Sequential Elimination vs. Instantaneous Voting

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Abstract

Voting procedures eliminating candidates sequentially and one-at-a-time based on repeated ballots are superior to well-known single-round voting rules: if voters are strategic the former will induce the Condorcet winner in unique equilibrium, whereas the latter may fail to select it. In addition, when there is no Condorcet winner the outcome of sequential elimination voting always belongs to the 'top cycle'. The proposed sequential family is quite large, including an appropriate adaptation of *all* single-round voting procedures. The importance of one-by-one elimination and repeated ballots for Condorcet consistency is further emphasized by its failure for voting rules such as *plurality runoff* rule, *exhaustive ballot* method, and *instant runoff* voting. ***JEL*** Classification Numbers: P16, D71, C72. **Key Words:** Sequential elimination voting, Condorcet winner, top cycle, weakest link voting, exhaustive ballot, instant runoff voting, Markov equilibrium, complexity aversion.

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1 Introduction

One way to assess a voting rule is to compare the outcome(s) it induces against the social planner's objective. However, in the absence of an all-powerful planner to impose his own ideal decision procedure, any voting rule must be ratified by a majority of the members of a society if it is going to function as a democratic way of reaching collective decisions. Condorcet (1785) had first epitomized an extreme version of this view by suggesting that whenever there is a candidate (or alternative) always preferred by a majority of voters against any other candidate on pairwise comparisons, such a candidate, later to become known as the *Condorcet winner*, must be elected by the chosen voting rule. This requirement, called *Condorcet consistency*, is "widely regarded as a compelling democratic principle" (sect. 4, ch. 9 of Moulin's book, 1988). Yet many voting mechanisms fail to satisfy this appealing property. As we shall see, almost all well-known voting rules fail to be Condorcet consistent if voters are assumed to vote strategically.

In this paper we will argue that voting procedures based on sequential elimination are superior to ones in which the winner is determined in a single-round voting, using the criterion of Condorcet consistency. We show that when voters behave strategically,[1] sequential elimination procedures may aggregate voter preferences better than single-round procedures. Roughly, in sequential elimination procedures the voting outcome is determined gradually allowing voters more influence on the outcome and preventing them from getting locked in a "bad" equilibrium, a non-Condorcet outcome (when a Condorcet winner exists), because of miscoordination. Even when the Condorcet winner does not exist, sequential elimination procedures have some nice features: the outcome always belongs to the 'top cycle', and is Pareto efficient when only three candidates are involved. In contrast, miscoordination tends to be pervasive in most single-round and various other voting and thus may fail to select the Condorcet winner.

The idea of sequential elimination voting is best conveyed by the following repeated application of the one-person-one-vote principle, although voters may be allowed to declare preferences in more complicated ways, say, by submitting a ranking. Voting takes place in rounds with all the voters simultaneously casting their votes in each successive round. In any round the candidate receiving the smallest

[1] The idea of strategic voting, as opposed to *sincere voting*, was popularized Farquharson (1969) and is also known as *sophisticated voting*.

number of votes is eliminated, with any tie involving the smallest number of votes broken by a deterministic tie-breaking rule. This process continues until all but one of the candidates have been eliminated. We call this the *weakest link voting.*[2] Actually, the weakest link voting is nothing but a natural multi-round extension of the plurality voting principle, with elimination of only the worst plurality loser in each round.[3] Similar one-by-one sequential elimination method can be adopted to extend any familiar single-round voting rule to its appropriate multi-round equivalent.[4]

Our results are as follows. When voters are strategic, the unique equilibrium (if an equilibrium exists) of a broadly defined sequential elimination voting game will select the Condorcet winner (Theorems 1–3). The sufficient condition for this to happen (and satisfied by sequential versions of all familiar single-round voting procedures, sequential binary voting, sequential runoff voting etc.) is indeed very weak: *For any group of majority voters, there always exists strategies coordinating which they can ensure that a candidate will not be eliminated in an ongoing*

[2]A noteworthy application of this voting rule is the selection of the 2012 Olympic Games host city winner: London emerged as the winner after Moscow, New York, Madrid and Paris were eliminated in that order in successive votes held over four rounds – the actual voting is summarized at the end of the Appendix; see also 6 July, 2005 news ("London beats Paris to 2012 Games") in *http://news.bbc.co.uk/sport1/hi/front_page/4655555.stm*. Another interesting application of the weakest link voting is the election of the leader of the Conservative Party in the United Kingdom in 2001. The voting rule used then involved two stages. In the first stage, a small number of (to be precise, five) candidates had put themselves up for selection who were all simultaneously voted upon by the party's parliamentary members in a sequence of rounds. In each round the candidate getting the smallest number of votes was eliminated. This process continued until only two candidates remained who then faced, in a second stage, the poll of a large number of party members. (In fact, in the latest Conservative party deliberations in 2005, the second stage is being considered to be even dropped, with the eliminations of candidates to be confined to the first stage only.) For an early account of the leadership contest procedure, see the explanation by Julian Glover in The Guardian, July 10, 2001 accessible at *http://politics.guardian.co.uk/Print/0,3858,4196604,00.html*. This voting procedure is only a recent innovation in democratically electing the party leader; on all other previous occasions, the Conservative Party leader used to be chosen on an ad hoc basis.

[3]The weakest link voting is very similar to *sequential runoff election* where alternatives are eliminated one-at-a-time but based on the voters submitting a full strict-order ranking of the remaining alternatives (instead of voting for a single alternative), eliminating in each round the alternative with the least number of first place votes.

[4]Of course, the elimination criterion will be different depending on the voting rule.

round.[5] In addition, if there is no Condorcet winner, we show that the elected candidate must be in the top cycle (Theorem 3). In contrast, we identify a large class of instantaneous (single-round) voting rules that fail to be Condorcet consistent (Theorem 4; Proposition 2).[6] This class includes all the popular voting rules such as plurality rule, approval voting, Borda rule and negative voting. We also highlight two specific features of the sequential voting method that are important for Condorcet consistency: *one-by-one elimination* and *repeated ballots*. For the first, we provide two counter examples: (*i*) a *plurality runoff* rule eliminating all but two candidates in the first round using plurality rule and then choosing the winner in a second ballot from the remaining two candidates using majority rule, is shown to fail Condorcet consistency (Proposition 3);[7] (*ii*) an *exhaustive ballot* method, which is same as the weakest link voting except that if at any round a candidate receives majority votes then that candidate immediately becomes the winner, also fails Condorcet consistency (Proposition 4). The importance of repeated ballots is shown by the failure of Condorcet consistency of the *instant runoff voting* rule which is based on a *single ballot*(Proposition 5).[8]

Both the positive and negative results on Condorcet consistency in this paper should be viewed as taking an important issue further that has only been intermittently studied in the literature. One strand of the literature analyzing the issue of Condorcet consistency under sophisticated voting focus on sequential binary voting and its variants (McKelvey and Niemi, 1978; Dutta and Pattanaik, 1985; Dutta and Sen, 1993; and Dutta et al., 2002). An alternative focus on the same issue makes the assumption of sincere voting (see ch. 9 of Moulin's book (1988) for example).

[5]This will be the case, if, for example, the voting rule states that a candidate receiving a majority of the top rank cannot be eliminated.

[6]Roughly, *instantaneous voting* means voting takes place only once but the winner may be selected in one or more rounds of eliminations.

[7]Its extended version, sequential runoff voting, is Condorcet consistent.

[8]Instant runoff voting rule, alternatively known in the literature as *alternative vote method* and *full preferential voting* (see Moulin, 1988, ch. 9), is as follows. Voters express their rankings of candidates in a single ballot. If a candidate wins the majority of the top rank then that candidate becomes the winner. Otherwise, the candidate receiving the smallest number of top-rank votes is eliminated, and a fresh count is taken. In this count if the eliminated candidate was the top-ranked candidate for some of the voters, then for those voters their second-most favorite candidates become their respective top-ranked candidate in the second count. The vote counting (using transferrable votes) continues until some candidate wins the majority of the top rank from the remaining candidates.

However, very little is available by way of characterization of Condorcet consistent voting rules.[9] Thus, our analysis should be viewed as filling an important gap in the voting literature. Another major issue in the voting literature (Dekel and Piccione, 2000; Strumpf, 2002; Battaglini, 2005) with which our work bears some similarity is whether sequential voting is any better than simultaneous voting in aggregating information that might be dispersed among the electorates. Our main concern, however, is *preference aggregation* rather than information aggregation. Also, sequential *elimination* voting in this paper is very different from sequential voting considered by these authors; our voters cast their votes simultaneously and repeatedly in successive rounds.

The paper is organized as follows. In the next section we formally describe the voting rules and the related equilibrium solution concepts. Section 3 contains results on sequential elimination voting. In section 4, we analyze single-round and some other popular voting mechanisms. Section 5 concludes. Most proofs are relegated to an Appendix.

2 The Voting Rules and Equilibrium Solutions

Much of our insight about sequential elimination voting can be gained by studying the *weakest link voting*, so we start with this particular voting rule.

The weakest link voting

Formally, the weakest link voting proceeds as follows. The voting takes place in a finite number of stages. The set of candidates is denoted as $\mathcal{K}$ with cardinality k, and the voter set is denoted as $\mathcal{N}$ with cardinality n, both k and n at least three, and $\mathcal{K} \cap \mathcal{N} = \emptyset$. Each voter has a strict, ordinal preference ordering over the candidates. A single winner is selected after completion of $k-1$ voting stages. In stage 1, the starting point of the weakest link game, all candidates are simultaneously voted upon by the voters. The candidate receiving the smallest number of votes drops out from the competition and the remaining $k-1$ candidates proceed to a similar stage 2 voting; any tie is broken by a single deterministic tie-breaking rule ranking all k candidates. This procedure continues until all but one of the candidates have been eliminated.

[9] Dutta and Sen (1993) is on implementation of Condorcet *social choice functions*.

Two examples

Example 1. Consider three voters, and four candidates: $\{w, x, y, z\}$. The voters' preferences are as follows:

$$1: \quad y,\ x,\ z,\ w$$
$$2: \quad z,\ x,\ y,\ w$$
$$3: \quad w,\ x,\ z,\ y.$$

Note that candidate x is the Condorcet winner. Assume first that the weakest link rule is being used and all three voters vote sincerely so that at each voting stage each votes for his most preferred candidate among the surviving ones. Clearly, x will be eliminated at the first stage (regardless of the tie-breaking rule) and will not emerge as the winner. In contrast, assume now that the three voters behave strategically. Without defining formally a solution concept at this stage, we will argue informally why strategic behavior must result in x winning. We will invoke a backward induction argument: Consider first the last stage in which only two candidates remain. If x is one of these candidates then clearly x will win at this stage because two out of three voters prefer x to the other candidate (and voting for x weakly dominates voting for the other candidate). Similar argument implies that if the two candidates at the last stage are (z, w), then z wins. If they are (z, y) then z wins as well and for (y, w) candidate y emerges the winner. Consider now the second-to-last stage of voting: If the three surviving candidates at this stage are (y, z, w), then z must emerge the winner. This is because given the possible continuations (as specified above), if z survives this stage he will become the eventual winner and otherwise y will be the winner. So, voting for z is the only (weakly) undominated choice for voters 2 and 3. Hence z will not be eliminated at this stage and will emerge the winner. A similar argument implies that if x is one of the remaining candidates at the second-to-last stage then he must emerge the eventual winner. We now consider the first stage of the voting. Given the possible continuations as described above, if x is eliminated at the first stage then z will emerge the winner, whereas x will emerge the winner if any of the other three candidates is eliminated. Hence, the only weakly undominated choice for voters 1 and 3 is to vote for x. Thus x will survive and will emerge the winner.

