

# On the Bayesian Foundation of Dominant Strategy Mechanisms

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**Abstract**

## 1 Introduction

Harsanyi's model for games with incomplete information has been a key practical tool for much of economic analysis. Among the most important applications of the Harsanyi (1967-68) formulation has been to the theory of mechanism design. A mechanism designer, in order to implement her desired outcome requires the private information held by some agents. Applying Harsanyi's approach, all relevant private information held by an agent is summarized by the agent's *type*, and the principal's problem is reduced to providing incentives for agents to truthfully reveal their type.

Mertens and Zamir (1985) (see also Brandenburger and Dekel (1993)) demonstrated that the Harsanyi formulation is without loss of generality in the following sense. There is no incomplete information scenario that cannot be modeled using Harsanyi's approach. In fact, given the primitive payoff relevant data, there exists a *universal type space* in which every conceivable state of private information is represented by a distinct type. Thus, one can analyze all problems of incomplete information using this particular universal Harsanyi model. But notice that *only* this universal model can be used without loss of generality. Any smaller type space implicitly embeds some assumption about the nature of incomplete information.

In practical applications of the theory the universal type space is almost never used. It is simply too complicated to be tractable even in the simplest of cases. Instead, analysis is

carried out in the context of smaller, easily definable type spaces. A common methodological view among game theorists is that in yielding to this practical necessity, the analyst must be conscious of the trade-off between tractability and generality, particularly the possibility that models constructed for simplicity may unnoticably entail strong assumptions that deliver special results.<sup>1</sup>

These concerns are equally germane to the theory of mechanism design. Indeed, it has been argued in the literature (Neeman (2002) and Bergemann and Morris (2002)) that commonly used small type spaces are at the heart of strong results such as full surplus extraction in, for example, auction theory (Cr mer and McLean (1988)). Whether or not these arguments are compelling, it cannot be disputed that the unexamined use of small type spaces rules out any analysis of the extent to which some specific result holds more generally across the full range of incomplete information environments.

There are at least two short-cuts that have been taken in the literature to address these concerns without sacrificing tractability. The first is to make the nongeneric assumption of independent distribution whenever possible. While incentive compatibility in general has no bite when types are correlated, the problem of optimal mechanism design becomes interesting again in the hairline case of independently distributed types.

Another short-cut is to strengthen the solution concept. In particular, while interim incentive compatibility (hereafter interim IC or iIC) is very permissive within commonly used simple type spaces,<sup>2</sup> dominant strategy incentive compatibility (hereafter dominant strategy IC, or dSIC) remains consistent with the intuitive postulates of information rent, the tradeoff between efficiency and optimality, etc.<sup>3</sup>

An informal motivation for this short-cut is the following. Take the problem of optimal auction design as an example. The auctioneer may have confidence in her estimate of the distribution  $\nu$  of the bidders' valuations, perhaps based on data from similar auctions in the past, but does not have reliable information on which to base a conjecture about the bidders' beliefs (including their beliefs about one another's valuations, their beliefs about these beliefs, etc.), as these are arguably never observed. Given her lack of information on bidders' beliefs, she may consider many different combinations of valuation and belief as probable. The set of probable combinations, from her point of view, may be much larger

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<sup>1</sup>Myerson (1991) neatly articulates this perspective.

<sup>2</sup>In the literature on mechanism design, the term of "Bayesian incentive compatibility" is usually used. In this paper, we follow the convention of Bergemann and Morris (2002) and adopt the terminology of "interim incentive compatibility," partly to emphasize that agents' beliefs do not necessarily come from the Bayesian updating of any (common) priors. Bergemann and Morris (2002) also use the terminology of "ex post incentive compatibility" instead of "dominant strategy incentive compatibility" even in private-value settings (these two concepts are different in interdependent-value settings). In this paper, however, we continue to use the more traditional terminology of "dominant strategy incentive compatibility," partly to remind the reader that we are focusing on private-value settings only.

<sup>3</sup>We use the phrase "incentive compatible" to include individual rationality. This is important because dominant strategy incentive compatibility alone is not enough to rule out full-surplus-extraction mechanisms. See Cr mer and McLean (1988).

than those captured by commonly used small type spaces, so much so that she actually find a dominant strategy auction being optimal.

The goal of this paper is to examine this informal motivation.

While suggestive, the above informal motivation do not yet constitute a formal rationale for an expected-revenue-maximizing auctioneer to choose a dominant strategy mechanism. After all, any formal foundation must first explain how the auctioneer chooses among mechanisms when she faces uncertainty over bidders' higher-order beliefs. One approach is to think of this auctioneer as a rational *Bayesian* decision maker: she first forms a belief  $\mu$  about bidders' Harsanyi types, and then chooses a  $\mu$ -optimal interim incentive compatible mechanism. This *Bayesian* approach requires us to identify a belief  $\mu$  that would rationalize the auctioneer's choice of a dominant strategy mechanism.

In other words, the following question is crucial to a *Bayesian* foundation of dominant strategy mechanisms: *given any distribution  $\nu$  over bidders' valuations, is there always a belief  $\mu$  consistent with  $\nu$  against which the optimal interim IC mechanism is also dominant strategy IC?*

Our answer is negative, and the proof is by counterexample. In Section 3, we provide an auction example, which only involves two bidders and two possible valuations for each bidder, where the dominant strategy mechanism can *never* be rationalized by *any* belief about bidders' Harsanyi types. In fact, for any belief  $\mu$  about bidders' Harsanyi types, the auctioneer's  $\mu$ -expected revenue generated by an optimal interim IC mechanism is uniformly bounded away from the  $\mu$ -expected revenue generated by any dominant strategy mechanism.

The negative result in Section 3 notwithstanding, when one imposes enough restrictions on the environments, dominant strategy mechanisms can still be proved to be rationalizable. The second half of this paper hence seeks to shed some insights on the qualitative features of those beliefs that rationalize dominant strategy mechanisms. In particular, we are interested in whether or not dominant strategy mechanisms, whenever they can be rationalized by some beliefs, can always be rationalized by some *common-prior beliefs*. This question is especially interesting given the prevalence of the common-prior assumption in the theory of mechanism design.

Our answer to this second question is also negative. In Section 4, we present a special case of the optimal auction design problem—namely single-object private-value auction with two bidders, each bidder has two possible valuations, and no bidder is known for sure to have a higher valuation—where dominant strategy mechanisms can be proved to be always rationalizable. We show that, however, in fairly general situations, those rationalizing beliefs necessarily involve the auctioneer believes that the bidders hold “wrong” beliefs.

Section 2 provides the preliminaries of this paper. Section 5 concludes.

## 2 Preliminaries

### 2.1 Types

Although the questions we address here are related to the broader theory of mechanism design, all the examples used in this paper take the form of an optimal auction design problem.

A single unit of an indivisible object is up for sale. Two bidders with privately known valuations will compete for the object. We model this by supposing that bidder  $i$  is described by a *payoff-relevant* type, his valuation  $v_i$ , which is commonly known to be an element of a finite set  $V_i \subset \mathbf{R}$ . Let  $V = V_1 \times V_2$ . In all the examples used in this paper, each  $V_i$  will consist of two possible valuations,  $0 \leq \underline{v}_i < \bar{v}_i$ . A bidder with valuation  $v_i$  receives expected utility  $p_i v_i - t_i$  if the probability with which he will be awarded the object is  $p_i \in [0, 1]$  and if his expected monetary payment is  $t_i$ .

To characterize the (equilibrium) behavior of the bidders who compete in some given auction mechanism, it is not enough to specify the bidder's possible payoff-relevant types or even the probability distribution from which they are drawn. In addition, we must also specify their beliefs about the valuations of their opponents (called the *first-order* beliefs), their beliefs about one another's first-order beliefs (*second-order* beliefs), etc.

We wish to consider a formulation of the optimal auction problem which avoids implicit assumptions on higher-order beliefs. The way to do this is to first consider the *universal* belief space in which for every conceivable (coherent) hierarchy of higher-order beliefs there is a representative "belief type." This prevents the modeler from implicitly building in any assumptions about the connections between beliefs among bidders and across orders. Then a "type" consists of a payoff-relevant type together with a belief type. The universal type space is the set of all such types. Finally, the auctioneer's uncertainty is described by some probability measure over the universal type space. Any assumption connecting the auctioneer's beliefs with those of the bidders is then made explicit by the choice of this probability measure.

Specifically, we construct the universal belief space from the basic payoff-relevant data as follows (the construction is standard, see Mertens and Zamir (1985) and Brandenburger and Dekel (1993) for details). To begin with, whenever  $X$  is a metric space, we treat  $X$  as a measurable space with the Borel  $\sigma$ -algebra and let  $\Delta X$  be the space of all Borel probability measures on  $X$  endowed with the weak topology.

The set of possible first-order beliefs for bidder  $i$  is

$$\mathcal{J}_i^1 := \Delta V_{-i}$$

and the set of all possible  $k$ th-order beliefs is

$$\mathcal{J}_i^k := \Delta(V_{-i} \times \mathcal{J}_{-i}^{k-1}).$$

Because the set  $\Delta X$  is compact metric whenever  $X$  is, by induction each  $\mathcal{T}_i^k$  is a compact metric space. The projections  $\phi_i^k : \mathcal{T}_i^k \rightarrow \mathcal{T}_i^{k-1}$ , defined inductively by  $\phi_i^2(\tau_i^2)(v_{-i}) = \tau_i^2(\{v_{-i}\} \times \mathcal{T}_{-i}^1)$ , and for each measurable subset  $\{v_{-i}\} \times B \subset V_{-i} \times \mathcal{T}_{-i}^{k-2}$ ,

$$\phi_i^k(\tau_i^k)(\{v_{-i}\} \times B) = \tau_i^k(\{v_{-i}\} \times [\phi_{-i}^{k-1}]^{-1}(B)),$$

demonstrate that each  $k$ th-order belief for  $i$  implicitly defines beliefs at lower orders as well.

