $bE_{\tau,\tau}\left[\lambda|\, x, \underline{x}^*+\right] > c$, and $\tau_1^*(x) = 0$ otherwise. Lemma 6 has shown that if

$bE_{\tau,\tau}\left[\lambda|\, x, \underline{x}^{*}+\right] > c$ and the opponent adopts the stopping strategy τ_1^*, each player endowed with signal x who happens to be in the race at time t finds it optimal to leave the race at time $\tau_1^*(x) > 0$. So, in order to conclude that the strategy τ_1^* is the unique SMPBE, we are only left to determine the optimal decision at the very beginning of the game, i.e. at time $t = 0$.

Lemma 8. *Suppose that player B plays the first-quitter stopping strategy τ_1^*. If player A holds a signal $x \leq \underline{x}^*$, then at time 0 she optimally chooses not to enter the game. Whereas if $x > \underline{x}^*$, then player A enters the game at time 0, and optimally selects the time-consistent stopping time $\tau_1^*(x)$.*

The following Proposition summarizes our findings for equilibrium play before the opponent quits and, together with Proposition 2, fully characterizes the unique SMPBE $\{\tau_1^*, \tau_2^*\}$.

Proposition 4. ***(Equilibrium Play Before the Opponent Has Quit)*** *The unique SMPBE quitting time, conditional on a private signal x and on the opponent still being in the game, is*

$$\tau_1^*(x) = \begin{cases} 0 & \text{if } x < \underline{x}^* \\ \tau_1(x) & \text{if } x \geq \underline{x}^*, \end{cases}$$

where $\tau_1(x)$ is the unique increasing solution of

$$c = b(\beta+1)\frac{(\alpha+2\tau-x)^{-\beta-2} - (\alpha+2\tau-2x)^{-\beta-2}}{(\alpha+2\tau-x)^{-\beta-1} - (\alpha+2\tau-2x)^{-\beta-1}},$$

and $\underline{x}^* < 0$ is the unique root of $\tau_1(\underline{x}^*) = 0$, namely

$$c = b(\beta+1)\frac{(\alpha-\underline{x}^*)^{-\beta-2} - (\alpha-2\underline{x}^*)^{-\beta-2}}{(\alpha-\underline{x}^*)^{-\beta-1} - (\alpha-2\underline{x}^*)^{-\beta-1}}.$$

6. Welfare Comparison

We may now compare the team solution to the equilibrium outcome, and investigate the welfare properties of the unique SMPBE. We will show that the absence of information-sharing may induce in equilibrium either over-experimentation or under-experimentation, relative to the efficient team's policy, depending on the agreement of the private signal realizations x and y. Specifically, equilibrium play features excess experimentation if x and y are close enough, suboptimal experimentation if x and y are sufficiently far apart. In the analysis presented in this section, without loss of generality, we let $x \leq y$ so that player A is the first to quit in equilibrium, and to avoid triviality we assume that $y > \underline{x}^*$, so that at least one player participates.

Suppose that the more optimistic private signal y is sufficiently close to the other signal x (and that in particular $x > \underline{x}^*$). Our "winner's curse" effect implies that player A's stopping time is optimal, given her own signal x and the information that player B's signal y is larger than x. When player A quits the game, she reveals her signal x to B. If y is in fact very close to x, player B learns that the players have illuded each other into remaining too long in the race. Hence, B immediately follows A suit and quits the game, regretting not having left before. When regret occurs, there is clearly excessive experimentation in equilibrium because both projects are stopped too late.

Lemma 9. *Suppose that $y \geq x$. If*

$$\tau_1^*(x) > T^*(x, y) = \frac{1}{2}\left[(\beta + 2)\,\frac{b}{c} - \alpha + x + y\right] \tag{6.1}$$

then both players' equilibrium experimentation durations are longer than the efficient duration:

$$\tau_2^*(y, \tau_1^*(x)) = \tau_1^*(x) > T^*(x, y).$$

Since $T^*(x, y)$ is increasing in both x and y, given that $y \geq x$, the inequality in Equation (6.1) holds if x and y are sufficiently close. The source of the inefficiency is herding, as the first player to quit puts too much weight on the public information that the opponent is still in the game, and too little weight on her own private information. As is standard in R&D races literature, this inefficiency manifests itself as duplication costs.

More complex is the case when the private signals x and y are sufficiently spread apart. In this case player A underestimates the opponent's signal y and quits too soon. This is the case when

$$\tau_1^*(x) < T^*(x, y) \tag{6.2}$$

which, given that $y \geq x$, is true when x is small enough given y. While clearly player A's equilibrium amount of experimentation is too low, the remaining player B may stay in the race longer than $T^*(x, y)$ and possibly "make up" for player A's underexperimentation. In order to answer this question, we need to distinguish two cases depending on whether or not player A has entered the race at all. The least interesting case occurs when $x \leq \underline{x}^*$ so that $\tau_1^*(x) = 0$, and A does not enter the race. In such a case player B only learns that $x \leq \underline{x}^*$ but cannot precisely figure out x, hence she plays:

$$\tau_2^*(y, 0) = \frac{b}{c}\,(\beta + 1) + y + \underline{x}^* - \alpha.$$

Therefore $\tau_2^*(y,0) \le T^*(x,y)$ if and only if $\frac{b}{c}\beta + y + 2\underline{x}^* - \alpha \le x \le \underline{x}^*$. This configuration identifies a strong but somewhat special instance of underexperimentation. Due to imperfect equilibrium information transmission when the most pessimist player does not even enter the race, both players wind up stopping too soon in equilibrium.[9]

The most interesting situation is the one where $x \ge \underline{x}^*$, so that player A initially enters the race, but then prematurely exits. In this case, player B correctly infers the signal x upon seeing A quitting the game. As a result, player B will surely stay in the race longer than the team $T^*(x,y)$. Since $\tau_1^*(x) < T^*(x,y)$, in fact:

$$\tau_2^*(y,\tau_1^*(x)) = \frac{b}{c}(\beta+2)+g(\tau)+y-\alpha-\tau_1^*(x) > \frac{b}{c}(\beta+2)+x+y-\alpha-T^*(x,y) = T^*(x,y).$$

The first question is whether or not player B will make up for A's too short experimentation duration, or she would even exceeds optimal joint duration. In our Gamma-Poisson model, B's equilibrium stopping time $\tau_2^*(y,\tau)$ is linear in signals and in A's exit time τ. As a consequence, player B's extended duration in the race *exactly offsets* the premature quitting by player A.