The fact that x fails to emerge as the winner if voters vote sincerely and x is the winner when voters vote strategically has the following interesting implication.

Consider the following simultaneous version of the weakest link rule, where the three voters are required to submit once at the outset a preference profile[10] based on which the winner will be determined according to the weakest link rule. Assume further that the tie-breaking rule puts x last in the elimination priority. If we consider the resulting strategic form game among the three voters (with preference profiles being the strategies), then it can be easily verified that voting sincerely is a Nash equilibrium with undominated strategies. Hence x may fail to emerge as the winner when voters behave strategically in the simultaneous version of the weakest link, but in its sequential version strategic behavior guarantees it winning.

This example highlights the advantage of a voting procedure in which voters are called to step in repeatedly in submitting their preferences over ones that involve a single stage. As we will show later, this advantage goes beyond the weakest link rule. Indeed we will introduce a large class of sequential elimination procedures which guarantee the selection of the Condorcet winner in strategic voting and will also argue that almost all well-known single-round voting procedures fail to have this property. ||

Example 2. We will use the recently held selection of London as the winner of the 2012 Olympic Games' host city, to explain some interesting voting pattern (at the very end in the Appendix the actual votes are summarized). Of particular interest is the inconsistency regarding Madrid: total votes in favor of Madrid dropped from 32 in the second round to 31 in the third round, when Madrid got eliminated from the competition. The press naively explained this voting pattern by claiming that the voters had changed their minds about Madrid. The insight that we can offer to this story is that this explanation may be completely wrong. The apparent inconsistency could be mainly due to the *strategic voting* by the voting members. One can construct consistent preferences for the voters under which the equilibrium behavior in the weakest link voting will generate exactly the voting pattern that was observed. ||

Next we begin to describe our equilibrium procedure with respect to the weakest link voting game. The same procedure can be applied to other one-by-one sequential elimination voting and a few other potentially multi-round (but not necessarily one-by-one) elimination voting games with minor modifications, although we will not

[10] A voter's preference profile is a mapping from the set of available candidates in any elimination round to a single candidate for non-elimination in that round.

elaborate on these modifications. In the extreme, the solution will also apply to all single-round voting.

The equilibrium

The equilibrium outcome of the weakest link voting follows successive eliminations of strategies failing subgame perfection and weak non-domination. In addition, voters will be assumed to use only Markov strategies. These are explained next.

In any final stage subgame (i.e., at stage $k-1$), the voters' strategies must be a Nash equilibrium and must not be (weakly) dominated when considered specifically with respect to the subgame;[11] all other strategies are eliminated. Then in the subgames starting with stage $k-2$, only the strategies that survived eliminations at stage $k-1$ are considered. In any of these subgames, again, the voters' strategies from the restricted set must be a Nash equilibrium and must not be dominated along the subgame, where the *permissible strategies* of the voters with respect to which the weak-domination check is carried out are the strategies that have survived backward eliminations up to that stage. We follow this backward-elimination procedure all the way to stage 1. In general, in the subgames following a stage, irrespective of whether such subgames are reached or not, no voter will use strategies that fail to survive backward eliminations.

The voters adopt only *Markov* strategies, that is, the strategies at any stage onwards depend only on the candidates who have survived up to that stage and *not* on the specific history leading up to it.

We now develop the equilibrium procedure more formally.

The voters' strategies are easier to describe with respect to *histories.* A *history*, h, associated with any particular node at any stage of the successive elimination process is a complete description of the *actual* voting decisions leading up to that node. Given a deterministic tie-breaking rule, any history h uniquely defines a subgame $\Gamma(h)$ determining the set of candidates $C \subseteq \mathcal{K}$ who have survived at the end of h; $\Gamma(h_0)$ denotes the entire game where h_0 denotes the null history.

Let $\mathcal{H}_C = \{h | C \text{ is the set of remaining candidates}\}$, and $\mathcal{H} = \bigcup_{C \subseteq \mathcal{K}} \mathcal{H}_C$ be the set of all histories.

[11] While a particular strategy may not be dominated in the entire game, it could still be dominated when restricted to a (smaller) subgame.

Define the set of (pure) strategies of voter i by $S_i = \{s_i : \mathcal{H} \to \mathcal{K} \text{ s.t. } s_i(h) \in C \text{ if } h \in \mathcal{H}_C\}$.

For any h, let $C(h)$ be the set of candidates left with cardinality $n(h)$, so that history h involves the first $k - n(h)$ stages of eliminations. Also, let $S_i(h)$ be the restriction of S_i to the subgame $\Gamma(h)$. We want to define $S(h) = \Pi_i S_i(h)$ to be the product set of strategies in the subgame $\Gamma(h)$. Thus,

$$\begin{array}{rl} \text{if} & s_i \in S_i(h), \\ \text{then} & s_i : \tilde{\mathcal{H}}(h) \to C(h) \ \text{ s.t. } \ s_i(h') \in C' \quad \forall h' \in \mathcal{H}_{C'} \cap \tilde{\mathcal{H}}(h) \neq \emptyset \\ & \text{where } \tilde{\mathcal{H}}(h) = \left\{h' \in \mathcal{H} | h' = (h, h' \backslash_{1,\ldots,k-n(h)}) \ \text{ for any history } \ h'\right\} \ \text{ is} \\ & \text{the set of histories that follow } h, \text{ with } h' \backslash_{1,\ldots,k-n(h)} \text{ denoting actual} \\ & \text{voting decisions associated with } h' \text{ except in the first } k-n(h) \text{ stages.} \end{array}$$

Define, inductively, a profile of strategies $s^* \in \prod_i S_i$ to be an equilibrium, as follows.

At any history h s.t. $n(h) = 2$, $s^*(h)$ is a Nash equilibrium (in short, N.E.) in the subgame $\Gamma(h)$ <u>and</u> not weakly dominated in this subgame. That is,

$$\begin{array}{rrcl} & s^*(h) & \in & S(h); \\ \text{(N.E.)} & \pi_i(s_i^*(h), s_{-i}^*(h)) & \geq & \pi_i(s_i', s_{-i}^*(h)) \quad \forall s_i' \in S_i(h); \end{array}$$

$$\left. \begin{array}{rl} \text{(Weak non-domination)} & \nexists s_i' \in S_i(h) \ \text{ s.t.} \\ & \pi_i(s_i', s_{-i}') \geq \pi_i(s_i^*(h), s_{-i}') \quad \forall s_{-i}' \in S_{-i}(h), \text{ and} \\ & \pi_i(s_i', s_{-i}') > \pi_i(s_i^*(h), s_{-i}') \quad \text{for some } s_{-i}' \in S_{-i}(h). \end{array} \right\} \quad (1)$$

Suppose $s^*(h')$ is defined for all h' s.t. $n(h') \leq j-1$. We now want to define $s^*(h)$ for h s.t. $n(h) = j$.

Let $\hat{S}(h) = \{s \in S(h) | \ s(h') = s^*(h') \quad \forall h' \in \tilde{\mathcal{H}}(h) \text{ and } n(h') < j\}$; that is, the strategies in $\hat{S}(h)$ restrict the voters to choose only their *equilibrium* strategies in the subgames $\Gamma(h')$, where the equilibrium is already defined. Now let $\widehat{\Gamma}(h)$ be the (reduced) subgame $\Gamma(h)$ when the strategies are restricted to the set $\hat{S}(h)$. Then $s^*(h)$ is a N.E. in the subgame $\widehat{\Gamma}(h)$ that is not weakly dominated. That is,

$$\begin{array}{rcl} s^*(h) & \in & \hat{S}(h); \\ \pi_i(s_i^*(h), s_{-i}^*(h)) & \geq & \pi_i(s_i', s_{-i}^*(h)) \quad \forall s_i' \in \hat{S}_i(h); \end{array}$$

$$\left.\begin{array}{c}\nexists s_i' \in \hat{S}_i(h) \text{ s.t.} \\ \pi_i(s_i', s_{-i}') \geq \pi_i(s_i^*(h), s_{-i}') \quad \forall s_{-i}' \in \hat{S}_{-i}(h), \text{ and} \\ \pi_i(s_i', s_{-i}') > \pi_i(s_i^*(h), s_{-i}'), \text{ for some } s_{-i}' \in \hat{S}_{-i}(h). \end{array}\right\} \tag{2}$$

By iterating backwards all the way to the first stage, we obtain $s^*(h_0)$.

Now if we let $\widehat{\Gamma}(h) = \Gamma(h)$ for any h such that $n(h) = 2$ then we can define an **equilibrium** s^* as a subgame perfect equilibrium that is not weakly dominated in any subgame $\widehat{\Gamma}(h)$ for any h (thus it satisfies inductively condition (1) when $n(h) = 2$ and condition (2) when $n(h) > 2$). ||

We define the equilibrium, s^*, to be a **Markov equilibrium**, if the voters are restricted to use only Markov strategies. That is, each i selects strategies only from

$$S_i^M = \{s_i : \mathcal{H} \to \mathcal{K} \text{ s.t. } \forall h \in \mathcal{H}_C,\ s_i(h) \in C; \text{ and} \\ \text{if } C(h) = C(h') \text{ then } s_i(h) = s_i(h')\}.$$

This completes the formal equilibrium procedure.

Remarks. Our backward-elimination procedure differs, it must be noted, from the more familiar procedure of iterative elimination of (weakly) dominated strategies in one important aspect: while in the latter approach the weak-domination check is carried out in relation to the entire game, ours is only along the subgames.[12] Iterative elimination on its own, or even in combination with subgame perfection, is unlikely to solve the miscoordination problems that result in undesirable outcomes. It is well-known in other voting contexts that iterative elimination can have very little elimination power.[13]

Our equilibrium procedure can be considered with or without the assumption of Markov strategies. Later we justify the use of the Markov assumption for our sequential elimination voting.

Single-round voting

To contrast the weakest link and other one-by-one sequential elimination voting rules, we will also consider single-round voting rules such as plurality rule, approval

[12] Moulin (1979) formally analyzed the iterative elimination procedure to generalize the concept of sophisticated voting and applied it to a significant class of voting – voting by veto, kingmaker and voting by binary choices.

[13] For example, Dhillon and Lockwood (2004) have shown in the case of plurality voting (see their Lemma 1 and the related discussion) that the strategy of voting *any* candidate other than one's lowest-ranked candidate will survive iterative eliminations of weakly dominated strategies.

voting, Borda voting and negative voting. For these voting rules the equilibrium solution concept is *undominated Nash*, i.e., the voters' strategies must be a Nash equilibrium and must not be (weakly) dominated. Note that our twin requirements of subgame perfection and non-domination (independently of the Markov strategy assumption) boil down to the equilibrium definition for single-round voting rules. Thus, the comparisons to be made in section 3 between sequential elimination voting and single-round voting are based on the same benchmark solution concept.

