Group Decision Making in the Shadow of Disagreement

Kfir Eliaz[1] Debraj Ray[2] Ronny Razin[3]

May 2003, Preliminary and incomplete.

[1] New York University
[2] New York University and Instituto de Analisis Economico (CSIC)
[3] New York University

1 Introduction

In many situations, a group of individuals must decide between alternative courses of action, in a context where disagreement (not implementing a choice at all) is the worst possible outcome for all concerned. A government may need to formulate a long-run response to terrorism: individuals may (strongly) disagree over the nature of an appropriate response, but everyone would deplore complete inaction. An academic department may need to make an offer to one of several candidates. Again, individuals differ in their relative preferences but no one wants to see their slot taken away by the Dean because they couldn't agree on an offer. Similarly, a committee seeking to spend a budget by the end of the fiscal year may disagree on goals, but the worst outcome would be to have the funds evaporate owing to lack of use.

The goal of this paper is to study a particular formulation of this situation, which we believe to be representative of many real-world scenarios.[1] We proceed as follows.

A group of n agents must make a joint choice from a set of two alternatives. Each agent must either name an alternative — A or B — or she can declare "neutrality", in that she agrees to be counted, in principle, for either side. Once this is accomplished, we tally declarations for each alternative, *including the number of neutral announcements*. If, for an alternative, the resulting total is no less than some exogenously given supermajority, we shall call that alternative *eligible*.

Because neutral annoucements are allowed for and tallied on both sides, all sorts of combinations are possible: exactly one alternative may be eligible, or both, or neither. If *exactly* one alternative is eligible, that alternative is implemented. If both are eligible — as will typically be the case when there are a large number of neutrals — one alternative is picked and implemented at random. If neither is eligible — which will happen if there is a fierce battle to protect one's favorite alternative — then no alternative is picked: the outcome is disagreement.

The objective of the paper is to set up this model and study its equilibria.

Several features of the model deserve comment. First, while the specific formulation is cast in terms of a voting model, we do not necessarily have voting in mind. The exogenously given supermajority may or may not amount to full consensus or unanimity, and in any case is to be interpreted as some preassigned degree of consensus that the group needs to achieve. In addition, one might view the neutrality announcement as the delegation of one's ballot to an impartial

[1] Thus it is not an axiomatic description of a normative or quasi-normative solution that we are after, as in Nash bargaining, nor so we seek to implement a particular solution correspondence by the choice of a mechanism.

arbitrator, who appreciates the anxiety of all concerned to avoid disagreement, and is therefore interested in implementing some outcome. We find this option of declaring neutrality particularly appealing and natural in the description of the situations we have in mind. Yet, the reader should be careful to note that neutrality and abstention are not the same things.

Next, we are interested in the "intensity" of preference for one alternative over the other, and how this enters into the decision to be neutral, or to fight for one's favorite outcome. Specifically, we permit each person's valuations to be independent (and private) draws from a distribution, and allow quite generally for varying cardinal degrees of preference. A corollary of this formulation is that *others* are not quite sure of how strongly a particular individual might feel about an outcome and therefore about how that individual might behave. This is one way in which uncertainty enters the model.

Uncertainty also plays an additional role, in that no one is sure how many people favor one given alternative over the other. To be sure, we assume that there is a common prior — represented by an independent probability p — that an individual will favor one alternative (call it A) over the other (call it B). Without loss of generality take $p \leq 1/2$. If, in fact, $p < 1/2$, one might say that it is commonly known that people of "type A" are in a minority, or more precisely in a stochastic minority.

A major goal of the paper is to study equilibria that "favor" one side: either the minority or the majority. It is intuitive — and we develop this formally in the analysis — that in any equilibrium, each individual will use a cutoff rule: there will exist some critical relative intensity of preference (for A over B or *vice versa*) such that the individual will announce her favorite outcome if intensities exceed this threshold, and neutrality otherwise. If the cutoff is lower, then an individual may be viewed as being more "aggressive": she announces her own favorite outcome more easily (and risks disagreement with greater probability). Thus, equilibria in which an individual of the majority type uses a lower cutoff than an individual of the minority type may be viewed as favoring the majority: we call them *majority equilibria*. Likewise, equilibria in which the minority type uses a lower cutoff will be called *minority equilibria*.

One might use a parallel from the Battle of the Sexes (after all, in some sense, our model is an enriched version of that game) to search for particular majority or minority equilibria. For instance, might one not be able to sustain an equilibrium in which all members of a particular type are "fully aggressive" (using the lowest possible cutoff) while their opponents all timidly declare neutrality, regardless of valuation? The answer is that such a configuration is indeed an equilibrium. But, as we argue in detail in Section 3.2.2, this equilibrium fails a weak robustness or stability criterion. If the compatriots of, say, a type-A individual *do* announce neutrality for a huge range of relative valuations (rather than the *entire* range), it will push an individual type-A person to announce A for a large range of valuations, thus rendering the "perfect neutrality" cutoff unstable to the tiniest perturbations. As we shall see in Section 3.2.2, uncertainty about group sizes plays a central and indispensable role in this result, though this is not the only indispensable role played by uncertainty in this model.

Nevertheless, Proposition 1 establishes that a *majority* equilibrium — one satisfying the robustness criterion just described — always exists. In this equilibrium, both sides use "interior" cutoffs, but the majority uses a more aggressive cutoff than the minority. This is an interesting manifestation of the "tyranny of the majority". Not only are the majority greater in number (or at least stochastically so), they are also more vocal in expressing their opinion. In response — and fearing disagreement — the minority are more cowed towards neutrality. So in majority equilibrium, group outcomes are doubly shifted towards the majority view, once through numbers, and once through greater voice.

We then turn to minority equilibria. Given the refinement described two paragraphs ago, such equilibria do not generally exist; indeed, it is easy enough to find examples of nonexistence. Yet Proposition 2 establishes the following result: if the required supermajority μ is not unanimity (i.e., $\mu < 1$), and if the size of the stochastic minority p exceeds $1 - \mu$, then for all sufficiently large population sizes, a minority equilibrium must exist.

How large is large? To be sure, the answer must depend on the model specifics, but our computations suggest that in reasonable cases, population sizes of 8-10 (certainly less than the size of a jury!) are enough for existence.

We found this result remarkable, though we confess that we do not understand it fully. In part, it is intuitive. As population size increases, the two types of uncertainty that we described — uncertainty about type and uncertainty regarding valuation intensity — tend to diminish under the strength of the Law of Large Numbers. This would do no good if $p < 1 - \mu$, for then the minority would neither be able to win, nor would it be able to block the majority. But if p exceeds $1 - \mu$, the minority acquires "credibility" to block the wishes of the majority. This is sufficient to generate the existence of a minority equilibrium.

For two reasons, this is not a complete explanation. First, credible blocking is not tantamount to a credible *win*. Indeed, it is easy to see that as μ goes up, the minority find it easier to block but also harder to win. So the previous result must *not* be viewed as an assertion that the minority is "better protected" by an increase in μ. [Indeed, as Example 2 in Section 6 makes clear, this is not true.] Nevertheless, insofar as existence is concerned, the fact that $p > 1 - \mu > 0$ guarantees existence for large population sizes.

Second, the case of unanimity remains open. The techniques used to prove Proposition 2 do not work in that case, and indeed we conjecture that the result is false. That is, once can write down a group decision model with unanimity in which a minority equilibrium never exists, no matter what the population size is. We report on this conjecture in Section 6. So blocking credibility alone does not translate into the existence of a minority equilibrium in the unanimity case.

The next main result in the paper studies minority equilibrium. Recall that in the majority equilibrium, the majority group will have a greater chance of implementing its preferred outcome on two counts: greater voice, and greater number. Obviously, this synergy is reversed for the minority equilibrium: there, the minority have greater voice, yet they have smaller numbers. One might expect the net effect of these two forces to result in some ambiguity. The intriguing content of Proposition 3 is that in a minority equilibrium, *the minority must always implement its favorite action with greater probability than the majority.* Voice more than compensates for number.

Our paper thus suggests that in group decision-making the outcomes tend to be invariably biassed in one direction or another. In majority equilibrium this is obvious. But it is also true of minority equilubrium. This lends some support to the view that group decision-making tends to have an extreme character of its own, something that this model does share (but for subtler reasons) with the battle of the Sexes.