Lemma 10. *Suppose that $y > x > \underline{x}^*$. If $\tau_1^*(x) < T^*(x,y)$. Then the total equilibrium experimentation duration (conditional on no prize arrival) exactly equals the optimal team duration:*

$$\tau_2^*(y,\tau_1^*(x)) + \tau_1^*(x) = \frac{b}{c}(\beta+2) + x + y - \alpha = 2T^*(x,y).$$

While total experimentation durations coincide in the equilibrium and in the team solution, a key difference remains. The team solution dictates that the two projects should be *synchronized and stopped simultaneously.* In equilibrium this does not happen as player A exits too soon. As a result, in this case the only difference between the SMPBE strategies and the optimal team's policy consists of a "postponement" of player A's experimentation duration. Since the team is impatient, we claim that this postponement corresponds to suboptimally slowing down the joint rate of innovation arrival. More precisely, we shall devote the remainder of the analysis to show that this postponement corresponds to an instance of suboptimal equilibrium experimentation, whose interpretation is two-fold: both the expected PDV of experimentation costs and the total expected discounted benefit of experimentation are smaller in equilibrium than in the team solution. We decompose the expression for *ex ante* team's welfare $W(T_1,T_2|x,y)$, conditional on stopping the first project at time T_1 and the second one at time T_2, as the difference between expected

[9] Conversely, $\tau_2^*(y,0) > T^*(x,y)$ if and only if $\frac{b}{c}\beta + y + 2\underline{x} - \alpha > x$. As it will be made clear soon, as long as the discount factor r is strictly positive, this induces equilibrium under-experimentation if $\tau_2^*(y,0)$ is close enough to $2T^*(x,y)$, and equilibrium over-experimentation otherwise.

discounted reward, including all possible cases of arrival or non-arrival of the prize, and the expected PDV of costs:

$$W(T_1, T_2|x, y) = B(T_1, T_2|x, y) - C(T_1, T_2|x, y)$$

where the benefits from investment in research are

$$\begin{aligned} B(T_1, T_2|x, y) &= b \int_\Lambda \left[\int_0^{T_1} 2f(s|\lambda)(1 - F(s|\lambda)) e^{-rs} ds \right. \\ &\quad \left. + (1 - F(T_1|\lambda)) \int_{T_1}^{T_2} f(s|\lambda) e^{-rs} ds \right] \pi_{0,0}(\lambda|x, y) d\lambda \end{aligned}$$

and the costs are:

$$\begin{aligned} C(T_1, T_2|x, y) = c \int_\Lambda \left[\int_0^{T_1} 2f(s|\lambda)(1 - F(s|\lambda)) \int_0^s 2e^{-rv} dv ds + (1 - F(T_1|\lambda))^2 \int_0^{T_1} 2e^{-rv} dv \right. \\ \left. + (1 - F(T_1|\lambda)) \int_{T_1}^{T_2} f(s|\lambda) \int_{T_1}^{s} e^{-rv} dv ds + (1 - F(T_1|\lambda))(1 - F(T_2|\lambda)) \int_{T_1}^{T_2} e^{-rv} dv \right] \pi_{0,0}(\lambda|x, y) d\lambda. \end{aligned}$$

We then proceed to show the following result:

Lemma 11. *If $y > x > \underline{x}^*$, and $\tau_1^*(x) < T^*(x, y)$, then both the expected PDV of research costs and of the prize are smaller in equilibrium than in the team solution: $C(\tau_1^*(x), \tau_2^*(y, \tau_1^*(x))|x, y) < C(T^*(x, y), T^*(x, y)|x, y)$ and hence $B(\tau_1^*(x), \tau_2^*(y, \tau_1^*(x))|x, y) < B(T^*(x, y), T^*(x, y)|x, y)$.*

We recall that the source of this key result is a failure of information aggregation. Private information can only be revealed credibly by quitting decisions, therefore (when playing monotonic strategies) it may flow only from the initially more pessimistic player to her opponent, and never vice versa. The analysis of this section is summarized in the following:

Proposition 5. ***(Equilibrium Under- and Over-Experimentation)*** *Given the unique SMPBE (τ_1^*, τ_2^*) with entry signal threshold $\underline{x}^*$ and the team's solution T^*, there exists an increasing continuous function $\xi : (\underline{x}^*, 0] \to (\underline{x}^*, 0]$, implicitly defined by*

$$\tau_1^*(x) = T^*(x, \xi(x)) = \frac{1}{2} \left[(\beta + 2) \frac{b}{c} - \alpha + x + \xi(x) \right],$$

such that $\xi(x) > x$ for any x, and that for any $y > x$:

1. *if $y > \xi(x)$, namely if the two private signals disagree sufficiently, then equilibrium experimentation is suboptimal: both the total expected discounted experimentation costs and the total expected discounted prize are strictly smaller in equilibrium than in the team's solution.*

2. *if instead* $y < \xi(x)$, *then equilibrium experimentation is excessive, so that total expected discounted experimentation costs are strictly larger in equilibrium than in the team's solution.*

7. Discussion and Extensions

Our construction can be extended in several dimensions. In this section we discuss some the most interesting ones. It would go beyond the scope of this single paper to cover all cases thoroughly. While we have some formal arguments to validate robustness, whose proofs are available upon request, we also present reasons to expect our results to survive qualitatively intact, and we trace a map for future research. We remark that our welfare analysis has been so far confined to *conditional* statements, given private information draws. In future research we will address the question of unconditional equilibrium efficiency, at two levels: for a given project promise λ we can average over private information draws using their likelihood, and finally we can average over projects λ drawn from the prior belief distribution.

Deterministic Accumulation of Knowledge over Time. Our model inherits from differential game models of R&D races the assumption that innovation arrival is governed by a Poisson process of parameter λ. This is equivalent to say that the unknown innovation hazard rate $L(t,\lambda) \equiv f(t|\lambda)/[1-F(t|\lambda)]$ is constant over time (and equal to λ). Then, it is natural to postulate conjugate Gamma prior and negative exponentially distributed signals. It is not too difficult, however, to extend our equilibrium construction when the innovation hazard rates $L(t,\lambda)$ of unknown parameter λ is not constant over time. The engine of our equilibrium characterization is Theorem A.1 in the Appendix. Its power goes well beyond the Gamma-Poisson specification of our model.

Make the following mild regularity assumptions: the distribution of λ has connected support Λ (with $\underline{\lambda} = \inf \Lambda$ and $\bar{\lambda} = \sup \Lambda$), the hazard rate of the prize $L(t,\lambda)$ is continuous in t, λ, and strictly increasing in λ for every t, with $L(t,\bar{\lambda}) > c/b > L(t,\underline{\lambda})$, and the density $h(x|\lambda)$ of private signals x is differentiable, with support $X \subseteq \mathbb{R}$, and $\lim_{x\to\sup X} h(x|\lambda) < \infty$. In this environment, it can be shown that our equilibrium characterization presented in Propositions 2 and 4 holds if the following two substantive restrictions are met. First, the signal density $h(x|\lambda)$ is log-supermodular, i.e. the ratio $h'(x|\lambda)/h(x|\lambda)$ is strictly increasing in λ. Second, the expected hazard rates $\mathbb{E}_{t,t'}[L(t,\lambda)|\, x, y]$ and $\mathbb{E}_{t,t'}[L(t,\lambda)|\, x, y+]$ are strictly decreasing in t for any fixed signals x, y and t'; and they both attain the lower bound $L(t,\underline{\lambda})$ in the limit as $t \to \infty$. The first condition is reminiscent of log-supermodularity conditions for equilibrium existence derived in Athey (1999) and in Reny

(1999). The second condition requires that for any fixed pair of signals x, y, as time goes by and the innovation does not arrive, one becomes more and more pessimistic about the promise of the project, both in the case that one knows precisely both signal realizations x and y, or that one only knows one signal realization to be precisely x while that the other signal realization is larger than y. We underline that this is not unduly strong. Among other things, it does not require that the *actual* innovation hazard rate $L(t, \lambda)$ be decreasing over time. As knowledge accumulate throughout the R&D process, the true hazard rate of an innovation is likely to be increasing over time; but it is also quite likely that one becomes less optimistic about its feasibility as time goes by and the innovation does not materialize.

