

# Unraveling of Dynamic Sorting

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**ABSTRACT:** We consider a two-sided, finite-horizon model of search and matching with heterogeneous types on both sides of the market. The quality of the pool of potential matches deteriorates as agents who have found mutually agreeable matches exit the market. With automatic participation of all agents in each round, the market performs a sorting function in that more attractive types of agents have a chance to meet and match with their peers in earlier rounds of the market. If agents incur an arbitrarily small cost in order to participate in each round, however, the market completely loses its sorting function as all agents rush to participate in the first round and match with anyone they meet.

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## 1. Introduction

Search takes time, and trading opportunities may change as time passes. The possibility of vanishing trading opportunities introduces an element of urgency in the search process that is absent in search models with a stationary environment. Consider, for example, a shopper who comes across an item on clearance sale. He may choose to do some more comparison shopping before deciding whether to buy that item, as is described in conventional models of search (e.g., McCall 1970; Rothschild 1973), but there is a possibility that the item will be sold out when he comes back. In such a non-stationary environment, the celebrated result of the equivalence between search with recall and search without recall no longer holds. Our shopper may rush to a purchase without extensive search, especially if the shop is crowded with fellow customers. One can even imagine a shopping frenzy that arises as a result of this type of behavior. Indeed, trading opportunities can themselves be shaped by the search process. The relationship between search and evolving trading opportunities introduces an interesting element in the dynamics of search that has not been adequately analyzed in the existing literature. Our paper attempts to make a first cut at this kind of interactions.

We analyze the search process in the context of matching markets, in which people who have found an acceptable match leave the market and hence affect the pool of potential partners available to subsequent searchers. Burdett and Coles (1997) and Shimer and Smith (2000a) have characterized steady state search equilibrium in such markets, but the focus on steady states does not give due recognition to the often non-stationary nature of these markets.<sup>1</sup> A number of entry-level labor markets for professionals (e.g., academic economists, hospital interns, federal law clerks) are organized around annual recruitment cycles. Employers who fail to fill a vacancy or job-seekers who fail to secure a position by the end of one cycle have to bear a substantial cost. These labor markets are best viewed

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<sup>1</sup> Smith (1995) first studied an infinite horizon matching model with no entry, where non-steady state dynamics is driven by temporary matches that are formed because finding acceptable mates takes time and waiting is costly in terms of foregone production. Shimer and Smith (2000b) examine the possibility that efficient search and matching requires non-stationarity.

as having a definite terminal date; the fact that there will be another recruitment cycle the following year does not make the market environment stationary.

The study of the operation of matching markets and its associated coordination problems has been pioneered in a series of papers by Alvin Roth and his coauthors (e.g., Roth, 1984, Roth and Xing, 1994). They describe how participants on both sides of such markets often try to arrange interviews and make contract offers ahead of an agreed upon starting date, or rush to sign contracts ahead of their competitors. Two types of inefficiencies can occur in the rush to sign contracts. This first type of inefficiency involves mismatches because information about quality of applicants and about positions is not yet available when participants sign early contracts. This type of inefficiency has been studied by Li and Rosen (1998), Li and Suen (2000), and Suen (2000). Unraveling, or early contracting, occurs in this type of models because it provides insurance benefits to risk-averse participants in spite of sorting inefficiency due to lack of match information.

Another type of sorting inefficiency occurs when the rush to contract ahead of competitors causes congestion of proposals and decisions at the early stages of the market. In their study of the market for clinical psychologists, Roth and Xing (1997) demonstrate that market participants may strategically choose to match with less desirable partners lest the pool of acceptable matching partners dries up quickly. We argue in this paper that, under certain conditions, this type of behavior can result in a concentration of activities in the early market and an almost total collapse of trade in the later stages of the market. Since market participants cannot consider more than a few choices simultaneously, the congestion of search and contracting in the early stages of the market results in reductions in the scope of the market and in the efficiency of sorting. Even if a wider and more prolonged search is beneficial to both sides of the market from a collective point of view, the competitive pressure to contract early may confine market participants to localized search. This is illustrated in a recent paper by Niederle and Roth (2001). Using data from the entry-level market for American gastroenterologists, they show that unraveling reduces the market scope in that gastroenterologists are more likely to be employed at the same hospital in which they were residents than they were when a centralized clearinghouse was in use. Presumably sorting is more efficient in a national market than in segmented local

markets, so the result of Niederle and Roth demonstrates the loss of sorting efficiency when congestion occurs as a result of rushing to contract early.

Our paper uses a stylized model of search that illustrates how search decisions interact with evolving matching opportunities in a matching market with a finite horizon. A job market operates in two rounds. Applicants differ in a one-dimensional productive attribute, called “type,” and so do firms. We assume the production function exhibits complementarity between worker type and firm type, so there are benefits from matching high type workers with high type firms. In the first round, applicants and firms meet each other randomly and they decide whether or not to form a match. If they do, they get their payoffs and withdraw from the market. Otherwise, they proceed to the second (and last) round. Those who proceed to the second round again meet each other randomly, and since this is the last round, they match with whomever they meet. One of our objectives is to investigate whether there will be “excessive” search and matching in the first round.

We consider two cases of the job market. In the first case, there is no search cost. Because search is free, everybody participates in the first round market. Equilibrium involves a uniform threshold such that an applicant accepts an offer from a firm if the latter’s type exceeds the threshold, and waits for the second round otherwise. If all applicants and firms follow this strategy, types lower than the threshold will not find a match and will participate in the second round market. Furthermore, some types higher than the threshold will not be lucky enough to find an acceptable match and will also participate in the second round market. When the expected type of all those who will be in the second round market equals the acceptance threshold, we have an equilibrium. In this equilibrium, the job market performs a sorting function by giving higher types a chance to meet with their peers and realize their higher match values. The sorting function is admittedly crude due to the kind of search frictions we have imposed, but it turns out to be socially efficient in that it maximizes the expected total match values realized in two rounds of random matching.

In the second case of the job market, applicants and firms have to incur a small cost in order to participate in each round of the market. It is apparent that the original equilibrium we consider will not survive. To begin, agents of type lower than the threshold have no

reason to pay the cost to be in the first round market since they face zero probability of forming a match. As these types withdraw, higher types that still participate in the first round market now have greater chances of meeting their peers and they exit the market in greater numbers. As a result, the pool of the first round market improves while the pool of the second round market worsens. But this division into an “elite” upmarket in the first round and a downmarket in the second round cannot be an equilibrium, because the best types in the downmarket have incentives to join the upmarket knowing that they are acceptable because they are the best in the downmarket pool. As more of the best types from the second round market join the first round market, the pool in the second round worsens further, which lowers the acceptance threshold in the first round still further. If the participation cost is arbitrarily small, the result is that in the only equilibrium the market completely loses its sorting function as all agents rush to participate in the first round and match with any one they meet. There is a total collapse of the second round market. Needless to say, such unraveling outcome is the most inefficient, even though the participation cost is arbitrarily small.

## **2. Efficiency with Automatic Participation**

To analyze how the search process interacts with matching opportunities over time, we consider a finite horizon two-sided matching market where there is no infusion of new agents in the relevant horizon. Matching can occur in any of the several matching rounds, but agents leave the market once they form a match. Thus the distribution of agents change endogenously over time. Agents are forward looking and their decisions on whether to search or to form a match will depend on their expectations about future matching opportunities. The flavor of the interactions that arise in this type of settings can be conveniently conveyed in a market with two matching rounds. The extension to multiple matching rounds will be described in Section 4.

Agents on each side of the market differ in a one-dimensional productive characteristic, called “type.” Types of agents on the two sides of the market are distributed continuously and symmetrically on the support  $[a, b] \subset (0, \infty)$ , with density  $f$  and distribution  $F$ . We

assume complementarity between agents' types. In particular, match value to a type  $x$  agent, if matched with a type  $y$  agent on the other side of the market, is  $xy$ . We also rule out side payment between any two agents that have matched. Agents who fail to find a match at the end of all matching rounds suffer a large cost, which we normalize by assuming that such agents get a payoff of 0 regardless of type.<sup>2</sup> Agents are risk-neutral, and do not discount between the two rounds.<sup>3</sup>

In this section, we consider the case of no search cost. Therefore unmatched agents automatically participate, or search, in each round of market. The search technology assumed in this paper is primitive: if the type distribution function is  $G$ , then the probability that any type  $x$  agent meets a type  $y$  or lower from the other side of the market is simply given by  $G(y)$ . In other words, the search technology in our model is random matching.<sup>4</sup>

Since an unmatched agent gets a payoff of 0, agents accept all random matches in the second (and last) round of the market. Anticipating this, an agent of type  $x$  agrees to match with  $y$  in the first round if and only if  $y \geq m$ , where  $m$  is the symmetric expected type in the second round. This implies a uniform threshold  $m$  for all types of agents. Note that any type lower than  $m$  is rejected by other types lower than  $m$ , as well as by all types higher than  $m$ .

The expected type in the second round market,  $m$ , is determined by the distribution of types that remains unmatched after the first round. Let  $k$  represent the first round acceptance threshold. Since a pair of agents leave the market only when both are of type greater than  $k$ , the size of the second round market is:

$$R(k) = 1 - (1 - F(k))^2.$$

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<sup>2</sup> Our results are unaffected as long as the unmatched payoff is lower than the lowest match payoff for every type.

<sup>3</sup> Adding a discount factor does not change any of our conclusions. Further, since production by agents takes place only after the conclusion of the job market, regardless of whether matches are formed in the first round or in the second round, it is reasonable to assume no discounting in our setup.

<sup>4</sup> Montgomery (1991), Lagos (2000), and Shimer (2001), among others, have considered models with more realistic search frictions. We adopt the simplistic random matching technology here because it makes the evolution of distribution of types analytical more tractable. In the search and matching literature, random matching technology is sometimes referred to as "linear," as opposed to "quadratic" (e.g. Smith, 1995). With a quadratic search technology, the matching payoff of any agent is unaffected by the matching decision of agents with whom he is not willing to match. This rules out congestion or crowding out match externalities that are crucial for our results.