3 Sequential (elimination) voting

In this section and the rest of the paper, the voters are always understood to be strategic unless otherwise specified.

3.1 Condorcet consistency

We start with a key result on the weakest link voting and generalize to a class of sequential elimination voting.

Theorem 1 *Suppose the voters use Markov strategies. Then the weakest link voting with a deterministic tie-breaking rule is Condorcet consistent.*

Proof. Suppose there is a Condorcet winner z, but the weakest link voting game has a Markov equilibrium that results in some other candidate z_1 $(\neq z)$ as the ultimate winner. Order the stages of elimination of $(k-1)$ candidates as follows:

The winner	z_1
Stage k-1	$z_1,\ z_2$
Stage k-2	$z_1,\ z_2,\ z_3$
Stage k-3	$z_1,\ z_2,\ z_3,\ z_4$
	.
	.
Stage 1	$z_1,\ z_2,\ z_3,\ z_4, \ldots, z_k$.

Thus in stage j, the eliminated candidate is labelled as candidate z_{k-j+1}, $j = 1, 2, \ldots, k-1$.

Initially, consider stage $k-1$ and suppose $z_2 = z$. Given that z_2 is the Condorcet winner, clearly for any voter who prefers z_2 over z_1, and there will be a majority of such voters, the strategy of choosing z_1 is weakly dominated in the stage-$(k-1)$ subgame: choosing z_2 instead will produce no worse and sometimes a better outcome (when z_2 wins the majority vote). (In fact, sincere voting by all voters is the only Nash equilibrium that is also undominated in this final stage subgame.) Thus if the Condorcet winner z survives up to stage $k-1$, he must be the ultimate winner so that $z_1 = z$; $z_2 = z$ is not possible.

Now suppose the following hypothesis is true:

If candidate z survives up to stage j on- or off-the-equilibrium path, then he will also survive the remaining stages and become the ultimate winner.

We then prove that having proceeded to any stage-$(j-1)$ subgame with only $k-j+2$ candidates left,[14] candidate z will also survive stage $j-1$ to move up to stage j and thus become the ultimate winner.

Suppose on the contrary that $z_{k-j+2} = z$. Consider those voters who prefer z_{k-j+2} over z'_1, where z'_1 is going to be the ultimate winner if z is eliminated in stage-$(j-1)$ voting.[15] By definition of z, these voters will form a majority. Consider any such voter's strategy in the stage-$(j-1)$ subgame, where z'_1 is to become the ultimate winner. Suppose the representative voter were to choose some $z' \neq z$. This, we claim, is not possible. If the voter switches his vote from z' to z and z is *not* eliminated, which will be the case if all who prefer z over z'_1 vote for z, then z survives, by hypothesis, all the subsequent stages and becomes the ultimate winner that is *better* than z'_1; if, on the other hand, z *is* eliminated then by the Markov property of the voters' strategies z'_1 becomes the ultimate winner. Thus, in the subgame z will weakly dominate z' so that the representative voter must vote for z only. This implies a majority of voters would vote for z, contradicting that $z_{k-j+2} = z$.

We already proved our hypothesis for $j = k-1$. So use an induction argument to conclude that the weakest link voting with a deterministic tie-breaking rule will not admit, in equilibrium, any non-Condorcet (winner) outcome. To show that the Condorcet winner z will be the ultimate winner, one has to show that there

[14]This subgame can be on- or off-the-equilibrium path.

[15]$z'_1 = z_1$ if this subgame is on the equilibrium path, and otherwise z'_1 can be some other candidate.

exists an equilibrium in the weakest link voting game whenever there is a Condorcet winner. In the Appendix, we establish an existence result that does not rely on any assumption about the Condorcet winner (Theorem 6).

Our argument so far does not make any reference to the tie-breaking rule, thus the weakest link voting is Condorcet consistent for any arbitrary deterministic tie-breaking rule. **Q.E.D.**

Remarks. The procedure of eliminating the weakest link, if extended naturally to eliminate all but one candidate in a single round of voting (as opposed to successive eliminations), translates into one-shot plurality voting. Plurality voting will be later shown to fail Condorcet consistency (Theorem 4). Thus, Theorem 1 illustrates the distinct advantage of the sequential elimination procedure over a single-round elimination (of plurality rule). Later, based on Theorem 2, similar parallels can be made between all well-known single-round voting rules and their sequential counterparts.

A weaker version of the Markov property would suffice for Theorem 1 proof. All we require is that the strategies do not depend on the history through the specific configuration of votes that led to the particular candidates' eliminations. However, the strategies can still depend on the order in which the candidates were eliminated. In fact, if we assume that the votes are not revealed between stages but only the identity of the eliminated candidate at each stage is announced, then we do not need the Markov property. Also, the assumption that the voters use only Markov strategies could be considered a limitation of Theorem 1. In the Appendix (Theorem 5) we provide a formal justification for this assumption.

Theorem 1 should be viewed in combination with Theorem 6 (stated and proved in the Appendix) that shows that there always exists an equilibrium of the weakest link voting game for our solution concept defined in section 2. The equilibrium existence result in Theorem 6 is particularly important (although we prefer it to be put in the Appendix because of the technical nature of the proof), because there could be subgames off-the-equilibrium path without a Condorcet winner (among the remaining candidates) and it is by no means clear that our solution concept would necessarily yield an equilibrium in such subgames. ||

Next we define a rather general *sequential* process of elimination with the only restriction that in each round only one candidate will be eliminated.

Sequential voting.[16] Players vote in $k-1$ rounds. At each stage the voters simultaneously vote and one candidate is removed. The winner is the candidate who survives the last stage. If C is the set of candidates left at any stage $j < k$ then a vote for voter i at that stage consists of choosing an element of the set $A_i(C, j)$; moreover if each i chooses $a_i \in A_i(C, j)$ at this stage then we shall denote the eliminated candidate by $e(a_1, ..., a_n) \in C$. So in stage 1 each voter i chooses $a_i \in A_i(\mathcal{K}, 1)$. We now specify the following property to define a class of sequential voting rules.

Non-elimination property: For any stage j, any remaining set of candidates C, any $k \in C$ and any majority of voters $\mu_1, ..., \mu_{n^*}$ (where $n^* = \begin{cases} n/2+1 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$) there exist votes $a^k_{\mu_1} \in A_{\mu_1}(C, j), a^k_{\mu_2} \in A_{\mu_2}(C, j), .., a^k_{\mu_{n^*}} \in A_{\mu_{n^*}}(C, j)$ such that $e(a^k_{\mu_1}, a^k_{\mu_2}, .., a^k_{\mu_{n^*}}) \neq k$. Thus at any stage with C the set of remaining candidates, any majority can ensure that any specific candidate k is not eliminated.[17]

All sequential voting rules satisfying the non-elimination property constitute the family $\mathcal{F}$. $||$

We further like to emphasize the following aspects of our sequential voting. The voting cast by the voters is quite general. Voters can express their preference for only a single candidate or more than one candidate by submitting a ranking, and the ranking can be strict or weak; in fact, the preference submission could be more abstract than a simple ranking of candidates. The candidates considered for elimination in any particular round can be all the remaining candidates or a subset of the candidates.[18]

Now go back to the proof of Theorem 1. It is not difficult to see that the arguments there will apply equally for the entire family $\mathcal{F}$. In particular, the non-elimination property comes to equal effect in sustaining the Condorcet winner as follows. Assume the hypothesis for stage j is true and consider stage-$(j-1)$ subgame.[19] Suppose if z is eliminated in stage $j-1$ then z'_1 becomes the ultimate winner. There will be a majority of voters $\mu_1, ..., \mu_{n^*}$ who prefer z over z'_1. More-

[16]By sequential voting, we mean sequential elimination of candidates.

[17]This non-elimination property implies *majority rule* in stage $k-1$.

[18]In particular, the voting could be over pairs of candidates in successive rounds as in *sequential binary voting*, an extensively studied subject.

[19]Clearly, the hypothesis is true for $j = k-1$ because non-elimination property implies majority rule in stage $k-1$.

over, by the non-elimination property there exist votes $a^z_{\mu_1} \in A_{\mu_1}(C, j-1), a^z_{\mu_2} \in A_{\mu_2}(C, j-1), .., a^z_{\mu_{n^*}} \in A_{\mu_{n^*}}(C, j-1)$ such that $e(a^z_{\mu_1}, a^z_{\mu_2}, .., a^z_{\mu_{n^*}}) \neq z$.

Now suppose any such voter μ_i chooses $a'_{\mu_i} \neq a^z_{\mu_i}$ resulting in z being eliminated. This, we claim, is not possible. If the voter instead chose $a^z_{\mu_i}$ in place of a'_{μ_i} and z is *not* eliminated, then by hypothesis z will progress to become the ultimate winner, which is a better outcome than z'_1; on the other hand, if z *is* eliminated then by Markov property z'_1 is the ultimate winner. Thus, choosing $a^z_{\mu_i}$ weakly dominates a'_{μ_i}. Hence, z cannot be eliminated in the stage-$(j-1)$ subgame and an induction argument would induce z to become the ultimate winner. Thus, we can state the following result:

Theorem 2 *All sequential voting rules in the family $\mathcal{F}$ will be Condorcet consistent.*

Note that the Markov assumption for Theorem 2, like Theorem 1, is justifiable as we have argued in the Appendix (Theorem 5).

Remark. *The required non-elimination condition for voting rules to belong to the family $\mathcal{F}$ is indeed very weak. The sequential (one-by-one elimination) versions of all single-round voting rules, plus sequential binary voting, sequential runoff voting etc. satisfy this property and thus belongs to the family $\mathcal{F}$.*

So far our analysis is based on the assumption that a Condorcet winner exists. The structure of equilibrium in the absence of a Condorcet winner should be of interest. The next result applies to the sequential family $\mathcal{F}$, with or without a Condorcet winner.

Before stating our result let us clarify some notations. Given the voters' strict preference ordering over candidates, a binary comparison operator T defines a candidate x to be *majority preferred* over another candidate y, written as xTy, if the number of voters preferring x over y exceeds the the number of voters preferring y over x. The operator T will be a *majority tournament* if either n is odd or a deterministic tie-breaking rule breaks the ties.

Top Cycle. $TC(\mathcal{K}) = \{x \in \mathcal{K} :$ for all candidates $y \neq x,$ either xTy or there exist $x_1, x_2, \ldots, x_\tau$ candidates such that $xTx_1T \ldots Tx_\tau Ty.\}$

Theorem 3 *Suppose strategy s^* is a Markov equilibrium. Then in any stage with remaining candidates Ω, w is the winner in the subgame (with candidates Ω) of any sequential elimination voting in the family $\mathcal{F}$ only if w is a member of the top cycle $TC(\Omega)$.*

Proof. We will use induction according to the number of remaining candidates.

Suppose Ω consists of two candidates. Then the result is trivially true. Now assume the following hypothesis:

Theorem 3 is true for any subgame with remaining candidates Ω of cardinality $\tau - 1$.