If the cross-partial derivative of the hazard rate function $L(t, \lambda)$ is non-positive (note that in the Poisson case it is exactly to zero), our main welfare prediction extends in qualitative terms provided that the team's optimal solution is to start either no or both projects, as in our model. While we have not formally ruled it out, the possibility that the team's optimal policy entails activating just one project appears quite unlikely. Since the two prize arrivals are independent, keeping two projects open instead of one is equivalent to doubling the intensity of experimentation. As the team is impatient and discounts the future, whenever the marginal trade-off between flow costs and expected flow value is positive, it is plausible that the team will choose to pursue the prize at the maximal intensity available. This suggests that (possibly after making further regularity assumptions ruling out implausible circumstances) our result that the team optimally chooses to switch off both projects at the same time extends also to this more general environment.

Random Accumulation of Private Information over Time. Knowledge about the project may also arrive randomly as the race unfolds, and therefore remain private. A simple way to capture this phenomenon within our Gamma-Poisson model is to assume that, after the initial signal $z_i \leq 0$, each firm i who remains in the race may also observe "good news" as time goes by. Each good news is the arrival of a known event at uncertain times, following a Poisson process of parameter $K\lambda$, where $K > 1$. Therefore, the more promising the project, the higher λ and the more frequently good news accrue, typically before the innovations itself. For any pair of signals x, y, any time t, and any number n of good-news accrued to date t, the posterior distribution of λ is again a Gamma. The calculation of the marginal value of waiting is carried on in analogous way as in the present model. The relevant events are (i) the arrival of prize, (ii) the exit of the opponent, (iii) the arrival of good news. Each of these events corresponds to a flow benefit that one weighs against the flow cost of remaining in the race.

Just like in our model, a symmetric monotonic equilibrium strategy consists of two stopping time functions. For any pair (x, n) representing private information, the first stopping time function τ_1 prescribes to leave the race at time $\tau_1(x, n)$ as long as the opponent is still in the race, whereas given that the opponent left the race at time τ, equilibrium exit is prescribed by stopping time function $\tau_2(x, n, \tau)$. As in our model, the equilibrium displays a winner's curse property as firms herd on each other as long as they both stay in the race. Notice that private information (x, n) affects the equilibrium stopping time τ_1 only through on posterior beliefs about λ: therefore, by definition of good news τ_1 is monotonically increasing in x and n. For any time τ there is a unique decreasing function ξ_τ such that for any n, $\tau_1(\xi_\tau(n), n) = \tau$. When the opponent is still in the race, the only information available is that her private information (x, n) satisfies $x > \xi_\tau(n)$; when the opponent leaves the race he reveals that $x = \xi_\tau(n)$ and hence he delivers a discrete lump of bad news. Albeit x and n cannot be separately identified–the opponent could have been initially very pessimistic, but then received plenty of interim good news before quitting, or vice versa–*all one needs to infer from the opponent's quit is the opponent's posterior beliefs about* λ, *which are in 1:1 correspondence with the stopping time*. All of our results then survive qualitatively. If one firm exits prematurely either because its initial signal is very bad, or because it is unlucky and receives no good news at the beginning of the race, then the rate of innovation is inefficiently slowed down as valuable experimentation activity is postponed in the future.

Partially Independent Values. Another feasible extension consists of allowing for interdependent values, instead of pure common value. The two firms address the same research project, each choosing a different approach to find a solution. The arrival rate of invention $i = A, B$ is $\lambda_i = \lambda + \varepsilon_i$, where $\lambda \sim Ga(\alpha, \beta)$ and $\varepsilon_i \sim Ga(\alpha, \beta_i)$, with ε_A and ε_B independent. Here λ measures the "promise" of the project, and the idiosyncratic components ε_i measure the specific promise of the approach chosen by player i. If $\beta = 0$ then $\lambda = 0$ a.s. and the two projects are independent, so no informational spillovers occur in the game; if $\beta_A = \beta_B = 0$ then $\lambda_i = \lambda$ a.s. and we are back to our common value case. Each private signal $z_i \leq 0$ is drawn from the distribution $e^{\lambda_i z_i}$, so it is informative about own line of research and partially about the competitor's. For any pair of values of $\varepsilon_A, \varepsilon_B$ the game can be transformed into ours. Allowing for uncertainty about ε_A, ε_B requires for each player a further integration over this additional dimension of uncertainty. This is simple due to the properties of Gamma distributions, which imply $\lambda_i \sim Ga(\alpha, \beta + \beta_i)$. It is natural to conjecture that a unique symmetric monotone equilibrium exists, which features the winner's curse, and a new threshold function $\bar{\xi}(\cdot)$ depending also on β_A, β_B

determines the same qualitative welfare implications.

Variable Experimentation Intensity. The most difficult extension allows for variable intensity of R&D investment. This requires abandoning the relatively comfortable environment of stopping games, to venture in the more technically involved class of differential games. Our model could be augmented, following Reinganum (1981), by assuming that at each moment in time t, each firm i pays cost $C\left(u^i\left(t\right)\right)$, if it wants to select flow experimentation intensity $u^i\left(t\right)$ to enjoy innovation hazard rate of $u^i\left(t\right)\lambda$, where C is an increasing and convex function.[10] Again assuming that exit is irreversible, we would say that each firm may only observe if the opponent is engaging in the R&D race, but cannot observe precisely how many resources she is allocating to it.

As in our stopping game environment, the value of experimentation is larger, the larger is the expected hazard rate of the prize. Both in the team's problem and in equilibrium, the optimal experimentation intensity, given all available information, is determined by equating the flow marginal costs with expected continuation marginal benefits. Unlike in our stopping game environment, however, also higher-order moments of beliefs with respect to the unknown hazard rate are likely to play a role. For example, the higher the variance, the larger is the marginal value of gathering information, and hence the larger the optimal experimentation flow. While the first-moment effect suggests that the optimal amount of experimentation is increasing in the signals and decreasing in time, this is counteracted by the effect of the variance of beliefs, which is decreasing in time and increasing in signals.

Despite this complication, we feel that it is likely that a monotonic equilibrium akin to the ones we study in this paper exists (possibly under reasonable regularity conditions). In this differential game environment, a monotonic equilibrium strategy is defined as a pair of policy functions decreasing over time and increasing in signals. The first one ζ_1 prescribes the experimentation intensity $\zeta_1\left(x,t\right)$ at any time t as long as the opponent is still in the race, whereas given that the opponent left the race at time τ, equilibrium intensity is prescribed by $\zeta_2\left(x,\tau,t\right)$, with ζ_2 increasing in τ. Due to the monotonicity requirements, each signal x identifies a unique stopping time τ_1 such that $\zeta_1\left(x,\tau_1\right)=0$, and this relation is strictly increasing.

In such an equilibrium, information aggregation would be analogous to our stopping game equilibrium. If the opponent is still in the race at time τ, each player only knows

[10] The closest contribution appearing in the literature is Malueg and Tsutsui (1999) who allow for unknown hazard rate λ but not for private information. Their calculation techniques hinge on the belief over λ having a Gamma distribution. This is incompatible with equilibrium beliefs for the continuous-signal case, since inference under asymmetric information yields a left-truncated distribution. As is typically the case in these continuous action games of private information, resorting to a discrete signal construction is likely to disturb pure-strategy existence.

that her opponent's signal y is strictly larger than the unique z such that $\zeta_1(z, \tau) = 0$. If the opponent leaves the race at τ, the player learns that y is exactly equal to the unique z such that $\zeta_1(z, \tau) = 0$. As a result, this monotonic equilibrium displays an analogous winner's curse to the one we identify in this paper.