Then,  $m$  is determined by  $k$  according to:

$$m(k) = \int_a^b x dG(x; k),$$

where  $G(x; k)$  is the distribution of types in the second round, given by

$$G(x; k)R(k) = \begin{cases} F(x), & \text{if } x \leq k; \\ F(k) + (F(x) - F(k))F(k), & \text{if } x > k. \end{cases}$$

Since  $R(a) = 0$ , the above definition does not cover  $m(a)$ . Instead, we define<sup>5</sup>

$$m(a) = \lim_{k \rightarrow a} m(k).$$

We can verify that  $G(x; k)$  stochastically dominates  $G(x; k')$  if  $k > k'$ . It follows that  $m'(k) > 0$  for any  $k \in (a, b)$ .

DEFINITION 1. A threshold type  $k^e$  is an equilibrium if  $k^e = m(k^e)$ .

Starting from an acceptance threshold  $k$  in the first round, if the resulting expected type  $m(k)$  in the second round falls below  $k$ , then  $k$  is too high to be justified. On the other hand, if  $m(k)$  exceeds  $k$ , agents are not picky enough in the first round. A rational expectations equilibrium occurs when the expected type  $m(k)$  that results from an acceptance threshold  $k$  precisely justifies  $k$ . Our first result is that an equilibrium always exists in our model.

PROPOSITION 1. An equilibrium  $k^e \in (a, b)$  exists.

PROOF. By definition, we have:

$$m(b) = \int_a^b x dF(x) < b;$$

$$m(a) = \lim_{k \rightarrow a} \left( \int_a^k \frac{xf(x)}{R(k)} dx + \int_k^b \frac{xf(x)}{2 - F(k)} dx \right) = \frac{1}{2}a + \frac{1}{2} \int_a^b xf(x) dx > a.$$

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<sup>5</sup> As can be seen from the following proof of Proposition 1, our definition of  $m(a)$  rules out  $k = a$  as an equilibrium. Letting  $m(a) = a$  would make  $k = a$  an equilibrium, but this equilibrium would not be robust, if with some probability agents who are indifferent between accepting their match in the first round and waiting for the second round "tremble" and reject their match.

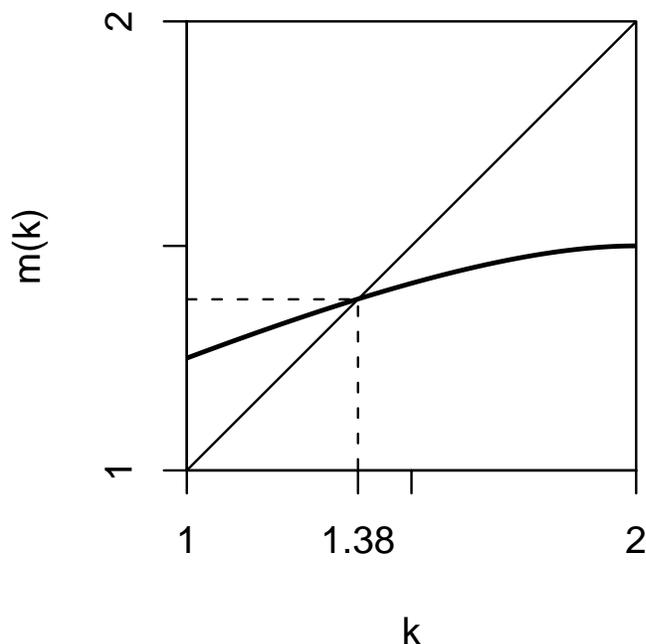


Figure 1

Since  $m(k)$  is a continuous function, by the Intermediate Value Theorem, an equilibrium  $k^e \in (a, b)$  exists. *Q.E.D.*

Figure 1 shows the case of uniform type distribution on  $[1, 2]$ . There is a unique equilibrium at  $k^e = 1.38$  in this case. However, since  $m'(k) > 0$ , expectations about the prospects in the second round market can be self-fulfilling and multiple equilibria may occur. The type distribution  $F$  may be concentrated around types just above  $k^e$ , such that further increases in  $k$  from  $k^e$  lead to sharp increases in the second round expected type. Such increases may then be self-fulfilling, resulting in another equilibrium with a first round acceptance threshold  $k$  higher than  $k^e$ . The issue of multiple equilibria is certainly interesting, but is orthogonal to the purpose of the present paper.<sup>6</sup> The following proposition uses a log-concavity condition on the type distribution to rule out multiple equilibria.<sup>7</sup>

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<sup>6</sup> Li and Suen (2001) deal with the issue of multiple equilibria in an early contracting model based on the trade-off between insurance benefits and sorting inefficiency.

<sup>7</sup> A log-concavity condition is used in Burdett and Coles (1997) to establish existence of a stationary search equilibrium. Their uniqueness result is obtained under a stronger condition on the type distribution.

PROPOSITION 2. *If the distribution of types  $F(x)$  is log-concave, then there is a unique equilibrium.*

PROOF. A sufficient condition for uniqueness of equilibrium is that  $m'(k) < 1$  for all  $k$ . To check that this condition is indeed satisfied, note that  $m(k)$  can be written as:

$$m(k) = w(k)q(k) + (1 - w(k))Q(k), \quad (1)$$

where  $q(k) = \mathbb{E}[x \mid x < k]$ ,  $Q(k) = \mathbb{E}[x \mid x \geq k]$ , and  $w(k) = F(k)/R(k)$ . Thus,  $m(k)$  is a weighted average of the conditional mean below  $k$  and the conditional mean above  $k$ . Take derivative of equation (1), we get

$$m'(k) = w(k)q'(k) + (1 - w(k))Q'(k) + w'(k)(q(k) - Q(k)).$$

Under the assumption of log-concavity of  $F$ ,  $q'(k) < 1$  and  $Q'(k) < 1$  (see An, 1998). Furthermore,  $w'(k) = f(k)F^2(k)/R^2(k) > 0$  and  $q(k) < Q(k)$ . Thus,  $m'(k) < 1$ .

*Q.E.D.*

In an equilibrium with first round threshold  $k^e$ , the market performs a sorting function by giving types higher than  $k^e$  a chance to meet with their peers and realize their higher match values. The sorting is crude due to the kind of search frictions we have imposed. But can it be improved without changing the random matching search technology? In particular, is the equilibrium acceptance threshold  $k^e$  optimal from the point of view of maximizing the expected total match values realized in two rounds of random matching? In other words, are agents in the market too selective, or do they rush to match in the first round?

To answer the above efficiency question, consider a social planner's problem of choosing a threshold type  $k$  to maximize the total match values in the two rounds.<sup>8</sup> In the first

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For our next result, log-concavity of the function  $\int_a^x F(t)dt$  suffices to guarantee uniqueness of equilibrium. The proof is available from the authors. We use a stronger condition in this paper, namely log-concavity of  $F(x)$ , in order to simplify the proof.

<sup>8</sup> This formulation of the social planner's problem puts restrictions on what the planner can do. In particular, the planner cannot choose different acceptance thresholds for different types. If the planner can tax or subsidize people who form a match in the first round market, this restriction amounts to confining the planner to choosing a proportional tax or subsidy rate that is uniform for all agents.

round “elite” market, only those with types higher than  $k$  will match. So if  $Q(k)$  is the conditional mean of types above  $k$ , expected match value from the first round market is  $Q^2(k)$ . Similarly, since the mean type in the second round market is  $m(k)$ , expected match value from the second round market is  $m^2(k)$ . Thus, the total match value for one side of the market is:

$$V(k) = (1 - R(k))Q^2(k) + R(k)m^2(k).$$

In the decentralized market of the first round, the matching decisions of individual agents have external effects on the agents who remain in the second round market. By choosing the threshold  $k$  appropriately, the socially planner can potentially internalize the externalities and increase total match value. The next result shows that this is not the case: the planner can do no better than the decentralized market.

**PROPOSITION 3.** *If  $k^*$  solves the social planner’s problem, then  $k^*$  is an equilibrium.*

**PROOF.** Taking derivative of  $V(k)$  with respect to  $k$ , we get:

$$V'(k) = 2(1 - R)QQ' + 2Rmm' - R'(Q - m)(m + Q). \quad (2)$$

Now, notice that for all values of  $k$ ,  $m(k)$  and  $Q(k)$  also satisfy the relationship,

$$(1 - R(k))Q(k) + R(k)m(k) = m^u, \quad (3)$$

where  $m^u$  is the unconditional mean of the distribution  $F$  of types. So we can differentiate equation (3) with respect to  $k$  to get

$$(1 - R)Q' + Rm' - R'(Q - m) = 0. \quad (4)$$

Substitute this into equation (2) to get

$$\begin{aligned} V'(k) &= 2(1 - R)QQ' + 2Rmm' - ((1 - R)Q' + Rm')(m + Q) \\ &= (Q - m)((1 - R)Q' - Rm') \end{aligned} \quad (5)$$

From equation (4), we also have

$$\begin{aligned} Rm' - (1 - R)Q' &= R'(Q - m) - 2(1 - R)Q' \\ &= 2f(1 - F)(Q - m) - 2(1 - R)(f/(1 - F))(Q - k) \\ &= 2f(1 - F)(k - m). \end{aligned} \quad (6)$$

Substitute (6) into (5) gives

$$V'(k) = 2(Q - m)f(1 - F)(m - k).$$

Since  $V'(a) > 0$  and  $V'(b) < 0$ , the optimal threshold  $k^*$  is an interior solution and satisfies the first-order condition  $V'(k^*) = 0$ . Thus,  $m(k^*) = k^*$  and  $k^*$  is be an equilibrium threshold in the decentralized market. *Q.E.D.*

Equation (2) in the proof of Proposition 3 shows that raising the first round acceptance threshold  $k$  has two counteracting effects on the expected match value. On one hand, since  $Q'(k) > 0$  and  $m'(k) > 0$ , an increase in the acceptance threshold from its equilibrium value improves the quality of the pool of agents in both the first round “elite” market and in the remaining second round market.<sup>9</sup> This suggests that agents in the decentralized market will not be selective enough in their choice of matching partners in the first round. On the other hand, since  $R'(k) > 0$ , raising the first-round acceptance threshold increases the size of the second round market, where expected match value is lower than in the first round “elite” market. An agent of type  $x > k^e$  who accepts a marginal partner of type slightly lower than  $k^e$  will bear a small cost, but will confer a discrete increase in payoff to that marginal type because it gives the marginal type a chance to match with higher types. In the absence of side payment between agents, this external benefit does not enter into the matching decisions and this suggests that agents will be too selective in the first round. Proposition 3 establishes that at the point  $k = m(k)$  these two effects exactly cancel each other, so that the decentralized equilibrium is also socially efficient.