We want to show that the result is also true for any subgame with τ remaining candidates. Suppose not. Then there is a subgame with remaining candidates $\tilde{\Omega}$ of cardinality τ such that w wins (i.e., first z_1 is eliminated, followed by successive eliminations of $z_2, \ldots, z_{\tau-1}$ and then w wins) and $w \notin TC(\tilde{\Omega})$. This implies there exists some $y \in \tilde{\Omega}$ such that

$$y\ T\ w \ \ \underline{\text{and}} \text{ it is } \textit{not} \text{ the case that } \ w\ T\ x_1\ T\ x_2\ T\ \ldots T\ x_\ell\ T\ y. \tag{3}$$

Next we establish two intermediate claims.

Claim 1: y must be the eliminated candidate in this stage (i.e., $z_1 = y$).

If not, $z_1 \neq y$ is eliminated. The remaining candidate set is $\tilde{\Omega} \setminus z_1$ and w wins, which implies by hypothesis $w \in TC(\tilde{\Omega} \setminus z_1)$. But then there will be a (direct or an indirect) chain such that $w\ T\ x_1\ T\ x_2\ T\ \ldots T\ x_\ell\ T\ y$, contradicting (3). ||

Next consider any candidate $a \neq y,\ a \in \tilde{\Omega}$ and any subgame Γ_y with the remaining candidates $\tilde{\Omega} \setminus a$ (i.e., any off-equilibrium subgame where y is not eliminated). Denote the winner of this subgame by $\hat{w}$. By hypothesis, $\hat{w} \in TC(\tilde{\Omega} \setminus a)$.

Claim 2: In the subgame Γ_y, it must be that $\hat{w} = y$.

First note that, $y \in \tilde{\Omega} \setminus a$ implies that $w \notin TC(\tilde{\Omega} \setminus a)$, and hence by hypothesis $\hat{w} \neq w$.

Now suppose contrary to Claim 2, $\hat{w} \neq y$. Since $\hat{w} \in TC(\tilde{\Omega} \setminus a)$, it must be that

$$\hat{w}\ T\ \ldots T\ y. \tag{4}$$

Also, since by Claim 1 and hypothesis $w \in TC(\tilde{\Omega} \setminus y)$, and $\hat{w} \neq w$ (as established above), therefore

$$w\ T\ \ldots T\ \hat{w}. \tag{5}$$

Now (4) and (5) together imply

$$w\, T\ \dots T\ \hat{w}\, T\ \dots T\ y,$$

but this contradicts (3). So Claim 2 must be true. $||$

The rest of the proof is the same as in the case of having a Condorcet winner, as follows. In the subgame with remaining candidates $\tilde{\Omega}$, consider any voter i such that $y \succ_i w$; there will be a majority of such voters because $y\ T\ w$. Denote these majority voters (and voter i is one of them) by $\mu_1, \mu_2, ..., \mu_{n^*}$. Denote i's proposed equilibrium vote at this stage (with candidates $\tilde{\Omega}$) by $a_i \neq a_i^y$ where $a_i, a_i^y \in A_i(\tilde{\Omega}, j-1)$ and $e(a_{\mu_1}^y, a_{\mu_2}^y, .., a_{\mu_{n^*}}^y) \neq y$, with the last condition guaranteed by the non-elimination property of $\mathcal{F}$. But then for such i voting for a_i^y weakly dominates voting for any other a_i: either y is eliminated in which case w wins (by Claim 1); if y is not eliminated then y will win (by Claim 2). But then the majority voters, $\mu_1, \mu_2, ..., \mu_{n^*}$, would vote for non-elimination of y, following which y becomes the winner (by Claim 2). Thus, $w \notin TC(\tilde{\Omega})$ cannot be the winner. **Q.E.D.**

The top cycle equilibrium property of our sequential elimination voting is a nice feature. In a related context of binary voting games, McKelvey and Niemi (1978) had shown a similar result. However, the class of binary voting games is neither a subset nor a superset of our sequential elimination voting games, thus the result in Theorem 3 should be of interest. While the top cycle set is a familiar bound characterizing voting equilibria in some contexts (see sect. 3 of Dutta et al. (2002) on this), it is not the most desirable bound, however, as it can admit Pareto inefficient outcome. So we further ask the question of Pareto efficiency with respect to our family of sequential elimination voting. As can be expected, given that the class of sequential elimination voting is quite large, Pareto efficiency fails as we show in an example in the Appendix for the weakest link voting.[20] On the other hand, Pareto efficiency may hold for other voting rules within the sequential elimination family; for instance, sequential binary voting is known to be Pareto efficient. We could establish only a limited result on Pareto efficiency, summarized in the following proposition.

Proposition 1 *For three candidates and an odd but arbitrary number of voters (≥ 3), any sequential elimination voting in the family $\mathcal{F}$ is Pareto efficient.*

[20]Obviously, for such a result to occur it must be that the voter preferences do not admit a Condorcet winner; Condorcet winner, when it exists, *is* Pareto efficient.

Characterization of Pareto efficient voting rules more generally under strategic voting is an interesting and difficult issue that we hope will be addressed in future research. Dutta and Pattanaik (1985) show Pareto efficiency for group decision rules based on pairwise comparisons. Moulin (1988) shows Pareto optimality for a special type of sequential binary voting using majority comparison (called the multistage elimination tree), but under the assumption of sincere voting (see sect. 4, of ch. 9).

4 Single-round and some other voting mechanisms

In this section we look at voting rules that differ from the sequential family $\mathcal{F}$ in two important respects: either (1) the elimination of candidates is *not* one-by-one, or (2) the elimination which may even be sequential is through a *single ballot*, or both. This complementary class[21] includes all single-round voting, a plurality runoff rule, the exhaustive ballot method, and instant runoff voting (also known as the alternative vote method).

Let us start by examining the Condorcet consistency property (or the lack of it) for a class of single-round voting rules where the voters simultaneously submit a strict ranking of candidates and the voting rule elects one candidate as the clear winner (that is, only a deterministic tie-breaker is invoked, if at all). While submission of strict-order ranking would rule out *approval voting*,[22] included are voting rules that ask for submission of candidates of a particular rank (such as plurality rule, negative voting etc.): the mapping from submitted preference profiles to a winner may well ignore part of the information.

Definition 1 *A voting rule, v, is* <u>*responsive*</u> *if it satisfies the following two conditions:*

1. *Consider any three-candidates voting problem and any pair of candidates x, y. For any preference submission strategy R^i by any voter i such that $x \succ y$, there exists a profile of preference submission strategies $R^{N\setminus\{i\}}$ by the remaining voters such that combining R^i and $R^{N\setminus\{i\}}$ elects x as the winner, and*

[21] It is also worth noting that, this complementary class includes any sequential elimination voting that fails the *non-elimination* property of $\mathcal{F}$.

[22] Approval voting asks voters to partition candidates into 1's (i.e., the candidates one approves) and 0's (the candidates one disapproves).

combining $\hat{R}^i$ and $R^{N\setminus\{i\}}$ elects y as the winner where $\hat{R}^i$ is any submission by voter i such that $y \succ x$.

2. *Consider any three-candidates voting problem. For any voter i and any pair of candidates x, y, there exists a profile of preference submission strategies $R^{N\setminus\{i\}}$ by the remaining voters such that if i places x <u>at the top</u> and y <u>at the bottom</u> candidate x is elected, and if i places x <u>second</u> and y <u>at the bottom</u> candidate y is elected.*

Definition 2 *A voting rule v is <u>scale invariant</u> if replicating the set of voters with their submitted preferences by any multiple will not alter the winner.*

The meaning of scale invariance should be quite clear. Responsiveness means that for any pair of candidates any voter can become pivotal in determining the winner among the two candidates for some strategy profile of the remaining voters.

Theorem 4 *Suppose the number of voters, n, is odd. For any single-round voting rule, v, with voters submitting a full, strict-order ranking, if v satisfies responsiveness and scale invariance then v is <u>not</u> Condorcet consistent.*

In the Appendix we verify that plurality rule, negative voting, Borda rule, Copeland rule and Simpson rule will all satisfy responsiveness and scale invariance conditions, thus coming under the scope of Theorem 4. It is also worth pointing out that under sincere voting both Copeland and Simpson rules *are* Condorcet consistent (Moulin, 1988, ch. 9). Thus, strategic considerations could make things worse for voters as a whole.

Proof of Theorem 4. We need to show that for some preference profile admitting a Condorcet winner, there exists a Nash equilibrium with weakly undominated strategies such that a non-Condorcet winner is elected. We will prove the assertion for the case of only three candidates $A = \{a, b, c\}$.[23]

We start by showing that reporting one's true preference is never a weakly dominated strategy. Without any loss of generality assume that agent i has the

[23] The restriction to three candidates in this proof can be easily relaxed by assuming an intuitive property, as follows. *A voting rule satisfies <u>weak IIA</u> if the following hold: For every three-candidates problem $A = \{a, b, c\}$ with preference orderings R^N and any k candidates preference orderings $\tilde{R}^N$ with $\tilde{R}^N|_{\{a,b,c\}} = R^N$ and such that any candidate $x \neq a, b, c$ is ranked below a, b, c in $\tilde{R}^N$, the same candidate is elected for both $\tilde{R}^N$ and R^N.*

preference relation R which is $a \succ b \succ c$. Suppose by way of contradiction that there is a nontruthful submission strategy R' that weakly dominates the submission of R. First consider the top position in R' and suppose b is placed at the top. Applying condition 1 in Definition 1 with respect to a and b, conclude that b cannot be at the top of R'. Similarly c cannot be at the top of R'. Therefore a must be placed at the top of R'. Next consider the second position in R' and suppose c is placed second, so that b is placed at the bottom of R'. Now apply condition 1 again with respect to b and c to conclude that c cannot be in the middle. Therefore the second position in R' is occupied by b, hence $R = R'$, which is a contradiction.

Consider now a voting rule which is *not* Condorcet consistent with respect to sincere voting and let R^N be the preference profile in which Condorcet consistency is violated. By the arguments above each voter submitting his true preference is not using a dominated strategy. Consider now a sufficiently large replica of the voting game with everybody submitting the true preferences (so that the *scale invariance* of Definition 2 applies) and such that unilateral deviation does not alter the outcome. Then the corresponding strategy combination is a Nash equilibrium with undominated strategies yielding a candidate which is not a Condorcet winner.