Turning to welfare analysis, the two effects identified in our simpler stopping game would still hold in the monotonic equilibrium. When signals are very close, the winner's curse would determine excessive experimentation durations, and when signals are very far apart, the pessimistic firm would leave the race too early. It is also still the case that, as soon as one firm leaves the race, the remaining one acts fully informed. But the analysis is made more complex by the possibility of counteracting effects. First, the expected hazard rate has an additional impact on experimentation intensity. If the signals are very spread apart, the optimistic firm would expect that the opponent also holds a good signal, and vice versa. As long as both firms are still in the race, while the pessimistic firm would choose to experiment less than the optimal team's choice, the optimistic firm would choose to experiment more than the team, and the total effect remains to be verified. The second complication is introduced by cost convexity (decreasing returns to scale). Given aggregate information, the least costly team's experimentation policy requires to equalize experimentation intensity across the two facilities. Equilibrium investment rates are different across firms, making it costlier to achieve the same aggregate experimentation intensity as the team. The last complication we identify is the effect of variance over experimentation intensity: this is relevant because, as long as both firms are still in the race, the team's information is more precise than the players.

Overall, the above considerations suggest the following equilibrium welfare properties. Once one firm has left the race, the remaining firm acts fully informed. Because of convex costs, it chooses experimentation intensities that are smaller than the team's optimal solution in the same contingencies.[11] As long as both firms are still in the race, however, the effect of convex costs is counteracted by the fact that the team, holding more precise information, is likely to hold less pressing incentives to gather information through experimentation. If the first firm leaves the race much earlier than in the team's optimal solution, then the second property is likely dominated by the first one. This suggests that our main welfare result extends to the case of variable experimentation: when signals are spread apart enough, overall equilibrium underexperimentation takes place.

[11] The effect on experimentation costs is however undetermined, as it may be that in equilibrium firms experiment less than it is optimal, but at a higher cost than the team's.

A. Appendix

We first prove a technical result that will prove useful many times through the paper.

Theorem A.1. *Let* $q\left(\lambda,\vec{\theta}\right):\Lambda\times\Re^n\rightarrow\Re_+$*, differentiable in* $\vec{\theta}$ *and integrable in* λ *with* $\int_\Lambda q\left(\lambda,\vec{\theta}\right)d\lambda\in(0,\infty)$*. Then the c.d.f. defined by:*

$$\varphi\left(L,\vec{\theta}\right)\equiv\frac{\int_{\underline{\lambda}}^{L} q\left(\lambda,\vec{\theta}\right)d\lambda}{\int_\Lambda q\left(\lambda',\vec{\theta}\right)d\lambda'}$$

for every $L\in\left(\underline{\lambda},\bar{\lambda}\right)$ *is stochastically strictly increasing in* $\theta_i\in\vec{\theta}$ *if* $q\left(\lambda,\vec{\theta}\right)$ *is log-supermodular in* (λ,θ_i)*, i.e. if*

$$\frac{\partial\log q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}$$

is strictly increasing in λ*.*

Proof. We want to show that for every $L\in\left(\underline{\lambda},\bar{\lambda}\right)$

$$0>\frac{\partial}{\partial\theta_i}\frac{\int_{\underline{\lambda}}^{L} q\left(\lambda,\vec{\theta}\right)d\lambda}{\int_\Lambda q\left(\lambda,\vec{\theta}\right)d\lambda}=\frac{\int_{\underline{\lambda}}^{L}\frac{\partial q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}d\lambda}{\int_\Lambda q\left(\lambda,\vec{\theta}\right)d\lambda}-\frac{\int_{\underline{\lambda}}^{L} q\left(\lambda,\vec{\theta}\right)d\lambda\int_\Lambda\frac{\partial q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}d\lambda}{\left[\int_\Lambda q\left(\lambda,\vec{\theta}\right)d\lambda\right]^2}$$

using

$$\int\frac{\partial q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}d\lambda=\int\frac{\partial\log q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}q\left(\lambda,\vec{\theta}\right)d\lambda$$

the claim reads

$$\int_\Lambda\frac{\partial\log q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}\frac{q\left(\lambda,\vec{\theta}\right)}{\int_\Lambda q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda>\int_{\underline{\lambda}}^{L}\frac{\partial\log q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}\frac{q\left(\lambda,\vec{\theta}\right)}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda$$

A sufficient condition for the latter inequality is that the RHS be strictly increasing in L. Since the RHS is differentiable in L, it suffices that

$$\begin{aligned}0&<\frac{\partial}{\partial L}\left[\int_{\underline{\lambda}}^{L}\frac{\partial\log q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}\frac{q\left(\lambda,\vec{\theta}\right)}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}d\lambda\right]\\&=\frac{\partial\log q\left(L,\vec{\theta}\right)}{\partial\theta_i}\frac{q\left(L,\vec{\theta}\right)}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'}-q\left(L,\vec{\theta}\right)\int_{\underline{\lambda}}^{L}\frac{\partial\log q\left(\lambda,\vec{\theta}\right)}{\partial\theta_i}\frac{q\left(\lambda,\vec{\theta}\right)}{\left[\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right)d\lambda'\right]^2}d\lambda\end{aligned}$$

or

$$\frac{\partial \log q\left(L,\vec{\theta}\right)}{\partial \theta_i} > \int_{\underline{\lambda}}^{L} \frac{\partial \log q\left(\lambda,\vec{\theta}\right)}{\partial \theta_i} \frac{q\left(\lambda,\vec{\theta}\right)}{\int_{\underline{\lambda}}^{L} q\left(\lambda',\vec{\theta}\right) d\lambda'} d\lambda.$$

But, since $\int_{\underline{\lambda}}^{L} \frac{q(\lambda,\vec{\theta})}{\int_{\underline{\lambda}}^{L} q(\lambda',\vec{\theta}) d\lambda'} d\lambda = 1$, this follows from the assumption that the LHS is strictly increasing in λ. ■

Proof of Lemma 1. It suffices to show that for every x, y, t, t', the c.d.f. associated with the posterior beliefs $\pi_{t,t'}(\lambda|x, y+)$ are stochastically strictly decreasing in t and in t' and strictly increasing in x and in y. We prove all these results as corollaries of Theorem A.1, a general result proved in the Appendix. Let $\vec{\theta} = (x, y, t, t')$, and

$$q\left(\lambda,\vec{\theta}\right) = \pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)],$$

so that

$$\varphi\left(L,\vec{\theta}\right) = \int_{\underline{\lambda}}^{L} \pi_{t,t'}(\lambda|x, y+) d\lambda.$$