One might criticize Proposition 3 on the grounds that it may be empty. Minority equilibrium typically exist for large population sizes, but for such equilibria the probability of disagreement should be very large or approaching unity. [For instance, suppose that μ is very close to unity. Wouldn't all outcomes be blocked?] Of course, this sort of argumentation neglects the strategic nature of decsion-making in this model. Individual cutoffs vary endogenously with population size, after all. Indeed, Proposition 4 establishes that the probability of disagreement is not only strictly less than unity in all equilibria and for all population sizes, *it is bounded away from one as the population size goes to infinity.*[2] Therefore Proposition 3 has a force that does not fade with increasing population size.

[2] Once again, we need to assume that $\mu < 1$. The unanimity case is discussed in Section xxx.

1.1 Literature review

[To be added]

2 The Model

2.1 The Group Choice Problem

A group of n agents must make a joint choice from a set of two alternatives, which we denote by A and B. The rules of choice are described as follows:

[1] Each agent must either name an alternative — A or B — or she can declare "neutrality", in that she agrees to be counted, in principle, for either side.

[2] If the total number of votes for an alternative plus the number of neutral votes is no less than some exogenously given supermajority m $(> n/2)$, then we shall call that alternative *eligible*.

[3] If no alternative is eligible, no alternative is chosen: a state D (for "disagreement") is the outcome.

[4] If a single alternative is eligible, then that alternative is chosen.

[5] If *both* alternatives are eligible, A or B are chosen with equal probability.

2.2 Valuations

Normalizing the value of disagreement to zero, each individual will have valuations (v_A, v_B) over A and B. These valuations are random variables, and we assume they are private information. Use the notation (v, v'), where v is the valuation of the favorite outcome $(\max\{v_A, v_B\})$, and v' is the valuation of the remaining outcome $(\min\{v_A, v_B\})$. An individual will be said to be of *type* A if $v = v(A)$, and of *type* B if $v = v_B$. [The case $v_A = v_B$ is unimportant as we will rule out mass points below.]

Our first restriction is

[A.1] Each individual prefers either outcome to disagreement. That is, $(v, v') \gg 0$ with probability one.

In Section 6 we explore the consequences of dropping the assumption that disagreement is worse than either alternative.

In what follows we shall impose perfect symmetry across the two types *except* for the probability of being one type or the other, which we permit to depart from 1/2. [The whole idea, after all, is to study majorities and minorities.]

[A.2] A person is type A with (iid) probability $p \in [0, 1/2]$, and is type B otherwise. Regardless of specific type, however, (v, v') are chosen independently and identically across agents.

2.3 The Game

First, each player is (privately) informed of her valuation (v_A, v_B). Conditional on this information she decides to announce either A or B, or simply remain neutral and agree to be counted in any direction that facilitates agreement. Because an announcement of the opposite alternative (to a player's type) is weakly dominated by a neutral stance, we presume that each player either decides to vote her own type, or to be neutral.[3] The rules in Section 2.1 then determine expected payoffs.

3 Equilibrium

3.1 Cutoffs

Consider a player of a particular type, with valuations (v, v'). Define $q \equiv n - m$. Notice that our player only has an effect on the outcome of the game (that is, she is pivotal) in the event that there are *exactly* q other players announcing her favorite outcome. For, suppose there are more than q such announcements, say for A. Then B cannot be eligible, and whether or not A is eligible, our player's announcement cannot change this fact. So our player has no effect on the outcome. Likewise, if there are strictly less than q announcements of A, then B is eligible whether or not A is, and our player's vote (A or neutral) cannot change the status of the latter.

Now look at the pivotal events more closely. One case is when there are precisely q announcements in favor of A, and $q+1$ or more announcements favoring B. In this case, by staying neutral our agent ensures that B is the only eligible outcome and is therefore chosen. By announcing A she guarantees that neither outcome is eligible, so disagreement ensues. In short, by switching her announcement from neutral to A, our agent creates a personal loss of v'.

In the second case, there are q announcements or less in favor of B. In this case, by going neutral our agent ensures that A and B are both eligible, so the outcome is an equiprobable

[3] For a similar reason we need not include the possibility of abstention. Abstention (as opposed to neutrality) simply increases the probability of disagreement, which all players dislike by assumption.

choice of either A or B. On the other hand, by announcing A, our agent guarantees that A is the *only* eligible outcome. Therefore by switching in this instance from neutral to announcing A, our agent creates a personal gain of $v - (v + v')/2$.

To summarize, let P^+ denote the probability of the former pivotal event (q compatriots announcing A, $q + 1$ or more announcing B) and P^- the probability of the latter pivotal event (q compatriots announcing A, q or less announcing B). It must be emphasized that these probabilities are not exogenous. They depend on several factors, but most critically on the strategies followed by the other agents in the group. Very soon we shall look at this dependence more closely, but notice that even at this preliminary stage we can see that our agent must follow a *cutoff rule*. For announcing A is weakly preferred to neutrality if and only if

$$P^-[v - (v + v')/2] \geq P^+ v'.$$

Define $u \equiv \frac{v-(v+v')/2}{v'}$. Note that (by [A.1]) u is a well-defined random variable. Then the condition above reduces to

$$P^- u \geq P^+, \tag{1}$$

which immediately shows that our agent will follow a cutoff rule using the variable u.

Notice that we include the extreme rules of always announcing neutrality (or always announcing one's favorite action) in the family of cutoff rules. [Simply think of u as a nonnegative extended real.] If a cutoff rule does not conform to one of these two extremes, we shall say that it is *interior*.

By [A.2], the variable u has the same distribution no matter which type we are referring to. We assume

[A.3] u is distributed according to the cdf F, with strictly positive density f on $(0, \infty)$.

3.2 Symmetric Equilibrium

In this paper, we study symmetric equilibria: those in which individuals of the same type employ identical cutoffs.

3.2.1 Symmetric Cutoffs

Assume, then, that all A-types use the cutoff u_A and all B-types use the cutoff u_B. We can now construct the probability that a *randomly chosen* individual will announce A: she must be of type A, which happens with probability p, and she must want to announce A, which happens with probability $1 - F(u_A)$. Therefore the overall probability of announcing A, which we denote by λ_A, is given by

$$\lambda_A \equiv p[1 - F(u_A)].$$

Similarly, the probability that a randomly chosen individual will announce B is given by

$$\lambda_B \equiv (1 - p)[1 - F(u_B)].$$

With this notation in hand, we can rewrite the cutoff rule (1) more explicitly. First, add P^- to both sides to get

$$P^-(1 + u) \geq P^+ + P^-.$$

Assuming that we are studying this inequality for a person of type A, the right-hand side is the probability that exactly q individuals announce A, while the left-hand side is the joint probability that exactly q individuals announce A *and* no more than q individuals announce B. With this in mind, we see that the cutoff u_A must solve the equation

$$\binom{n-1}{q}\lambda_A^q \sum_{k=0}^{q} \binom{n-1-q}{k} \lambda_B^k (1 - \lambda_A - \lambda_B)^{n-1-q-k}(1 + u_A) = \binom{n-1}{q}\lambda_A^q (1 - \lambda_A)^{n-1-q}. \quad (2)$$

Likewise, the cutoff u_B solves

$$\binom{n-1}{q}\lambda_B^q \sum_{k=0}^{q} \binom{n-1-q}{k} \lambda_A^k (1 - \lambda_A - \lambda_B)^{n-1-q-k}(1 + u_B) = \binom{n-1}{q}\lambda_B^q (1 - \lambda_B)^{n-1-q}. \quad (3)$$

We will sometimes refer to these cutoffs as "best responses", though it should be clear that u_A embodies not just a "response" by an individual but also an equilibrium condition: that this individual response is equal to the cutoff employed by all compatriots of the same type.

3.2.2 A Simple Refinement

At this stage, an issue arises which we would do well to deal with immediately. It is that a symmetric cutoff of ∞ is always a best response for any type to any cutoff employed by the other type, provided that $q > 0$. This is easy enough to check: if no member in group A is prepared to declare A in any circumstance, then no A-type will find it in her interest to do so as well. This is because (with $q > 0$) no such individual is ever pivotal.