A universal belief type for bidder  $i$  is a sequence (or *hierarchy*)  $\tau_i = (\tau_i^1, \tau_i^2, \dots)$  satisfying  $\tau_i^k \in \mathcal{T}_i^k$  and the *coherency* condition that  $\phi_i^k(\tau_i^k) = \tau_i^{k-1}$ . The universal belief space for bidder  $i$  is then the set  $\mathcal{T}_i^* \subset \prod_{k=1}^{\infty} \mathcal{T}_i^k$  of all such coherent hierarchies. The product space endowed with the product topology is compact. Since the set of coherent hierarchies is closed, the universal belief space is compact. By Mertens and Zamir (1985) and Brandenburger and Dekel (1993), there is a homeomorphism between  $\mathcal{T}_i^*$  and  $\Delta(V_{-i} \times \mathcal{T}_{-i}^*)$  and thus the latter is compact. Let  $g_i : \mathcal{T}_i^* \rightarrow \Delta(V_{-i} \times \mathcal{T}_{-i}^*)$  be such a mapping, and let  $f_i : \mathcal{T}_i^* \rightarrow \Delta V_{-i}$  be the marginal distribution of  $g_i$  over  $V_{-i}$ .

A *type* is a pair  $\omega_i = (v_i, \tau_i)$ . A type space is a set  $\Omega = \Omega_1 \times \Omega_2$ , where  $\Omega_i \subset V_i \times \mathcal{T}_i$ . In this paper, we are mainly interested in two kinds of type space.

The *universal type space*  $\Omega^*$  is the type space where each  $\Omega_i^* = V_i \times \mathcal{T}_i^*$ . Notice that every type space is a subset of the universal type space. Let  $\mathcal{T}^* = \mathcal{T}_1^* \times \mathcal{T}_2^*$ . For any  $v \in V$ , we shall write  $\Omega^*(v)$  for the open subset  $\{v\} \times \mathcal{T}^* \subset \Omega^*$ .

Another kind of type space, used almost without exception in the literature of mechanism design, is the *naive type space*  $\Omega^\nu$  generated from some distribution  $\nu$  over the set of payoff-relevant types  $V$ .<sup>4</sup> Specifically, this means that bidder  $i$ 's first-order belief is a function of his valuation  $v_i$  and is given by the conditional distribution  $\tau_i^1(v_i) = \nu(\cdot | v_i)$ . Furthermore, since bidder  $-i$ 's first-order beliefs are  $\tau_{-i}^1(v_{-i}) = \nu(\cdot | v_{-i})$ , bidder  $i$ 's second-order beliefs can be computed from  $\nu$  as well. In particular, bidder  $i$  believes that with probability

$$\tau_i^2(\gamma) := \nu([\tau_{-i}^1]^{-1}(\gamma) | v_i),$$

bidder  $-i$  has first-order belief  $\gamma$ . Similarly, all higher-order beliefs can be inductively derived from  $\nu$ .

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<sup>4</sup>This terminology is due to Bergemann and Morris (2002).

## 2.2 Mechanisms

We consider direct revelation mechanisms.<sup>5</sup> A direct revelation auction mechanism for type space  $\Omega$  is a game form in which the bidders simultaneously announce their types from the corresponding set  $\Omega_i$ , and the object is allocated and monetary transfers enforced as a function of their announcements. Formally, an auction  $\Gamma = (p, t)$  is defined by two functions,  $p : \Omega \rightarrow [0, 1] \times [0, 1]$  and  $t : \Omega \rightarrow \mathbf{R}^2$ . The allocation rule  $p$  specifies the probabilities  $p_i(\omega)$  with which each bidder  $i$  will receive the object. The allocation rule is restricted to be feasible:  $\sum_{i=1,2} p_i(\omega) \leq 1$ . The transfer rule  $t$  defines payments  $t_i(\omega)$  made from bidder  $i = 1, 2$  to the auctioneer. Denote by  $\bar{t}(\omega)$  the sum  $t_1(\omega) + t_2(\omega)$ .

The auctioneer's problem is to choose an incentive compatible mechanism which generates the highest possible expected revenue. The revenue possibilities thus depend on the type space as well as the definition of incentive compatibility. The focus of this paper is the connection between optimal mechanisms for different type spaces and different definitions of incentive compatibility.

**Definition 1** *An auction  $\Gamma$  is dominant strategy incentive compatible with respect to the naive type space  $\Omega^\nu$  (or simply dsIC) if for each bidder  $i$  and type profile  $\omega \in \Omega^\nu$ ,*

$$\begin{aligned} p_i(\omega)v_i - t_i(\omega) &\geq 0, \quad \text{and} \\ p_i(\omega)v_i - t_i(\omega) &\geq p_i(\hat{\omega}_i, \omega_{-i})v_i - t_i(\hat{\omega}_i, \omega_{-i}), \end{aligned}$$

for any alternative type  $\hat{\omega}_i \in \Omega_i^\nu$ .

Since  $|\Omega_i^\nu| = |V_i|$ , and since the incentive compatibility constraints for dsIC depend only on valuations, an auction is dominant strategy IC with respect to a naive type space  $\Omega^\nu$  if and only if it is dominant strategy IC with respect to any other naive type space  $\Omega^{\nu'}$ . So we can always discuss whether an auction is dominant strategy IC with respect to the naive type space without referring to the specific distribution  $\nu$  from which the naive type space is generated.

**Definition 2** *An auction  $\Gamma$  is interim incentive compatible with respect to the universal type space  $\Omega^*$  (or simply iIC) if for each bidder  $i$  and type  $\omega_i \in \Omega_i^*$ ,*

$$\begin{aligned} \int_{\Omega_{-i}^*} [p_i(\omega)v_i - t_i(\omega)] g_i(\tau_i)(d\omega_{-i}) &\geq 0, \quad \text{and} \\ \int_{\Omega_{-i}^*} [p_i(\omega)v_i - t_i(\omega)] g_i(\tau_i)(d\omega_{-i}) &\geq \int_{\Omega_{-i}^*} [p_i(\hat{\omega}_i, \omega_{-i})v_i - t_i(\hat{\omega}_i, \omega_{-i})] g_i(\tau_i)(d\omega_{-i}), \end{aligned}$$

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<sup>5</sup>According to revelation principle, the proposition that there is no loss of generality in restricting attention to incentive compatible direct revelation mechanisms, can be shown by standard arguments to hold for all type spaces and all definitions of incentive compatibility considered here.

for any alternative type  $\hat{\omega}_i \in \Omega_i^*$ .

Notice that any auction  $\Gamma$  that is dominant strategy IC with respect to the naive type space can be extended naturally into a mechanism that is interim IC with respect to the universal type space in a straightforward manner. We shall abuse notation and use  $\Gamma$  to denote this natural extension as well.

## 2.3 A Bayesian Auctioneer

In this paper, we assume that the auctioneer is a Bayesian decision maker, and hence decision making under uncertainty over bidders' higher-order beliefs can be modelled as an optimization problem given a certain belief over the universal type space.

Specifically, let  $\mu$  be the auctioneer's belief over  $\Omega^*$ . For any interim IC auction  $\Gamma$ , the auctioneer's  $\mu$ -expected revenue is defined as  $R_\mu(\Gamma) = \int_{\Omega^*} \bar{t} \mu(d\omega)$ .

For any distribution  $\nu$  over  $V$ , let  $\mathcal{M}(\nu)$  denote the compact subset of beliefs  $\mu \in \Delta\Omega^*$  with marginal distribution over  $V$  equal to  $\nu$ . There is a unique element  $\nu^*$  in this subset that concentrates on the naive type space  $\Omega^\nu$  generated by  $\nu$ . In this paper, we always take the distribution  $\nu$  over  $V$  as given, as these data are arguably easier to estimate from past auctions. However, contrary to the standard approach, we do not insist on the auctioneer holding the belief  $\nu^*$ , but allow her belief to be anything in the subset  $\mathcal{M}(\nu)$  instead.

Notice that if an auction  $\Gamma$  is dominant strategy IC with respect to the naive type space, then for any belief  $\mu \in \mathcal{M}(\nu)$  that is consistent with the distribution  $\nu$ , the  $\mu$ -expected revenue of  $\Gamma$ —or, more precisely, its natural extension into the universal type space—depends only on the distribution  $\nu$ . Hence we can write  $R_\mu(\Gamma)$  as  $R_\nu(\Gamma)$  without ambiguity.

**Definition 3** *Given any distribution  $\nu$  over  $V$ , the optimal dominant strategy IC revenue is defined as*

$$V^D(\nu) := \sup_{\Gamma \text{ is dsIC}} R_\nu(\Gamma).$$

**Definition 4** *Given any belief  $\mu$  over  $\Omega^*$ , the optimal interim IC revenue is defined as*

$$V^I(\mu) := \sup_{\Gamma \text{ is iIC}} R_\mu(\Gamma).$$

## 2.4 The Problem

Take any distribution  $\nu$  over  $V$  as given, if we follow the standard approach and insist on the auctioneer holding belief  $\nu^*$ , then generically it is irrational for the auctioneer to use a dominant strategy auction; i.e., generically we will have  $V^I(\nu^*) > V^D(\nu)$  (Cr mer and McLean (1988)).