Consider next voting rules which are Condorcet consistent in sincere voting, i.e., if a Condorcet winner exists with respect to the reported preferences then that candidate is elected. Consider a true preference profile R^N for which candidate a is a Condorcet winner. Since the number of voters is odd, we can assume without loss of generality that there exists a majority S preferring b to c. Consider a representative voter i from the set S and suppose w is ranked bottom according to his true preference relation. Clearly $w \neq b$. We assert that, for i the strategy of submitting b at the top and w at the bottom is not weakly dominated: first, note that by condition 2 in Definition 1 there is a joint strategy for $N \setminus \{i\}$ such that b is elected if i submits b at the top and w at the bottom, and w is elected if i submits b in the middle and w at the bottom; then, in comparing i's strategy of placing b top, w bottom with any of his remaining strategies, condition 1 guarantees that there is always some strategy profile for $N \setminus \{i\}$ such that the former yields i a strictly higher payoff. Consider now any strategy combination in which every member of S puts b at the top and his least-preferred candidate at the bottom (which is just shown to be undominated), and the rest submit any undominated strategy. Since there is a majority submitting b at the top, b must be a Condorcet winner with respect to the submitted preferences. Since the voting system is Condorcet consistent with respect to sincere voting, b must be elected. Consider now a sufficiently large replica of the voting games with the submitted strategies (so that the *scale invariance* of Definition 2 applies) for which unilateral deviation does not alter the outcome. Then the corresponding strategy combination is a Nash equilibrium with undominated strategies, yielding the candidate b. But a is the Condorcet winner with respect to true preferences. **Q.E.D.**

Another single-round voting rule that might be of interest is *approval voting* (Brahms and Fishburn, 1978; Myerson 2002): each voter partitions the candidates into "equally good" and "equally bad" ones by giving candidates in the first category 1's and the second category 0's, and the candidate with a maximal number of votes is elected. Because approval voting does not allow strict-order submissions, Theorem 4 does not apply.

Proposition 2 *Approval voting is not Condorcet consistent.*

So far in this section we have considered only single-round voting for which the candidates (except the winner) are all eliminated simultaneously. Next we consider voting rules that do not belong to either the class of single-round voting considered above or the sequential (elimination) family of section 3. Obviously one can think of many voting rules that come under a third complementary group. We are not going to make any general observation here. Instead, we consider three specific voting rules from this category – *plurality runoff rule*,[24] *exhaustive ballot method*, and *instant runoff voting* – which are not entirely uncommon and demonstrate key characteristics not exhibited by the first two voting families; see section 1 for descriptions of these voting rules. We use these mainly to indicate why both *one-by-one elimination* and *repeated ballots* are potentially important for Condorcet consistency. Both plurality runoff rule and exhaustive ballot method share features of weakest link voting but fail one-by-one elimination requirement;[25] instant runoff voting with elimination procedure roughly similar to weakest link voting (except for a majority vote trigger) fails both requirements of one-by-one elimination and repeated ballots; and all three voting rules fail Condorcet consistency as the following propositions show.

[24] This voting rule is also known under alternative names such as two-ballot, double ballot, second ballot, majority runoff, and two round system.

[25] In fact, the plurality runoff rule is even closer to 'sequential runoff election'. The latter is a full drawn-out version of the former; see section 1.

Proposition 3 *The plurality runoff rule is not Condorcet consistent.*

Proposition 4 *The exhaustive ballot method is not Condorcet consistent.*

Proposition 5 *The instant runoff rule is not Condorcet consistent.*

5 Concluding remarks

One potential downside of sequential elimination procedures is that they take longer to run and that they are organizationally more elaborate. Interestingly we can avoid the first downside by calling voters to step in only one time in the sequential elimination procedure at which they submit a full strategy of the sequential game (instead of submitting a vote or a ranking). Let us go back to our example in section 2 with its weakest link rule. Instead of calling agents stage-by-stage to submit their votes, we can ask them to submit a strategy for the sequential game of the weakest link at the outset. Formally such a strategy is a mapping f that maps for each subset of candidates $S \subset \{x, y, w, z\}$, a single candidate $f(S)$ in S. Given a strategy profile by all voters we can implement the weakest link rule as if voters stepped in stage-by-stage. The game now becomes a normal form game, which is strategically equivalent to the sequential one, and the Condorcet winner is guaranteed to be the outcome in an equilibrium (which is now Nash with undominated strategies). What is then the difference between the sequential version of the weakest link and its one-stage version (in which voters submit a ranking one time at the outset). Evidently the only difference is with the set of available strategies in the normal form game. Note that every ranking strategy can be embedded in a mapping strategy f of the kind we described above but not vice-versa. There are mapping strategies which do not correspond to a ranking. Hence the sequential elimination version allows voters a larger set of strategies. We shall say that a strategy mapping f satisfies the *IIA* (independent of irrelevant alternatives) property if it satisfies the following: for every two sets of candidates $S, T \subset \{x, y, w, z\}$ such that $T \subset S$ and $f(S) \in T$ we have $f(S) = f(T)$. It can be easily verified that a strategy mapping satisfies the *IIA* property if and only if it is a ranking strategy. Hence the difference between the sequential version of the weakest link voting and the one-stage version can be viewed as the difference between allowing voters to submit any strategy mapping and restricting them to submit only ones that satisfy *IIA*. As it turns out such a restriction can be quite binding from a strategic point of view and can result in less desirable voting outcomes.

Our interest in Condorcet consistent voting rules is borne out of the fact that voting mechanisms seem to be the most natural and decentralized way of reaching collective decisions. Substantial research in the important literature of implementation theory investigate questions of how to achieve desirable social objectives. Pareto efficiency, while a very weak requirement in many environments, is still one of the most noncontroversial criteria by which any social decision procedure can be judged. So ensuring Condorcet consistency, a necessary part of the broader objective of achieving Pareto efficiency, should be an important task of any collective decision process.

Appendix

Justifying the use of Markov strategies.

Recall, S_i is the strategy set of voter i with $s_i : \mathcal{H} \to \mathcal{K}$ s.t. $s_i(h) \in C \ \ \forall h \in \mathcal{H}_C$. Also, let $S = \Pi_i S_i$.

Definition 3 *A strategy $s_i \in S_i$ is more complex than another strategy $s'_i \in S_i$ if $\exists C$ s.t.*

(*i*) $s_i(h) = s'_i(h) \ \ \forall h \notin \mathcal{H}_C$;

(*ii*) $s'_i(h) = s'_i(h') \ \ \forall h, h' \in \mathcal{H}_C$;

(*iii*) $s_i(h) \neq s_i(h')$ *for some* $h, h' \in \mathcal{H}_C$.

The above ordering of complexity is only a partial ordering. Nevertheless, it will prove a powerful one for our purpose. Based on this ordering let us introduce an equilibrium definition, which is a further refinement of our earlier definition of *equilibrium solution.*

Definition 4 *A strategy profile $s \in S$ will be called a simple equilibrium of the weakest link voting game, if*

(*i*) $s \in S^*(h_0)$;

(*ii*) $\nexists i \in \mathcal{N}$ *s.t.* $\exists s'_i \in S_i$ *s.t.* $\pi_i(s'_i, s_{-i}) = \pi_i(s_i, s_{-i})$,
and s_i *is more complex than* s'_i, *where* $\pi_i(\cdot, \cdot)$ *is the payoff function of voter* i.

Note that while the definition of *simple* equilibrium allows history-dependent (i.e., non-Markov) strategies,[26] the second condition reflects the implicit assumption that the voters are averse to complexity (as in Definition 4) *unless* it helps to increase their payoffs. Thus, simplicity of the simple equilibrium is a very weak, and in our view plausible, requirement for any descriptive analysis. We can therefore use the simplicity criterion for equilibrium selection.

Theorem 5 *Any simple equilibrium is also a Markov equilibrium.*

Proof. Suppose $s \in S$ is a simple equilibrium but not a Markov equilibrium. Then there exists some i, C and $h, h' \in \mathcal{H}_C$ s.t. $s_i(h) \neq s_i(h')$. Clearly, if $\mathcal{H}_C \cap E \neq \emptyset$ where E is the equilibrium path corresponding to the simple equilibrium s, then $\mathcal{H}_C \cap E$ is *unique*; that is, C happens on the equilibrium path at most once. Now consider another strategy $s_i' \in S_i$ s.t.

$$\begin{array}{rcl} s_i'(h) & = & s_i(h) \quad \forall h \notin \mathcal{H}_C; \\ \forall h \in \mathcal{H}_C, \quad s_i'(h) & = & \begin{cases} s_i(\mathcal{H}_C \cap E) & \text{if } \mathcal{H}_C \cap E \neq \emptyset, \\ a \in C & \text{if } \mathcal{H}_C \cap E = \emptyset, \end{cases} \end{array}$$

where a denotes any arbitrary element.

It is easy to see that s_i' is simpler than s_i. Moreover, because s_i' differs from s_i only for histories in $\mathcal{H}_C$ that are off-the-equilibrium path, (s_i', s_{-i}) will result in the same winner as the equilibrium s, so that $\pi_i(s_i', s_{-i}) = \pi_i(s_i, s_{-i})$. Hence, s cannot be a simple equilibrium – a contradiction. **Q.E.D.**

Proof of Proposition 1. Consider three candidates $C = \{x, y, z\}$, and an odd (but otherwise) arbitrary number of voters. Suppose x is the winner of any sequential elimination voting satisfying the non-elimination property and x is Pareto dominated by z. Given strict preference ordering by voters, it must be that $z \succ_i x$ for all i. But because x must be in the top cycle (by Theorem 3), either $x\ T\ z$ or $x\ T\ y\ T\ z$. But the first one is not possible. Therefore it must be that $x\ T\ y$, combining which with the fact that $z \succ_i x$ for all i and the number of voters is odd implies that $z\ T\ y$, contradicting $x\ T\ y\ T\ z$. **Q.E.D.**

[26] $S^*(h_0)$ does not employ the Markov assumption.

Proof of Proposition 2. Consider a 3 voters, 3 candidates scenario, where the voters' ranking of the candidates are as follows:

$$\begin{aligned} 1: &\quad z_1, z_2, z_3 \\ 2: &\quad z_3, z_1, z_2 \\ 3: &\quad z_2, z_1, z_3. \end{aligned}$$

z_1 is the Condorcet winner. Suppose, as before, the tie-breaker is: z_2, z_3, z_1. The proposed equilibrium strategies under approval voting are as follows:

$$\begin{aligned} 1. &\quad 1,1,0 \\ 2. &\quad 0,0,1 \\ 3. &\quad 0,1,0, \end{aligned}$$

where the vote points are respectively for z_1, z_2 and z_3. Denote this strategy profile by s^*.

s^* will pick z_2. It is obvious that s^* is a Nash equilibrium. Next we verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy in s^*.