Hence the "full neutrality cutoff" $u = \infty$ is alaways a best response. But it is an unsatisfactory best response. The reason is that if the compatriots of, say, a type-A individual *do* announce A for a tiny range of very high u-values, it will push an individual type-A person to announce A for a large range of u-values, thus rendering the cutoff $u_A = \infty$ "unstable".

First let us give an intuitive argument for this. Consider an individual of type A, and let us entertain a small perturbation in the strategy of her compatriots: they use a very large cutoff, but not an infinite one. Now, in the event that our agent is pivotal, it must be that her group is very large with high probability, because her compatriots are only participating to a tiny extent, and yet there are q participants in the pivotal case. This means that group A is likely to win (conditional on the pivotal event), and our individual will want to declare A for a large range of her u-values. This shows the "instability" of the cutoff $u_A = \infty$.

This argument has a clean counterpart in the formal analysis. Once we allow for compatriots (say, of type A) to use any interior cutoff u_A, we have $\lambda_A > 0$, so that (2) reduces to the simpler form

$$\sum_{k=0}^{q} \binom{n-1-q}{k} \lambda_B^k (1-\lambda_A-\lambda_B)^{n-1-q-k}(1+u'_A) = (1-\lambda_A)^{n-1-q}. \tag{4}$$

where we're denoting our individual's cutoff by u'_A as a reminder that we haven't imposed the symmetry condition yet.

If we divide λ_B by $1-\lambda_A$, we form the probability that a randomly chosen person announces B conditional on her not announcing A. Let's call this probability π:

$$\pi \equiv \frac{\lambda_B}{1-\lambda_A}.$$

With this notation, (4) may be rewritten as

$$\frac{1}{1+u'_A} = \sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k}, \tag{5}$$

where m, it will be recalled, is the size of the supermajority ($n-q$ in other words). Now imagine that all compatriots have a very large cutoff, so that u_A is very big. Then λ_A is close to zero, so that $\pi \simeq \lambda_B$. So, by (5), u'_A is bounded. This means that the full-neutrality response is not robust to small perturbations away from full neutrality.

These arguments are *a fortiori* true in the special case of unanimity: $q = 0$. Indeed, it is easy to check that full neutrality is *never* a best response in this case, so no robustness arguments need to be invoked.

3.2.3 Equilibrium Conditions

In summary, then, the arguments of the previous section permit us to rewrite the equilibrium conditions (2) and (3) as follows:

$$\alpha(u_A, u_B) \equiv (1+u_A) \sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k} = 1, \tag{6}$$

and

$$\beta(u_A, u_B) \equiv (1+u_B) \sum_{k=0}^{q} \binom{m-1}{k} \sigma^k (1-\sigma)^{m-1-k} = 1, \tag{7}$$

where $m = n - q$, $\pi \equiv \lambda_B/1-\lambda_A$, and $\sigma \equiv \lambda_A/1-\lambda_B$.

We dispose immediately of a simple subcase: the situation in which there is simple majority and n is odd, so that q precisely equals $(n-1)/2$. The following result applies:

Observation 1 *If $q = (n-1)/2$, there is a unique equilibrium which involves $u_A = u_B = 0$.*

To see why this must be true, consult (6) and (7). Notice that when $q = (n-1)/2$, it must be that $m - 1 = n - q - 1 = q$. So the best responses must equal zero no matter what the size of the other group's cutoff. In words, there is no cost to announcing one's favorite outcome in this case. Recall that the only conceivable cost to doing so is that disagreement might result, but in the pivotal case of concern to any player, there are q compatriots announcing the favorite outcome, which means there are no more than $n - 1 - q = q$ opposing announcements. So disagreement is not a possibility.

In the remainder of the paper, then, we concentrate on the case in which a genuine supermajority is called for:

[A.4] $q < (n-1)/2$.

The following observations describe the structure of response functions in this situation. [A.1]–[A.4] hold throughout.

observation 2 *A symmetric response u_i is uniquely defined for each u_j, and declines continuously as u_j increases, beginning at some positive finite value when $u_j = 0$, and falling to zero as $u_j \to \infty$.*

observation 3 *Consider the point at which type A's response crosses the 45^0 line, or more formally, the value $\bar{u}$ at which $\alpha(\bar{u}, \bar{u}) = 1$. Then type B's best response cutoff to $\bar{u}$ is lower than $\bar{u}$, strictly so if $p < 1/2$.*

While the detailed computations that support these observations are relegated to the Appendix, a few points are to be noted. First, complete neutrality is never a (robust) best response even when members of the other group are *always* announcing their favorite alternative. The argument for this is closely related to the remarks made in Section 3.2.2 and we shall not repeat them here. On the other hand, "full aggression" — $u = 0$ — is *also* never a best response except in the limiting case as the other side tends to complete neutrality. These properties guarantee that every equilibrium (barring those excluded in Section 3.2.2) employs interior cutoffs.

observation 3 requires some elaboration. It states that *at the point where the best response of Group A leaves both sides equally aggressive* (so that $u_A = u_B = \overline{u}$), group B's best response leads to greater aggression. The majority takes greater comfort from its greater number, and therefore are more secure about being aggressive. There is less scope for disagreement. However, note the emphasized qualification above. As we shall see later, it will turn out to be important.

Figure **??** provides a graphical representation. Each response function satisfies observation 2, and in addition observation 3 tells us that the response function for B lies above that for A at the 45^0 line. We have therefore established the following proposition.

Proposition 1 *An equilibrium exists in which members of the stochastic majority — group B — behave more aggressively than their minority counterparts:* $u_B < u_A$.

Proposition 1 captures an interesting aspect of the "tyranny of the majority". Not only are the majority greater in number (at least stochastically so in this case), they are also more vocal in expressing their opinion. So group outcomes are doubly shifted — *in this particular equilibrium* — towards the majority view, once through numbers, and once through greater voice.[4] We will call such an equilibrium a *majority equilibrium*.

4 Minority Equilibria

4.1 Existence

Figure **??**, which we used in establishing Proposition 1, is drawn from actual computation. We used the exponential distribution $F(u) = 1 - \exp -2.2u$ to describe draws of u and the following

[4]Notice that this model has no voting costs so that free-riding is not an issue. Such free-riding is at the heart of the famous Olson paradox (see Olson [1965]), in which small groups may be more effective than their larger counterparts.

additional parameters: $n = 4$, $q = 1$ and $p = 0.4$. Under this specification, there is, indeed, a unique equilibrium and (by Proposition 1) it must be the majority equilibrium.

Further experimentation with these parameters leads to an interesting outcome. When n is increased, the response curves appear to “bend back" and intersect yet again, this time above the 45^0 line (see Figure **??**). A *minority equilibrium* (in which $u_A < u_B$, so that the minority are more aggressive) makes its appearance. For this example, it does so when there are 12 players.

The bending-back of response curves to generate a minority equilibrium appeared endemic enough in the computations, that we decided to probe further. To do this, we study large populations in which the ratio of q to n is held fixed at $\nu \in (0, 1/2)$. More precisely,we look at sequences $\{n, q\}$ growing unboundedly large so that q is one of the (at most) two integers closest to νn. We obtain the following analytical confirmation of the simulations:

Proposition 2 *Assume that $0 < \nu < p < 1/2$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the (at most) two integers closest to νn. Then there exists a finite N such that for all $n \geq N$, a minority equilibrium must exist.*

Several comments are in order. First, if there is a minority equilibrium, there must be at least two of them, because of the end point restrictions implied by Observations 1 and 2. Some of these equilibria will suffer from stability concerns similar to those discussed in Section 3.2.2. But there will always be other minority equilibria that are “robust” in this sense.[5]

Second, it might be felt that the threshold N described in Proposition 2 may be too large for “reasonable” group sizes. Our simulations reveal that this is not true. For instance, within the exponential class of valuation distributions, the threshold at which a minority equilibrium appears is typically around $N = 10$ or thereabouts, which is by no means a large number.

Third, the qualification that $\nu > 0$ is important. The unanimity case, with $q = 0$ is delicate. We return to this issue in Section 6. The case $p \leq \nu$, which we do not treat here, is also of interest. See Section 6 for further discussion.

