Electoral Competition with Privately Informed Candidates*

(PRELIMINARY)

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Abstract

We consider a model of elections in which two office-motivated candidates are uncertain about the location of the median voter's ideal point, and in which candidates receive private signals about the location of the median prior to taking policy positions. Assuming signal spaces are finite, we show that there is at most one pure strategy equilibrium and give a sharp characterization, if one exists: after receiving a signal, each candidate locates at the median of the distribution of the median voter, conditional on the other candidate receiving the *same* signal. A candidate's position, conditional on his/her signal, is therefore a biased estimate of the true median, with candidate positions tending to the extremes of the policy space. We give sufficient conditions for existence of pure strategy equilibria. Though the electoral game gives the candidates discontinuous payoffs, we prove existence and upper hemicontinuity of mixed strategy equilibria generally, and we characterize mixed strategy equilibria in a special case of the model. Finally, we consider the consequences of electoral competition for the welfare of the median voter.

1 Introduction

Remark to us: I haven't reworked the intro, but here are the main points of the paper, as they occur to me right now.

- We establish a general model of elections that captures the usual Downsian model and a large class of probabilistic voting models as special cases.
- We analyze a previously untouched (except for Chan) aspect of polling, namely, the effect on candidate platforms.
- We prove uniqueness and fully characterize the pure strategy equilibrium of the model, if one exists. Our logic generalizes that of Downs and Calvert (in a probabilistic voting model), who consider models of symmetric information.
- We show that the logic does *not* extend in the expected way, and that private information leads to a tendency to take more extreme policy positions.
- We give a new explanation for "platform divergence," a phenomenon that almost all models have a hard time explaining.
- We show that pure strategy equilibria in symmetric information models, including the Downsian model, can be very sensitive to the addition of asymmetric information: arbitrarily small perturbations can lead to non-existence of pure strategy equilibrium.
- We extend the mixed strategy equilibrium existence result of Ball (19??) to a model of asymmetric information.
- We prove upper hemicontinuity of mixed strategy equilibria, providing a robustness result for symmetric information models: even if pure strategy equilibria of the Downsian or Calvert models disappear when a little asymmetric information is added, there will be mixed strategy equilibria, and all of them will be near the original equilibrium.
- We give a uniqueness result and explicit solution for mixed strategy equilibria in a special case of the model, allowing us to do comparative statics.
- We consider welfare issues.

The most familiar and fundamental prediction in formal political theory is that of platform convergence. The classical framework for this prediction (Black (1958), Downs (1957)) considers a political contest between two rational, well-informed candidates who care only about holding office, and who face a well-informed electorate which cares only about the platform that the winning candidate will adopt. The so-called "median voter theorem" details that in this environment both office seekers should propose the platform that corresponds to the median voter's "ideal" platform.

This prediction extends immediately to office-motivated candidates who care only about winning, but who are uncertain about the location of the median voter: in the equilibrium of this probabilistic voting game, both office seekers propose the platform that corresponds to the median of the distribution over the median voter's location.

The intuition underlying the median voter theorem is at once both transparent and compelling: If one candidate does not locate at the median, then the other candidate maximizes his probability of winning by locating near the first candidate, but marginally closer to the median, thereby winning with a probability exceeding one-half.

The qualitative result of platform convergence extends to models with additional candidate heterogeneity. In a probabilistic voting environment featuring candidates who care about the policy adopted by the winning candidate, candidates will adopt platforms that are closer to the median than are their most-preferred platforms, as they trade-off a less-preferred platform for an increased probability of winning. In dynamic models with policy-motivated candidates who cannot commit to their platforms, and voters who use an incumbent's policy location to update about his ideology (and hence likely future policy locations), an incumbent must take a position sufficiently close to the median voter's preferred position in order to win re-election (Duggan (2000), Banks and Duggan (2001), Bernhardt, Hughson and Dubey (2002)).

This paper shows that this apparently robust result of platform convergence is over-turned if candidates have access to private signals about the likely location of the median voter. We introduce private polling by candidates into an otherwise canonical probabilistic voting model. The two candidates do not know the location of the median voter, but receive private signals about the median voter's true location. Given their signals, candidates then simultaneously choose platforms, and the candidate who locates closest to the median wins; in the event that both candidates choose the same platform, each has an equal chance of winning.

We first show that if a pure strategy equilibrium exists, then it is unique. We then show that although allowing candidates to commission private polls that provide them information about the median voter's location seems innocuous, strategic location choices by candidates are radically altered. This is most starkly highlighted when candidates receive binary signals — "left" or "right" — about the likely median. Then, as long as the probability candidates receive the same signed signal exceeds 0.5, the pure strategy equilibrium exists and in it, a candidate receiving a left (right) signal will locate at the conditional median *given that both candidates receive left (right) signals.* Since the median given two "left" signals (of possibly different qualities) is more extreme than the median given a single "left" signal — it follows that in a pure strategy equilibrium, candidates locate further from the *ex ante* median than their own polling information suggests is the likely location of the median voter. That is, candidates "bias" their locations in the direction of their private signals, so that platforms are more dispersed than their information.

The qualitative nature of this result is robust. If candidates have access to identical polling technologies (of course, signal *realizations* can differ), then a pure strategy equilibrium exists if candidates are sufficiently likely to receive the same signal realization; and in this pure strategy equilibrium, a candidate receiving a signal s, will locate at the median of the distribution over the median voter's location *given that the other candidate also receives a signal of* s. While private polling information necessarily leads to some dispersion in platforms because candidates receive different signals, the strategic response of the candidates to this information is to increase further this platform dispersion. To adapt Barry Goldwater's campaign slogan, "Extremism in the pursuit of victory is no vice."