The efficiency result may appear surprising, and one naturally wonders to what extent it depends on our strong symmetry assumptions that the two sides of the market have the same type distributions and that the match value function takes the symmetric product form of  $xy$ . It turns out that the symmetry assumptions are not crucial for our efficiency result. To see this, suppose the match value function is still  $xy$  but the two sides have different type distributions,  $F_x$  and  $F_y$ . For any multiplicatively separable match value

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<sup>9</sup> Transferring marginal students from an elite university to a bad college raises the average student quality in both schools.

function (with positive cross derivatives), we can always redefine the types so that the match value function takes the form  $xy$ . In this asymmetric random matching model, an equilibrium is given by two acceptance thresholds  $k_x$  and  $k_y$ , such that in the first round market  $k_x$  is the marginal  $x$ -type that  $y$ -agents are willing to accept, and  $k_y$  is the marginal  $y$ -type that  $x$ -agents are willing to accept. In equilibrium matches are formed in the first round market when types  $x \geq k_x$  and  $y \geq k_y$  meet with each other, with  $k_x$  equal to the expected type of  $x$ -agents in the second round and  $k_y$  equal to the expected type of  $y$ -agents. The following proposition shows that if a social planner can choose two acceptance thresholds for the first round market, one for each side, then equilibrium thresholds will be chosen. The proof is similar to that of Proposition 3 and is relegated to the Appendix.

**PROPOSITION 4.** *If  $k_x^*$  and  $k_y^*$  solves the social planner's problem in the asymmetric model, then  $k_x^*$  and  $k_y^*$  is an equilibrium.*

Dynamic sorting allows higher types to meet with each other in the first round. How much is the gain from the optimal dynamic sorting relative to pure random matching? In our example of uniform type distribution on  $[1, 2]$ , pure random matching, which is equivalent to dynamic sorting with an acceptance threshold  $k$  equal to either  $a$  or  $b$ , gives an expected match value of  $V^0 = 2.25$ , while efficient sorting with  $k^e = 1.38$  gives an expected match value of  $V^* = 2.272$ . The percentage gain from the optimal dynamic sorting seems small, less than 1%, but this is only because of the rather limited extent of complementarity in this example. The gain is significantly greater, if either the support of the types is wide ( $[1, 10]$  instead of  $[1, 2]$ ), or the match value function exhibits “increasing returns” to types ( $x^2y^2$  instead of  $xy$ ). To isolate the sorting gains from complementarity effects that may arise from rescaling the types, a more accurate measure of sorting efficiency is needed. In our numerical example with uniform type distribution on  $[1, 2]$  and match value function  $xy$ , the total match value from perfect positive assortative matching is only  $V^\infty = 2.333$ . This suggests that we measure dynamic sorting efficiency by  $(V^* - V^0)/(V^\infty - V^0)$ , which implies a relative gain of 23.5% from the optimal dynamic sorting. In Section 4, we show that having more rounds of matching further improves the sorting efficiency.

Though our optimality results may not be completely general given other types of search technologies, we want to stress that multiple matching rounds allow higher quality

agents to meet one another in an “elite” early market. Under complementarity of types, the early market improves the chance of assortative matching. Such sorting function of the early market, however, is rather precarious; it hinges on our assumption of zero search cost. In the following section, we show how sorting in the early market will unravel when search costs are introduced.

### 3. Inefficiency with Endogenous Participation

In the model of the previous section, agents do not “search.” They appear in the first round market even if they have no chance of forming a match. This seems innocuous if there is no cost of participating in the market. But by appearing in the market without any prospect of getting matched, agents of lower types impose a negative “search” externality on others who intend to match. Ironically, such negative externality turns out to be necessary for the market to perform the sorting function in the early round. High type agents who happen to meet the low type agents in the first round will have to try their luck again in the second round market. The externality imposed by low type agents therefore helps preserve the quality of the pool in the second round market. Without this externality effect, matching opportunities in the second round market will deteriorate and this can lead to a collapse of the second round market. In this section we assume that there is a small and uniform participation cost  $c$  in each market that is additive in an agent’s payoff. Our aim is to investigate whether endogenous participation will cause adverse selection that leads to the unraveling of the second round market.

With participation cost, agents make sequential decisions on participation and matching. Conditional on participation, the matching decisions of the agents in the first round market are determined by their expectation of the average type in the second round market. The participation decision depends both on the expected type in the second round and on the prospects of finding a mutually acceptable match in the first round. We now show that participation in the first round market is determined by a threshold rule.

LEMMA 1. *There exists a threshold  $l \in [a, b]$  such that agents of types higher than  $l$  participate in the first round market, and types lower than  $l$  wait for the second round market.*

PROOF. Let  $m$  be the expected match type in the second round market. Then, in the first round market, conditional on participation, a match between type  $x$  and type  $y$  is mutually agreeable if and only if  $xy \geq xm - c$  and  $xy \geq ym - c$ . Consider the participation decision in the first round by an agent of type  $x$ . It is optimal for type  $x$  agent to participate in the first round market if

$$\mathbb{E}_y[p(x, y)xy + (1 - p(x, y))(xm - c)] - c \geq xm - c,$$

where the expectation is taken with respect to the distribution of types  $y$  that participate in the first round market, and  $p(x, y)$  is the probability that agents of types  $x$  and  $y$  form a match. The above inequality can be written as:

$$\mathbb{E}_y[p(x, y)y + (1 - p(x, y))(m - c/x)] \geq m.$$

Any type  $x' > x$  agent can follow the same acceptance strategy of type  $x$ , and can guarantee that  $p(x', y) = p(x, y)$  for any  $y$  by rejecting any type  $y$  that is willing to accept type  $x'$  but not type  $x$ . Since  $m - c/x$  is increasing in  $x$ , the above strategy implies that it is optimal for type  $x'$  to participate. *Q.E.D.*

The presence of search cost changes the character of the matching decision. An agent who rejects a match in the first matching round will have to incur the search cost again to participate in the second round. Since agents of higher types have relatively more to gain from finding a good match, they tend to be more willing to incur the uniform cost  $c$ . Unlike the model of Section 2, therefore, acceptance thresholds differ for different participating types in the first round market. To describe the mutual acceptance interval for any potential participating type  $x$ , fix an expected type  $m \in [a, b]$  in the second round, and let

$$u(x) = m - \frac{c}{x}.$$

When  $u(x)$  lies between  $a$  and  $b$ , it represents the lowest type that type  $x$  is willing to accept. Similarly, define

$$v(x) = \frac{c}{m - x}.$$

When  $v(x)$  lies between  $a$  and  $b$ , it is the highest type that is willing to accept type  $x$ . The following properties of  $u(x)$  and  $v(x)$  are immediate: (i)  $u(x)$  is increasing and concave, and  $v(x)$  is increasing and convex; (ii) there are at most two intersections of  $u(x)$  and  $v(x)$ ; (iii)  $u(x) = x = v(x)$  at any intersection  $x$ ; and (iv)  $u(x) < x$  if and only if  $v(x) > x$ . If the threshold for participation in the first round market is  $l$ , a match between participating types  $x$  and  $y$  is mutually acceptable if and only if

$$\min\{v(x), b\} \geq y \geq \max\{u(x), l\}. \quad (7)$$

The pool of agents in the second round market therefore consists of all those with type lower than  $l$ , as well as those with  $x \geq l$  but whose random encounter in the first round does not satisfy equation (7).

Lemma 1 and the matching rule (7) suggest the following definition of equilibrium in a two-round random matching model with participation cost:

**DEFINITION 2.** *An equilibrium is a participation threshold  $l^e \in [a, b]$  and an expected type  $m^e \in [a, b]$  for the second round market, such that (i) given  $m^e$ , any type  $x \geq l^e$  prefers participating in the first round market and any type  $x < l^e$  prefers waiting for the second round market; and (ii) given  $l^e$  and  $m^e$ , the expected type in the second round market resulting from the participation decision and the optimal matching rule is precisely  $m^e$ .*

The rest of this section is devoted to characterizing equilibrium in the market and examining how equilibrium changes when search cost becomes arbitrarily small. We assume that

$$c < \min\{a^2, a(m^u - a)\},$$

where  $m^u$  is the unconditional mean of the type distribution function  $F$ .<sup>10</sup> The condition  $c < a^2$  ensures that the lowest type agent will participate in the matching market at least once, and  $c < a(m^u - a)$  implies that participation in pure random matching is worthwhile even if the lowest type agent can get the perfect matching payoff for free.

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<sup>10</sup> The above assumption reduces the number of cases we need to consider. A result analogous to Lemma 2 below can be obtained if  $0 < c < a^2$ .

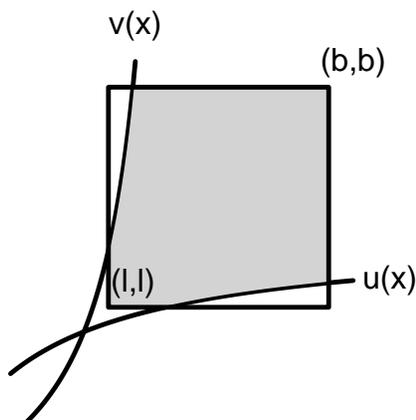


Figure 2

Since we are interested in the equilibrium for small participation costs, these assumptions are not restrictive.

Now, consider how to analyze the first part of the equilibrium condition. We show that there is a well-defined function  $h(m)$  that gives the participation threshold  $l$ , such that given  $m$  any types higher than  $h(m)$  are better off participating in the first round market, and types lower than  $h(m)$  are better off waiting for the second round market.