An informal argument for studying dominant strategy mechanisms is that that the mechanism designer holds belief  $\nu$  over  $V$  does not imply she must then hold belief  $\nu^*$  over  $\Omega$ . In fact, her belief may actually be very different from  $\nu^*$ . She may actually consider many different combinations of payoff-relevant types and belief types as possible, so much so that the support of her belief may actually be very different from the naive type space  $\Omega^\nu$ . It may actually turn out that it is rational for her to use a dominant strategy mechanism given how little information she has on agents' higher-order beliefs.

To examine this informal argument, we ask the following question: for any given distribution  $\nu$  over  $V$ , is it always possible to find a belief  $\mu \in \mathcal{M}(\nu)$  such that  $V^I(\mu) = V^D(\nu)$ ? If such a belief exists, we shall say that it *rationalizes* the auctioneer's choice of a dominant strategy auction, just like how  $\nu^*$  would have rationalized her choice of the Crémer-McLean full-surplus-extraction mechanism.

Our answer will be negative, and the proof is by counterexample (Section 3).

This negative result nevertheless does not rule out the possibility that, in some restrictive enough special case, dominant strategy mechanisms are still rationalizable by some beliefs. So we also seek to shed some insights on the qualitative features of those rationalizing beliefs, if they ever exist (Section 4). In particular, we are interested in whether or not dominant strategy mechanisms, if they can ever be rationalized, can always be rationalized by *common-prior beliefs*. A common-prior belief  $\mu$  is a probability measure over  $\Omega^*$  such that, for every type profile within its support, bidders' beliefs over their opponents' types are conditional probabilities derived from  $\mu$ . An example of common-prior beliefs is once again  $\nu^*$ . If the auctioneer holds a common-prior belief, she believes that both bidders share the same prior as  $\mu$ , and derive their beliefs by Bayesian updating. If the auctioneer's belief is not a common-prior belief, she believes that the bidders sometimes hold “wrong” beliefs about each other's types.

For any subset  $A \in \Omega_i^*$ , we shall write  $\mu(A)$  as a short hand for  $\mu(A \times \Omega_{-i}^*)$ . In other words, we abuse notation and use the same notation for a probability measure as well as its marginal distributions.

**Definition 5** *A belief  $\mu$  over  $\Omega^*$  is a common-prior belief if for any measurable subsets  $A \subset \Omega_i^*$  and  $B \subset \Omega_{-i}^*$ ,*

$$\int_A g_i(\tau_i)(B) \mu(d\omega_i) = \mu(A \times B).$$

Our answer to this second question will also be negative, and the proof is also by counterexample (Section 4).



### 3 On the Lack of Bayesian Foundation

In this section, we shall provide an example that has the following property. No matter what the mechanism designer thinks the agents' beliefs are, it is never justified to use a dominant strategy mechanism. As it will be clear from the proof, this example is robust to perturbations.

The example is one of single-object private-value auction. Consider an auctioneer facing two bidders. The auctioneer has one indivisible object to sell. The object is worth nothing to the her. Each of the two bidders has two possible valuations on the object, and their valuations are correlated as described in the distribution  $\nu$  (Figure 1).

	$v_1 = 5$	$v_1 = 10$
$v_2 = 4$	1/6	0
$v_2 = 2$	1/3	1/2

Figure 1: The distribution  $\nu$ .

The optimal dominant strategy IC auction is depicted in Figure 2. We use “ $\alpha = i$ ” as a shorthand for “allocating the object to bidder  $i$ ” (i.e.,  $p_i = 1$  and  $p_{-i} = 0$ ).

	$v_1 = 5$	$v_1 = 10$
$v_2 = 4$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$v_2 = 2$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$

Figure 2: The optimal dominant strategy IC auction  $\Gamma$ .

Notice that the valuation of bidder 1 is always higher than that of bidder 2. However, conditional on bidder 2 having low valuation (i.e., conditional on the second row of the matrix), the  $\mu$ -conditional probability that bidder 1 is of a high-valuation type is so high, so much so that the auctioneer chooses to sell to bidder 2 instead when bidder 1 has low valuation. This is the same logic as monopoly pricing, where a monopolist refuses to sell to low-valuation buyers. Conditional on bidder 1 having low valuation (i.e., conditional on the first column), since the auctioneer is selling to bidder 2 when his valuation is low, she also needs to sell to bidder 2 when his valuation is high in order to satisfy dominant strategy incentive compatibility.

**Proposition 1** *The optimal dominant strategy IC auction  $\Gamma$  depicted in Figure 2 cannot be rationalized by any belief  $\mu$  of the auctioneer that is consistent with the distribution  $\nu$  depicted in Figure 1.*

In the remainder of this section we will present the proof of Proposition 1. In Appendix A we prove the following stronger result.

**Proposition 2** *For the distribution  $\nu$  depicted in Figure 1, the optimal interim IC revenue is uniformly bounded away from the optimal dominant strategy IC revenue; i.e.,*

$$\inf_{\mu \in \mathcal{M}(\nu)} \sup_{\Gamma \text{ is interim IC}} R_{\mu}(\Gamma) > V^D(\nu).$$

To prove Proposition 1, fix any belief  $\mu \in \mathcal{M}(\nu)$  that rationalizes the optimal dominant strategy IC auction  $\Gamma$ , we shall prove that there exists another interim IC auction that generates higher  $\mu$ -expected revenue than  $\Gamma$  does. This would contradict the assumption that  $\mu$  rationalizes  $\Gamma$  and complete the proof.

We shall break the proof into several lemmata. For the purpose of this proof, it suffices to work only with bidder 2's first-order beliefs in order to arrive at a contradiction. So, for notational convenience, we shall summarize bidder 2's belief type  $\tau_2$  by one single number, which is his first-order belief that bidder 1 has high valuation. Specifically, for any type  $\omega_2 = (v_2, \tau_2)$  of bidder 2, if  $v_2 = 4$ , we shall use  $a$  to denote  $f_2(\tau_2)(v_1 = 10)$ ; and if  $v_2 = 2$ , we shall use  $b$  to denote  $f_2(\tau_2)(v_1 = 10)$ . For any (measurable) subset  $A \subset [0, 1]$ , we shall use " $a \in A$ " to denote the event  $\{\omega_2 = (v_2, \tau_2) : v_2 = 4, f_2(\tau_2)(v_1 = 10) \in A\}$ ; similarly for the notation " $b \in B \subset [0, 1]$ ."

The first lemma says that, conditional on any  $\mu$ -non-null subset of low-valuation types of bidder 2, the  $\mu$ -conditional-probability that bidder 1 having high valuation cannot be too low, otherwise the auctioneer can improve upon  $\Gamma$  by selling to some low-valuation types of bidder 1.<sup>6</sup>

**Lemma 1** *For any  $x \in (0, 1]$  such that  $\mu(b = x) = 0$ , if  $\mu(b < x) > 0$ , then  $\mu(v_1 = 10 | b < x) \geq 3/8$ .*

**Proof:** Suppose there exists  $x \in (0, 1]$  such that  $\mu(b < x) = \mu(b \leq x) > 0$ , and yet  $\mu(v_1 = 10 | b < x) < 3/8$ . Consider the modified auction  $\Gamma(x)$  as depicted in Figure 3.

To see that  $\Gamma(x)$  continues to be interim IC, notice that (i) truth-telling continues to be a dominant strategy of bidder 1, (ii) low-valuation types of bidder 2 always have zero rent regardless of what they announce, and (iii) high-valuation types of bidder 2 would not announce the (newly added) message " $b < x$ " as that gives them zero rent.

The only difference between  $\Gamma(x)$  and  $\Gamma$  is in the ( $\mu$ -non-null) event of  $b < x$ , in which case  $\Gamma(x)$  generates  $\mu$ -expected revenue of  $5\mu(v_1 = 5 | b < x) + 5\mu(v_1 = 10 | b < x) = 5$ ,

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<sup>6</sup>In Lemma 1 (and similarly in Lemmata 2-4), the seemingly redundant requirement of  $\mu(b = x) = 0$  is a null-boundary property used only in the proof of Proposition 2.

	$v_1 = 5$	$v_1 = 10$
$a \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b \geq x$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b < x$	$\alpha = 1, t_1 = 5, t_2 = 0$	$\alpha = 1, t_1 = 5, t_2 = 0$

Figure 3: The modified auction  $\Gamma(x)$ .

whereas  $\Gamma$  only generates  $\mu$ -expected revenue of  $2\mu(v_1 = 5|b < x) + 10\mu(v_1 = 10|b < x) < 2(5/8) + 10(3/8) = 5$ , contradicting the assumption that  $\mu$  rationalizes  $\Gamma$ . ■

The second lemma says that for any low-valuation type of bidder 2 that the auctioneer perceives as possible, his first-order belief  $b$  also cannot be too low, otherwise his belief would be too “wrong” (relative to the auctioneer’s belief), so much so that the auctioneer can improve upon  $\Gamma$  by betting against him.