Finally, we provide some intuition as to why minority existence is guaranteed for large n but not so for small n. Observe that when n is “small", there are two sorts of uncertainties that plague any player. She does not know how many people there are of her type, and she is uncertain about the realized distribution of valuations. Both these uncertainties are troublesome in that they may precipitate costly disagreement. The possibility of disagreement is lowered by more and more

[5] Once again, this follows from the end-point restrictions.

people adopting a neutral stance, though after a point it will be lowered sufficiently so that it pays individuals to step in and announce their favorite outcome. For a member of the stochastic majority, this point will be reached earlier, and so a majority equilibrium will always exist.

On the other hand, when n is large, these uncertainties go away or at any rate are reduced. Now the expectation that the minority will be aggressive can be credibly self-fulfilling, because the expectation of an aggressive strategy can be more readily transformed into the expectation of a winning outcome.

Notice — as discussed in the Introduction — how this model is akin to but simultaneously much richer than the symmetric Battle of the Sexes. By permitting different and heterogeneous valuations, as well as different group sizes, we obtain a more nuanced description of when the double equilibrium actually comes into being.

4.2 Minorities Win in Minority Equilibrium

In this section we address the distinction between an equilibrium in which one group *behaves* more aggressively, and one in which that group *wins* more often. For instance, in the majority equilibrium the majority fights harder *and* wins more often than the minority does. [It cannot be otherwise, the majority are ahead both in numbers and aggression.] But there is no reason to believe that the same is true of the minority equilibrium. The minority may be more aggressive, but the numbers are not on their side.

However, a remarkable property of this model is that a minority equilibrium *must involve the minority winning with greater probability than the majority.* Provided that a minority equilibrium exists, aggression must compensate for numbers.

Proposition 3 *In a minority equilibrium, the minority outcome is implemented with greater probability than the majority outcome.*

This framework therefore indicates quite clearly how group behavior in a given situation may be swayed both by majority and minority concerns. When the latter occurs, it turns out that we have some kind of "tyranny of the minority": they are so vocal that they actually swing outcomes (in expectation) to their side.

5 The Probability of Disagreement

In the previous section we established the existence of a minority equilibrium. However, existence was guaranteed only for large n. Hence, for this result to be meaningful it must be the case that for large n, the probability of disagreement is bounded away from one. Our next proposition proves exactly that.

Proposition 4 *Assume that $0 < \nu < p < \frac{1}{2}$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the (at most) two integers closest to νn. Then the probability of disagreement is bounded away from one.*

The intuition for this result is the following. Suppose that the probability of disagreement is high. Then the probability that each group is blocking the supermajority of its rival is also high. We show that if group A, for example, is blocking group B, then the latter will be discouraged from casting a B vote: doing so will most likely lead to disagreement, while casting a neutral vote ensures an agreement on A.

To formalize the above intuition, we do the following. We begin by showing that if the probability of each group blocking its rival goes to one, then there must be a lower bound on the probabilities with which a random individual votes for A or B, i.e. λ_A or λ_B respectively. Next, we show that this implies that π and σ are bounded below by $\frac{q}{m-1}$. Recalling the equilibrium equations, it follows that the cutoffs must be going to infinity in contradiction to our first step.

What allows individuals to agree, even when there are great many of them, is the option to remain neutral. This can be seen if we analyze a restricted version of our model in which individuals have only two options: A or B. We carry out this analysis in Section 6.1. There, we show that Proposition 4 ceases to hold.

6 Extensions

6.1 No Neutrality

In our opinion, when faced with impending disagreement, the option of a neutral stance is very natural. This is why we adopted this specification in our basic model. [As discussed already, neutrality is not to be literally interpreted as a formal announcement, but rather as a willingness to delegate one's vote to an impersonal arbiter who is pledged to achieving agreement if at all possible.] Nevertheless, it would be useful to see if the insights of the exercise are broadly preserved if announcements are restricted to be either A or B.

We can quickly sketch such a model. An individual is now pivotal under two circumstances. In the first event, the number of people announcing her favorite outcome is exactly q, which we assume to be less than $(n-1)/2$.[6] By announcing her favorite, then, disagreement is the outcome, while an announcement of the other alternative would lead to that alternative being implemented. The loss, then, from voting one's favorite in this event is precisely v' (recall that the disagreement payoff is normalized to zero). In the second event, the number of people announcing the alternative is exactly q. By announcing her favorite, she guarantees its implementation, while the other announcement would lead to disagreement. So the gain from voting one's favorite in this event is v. Consequently, an individual will announce her favorite if

$$\Pr(\text{exactly } q \text{ others vote for alternative})v \geq \Pr(\text{exactly } q \text{ others vote for favorite})v'.$$

Define $w \equiv v/v'$. Then equilibrium cutoffs w_A and w_B are given by the conditions

$$w_A \Pr(|A| = q) \geq \Pr(|B| = q) \tag{8}$$

and

$$w_B \Pr(|B| = q) \geq \Pr(|A| = q) \tag{9}$$

where $|A|$ and $|B|$ stand for the number of A- and B-announcements out of $n-1$ individuals, and where equality must hold in each of the conditions provided the corresponding cutoff strictly exceeds 1, which is the lower bound for these variables.

In this variation of the model, it is obvious that at least one group must be "fully aggressive" (i.e., its cutoff must equal one).[7] Moreover, as long as we are in the case $q < (m-1)/2$, *both* groups cannot *simultaneously* be "fully aggressive": one of the cutoffs must strictly exceed unity.

So, in contrast to our model, in which all (robust) equilibria are fully interior, the equilibria here are at "semi-corners". Nevertheless, one can study majority and minority equilibria in a parallel manner.

Specifically, to unearth a majority equilibrium, set $w_B = 1$. Then, using the equality version of (8), it must be the case that

$$w_A = \left(\frac{pF(w_A) + (1-p)}{p[1 - F(w_A)]}\right)^{n-1-2q} \tag{10}$$

[6] The case $q = (n-1)/2$ is exactly the same as in Observation 1 for the main model. No matter what the valuations are, each individual will announce her favorite outcome.

[7] Simply examine (8) and (9) and note that both right-hand sides cannot strictly exceed one.

Likewise, to examine possible minority equilibria, set $w_A = 1$. Then use the equality version of (9) to assert that

$$w_B = \left(\frac{p + (1-p)F(w_B)}{(1-p)[1 - F(w_B]} \right)^{n-1-2q} \tag{11}$$

in any minority equilibrium.

It is easy to use (10) to conclude that

Observation 4 *[1] A majority equilibrium always exists, and moreover there exists at least one which is robust in the sense of Section 3.2.2. [2] A (robust) minority equilibrium may or may not exist, but it does exist if (n, q) are sufficiently large. [3] In any minority equilibrium, the minority outcome is implemented with greater probability than the majority outcome.*

So the broad contours of our model can be replicated in this special case. This is reassuring, because it reassures us of the robustness of the results. At the same time this allows us to highlight the main implication of allowing voters to remain neutral: absent neutrality voters may be locked in situations in which they are almost certain to disagree. This is formalized in the next result.

Observation 5. *Assume $0 < \nu < p < \frac{1}{2}$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the two integers closest to νn. There exists a sequence of semi-corner equilibria for which the probability of disagreement coverges to one.*

The above result demonstrates the importance of being neutral: Neutrality allows the players to avoid disagreement. Recall that Proposition 4 establishes that with neutrality, the probability of disagreement at every interior equilibrium is bounded away from one. Once the option of neutrality is taken away, the probability that players reach a disagreement (at any interior equilibrium) goes to one.

6.2 Comparative statics on the supermajority rule

Up to this point we have focused on group size as a determinant of the existence of minority equilibria. A related question is whether existence is affected by variations in the supermajority rule.

Common intuition suggests that a higher supermajority requirement facilitates the emergence of a minority equilibrium. Indeed, the comparative politics literature compares different political systems and motivates what has been termed "consensus systems" (Lipjiart (1985)) by the desire to protect minorities from the tyrany of the majority.

Our model allows us to investigate the hypothesis discussed above. We have already shown that when the supermajority is relatively high (i.e., $\frac{m}{n} > 1-p$), the existence of minority equilibria is guaranteed for large n. The question is, whether the minority can get its way when the supermajority requirement is relaxed. This is the subject of our next proposition.