LEMMA 2. *Suppose that  $c < \min\{a^2, a(m^u - a)\}$ . Then, there exists  $\hat{m} \in (a + c/b, a + c/a)$  such that  $h(m) = a$  for any  $m \in [a, \hat{m}]$ ,  $h(m)$  is strictly between  $a$  and  $m$  and increases for any  $m \in (\hat{m}, b)$ , and  $h(b) = b$ .*

The proof of Lemma 2 is in the Appendix. We illustrate the intuition behind this result with Figure 2, which shows the functions  $u(x)$  and  $v(x)$  for a given value of  $m$ . Also shown is a square box  $[l, b] \times [l, b]$  which represents the pool of agents participating in the first round market.<sup>11</sup> Random encounters that fall in the shaded region result in matches in the first round. A moment's reflection suggests that any  $l$  below the intersection of  $u(x)$  and  $v(x)$  cannot be the participation threshold: such a type has no prospect for finding a mutually agreeable match in the first round and is better off not participating. We can therefore focus on type  $l$  above the intersection point. If  $l$  is strictly greater than  $a$ , then  $h(m) = l$  if and only if type  $l$  is indifferent between participating and not participating in

<sup>11</sup> The assumption  $c < \min\{a^2, a(m^u - a)\}$  implies that the two functions  $u(x)$  and  $v(x)$  intersect at most once in  $[a, b] \times [a, b]$ . See the proof of Lemma 2 in the appendix.

the first round market. Notice that when  $m$  falls, the  $u(x)$  curve shifts up while the  $v(x)$  curve shifts down. This increases the benefit from participating in the first round market as the probability of finding a match in the first round rises. The participation threshold  $h(m)$  therefore decreases as  $m$  falls. When  $m$  is very low (say,  $m \leq a + c/b$ ), the  $u(x)$  and  $v(x)$  curves are so far apart that the whole box  $[l, b] \times [l, b]$  lies between them even if  $l = a$ . Any type is better off participating than not participating. It follows that  $h(m) = a$ .<sup>12</sup>

We next analyze the second part of the equilibrium definition. For this we need to calculate the distribution of the pool of agents who will be in the second round market. Fix any participation threshold  $l$  and expected type  $m$ . As in the proof of Lemma 2, there are three cases regarding  $m$ . The first case is  $m \in (a + c/a, b]$ ; the other two cases are similar. In the first case, we can define  $\bar{l} = m - c/b$  such that  $v(\bar{l}) = b$ , and  $\underline{l} = \frac{1}{2}(m + \sqrt{m^2 - 4c})$  such that  $u(\underline{l}) = v(\bar{l})$ . Then, for  $l \in [\underline{l}, \bar{l}]$ , the set of mutually agreed matches in round 1 is as depicted in Figure 2. The corresponding distribution  $G(x; m, l)$  of types that remain in the second round market is given by

$$G(x; m, l)R(m, l) = \begin{cases} F(x), & \text{if } x \leq l; \\ F(l) + \frac{1}{1-F(l)} \int_l^x (1 - F(\min\{v(x), b\}) + F(\max\{u(x), l\}) - F(l))dF(x), & \text{if } x > l \end{cases}$$

where  $R(m, l)$  is the size of the market in the second round, given by

$$R(m, l) = F(l) + \frac{1}{1-F(l)} \int_l^b (1 - F(\min\{v(x), b\}) + F(\max\{u(x), l\}) - F(l))dF(x).$$

If  $l \in [a, \underline{l}]$ , any participating type below the intersection point  $\underline{l}$  have no prospect of being matched in the first round. The distribution of the types in the second round market is given by

$$G(x; m, l)R(m, l) = \begin{cases} F(x), & \text{if } x \leq \underline{l}; \\ F(\underline{l}) + \frac{1}{1-F(\underline{l})} \int_{\underline{l}}^x (1 - F(\min\{v(x), b\}) + F(\max\{u(x), l\}) - F(\underline{l}))dF(x), & \text{if } x > \underline{l} \end{cases}$$

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<sup>12</sup> Indeed  $h(m) = a$  even if  $m$  is slightly above  $a + c/b$ . See the proof of Lemma 2 in Appendix for details.

where  $R(m, l)$  is given by

$$R(m, l) = F(l) + \frac{1}{1 - F(l)} \int_l^b (1 - F(\min\{v(x), b\}) + F(\max\{u(x), l\}) - F(l)) dF(x).$$

Finally, if  $l \in (\bar{l}, b]$ , since  $v(x) > u(x)$  for any such  $l$ , everyone who participate in the first round market will get matched. Only types  $x < l$  remain in the second round market. Therefore the distribution is given by

$$G(x; m, l)F(l) = \begin{cases} F(x), & \text{if } x \leq l; \\ F(l), & \text{if } x > l. \end{cases} \quad (8)$$

Note that in this case the distribution  $G(x; m, l)$  is just the lower truncation of the original distribution  $F$  at the point  $l$ ; it does not depend on the value of  $m$ . The second equilibrium condition in Definition 2 can now be formally represented by the equation

$$m^e = \int_a^b x dG(x; m^e, l^e).$$

The next lemma shows that for any  $l$  the equation  $m = \int_a^b x dG(x; m, l)$  admits at least one solution in  $m$ , corresponding to the case where all types above  $l$  are matched with probability 1.

**LEMMA 3.** *For any  $l \in [a, b]$ ,  $m = \int_a^b x dG(x; m, l)$  is satisfied if  $m = \mathbb{E}[x \mid x < l]$ . Further, there is no other solution to the equation in the region where  $m \leq l + c/b$ .*

**PROOF.** The function  $q(l) = \mathbb{E}[x \mid x < l]$  is increasing in  $l$  and satisfies  $q(l) < l$  for any  $l > a$ . At  $l = a$ , the limit of  $q(l)$  is  $a$ . Thus, if we let  $m = q(l)$ , then for any  $l \in [a, b]$ , we have  $l > m - c/b = \bar{l}$ . This means the distribution  $G(x; m, l)$  of types in the second round market is given by equation (8), which is a lower truncation of the original distribution  $F(x)$ . By definition then,  $(l, m)$  satisfies  $m = \int_a^b x dG(x; m, l)$ . For the second part of the statement, note that  $m \leq l + c/b$  implies that the distribution  $G(x; m, l)$  is given by equation (8), which does not depend on  $m$ . *Q.E.D.*

Putting Lemmas 2 and 3 together, we can establish the existence of an equilibrium.

**PROPOSITION 5.** *For any participation cost  $c > 0$ ,  $l^e = a$  and  $m^e = a$  is an equilibrium.*

PROOF. By Lemma 2,  $h(a) = a$ . By Lemma 3,  $l = a$  and  $m = a$  satisfies the equation  $m = \int_a^b x dG(x; m, l)$ . Q.E.D.

Without participation cost, we have identified in Section 2 an equilibrium with full participation and a uniform acceptance threshold  $k^e > a$  such that all types higher than  $k^e$  accept each other. This equilibrium unravels when participation cost is positive. First, types lower than  $k^e$  have no reason to pay the cost to be in the first round market since they face zero probability of forming a match. As these types withdraw, higher types that still participate in the first round market now see greater chances of meeting their peers. As a result, the pool of the first round market improves while the pool of the second round market worsens. But this will give incentives for the best types in the second round market to join the first round market, knowing that they are acceptable because they are the best in the second round pool. As more of the best types from the second round market join the first round market, the pool in the second round worsens further, which lowers the acceptance threshold in the first round still further. The result of this “unraveling” is the equilibrium established in Proposition 5. In such an equilibrium, the second round market ceases to operate as agents rush to form matches in the first round with anyone they happen to meet. The sorting function afforded by multiple rounds of search collapses under adverse selection.

Proposition 5 establishes full unraveling as an equilibrium outcome, but it does not rule out the existence of other equilibria with partial participation in the first round market. Of particular interest is whether there exists another equilibrium in the neighborhood of the equilibrium with no search cost when search costs becomes arbitrarily small. The following result states that the answer is no. Indeed, the full unraveling outcome, with  $l^e = a$  and  $m^e = a$ , is the only equilibrium when the participation cost is arbitrarily small.

**PROPOSITION 6.** *When participation cost  $c$  is sufficiently small but positive,  $l^e = a$  and  $m^e = a$  is the only equilibrium.*

Proposition 6 is proved in Appendix. We illustrate the result with a numerical example based on the uniform type distribution on  $[1, 2]$ . In this case, besides the sequence of equilibria with  $l^e = 1$  and  $m^e = 1$ , there exists another sequence of equilibria with

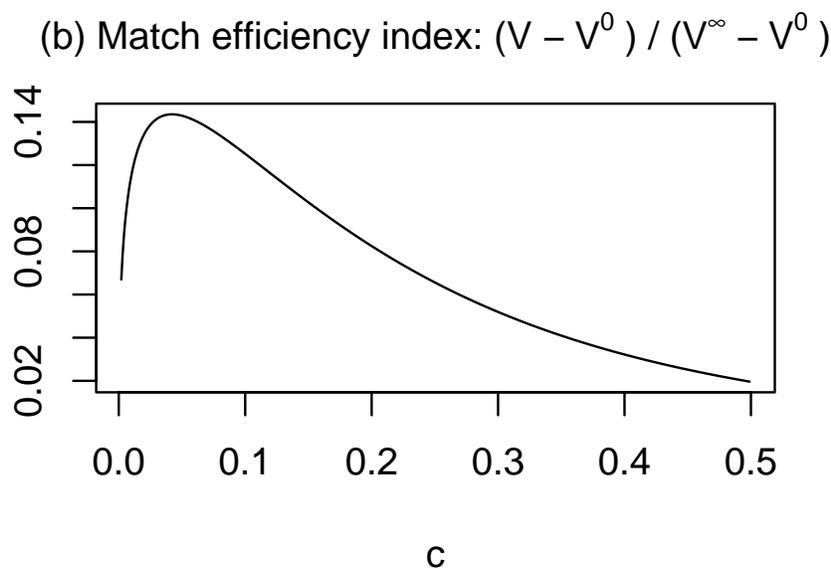
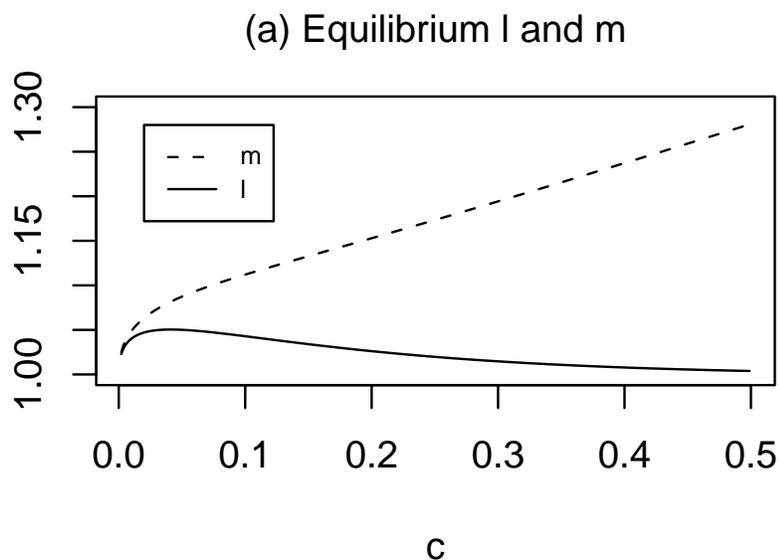