Since

$$\frac{\partial \log q\left(\lambda,\vec{\theta}\right)}{\partial t} = -\frac{f(t|\lambda)}{1 - F(t|\lambda)} = \lambda, \ \frac{\partial \log q\left(\lambda,\vec{\theta}\right)}{\partial t'} = -\frac{f(t'|\lambda)}{1 - F(t'|\lambda)} = -\lambda$$

and

$$\frac{\partial \log q\left(\lambda,\vec{\theta}\right)}{\partial x} = \frac{h'(x|\lambda)}{h(x|\lambda)} = \lambda, \ \frac{\partial \log q\left(\lambda,\vec{\theta}\right)}{\partial y} = -\frac{h(y|\lambda)}{1 - H(y|\lambda)} = \frac{\lambda e^{\lambda x}}{1 - e^{\lambda x}}$$

are strictly increasing in λ, all monotonicity results follow from Theorem A.1. The result that $\lim_{t\to\infty} \mathbb{E}_{t,t'}[\lambda|\, x, y+] = 0$ follows from:

$$\begin{aligned} & \lim_{\tau\to\infty} \pi_{\tau,\tau}(\lambda > \varepsilon|x, y+) \\ = & \lim_{\tau\to\infty} \frac{\int_{\varepsilon}^{\infty} e^{-\alpha\lambda}\lambda^{\beta-1}\lambda e^{\lambda x}\left(1 - e^{\lambda y}\right) e^{-\lambda 2\tau} d\lambda}{\int_{\varepsilon}^{\infty} e^{-\alpha\lambda'}\lambda'^{\beta-1}\lambda' e^{\lambda' x}\left(1 - e^{\lambda' y}\right) e^{-\lambda 2\tau} d\lambda'} = 0. \ ■ \end{aligned}$$

Proof of Lemma 2. To show the first claim, we need to show that for any $L \in \left(\underline{\lambda}, \bar{\lambda}\right)$,

$$\begin{aligned} & \frac{\int_{\underline{\lambda}}^{L} \pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)] d\lambda}{\int_{\Lambda} \pi(\lambda') h(x|\lambda') [1 - H(y|\lambda')] [1 - F(t|\lambda')] [1 - F(t'|\lambda')] d\lambda'} \\ < & \frac{\int_{\underline{\lambda}}^{L} \pi(\lambda) h(x|\lambda) h(y|\lambda) [1 - F(t|\lambda)] [1 - F(t'|\lambda)] d\lambda}{\int_{\Lambda} \pi(\lambda') h(x|\lambda') h(y|\lambda') [1 - F(t|\lambda')] [1 - F(t'|\lambda')] d\lambda'}, \end{aligned}$$

this follows from

$$\int_{\underline{\lambda}}^{L} \frac{h(y|\lambda)}{1 - H(y|\lambda)} \frac{\pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)]}{\int_{\underline{\lambda}}^{L} \pi(\lambda) h(x|\lambda) [1 - H(y|\lambda)] [1 - F(t|\lambda)] [1 - F(t'|\lambda)] d\lambda} d\lambda$$

$$\equiv \mathbb{E}_{t,t'}\left[\left.\frac{\lambda e^{\lambda y}}{1-e^{\lambda y}}\right| t_0 \geq t, t_1 \geq t', x_0 = x, x_1 \geq y, \lambda \leq L\right]$$

$$> \mathbb{E}_{t,t'}\left[\left.\frac{\lambda e^{\lambda y}}{1-e^{\lambda y}}\right| t_0 \geq t, t_1 \geq t', x_0 = x, x_1 \geq y\right]$$

$$\equiv \int_\Lambda \frac{h(y|\lambda')}{1-H(y|\lambda')} \frac{\pi(\lambda')h(x|\lambda')[1-H(y|\lambda')][1-F(t|\lambda')][1-F(t'|\lambda')]}{\int_\Lambda \pi(\lambda')h(x|\lambda')[1-H(y|\lambda')][1-F(t|\lambda')][1-F(t'|\lambda')]\,d\lambda'}d\lambda',$$

where the inequality follows since

$$\frac{\partial}{\partial\lambda}\left(\frac{\lambda e^{\lambda y}}{1-e^{\lambda y}}\right) = e^{\lambda y}\frac{\left(1-e^{\lambda y}+\lambda y\right)}{\left(1-e^{\lambda y}\right)^2} < 0 \text{ for any } \lambda \text{ and } y.$$

The proof of the second inequality is analogous. ■

Proof of Lemma 4. First we expand the derivative as in the analysis determining $T_2^*(x,y,T_1)$:

$$\frac{dW_{1,t}(T_1|x,y)}{dT_1} = \int_\Lambda \frac{d}{dT_1}\hat{U}_t(T_1|\lambda)\,\pi_{t,t}(\lambda|x,y)\,d\lambda$$

$$= \int_\Lambda \left\{ \frac{2f(T_1|\lambda)(1-F(T_1|\lambda))}{(1-F(t|\lambda))^2}\left[\int_t^{T_1}(-2c)\,e^{-r(v-t)}dv + e^{-r(T_1-t)}b\right]\right.$$

$$-\frac{2f(T_1|\lambda)(1-F(T_1|\lambda))}{(1-F(t|\lambda))^2}\left[\int_t^{T_1}(-2c)\,e^{-r(v-t)}dv + e^{-r(T_1-t)}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1)\right]$$

$$\left.+\frac{(1-F(T_1|\lambda))^2}{(1-F(t|\lambda))^2}\left[(-2c)\,e^{-r(T_1-t)} + \frac{d}{dT_1}\left(e^{-r(T_1-t)}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1)\right)\right]\right\}\pi_{t,t}(\lambda|x,y)\,d\lambda$$

simplifying the term in $2c$ and collecting the discounting term

$$= e^{-r(T_1-t)}\int_\Lambda \left\{\frac{2f(T_1|\lambda)(1-F(T_1|\lambda))}{(1-F(t|\lambda))^2}[b - W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1)]\right.$$

$$\left.+\frac{(1-F(T_1|\lambda))^2}{(1-F(t|\lambda))^2}\left[-2c - rW_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1) + \frac{d}{dT_1}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1)\right]\right\}\cdot$$

$$\cdot\pi_{t,t}(\lambda|x,y)\,d\lambda$$

using the definitions of $\pi_{t,t}(\lambda|x,y)$, $\pi_{T_1,T_1}(\lambda|x,y)$

$$\frac{d}{dT_1}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1) \propto \int_\Lambda \left[\frac{f(T_1|\lambda)}{(1-F(T_1|\lambda))}(b - W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1))\right]\pi_{T_1,T_1}(\lambda|x,y)$$

$$+\int_\Lambda \left[-\frac{r}{2}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1) + \frac{1}{2}\frac{d}{dT_1}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1) - c\right]\pi_{T_1,T_1}(\lambda|x,y)\,d\lambda$$

$$\propto b\mathbb{E}_{T_1,T_1}[\lambda|\,x,y] - c - W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1)\left(\frac{r}{2} + \mathbb{E}_{T_1,T_1}[\lambda|\,x,y]\right)$$

$$+\frac{1}{2}\frac{d}{dT_1}W_{2,T_1}(T_2^*(x,y,T_1)|x,y,T_1). \tag{A.1}$$

Using the expression for $W_{2,T_1}\left(T_2^*(x,y,T_1)|x,y,T_1\right)$, and the envelope theorem to ignore the effect of T_1 on this value through $T_2^*(x,y,T_1)$,

$$\frac{d}{dT_1}W_{2,T_1}\left(T_2^*(x,y,T_1)|x,y,T_1\right)=\frac{d}{dT_1}\mathbb{E}_{T_1,T_1}\left[U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]$$