Each voter can choose one of the following 8 strategies:

$$\begin{array}{lll} 1,0,0; & 0,1,0; & 0,0,1; \\ 1,1,0; & 0,1,1; & 1,0,1; \\ 1,1,1; & 0,0,0. & \end{array}$$

First consider the strategy of voter 1. It is easy to see that

$$\begin{array}{lllll} 1. & 1,0,0 & & 1,1,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 0,1,1 & & 0,1,1 \end{array} \qquad \begin{array}{lllll} 1. & 0,1,0 & & 1,1,0 \\ 2. & 1,0,0 & \prec & 1,0,0 \\ 3. & 0,0,1 & & 0,0,1 \end{array}$$

$$\begin{array}{lllll} 1. & 0,0,1 & & 1,1,0 \\ 2. & 0,1,1 & \prec & 0,1,1 \\ 3. & 0,0,1 & & 0,0,1 \end{array} \qquad \begin{array}{lllll} 1. & 0,1,1 & & 1,1,0 \\ 2. & 1,0,0 & \prec & 1,0,0 \\ 3. & 0,0,1 & & 0,0,1 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,1 & & 1,1,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 1,0,0 & & 1,0,0 \end{array} \qquad \begin{array}{lllll} 1. & 1,1,1 & & 1,1,0 \\ 2. & 0,1,1 & \prec & 0,1,0 \\ 3. & 0,0,1 & & 0,0,1 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,0 & & 1,1,0 \\ 2. & 1,0,0 & \prec & 1,0,0 \\ 3. & 0,0,1 & & 0,0,1. \end{array}$$

For voter 2,

$$\begin{array}{lllll} 1. & 1,0,1 & & 1,0,1 \\ 2. & 1,0,0 & \prec & 0,0,1 \\ 3. & 1,0,0 & & 1,0,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,1 & & 1,0,1 \\ 2. & 0,1,0 & \prec & 0,0,1 \\ 3. & 0,1,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,1 & & 1,0,1 \\ 2. & 1,1,0 & \prec & 0,0,1 \\ 3. & 0,1,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 0,0,1 & & 0,0,1 \\ 2. & 0,1,1 & \prec & 0,0,1 \\ 3. & 0,1,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,1 & & 1,0,1 \\ 2. & 0,1,0 & \prec & 0,0,1 \\ 3. & 0,1,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 0,0,0 & & 0,0,0 \\ 2. & 1,1,1 & \prec & 0,0,1 \\ 3. & 0,1,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,1 & & 1,0,1 \\ 2. & 0,0,0 & \prec & 0,0,1 \\ 3. & 0,1,0 & & 0,1,0. \end{array}$$

Finally for voter 3,

$$\begin{array}{lllll} 1. & 1,1,0 & & 1,1,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 1,0,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,1,0 & & 1,1,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 0,0,1 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,0 & & 1,0,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 1,1,0 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,0 & & 1,0,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 0,1,1 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,1,0 & & 1,1,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 1,0,1 & & 0,1,0 \end{array}$$

$$\begin{array}{lllll} 1. & 1,0,0 & & 1,0,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 1,1,1 & & 0,1,0 \end{array}$$

$$\begin{array}{llcl} 1. & 1,0,0 & & 1,0,0 \\ 2. & 0,0,1 & \prec & 0,0,1 \\ 3. & 0,0,0 & & 0,1,0. \end{array}$$

Thus, s^* is an equilibrium. Hence, approval voting is not Condorcet consistent. **Q.E.D.**

Proof of Proposition 3. Consider a 3 voters, 4 candidates scenario, where the voters' ranking of the candidates are as follows:

$$\begin{array}{ll} 1: & z_1, z_4, z_2, z_3 \\ 2: & z_1, z_2, z_3, z_4 \\ 3: & z_3, z_4, z_2, z_1. \end{array}$$

z_1 is the Condorcet winner. Also, $z_2 \, T \, z_3 \, T \, z_4 \, T \, z_2$.

Consider the tie-breaker: z_2, z_3, z_4, z_1. Under plurality-runoff rule, an equilibrium strategy profile is

$$(z_2, z_3, z_4)$$

in stage 1, followed by sincere voting in stage 2. In stage 1, z_1 and z_4 are eliminated, so that z_2 is picked as the ultimate winner.

Given that sincere voting in stage 2 constitute a Nash equilibrium that is also weakly undominated, we only need to check that the proposed stage 1 strategies will be Nash equilibrium and weakly undominated. Checking for Nash equilibrium is straightforward. So we will only verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy.

The votes by voters 1 and 2 are the unique best responses, thus also weakly undominated. So let us consider voter 3's strategy. Let voters 1 and 2 choose in stage 1 respectively z_1 and z_3. If voter 3 chooses z_4 the outcome is z_3; on the other hand, if voter 3 chooses z_1 or z_3 the outcome is z_1, and if he chooses z_2 the outcome is z_2, and both are worse compared to z_3.

Thus, plurality-runoff rule is not Condorcet consistent. **Q.E.D.**

Proof of Proposition 4. There are three types of voters, A, B and C, with 3 voters of each type. There are three alternatives with the following preferences:

$$A: \quad x, y, z$$

B : y, z, x

C : x, z, y.

The Condorcet winner is x.

Consider the following strategy profile: In round 1, type A voters vote for x and types B and C vote for z. In round 2, if reached, each type vote for the alternative (from the remaining two) which he prefers most.

The above strategy induces z as the winner. We claim that this strategy will be an equilibrium. That round 2 voting satisfies the equilibrium conditions is trivial. So consider round 1 voting. First, note that no player can gain by deviating unilaterally in this round (this is because each type has three voters). It thus remains to argue that no weakly dominated strategies are used in round 1. As type-A voters vote for their top-ranked candidate, clearly the strategy is undominated. So we need to argue that voter types B and C are not using weakly dominated strategies in round 1. First consider type-C voters. Let the strategy combination in round 1 be as follows: all A-type vote for y, all B-type vote for z, two C-type vote for z and one C-type votes for x. This leads to z being elected. If on the other hand one of the two C-type voters who voted for z now switches to either y or x, then x will be eliminated and y is the ultimate outcome, which is worse for a type-C voter. Consider now type-B voters. Let the strategy combination in round 1 be follows: all A-type vote for x, all C-type vote for z, one of B-type votes for x and the other two vote for z. For this profile the outcome is z. If, however, one of the voters who earlier voted for z now switches to either y or x, then y will be eliminated and x is the ultimate outcome, which is worse for a type-B voter. **Q.E.D.**

Proof of Proposition 5. Consider a replicated version of the preferences in section 2 example involving three types of voters A, B and C, with 3 voters of each type:

A : y, x, z, w

B : z, x, y, w

C : w, x, z, y.

The Condorcet winner is x.

First we claim that in equilibrium (assuming an equilibrium exists[27]) each voter will place his top-ranked candidate ahead of others. Without loss of generality

[27] If there is no equilibrium then our Proposition trivially holds.

assume that this voter's (whom we denote voter i) true preference is: y, x, z, w. Note that x cannot be placed top in R_i. Otherwise, consider the remaining 8 voters and assume that four of them submit y, x, z, w and four submit x, z, y, w. Then clearly placing y at the top (which will lead to y) is better than placing x at the top (which leads to x). Using the same argument, conclude that neither z nor w can be placed top in R_i. Hence it must be that y is placed top in R_i.

Given that each voter will place his top-ranked candidate ahead of others, x will be eliminated in the first count. Thus, the instant runoff rule will not be Condorcet consistent. **Q.E.D.**

Theorem 6 *Assume $N \geq 2K - 1$. Then in the weakest link game there exists s^* which is SPE and not weakly dominated.*

Proof. To prove existence we need to show that there exists a strategy profile s^* such that at each stage it is Nash and undominated assuming that all players play according to s^* in any later stages.

Denote v to be a generic voter. Define s^* inductively in subgames with a given number of candidates as the inductive variable, as follows.

Let, $\forall h$ s.t. $k(h) = 2$ ($k(h)$ denotes the number of voters at h), voter v choose *sincere voting* as his strategy, $s_v^*(h)$. Clearly such a strategy profile is an undominated Nash equilibrium in this last stage.

Now suppose for all h' such that $k(h') \leq J - 1$, $s^*(h')$ is defined.

We need to define a profile of choices for all voters $s^*(h)$ $\forall h$ s.t. $k(h) = J$ such that $s^*(h)$ is an undominated Nash equilibrium assuming that all follow $s^*(h')$ $\forall k(h') \leq J - 1$.

Fix any h s.t. $k(h) = J$. Let $C = \{c^1, ..., c^J\}$ be the set of candidates at h. Without any loss of generality assume that c^i is higher in the tie-breaking rule than c^j if and only if $i < j$.

Also let $\sigma(c)$ be the winner if c is eliminated at the start of play of the subgame $\Gamma(h)$,[28] and $M_v = \arg\max_{c \in C} \pi_v(\sigma(c))$, where $\pi_v(\cdot)$ is voter v's payoff resulting from the winning candidate. Note that M_v need not be unique. Finally, let $M_v^c = C \backslash M_v$. First note the following result.

Lemma 1 *In the subgame $\Gamma(h)$, any $c \in M_v^c$ is not weakly dominated for v.*

[28] $\sigma(c)$ is unique by the assumption that the voters use Markov strategies.

Proof of Lemma 1. Fix $c \notin M_v$ and any $c' \neq c$; c' may or may not belong to M_v. We want to argue that switching his vote from c to c' would be worse for voter v for at least one profile of other voters' votes.

If the tie-breaker places *some* $\hat{c} \in M_v$ ahead of c and $\hat{c} \neq c'$, let the distribution of votes be as follows:

voter v: c;
others: $w(\hat{c}) = 0$,
$w(c) = 0$,
and $w(\tilde{c}) > 0, \ \forall \tilde{c} \neq \hat{c}, c$,

where $w(\cdot)$ denotes the number of votes in favor of a candidate. Distribution of votes as above leads to the elimination of $\hat{c}$. However, if v switches to c' while the rest stay with their votes as above, candidates $\hat{c}$ and c will be tied with minimal votes and by the tie-breaker c will be eliminated, which is worse for voter v. If $\hat{c} = c'$, the argument holds with even greater force as c would be eliminated (as v switches to c') without having to invoke the tie-breaker.

If the tie-breaker is such that c is placed ahead of *all* $\hat{c} \in M_v$, let the distribution of votes be as follows:

voter v: c;
others: $w(\check{c}) = 1$ for some $\check{c} \in M_v$,
$w(c) = 0$,
and $w(\tilde{c}) > 1, \ \forall \tilde{c} \neq \check{c}, c$,

which leads to the elimination of $\check{c}$. However, if v switches to c' while the rest stay with their votes as above, c will be unique with minimal votes and therefore be eliminated, which is worse for voter v. This completes the proof of Lemma 1. ||

Next for any $k = 1, .., J$ we define the following property.

Definition 5 *Any $k \in \{2, .., J\}$ satisfies property α if there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, .., u_{k-1}, v_{k-1})$ consisting of $2(k-1)$ different voters such that*

$$c^j \in M_v^c \ \text{ for } \ v = u_j, v_j \text{ for all } \ j < k. \tag{6}$$

Also, to simplify the exposition we assume that $k = 1$ always satisfies property α.