Proposition 5 *Assume that $0 < p < \nu < \frac{1}{2}$. Consider any sequence $\{n, q\}$ such that $n \to \infty$ and q is one of the (at most) two integers closest to νn. Then there exists a finite N such that for all $n \geq N$, a minority equilibrium does not exist.*

Taken together, Propositions 2 and 5 may suggest a monotonic relation between the supermajority requirement and the "power" of the minority. However, it is not clear whether this is true in our model. To see why, consider an A voter and the equilibrium equation (6). Holding B's cutoff fixed, as q decreases, A's cutoff increases, i.e., the group fights less aggressively. This follows from the fact that as q decreases, the probability, that the B-types would block A from being chosen, increases. Because the above effect of lowering q applies to both groups, it is not clear which group benefits from this change. To demonstrate the ambiguous effect of lowering q consider the following example.

Example 2. Let $n = 1,000$ (in light of Proposition 3 we intentionaly pick a large n), $p = 0.4$ and consider the distribution function $F(u) = 1 - \frac{1}{\sqrt{\ln(u+e)}}$. For $q = 300$ there exists a minority equilibrium $u_A \simeq 1.35$ and $u_B \simeq 80$. However, for $q = 10$ there exists no minority equilibrium.

The above example seems to suggest that for some distribution functions a minority equilibrium may not exist when the supermajority requirement is close to unanimity.

Conjecture 1 *Suppose $m = n$. There exists a family of distribution functions for which a minority equilibrium does not exists for any n.*

6.3 Certain group size

In this section we investigate the implication of uncertainty regarding the size of one's group. For this purpose, we modify our model by assuming that it is common knowledge that there are n_A individuals of type A and $n_B > n_A$ individuals of type B. We retain all our other assumptions.

The first observation we make is that our arguments in Section 3.2.2. do not apply to this new model. To see why, consider the case when all B types are voting for B, whereas only extreme A-types are voting for A. When an A-type knows exactly how many B-types there are, he realizes that he can only create a disagreement by voting for A. Therefore, when group sizes are known, the two corner equilibria are robust (in the sense of Section 3.2.2). This suggests that the corner equilibria are unnatural in the following sense: when faced with some uncertainty about group sizes, some individuals may still put up a fight.

Our second observation relates to the importance of group size in the emergence of minority equilibria. Potentially, the existence of minority equilibria in our original model may be due to two types of uncertainties that are relaxed in large groups. First, as the number of individuals in the group increases, voters have a more accurate estimate of the proportion of their types in the group. Second, as the population increases, each individual has a better picture of the distribution of intesities among his compatriots.

To better understand the effect of the first type of uncertainty, we focus now on a model in which the numbers of A types and B types are known. We ask whether in this new model, minority equilibria can arise for any n. It can easily be shown that the equilibrium cutoff for one type depend only on the equilbrium cutoff of the other type. More precisely, an equilibrium (u_A, u_B) satisfies the following equations,

$$\begin{aligned}(1+u_A)\sum_{k=0}^{q}\binom{n_B}{k}(F(u_B))^{n_B-k}(1-F(u_B))^k &= 1\\(1+u_B)\sum_{k=0}^{q}\binom{n_A}{k}(F(u_A))^{n_A-k}(1-F(u_A))^k &= 1\end{aligned}$$

where $n_A < n_B$ are the number of individuals of type A and B respectively.

It is straightforward to construct examples in which there does not exist a minority equilibrium for small n_A and n_B. For instance, take $F(u) = 1 - \frac{1}{\sqrt{\ln(u+e)}}$, $n_A = 2$, $n_B = 3$ and $q = 1$. For these values there exists a unique interior majority equilibrium, $u_A \approx 250$ and $u_B \approx 0.22$. However, using arguments similar to those employed in Proposition 2 and 4, one can show that for large n a minority equilibrium exists and the probability of disagreement is bounded away from one. By simple stochastic dominance arguments, it can be shown that in any minority equilbrium the minority wins more often.

We conclude that certainty regarding the numbers of A and B types is not sufficient to generate a minority equilibrium; even when the numbers of A and B types are known, we still need n to be sufficiently large for the minority to prevail.

6.4 Types who prefer disagreement to the rival alternative

Suppose there exist types who rank disagreement above their second best alternative. Clearly, voting for the preferred alternative is weakly dominant for these types. Hence, in any interior equilibrium these individuals would vote their type. In this sense, incorporating these voters into our model is equivalent to adding aggregate noise. We believe that if the proportion of such types is sufficiently low, all of our results continue to hold.

7 Appendix

Proof of observation 2. For concreteness, set $i = A$ and $j = B$. Fix any $u_B \in [0, \infty)$. Recall that

$$\pi = \frac{\lambda_B}{1 - \lambda_A} = \frac{(1-p)[1 - F(u_B)F(u_B)]}{1 - p[1 - F(u_A)F(u_A)]},$$

so that π is continuous in u_A, with $\pi \to 1 - F(u_B)$ as $u_A \to 0$, and $\pi \to (1-p)[1 - F(u_B)]$ as $u_A \to \infty$. Consequently, recalling (6) and recalling that $q < (n-1)/2$, we see that $\alpha(u_A, u_B)$ converges to a number strictly less than one as $u_A \to 0$, while it becomes unboundedly large as $u_A \to \infty$. By continuity, then, there exists some u_A such that $\alpha(u_A, u_B) = 1$, establishing the existence of a cutoff.

To show uniqueness, it suffices to verify that α is strictly increasing in u_A. Because the expression $\sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k}$ must be decreasing in π, it will suffice to show that π itself is declining in u_A, which is a matter of simple inspection.

To show that the response u_A strictly decreases in u_B, it will therefore be enough to establish that α is also increasing in u_B. Just as in the previous paragraph, we do this by showing that π is decreasing in u_B, which again is a matter of elementary inspection.

Finally, we observe that $u_A \downarrow 0$ as $u_B \uparrow \infty$. Note that along such a sequence, $\pi \to 0$ regardless of the behavior of u_A. Consequently, $\sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k}$ converges to 1 as $u_B \uparrow \infty$. To maintain the equality (6), therefore, it must be the case that $u_A \downarrow 0$.

Of course, all these arguments hold if we switch A and B. ∎

Proof of Observation 3 . Let $\overline{u}$ be defined as in the statement of this Observation. Define $\overline{\lambda_A} \equiv p[1 - F(\overline{u})]$ and $\overline{\lambda_B} \equiv (1-p)[1 - F(\overline{u})]$. Then

$$(1 + \overline{u}) \sum_{k=0}^{q} \binom{m-1}{k} \overline{\pi}^k (1 - \overline{\pi})^{m-1-k} = 1, \tag{12}$$

where $\overline{\pi} \equiv \overline{\lambda_B}/(1-\overline{\lambda_A})$. Now recall that σ in (7) is defined by $\sigma = \frac{\lambda_A}{1-\lambda_B}$, so that if we consider the corresponding value $\overline{\sigma}$ defined by setting $u_A = u_B = \overline{u}$, we see that

$$\overline{\sigma} \leq \overline{\pi} \text{ if and only if } \overline{\lambda_A}(1-\overline{\lambda_A}) \leq \overline{\lambda_B}(1-\overline{\lambda_B}).$$

But $\lambda_A \leq 1/2$ (because $p \leq 1/2$), so that the second inequality above holds if and only if $\overline{\lambda_A} \leq \overline{\lambda_B}$, and this last condition follows simply from the fact that $p \leq 1/2$.

So we have established that $\overline{\sigma} \leq \overline{\pi}$. It follows that

$$\sum_{k=0}^{q} \binom{m-1}{k} \overline{\pi}^k (1-\overline{\pi})^{m-1-k} \leq \sum_{k=0}^{q} \binom{m-1}{k} \overline{\sigma}^k (1-\overline{\sigma})^{m-1-k}$$

and using this information in (12), we must conclude that

$$\beta(\overline{u}, \overline{u}) = (1+\overline{u}) \sum_{k=0}^{q} \binom{m-1}{k} \overline{\sigma}^k (1-\overline{\sigma})^{m-1-k} \geq 1, \tag{13}$$

Recalling that β is increasing in its first argument (see proof of observation 1 **??**), it follows from (13) that type B's best response to $\overline{u}$ is no bigger than $\overline{u}$.