**Lemma 2** *Let  $\underline{b} = \sup\{x \in [0, 1] : \mu(b < x) = 0\}$ . Then  $\underline{b} \geq 3/13$ .*

**Proof:** Suppose  $\underline{b} < 3/13$ . Then pick any  $x$  in between  $\underline{b}$  and  $3/13$  such that  $\mu(b = x) = 0$ ,<sup>7</sup> and consider the modified auction  $\Gamma'(x)$  as depicted in Figure 4.

	$v_1 = 5$	$v_1 = 10$
$a \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b \geq x$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b < x$	$\alpha = 0, t_1 = 0, t_2 = -2$	$\alpha = 1, t_1 = 10, t_2 = 2(1 - x)/x$

Figure 4: The modified auction  $\Gamma'(x)$ .

To see that  $\Gamma'(x)$  continues to be interim IC, notice that (i) truth-telling continues to be a dominant strategy of bidder 1, (ii) low-valuation types of bidder 2 would have strict incentive to announce the (newly added) message “ $b < x$ ” if and only if the resulting rent of  $2(1 - b) - [2(1 - x)/x]b = 2(1 - b/x)$  is positive, or equivalently if and only if  $b < x$ , and (iii) high-valuation types of bidder 2 would not announce the (newly added) message “ $b < x$ ” as that gives them rent of  $2(1 - a) - [2(1 - x)/x]a = 2(1 - a/x)$ , which is lower than the rent of  $2(1 - a)$  if they tell the truth.

<sup>7</sup>It is always possible to pick such an  $x$ , as any distribution over  $[0, 1]$  can have at most countably many mass points.

The only difference between  $\Gamma'(x)$  and  $\Gamma$  is in the ( $\mu$ -non-null) event of  $b < x$ , in which case  $\Gamma'(x)$  collects from bidder 2 an  $\mu$ -expected amount of

$$\begin{aligned}
& (-2)\mu(v_1 = 5|b < x) + [2(1-x)/x]\mu(v_1 = 10|b < x) \\
\geq & (-2)(5/8) + [2(1-x)/x](3/8) \\
= & 3/(4x) - 2 \\
> & [3/4(3/13)] - 2 \\
= & 5/4
\end{aligned}$$

(where the first inequality follows from Lemma 1), whereas  $\Gamma$  only collects from bidders 2 an  $\mu$ -expected amount of  $2\mu(v_1 = 5|b < x) \leq 2(5/8) = 5/4$ , contradicting the assumption that  $\mu$  rationalizes  $\Gamma$ .  $\blacksquare$

The third lemma says that for any high-valuation type of bidder 2 that the auctioneer perceives as possible, his first-order belief  $a$  also cannot be too low, otherwise beliefs held by high- and low-valuation types of bidder 2 would be too different, so much so that the auctioneer can improve upon  $\Gamma$  by introducing Cr mer-McLean-kind of bets to separate these types and relax incentive compatibility constraints.

**Lemma 3** *Let  $\underline{a} = \sup\{y \in [0, 1] : \mu(a < y) = 0\}$ . Then  $\underline{a} \geq 1/11$ .*

**Proof:** Suppose  $\underline{a} < 1/11$ . Then pick any  $y$  in between  $\underline{a}$  and  $1/11$  such that  $\mu(a = y) = 0$ . Notice that  $y < 1/11$  implies  $y < 3y/(2y + 1) < 3/13$ , and hence we can also pick an  $x$  in between  $3y/(2y + 1)$  and  $3/13$  such that  $\mu(b = x) = 0$ . Consider the modified auction  $\Gamma(x, y)$  as depicted in Figure 5.

	$v_1 = 5$	$v_1 = 10$
$a < y$	$\alpha = 1, t_1 = 5, t_2 = -2x(1-y)/(x-y)$	$\alpha = 1, t_1 = 5, t_2 = 2(1-x)(1-y)/(x-y)$
$a \geq y$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b < x$	$\alpha = 1, t_1 = 5, t_2 = -2x(1-y)/(x-y)$	$\alpha = 1, t_1 = 5, t_2 = 2(1-x)(1-y)/(x-y)$
$b \geq x$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$

Figure 5: The modified auction  $\Gamma(x, y)$ .

To see that  $\Gamma(x, y)$  continues to be interim IC, notice that (i) truth-telling continues to be a dominant strategy of bidder 1, (ii) low-valuation types of bidder 2 would have strict incentive to announce the (newly added) message “ $b < x$ ” if and only if the resulting rent of  $[2x(1-y)/(x-y)](1-b) - [2(1-x)(1-y)/(x-y)]b = 2(1-y)(x-b)/(x-y)$  is positive, or equivalently if and only if  $b < x$ , and (iii) high-valuation types of bidder 2 would have

strict incentive to announce the (newly added) message “ $a < y$ ” if and only if the resulting rent of  $[2x(1 - y)/(x - y)](1 - a) - [2(1 - x)(1 - y)/(x - y)]a = 2(1 - y)(x - a)/(x - y)$  is strictly higher than the truth-telling rent of  $2(1 - a)$ , or equivalently if and only if  $a < y$ .

Since the event of  $b < x$  is a  $\mu$ -null event by Lemma 2, the only real difference between  $\Gamma(x, y)$  and  $\Gamma$  is in the ( $\mu$ -non-null) event of  $a < y$ , in which case  $\Gamma(x, y)$  generates  $\mu$ -expected revenue of

$$\begin{aligned}
& 5 - 2x(1 - y)/(x - y) \\
&= 5 - 2(x - y + y)(1 - y)/(x - y) \\
&= 5 - 2(1 - y) - 2y(1 - y)/(x - y) \\
&> 5 - 2(1 - y) - 2y(1 - y)(2y + 1)/[3y - y(2y + 1)] \\
&= 5 - 2(1 - y) - 2y(1 - y)(2y + 1)/[2y(1 - y)] \\
&= 2,
\end{aligned}$$

whereas  $\Gamma$  only generates  $\mu$ -expected revenue of 2, contradicting the assumption that  $\mu$  rationalizes  $\Gamma$ . ■

Finally, the fourth lemma says that, if  $\underline{a} \geq 1/11$ , then for any high-valuation type of bidder 2 that the auctioneer perceives as possible, his belief would be too “wrong,” so much so that the auctioneer can improve upon  $\Gamma$  by betting against him, contradicting the assumption that  $\mu$  rationalizes  $\Gamma$ . This would complete the proof of the proposition.

**Lemma 4** *If  $\underline{a} \geq 1/11$ , then  $\mu$  does not rationalize  $\Gamma$ .*

**Proof:** Suppose  $\underline{a} \geq 1/11$ . Consider the modified auction  $\Gamma'$  as depicted in Figure 6.

	$v_1 = 5$	$v_1 = 10$
$a \geq 1/12$	$\alpha = 2, t_1 = 0, t_2 = 123/61$	$\alpha = 2, t_1 = 0, t_2 = 233/61$
$a < 1/12$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$
$b \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 2$	$\alpha = 1, t_1 = 10, t_2 = 0$

Figure 6: The modified auction  $\Gamma'$ .

To see that  $\Gamma'$  continues to be interim IC, notice that (i) truth-telling continues to be a dominant strategy of bidder 1, (ii) low-valuation types of bidder 2 would not announce the (newly added) message “ $a \geq 1/12$ ” as that gives them strictly negative rent regardless of what bidder 1 announces, and (iii) high-valuation types of bidder 2 would have weak incentive to announce the (newly added) message “ $a \geq 1/12$ ” if and only if the resulting rent of  $(4 - 123/61)(1 - a) + (4 - 233/61)a$  is weakly higher than their original rent of  $2(1 - a)$ , or equivalently if and only if  $a \geq 1/12$ .

Since the event  $a < 1/12 < 1/11$  is a  $\mu$ -null event by assumption, the only real difference between  $\Gamma'$  and  $\Gamma$  is in the ( $\mu$ -non-null) event of  $a \geq 1/12$ , in which case  $\Gamma'$  generates  $\mu$ -expected revenue of  $123/61 > 2$ , whereas  $\Gamma$  only generates  $\mu$ -expected revenue of 2. This proves that  $\mu$  does not rationalize  $\Gamma$ . ■

## 4 On the Role of “Wrong” Beliefs

The negative result in Section 3 notwithstanding, when one imposes enough restrictions on the environments, dominant strategy mechanisms can still be proved to be rationalizable. This section hence seeks to shed some insights on the qualitative features of those beliefs that rationalize dominant strategy mechanisms. In particular, we are interested in whether or not dominant strategy mechanisms, if they can ever be rationalized, can always be rationalized by some *common-prior beliefs*. This question is especially interesting given the prevalence of the common-prior assumption in the theory of mechanism design.

We first describe a particular special case where dominant strategy mechanisms can be proved to be always rationalizable. (Notice how the example in Section 3 violates one of the conditions in Proposition 3.)

**Proposition 3** *In the two-bidder two-valuation special case of the optimal auction design problem, if no bidder is known for sure to have a higher valuation (i.e.,  $\min\{\bar{v}_1, \bar{v}_2\} > \max\{v_1, v_2\}$ ), then, for any given distribution  $\nu$  over bidders’ valuations, there exists a belief  $\mu$  consistent with  $\nu$  against which an optimal interim IC auction is dominant strategy IC.*

We leave the proof to Appendix B, as our main interest here is in the qualitative features of those rationalizing beliefs. In the proof of Proposition 3, all the rationalizing beliefs involve the auctioneer believing that bidders hold “wrong” beliefs. However, this by itself does not prove that dominant strategy mechanisms *cannot* be rationalized by any common-prior belief. Therefore, we shall present an example where this is indeed the case.