$$\begin{aligned}
&= \frac{d}{dT_1}\int_\Lambda\left[\int_{T_1}^{T_2^*(x,y,T_1)}\frac{f(s|\lambda)}{1-F(T_1|\lambda)}\left[\int_{T_1}^{s}(-c)\,e^{-r(v-T_1)}dv+e^{-r(s-T_1)}b\right]ds\right.\\
&\quad\left.+\frac{1-F\left(T_2^*(x,y,T_1)|\lambda\right)}{1-F(T_1|\lambda)}\int_{T_1}^{T_2^*(x,y,T_1)}(-c)\,e^{-r(v-T_1)}dv\right]\pi_{T_1,T_1}(\lambda|x,y)\,d\lambda
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{T_1,T_1}\left[-\lambda b+\frac{1-F\left(T_2^*(x,y,T_1)|\lambda\right)}{1-F(T_1|\lambda)}c+c\int_{T_1}^{T_2^*(x,y,T_1)}\frac{f(s|\lambda)}{1-F(T_1|\lambda)}|x,y\right]\\
&\quad+r\mathbb{E}_{T_1,T_1}\left[\int_{T_1}^{T_2^*(x,y,T_1)}\frac{f(s|\lambda)}{1-F(T_1|\lambda)}\left[\int_{T_1}^{s}(-c)\,e^{-r(v-T_1)}dv+e^{-r(s-T_1)}b\right]ds|x,y\right]\\
&\quad+r\mathbb{E}_{T_1,T_1}\left[-c\frac{1-F\left(T_2^*(x,y,T_1)|\lambda\right)}{1-F(T_1|\lambda)}\int_{T_1}^{T_2^*(x,y,T_1)}e^{-r(v-T_1)}dv|x,y\right]\\
&\quad+\mathbb{E}_{T_1,T_1}\left[\lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]+\int_\Lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)\frac{d\pi_{T_1,T_1}(\lambda|x,y)}{dT_1}d\lambda
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{T_1,T_1}\left[-\lambda b+c|x,y\right]+rW_{2,T_1}\left(T_2^*(x,y,T_1)|x,y,T_1\right)\\
&\quad+\mathbb{E}_{T_1,T_1}\left[\lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]+\int_\Lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)\frac{d\pi_{T_1,T_1}(\lambda|x,y)}{dT_1}d\lambda
\end{aligned}$$

Replacing this expression for $dW_{2,T_1}\left(T_2^*(x,y,T_1)|x,y,T_1\right)/dT_1$ into (A.1) and collecting terms

$$\begin{aligned}
&\frac{dW_{1,t}(T_1|x,y)}{dT_1}\propto\frac{1}{2}\left\{b\mathbb{E}_{T_1,T_1}\left[\lambda|\,x,y\right]-c\right\}-\mathbb{E}_{T_1,T_1}\left[\lambda|\,x,y\right]\mathbb{E}_{T_1,T_1}\left[U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]\\
&+\frac{1}{2}\mathbb{E}_{T_1,T_1}\left[\lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]+\frac{1}{2}\int_\Lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)\frac{d\pi_{T_1,T_1}(\lambda|x,y)}{dT_1}d\lambda. \quad \text{(A.2)}
\end{aligned}$$

Using further manipulations

$$\frac{d\pi_{T_1,T_1}(\lambda|x,y)}{dT_1}=-2\lambda\pi_{T_1,T_1}(\lambda|x,y)+2\pi_{T_1,T_1}(\lambda|x,y)\,\mathbb{E}_{T_1,T_1}\left[\lambda|x,y\right]$$

and replacing into (A.2)

$$\begin{aligned}
\frac{dW_{1,t}(T_1|x,y)}{dT_1}\quad\propto\quad&\frac{1}{2}\left\{b\mathbb{E}_{T_1,T_1}\left[\lambda|\,x,y\right]-c\right\}-\mathbb{E}_{T_1,T_1}\left[\lambda|\,x,y\right]\mathbb{E}_{T_1,T_1}\left[U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]\\
&+\frac{1}{2}\mathbb{E}_{T_1,T_1}\left[\lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]-\mathbb{E}_{T_1,T_1}\left[\lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]\\
&+\mathbb{E}_{T_1,T_1}\left[\lambda|\,x,y\right]\mathbb{E}_{T_1,T_1}\left[U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]
\end{aligned}$$

$$\begin{aligned}
&\propto \frac{1}{2}\left\{b\mathbb{E}_{T_1,T_1}\left[\lambda|\,x,y\right]-c-\mathbb{E}_{T_1,T_1}\left[\lambda U_{2,T_1}\left(T_2^*(x,y,T_1)|\lambda\right)|x,y\right]\right\}\\
&\propto -c+\mathbb{E}_{T_1,T_1}\left[\lambda\left(b-U_{2,T_1}(T_2^*(x,y,T_1)|\lambda)\right)|x,y\right].\ \blacksquare
\end{aligned}$$

Proof of Lemma 5. First we calculate

$$\begin{aligned}
U_{2,t}\left(T|\lambda\right) &= \int_t^T \frac{f\left(s|\lambda\right)}{1-F\left(t|\lambda\right)}\left[\int_t^s \left(-c\right)e^{-r(v-t)}dv+e^{-r(s-t)}b\right]ds+\frac{1-F\left(T|\lambda\right)}{1-F\left(t|\lambda\right)}\int_t^T\left(-c\right)e^{-r(v-t)}dv.\\
&= \int_t^T \lambda e^{-\lambda(s-t)}\left[\int_t^s\left(-c\right)e^{-r(v-t)}dv+e^{-r(s-t)}b\right]ds-e^{-\lambda(T-t)}\int_t^T\left(-c\right)e^{-r(v-t)}dv\\
&= \left(\lambda b-c\right)\frac{1-e^{-(\lambda+r)(T-t)}}{\lambda+r}.
\end{aligned}$$

Using this expression in the claim of Lemma 4, for every $T_1\in(t,T^*(x,y)]$ we obtain:

$$\frac{dW_{1,t}\left(T_1|x,y\right)}{dT_1}\propto -c+\mathbb{E}_{T_1,T_1}\left[\lambda\left(b-\left(\lambda b-c\right)\frac{1-e^{-(\lambda+r)\left(T_2^*(x,y,T_1)-T_1\right)}}{\lambda+r}\right)|x,y\right]$$

using $c=\mathbb{E}_{T_1,T_1}\left[c|x,y\right]$, some algebra and $T^*(x,y)>0$,

$$\begin{aligned}
&= \mathbb{E}_{T_1,T_1}\left[\left(\lambda b-c\right)\frac{r+\lambda e^{-(\lambda+r)\left(T_2^*(x,y,T_1)-T_1\right)}}{\lambda+r}|x,y\right]\\
&= \int_\Lambda \left(\lambda b-c\right)\frac{r+\lambda e^{-(\lambda+r)\frac{b}{c}(\beta+2)+x+y-\alpha-2T_1}}{r+\lambda}\frac{e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}}{\Gamma\left(\beta+2\right)\left(\alpha-x-y+2T_1\right)^{-\beta-2}}d\lambda\\
&\propto S(T_1)\equiv\int_\Lambda\left(\lambda\frac{b}{c}-1\right)\frac{r+\lambda e^{-(\lambda+r)\left(\frac{b}{c}(\beta+2)+x+y-\alpha-2T_1\right)}}{\lambda+r}e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}d\lambda,
\end{aligned}$$

where note that $S(T^*(x,y))=0$ by construction.