Finally, observe that all these arguments apply with strict inequality when $p < 1/2$. ∎

Proof of Proposition 1. For each $u_B \geq 0$, define $\phi(u_B)$ by composing best responses: $\phi(u_B)$ is A's best response to B's best response to u_B. By observation 2, we see that A's best response is a positive, finite value when $u_B = 0$, and therefore so is B's response to this response. Consequently, $\phi(0) > 0$. On the other hand, A's best response is precisely $\overline{u}$ when $u_B = \overline{u}$, and by Observation 3 we must conclude that $\phi(\overline{u}) < \overline{u}$. Because ϕ is continuous (Observation 2 again), there is $u_B^* \in (0, \overline{u})$ such that $\phi(u_B^*) = u_B^*$. Let u_A^* be type A's best response to u_B^*. Then it is obvious that (u_A^*, u_B^*) is an equilibrium. Because $u_B^* < \overline{u}$, we see from Observation 2 that $u_A^* > \overline{u}$. We have therefore found a majority equilibrium. ∎

Proof of Proposition 2. Consider any sequence $\{n, q\}$ as described in the statement of the proposition. Because $p > \nu$, there exists a cutoff $\overline{u}_A > 0$ and a finite n^* such that for all $n \geq n^*$,

$$\overline{\lambda}_A \equiv p\left[1 - F\left(\overline{u}_A\right)\right] > \frac{q}{n-1} \simeq \nu. \tag{14}$$

Note that there is also an associated sequence $\{m\}$ defined by $m \equiv n - q$.

We break the proof up into several steps.

Step 1. We claim that there exists an integer M such that for each $m \geq M$ there is $u_B^m < \infty$ that solves the following equation:

$$\sum_{k=0}^{q} \binom{m-1}{k} (\pi_m)^k (1-\pi_m)^{m-1-k} = \frac{1}{1+\overline{u}_A} \tag{15}$$

where

$$\pi_m \equiv \frac{\lambda_B^m}{1-\overline{\lambda}_A}$$

and

$$\lambda_B^m \equiv (1-p)\left[1-F\left(u_B^m\right)\right].$$

We prove this claim. Note that for all $n \geq n^*$, $1-p > p > q/(n-1)$, so that

$$\overline{\pi} \equiv \frac{(1-p)(n-1)}{m-1} > \frac{q}{m-1} \simeq \frac{\nu}{1-\nu}$$

for all $n \geq n^*$. Consequently, by the Strong Law of Large Numbers (SLLN),

$$\sum_{k=0}^{q} \binom{m-1}{k} \overline{\pi}^k \left(1-\overline{\pi}\right)^{m-1-k} \to 0$$

as m and q grow to infinity. It follows that there exists M such that for all $m \geq M$ (and associated q),

$$\sum_{k=0}^{q} \binom{m-1}{k} \overline{\pi}^k \left(1-\overline{\pi}\right)^{m-1-k} < \frac{1}{1+\overline{u}_A}. \tag{16}$$

For such m, provisionally consider $u_B^m = 0$. Then

$$\frac{\lambda_B^m}{1-\overline{\lambda}_A} = \frac{1-p}{1-p\left[1-F\left(\overline{u}_A\right)\right]},$$

and using this in (14), we conclude that

$$\pi_m = \frac{\lambda_B^m}{1-\overline{\lambda}_A} = \frac{1-p}{1-p\left[1-F\left(\overline{u}_A\right)\right]} > \frac{(1-p)\,(n-1)}{m-1} = \overline{\pi}.$$

Combining this information with (16), we see that if $u_B^m = 0$, then

$$\sum_{k=0}^{q} \binom{m-1}{k} \pi_m^k \left(1-\pi_m\right)^{m-1-k} < \frac{1}{1+\overline{u}_A}. \tag{17}$$

Next, observe that if u_B^m is chosen very large, then λ_B^m and consequently π_m are both close to zero, so that $\sum_{k=0}^{q} \binom{m-1}{k} \pi_m^k \left(1-\pi_m\right)^{m-1-k}$ is close to unity. It follows that for such u_B^m,

$$\sum_{k=0}^{q} \binom{m-1}{k} \pi_m^k \left(1-\pi_m\right)^{m-1-k} > \frac{1}{1+\overline{u}_A}. \tag{18}$$

Combining (17) and (18) and noting that the LHS of (15) is continuous in u_B^m, it follows that for all $m \geq M$ there exists $0 < u_B^m < \infty$ such that the claim is true.

Step 2. One implication of (15) in Step 1 is the following assertion: as $(m, q) \to \infty$,

$$\pi_m \to \nu/(1-\nu) \in (0,1), \text{ and in particular, } u_B^m \text{ is bounded.} \tag{19}$$

To see why, note that $\frac{1}{1+\overline{u}_A} \in (0,1)$. Using (15) and SLLN, it must be that $\pi_m \to \nu/(1-\nu) \in (0,1)$ as $(m,q) \to \infty$. Recalling the definition of π_m it follows right away that u_B^m must be bounded.

Step 3. Next, we claim there exists an integer M^* such that

$$\text{For all } m \geq M^*, u_B^m > \overline{u}_A. \tag{20}$$

To establish this claim, note first, using (14), that

$$p\left[1 - F\left(\overline{u}_A\right)\right] > \frac{q}{n-1} = \frac{\frac{q}{m-1}}{1 + \frac{q}{m-1}} > \frac{\frac{q}{m-1}}{\frac{1-p}{p} + \frac{q}{m-1}}$$

where the last inequality follows from the assumption that $p \in \left(0, \frac{1}{2}\right)$, so that $\frac{1-p}{p} > 1$. A sinmple rearrangement of this inequality shows that

$$\frac{(1-p)\left[1 - F\left(\overline{u}_A\right)\right]}{1 - p\left[1 - F\left(\overline{u}_A\right)\right]} > \frac{q}{m-1} \simeq \frac{\nu}{1-\nu} \tag{21}$$

Now suppose, contrary to the claim, that $u_B^m \leq \overline{u}_A$ along some subsequence of m. Then on that subsequence,

$$\pi_m = \frac{\lambda_B^m}{1 - \overline{\lambda_A}} = \frac{(1-p)\left[1 - F\left(u_B^m\right)\right]}{1 - p\left[1 - F\left(\overline{u}_A\right)\right]} \geq \frac{(1-p)\left[1 - F\left(\overline{u}_A\right)\right]}{1 - p\left[1 - F\left(\overline{u}_A\right)\right]} \tag{22}$$

Combining (21) and (22), we may conclude that along the subsequence of m for which $u_B^m \leq \overline{u}_A$,

$$\inf_m p_m > \frac{\nu}{1-\nu},$$

which contradicts (19) of Step 2.

To prepare for the next step, let $\hat{u}_B^m$ denote the best response of the B-types to $u_A = \overline{u}_A$. That is,

$$\frac{1}{1 + \hat{u}_B^m} = \sum_{k=0}^{q} \binom{m-1}{k} \sigma_m^k \left(1 - \sigma_m\right)^{m-1-k}, \tag{23}$$

where

$$\sigma_m \equiv \frac{\overline{\lambda}_A}{1 - \hat{\lambda}_B^m}$$

and

$$\hat{\lambda}_B^m \equiv (1-p)\left[1 - F\left(\hat{u}_B^m\right)\right]$$

Step 4. There is an integer M^{**} such that for all $m \geq M^{**}$ $\hat{u}_B^m > u_B^m$.