The logical underpinnings are quite simple: the probability that a candidate wins given both his platform choice and that of his opponent varies continuously with his platform choice *except* when the candidates choose the same platform. Then, a candidate receives a discontinuous jump in his probability of winning if his platform is marginally closer to the median given *both* of their signal realizations. But such jumps in winning probabilities cannot be consistent with equilibrium platform selections by both candidates. It follows that the only possible platforms on which candidates place a positive probability mass following a signal s are those that correspond to the median of the distribution over the median voter's location given signal s, and some signal t for the other candidate: only then does the probability of winning vary continuously with the candidate's platform choice.

The strategic impact of private information on candidate location has a direct voting analog. Here, candidates consider the information associated with both candidates choosing the same platform; voters consider the information associated with being pivotal, so that their vote determines discontinuously the identity of the winning candidate.

We provide a rich characterization of equilibrium strategies. We provide sufficient conditions for the existence of a pure strategy equilibrium for several distinct informational envirnments. We next show that if the pure strategy equilibrium does not exist (essentially *if* receiving a signal s does not make it sufficiently likely that the other candidate received some signal t), then a mixed strategy equilibrium does exist. Further, if candidates have access to a common polling technology then there exists an equilibrium in which candidates adopt the same (possibly mixed) strategy. We show that equilibrium payoffs vary continuously with (changes in) the environment.

We then consider a specific environment in which candidates obtain polling information about the electorate's views at some time prior to the election, and must then select platforms, recognizing that the median voter's preferred position may shift by the election's date. For example, after platforms have been selected, a weakening economy may change the median voter's view about the appropriateness of increased fiscal spending; or terrorist atacks may alter the median voter's views about the appropriateness of increased civil rights restrictions. We explicitly solve for equilibrium strategies when candidates have access to a common polling technology, receive conditionally independent signals, and the shift in the median voter's position after platforms have been chosen is uniformly distributed. This explicit solution is important, because even when only a (purely) mixed strategy equilibrium exists, it allows us to determine when candidates tend to locate more extremely than their signals suggest is appropriate.

Finally, we use these findings to interpret a seemingly puzzling complaint in both the public press, and academic research. To whit, it is generally recognized that improved polling techniques lead to a convergence in candidate platforms, as candidates try to hone in on the median voter's preferred platform given their current information. Such convergence is often criticized as being bad for the median voter, and for voters in general — candidates are collectively too similar, there is “not enough choice” between candidates, and “they are all the same”.

Yet, on first pass, it is hard to understand why and how it can hurt the median voter for candidates to target his likely location with increased precision; and it is hard to understand why this could reduce the welfare of risk averse voters with other ideologies who care about electoral outcomes rather than the location of the candidate for whom they vote, because such improved targetting is apparently variance-reducing.

We explore this issue within the environment in which candidates obtain polling information about the electorate's views at some time prior to the election, and must then select platforms, recognizing that the median voter's preferred position may shift by the election's date. Within this context, we reconcile the view that more accurate polling can harm welfare. Increased polling accuracy allows each candidate to target more accurately the median voter's current position; but the median voter's position may change in the interim. With better polling, each single candidate's platform is closer in expectation to the median voter's ultimate ideal platform, but the expected separation in their platforms is reduced. Since the winning platform is the one that the median voter likes best, this reduced dispersion in platforms is welfare harming. The optimal amount of polling noise is always intermediate — it is never optimal to have perfectly informative signals, in which case there is no choice amongst candidates; and it is never optimal to have perfectly uninformative signals. Qualitatively, we show that the more by which the median voter's views may change in the interim between when platforms are chosen and the election date, the noisier is the optimal polling technology.

2 The Model

Let two political candidates, A and B, simultaneously choose policy platforms, x and y, on the real line, $\Re$. For simplicity, we model the electorate as a cut point, μ, that determines which of the two candidates wins: candidate A wins the election if $|x-\mu| < |y-\mu|$ and loses if the inequality is reversed; if $|x-\mu| = |y-\mu|$, then we assume the election is decided by a fair coin toss, so both candidates win with probability one half. Assuming symmetric utilities, this allows us to capture representative voter models and, as long as a median is uniquely defined, models with multiple voters. This would be the case with an odd number of voters and with a continuum of voters with ideal points distributed according to a density with convex support. In such models, policy z will be majority-preferred to w if and only if it is preferred by the median voter.

The location of the cut point, μ, is unobserved by the candidates, but the candidates receive private signals, s and t, of the true location of μ. Let S denote the finite set of possible signals for A, and let T denote the finite set of possible signals for B. We suppose the candidates have a common prior distribution on $\Re \times S \times T$, where the distribution of μ conditional on s and t is denoted $F_{s,t}$, and the marginal probability of signal pair (s,t) is $P(s,t)$. We denote the marginal probabilities of signal s by $P(s)$ and signal t by $P(t)$, and we assume throughout that $P(s) > 0$ and $P(t) > 0$ for all $s \in S$ and all $t \in T$, allowing us to define conditional probabilities, denoted $P(\cdot|s)$ and $P(\cdot|t)$, using Bayes rule. The model is completely general with respect to correlation between the candidates' signals, allowing for conditionally independent signals and perfectly correlated signals as special cases.