Figure 3

incomplete unraveling for  $c \geq 0.0016$ . The solution values of such a sequence are displayed graphically in Figure 3, panel (a). As search cost  $c$  decreases, the participation threshold  $l$  first rises and then falls. The reason for this non-monotonic relationship is that there are two opposing effects of a reduction in search cost. On one hand, a lower search cost tends to directly encourage more people to participate in the first round, therefore lowering the participation threshold. On the other hand, a smaller  $c$  means that the  $u(x)$  and  $v(x)$

functions are closer to each other. In other words, since searching again in the second round also becomes less costly, agents become more picky conditional on participation. This tends to reduce the probability of finding a suitable match for the marginal type, thereby discouraging them from participating in the first round. Indeed when  $c$  is very small ( $c < 0.0016$  in our example), the  $u(x)$  function becomes almost horizontal while the  $v(x)$  function becomes almost vertical. Almost all participating agents in the first round market will be accepted with probability 1, which renders the second round market a lemons market. Equilibrium with partial unraveling (i.e.,  $l > 1$ ) ceases to exist and the only equilibrium is one with full unraveling.

Panel (a) of Figure 3 also indicates that, for all values of  $c$ , the equilibrium second-round expected type  $m$  is never close to the value of  $m(k^e) = 1.38$  that we find in the model without search cost. Equilibria in these two models are not directly comparable as the match patterns are different, so we use the efficiency measure introduced in Section 2. In our example of uniform type distribution on  $[1, 2]$ , pure random matching (or the full unraveling equilibrium) gives an expected match value of  $V^0 = 2.25$ , efficient sorting with  $k^e = 1.38$  gives an expected match value of  $V^* = 2.272$ , and the perfect sorting gives a total match value of  $V^\infty = 2.333$ . We use  $V^0$  and  $V^\infty$  as benchmarks to calculate the sorting efficiency measure,  $(V - V^0)/(V^\infty - V^0)$ , where  $V$  is the equilibrium expected match value (before subtracting search costs). The result is shown in panel (b) of Figure 3. The value of the sorting measure ranges between 1.96% and 14.35%, significantly smaller than the value 23.5% achieved by the optimal dynamic sorting in Section 2. Regardless of whether there is full or partial unraveling, endogenous participation produces a significant deterioration in sorting efficiency.

#### 4. Multiple Matching Rounds

In this section we extend the analysis to the case of more than two matching rounds. Such an extension is useful because it illustrates the kind of analytic techniques that may be needed in an environment of non-stationary matching opportunities. The main result from this exercise is that adding more matching rounds adds to the contrast between automatic

participation and endogenous participation: sorting efficiency improves in the model with automatic participation, while full unraveling remains an equilibrium in the model with endogenous participation.

Consider first the case of automatic participation (no search cost). To fix idea, suppose there are three instead of two matching rounds. In the last matching round, since the value of being unmatched is zero, any pair of agents will match. Thus the threshold for acceptance is  $k_3 = a$ . In the second matching round, the distribution of types depends on agents' choice in round 1. We can denote this distribution by  $G_2$ . The analysis in Section 2 shows that, regardless of the distribution of types, there is a threshold  $k_2 \in (a, b)$  such that only agents with type greater than or equal to  $k_2$  are accepted. Since  $k_2$  is also equal to the expected type in the third round, the expected payoff in the second round to an agent of type  $x$  is

$$\begin{cases} x \left( G_2(k_2)k_2 + \int_{k_2}^b x dG_2(x) \right), & \text{if } x \geq k_2; \\ x k_2. & \text{if } x < k_2. \end{cases}$$

Note that the payoff is discontinuous at  $x = k_2$ . In round 1, therefore, different agents will have different acceptance thresholds in the same market. In particular, agents of type  $x \geq k_2$  will accept any type  $y \geq k_1$  where

$$k_1 = G_2(k_2)k_2 + \int_{k_2}^b x dG_2(x), \quad (9)$$

while agents of type  $x < k_2$  is willing to accept any type  $y \geq k_2$ . Because of the discontinuity in the payoff function, however, no one is willing to accept agents of type  $x < k_2$  in the first round. A pair of agents will match in round 1 if and only if  $x \geq k_1$  and  $y \geq k_1$ . So, even though there are more than one acceptance thresholds adopted by different agents in the first round market, only the threshold  $k_1$  in equation (9) matters. In other words we have described an equilibrium in which there is effectively a uniform acceptance threshold in each round, with  $k_1 > k_2 > k_3$ , and the distribution of types in different rounds evolves accordingly.

The above discussion leads to a general definition of equilibrium with  $T$  matching rounds. For each  $t = 1, \dots, T$ , let  $G_t$  be the symmetric type distribution in round  $t$ , and  $k_t$  be the threshold type. In the first round of the market,  $G_1$  is just  $F$ , the initial type distribution.

DEFINITION 3. A sequence of threshold types  $k_1, k_2, \dots, k_T = a$  and a sequence of type distributions  $G_1 = F, G_2, \dots, G_T$ , are an equilibrium if (i) for any  $t = 1, 2, \dots, T - 1$ ,

$$G_{t+1}(x)R_{t+1}(k_t) = \begin{cases} G_t(x), & \text{if } x \leq k_t; \\ G_t(k_t) + (G_t(x) - G_t(k_t))G_t(k_t), & \text{if } x > k_t; \end{cases} \quad (10)$$

where  $R_{t+1}(k_t) = 1 - (1 - G_t(k_t))^2$  is the size of the round  $t + 1$  market; and (ii) for any  $t = 1, \dots, T - 1$ ,

$$k_t = G_{t+1}(k_{t+1})k_{t+1} + \int_{k_{t+1}}^b x dG_{t+1}(x). \quad (11)$$

According to the above definition, in each round  $t$ , there is a uniform acceptance threshold  $k_t$  such that only types higher than  $k_t$  have a positive probability of being matched. Further, the second equilibrium condition above implies that  $k_t > k_{t+1}$  for each  $t$ . Any equilibrium involves a decreasing sequence of acceptance thresholds  $k_1, \dots, k_T$ , so that agents in equilibrium become increasingly less picky as matching proceeds over time. Also, the sequence of type distributions is ordered by stochastic dominance:  $G_t$  first order stochastically dominates  $G_{t+1}$  for each  $t = 1, \dots, T - 1$ . Finally, Definition 3 assumes that the market does not end before the final round  $T$ . The justification for this follows the same logic as Proposition 1. If the market were to end in round  $t < T - 1$ , with  $k_t = a$ , then for any type distribution  $G_t$  at the beginning of round  $t$ , the expected match type from waiting for another round would be given by

$$\frac{1}{2}a + \frac{1}{2} \int_a^b x G_t(x) dx,$$

which is greater than  $a$ . Thus, agents who were accepting types marginally higher than  $a$  in round  $t$  market were not making the optimal decision, implying that the market cannot end in round  $t$ .

PROPOSITION 7. (i) An equilibrium in the  $T$ -round model exists with  $k_1 < b$ . (ii) If the initial distribution of types  $F$  is log-concave, then there is a unique equilibrium.

The proof of Proposition 7 is rather involved and is relegated to Appendix. The main technical difficulty lies in the fact that the matching decisions are determined by a

backward induction through equation (11), while the evolution of matching opportunities is determined by a forward induction through equation (10). We overcome this difficulty by introducing an algorithm that iterates back and forth between equations (10) and (11), resulting in a fixed point in  $k_1$ .

What can we say about the sorting efficiency of an equilibrium with multiple matching rounds? For any sequence of acceptance thresholds  $k_1, \dots, k_{T-1}$ , the expected total match value is given by:

$$\sum_{t=1}^T \prod_{s=T-t}^{T-1} R_{T-s+1}(k_{T-s}) \left( \int_{k_t}^b x dG_t(x) \right)^2,$$

where  $R_1(k_0) = 1$ , and where the sequence of type distributions  $G_1 = F, G_2, \dots, G_T$  satisfy equation (10). It is clear that having more than two matching rounds improves sorting efficiency: even if the thresholds  $k_1, \dots, k_{T-1}$  are constrained to be the same, by setting these thresholds to equal  $k^e$  (the two-round equilibrium threshold), multiple matching rounds afford more chances for higher types to meet one another. To be sure, the acceptance thresholds  $k_1, \dots, k_{T-1}$  need not be the same, and this allows finer sorting of agents into  $T$  classes instead of just two classes.<sup>13</sup> Indeed we can make a stronger claim. Imagine that a social planner chooses a sequence of acceptance thresholds  $k_1, \dots, k_T$  to maximize the expected total match value. If the optimal sequence is a decreasing sequence, then it is an equilibrium in the  $T$ -round matching model. The proof of this result has to restrict attention to sequences of decreasing thresholds, but we conjecture that the equilibrium is efficient among all sequences. Since the objective of the social planner is to improve sorting efficiency, it makes little sense for the planner to allow two agents to match in one round while disallowing the same two types of agents from matching in later rounds. The following statement is proved in the Appendix.

**PROPOSITION 8.** *There is an equilibrium sequence of thresholds,  $k_1, \dots, k_T$ , that maximizes the expected total match value among all decreasing sequences.*

With more rounds of matching, dynamic sorting becomes more efficient. In our example of uniform type distribution on  $[1, 2]$ , pure random matching, which is equivalent to

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<sup>13</sup> The notion of “classes” here is different from that in Burdett and Coles (1977). In their model, agents of class  $i$  only match with agents in the same class. In our model, agents of class  $t$  (i.e.,  $x \in [k_t, k_{t-1})$ ) match with agents of class  $s$  or above in round  $s \geq t$ .

dynamic sorting with a single round, gives an expected match value of  $V^0 = 2.25$ , while dynamic sorting with two rounds and the optimal acceptance threshold  $k^e = 1.38$  in the first round gives an expected match value of  $V^* = 2.272$ . With three rounds of matching, the optimal (and the unique equilibrium) acceptance thresholds are  $k_1 = 1.48$  in the first round and  $k_2 = 1.32$  in the second round. The resulting expected match value is  $V^{**} = 2.284$ . According to the efficiency measure introduced in Section 2, in this example a matching market with three rounds achieves the efficiency level of  $(V^{**} - V^0)/(V^\infty - V^0)$ , which represents 40.7% of the available efficiency gain (the difference between perfect sorting and random matching). This gain is quite impressive compared to the efficiency gain of 23.5% with two rounds of matching.