In the example below, the optimal dominant strategy mechanism, although rationalizable according to Proposition 3, can never be rationalized by any common-prior belief. As the proof should make it clear, this necessity of “wrong” beliefs is a robust phenomenon.

Consider an example that satisfies the conditions in Proposition 3 (and hence the optimal dominant strategy mechanism is rationalizable), with bidders’ valuations correlated according to the distribution  $\nu$  (Figure 7).

The optimal dominant strategy IC auction is depicted in Figure 8.<sup>8</sup>

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<sup>8</sup>We follow the convention introduced in Section 3, and use “ $\alpha = i$ ” as a shorthand for “allocating the object to bidder  $i$ .”

	$v_1 = 4$	$v_1 = 9$
$v_2 = 11$	3/10	1/10
$v_2 = 5$	3/10	3/10

Figure 7: The distribution  $\nu$  of bidders' valuations.

	$v_1 = 4$	$v_1 = 9$
$v_2 = 11$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 2, t_1 = 0, t_2 = 11$
$v_2 = 5$	$\alpha = 0, t_1 = 0, t_2 = 0$	$\alpha = 1, t_1 = 9, t_2 = 0$

Figure 8: The optimal dominant strategy auction  $\Gamma$ .

**Proposition 4** *Although the optimal dominant strategy IC auction  $\Gamma$  depicted in Figure 8 is rationalizable, it can never be rationalized by any common-prior belief that is consistent with the distribution  $\nu$  depicted in Figure 7.*

**Proof:** Fix any common-prior belief  $\mu \in \mathcal{M}(\nu)$  that rationalizes the optimal dominant strategy IC auction  $\Gamma$ , we shall prove that there exists another interim IC auction that generates higher  $\mu$ -expected revenue than  $\Gamma$  does. This would contradict the assumption that  $\mu$  rationalizes  $\Gamma$ .

Once again, it suffices to work only with bidder 2's first order beliefs in order to arrive at a contradiction. So, following the convention in Section 3, we shall continue to use  $a$  ( $b$ ) to denote the first-order belief of a high-valuation (low-valuation) type of bidder 2 that bidder 1 has high valuation. Define  $\underline{b}$  similarly as in Lemma 2.

First, observe that  $\underline{b} \geq 4/9$ . Suppose, on the contrary,  $\underline{b} < 4/9$ . Then pick any number  $z$  in between  $\underline{b}$  and  $4/9$ , and consider the modified auction  $\Gamma(z)$  as depicted in Figure 9.

	$v_1 = 4$	$v_1 = 9$
$a \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 2, t_1 = 0, t_2 = 11$
$b \geq z$	$\alpha = 0, t_1 = 0, t_2 = 0$	$\alpha = 1, t_1 = 9, t_2 = 0$
$b < z$	$\alpha = 1, t_1 = 4, t_2 = 0$	$\alpha = 1, t_1 = 4, t_2 = 0$

Figure 9: The modified auction  $\Gamma(z)$ .

It is obvious that  $\Gamma(z)$  continues to be interim IC. The only difference between  $\Gamma(z)$  and

$\Gamma$  is in the ( $\mu$ -non-null) event of  $b < z$ , in which case  $\Gamma(z)$  generates  $\mu$ -expected revenue of 4, whereas  $\Gamma$  only generates  $\mu$ -expected revenue of  $9\mu(v_1 = 9|b < z) < 9z < 9(4/9) = 4$ , where the first inequality comes from the fact that  $\mu$  is a common-prior belief. Since this would have contradicted the assumption that  $\mu$  rationalizes  $\Gamma$ , we must have  $\underline{b} \geq 4/9$ .

Then, consider the modified auction  $\Gamma''$  as depicted in Figure 10.

	$v_1 = 4$	$v_1 = 9$
$a \in [0, 1]$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 1, t_1 = 9, t_2 = -15/2$
$b \geq 4/9$	$\alpha = 2, t_1 = 0, t_2 = 11$	$\alpha = 1, t_1 = 9, t_2 = -15/2$
$b < 4/9$	$\alpha = 0, t_1 = 0, t_2 = 0$	$\alpha = 0, t_1 = 0, t_2 = 0$

Figure 10: The modified auction  $\Gamma''$ .

To see that  $\Gamma''$  continues to be interim IC, it suffices to observe that, for low-valuation types of bidder 2 with  $b \geq 4/9$ , truth-telling gives them a non-negative rent of  $(5 - 11)(1 - b) + (15/2)b \geq (-6)(5/9) + (15/2)(4/9) = 0$ .

Since  $b < 4/9$  is a  $\mu$ -null event,  $\Gamma''$  generates  $\mu$ -expected revenue of  $9(4/10) + 11(6/10) - (15/2)(4/10) = 72/10$ , whereas  $\Gamma$  only generates  $\mu$ -expected revenue of  $9(3/10) + 11(4/10) = 71/10$ . This proves that  $\mu$  does not rationalize  $\Gamma$ . ■

However, according to Proposition 3, the optimal dominant strategy auction  $\Gamma$  can be rationalized by a belief  $\mu$  that involves the auctioneer believing that bidders hold “wrong” beliefs. To construct one such belief, we extend the convention in Section 3 and use  $a_i$  ( $b_i$ ) to denote the first-order belief of a high-valuation (low-valuation) type of bidder  $i$  that bidder  $-i$  has high valuation.

Consider a belief  $\mu$  of the auctioneer such that its marginal distribution over bidders’ valuations and first-order beliefs is as depicted in Figure 11.

	$b_1 = 2/5$	$a_1 = 1/4$
$a_2 = 1/4$	$3/10$	$1/10$
$b_2 = 2/5$	$3/10$	$3/10$

Figure 11: The auctioneer’s belief  $\mu$ .

The auctioneer’s belief  $\mu$  has a 4-point support: for every bidder  $i$ , every payoff-relevant type is associated with only one possible belief type. The construction of bidders’ higher-order beliefs is by induction. Specifically, for a low-valuation type of bidder 1, his second-order belief assigns probability  $2/5$  ( $3/5$ ) to bidder 2 having high (low) valuation and holding



first-order belief  $a_2 = 1/4$  ( $b_2 = 2/5$ ), and a high-valuation (low-valuation) type of bidder 2 has a third-order belief that assigns probability  $3/4$  ( $3/5$ ) to bidder 1 having low valuation and having such a second-order belief, and so on.

It is obvious that this belief  $\mu$  of the auctioneer is consistent with the distribution  $\nu$ . However, conditional on the event that bidder  $i$  has low valuation, the auctioneer and bidder  $i$  assign different probabilities on bidder  $-i$  having high valuation: while the auctioneer assigns probability  $1/2$ , bidder  $i$  only assigns probability  $2/5$ . In other words, the auctioneer believes that bidder  $i$ 's first-order belief is "wrong."

While a full-blown proof that the above belief  $\mu$  rationalizes the optimal dominant strategy auction  $\Gamma$  in Figure 8 is contained in Appendix B, we shall provide some intuition here on why it is able to achieve something that a common-prior belief cannot.

Given the above belief  $\mu$ , there are two obvious candidate routes to improve upon the optimal dominant strategy auction  $\Gamma$  in Figure 8. First, since the auctioneer believes that the low-valuation type of bidder  $i$  holds "wrong" belief, the auctioneer may profit from betting against him on bidder  $-i$ 's types. Second, since high- and low-valuation types of bidder  $i$  hold different beliefs, the auctioneer may profit from separating these two types by Crémer-McLean-kind of bets and relaxing incentive compatibility constraints. We shall see that either route is fruitless.

First, consider introducing any bet  $(x, y)$  on bidder 2's type, where  $x$  and  $y$  are the amount bidder 1 pays the auctioneer in the events bidder 2 has low and high valuations respectively. If the bet is acceptable to both the auctioneer and the low-valuation type of bidder 1, we must have

$$\begin{aligned} (1/2)x + (1/2)y &\geq 0, \quad \text{and} \\ (3/5)(-x) + (2/5)(-y) &\geq 0, \end{aligned}$$

with at least one inequality strict unless  $x = y = 0$ . But then the high-valuation type of bidder 1 would find the bet acceptable as well, as

$$(3/4)(-x) + (1/4)(-y) = (5/2)[(3/5)(-x) + (2/5)(-y)] + (3/2)[(1/2)x + (1/2)y],$$

which is strictly bigger the zero rent for the high-valuation type of bidder 1 under the auction  $\Gamma$ . With both high- and low-valuation types of bidder 1 accepting such a bet, such a bet turns sour for the auctioneer, as

$$(3/5)(-x) + (2/5)(-y) \leq 0,$$

and this explains why the first route is fruitless.

Second, consider introducing any Crémer-McLean-kind of bet to separate the high- and low-valuation types of bidder 1 and relax the downward incentive compatibility constraint. Once again, let  $(x, y)$  be such a bet on bidder 2's type. Suppose the bet is successful in

the sense that the auctioneer can now sell to the low-valuation type of bidder 1 without the need to leave extra rent for the high-valuation type of bidder 1 (as she needed to before the introduction of such a bet that relaxes the downward incentive compatibility constraint), then we must have

$$\begin{aligned} (3/5)(4-x) + (2/5)(-y) &\geq 0, & \text{and} \\ (3/4)(9-x) + (1/4)(-y) &\leq 0, \end{aligned}$$

where the first (second) inequality follows from the individual rationality (incentive compatibility) constraint of the low-valuation (high-valuation) type of bidder 1. However, these together imply that any bet like this is too good to be profitable for the auctioneer, as

$$(1/2)x + (1/2)y = (2/3)[(3/4)(-x) + (1/4)(-y)] - (5/3)[(3/5)(-x) + (2/5)(-y)] \leq -1,$$

and this explains why the second route is fruitless as well.