We assume throughout the following typical regularity conditions on the conditional distributions: for all $s \in S$ and all $t \in T$, $F_{s,t}$ is continuous, and, for all $a, b, c \in \Re$ with $a < b < c$, $0 < F_{s,t}(a)$ and $F_{s,t}(c) < 1$ implies $F_{s,t}(a) < F_{s,t}(b) < F_{s,t}(c)$. Thus, $F_{s,t}$ admits a density, denoted $f_{s,t}$, with convex support. We denote the uniquely defined median of $F_{s,t}$ by $m_{s,t}$. We write the distribution of μ conditional on $s \in S$ and $T' \subseteq T$ as

$$F_{s,T'}(x) \quad = \quad \sum_{t \in T'} \frac{P(t|s)}{P(T'|s)} F_{s,t}(x),$$

when $P(T'|s) > 0$. We denote the median of $F_{s,T'}$, which is uniquely defined, by $m_{s,T'}$. We write F_s for $F_{s,T}$, the distribution of the cutpoint conditional on s, and we write m_s for $m_{s,T}$, the median of this distribution. We define the notation F_t and m_t similarly. We say signal realizations s and s' are *conditionally equivalent* if, for all t, we have $F_{s,t} = F_{s',t}$, and similarly for signals t and t'. Note that conditional equivalence does not restrict the posteriors on T conditional on s and s', but $F_{s,T'} = F_{s',T'}$ nevertheless holds for all $T' \subseteq T$ if s and s' are conditionally equivalent.

Remark to us: We can allow for distributions with finitely many discontinuity points. Our mixed strategy existence and continuity results will go through unchanged. Our pure strategy equilibrium characterization will require an additional restriction (though we can still relax continuity a bit).

The probability that A wins, given distinct platforms x and y and conditional on s and t, is

$$F_{s,t}((x+y)/2) \text{ if } x < y; \quad 1 - F_{s,t}((x+y)/2) \text{ if } y < x;$$

and $1/2$ if $x = y$. The probability that B wins is defined symmetrically and is just one minus the probability that A wins. This defines a Bayesian game between the candidates, where pure strategies for the candidates are vectors $X = (x_s)$ and $Y = (y_t)$ and a candidate's payoff is the probability of winning the election. Given pure strategies X and Y, candidate A's interim expected payoff conditional on signal s is

$$\begin{aligned}
&\Pi_A(X, Y|s) \\
&\quad = \sum_{t \in T : x_s < y_t} P(t|s) F_{s,t}((x_s + y_t)/2) + \sum_{t \in T : y_t < x_s} P(t|s)(1 - F_{s,t}((x_s + y_t)/2)) \\
&\qquad + (1/2) \sum_{t \in T : x_s = y_t} P(t|s),
\end{aligned}$$

where we use the usual notation for conditioning on signals. Candidate B's interim expected payoff is defined symmetrically and is just one minus A's expected payoff. A *pure strategy Bayesian equilibrium* is a pair (X, Y) such that

$$\Pi_A(X, Y|s) \quad \geq \quad \Pi_A(X', Y|s)$$

for all $s \in S$ and all X', and

$$\Pi_B(X, Y|t) \quad \geq \quad \Pi_B(X, Y'|t)$$

for all $t \in T$ and all Y'. This formalizes the idea that the candidates' campaign platforms are optimal given all information available to them.

At times, we will make use of several conditions. The first four define our "canonical model" of polling, where the candidates employ identical polling technologies and signals exhibit a natural ordering structure. The first three impose symmetry on the model.

(C1) $S = T$.

Thus, we may use the same set I, with elements i, j, etc., to index these sets. We then write simply $P(i, j)$ for $P(s_i, t_j)$, $F_{i,j}$ for F_{s_i,t_j}, and so on. Condition (C1) is not restrictive in itself, because realizations can be renamed and redundant ones may be added to S or T to achieve the required equality, but the next condition adds substance to this indexing.

(C2) For all $i, j \in I$, $P(i,j) = P(j,i)$ and $F_{i,j} = F_{j,i}$.

In words, we can identify the ith signal of candidate A with the ith signal of candidate B, in the sense that they are equally informative. The next condition is extremely weak: if one candidate receives one signal, then it must be possible that the other also receives it.

(C3) For all $i \in I$, $P(i,i) > 0$.

While our model allows for asymmetries between the candidates, it is natural to expect the candidates to have equal access to polling technology, in which case (C1)-(C3) are appropriate. In that case, we will be interested in equilibria in which candidates use their information similarly: we define a *symmetric* pure strategy Bayesian equilibrium as an equilibrium (X, Y) such that $x_i = y_i$ for all $i \in I$.

The last condition defining the canonical model imposes a natural ordering structure on the candidates' signals. Under (C1)-(C3), which the condition presumes, we write $m_{i,K}$ for the median of the distribution of μ conditional on the ith signal of one candidate and the other candidate receiving a signal indexed by an element of K. We say an ordering $<$ of I *preserves conditional equivalence* if, for all $i, j \in I$, $i \neq j$ if and only if i and j are not conditionally equivalent.

(C4) There is an ordering $<$ of I that preserves conditional equivalence and such that, for all $i, j \in I$ all $K \subseteq I$ with $P(K|i) > 0$ and $P(K|j) > 0$, $i < j$ implies $m_{i,K} < m_{j,K}$.