The improvement in sorting efficiency afforded by multiple matching rounds does not extend to the case with endogenous participation. A simple induction argument makes this point clear. In round  $T - 1$ , if the market has not already ended, our two-round unraveling result in Section 3 applies: there is an equilibrium in which all agents who are still participants in the round  $T - 1$  market accept anyone they meet. But then in round  $T - 2$ , agents should anticipate that the market will close in the next round. So round  $T - 2$  is just like the next-to-last round. Our two-round unraveling result again applies, and so on. Thus, for any finite  $T$ , the only equilibrium with  $T$  matching rounds when search cost per round is arbitrarily small is that the market operates only for the first round in which all agents participate and accept whomever they meet. When agents choose when to search, adding more matching rounds only serves to hasten the date of search and contracting for all market participants. The unraveling of contracting date does not come with any increase in matching efficiency. Regardless of how many potential matching rounds there are, our result of the total collapse of the sorting function under endogenous participation applies.

## 5. Conclusions

Economists have long recognized that in a matching market both matching decisions and search decisions involve externalities and can cause market inefficiency. The existing literature (Diamond 1982; Mortensen 1982; Hosios 1990) has focused on the search externalities

by assuming homogeneity on the two sides of the market. The research on the search externalities culminates in the so-called Hosios (1990) condition for search efficiency, which requires the equality of an agent's bargaining power and the elasticity of the matching function. A recent paper by Shimer and Smith (2001) examines the implications of search and matching externalities in a dynamic model with heterogeneous agents. The Hosios condition does not hold in the model of Shimer and Smith: in the decentralized market attractive types search too little and match too readily, while unattractive types search too much and match too infrequently. In a different setup with posted prices and directed search, Shi (2001) finds efficiency in his dynamic matching model with heterogeneous agents.

The papers on search and matching inefficiencies mentioned above focus on steady-state stationary analysis, which greatly reduces the distributional complexity of search and matching dynamics. Our model is motivated by the concern that the steady state need not be the relevant horizon in many market for entry level professional workers. We posit that the dynamics in this kind of markets are better captured by a finite horizon model with no replacement of the types that have formed matches and left the market.

Focusing on the interactions between a fixed and non-replaceable set of agents heightens the uncertainties inherent in matching markets with frictions. Agents of higher types can no longer count on the infusion of new agents if they refuse to match with lower types in the early market. The situation is somewhat akin to that in a bank run (Diamond and Dybvig 1983), in which depositors lay claims on a fixed amount of bank reserves, or in a buying frenzy (DeGraba 1995), in which potential buyers do not wait for further information lest the good becomes sold out. The unraveling result in this paper also shares some similarities with Akerlof's (1970) story of the market for lemons. In both stories, the externalities present in the participation decisions of the agents combine with heterogeneity in the market to create adverse selection in the quality of potential trades in the market. In Akerlof's model, adverse selection arises because of private information about the quality of the good being traded. In comparison, adverse selection arises in our model because trading opportunities for those who wait directly depend on search and matching activities in the early market.

The endogenous evolution of trading opportunities poses difficult problems for equilibrium analysis. In Diamond and Dybvig (1983) depositors care only about the total amount of reserves available; in DeGraba (1995) consumers care only about the total stock of the good available. Characterizing trading opportunities in matching markets is a lot more complicated because agents are heterogeneous and sorting is important. We are able to make some progress on this front by making simplifying assumptions about the search technology and the match value function, and by ruling out side transfers. Relaxing these assumptions in a tractable way is a challenge that lies ahead.

## Appendix

PROOF OF PROPOSITION 4. For any pair of thresholds  $k_x$  and  $k_y$ , let

$$R(k_x, k_y) = 1 - (1 - F_x k_x)(1 - F_y(k_y))$$

be the size of second round market. Define

$$Q_x(k_x) = E[x \mid x \geq k_x];$$

$$Q_y(k_y) = E[y \mid y \geq k_y].$$

Let  $m_x(k_x, k_y)$  be the mean of  $x$ -agents in the second round market, and define  $m_y(k_x, k_y)$  similarly. An equilibrium is characterized by  $k_x^e$  and  $k_y^e$  such that

$$k_x = m_x(k_x, k_y);$$

$$k_y = m_y(k_x, k_y).$$

The planner's problem is to choose  $k_x$  and  $k_y$  to maximize the total match value  $V(k_x, k_y)$ , given by

$$(1 - R(k_x, k_y))Q_x(k_x)Q_y(k_y) + R(k_x, k_y)m_x(k_x, k_y)m_y(k_x, k_y).$$

Taking derivatives, we have

$$\frac{\partial V}{\partial k_x} = (1 - R)Q'_x Q_y + Rm_y \frac{\partial m_x}{\partial k_x} + Rm_x \frac{\partial m_y}{\partial k_x} + (m_x m_y - Q_x Q_y) \frac{\partial R}{\partial k_x}.$$

But we know that for any  $k_x$ ,

$$(1 - R)Q_x + Rm_x = m_x^u,$$

where  $m_x^u$  is the unconditional mean of  $x$ . Differentiate the above with respect to  $k_x$  to get

$$(1 - R)Q'_x + R \frac{\partial m_x}{\partial k_x} + (m_x - Q_x) \frac{\partial R}{\partial k_x} = 0.$$

Similarly, if  $m_y^u$  is the unconditional mean of  $y$ , then for any  $k_y$  we have

$$(1 - R)Q_y + Rm_y = m_y^u.$$

Differentiate with respect to  $k_x$  to get

$$R \frac{\partial m_y}{\partial k_x} + (m_y - Q_y) \frac{\partial R}{\partial k_x} = 0.$$

We can therefore follow similar procedures as those in the proof of Proposition 3 to get

$$\begin{aligned} \frac{\partial V}{\partial k_x} &= (Q_y - m_y) \left( (1 - R)Q'_x - (Q_x - m_x) \frac{\partial R}{\partial k_x} \right) \\ &= (Q_y - m_y) \left( (1 - R)(Q_x - k_x)f_x / (1 - F_x) - (Q_x - m_x)f_x(1 - F_y) \right) \\ &= (Q_y - m_y)f_x(1 - F_y)(m_x - k_x), \end{aligned}$$

where  $f_x$  is the density function corresponding to  $F_x$ . This implies that the equilibrium threshold  $k_x^e$  is the optimal choice for the threshold of the  $x$ -agents. Similarly,  $\partial V / \partial k_y = 0$  at the equilibrium threshold  $k_y^e$ . Q.E.D.

#### PROOF OF LEMMA 2.

(i) Assume  $m \in (a + c/a, b]$ . (Note that by assumption  $c < a(m^u - a)$  so  $a + c/a < b$ .) Then,  $m^2 > 4c$  and  $u(a) > a > v(a)$ . By the properties of  $u(x)$  and  $v(x)$ , there is exactly one intersection between  $u(x)$  and  $v(x)$  in  $[a, b]$ , given by  $\underline{l} = \frac{1}{2}(m + \sqrt{m^2 - 4c})$ . Define  $\bar{l} = m - c/b$  such that  $v(\bar{l}) = b$ . If type  $l \leq \underline{l}$  is the threshold participation type, then since  $u(l) \geq v(l)$ , type  $l$  will not accept any type that is willing to accept type  $l$ . Since participation is costly, such type cannot be the threshold. If some type  $l \in (\underline{l}, \bar{l}]$  is the threshold, the difference between the participation payoff and the waiting payoff for  $l$  is

$$-c + \int_l^{v(l)} lx \frac{f(x)}{1 - F(l)} dx + \frac{1 - F(v(l))}{1 - F(l)}(lm - c) - (lm - c).$$

Thus, type  $l$  prefers participating to waiting if

$$\int_l^{v(l)} xf(x)dx + (1 - F(v(l))) \left(m - \frac{c}{l}\right) - m(1 - F(l)) \geq 0.$$

The left-hand-side is increasing in  $l$  because  $m > \bar{l} \geq l$  and  $v(l) > u(l)$ . Further, at  $l = \underline{l}$ , since  $v(\underline{l}) = \underline{l}$ , the left-hand-side of the above inequality is negative. Similarly, if some type  $l \in (\bar{l}, b]$  is threshold, then participating is optimal for type  $l$  if

$$\int_l^b \frac{xf(x)}{1 - F(l)} dx - m \geq 0.$$

The left-hand-side of the above inequality is increasing in  $l$ , is positive at  $l = b$  when  $m < b$  and approaches 0 from below if  $m = b$ . Thus, for  $m \in (a + c/a, b]$ , there is a unique threshold type  $h(m) \in (\underline{l}, b]$  which is indifferent between participating and waiting, such that by Lemma 1, all types above  $h(m)$  prefer participating and all types below prefer waiting. Note that  $h(m) \leq m$ , with equality only if  $m = b$ , because either  $h(m) \leq \bar{l} < m$ , or  $h(m) > \bar{l}$  in which case  $\int_l^b xf(x)dx/S(l) > l$ . Further, we can verify that the left-hand-side of each participation condition above is strictly decreasing in  $m$ , implying that  $h(m)$  is strictly increasing in  $m$ .