Differentiating $S(T_1)$ we obtain:

$$S'\left(T_1\right)=\frac{d}{dT_1}\left[\int_\Lambda\left(\lambda\frac{b}{c}-1\right)\frac{r+\lambda e^{-(\lambda+r)\left(\frac{b}{c}(\beta+2)+x+y-\alpha-2T_1\right)}}{\lambda+r}e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}d\lambda\right]$$

$$\begin{aligned}
&= \int_\Lambda\left(\lambda\frac{b}{c}-1\right)\frac{-2r\lambda+2r\lambda e^{-(\lambda+r)\left(\frac{b}{c}(\beta+2)+x+y-\alpha-2T_1\right)}}{\lambda+r}e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}d\lambda\\
&= 2r\int_\Lambda\left(\lambda\frac{b}{c}-1\right)\frac{r+\lambda e^{-(\lambda+r)\left(\frac{b}{c}(\beta+2)+x+y-\alpha-2T_1\right)}}{\lambda+r}e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}d\lambda\\
&\quad +2r\int_\Lambda\left(\lambda\frac{b}{c}-1\right)\frac{-\lambda-r}{\lambda+r}e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}d\lambda\\
&= 2rS(T_1)-2r\int_\Lambda\left(\lambda\frac{b}{c}-1\right)e^{-\lambda(\alpha-x-y+2T_1)}\lambda^{\beta+1}d\lambda\\
&= 2rS(T_1)-2r\frac{\Gamma\left(\beta+2\right)}{\left(\alpha-x-y+2T_1\right)^{\beta+2}}\left[\frac{\left(\beta+2\right)}{\alpha-x-y+2T_1}\frac{b}{c}-1\right].
\end{aligned}$$

So, finally we obtain the differential equation

$$S'(T_1) = 2rS(T_1) - 2r\frac{\Gamma(\beta+2)}{(\alpha - x - y + 2T_1)^{\beta+3}}2(T^*(x,y) - T_1) \tag{A.3}$$

where clearly $(\alpha - x - y + 2T_1)^{\beta+3} > 0$. We use (A.3) to prove the claim. First, we exclude an optimal interior (positive) $T_1^*(x,y) \in (0, T^*(x,y))$. By contradiction: if there is an optimal interior $T_1 = T_1^*(x,y) \in (0, T^*(x,y))$, it must satisfy the NFOC

$$\frac{dW_{1,t}(T_1|x,y)}{dT_1} = 0 \Rightarrow S(T_1) = 0.$$

But then, $S(T_1^*(x,y)) = 0$ and $T_1^*(x,y) < T^*(x,y)$ in (A.3) together imply $S'(T_1^*(x,y)) < 0$. By continuity there exists $\varepsilon > 0$ such that $S(T_1^*(x,y) + \varepsilon) < 0$. Then by smoothness of S (many times differentiable) and $S(T^*(x,y)) = 0$ there exists a $T_1' \in (T_1^*(x,y), T^*(x,y))$ such that $S'(T_1') = 0 > S(T_1')$; but from (A.3) $T_1' < T^*(x,y)$ and $S(T_1') < 0$ imply $S'(T_1') < 0$ a contradiction. Second, we exclude a corner solution at $T_1 = T_1^*(x,y) = 0$. For this, we would require

$$\frac{dW_{1,t}(0|x,y)}{dT_1} \leq 0 \Rightarrow S(0) \leq 0$$

which, together with $T^*(x,y) > 0 = T_1^*(x,y)$ and (A.3), imply $S'(0) < 0$. But then the function $S(t)$ keeps declining at increasing rate from the initial value $S(0) \leq 0$ (formally the differential equation for S is exploding downward) as t rises from 0, contradicting $S(T^*(x,y)) = 0$. ■

Proof of Proposition 3. We consider the problem of player A who contemplates stopping second at time τ^A, player $B's$ problem being symmetric. In the same manner as in the proof of Proposition 3, we compute

$$\begin{aligned}
&\mathbb{E}\left[\lambda|\, z_A = x, z_B \leq y, t_A \geq t, t_B \geq t'\right] \\
= &\int_\Lambda \lambda \frac{\pi(\lambda)\, h(x|\lambda)\, H(y|\lambda)\left[1 - F(t|\lambda)\right]\left[1 - F(t'|\lambda)\right]}{\int_\Lambda \pi(\lambda')\, h(x|\lambda')\, H(y|\lambda')\left[1 - F(t|\lambda')\right]\left[1 - F(t'|\lambda')\right] d\lambda'} d\lambda \\
= &\int_\Lambda \lambda \frac{e^{-\alpha\lambda}\lambda^{\beta-1}\lambda e^{\lambda x} e^{\lambda y} e^{-\lambda t} e^{-\lambda t'}}{\int_\Lambda e^{-\alpha\lambda'}\lambda'^{\beta-1}\lambda' e^{\lambda' x} e^{\lambda' y} e^{-\lambda' t} e^{-\lambda' t'} d\lambda'} d\lambda = \int_\Lambda \frac{\lambda^{\beta+1} e^{-\lambda(\alpha+t+t'-x-y)}}{\Gamma(\beta+1)(\alpha + t + t' - x - y)^{-\beta-1}} d\lambda \\
= &\frac{\Gamma(\beta+2)(\alpha + t + t' - x - y)^{-\beta-2}}{\Gamma(\beta+1)(\alpha + t + t' - x - y)^{-\beta-1}} = \frac{\beta+1}{\alpha - x - y + t + t'}.
\end{aligned}$$

If the FOC has a positive solution, this is unique and a global maximum of the value function. Therefore, the desired stopping time is the maximum of 0 and the solution τ^A to $c = b\mathbb{E}\left[\lambda|\, z_A = x, z_B \leq g(0), t_A \geq \tau^A, t_B \geq 0)\right]$. ■

Proof of Lemma 6 From Proposition 2, we have

$$\tau_2^*(x, \tau) = \max\{\tau, \frac{b}{c}(\beta+2) + x + g(\tau) - \alpha - \tau\}.$$

Since $g(\tau) \leq 0$, it follows that $\frac{b}{c}(\beta+2) + x + g(\tau) - \alpha - \tau \leq \frac{b}{c}(\beta+2) + x - \alpha - \tau$. For any x, there is $\bar{\tau}$ large enough such that for any $\tau \geq \bar{\tau}$, $\frac{b}{c}(\beta+2) + x - \alpha - \tau \leq \tau$; hence $\tau_2^*(x,\tau) = \tau$, $W_{2,\tau}(\tau_2^*(x,\tau)|x, g(\tau), \tau) = 0$ and

$$\begin{aligned} V_{1,t}'(\tau|x) &\propto b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)+\right] + W_{2,\tau}(\tau_2^*(x,\tau)|x, g(\tau), \tau)\mathbb{E}_{\tau,\tau}\left[\left.\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)}\right| x, g(\tau)+\right] - c \\ &= b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)+\right] - c \leq b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, 0\right] = \frac{\beta+2}{\alpha - x + 2\tau} \to 0 \text{ for } \tau \to \infty. \end{aligned}$$

where the inequality follows by Lemma 2.