This condition is natural if "higher" signals are correlated with higher cut points, as we would expect. An implication is that, given $i < j$,

$$m_{i,\{i\}} < m_{j,\{i\}} = m_{i,\{j\}} < m_{j,\{j\}},$$

so that $i < j$ implies $m_{i,i} < m_{j,j}$. Under (C4), given signal i, we denote by $i + 1$ the next signal realization according to $<$.

Remark to us: Since I'm using $i < j$ to indicate that two signals aren't conditionally equivalent, I need to use notation different from the usual $<$.

Assuming (C1)-(C3), we say *symmetric information* holds if, for all $i, j \in I$ with $P(i,j) > 0$, we have

$$P(i,k)F_{i,k} \quad = \quad P(j,k)F_{j,k}$$

for all $k \in I$. Note the implication that $P(i,k) = P(j,k)$. Thus, if signal realizations i and j are consistent, then one candidate's information following i is exactly that of the other following j. Under symmetric information, I may be partitioned into equivalence classes as

$$I(i) \;=\; \{j \in I \mid P(i,j) > 0\},$$

and these have the property that, for all $j \in I(i)$ and all $k \in I$, $F_{i,k} = F_{j,k}$, i.e., i and j are conditionally equivalent. Thus, following consistent signal realizations, the distribution of μ is actually common knowledge. Moreover, we see that, for all $j \in I(i)$, $P(i,j) = 1/|I|^2$ and $P(j|i) = 1/|I|$. Note that symmetric information holds trivially if information is complete, i.e., $P(i|i) = 1$ for all $i \in I$. The electoral game with set $I(i)$ of signal realizations may be analyzed independently. That is, (X,Y) is a pure strategy Bayesian equilibrium if and only if, for each equivalence class I', the restricted strategies $(X_{I'}, Y_{I'}) = (x_i, y_j)_{i,j \in I'}$ are an equilibrium of the restricted game with payoffs

$$\begin{aligned}\Pi_A(X_{I'}, Y_{I'}|i) \;=\;& \frac{|\{j \in I' \mid x_i < y_j\}|}{|I'|} F((x_i + y_j)/2) \\ &+ \frac{|\{j \in I' \mid y_j < x_i\}|}{|I'|} (1 - F((x_i + y_j)/2)) \\ &+ \frac{|\{j \in I' \mid x_i = y_j\}|}{2|I'|}\end{aligned}$$

for all $i \in I'$, and likewise for candidate B.[1] We call this the *component game* of I'. While some of our results are general, some of them will apply to symmetric information games by decomposing them into component games that satisfy our assumptions.

Symmetric information essentially implies (C4). Let $F^1, F^2, \ldots, F^l$ be the distributions corresponding to the component games, and let m^k be the median of F^k. As long as these medians are distinct, i.e., $k \neq k'$ implies $m^k \neq m^{k'}$, then the elements of I can be ordered so that lower signal realizations correspond to component games with lower medians, fulfilling (C4).

It is instructive to define A's *ex ante* expected payoff, the candidate's

[1] In fact, because the posteriors above are independent of i, we may view every pure strategy Bayesian equilibrium of the restricted game as a mixed strategy equilibrium of a complete information game between the candidates.

expected payoff before receiving any signal, as

$$\Pi_A(X,Y) \;=\; \sum_{s\in S} P(s)\Pi_A(X,Y|s),$$

with candidate B's *ex ante* expected payoff, $\Pi_B(X,Y)$, defined similarly. Clearly,

$$\Pi_A(X,Y) + \Pi_B(X,Y) \;=\; 1$$

for all X and Y. Note that (X,Y) is a pure strategy Bayesian equilibrium if and only if

$$\Pi_A(X,Y') \;\geq\; \Pi_A(X,Y) \;\geq\; \Pi_A(X',Y)$$

for all X' and all Y'. Under $(C1)-(C3)$, the *ex ante* payoffs of the candidates are symmetric, in the sense that

$$\Pi_A(X,Y) \;=\; \Pi_B(Y,X)$$

for all X and Y. Thus, we may view the pure strategy Bayesian equilibria of electoral competition as equilibria of a two-player, symmetric, constant sum game. A consequence of the constant sum property is "interchangeability" of equilibria: if (X,Y) and (X',Y') are equilibria, then so are (X,Y') and (X',Y). With symmetry, if (X,Y) is an equilibrium, then so is (Y,X).

Example: The Downsian and Probabilistic Voting Models.

[I have to fill this in.]

Example: The Stacked Uniform Model. In the canonical model, we decompose the cut point μ as $\mu = \alpha + \beta$, where α and β are independent random variables, with α uniformly distributed on $[-a,a]$ and β a discrete random variable, with support on $b_1 < b_2 < \cdots < b_c$. Let $Q(k)$ denote the probability of b_k. We assume that the candidates share the same set of signal realizations, so (C1) is satisfied, and that their signals depend stochastically on the the realization of β, where $Q(i,j|k)$ is the probability, conditional on b_k, that the candidates receive the ith and jth signals. Then

$$P(i,j) \;=\; \sum_{k=1}^{c} Q(i,j|k)Q(k),$$

so (C3) is satisfied if, for all $i,j\in I$, there is some k such that $Q(i,j|k)>0$. Conditional on b_k, the distribution of μ is uniform on $[b_k-a, b_k+a]$, with piece-wise linear distribution

$$F_k(z) \quad = \quad \max\left\{\min\left\{0, \frac{x - b_k + a}{2a}\right\}, 1\right\}.$$

By Bayes rule, the probability of b_k conditional on signals i and j is

$$Q(k|i,j) \quad = \quad \frac{Q(i,j|k)Q(k)}{P(i,j)}$$

when well-defined, and the distribution of μ conditional on signals i and j is

$$F_{i,j}(z) \quad = \quad \sum_{k=1}^{c} Q(k|i,j)F_k(z).$$

Thus, (C2) is satisfied if $Q(i,j|k) = Q(j,i|k)$ for all i, j, and k. This special case of our model describes a situation in which there is a finite number of preliminary cut point locations, a finite number of possible signals about these preliminary locations generated by candidate polling, and a uniform disturbance to the preliminary cut points.