Lemma 2 *Consider any $k \leq J$. Suppose that k satisfies property α and $k+1$ does not satisfy property α. Then there exist a choice profile $s^*(h)$ that is Nash and is not weakly dominated.*

Proof of Lemma 2. Given that k satisfies property α, there exists a set of voters Ω (that is empty if $k = 1$) consisting of $2(k-1)$ different voters $(u_1, v_1, u_2, v_2, .., u_{k-1}, v_{k-1}) \subset \mathcal{N}$ such that

$$c^j \in M_v^c \text{ for } v = u_j, v_j \text{ for all } j < k \tag{7}$$

and there exists a set of voters $V \subset \mathcal{N} \backslash \Omega$ such that

$$\mid V \mid = N - 2(k-1) - 1 \tag{8}$$

and

$$c^k \in M_v \text{ for any } v \in V. \tag{9}$$

Let

$$C^k = \left\{ c \in C \,\middle|\, \sigma(c) = \sigma(c^k) \right\} \quad \text{and} \quad \overline{C}^k = \{C \backslash C^k\} \cap \{c^{k+1}, .., c^J\}.$$

Since the preferences of each agent is strict it follows that

$$\overline{C}^k \subset M_v^c \text{ for any } v \in V. \tag{10}$$

Also since $\mid V \mid = N - 2(k-1) - 1$, $\mid \overline{C}^k \mid \leq J - k$ and by assumption $N \geq 2K - 1 \geq 2J - 1$ and $k \leq J$, it follows that the number of voters in V is at least twice the number of candidates in $\overline{C}^k$. But this implies that there exists a choice profile $\{s_v^*(h)\}_{v \in V}$ such that

$$s_v^*(h) \in \overline{C}^k \text{ for each } v \in V, \tag{11}$$

$$|\{v \in V \,| s_v^*(h) = c\}| \geq 2 \text{ for each } c \in \overline{C}^k. \tag{12}$$

(The second condition says that each candidate $c \in \overline{C}^k$ receive at least two votes). Next set the choice $s_v^*(h)$ of each $v \in \Omega$ to be such that

$$s_v^*(h) = c^j \text{ for } v = u_j, v_j. \tag{13}$$

Finally, denote the remaining voter $\mathcal{N} \backslash \{V \cup \Omega\}$ by x and set the choice of voter x to be such that

$$\begin{array}{ll} s_x^*(h) \in M_x^c \backslash c^k & \text{if } M_x^c \backslash c^k \text{ is not empty} \\ s_x^*(h) \in c^k & \text{otherwise.} \end{array} \tag{14}$$

Now by Lemma 1 and conditions (7), (10), (11), (13) and (14) the choice $s_v^*(h)$ is not dominated in this round for any voter v. Next we show that $s^*(h) = \{s_v^*(h)\}_{v \in \mathcal{N}}$ is Nash. There are two possible cases.

Case A. $M_x^c \neq c^k$**.** First, note that by (12) and (13), in this round each candidate $c \in \overline{C}^k \cup \{c^1, .., c^{k-1}\}$ receives at least two votes, c^k receives zero, and any other $c' \in C^k \cap \{c^{k+1}, \ldots, c^J\}$ receives at most one vote. This means that some candidate $c \in C^k$ is eliminated and $\sigma(c^k)$ will be the final winner. Moreover, since c^k receives zero vote it must be that the eliminated candidate $c^e \in C^k$ receives zero vote and $e \geq k$.

Since, by (9), $\sigma(c^k)$ is a best outcome for each $v \in V$ it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Moreover, each voter $v \in \Omega$ cannot change the final outcome $\sigma(c^k)$ by changing its action because the choice $s_v^*(h) \in \{c^1, .., c^{k-1}\}$ receives at least two votes, the eliminated candidate c^e has zero vote and $e \geq k$. Finally, voter x cannot change the final outcome $\sigma(c^k)$ by changing its action because either the choice $s_x^*(h) \in \{c^1, .., c^{k-1}\} \cup \overline{C}^k$, in which case $s_x^*(h)$ receives at least three votes and as before c^e has zero vote, or $s_v^*(h) \in C^k \cap \{c^{k+1}, \ldots, c^J\}$ in which case $s_x^*(h)$ receives one vote and any deviation results in some candidate in the set C^k to be eliminated.

Case B. $M_x^c = c^k$**.** Then for each $c' \neq c^k, c' \in M_x$. Therefore

$$\forall c', c'' \neq c^k, \;\; \sigma(c') = \sigma(c''). \tag{15}$$

This implies that $\overline{C}^k = \{c^{k+1}, .., c^J\}$. This together with (12) and (13) imply that in this round each candidate $c \neq c^k$ receives at least two votes, c^k receives one vote (the vote of x), c^k is eliminated and $\sigma(c^k)$ will be the final winner. As in the previous case, since this is a best outcome for each $v \in V$ it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Next note that for each voter $v = u_j, v_j$ for $j < k$ we have $s_v^*(h) = c^j \in M_v^c$ and thus $c^k \in M_v$. Therefore, eliminating c^k is also the best outcome for any $v \in \Omega$. Finally, note that voter x cannot change the final outcome $\sigma(c^k)$ by changing its action because every $c \neq c^k$ receives two votes. $||$

Lemma 3 *Suppose that J satisfies property α. Then there exist a choice profile $s^*(h)$ that is Nash and is not weakly dominated.*

Proof of Lemma 3. Given that J satisfies property α, there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, .., u_{J-1}, v_{J-1})$ consisting of $2(J-1)$ different voters such that

$$c^j \in M_v^c \;\text{ for }\; v = u_j, v_j \text{ for all } \; j \leq J. \tag{16}$$

Set the choice profile $\{s_v^*(h)\}_{v\in\Omega}$ to be such that

$$s_v^*(h) = c^j \text{ if } v = u_j, v_j. \tag{17}$$

Also partition the remaining agents as follows:

$$\begin{aligned} \Gamma^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c = c^J\right\} \\ \overline{\Gamma}^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c \neq c^J\right\}. \end{aligned}$$

Let the choice profile $\{s_v^*(h)\}_{v\in\mathcal{N}\backslash\Omega}$ to be such that

$$s_v^*(h) \in \begin{cases} c^J & \text{if } \ v \in \Gamma^J \\ M_v^c\backslash c^J & \text{if } \ v \in \overline{\Gamma}^J \end{cases}$$

and

$$\forall c, c' \neq c^J \ \text{ s.t. } \ \sigma(c) = \sigma(c'), |n(c) - n(c')| \leq 1 \tag{18}$$

where

$$n(c) = |\{v \in V\,| s_v^*(h) = c\}| \quad \text{for any } \ c.$$

(Since (18) is used only when Γ^J is non-empty, the strategy profile is written so that (18) is required to be satisfied only if Γ^J is non-empty. See below the explanation following (20) why (18) is feasible.) By Lemma 1 $s^*(h)$ is not weakly dominated. Next we show that it is a Nash equilibrium.

Case A. Γ^J is empty. Then every $c \neq c^J$ receives at least two votes, c^J receives no votes and is eliminated. This together with c^J having the lowest rank in the tie-breaking rule imply that no player can change the final outcome by changing their choices and thus $s^*(h)$ constitutes a a Nash equilibrium.

Case B. Γ^J is non-empty. Then

$$\forall c', c'' \neq c^J, \ \ \sigma(c') = \sigma(c'') \neq \sigma(c^J). \tag{19}$$

Then since for each $v \notin \Gamma^J$ there exists a $c \neq c^J$ such that $c \in M_v^c$, it follows that

$$\forall v \notin \Gamma^J, \ \ c^J \in M_v. \tag{20}$$

(Note that if Γ^J is non-empty, (18) is possible because of the following reasons. Each c^j, $j < J$ receives two votes from set of voters Ω. The only other voters that vote for the candidates c^j, $j < J$ is the set $\overline{\Gamma}^J$. Because Γ^J is non-empty it follows from (19) that for each $v \in \Gamma^J$, $M_v^c = \{c^1, \ldots, c^{J-1}\}$; therefore votes by the members of $\overline{\Gamma}^J$ can be arranged so that (18) is satisfied: the first member of $\overline{\Gamma}^J$ votes for c^1, the second for c^2 etc. until the $(J-1)$st member votes for c^{J-1}, the J-th member for c^1, $(J+1)$st for c^2 etc.)

Now there are two possibilities.

Subcase 1: Candidate c^J is eliminated. Then, by (20), this is the best outcome for any $v \notin \Gamma^J$ and therefore, each such v is choosing his optimal action. Moreover, each $v \in \Gamma^J$ cannot change the outcome by deviating from $s_v^*(h)$ because $s_v^*(h) = c^J$ and c^J is the candidate that is eliminated.

Subcase 2: Some $c \neq c^J$ is eliminated. Then, by the tie-breaking rule

$$n(c) < n(c^J) \tag{21}$$

Next note that by (19) and the definition of Γ^J, this is the best outcome for any $v \in \Gamma^J$ and therefore, each such v is choosing his optimal action. Next we show that no voter $v \notin \Gamma^J$ can change the outcome by deviating. Suppose not; then some voter $v \notin \Gamma^J$ can deviate from $s_v^*(h) = c^j (\neq c)$ for some $j < J$ and change the final outcome $\sigma(c)$ by voting for another candidate. Since the outcome is changed, by (19), it must be that c^J is eliminated. This implies that

$$n(c) + 1 \geq n(c^J)$$

and

$$n(c^j) - 1 \geq n(c^J)$$

But this together with (19) imply that

$$n(c^j) > n(c) + 1$$

But this contradicts condition (18). Therefore no $v \notin \Gamma^J$ can change the final outcome by deviating. ||

The last two lemmas together establish that there exists a choice profile $s^*(h)$ that is Nash and is not weakly dominated. **Q.E.D.**

An Example of a winner in the weakest link voting that is Pareto dominated.

Background

We have the following from the existence proof.

Let, $\forall h$ s.t. $k(h) = 2$ ($k(h)$ denotes the number of candidates), voter v choose *sincere voting* as his strategy, $s_v^*(h)$. Clearly such a strategy profile is an undominated Nash equilibrium in this last stage.

Now suppose for all h' such that $k(h') \leq J - 1$, $s^*(h')$ is defined.

We need to define a profile of choices for all voters $s^*(h)$ $\forall h$ s.t. $k(h) = J$ such that $s^*(h)$ is an undominated Nash equilibrium assuming that all follow $s^*(h')$ $\forall k(h') \leq J - 1$.

Fix any h s.t. $k(h) = J$. Let $C = \{c^1, ..., c^J\}$ be the set of candidates at h. Without any loss of generality assume that c^i is higher in the tie-breaking rule than c^j if and only if $i < j$.

Also let $\sigma(c)$ be the winner if c is eliminated at the start of play of the subgame $\Gamma(h)$, and $M_v = \arg\max_{c \in C} \pi_v(\sigma(c))$, where $\pi_v(\cdot)$ is voter v's payoff resulting from the winning candidate. Note that M_v need not be unique. Finally, let $M_v^c = C \backslash M_v$.

We have already established the following result in the proof of Theorem 6 that we will also use for the counter-example:

In the subgame $\Gamma(h)$ with set of remaining candidates C, any $c \in M_v^c$ is not weakly dominated for v.

Example:

4 candidates, 9 voters of 3 different types A, B, C with the following preferences:

- A($\times$3): z_1, z_2, z_4, z_3
- B($\times$3): z_3, z_1, z_2, z_4
- C($\times$3): z_4, z_3, z_1, z_2.

The strategies in each subgame:

1. C consists of two candidates: sincere voting.
2. $C = z_1, z_2, z_3$. Here z_3 is the CC and therefore the winner.
3. $C = z_1, z_2, z_4$. Here z_1 is the CC and therefore the winner.

4. $C = z_1, z_3, z_4$. Here As choose z_1, Bs choose z_1 and Cs choose z_4 and z_3 is eliminated and z_1 ends up winning.

   Note that since no agent is pivotal the above choices are best responses. Moreover, in this case $z_1 \in M_A^c = \{z_1, z_2\}, z_1 \in M_B^c = \{z_1, z_3\}$ and $z_4 \in M_C^c = \{z_3, z_4\}$; therefore these choices are not weakly dominated.

5. $C = z_2, z_3, z_4$. Here As choose z_2, Bs choose z_2 and Cs choose z_4 and z_3 is eliminated and z_2 ends up winning.