Since

$$\begin{aligned} V_{1,t}'(\tau|x) &= b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)+\right] + W_{2,\tau}(\tau_2^*(x,\tau)|x, g(\tau), \tau)\mathbb{E}_{\tau,\tau}\left[\left.\frac{h(g(\tau)|\lambda)g'(\tau)}{1-H(g(\tau)|\lambda)}\right| x, g(\tau)+\right] - c \\ &\geq b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, g(\tau)+\right] - c, \end{aligned}$$

for any x such that $b\mathbb{E}_{0,0}\left[\lambda|\, x, g(0)+\right] > c$, and any $t \leq \tau$ small enough, it must be that $V_{1,t}'(\tau|x) > 0$ by continuity of $V_{1,t}'$. ■

Proof of Lemma 8. Consider the choice at time $t = 0$ of player A with a signal x. If she chooses not to enter the game and set $\tau = 0$, her payoff is $V_{1,0}(0|x) = 0$. If she chooses to enter the game, and set $\tau > 0$, then she will observe whether B enters the game or not. This allows us to write A's expected payoff for playing any $\tau > 0$ as:

$$V_{1,0}(\tau|x) = \Pr(z_B \leq \underline{x}|x) \lim_{t\downarrow 0} W_{2,t}(\tau_2^*(x,0)|z_A = x, z_B \leq \underline{x}, t_B \geq 0) + [1 - \Pr(z_B \leq \underline{x}|x)] \lim_{t\downarrow 0} V_{1,t}(\tau|x)$$

Suppose first that x is such that $bE_{\tau,\tau}\left[\lambda|\, x, \underline{x}+\right] \leq c$; and hence that the equilibrium prescription is $\tau_1(x) = 0$. Since this implies that for any $\tau \geq t > 0$, $V_t'(\tau|x) < 0$; it follows that for any $\tau > 0$, $\lim_{t\downarrow 0} V_{1,t}'(\tau|x) < 0$. Since

$$b\frac{\beta+1}{\alpha - x - \underline{x} + 2\tau} < b\frac{\beta+2}{\alpha - x - \underline{x} + 2\tau} = b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, \underline{x}\right] < b\mathbb{E}_{\tau,\tau}\left[\lambda|\, x, \underline{x}+\right],$$

where the last inequality is by from Lemma 2, it follows that $\tau_2^*(x,0) = 0$ by Proposition 3, and hence $\lim_{t\downarrow 0} W_{2,t}(\tau_2^*(x,0)|z_A = x, z_B \leq \underline{x}, t_B \geq 0) = 0$. So the player optimally chooses to follow the equilibrium prescription $\tau_1(x) = 0$. Second, suppose that x is such that $bE_{\tau,\tau}\left[\lambda|\, x, \underline{x}+\right] > c$; the player will comply with the equilibrium prescription $\tau_1(x) > 0$ because $\lim_{t\downarrow 0} W_{2,t}(\tau_2^*(x,0)|z_A = x, z_B \leq \underline{x}, t_B \geq 0) \geq 0$ and for any τ small enough $\lim_{t\downarrow 0} V_{1,t}'(\tau|x) > 0$. ■

Proof of Lemma 11. Consider the time-0 team's expected costs for adopting stopping times $T_1 \leq T_2$ (such that $T_1 + T_2 = 2T^*$) and conditional on any λ:

$$\begin{aligned} C(T_1, T_2|\lambda) = c\Bigg[&\int_0^{T_1} 2f(s|\lambda)(1 - F(s|\lambda)) \int_0^s 2e^{-rv} dv ds + (1 - F(T_1|\lambda))^2 \int_0^{T_1} 2e^{-rv} dv \\ &+ (1 - F(T_1|\lambda)) \int_{T_1}^{T_2} f(s|\lambda) \int_{T_1}^s e^{-rv} dv ds + (1 - F(T_1|\lambda))(1 - F(T_2|\lambda)) \int_{T_1}^{T_2} e^{-rv} dv\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= c\left[\int_0^{T_1} 2\lambda e^{-\lambda s}e^{-\lambda s}\int_0^s 2e^{-rv}dvds + e^{-2\lambda T_1}\left(\int_0^{T_1} 2e^{-rv}dv\right)\right.\\
&\quad \left. +e^{-\lambda T_1}\int_{T_1}^{T_2}\lambda e^{-\lambda s}\int_{T_1}^{s} e^{-rv}dvds + e^{-\lambda T_1}e^{-\lambda T_2}\int_{T_1}^{T_2} e^{-rv}dv\right]
\end{aligned}
$$

$$
\begin{aligned}
&= c\left[2\frac{1-e^{-2\lambda T_1}}{r} - \frac{4\lambda}{(2\lambda+r)\,r}\left(1-e^{-(2\lambda+r)T_1}\right) + \frac{2}{r}\left(e^{-2\lambda T_1} - e^{-(2\lambda+r)T_1}\right)\right.\\
&\quad +\frac{1}{r}\left(e^{-\lambda T_1}\left(e^{-(\lambda+r)T_1} - e^{-\lambda T_2 - rT_1}\right) - e^{-\lambda T_1}\frac{\lambda}{\lambda+r}\left(e^{-(\lambda+r)T_1} - e^{-(\lambda+r)T_2}\right)\right)\\
&\quad \left. +\frac{1}{r}\left(e^{-\lambda T_2 - (r+\lambda)T_1} - e^{-(\lambda+r)T_2 - \lambda T_1}\right)\right]
\end{aligned}
$$

multiplying by r/c, simplifying terms and substituting $2T^* - T_1$ for T_2,

$$
\frac{rC\left(T_1, 2T^* - T_1|\lambda\right)}{c} = \frac{2r}{2\lambda+r} - e^{-(2\lambda+r)T_1}\frac{r^2}{(\lambda+r)\,(2\lambda+r)} - \frac{r}{\lambda+r}e^{-(\lambda+r)2T^*+rT_1}
$$

Taking a derivative with respect to T_1,

$$
\frac{\partial C\left(T_1, 2T^* - T_1|\lambda\right)}{\partial T_1} \propto e^{-(2\lambda+r)T_1}\frac{r^2}{\lambda+r} - \frac{r^2}{\lambda+r}e^{-(\lambda+r)2T^*+rT_1} = e^{-(2\lambda+r)T_1}\frac{r^2}{\lambda+r}\left(1-e^{-(\lambda+r)2(T^*-T_1)}\right).
$$

This quantity is strictly positive if and only if $T_1 < T^*$, thereby implying that if $T_1 + T_2 = 2T^*$, then $C\left(T_1, T_2|\lambda\right)$ is maximized by setting $T_1 = T^* = T_2$.

If $y \geq x > \underline{x}$ and $\tau_1^*(x) < T^*(x,y)$, then, since $\tau_2^*\left(y, \tau_1^*(x)\right) = 2T^*(x,y) - \tau_1^*(x)$,

$$
C\left(\tau_1^*\left(x\right), \tau_2^*\left(y, \tau_1^*\left(x\right)\right)|\lambda\right) < C\left(T^*\left(x,y\right), T^*\left(x,y\right)|\lambda\right) \text{ for every } \lambda,
$$

and hence

$$
\int_\Lambda C\left(\tau_1^*\left(x\right), \tau_2^*\left(y, \tau_1^*\left(x\right)\right)|\lambda\right)\pi_{0,0}\left(\lambda|x,y\right)d\lambda < \int_\Lambda C\left(T^*\left(x,y\right), T^*\left(x,y\right)|\lambda\right)\pi_{0,0}\left(\lambda|x,y\right)d\lambda.
$$

This inequality, together with the inequality

$$
W\left(\tau_1^*\left(x\right), \tau_2^*\left(y, \tau_1^*\left(x\right)\right)|x,y\right) < W\left(T^*\left(x,y\right), T^*\left(x,y\right)|x,y\right)
$$

implied by the team's optimal stopping times $T_1 = T_2 = T^*\left(x,y\right)$, shows that

$$
B\left(\tau_1^*\left(x\right), \tau_2^*\left(y, \tau_1^*\left(x\right)\right)|x,y\right) < B\left(T^*\left(x,y\right), T^*\left(x,y\right)|x,y\right). \blacksquare
$$

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