A nice special case is three possible preliminary cut point locations and three possible signals: $m = 3$, $b_1 = -b$, $b_2 = 0$, $b_3 = b$, and $I = \{-1, 0, 1\}$. If $a > b$, then we have "single-plateaued" conditional densities, as in Figure 1, with density over the center range equal to $1/2a$.

[Figure 1 here.]

If $b < a$, then the distribution $F_{i,j}$ is as in Figure 2.

[Figure 2 here.]

Thus, the shape of the posterior distributions of μ is that of "stacked" uniform distributions with shifted supports, with the general shape of the distribution depending on the size of the discrete component relative to the breadth of the support of the disturbance term. Assuming $a > b$, and given $z \in [b - a, a - b]$, note that

$$\begin{aligned} F_{i,j}(z) \quad &= \quad Q(-1|i,j)\left(\frac{z + b + a}{2a}\right) + Q(0|i,j)\left(\frac{z + a}{2a}\right) \\ &\quad\quad +Q(1|i,j)\left(\frac{z + a - b}{2a}\right) \\ &= \quad \frac{1}{2} + \frac{z + Q(-1|i,j) - Q(1|i,j)}{2a}. \end{aligned}$$

This gives us a useful expression for the conditional distributions over a relevant range of cut point realizations. In particular, if the conditional median $m_{i,j}$ satisfies $b - a \leq m_{i,j} \leq a - b$, then it implies

$$m_{i,j} \quad = \quad (Q(1|i,j) - Q(-1|i,j))b,$$

giving a simple expression for the conditional median.

For general numbers of preliminary cutpoint locations and signal realizations, our equilibrium characterization results are sharpest for the case in which a is large, specifically, $a \geq b_c - b_1$. This implies that $b_c - a \leq b_1 + a$ and gives us single-plateaued conditional densities, as in Figure 1. Moreover, this restriction implies that conditional medians lie within the center interval: for all $i \in I$, $b_c - a \leq m_{i,i} \leq b_1 + a$. To see this, note that a lower bound for $m_{i,i}$ is given by setting $Q(1|i,i) = 1$, in which case $F_{i,i}$ is uniform with median $b_1 \geq b_c - a$. Therefore, $b_c - a \leq m_{i,i}$. An upper bound for $m_{i,i}$ is given by setting $Q(c|i,i) = 1$, in which case $F_{i,i}$ is uniform with median $b_c \leq b_1 + a$, and therefore $m_{i,i} \leq b_1 + a$, as claimed. Note that, as a consequence,

$$\begin{aligned} F_{i,j}(z) &= \sum_{k=1}^{c} Q(k|i,j)\left(\frac{z+a-b_k}{2a}\right) \\ &= \frac{1}{2} + \frac{z - \sum_{k=1}^{c} Q(k|i,j)b_k}{2a} \end{aligned}$$

for all $z \in [b_c - a, b_1 + a]$. Therefore, when $a \geq b_c - b_1$, we have

$$m_{i,j} \quad = \quad \sum_{k=1}^{c} Q(k|i,j)b_k,$$

generalizing the above expression for the conditional median and illuminating the joint restriction on Q and the b_k's imposed by (C4). This allows us to write

$$F_{i,j}(z) \quad = \quad \frac{a - m_{i,j} + z}{2a}$$

for all $z \in [b_c - a, b_1 + a]$, which implies

$$F_{i,j}(z) - F_{i,j}(w) \quad = \quad \frac{z-w}{2a}$$

for all $z, w \in [b_c - a, b_1 + a]$. Note that, assuming (C4), $m_{h,i} < m_{j,k}$ implies that $F_{h,i}(z) > F_{j,k}(z)$ for all $z \in [b_c - a, b_1 + a]$, i.e., that $F_{j,k}$ stochastically dominates $F_{h,i}$ over the interval.

Example: Shape-Invariant Conditional Model. We assume that the shape of the distribution of the cut-point conditional on pairs of signal realizations is the same regardless of the signal realizations. This is equivalent to identifying the median conditional on the signal realizations as a scale parameter of a family of shape-invariant distributions. Formally, we assume that

$$F_{i,j}(z + m_{i,j}) = F_{k,l}(z + m_{k,l}),$$

for all $z \in \Re$ and all $i, j, k, l \in I$. As is well known, the assumption that conditional distributions form a family of shape-invariant distributions is pervasive in classical statistics, the most celebrated example of such a family of distributions being homoschedastic econometric models with normal errors.