Of course we still have not fully exhausted all possible ways to improve upon the optimal dominant strategy auction  $\Gamma$ . But the proof in Appendix B will show that we actually have not left out anything, and the belief  $\mu$  in Figure 11 indeed rationalizes  $\Gamma$ .

## 5 Conclusion

In this paper, we treated the mechanism designer as a *Bayesian* decision maker and challenged the informal argument that a mechanism designer who does not know agents' beliefs may as well use dominant strategy mechanisms. Although this Bayesian approach is more in line with the literature of optimal mechanism design, this is definitely not the only way one can model the auctioneer's decision making problem under uncertainty. For example, one can model auctioneer as choosing among mechanisms using a max-min criterion. That is, one can think of her as choosing an interim IC mechanism that has the best worst-case performance:

$$\max_{\Gamma \text{ is interim IC}} \min_{\mu \in \mathcal{M}(\nu)} R_{\mu}(\Gamma),$$

given any distribution  $\nu$  over agents' payoff-relevant types. It is our future research agenda to examine any possible max-min foundations for dominant strategy mechanisms.

## Appendix A: Proof of Proposition 2

We first establish two lemmata.

**Lemma 5** *Suppose  $K$  is a compact topological space and that  $\mathcal{F}$  is a family of real-valued functions on  $K$  such that, for each  $x \in K$ , there is some  $f_x \in \mathcal{F}$  which is continuous at  $x$  and satisfies  $f_x(x) > 0$ . Then we have  $\inf_{x \in K} \sup_{f \in \mathcal{F}} f(x) > 0$ .*

**Proof:** For each  $x \in K$ , there exists an open neighborhood  $U_x$  such that, for each  $y \in U_x$ , we have  $f_x(y) > f_x(x)/2$ . The collection  $\{U_x : x \in K\}$  forms an open covering of the compact space  $K$ , and hence there exists a finite subcovering. Let  $\{U_{x_1}, \dots, U_{x_n}\}$  be a finite subcovering and let  $\varepsilon = \min\{f_{x_1}(x_1), \dots, f_{x_n}(x_n)\} > 0$ . For each  $x \in K$ , we have  $x \in U_{x_l}$  for some  $l = 1, \dots, n$  so that  $\sup_{f \in \mathcal{F}} f(x) \geq f_{x_l}(x) > f_{x_l}(x_l)/2 \geq \varepsilon/2 > 0$ . ■

**Lemma 6** Suppose  $\mathcal{O}_1, \dots, \mathcal{O}_n$  are disjoint open subsets of  $\Omega^*$  such that  $\mu(\cup \mathcal{O}_l) = 1$ , and  $t : \Omega^* \rightarrow \mathbf{R}$  is a bounded real function that is constant on each  $\mathcal{O}_l$ . Then the mapping

$$\mu' \rightarrow \int_{\Omega^*} t \mu'(d\omega)$$

is continuous at the point  $\mu$ .

**Proof:** Fix any  $\varepsilon > 0$ . Let  $\bar{t} > 0$  be an upper bound for  $|t|$ . The function  $\mu' \rightarrow \mu'(\mathcal{O}_l)$  is lower semi-continuous (see Aliprantis and Border (1999)), hence we can set

$$\delta = \frac{\varepsilon}{tn^2}$$

and find a neighborhood  $U$  of  $\mu$  such that, for all  $\mu' \in U$ ,  $\mu'(\mathcal{O}_l) > \mu(\mathcal{O}_l) - \delta$  for  $l = 1, \dots, n$ . Since  $\mu(\cup \mathcal{O}_l) = 1$ , it follows that  $\mu'(\mathcal{O}_l) < \mu(\mathcal{O}_l) + (n-1)\delta$  and  $\mu'(\Omega^* \setminus \cup \mathcal{O}_l) < \mu(\Omega^* \setminus \cup \mathcal{O}_l) + n\delta = n\delta$ .

We can write

$$\int_{\Omega^*} t d\mu' = \sum_{l=1}^n \mu'(\mathcal{O}_l)t(\mathcal{O}_l) + \int_{\Omega^* \setminus \cup \mathcal{O}_l} t(\omega) d\mu',$$

so that

$$\begin{aligned} & \sum_{l=1}^n \mu'(\mathcal{O}_l)t(\mathcal{O}_l) - \mu(\Omega^* \setminus \cup \mathcal{O}_l)\bar{t} \leq \int_{\Omega^*} t \mu'(d\omega) \leq \sum_{l=1}^n \mu'(\mathcal{O}_l)t(\mathcal{O}_l) + \mu'(\Omega^* \setminus \cup \mathcal{O}_l)\bar{t} \\ \implies & \sum_{l=1}^n [\mu(\mathcal{O}_l) - \delta]t(\mathcal{O}_l) - n\delta\bar{t} < \int_{\Omega^*} t \mu'(d\omega) < \sum_{l=1}^n [\mu(\mathcal{O}_l) + (n-1)\delta]t(\mathcal{O}_l) + n\delta\bar{t} \\ \implies & -\delta \sum_{l=1}^n t(\mathcal{O}_l) - n\delta\bar{t} < \int_{\Omega^*} t \mu'(d\omega) - \int_{\Omega^*} t \mu(d\omega) < (n-1)\delta \sum_{l=1}^n t(\mathcal{O}_l) + n\delta\bar{t} \\ \implies & -2n\delta\bar{t} < \int_{\Omega^*} t \mu'(d\omega) - \int_{\Omega^*} t \mu(d\omega) < n^2\delta\bar{t}. \end{aligned}$$

This proves that  $|\int_{\Omega^*} t \mu'(d\omega) - \int_{\Omega^*} t \mu(d\omega)| < \max\{2n\delta\bar{t}, n^2\delta\bar{t}\} = \varepsilon$ . ■

**Proof of Proposition 2** Notice that, for each of the mechanisms used in the proof of Proposition 1, the total transfer  $(t_1 + t_2)(\omega)$  satisfies the conditions of Lemma 6. For example,

consider the mechanism  $\Gamma(x)$  in Lemma 1. For any  $(v_1, v_2)$ , the set of universal type profiles in which the valuation pair is  $(v_1, v_2)$  is open in the product topology with  $\mu$ -null boundary. Moreover, since  $\mu(b = x) = 0$ , the event  $b < x$  is also open in the product topology with  $\mu$ -null boundary. Therefore, we can take  $\mathcal{O}_1, \dots, \mathcal{O}_6$  to be the interiors of the sets represented by the cells of the table in Figure 3. These open sets are disjoint, have  $\mu$ -null boundaries, and have total  $\mu$ -measure equal to 1 as required.

Thus, for any auctioneer's belief  $\mu$  that is consistent with the distribution  $\nu$ , there exists an interim IC auction  $\Gamma(\mu)$  such that  $R_\mu \Gamma(\mu) - V^D(\nu) > 0$ , and the mapping  $\mu' \rightarrow R_{\mu'} \Gamma(\mu) - V^D(\nu)$  is continuous at the point  $\mu' = \mu$ . We can hence apply Lemma 5, taking  $K = \mathcal{M}(\nu)$  and  $\mathcal{F} = \{R_{(\cdot)} \Gamma - V^D(\nu) : \Gamma \text{ is interim IC}\}$ .  $\blacksquare$

## Appendix B: Proof of Proposition 3

Assume the conditions of Proposition 3 are satisfied. Fix any distribution  $\nu$  over bidders' valuations. We shall construct a belief  $\mu \in \mathcal{M}(\nu)$  of the auctioneer against which an optimal interim IC auction is dominant strategy IC.

For  $i = 1, 2$ , define

$$a_i^* := \nu(\bar{v}_{-i} | \bar{v}_i), \quad \text{and } b_i^* := \nu(\bar{v}_{-i} | \underline{v}_i).$$

Consider a belief  $\mu$  of the auctioneer that has the following 4-point support: each bidder  $i$  has only two types,  $\bar{\omega}_i = (\bar{v}_i, \bar{\tau}_i)$  and  $\underline{\omega}_i = (\underline{v}_i, \underline{\tau}_i)$ . Both belief types of bidder  $i$  have the 2-point support of  $\{\underline{\omega}_{-i}, \bar{\omega}_{-i}\}$ , with their respective distributions as follows.

$$\begin{aligned} (1 - a_i) &:= g_i(\bar{\tau}_i)(\underline{\omega}_{-i}) = \nu(\underline{v}_{-i} | \bar{v}_i) = (1 - a_i^*), \\ a_i &:= g_i(\bar{\tau}_i)(\bar{\omega}_{-i}) = \nu(\bar{v}_{-i} | \bar{v}_i) = a_i^*, \\ (1 - b_i) &:= g_i(\underline{\tau}_i)(\underline{\omega}_{-i}) = \nu(\underline{v}_{-i}) = \nu(\bar{v}_i)(1 - a_i^*) + \nu(\underline{v}_i)(1 - b_i^*), \\ b_i &:= g_i(\underline{\tau}_i)(\bar{\omega}_{-i}) = \nu(\bar{v}_{-i}) = \nu(\bar{v}_i)a_i^* + \nu(\underline{v}_i)b_i^*. \end{aligned}$$

For each of these four type profiles,  $(\omega_1, \omega_2)$ , let  $\mu(\omega_1, \omega_2) = \nu(v_1, v_2)$ . This makes the belief  $\mu$  consistent with the distribution  $\nu$ .