To prove this claim, suppose on the contrary that $\hat{u}_B^m \leq u_B^m$ along some subsequence of m. [All references that follow are to this subsequence.] Then

$$\sigma_m = \frac{\overline{\lambda}_A}{1 - \hat{\lambda}_B^m} = \frac{p\left[1 - F\left(\overline{u}_A\right)\right]}{1 - (1-p)\left[1 - F\left(\hat{u}_B^m\right)\right]} \geq \frac{p\left[1 - F\left(\overline{u}_A\right)\right]}{1 - (1-p)\left[1 - F\left(u_B^m\right)\right]} = \frac{\overline{\lambda}_A}{1 - \lambda_B^m}. \tag{24}$$

Recall from (19), Step 2, that $\frac{\lambda_B^m}{1-\overline{\lambda}_A} \to \frac{\nu}{1-\nu}$. Therefore $\lambda_B^m \to \overline{\lambda}_B$, where $\overline{\lambda}_B \equiv \frac{\nu}{1-\nu}\left(1-\overline{\lambda}_A\right)$. Recall from (14) that $\overline{\lambda}_A > \nu$, , so that $\overline{\lambda}_B < \nu$ and in particular $\overline{\lambda}_B < \overline{\lambda}_A$. Because $p < 1/2$, so is $\overline{\lambda}_B$, and these last assertions permit us to conclude that $\overline{\lambda}_A\left(1-\overline{\lambda}_A\right) > \overline{\lambda}_B\left(1-\overline{\lambda}_B\right)$, or equivalently, that

$$\frac{\overline{\lambda}_A}{1-\overline{\lambda}_B} > \frac{\overline{\lambda}_B}{1-\overline{\lambda}_A}.$$

Using this information in (24) and recalling that $\lambda_B^m \to \overline{\lambda}_B$, we may conclude that

$$\liminf_{m\to\infty} \sigma_m \geq \frac{\overline{\lambda}_A}{1-\overline{\lambda}_B} > \frac{\overline{\lambda}_B}{1-\overline{\lambda}_A} = \frac{\nu}{1-\nu},$$

where the last equality is from (19). It follows from (23) that $\hat{u}_B^m \to \infty$. But this contradicts our supposition that $\hat{u}_B^m \leq u_B^m$ (that along a subsequence) because the latter is bounded; see (19) of Step 2.

To complete the proof of the proposition, define, for each $m \geq M^{**}$ and each $u_A \in (0, \overline{u}_A]$, $\psi^m(u_A)$ as the *difference* between B's best response to u_A and the value of u_B to which u_A is a best response. By Step 1 and Observation **??**, ψ^m is well-defined and continuous on this interval. Using Observation **??** yet again, it is easy to see that (for each m) $\psi^m(u_A) < 0$ for small values of u_A, while Step 4 assures us that $\psi^m(\overline{u}_A) > 0$. Therefore for each m, there is $\tilde{u}_A^m \in (0, \overline{u}_A)$ such that $\psi^m(\tilde{u}_A^m) = 0$. If we define $\tilde{u}_B^m$ to be the best response to $\tilde{u}_A^m$, it is trivial to see that $(\tilde{u}_A^m, \tilde{u}_B^m)$ constitutes an equilibrium.

Finally, note that

$$\tilde{u}_A^m < \overline{u}_A < u_B^m < \hat{u}_B^m < \tilde{u}_B^m,$$

where the second inequality follows from Step 3, the third inequality from Step 4, and the last inequality from the fact that the best response function is decreasing (Observation **??**). This means that $(\tilde{u}_A^m, \tilde{u}_B^m)$ is a minority equilibrium. ∎

Proof of Proposition 3. Recall (6) and (7) and note that $u_A < u_B$ in a minority equilibrium. It follows right away that $\sum_{k=0}^{q}\binom{m-1}{k}\pi^k(1-\pi)^{m-1-k} > \sum_{k=0}^{q}\binom{m-1}{k}\sigma^k(1-\sigma)^{m-1-k}$, so that $\pi < \sigma$. Expanding this inequality, we conclude that $\lambda_B(1-\lambda_B) < \lambda_A(1-\lambda_A)$. Because $\lambda_A < 1/2$, this means that $\lambda_B < \lambda_A$.

Now turn to the difference in probability that A (rather than B) is the outcome. Let $|A|$ (resp. $|B|$) denote the number of A (resp. B) announcements. Notice that

$$\Pr(|A| \geq q+1; |B| \leq q) = \sum_{k=q+1}^{n}\binom{n}{k}\sum_{j=0}^{q}\binom{n-k}{j}\lambda_A^k\lambda_B^j(1-\lambda_A-\lambda_B)^{n-k-j},$$

while

$$\Pr(|B| \geq q+1; |A| \leq q) = \sum_{k=q+1}^{n} \binom{n}{k} \sum_{j=0}^{q} \binom{n-k}{j} \lambda_B^k \lambda_A^j (1-\lambda_A-\lambda_B)^{n-k-j},$$

Let $\Delta \equiv \Pr(|A| \geq q+1; |B| \leq q) - \Pr(|B| \geq q+1; |A| \leq q)$. Then, combining the equalities above,

$$\Delta = \sum_{k=q+1}^{n} \binom{n}{k} \sum_{j=0}^{q} \binom{n-k}{j} [\lambda_A^k \lambda_B^j - \lambda_A^k \lambda_B^j](1-\lambda_A-\lambda_B)^{n-k-j} > 0,$$

because $\lambda_A > \lambda_B$ and $k > j$.

But it is easy to verify that

$$\Delta = \Pr(A \text{ wins}) - \Pr(B \text{ wins}),$$

and the proof of the proposition is complete. ∎

Proof of Proposition 4. Assume that $q < \frac{n-1}{2}$ (When $q = \frac{n-1}{2}$ the probability of disagreement is zero). Note that the probability of disagreement is equal to $\Pr(|A| > q, |B| > q)$. It suffices to show that $\Pr(A > q)$ and $\Pr(B > q)$ cannot both converge to one. To see this, note that

$$\Pr(|A| > q, |B| > q) \leq \min\{\Pr(|A| > q), \Pr(|B| > q)\}.$$

Suppose, to the contrary, that $\lim_{n\to\infty} \Pr(A > q) = 1$ and $\lim_{n\to\infty} \Pr(B > q) = 1$. The proof proceeds in two steps. In the first step we show that for large n both λ_A and λ_B are strictly above υ. Moreover, if either λ_A or λ_B converges to υ, then it converges at a rate slower than $\frac{1}{\sqrt{n}}$. In the second step we show that this implies that the equilibrium cutoffs, u_A and u_B, must be growing to infinity, in contradiction to step 1.

Step 1. $\lim_{n\to\infty} \frac{(\lambda_A-\upsilon)\sqrt{n}}{\sqrt{\lambda_A(1-\lambda_A)}} = \infty$ and $\lim_{n\to\infty} \frac{(\lambda_B-\upsilon)\sqrt{n}}{\sqrt{\lambda_B(1-\lambda_B)}} = \infty$.

We prove $\lim_{n\to\infty} \frac{|\lambda_A-\upsilon|\sqrt{n}}{\sqrt{\lambda_A(1-\lambda_A)}} = \infty$, similar arguments hold for λ_B.

Assume to the contrary that there exists a subsequence for which $\lim_{n\to\infty} \frac{(\lambda_A^{k_n}-\upsilon)\sqrt{n}}{\sqrt{\lambda_A(1-\lambda_A)}} = c$, where $-\infty \leq c < \infty$.

Let X_n denote the number of A announcements (i.e., $|A|$). By the Berry-Esseen Theorem (see ???), for some $\varepsilon < \Phi(-c)$, there exists an N such that for $n > N$

$$\Pr(X_n > q) = \Pr(\frac{X_n - n\lambda_A^{k_n}}{\sqrt{n\lambda_A^{k_n}(1-\lambda_A^{k_n})}} > \frac{-(\lambda_A^{k_n}-\upsilon)\sqrt{n}}{\sqrt{\lambda_A^{k_n}(1-\lambda_A^{k_n})}}) < 1 - \Phi(-c) + \varepsilon < 1$$

and this contradicts our premise that $\lim_{n\to\infty} \Pr(|A| > q) = 1$.

Recalling that $\pi = \frac{\lambda_B}{1-\lambda_A}$ and $\sigma = \frac{\lambda_A}{1-\lambda_B}$, it follows from step 1 that $\lim_{n\to\infty} \frac{(\pi - \frac{v}{1-v})\sqrt{n}}{\sqrt{\pi(1-\pi)}} = \infty$ and $\lim_{n\to\infty} \frac{(\sigma - \frac{v}{1-v})\sqrt{n}}{\sqrt{\sigma(1-\sigma)}} = \infty$.

Step 2. If $\lim_{m\to\infty} \frac{(\pi - \frac{v}{1-v})\sqrt{m-1}}{\sqrt{\pi(1-\pi)}} = \infty$ and $\lim_{m\to\infty} \frac{(\sigma - \frac{v}{1-v})\sqrt{m-1}}{\sqrt{\sigma(1-\sigma)}} = \infty$, then $u_A \longrightarrow \infty$ and $u_B \longrightarrow \infty$.

As in step 1 we provide a proof for u_A and similar arguments follow for u_B.