3 Pure Strategy Equilibrium

The main result of this section is a full characterization of the pure strategy Bayesian equilibria of the canonical model: if such an equilibrium exists, then it is unique, and, after receiving a signal, a candidate locates at the median of the distribution of μ, conditional on *both* candidates receiving that signal. A corollary, with a natural restriction on conditional medians, is that candidates tend to take policy positions that are extreme relative to their expectations of the cut point, given their information. In other words, the position of a candidate is a biased estimator of the cut point: candidates who receive high signals will overestimate the cut point, while those who receive low signals will underestimate it. We also take up the question of existence of pure strategy equilibrium, giving sufficient conditions for the binary signal and multi-signal models.

The characterization result relies on the following lemma, which is proved in the appendix in a somewhat more general form.

Lemma 3 *In the canonical model, let (X, Y) be a pure strategy Bayesian equilibrium. If $x_i = y_j$ for some $i, j \in I$ with $P(i, j) > 0$, then $x_i = y_j = m_{i,j}$.*

For the intuition behind this lemma, suppose that candidates A and B locate at the same point following two signal realizations, i and j, and suppose for simplicity that these are the only realizations after which they locate there. (The proof uses condition (C4) to rule out the remaining cases.) Then, conditional on those realizations, each candidate expects to win the election with probability one half. We claim that, if the location of the candidates is not the median conditional on i and j, then the payoff of either candidate will be increased by a small move toward that median: if A deviates in this way, for example, then the candidate's expected payoff given other realizations for B will vary continuously, but the payoff given realization j will jump discontinuously above one half. Therefore, a small enough deviation will increase A's payoff, something that is impossible in equilibrium.

The proof of the characterization is then quick work.

Remark to us: This result holds as long as, for all i and j with $i < j$ (so not conditionally equivalent), $P(i,j)F_{i,j}$ is continuous. This captures symmetric information models as a special case.

Theorem 1 *In the canonical model, if (X,Y) is a pure strategy Bayesian equilibrium, then $x_i = y_i = m_{i,i}$ for all $i \in I$.*

Proof: First, consider a symmetric equilibrium (X,Y), where $x_i = y_i$ for all $i \in I$. By (C4) and Lemma 3, we have $x_i = y_i = m_{i,i}$, as required. Now suppose there is an asymmetric equilibrium (X,Y), where $x_i \neq m_{i,i}$ for some $i \in I$, and define the strategy $Y' = X$ for candidate B. Then, by symmetry and interchangeability, (X,Y') is a symmetric Bayesian equilibrium with $x_i \neq m_{i,i}$, a contradiction. ∎

When symmetric information holds in the canonical model, we must have $m_i = m_{i,i}$. To see this, note that

$$F_i(m_i) = \sum_{j \in I} P(j|i) F_{i,j}(m_i) \;=\; \frac{1}{2}$$

and that, for all j with positive probability in the above sum, $P(j|i) = 1/I(i)$ and $F_{i,j}(m_i) = F_{i,i}(m_{i,i})$. Therefore,

$$\begin{aligned}
\sum_{j \in I} P(j|i) F_{i,j}(m_i) &= \sum_{j \in I(i)} \frac{1}{I(i)} F_{i,j}(m_i) \\
&= \sum_{j \in I(i)} \frac{1}{I(i)} F_{i,i}(m_i) \\
&= F_{i,i}(m_i),
\end{aligned}$$

establishing the claim. Thus, Theorem 1 has the following familiar implication.

Corollary 1 *In the canonical model with symmetric information, if* (X, Y) *is a pure strategy Bayesian equilibrium, then* $x_i = y_i = m_i$ *for all* $i \in I$.

Theorem 1 and Corollary 1 give a necessary, not a sufficient, condition for existence of a pure strategy equilibrium. It is clear — and will follow from later results — that, with symmetric information, the strategies specified in Corollary 1 do indeed form an equilibrium. The next example shows, however, that pure strategy equilibria do not exist generally. In fact, the example begins with an arbitrary symmetric information model and demonstrates arbitrarily close models with no pure strategy equilibria, illustrating the fragility of the pure strategy equilibria in the above corollary. We return to the issue of robustness of equilibria in our analysis of mixed strategy equilibria. An implication of results there is that, even if pure strategy equilibria cease to exist in models close to symmetric information, mixed strategy equilibria do exist and will necessarily be "close" to the pure strategy equilibrium of the original model.

Example: Non-existence of Pure Strategy Equilibrium. In the canonical model, let $I = \{-1, 0, 1\}$, let $P(i, j) = 1/9$ for all $i, j \in I$, and let $F_{i,j} = F$ for all $i, j \in I$. Thus, symmetric information holds, and the unique pure strategy Bayesian equilibrium is (X, Y) such that $x_i = y_j = m$ for all $i, j \in I$, where m is the median of F. Now let $\{\hat{F}^k\}$ and $\{\tilde{F}^k\}$ be sequences of distributions converging weakly to F defined by $\hat{F}^k(z) = F(z + 1/k)$ and $\tilde{F}^k(z) = F(z - 1/k)$ for all $z \in \Re$. Denoting the medians of these distributions by $\hat{m}^k$ and $\tilde{m}^k$, we have $\hat{m}^k = m - 1/k$ and $\tilde{m}^k = m + 1/k$ for all k. Now perturb the original specification of the game so that $F_{-1,-1} = \hat{F}^k$ and $F_{1,1} = \tilde{F}^k$. Thus, by Theorem 1, if there is a pure strategy equilibrium, then it is (X, Y) defined by $x_{-1} = y_{-1} = \hat{m}^k$, $x_0 = y_0 = m$, and $x_1 = y_1 = \tilde{m}^k$. This is not an equilibrium, however, because candidate A can deviate profitably to X' defined as X but with $x'_{-1} = 0$. To see this, note that, in the perturbed game, we have

$$\Pi_A(X, Y|-1) \quad = \quad \frac{1}{3}\left(\frac{1}{2} + F((\hat{m}^k + m)/2) + F((\hat{m}^k + \tilde{m}^k)/2)\right)$$

and

$$\Pi_A(X', Y|-1) \quad = \quad \frac{1}{3}\left((1 - \hat{F}^k((\hat{m}^k + m)/2)) + \frac{1}{2} + F((m + \tilde{m}^k)/2)\right).$$