Notice that, if  $\nu$  is a product measure, then  $\mu = \nu^*$ , and we are back to the classical setting of naive type space with independent distribution. It is well known that, in this classical setting, an optimal interim IC auction is dominant strategy IC. Therefore, in the rest of this proof, we shall without loss of generality assume that  $\nu$  is not a product measure.

For expositional simplicity, let's redefine the allocation rule  $p$  and the transfer rule  $t$  as mappings from this smaller type space  $\{\omega_1, \bar{\omega}_1\} \times \{\omega_2, \bar{\omega}_2\}$  to probabilities and payments, respectively. We say that an allocation rule  $p$  is iIC-implementable if there exists a transfer rule  $t$  such that the auction  $(p, t)$  is interim IC.

Notice that, if  $\nu$  is not a product measure, then each bidder  $i$  will have his two different

types holding two different beliefs. By simple application of Crémer and McLean (1988), any allocation rule  $p$  is iIC-implementable.

We shall break the proof into two steps. In step one, we shall fix an arbitrary (iIC-implementable) allocation rule  $p$ , and ask which auction  $(p, t)$  would maximize the auctioneer's  $\mu$ -expected revenue. This step of the proof will give us a function (equation (1)) that assigns to every (iIC-implementable) allocation rule  $p$  the auctioneer's maximum  $\mu$ -expected revenue. In step two, we shall maximize the auctioneer's maximum  $\mu$ -expected revenue over all (iIC-implementable) allocation rule  $p$ , and show that the optimal allocation rule  $p$  is monotonic (Figure 14). To complete the proof, we shall observe that if an optimal allocation rule is monotonic, then there exists a dominant strategy IC auction that implements the same allocation rule and generates the same  $\mu$ -expected revenue.

Fix any (iIC-implementable) allocation rule  $p$ , and consider the following optimization problem:

$$\max_{t(\cdot)} \sum_{i=1,2} \sum_{\omega_i=\underline{\omega}_i, \bar{\omega}_i} \mu(\omega_1, \omega_2) [t_1(\omega_1, \omega_2) + t_2(\omega_1, \omega_2)] \quad (\text{P1})$$

subject to, for  $i = 1, 2$ ,

$$(1 - a_i)[p_i(\bar{\omega}_i, \underline{\omega}_{-i})\bar{v}_i - t_i(\bar{\omega}_i, \underline{\omega}_{-i})] + a_i[p_i(\bar{\omega}_i, \bar{\omega}_{-i})\bar{v}_i - t_i(\bar{\omega}_i, \bar{\omega}_{-i})] \geq 0, \quad (\overline{IR}_i)$$

$$(1 - b_i)[p_i(\underline{\omega}_i, \underline{\omega}_{-i})\underline{v}_i - t_i(\underline{\omega}_i, \underline{\omega}_{-i})] + b_i[p_i(\underline{\omega}_i, \bar{\omega}_{-i})\underline{v}_i - t_i(\underline{\omega}_i, \bar{\omega}_{-i})] \geq 0, \quad (\underline{IR}_i)$$

$$(1 - a_i)[p_i(\bar{\omega}_i, \underline{\omega}_{-i})\bar{v}_i - t_i(\bar{\omega}_i, \underline{\omega}_{-i})] + a_i[p_i(\bar{\omega}_i, \bar{\omega}_{-i})\bar{v}_i - t_i(\bar{\omega}_i, \bar{\omega}_{-i})] \\ \geq (1 - a_i)[p_i(\underline{\omega}_i, \underline{\omega}_{-i})\bar{v}_i - t_i(\underline{\omega}_i, \underline{\omega}_{-i})] + a_i[p_i(\underline{\omega}_i, \bar{\omega}_{-i})\bar{v}_i - t_i(\underline{\omega}_i, \bar{\omega}_{-i})], \quad (\overline{IC}_i)$$

$$(1 - b_i)[p_i(\underline{\omega}_i, \underline{\omega}_{-i})\underline{v}_i - t_i(\underline{\omega}_i, \underline{\omega}_{-i})] + b_i[p_i(\underline{\omega}_i, \bar{\omega}_{-i})\underline{v}_i - t_i(\underline{\omega}_i, \bar{\omega}_{-i})] \\ \geq (1 - b_i)[p_i(\bar{\omega}_i, \underline{\omega}_{-i})\underline{v}_i - t_i(\bar{\omega}_i, \underline{\omega}_{-i})] + b_i[p_i(\bar{\omega}_i, \bar{\omega}_{-i})\underline{v}_i - t_i(\bar{\omega}_i, \bar{\omega}_{-i})]. \quad (\underline{IC}_i)$$

Notice that  $(\underline{IC}_i)$  never binds. Indeed, if  $b_i^* > b_i > a_i = a_i^*$  (respectively  $b_i^* < b_i < a_i = a_i^*$ ), then by increasing (respectively decreasing)  $t_i(\bar{\omega}_i, \bar{\omega}_{-i})$  by the amount  $(1 - a_i)\epsilon$  and decreasing (respectively increasing)  $t_i(\bar{\omega}_i, \underline{\omega}_{-i})$  by the amount  $a_i\epsilon$ , the auctioneer can relax  $(\underline{IC}_i)$  without affecting other constraints and the  $\mu$ -expected revenue.

It is also without loss of generality to focus on transfer rules where  $(\underline{IC}_i)$  holds with exact equality. Indeed, if  $(\underline{IC}_i)$  holds with strict inequality, then by reversing the adjustments described in the above paragraph, we can arrive at a transfer rule where  $(\underline{IC}_i)$  holds with exact equality without affecting other constraints and the  $\mu$ -expected revenue.

Since  $(\underline{IC}_i)$  never binds,  $(\underline{IR}_i)$  must bind.

Notice that  $(\overline{IC}_i)$  must bind. Suppose, on the contrary,  $(\overline{IC}_i)$  holds with strict inequality. If  $b_i^* > b_i > a_i = a_i^*$  (respectively  $b_i^* < b_i < a_i = a_i^*$ ), then by increasing (respectively decreasing)  $t_i(\underline{\omega}_i, \bar{\omega}_{-i})$  by the amount  $(1 - b_i)\epsilon$  and decreasing (respectively increasing)  $t_i(\underline{\omega}_i, \underline{\omega}_{-i})$  by the amount  $b_i\epsilon$ , the auctioneer can increase the  $\mu$ -expected revenue without affecting other constraints.

It is also without loss of generality to focus on transfer rules where  $(\overline{IR}_i)$  holds with exact equality. Indeed, if  $(\overline{IR}_i)$  holds with strict inequality, then by increasing  $t_i(\bar{\omega}_i, \cdot)$  by the amount  $\epsilon$ , increasing  $t_i(\underline{\omega}_i, \bar{\omega}_{-i})$  by the amount  $[(1 - b_i)/(a_i - b_i)]\epsilon$ , and increasing  $t_i(\underline{\omega}_i, \underline{\omega}_{-i})$  by the amount  $[b_i/(b_i - a_i)]\epsilon$ , we can arrive at a transfer rule where  $(\overline{IR}_i)$  holds with exact equality without affecting other constraints. Moreover, throughout these adjustments, the change of the  $\mu$ -expected revenue

$$\begin{aligned}
& \nu(\bar{v}_i)\epsilon + \nu(\underline{v}_i)[(1 - b_i^*)b_i/(b_i - a_i) + b_i^*(1 - b_i)/(a_i - b_i)]\epsilon \\
&= \nu(\bar{v}_i)\epsilon + \nu(\underline{v}_i)[(b_i - b_i^*)/(b_i - a_i)]\epsilon \\
&= \nu(\bar{v}_i)\epsilon + \nu(\underline{v}_i)[(\nu(\bar{v}_i)a_i + \nu(\underline{v}_i)b_i^* - b_i^*)/(\nu(\bar{v}_i)a_i + \nu(\underline{v}_i)b_i^* - a_i)]\epsilon \\
&= \nu(\bar{v}_i)\epsilon + \nu(\underline{v}_i)[(\nu(\bar{v}_i)(a_i - b_i^*)/(\nu(\underline{v}_i)(b_i^* - a_i))]\epsilon \\
&= \nu(\bar{v}_i)\epsilon + \nu(\underline{v}_i)[- \nu(\bar{v}_i)/\nu(\underline{v}_i)]\epsilon
\end{aligned}$$

is zero as well.

Using the results that, without loss of generality, both  $(\underline{IC}_i)$  and  $(\overline{IR}_i)$  holds with exact equality, we can solve for the  $t_i(\bar{\omega}_i, \cdot)$  part of the optimal transfer rule. In particular, conditional on bidder  $i$  having type  $\bar{\omega}_i$ , the auctioneer collects from bidder  $i$  an  $\mu$ -expected amount of

$$\begin{aligned}
& (1 - a_i^*)t_i(\bar{\omega}_i, \underline{\omega}_{-i}) + a_i^*t_i(\bar{\omega}_i, \bar{\omega}_{-i}) \\
&= (1 - a_i)t_i(\bar{\omega}_i, \underline{\omega}_{-i}) + a_it_i(\bar{\omega}_i, \bar{\omega}_{-i}) \\
&= (1 - a_i)p_i(\bar{\omega}_i, \underline{\omega}_{-i})\bar{v}_i + a_ip_i(\bar{\omega}_i, \bar{\omega}_{-i})\bar{v}_i,
\end{aligned}$$

which depends only on  $p(\bar{\omega}_i, \cdot)$ .