(ii) Next, assume  $m \in (a + c/b, a + c/a]$ . We have  $b > v(a) \geq a \geq u(a)$ . Thus, if  $u(x)$  and  $v(x)$  intersect each other on  $[a, b]$ , they intersect twice. A necessary condition for intersection is that  $\frac{1}{2}(m - \sqrt{m^2 - 4c}) \geq a$ , which requires that  $m \geq 2a$ . But since  $m \leq a + c/a$ , we would have  $c \geq a^2$ , contradicting the assumption of the lemma. Therefore,  $u(x) < v(x)$  for all  $x \in [a, b]$ . Using the same argument as in the case of  $m \in (a + c/a, b]$ , we can show that if

$$\int_a^{v(a)} xf(x)dx + (1 - F(v(a))) \left(m - \frac{c}{a}\right) - m < 0,$$

then there is a unique threshold type  $h(m) \in (a, m)$ , strictly decreasing in  $m$ , which is indifferent between participating and waiting, such that all types above  $h(m)$  prefer participating and all types below prefer waiting. If instead

$$\int_a^{v(a)} xf(x)dx + (1 - F(v(a))) \left(m - \frac{c}{a}\right) - m \geq 0,$$

no type can be indifferent between participating and waiting, and so  $h(m) = a$ . Define  $\hat{m} \in (a + c/b, a + c/a)$  such that the above inequality holds with equality. This is possible because, (i) the left-hand-side of the inequality is decreasing in  $m$ ; (ii) at  $m = a + c/b$ , we have  $v(a) = b$  and the left-hand-side is positive because  $c < a(m^u - a)$ ; and (iii) at  $m = a + c/a$ , we have  $v(a) = a$  and the left-hand-side is negative.

(iii) Finally, assume  $m \leq a + c/b$ . Then,  $v(a) \geq b$  and  $u(b) \leq a$ . By the properties of  $u(x)$  and  $v(x)$ , we have  $u(x) \leq a$  and  $v(x) \geq b$  for all  $x \in [a, b]$ . If any type  $l$  is the participation threshold, then participating is optimal for type  $l$  if

$$\int_l^b \frac{xf(x)}{1 - F(l)} dx - m \geq 0.$$

Since  $\int_l^b xf(x)dx/(1 - F(l))$  is increasing in  $l$ , by the assumption of the lemma,  $m \leq a + c/b$  implies that  $m < \int_l^b xf(x)dx/(1 - F(l))$  for any  $l$ . Any type  $l$  is better off participating. It follows that  $h(m) = a$  for any  $m \leq a + c/b$ . *Q.E.D.*

**PROOF OF PROPOSITION 6.** Suppose that there is an equilibrium other than  $l^e = a$  and  $m^e = a$  regardless of how small  $c$  is. Then, there is a sequence of equilibria with participation threshold  $l_c$  and expected type  $m_c$  for each  $c > 0$ . By Lemma 3, for any  $c > 0$ , there is no equilibrium with  $l$  and  $m$  other than the full unraveling equilibrium such that  $m \leq l + c/b$ . Therefore, the sequence of equilibria has  $m_c > l_c + c/b$  and hence  $m_c > l_c$  for each  $c$ . For any  $c$  that is sufficiently small, we have  $m_c > a + c/a$ . It follows from the proof of Lemma 2 that there is exactly one intersection of  $v(x)$  and  $u(x)$  in  $[a, b]$ , and  $l_c > \underline{l} = \frac{1}{2}(m_c + \sqrt{m_c^2 - 4c})$ . On the other hand, we already know from Lemma 3 that  $l_c < \bar{l} = m_c - c/b$ . Thus, when  $c$  is arbitrarily close to 0,  $l^c$  is arbitrarily close to  $m_c$ . Moreover, as  $c$  converges to 0, the distribution  $G(x; m_c, l_c)$  becomes arbitrarily close to

$$G(x; m^c, l^c)F(l^c) = \begin{cases} F(x), & \text{if } x \leq l^c; \\ F(l^c), & \text{if } x > l^c, \end{cases}$$

which is independent of  $m^c$ . From the second part of Definition 2, the above condition implies  $m^c = \mathbb{E}[x \mid x < l^c]$ , which is less than  $l^c$ , contradicting our earlier conclusion that  $m_c > l_c$ . *Q.E.D.*

PROOF OF PROPOSITION 7. For the following proof of Propositions 6 and 7, it is convenient to write  $\tilde{G}_t(x) = \int_a^x G_t(z)dz$ ,  $S_t(x) = 1 - G_t(x)$ , and  $\tilde{S}_t(x) = \int_x^b S_t(z)dz$  for any  $t$ . Also, for each round  $t$ , threshold  $k_t$  and type distribution  $G_t$ , let

$$m(k_t; G_t) = G_t(k_t)k_t + \int_{k_t}^b x dG_t(x),$$

so that the second equilibrium condition can be written as  $k_{t-1} = m(k_t; G_t)$ . Whenever confusion does not arise, we write  $m_t$  instead of  $m(k_t; G_t)$ .

(i) The existence of equilibrium can be shown with an induction argument. We know from Proposition 1 that for any initial type distribution  $G$  an equilibrium exists when  $T = 2$ . Suppose that an equilibrium exists in a model with  $T \geq 2$  rounds, and let  $k_1(T; G)$  be the largest equilibrium threshold in the first round market with the initial type distribution  $G$ . Then, consider the following algorithm for finding an equilibrium with  $T + 1$  rounds of the market in total and the initial type distribution  $F$ : start with a first round threshold type  $k_1 \in (k_1(2; F), b)$ ; set the type distribution  $G_1$  in the first round market to  $F$ ; use  $G_1$  and  $k_1$  to compute the type distribution  $G_2$  in round 2 according to the first equilibrium condition (10); use  $k_1$  and  $G_2$  to determine a round 2 threshold  $k_2$  from the second equilibrium condition (11). If  $k_2 = k_1(T; G_2)$ , then an equilibrium has been found by combining this particular  $k_1$  with the sequence of  $T$  thresholds that starts with the resulting  $k_2$ , with the resulting  $G_2$  as the initial type distribution.

The above process is well-defined, because for each  $k_1$  and  $F$ , the type distribution  $G_2$  in the second round is uniquely defined according to (10). Further, (11) uniquely defines a second round threshold  $k_2$  for any  $k_1 \geq k_1(2; F)$ . To see this latter point, rewrite the condition as follows:

$$k_1 = \int_a^b x dG_2(x) + \int_a^{k_2} (k_2 - x) dG_2(x).$$

Using integration by parts and equation (10) for  $G_2$ , noting that  $k_2 \leq k_1$ , we can further rewrite the above as

$$\tilde{F}(k_2) = \tilde{F}(k_1) - F(k_1)\tilde{S}(k_1).$$

Since  $k_1(2; F)$  is the largest equilibrium threshold in the first round market with  $T = 2$  and the initial type distribution  $F$ , we have

$$\tilde{F}(k_1) \geq F(k_1)\tilde{S}(k_1)$$

for any  $k_1 \geq k_1(2; F)$ , with equality if and only if  $k_1 = k_1(2; F)$ . Thus, the first equilibrium condition uniquely defines  $k_2$  for any  $k_1 \geq k_1(2; F)$ .

Now, from Definition 1 (or equivalently Definition 3) we know that  $k_2 = a$  when  $k_1 = k_1(2; F)$ , so  $k_2 < k_1(T; G_2)$  if we start the process with  $k_1 = k_1(2; F)$ . On the other hand, from (11) we have  $k_2 = b$  when  $k_1 = b$ , so  $k_2 > k_1(T; G_2)$  if we start with  $k_1 = b$ . Continuity of  $k_2$  and  $k_1(T; G_2)$  in  $k_1$  then implies that the algorithm yields at least one  $k_1 \in (k_1(T; G_2), b)$  such that  $k_2 = k_1(T; G_2)$ , which identifies an equilibrium with  $T + 1$  rounds from the induction assumption.

(ii) To prove the uniqueness of the equilibrium, consider the following algorithm for finding an equilibrium with  $T$  rounds of the market in total and the initial type distribution  $F$ : start with a first round threshold type  $k_1$ ; use the initial distribution  $G_1 = F$  to compute the type distribution  $G_2$  in round 2 from the first equilibrium condition (10); use  $k_1$  and  $G_2$  to determine a round 2 threshold  $k_2$  from the second equilibrium condition (11); use  $k_2$  and  $G_2$  to find  $G_3$ ; repeat this process for all  $t = 3, \dots, T - 1$ , until we find  $k_{T-1}$  and  $G_T$ . Suppose for now that this algorithm is well-defined; we will check this later. Since the algorithm defines a sequence of decreasing thresholds, we have  $g_t(x)/G_t(x) = f(x)/F(x)$  for all  $x \leq k_{t-1}$  and for each  $t = 2, \dots, T - 1$ , and is therefore a decreasing function due to log-concavity of  $F$ . If  $k_{T-1} = \int_a^b x dG_T(x)$ , we have found an equilibrium.

To show that there is a unique equilibrium, we need to compute the derivatives with respect to  $k_1$ . Recognizing that  $k_1$  determines both the sequence of thresholds  $k_t$  and the sequence of distributions  $G_t$ , we use the following iterative method. For each  $t = 1, \dots, T - 2$ , using integration by parts, we can rewrite (11) as follows:

$$\tilde{G}_t(k_{t+1}) = \tilde{G}_t(k_t) - G_t(k_t)\tilde{S}_t(k_t).$$

Since the algorithm determines a decreasing sequence of thresholds, we can write the above as:

$$\tilde{F}(k_{t+1}) = \tilde{F}(k_t) - F(k_t)\tilde{S}_t(k_t).$$

For  $t = 1$ , taking derivatives implies that

$$\frac{dk_2}{dk_1} = \frac{F(k_1)(1 + S(k_1)) - f(k_1)\tilde{S}(k_1)}{F(k_2)}.$$

Note that  $dk_2/dk_1 > 1$  if

$$F(k_2) < F(k_1) - f(k_1)\tilde{S}(k_1).$$

For  $t = 2, \dots, T - 1$ , we use another way of rewriting the second equilibrium condition for round  $t - 1$ , again with integration by parts, to get

$$k_{t-1} = k_t + \tilde{S}_t(k_t).$$

Combining the above two equations we have for  $t = 2, \dots, T - 1$

$$\tilde{F}(k_{t+1}) = \tilde{F}(k_t) - F(k_t)(k_{t-1} - k_t).$$

The above equation can be used to compute each  $dk_t/dk_1$  recursively, starting from  $dk_2/dk_1$ . Taking derivatives, we have for  $t = 2, \dots, T - 1$

$$F(k_{t+1})\frac{dk_{t+1}}{dk_1} = (2F(k_t) - f(k_t)\tilde{S}_t(k_t))\frac{dk_t}{dk_1} - F(k_t)\frac{dk_{t-1}}{dk_1}.$$

An equilibrium is defined by  $k_{T-1} = \int_a^b x dG_T(x)$ , or with similar manipulations as above, by

$$k_{T-2} = k_{T-1} + \frac{\tilde{F}(k_{T-1})}{F(k_{T-1})}.$$

The above can be viewed as an equation in  $k_1$ . Since  $F$  is log-concave,  $\tilde{F}/F$  is increasing, and it follows that a unique fixed-point in  $k_1$  exists if  $dk_{T-1}/dk_1 > dk_{T-2}/dk_1$ . Thus, if for any each  $t = 1, \dots, T - 2$ ,

$$G_t(k_{t+1}) < G_t(k_t) - g_t(k_t)\tilde{S}_t(k_t),$$

then we obtain  $dk_{t+1}/dk_1 > dk_t/dk_1$  recursively, starting from  $dk_2/dk_1 > 1$ , and therefore the equilibrium is unique.