   Note that since no agent is pivotal the above choices are best responses. Moreover, in this case $z_2 \in M_A^c = \{z_4, z_2\}, z_2 \in M_B^c = \{z_2, z_3\}$ and $z_4 \in M_C^c = \{z_3, z_4\}$; therefore these choices are not weakly dominated.

6. $C = z_1, z_2, z_3, z_4$. Here As choose z_4, Bs choose z_3 and Cs choose z_2 and z_1 is eliminated and z_2 ends up winning.

   Note that since no agent is pivotal the above choices are best responses. Moreover, in this case $z_4 \in M_A^c = \{z_1, z_4\}, z_3 \in M_B^c = \{z_1, z_2, z_3\}$ and $z_2 \in M_C^c = \{z_1, z_2, z_3\}$; therefore these choices are not weakly dominated.

The winner is z_2 and is Pareto dominated by z_1.

Verification of responsiveness and scale-invariance conditions (Theorem 4).

Plurality rule. For plurality rule, what matters is whether a candidate is placed *at the top*. Any other rank in strict-order submissions can be considered equivalent. Let us check condition 1. Consider a pair of candidates a, b and suppose some voter i votes for a. Of the others, let $(n-1)/2$ voters vote for a and the remaining $(n-1)/2$ voters vote for b. This means a will be elected. But if i were to vote for b then b is elected. Thus condition 1 is satisfied.

We check condition 2 with an eye also on exactly how the condition is used in Theorem 4 proof. Fix voter i with his true preference as $a \succ b \succ c$. Also assume that the tie-breaker places b ahead of c. Among the other voters let one voter vote for a, $(n-1)/2 - 1$ voters vote for b, $(n-1)/2$ voters vote for c. Given this, if i

votes for b by placing b at the top then b is elected (by invoking the tie-breaker); if i does not vote for b (by placing b not at the top) then c is elected. This verifies condition 2.

Negative voting. The *negative voting rule*[29] allows the voters to express only their least desired candidate by giving a point, 0, while giving the remaining candidates all 1's. The candidate with the highest total points, i.e. the candidate with the fewest 0's, wins.

For negative voting, the only thing that matters is whether a candidate is placed *at the bottom*. Any other rank in strict-order submissions can be ignored.

To check that negative voting comes under the scope of Theorem 4, we shall check only condition 1; the first part of Theorem 4 proof that applies to voting rules that are not Condorcet consistent under sincere voting can be invoked (negative voting is not Condorcet consistent under sincere voting).[30] So consider a pair of candidates a, b and suppose some voter i votes against b by placing him at the bottom. Let the remaining $(n-1)$ voters place c at the bottom. This means a is elected. But if i were to place a at the bottom then b is elected. Thus condition 1 is satisfied.

Borda rule. Check condition 1 with respect to the pair of candidates a, b. Voter i can place a ahead of b in one of three ways: (1) $a \succ b \succ c$; (2) $a \succ c \succ b$; (3) $c \succ a \succ b$. In each of these three cases, let $(n-1)/2$ voters submit $a \succ b \succ c$ and the remaining $(n-1)/2$ voters submit $b \succ a \succ c$. With i making any of three submissions listed above, candidate a will be elected, whereas i submitting $b \succ a \succ c$ or $b \succ c \succ a$ or $c \succ b \succ a$ would elect b.

To check condition 2, let i submit $b \succ a \succ c$. Let $R^{N \setminus i}$ be such that $(n-1)/2-1$ voters submit $b \succ c \succ a$ and $(n-1)/2+1$ voters submit $c \succ b \succ a$, so that the overall Borda scores are $BS(b) = (5/2)(n-1)+2 = BS(c)$, $BS(a) = n+1$; thus, assuming tie-breaker b, c, a, candidate b is elected. On the other hand, when i submits $a \succ b \succ c$ instead, the Borda scores change to: $BS(b) = (5/2)(n-1) + 1$, $BS(c) = (5/2)(n-1)+2$, $BS(a) = n+2$ so that c is elected, satisfying condition 2.

[29] It is also known as the anti-plurality rule (p. 231, Moulin, 1988). See also Myerson (2002).

[30] Consider a 3 voters, 3 candidates scenario, where the voters' ranking of the candidates are: z_1, z_2, z_3 (voter 1); z_2, z_3, z_1 (voter 2); z_1, z_2, z_3 (voter 3). The Condorcet winner is z_1. However, for the tie-breaker z_2, z_3, z_1, sincere voting would elect z_2.

Copeland rule. (Moulin 1988, ch. 9). First recall how Copeland rule is defined. Compare candidate a with every other candidate x. Score $+1$ if a majority prefers a to x, -1 if a majority prefers x to a, and 0 if it is a tie. Summing up those scores over all $x, x \neq a$, yields the Copeland score of a. A candidate with the highest such score, called a *Copeland winner*, is elected.

Now let us check condition 1 with respect to a, b. Again, voter i can consider placing a ahead of b as follows: (1) $a \succ b \succ c$; (2) $a \succ c \succ b$; (3) $c \succ a \succ b$. In each of these three cases, again let $(n-1)/2$ voters submit $a \succ b \succ c$ and the remaining $(n-1)/2$ voters submit $b \succ a \succ c$. With i making any of three submissions listed above, scores of a against b and a against c will be $+1$ each, so that the Copeland score of a is $+2$, electing a as the winner (Copeland scores of b and c will be respectively 0 and -2). Alternatively, if i submits $b \succ a \succ c$ or $b \succ c \succ a$ or $c \succ b \succ a$, Copeland score of b will be $+2$, electing b as the winner (Copeland scores of a and c will be respectively 0 and -2).

We now verify condition 2 for the candidates pair a, c with c always placed at the bottom. Let$(n-1)/2$ voters submit $a \succ c \succ b$ and $(n-1)/2$ other voters submit $c \succ b \succ a$. Now if i submits $a \succ b \succ c$, let us calculate Copeland scores. Comparing a, b yields a the score $+1$ and comparing a, c yields a the score $+1$, so candidate a's Copeland score is $+2$, electing a.

On the other hand, if i were to submit $b \succ a \succ c$ instead, Copeland scores are calculated as follows. Candidate a: comparison a, b yields a the score -1 and comparison a, c yields a the score $+1$, so a's Copeland score is 0. Candidate b: against a, b's score is $+1$, and comparison b, c yields b the score -1, so b's Copeland score is 0. Candidate c: c's score against a is -1, and against b the score is $+1$. So c's Copeland score is 0. Now using the tie-breaker c, b, a or c, a, b, the winner is c.

Simpson rule. (Moulin 1988, ch. 9). Recall how Simpson rule is defined. Consider candidate a, and for every other candidate x, compute the number $N(a, x)$ of voters preferring a to x. The Simpson score of a is the minimum of $N(a, x)$ over all $x, x \neq a$. A candidate with the highest such score, called a *Simpson winner*, is elected.

Let us now check condition 1 with respect to a, b. Voter i places a ahead of b in any of following three submissions: (1) $a \succ b \succ c$; (2) $a \succ c \succ b$; (3) $c \succ a \succ b$. Let $(n-1)/2$ voters submit $a \succ b \succ c$ and the remaining $(n-1)/2$ voters submit $b \succ a \succ c$.

If i adopts (1), $SSc(a) = (n-1)/2+1$ whereas $SSc(b) = (n-1)/2$ and $SSc(c) = 0$, thus Simpson winner is a. Alternatively, if i submits $b \succ a \succ c$ or $b \succ c \succ a$ or $c \succ b \succ a$, then $SSc(a) = (n-1)/2$, $SSc(b) = (n-1)/2+1$ and $SSc(c) \leq 1$; thus Simpson winner is b. Following a similar procedure it is easy to check that Simpson winner is a if i adopts (2) or (3), whereas Simpson winner is b if i submits $b \succ a \succ c$ or $b \succ c \succ a$ or $c \succ b \succ a$. Thus, condition 1 is satisfied.

We will now verify condition 2 for the candidates pair a, c with c always placed at the bottom. Let$(n-1)/2$ voters submit $a \succ c \succ b$ and $(n-1)/2$ other voters submit $c \succ a \succ b$. Now if i submits $a \succ b \succ c$, then $SSc(a) = (n-1)/2+1$, $SSc(b) = 0$, $SSc(c) = (n-1)/2$ so that Simpson winner is a. On the other hand, if i were to submit $b \succ a \succ c$ instead, Simpson scores are: $SSc(a) = (n-1)/2$, $SSc(b) = 1$, $SSc(c) = (n-1)/2$. Thus, with a tie-breaker placing c ahead of a, Simpson winner is c. ||

Note that, *for all of the voting rules verified above Definition 2 will also be satisfied.*

Breakdown of the 2012 Olympic Games Host City Competition Votes

(Source: $http://news.bbc.co.uk/sport1/hi/other_sports/olympics_2012/4656529.stm$)

First round

London 22

Paris 21

Madrid 20

New York 19

Moscow 15 (eliminated)

Second round

Madrid 32

London 27

Paris 25

New York 16 (eliminated)

Third round

London 39

Paris 33

Madrid 31 (eliminated)

Fourth round

London 54

Paris 50 (eliminated)

References

[1] M. Battaglini, Sequential voting with abstention, *Games and Economic Behavior* forthcoming.

[2] S.J. Brahms and P. Fishburn, Approval voting, *American Political Science Review* **72** (1978), 831-847.

[3] M. de. Condorcet, Essai sur l'application de l'analyse a la probabilite des decisions rendus a la pluralite des voix, Paris, 1785.

[4] E. Dekel and M. Piccione, Sequential voting procedures in symmetric binary elections, *Journal of Political Economy* **108** (2000), 34-55.

[5] A. Dhillon and B. Lockwood, When are plurality rule voting games dominance solvable? *Games and Economic Behavior*, **46** (2004), 55-75.

[6] B. Dutta, M.O. Jackson, and M. Le Breton, Voting by successive elimination and strategic candidacy, *Journal of Economic Theory* **103** (2002), 190-218.

[7] B. Dutta and P.K. Pattanaik, On enforcing socially best alternatives of binary group decision rules, *Social Choice and Welfare* **1** (1985), 283-293.

[8] B. Dutta and A. Sen, Implementing generalized Condorcet social choice functions via backward induction, *Social Choice and Welfare* **10** (1993), 149-160.

[9] R. Farquharson, "Theory of Voting," Yale University Press, New Haven, 1969.

[10] P.C. Fishburn, "The Theory of Social Choice," Princeton University Press, Princeton, 1973.

[11] R. McKelvey and R. Niemi, A multistage game representation of sophisticated voting for binary procedures, *Journal of Economic Theory* **18** (1978), 1-22.

[12] H. Moulin, Dominance solvable voting schemes, *Econometrica* **47** (1979), 1337-1352.

[13] H. Moulin, "Axioms of Cooperative Decision Making," Cambridge University Press, Cambridge, 1988.

[14] R. Myerson, Comparison of scoring rules in poisson voting games, *Journal of Economic Theory* **103** (2002), 219-251.

[15] K.S. Strumpf, Strategic competition in sequential election contests, *Public Choice* **111** (2002), 377-397.