Similarly, using the results that both  $(\underline{IR}_i)$  and  $(\overline{IC}_i)$  are binding, we can solve for the  $t_i(\underline{v}_i, \cdot)$  part of the optimal transfer rule. Rewrite  $(\underline{IR}_i)$  and  $(\overline{IC}_i)$  as

$$\begin{aligned}
(1 - b_i)t_i(\underline{\omega}_i, \underline{\omega}_{-i}) + b_it_i(\underline{\omega}_i, \bar{\omega}_{-i}) &= [(1 - b_i)p_i(\underline{\omega}_i, \underline{\omega}_{-i}) + b_ip_i(\underline{\omega}_i, \bar{\omega}_{-i})]\underline{v}_i =: \underline{T}_i, & (\underline{IR}_i) \\
(1 - a_i)t_i(\underline{\omega}_i, \underline{\omega}_{-i}) + a_it_i(\underline{\omega}_i, \bar{\omega}_{-i}) &= [(1 - a_i)p_i(\underline{\omega}_i, \underline{\omega}_{-i}) + a_ip_i(\underline{\omega}_i, \bar{\omega}_{-i})]\bar{v}_i =: \bar{T}_i. & (\overline{IC}_i)
\end{aligned}$$

Then the optimal  $\mu$ -expected revenue the auctioneer collects from a  $\underline{\omega}_i$ -type of bidder  $i$  is

$$\begin{aligned}
& \nu(\underline{v}_i)[(1 - b_i^*)t_i(\underline{\omega}_i, \underline{\omega}_{-i}) + b_i^*t_i(\underline{\omega}_i, \bar{\omega}_{-i})] \\
&= [\nu(\underline{v}_i)(1 - b_i^*) + \nu(\bar{v}_i)(1 - a_i) - \nu(\bar{v}_i)(1 - a_i)]t_i(\underline{\omega}_i, \underline{\omega}_{-i}) + [\nu(\underline{v}_i)b_i^* + \nu(\bar{v}_i)a_i - \nu(\bar{v}_i)a_i]t_i(\underline{\omega}_i, \bar{\omega}_{-i}) \\
&= [(1 - b_i) - \nu(\bar{v}_i)(1 - a_i)]t_i(\underline{\omega}_i, \underline{\omega}_{-i}) + [b_i - \nu(\bar{v}_i)a_i]t_i(\underline{\omega}_i, \bar{\omega}_{-i}) \\
&= \underline{T}_i - \nu(\bar{v}_i)\bar{T}_i,
\end{aligned}$$

which depends only on  $p(\underline{\omega}_i, \cdot)$ .

These two results together give us the maximum  $\mu$ -expected revenue given any fixed

(iIC-implementable) allocation rule  $p$ :

$$\begin{aligned}
& \sum_{i=1,2} \nu(\bar{v}_i)[(1 - a_i)p_i(\bar{\omega}_i, \underline{\omega}_{-i})\bar{v}_i + a_i p_i(\bar{\omega}_i, \bar{\omega}_{-i})\bar{v}_i] + \underline{T}_i - \nu(\bar{v}_i)\bar{T}_i \\
&= \sum_{i=1,2} \sum_{\omega_{-i}=\underline{\omega}_{-i}, \bar{\omega}_{-i}} \mu(\bar{\omega}_i, \omega_{-i})[p(\bar{\omega}_i, \omega_{-i}) - p(\underline{\omega}_i, \omega_{-i})]\bar{v}_i + \mu(\omega_{-i})p(\omega_i, \omega_{-i})\underline{v}_i. \tag{1}
\end{aligned}$$

It remains to maximize the  $\mu$ -expected revenue over all (iIC-implementable) allocation rule. Since any allocation rule is iIC-implementable when  $\nu$  is not a product measure, to maximize the auctioneer's  $\mu$ -expected revenue over all iIC-implementable allocation rules, all we need is to do pointwise optimization. Differentiating the  $\mu$ -expected revenue with respect to  $p_i(\omega_i, \omega_{-i})$  for every type profile  $(\omega_i, \omega_{-i})$ , we will obtain the counterparts of “virtual utilities” of bidder  $i$  as in the classical theory of mechanism design, which are summarized in Figure 12.

	$\underline{\omega}_i$	$\bar{\omega}_i$
$\bar{\omega}_{-i}$	$b_i \underline{v}_i - \nu(\bar{v}_i) a_i \bar{v}_i$	$\nu(\bar{v}_i) a_i \bar{v}_i$
$\omega_{-i}$	$(1 - b_i) \underline{v}_i - \nu(\bar{v}_i) (1 - a_i) \bar{v}_i$	$\nu(\bar{v}_i) (1 - a_i) \bar{v}_i$

Figure 12: Derivatives of the  $\mu$ -expected revenue with respect to  $p_i(\omega_i, \omega_{-i})$ .

Using the relation between  $\mu$  and  $\nu$ , we can rewrite Figure 12 as Figure 13.

	$\underline{\omega}_i$	$\bar{\omega}_i$
$\bar{\omega}_{-i}$	$\mu(\omega_i, \bar{\omega}_{-i}) \underline{v}_i - \mu(\bar{\omega}_i, \bar{\omega}_{-i}) (\bar{v}_i - \underline{v}_i)$	$\mu(\bar{\omega}_i, \bar{\omega}_{-i}) \bar{v}_i$
$\omega_{-i}$	$\mu(\omega_i, \underline{\omega}_{-i}) \underline{v}_i - \mu(\bar{\omega}_i, \underline{\omega}_{-i}) (\bar{v}_i - \underline{v}_i)$	$\mu(\bar{\omega}_i, \underline{\omega}_{-i}) \bar{v}_i$

Figure 13: Derivatives of the  $\mu$ -expected revenue with respect to  $p_i(\omega_i, \omega_{-i})$ .

Using the condition  $\bar{v}_i > \underline{v}_i > 0$  in Proposition 3, an optimal allocation rule  $p$  must have the property depicted in Figure 14, which essentially says that  $prob_i$  is monotonically non-decreasing in bidder  $i$ 's valuation.

Now consider the auction  $(p, t)$ , where  $p$  is an optimal allocation rule, and each  $t_i$  is as depicted in Figure 15.

When the allocation rule  $p$  satisfies the monotonicity property depicted in Figure 14, the auction  $(p, t)$  depicted in Figure 15 will be dominant strategy IC. It suffices to prove that the dominant strategy IC auction  $(p, t)$  depicted in Figure 15 generates the same  $\mu$ -expected revenue as an optimal interim IC auction does.

	$\underline{\omega}_1$	$\bar{\omega}_1$
$\bar{\omega}_2$	$p_1 = 0, p_2 = 1$	
$\underline{\omega}_2$		$p_1 = 1, p_2 = 0$

Figure 14: Monotonicity of an optimal allocation rule  $p$ .

	$\underline{\omega}_i$	$\bar{\omega}_i$
$\bar{\omega}_{-i}$	$p_i(\underline{\omega}_i, \bar{\omega}_{-i})\underline{v}_i$	$p_i(\bar{\omega}_i, \bar{\omega}_{-i})\bar{v}_i - p_i(\underline{\omega}_i, \bar{\omega}_{-i})(\bar{v}_i - \underline{v}_i)$
$\underline{\omega}_{-i}$	$p_i(\underline{\omega}_i, \underline{\omega}_{-i})\underline{v}_i$	$p_i(\bar{\omega}_i, \underline{\omega}_{-i})\bar{v}_i - p_i(\underline{\omega}_i, \underline{\omega}_{-i})(\bar{v}_i - \underline{v}_i)$

Figure 15: The transfer rule  $t_i$  in the auction  $(p, t)$ .

Taking  $\mu$ -expectation of the revenue generated by the auction  $(p, t)$  depicted in Figure 15, we have

$$\begin{aligned} & \sum_{i=1,2} \sum_{\omega_{-i}=\underline{\omega}_{-i}, \bar{\omega}_{-i}} \mu(\underline{\omega}_i, \omega_{-i}) p_i(\underline{\omega}_i, \omega_{-i}) \underline{v}_i \mu(\bar{\omega}_i, \omega_{-i}) [p_i(\bar{\omega}_i, \omega_{-i}) \bar{v}_i - p_i(\underline{\omega}_i, \omega_{-i}) (\bar{v}_i - \underline{v}_i)] \\ &= \sum_{i=1,2} \sum_{\omega_{-i}=\underline{\omega}_{-i}, \bar{\omega}_{-i}} \mu(\bar{\omega}_i, \omega_{-i}) [p(\bar{\omega}_i, \omega_{-i}) - p(\underline{\omega}_i, \omega_{-i})] \bar{v}_i + \mu(\omega_{-i}) p(\underline{\omega}_i, \omega_{-i}) \underline{v}_i, \end{aligned}$$

which is the same as (1). This proves that the dominant strategy IC auction  $(p, t)$  depicted in Figure 15 is an optimal interim IC auction.

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