It remains to argue that for any distribution  $G$ , any thresholds  $k > k'$ , such that  $k > k_1(2; G)$ ,  $k'$  is determined by

$$\tilde{G}(k') = \tilde{G}(k) - G(k)\tilde{S}(k),$$

and  $g(x)/G(x)$  is decreasing for any  $x < k$ , we have

$$G(k') < G(k) - g(k)\tilde{S}(k).$$

This condition is sufficient, because even though changes in  $k_1$  affect all distributions  $G_t$ , the stated condition applies to all  $G$ ,  $k$  and  $k'$  that are linked through the equilibrium conditions and is therefore stronger than  $G_t(k_{t+1}) < G_t(k_t) - g_t(k_t)\tilde{S}_t(k_t)$  for any each  $t = 1, \dots, T-1$ . To see why the condition is true, note that since  $g(x)/G(x)$  is decreasing in  $x$ , and we have

$$g(x) > G(x) \frac{g(k)}{G(k)}$$

for any  $x < k$ . Integrating from  $k'$  to  $k$  (note that  $k' < k$  by assumption) gives

$$G(k) - G(k') > (\tilde{G}(k) - \tilde{G}(k')) \frac{g(k)}{G(k)}.$$

Since  $\tilde{G}(k') = \tilde{G}(k) - G(k)\tilde{S}(k)$ , we have

$$G(k') < G(k) - g(k)\tilde{S}(k),$$

as desired.

To complete the proof of uniqueness, we argue that the above iterative algorithm is well-defined. This can be established by an induction argument. We know that the algorithm is well-defined for  $T = 2$  and the initial distribution  $F$  if we start with any  $k_1$ , by Proposition 2. For  $T = 3$  and the initial distribution  $F$ , we have seen from part (i) of the proof that the algorithm is well-defined if we start with  $k_1 \geq k_1(2; F)$ . Let  $k_1(3; F)$  be the resulting unique equilibrium threshold in the first round, and then the algorithm is well-defined for any  $k_1 \geq k_1(3; F)$ . Now, suppose that it is well-defined for any  $T \geq 3$  and  $F$  if we start with any  $k_1 \geq k_1(T; F)$ . Then, for the model with  $T + 1$  rounds, by the induction assumption the iterative algorithm is well-defined for the first  $T - 1$  rounds if we start with any  $k_1 \geq k_1(T; F)$ , and yields a unique sequence of  $k_2, \dots, k_{T-1}$ , with  $dk_{t+1}/dk_1 > dk_t/dk_1$  for each  $t = 1, \dots, T - 2$ , and the associated sequence of  $G_2, \dots, G_T$ , such that  $k_{T-1} \geq k_1(2; G_{T-1})$ . The latter inequality guarantees that the second equilibrium condition determines a unique  $k_T \geq a$ , which in turn determines  $G_{T+1}$  by the first equilibrium condition. Thus, the algorithm is well-defined in round  $T$  as well as in the first  $T - 1$  rounds, completing the induction argument. *Q.E.D.*

**PROOF OF PROPOSITION 8.** The efficiency of an equilibrium can be established with an induction argument. Fix a market with a total of  $T$  rounds and the initial distribution

$F$ . For any sequence of decreasing thresholds  $k_1, \dots, k_{T-1}$ , let  $G_t$ ,  $t = 2, \dots, T$ , be defined according to equation (10), starting from  $G_1 = F$ . Let  $V_T$  be the expected total match value for round  $T$ , with type distribution  $G_T$ :

$$V_T = \left( \int_a^b x dG_T(x) \right)^2.$$

For each  $t = 1, \dots, T - 1$ , recursively define the expected total match value from round  $t$  onward:

$$V_t = \left( \int_{k_t}^b x dG_t(x) \right)^2 + R_{t+1}(k_t)V_{t+1},$$

which, by integration by parts, can be more conveniently written as

$$V_t = \left( k_t S_t(k_t) + \tilde{S}_t(k_t) \right)^2 + R_{t+1}(k_t)V_{t+1}.$$

The social planner's objective is then to maximize  $V_1$ .

Assume that the planner chooses a decreasing sequence of thresholds. Our induction argument starts with the observation from Proposition 3 that for the two-round case, any optimal threshold satisfies equation (11). Now, assume that this holds for any  $T-1$  rounds, so that for each  $t = 2, \dots, T - 1$ , any sequence of decreasing thresholds that maximizes  $V_t$  satisfies the equilibrium condition that  $k_{t-1} = m_t$ . Then, for a sequence of thresholds  $k_1, \dots, k_{T-1}$  to maximize  $V_1$ , it is necessary that  $k_{t-1} = m_t$  for all  $t \geq 3$ , and that  $\partial V_1 / \partial k_1$ , evaluated at  $k_1, \dots, k_{T-1}$ , is equal to zero. We will show that these necessary conditions imply that  $k_1 = m_2$ , which establishes the proposition by induction.

To show  $k_1 = m_2$ , we recursively derive the expressions of  $V_1$  and  $\partial V_1 / \partial k_1$ , both evaluated at the optimal sequence of thresholds  $k_1, \dots, k_{T-1}$ . To start, from the induction assumption that  $k_{T-1} = m_T$ , we have

$$V_T = k_{T-1}^2.$$

To compute  $\partial V_T / \partial k_1$ , we rewrite  $V_T$  as

$$V_T = \left( k_{T-1} + \frac{G_{T-1}(k_{T-1})\tilde{S}_{T-1}(k_{T-1}) - \tilde{G}_{T-1}(k_{T-1})}{R_T(k_{T-1})} \right)^2.$$

Taking derivatives with respect to  $k_1$ , and evaluating at  $k_{T-1} = m_T$ , which, by integration by parts is equivalent to

$$G_{T-1}(k_{T-1})\tilde{S}_{T-1}(k_{T-1}) = \tilde{G}_{T-1}(k_{T-1}),$$

we find that  $\partial V_T/\partial k_1$  is given by

$$\frac{2k_{T-1}}{R_T(k_{T-1})} \left( G_{T-1}(k_{T-1}) \frac{\partial \tilde{S}_{T-1}(k_{T-1})}{\partial k_1} + \tilde{S}_{T-1}(k_{T-1}) \frac{\partial G_{T-1}(k_{T-1})}{\partial k_1} - \frac{\partial \tilde{G}_{T-1}(k_{T-1})}{\partial k_1} \right).$$

Since  $k_{T-1} < k_1$ , we have

$$\tilde{S}_{T-1}(k_{T-1}) \frac{\partial G_{T-1}(k_{T-1})}{\partial k_1} = \frac{\partial \tilde{G}_{T-1}(k_{T-1})}{\partial k_1},$$

and therefore

$$\frac{\partial V_T}{\partial k_1} = 2k_{T-1} \frac{G_{T-1}(k_{T-1})}{R_T(k_{T-1})} \frac{\partial \tilde{S}_{T-1}(k_{T-1})}{\partial k_1}.$$

Now, we can proceed to round  $T - 1$ , and so on. In recursively computing  $\partial V_t/\partial k_1$ , we treat the thresholds  $k_2, \dots, k_{T-1}$  as independent variables, and recognize that the choice of  $k_1$  affects only the sequence of distributions  $G_2, \dots, G_T$ . These recursive computations lead to

$$V_2 = m_2^2 - \sum_{t=2}^{T-1} 2k_t \tilde{S}_t(k_t) G_2(k_t),$$

and its derivative

$$\frac{\partial V_2}{\partial k_1} = 2m_2 \frac{\partial \tilde{S}_2(k_2)}{\partial k_1} + \sum_{t=2}^{T-1} 2k_t \tilde{S}_t(k_t) \frac{\partial S_2(k_t)}{\partial k_1}.$$

Since for each  $t = 2, \dots, T - 1$ ,

$$\frac{\partial S_2(k_t)}{\partial k_1} = f(k_1) S(k_1) \frac{G_2(k_t)}{R_2(k_1)},$$

and since  $m_2 = k_2 + \tilde{S}_2(k_2)$ , we have

$$\frac{\partial V_1}{\partial k_1} = -2f(k_1)(k_1 S(k_1) + \tilde{S}(k_1))k_1 + 2f(k_1)S(k_1)m_2^2 + 2R_2(k_1)m_2 \frac{\partial m_2}{\partial k_1}.$$

Using the definition of  $m_2$  and integration by parts, we can rewrite  $m_2$  as

$$m_2 = k_1 + \frac{1}{R_2(k_1)}(F(k_1)\tilde{S}(k_1) - \tilde{F}(k_1) + \tilde{F}(k_2)).$$

The derivative of  $m_2$  with respect to  $k_1$  is

$$\frac{\partial m_2}{\partial k_1} = \frac{f(k_1)}{R_2(k_1)}(\tilde{S}(k_1)F^2(k_1) + 2S(k_1)(\tilde{F}(k_1) - \tilde{F}(k_2))).$$

Substituting  $m_2$  and  $\partial m_2/\partial k_1$ , we have

$$\frac{\partial V_1}{\partial k_1} = \frac{2f(k_1)}{R_2(k_1)}(F(k_1)\tilde{S}(k_1) + S(k_1)(\tilde{F}(k_1) - \tilde{F}(k_2)))(m_2 - k_1).$$

Thus, the optimal first round threshold  $k_1$  satisfies  $k_1 = m_2$ , completing the induction argument. Q.E.D.

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