Recalling equation (6), we now show that the right hand side of this equation converges to zero. Let Y_n be the sum of successes from a binomial distribution with probability of success π and with $m-1$ draws. Therefore,

$$\begin{aligned}\sum_{k=0}^{q} \binom{m-1}{k} \pi^k (1-\pi)^{m-1-k} &= \Pr(Y_n \leq q) \leq \Pr(|Y_n - (m-1)\pi| \geq (m-1)\pi - q) \\ &< \frac{Var(Y_n)}{((m-1)\pi - q)^2} = \frac{1}{\left(\frac{(\pi - \frac{q}{m-1})\sqrt{m-1}}{\sqrt{\pi(1-\pi)}}\right)^2} \to 0,\end{aligned}$$

where the last inequality is by Chebyshev's inequality and the limit follows from the premise. Therefore, by (6) it must be that $u_A \longrightarrow \infty$. This implies that $\lambda_A \to 0$, in contradiction to step 1.

This concludes the proof. ∎

Proof of Observation 4. Define $\delta \equiv 1/(n-1-2q)$, and rewrite (8) to obtain

$$(1 + w_A^{\delta})[1 - F(w_A)] = 1/p. \tag{25}$$

Notice that when $w_A = 1$, the LHS of (25) equals 2, while the RHS is strictly greater than 2 (because $p < 1/2$). Now we consider two cases.

Case 1. There is some w such that the LHS of (25), evaluated at $w_A = w$, strictly exceeds $1/p$. In this case, the first intersection (not counting tangencies) of the function $(1+x^{\delta})[1-F(x)]$ with the value $1/p$ can easily be seen to be a robust equilibrium, along with the value $w_B = 1$.

Case 2. $(1 + x^{\delta})[1 - F(x)]$ is no more than $1/p$ for all x. In this case it can be verified that $w_A = \infty$ and $w_B = 1$ is the a majority equilibrium satisfying our robustness criterion.

This proves part [1]. For part [2], rewrite (9) as

$$(1 + w_B^{\delta})[1 - F(w_B)] = 1/(1-p). \tag{26}$$

Notice that when $w_B = 1$, the LHS of (26) equals 2, while the RHS is strictly smaller than 2 (because $p < 1/2$).

Now suppose that there is some w such that the LHS of (26), evaluated at $w_B = w$, is strictly less than $1/(1-p)$. In this case, study the *second* intersection (not counting tangencies) of the function $(1+x^\delta)[1-F(x)]$ with the value $1/(1-p)$, along with the value $w_B = 1$. If no such intersection exists, set $w_B = \infty$. It can be verified that we have found a robust equilibrium.

It remains to show that the condition in the first line in the previous paragraph is satisfied for all (n,q) large enough. To this end, fix some w such that $1-F(w) < 1/2(1-p)$. Now take (n,q) to infinity adn notice that $\delta \to 0$. Therefore w^δ converges to 1. It follows that for large (n,q),

$$(1+w^\delta)[1-F(w)] < 1/(1-p),$$

and we are done. ∎

Proof of Observation 4, part [3]. The probability that the minority outcome is implemented is given by

$$\Pr(|A| \geq m) = \sum_{k=m}^{n} \binom{n}{m} [p+(1-p)F(\omega_B)]^k [(1-p)(1-F(\omega_B))]^{n-k}$$

Similarly,

$$\Pr(|B| \geq m) = \sum_{k=m}^{n} \binom{n}{m} [(1-p)(1-F(\omega_B))]^k [p+(1-p)F(\omega_B)]^{n-k}$$

Thus, $\Pr(|A| \geq m) > \Pr(|B| \geq m)$ if and only if

$$\begin{aligned}
(1-p)(1-F(\omega_B)) &< p+(1-p)F(\omega_B) \\
(1-p)(1-2F(\omega_B)) &< p \\
1-2F(\omega_B) &< \frac{p}{1-p} \\
1-\frac{p}{1-p} &< 2F(\omega_B) \\
\frac{1}{2}\left(\frac{1-2p}{1-p}\right) &< F(\omega_B)
\end{aligned}$$

The last inequality can be written as follows:

$$\frac{1}{2(1-p)} > 1-F(\omega_B) \qquad ((*))$$

From (26) it follows that

$$1-F(\omega_B) = \frac{1}{(1-p)\left(1+\omega_B^\delta\right)}$$

where $\omega_B > 1$. Hence, $(1-p)\left(1+\omega_B^{\delta}\right) > 2(1-p)$, which implies (*).■

Proof of Observation 5. Let $\omega_B^* > 1$ be the solution to the following equation:

$$p + (1-p)F(\omega_B^*) = (1-p)\left[1 - F(\omega_B^*)\right]$$

We proceed in two steps.

Step 1. There exists a sequence of semi-corner minority equilibria that converges to $(1, \omega_B^*)$.

Fix ω_A at one. Note that when $\omega_B = \omega_B^*$ the RHS of (11) is smaller than the LHS. For any $\varepsilon > 0$, set $\omega_B = \omega_B^* + \varepsilon$. Because $\frac{p+(1-p)F\left(\omega_B^*+\varepsilon\right)}{(1-p)\left[1-F\left(\omega_B^*+\varepsilon\right)\right]} > 1$, there exists $N(\varepsilon) < \infty$ such that for all $n \geq N(\varepsilon)$, the LHS of (11) is strictly greater than its RHS. It follows that for all $n \geq N(\varepsilon)$, there exists an equilibrium $(1, \omega_B^n)$ where $\omega_B^n \in (\omega_B^*, \omega_B^* + \varepsilon)$. This establishes the first step.

Step 2. The probability of disagreement, along the sequence of equilibria that converge to $(1, \omega_B^*)$, converges to one.

Along the above sequence, the proportion of players voting for A in this equilibrium converges to $p + (1-p)F(\omega_B^*) > \nu$. Assume that the probability of disagreement along this sequence does not converge to one. Then the proportion of players voting for B must converge to zero. But this proportion converges to $(1-p)(1-F(\omega_B^*)) >> 0$.■

Proof of Proposition 5. Suppose on the contrary that a minority equilibrium (u_A^n, u_B^n) exists along some subsequence of n [all references that follow are to this subsequence]. Then $\lim_{n\to\infty}(u_A^n, u_B^n)$ is either (∞, ∞), $(0, \infty)$ or a pair of strictly positive but finite numbers (u_A^*, u_B^*). To prove that our supposition is wrong, we show that none of these limits are true.

Assume $(u_A^n, u_B^n) \to (\infty, \infty)$. Then $\lambda_A^n \to 0$ and $\lambda_B^n \to 0$. This implies that $\pi^n \to 0$ and $\sigma^n \to 0$. But this implies, by equations (6) and (7) and using SLLN, that $(u_A^n, u_B^n) \to (0, 0)$, a contradiction.

Assume $(u_A^n, u_B^n) \to (0, \infty)$. Then $\lambda_A^n \to p$ and $\lambda_B^n \to 0$, so that $\sigma^n \to p < \nu < \frac{q}{m-1}$. But using (7) and SLLN, this implies that $u_B^n \to 0$, a contradiction.

Assume $(u_A^n, u_B^n) \to (u_A^*, u_B^*)$, where both u_A^* and u_B^* are strictly positive and finite. Using SLLN and equations (6) and (7), it follows that π^n and σ^n must both converge to $\frac{q}{m-1}$. This means that $\lambda_A^n \to \lambda_A^*$ and $\lambda_B^n \to \lambda_B^*$ such that

$$\frac{\lambda_B^*}{1-\lambda_A^*} = \frac{\lambda_A^*}{1-\lambda_B^*}$$

This equlity holds only if $\lambda_A^* = \lambda_B^*$, or if $\lambda_A^* = 1 - \lambda_B^*$. Suppose the former is true. Then $\pi^n \to \pi^*$

where

$$\pi^* = \frac{\lambda_B^*}{1-\lambda_A^*} < \frac{\nu}{1-\upsilon} \simeq \frac{q}{m-1}$$

But the above inequlity implies, by (6) and SLLN, that $u_A^n \to 0$, a contradiction. Suppose next that $\lambda_A^* = 1 - \lambda_B^*$. But $1 - \lambda_B^* > p > \lambda_A^*$, a contradiction.■

References

[To be added]