By construction, $F((\hat{m}^k+m)/2) < 1/2$, so the second term in $\Pi_A(X',Y|-1)$ is strictly greater than the second term in $\Pi_A(X,Y|-1)$. The sum of the remaining terms in $\Pi_A(X',Y|-1)$ is weakly greater than the sum in $\Pi_A(X,Y|-1)$ if

$$F((m+\tilde{m}^k)/2) \quad \geq \quad \hat{F}^k((\hat{m}^k+m)/2).$$

Using $\hat{m}^k = m - 1/k$, $\tilde{m}^k = m + 1/k$, and the definition of $\hat{F}^k$, this reduces to $F(m+1/2k) \geq F(m+1/2k)$. Thus, the deviation to X' increases candidate A's expected payoff, and we conclude that there does not exist a pure strategy Bayesian equilibrium for games arbitrarily close to the original.

In many situations where signals can be ordered, it is reasonable to suppose that some signals indicate lower cut points and other signals indicate higher ones. Then Theorem 1 has the following consequence for the equilibrium positions of the candidates.

Corollary 2 *In the canonical model, assume that there exists $\bar{n} \in I$ such that $i < \bar{n}$ implies $m_{i,i} < m_i$ and $\bar{n} < i$ implies $m_i < m_{i,i}$. If (X,Y) is a pure strategy Bayesian equilibrium, then $x_i < m_i$ for $i < \bar{n}$ and $m_i < x_i$ for $i > \bar{n}$.*

We next determine conditions under which the pure strategy equilibrium characterized in Theorem 1 exists. We consider two kinds of environments. The first, and simplest, is the canonical model with two possible signals, where our condition is rather weak. The second is the multi-signal case, where we impose additional restrictions. In both cases, pure strategy equilibrium existence holds essentially if, conditional on each signal i, the probability that the other candidate received that signal is high enough.

Remark to us: Unlike our result for the multi-signal model, our result for the binary signal model assumes continuous conditional distributions.

Theorem 2 *(Binary Signals) In the canonical model, let $I = \{-1,1\}$. Then a sufficient condition for the existence of the unique pure strategy Bayesian equilibrium, where candidates locate at $m_{i,i}$, following signal $i \in I$, is that*

$$\begin{aligned} P(1|1)f_{1,1}((z+m_{1,1})/2) \quad &\geq \quad P(-1|1)f_{1,-1}((z+m_{-1,-1})/2) \\ P(-1|-1)f_{-1,-1}((z+m_{-1,-1})/2) \quad &\geq \quad P(1|-1)f_{1,-1}((z+m_{1,1})/2) \end{aligned}$$

for all $z \in [m_{-1,-1}, m_{1,1}]$.

Proof: We will show that (X, Y) is an equilibrium, where $x_i = y_i = m_{i,i}$ for all $i = -1, 1$. Consider candidate A's best response problem, conditional on signal 1. If the candidate deviates to $x \in [m_{-1,-1}, m_{1,1}]$, then the change in the candidate's interim expected payoff is

$$\begin{aligned}
&P(-1|1)[F_{1,-1}((m_{1,1} + m_{-1,-1})/2) - F_{1,-1}((x + m_{-1,-1})/2)] \\
&\quad +P(1|1)[F_{1,1}((x + m_{1,1})/2) - (1/2)] \\
&= \int_x^{m_{1,1}} [P(-1|1)f_{1,-1}((z + m_{-1,-1})/2) - P(1|1)f_{1,1}((z + m_{1,1})/2)]\, dz \\
&\leq 0.
\end{aligned}$$

Thus, the deviation does not increase the candidate's expected payoff. It is easily verified that deviations $x < m_{-1,-1}$ and $x > m_{1,1}$ are also unprofitable. A similar argument holds for signal $i = -1$, and a symmetric argument for candidate B establishes that (X, Y) is an equilibrium. ∎

The sufficient condition of Theorem 2 is reasonably weak. If signals are not negatively correlated, so that $P(1|1) \geq P(-1|1)$ and $P(-1|-1) \geq P(1|-1)$, then it is sufficient that

$$\begin{aligned}
f_{1,1}((z + m_{1,1})/2) &\geq f_{1,-1}((z + m_{-1,-1})/2) \\
f_{-1,-1}((z + m_{-1,-1})/2) &\geq f_{1,-1}((z + m_{1,1})/2)
\end{aligned}$$

for all $z \in [m_{-1,-1}, m_{1,1}]$. The first of these inequalities can be thought of as comparing $f_{1,1}$, shifted to the left by $m_{1,1}/2$, and $f_{1,-1}$, shifted to the left by $m_{-1,-1}/2$. The nature of the comparison may be clearer if it is rewritten as:

$$f_{1,1}\left(\frac{m_{1,1} - m_{-1,-1}}{2} + z\right) \geq f_{1,-1}(z)$$

for all $z \in [(3m_{-1,-1} - m_{1,1})/2, (m_{1,1} + m_{-1,-1})/2]$. Thus, the condition is that $f_{1,1}$, shifted to the left by $(m_{1,1} - m_{-1,-1})/2 > 0$, is greater than or equal to $f_{1,-1}$ over a given range. This is clearly true for the stacked uniform model with $a > b_c - b_1$, where conditional densities are all equal to $1/2a$ over the relevant range. The condition holds in the version of the shape-invariant conditional model in which $f_{1,-1}$ is the translation of $f_{1,1}$ with median $(m_{1,1} + m_{-1,-1})/2$. More generally, the condition holds if $f_{1,-1}$ has median near $(m_{1,1} + m_{-1,-1})/2$ and is somewhat more dispersed than $f_{1,1}$, as we would expect if identical signals decrease the variance of the cutpoint distribution and opposing signals offset each other.

To simplify our sufficiency argument for existence in the multi-signal case, we provide separate conditions on the distribution of the cut point conditional on signal realizations and on the priors over signal pairs. Our first restriction on conditional distributions is a regularity condition that reinforces the symmetry already present in the canonical model.

(C5) For all $i, j \in I$ with $P(i,j) > 0$, $m_{i,j} = (m_{i,i} + m_{j,j})/2$.

This is trivially satisfied under symmetric information. It is satisfied in the stacked uniform model, assuming $a \geq b_c - b_1$, if

$$Q(k|i,j) \quad = \quad \frac{Q(k|i,i) + Q(k|j,j)}{2},$$

or equivalently,

$$\frac{Q(i,j|k)}{P(i,j)} \quad = \quad \frac{1}{2}\left(\frac{Q(i,i|k)}{P(i,i)} + \frac{Q(j,j|k)}{P(j,j)}\right),$$

for each disturbance k.

We also consider a stochastic dominance-like restriction on the conditional distributions. Note that the property $m_{i,i} \leq m_{j,j} \leq m_{k,k}$, below, would also follow from (C4).

(C6) For all $i, j, k \in I$ with $i \leq j \leq k$, we have $m_{i,i} \leq m_{j,j} \leq m_{k,k}$, and

$$F_{i,j}((m_{i,i} + z)/2) \quad \geq \quad F_{j,k}((m_{k,k} + z)/2)$$

for all $z \in [m_{i,i}, m_{k,k}]$.

The meaning of the inequality in the latter condition may be more transparent if rewritten as:

$$F_{i,j}(z) \quad \geq \quad F_{j,k}\left(\frac{m_{k,k} - m_{i,i}}{2} + z\right)$$

for all $z \in [m_{i,i}/2, (m_{i,i} + m_{k,k})/2]$. Thus, the distribution conditional on signals j and k must dominate, when shifted to the left by the amount $(m_{k,k} - m_{i,i})/2 > 0$, the distribution conditional on signals i and j. Clearly, (C6) is stronger than stochastic dominance in that $F_{j,k}$ is shifted to the left, but it is weaker in that the inequality must hold only over a given range.

Again, the condition is trivially satisfied under symmetric information, and, as we discuss below, the condition is satisfied in the stacked uniform model with sufficiently large enough support for the noise term.

Our restriction on priors over signals formalizes the idea that, conditional on a candidate's own signal, the probability that the other candidate received the same signal is sufficiently high. In fact, the condition is somewhat weaker than that, because it only restricts "net" probabilities. The condition is stated for the canonical model.

(C7) For all $i \in I$,

$$\begin{aligned} \sum_{j \in I: j \leq i} P(j|i) &\geq \sum_{j \in I: j > i} P(j|i) \\ \sum_{j \in I: j < i} P(j|i) &\leq \sum_{j \in I: j \geq i} P(j|i). \end{aligned}$$

In words, for any signal i, it must be that i is a "median" of the distribution $P(\cdot|i)$ on I.[2] The condition is trivially satisfied under symmetric information, because, given any $i \in I$, the only signals j such that $P(j|i) > 0$ are conditionally equivalent to i. A stronger condition is that $P(i|i) \geq 1/2$ for all $i \in I$. It is clear that (C7) most restrictive for the "extremal" signals, for which $P(i|i) \geq 1/2$ is necessary, and that its restrictiveness depends on the total number of possible signals. In the binary signal model, for example, it is satisfied whenever signals are not negatively correlated. In the multi-signal model, it is clearly satisfied in the case of perfect correlation, where $P(i|i) = 1$ for all $i \in I$. Perfect correlation can be relaxed, of course, but the extent to which that is possible depends on the number of possible signals and the conditional distributions.

Theorem 3 *(Multiple Signals) In the canonical model, (C5)-(C7) are sufficient for the existence of the unique pure strategy Bayesian equilibrium, where candidates locate at $m_{i,i}$ following signal $i \in I$.*

Proof: We show that (X, Y) is an equilibrium, where $x_i = y_i = m_{i,i}$ for all $i \in I$. Without loss of generality, we focus on candidate B's best response problem after receiving signal j. Consider a deviation to strategy Y'. There

[2] Equivalently, we may write $\sum_{j:j \leq i} P(j|i) \geq 1/2$ and $\sum_{j:i \leq j} P(j|i) \geq 1/2$.

are two cases: $y'_j < m_{j,j}$ and $m_{j,j} < y'_j$. In the first case, define

$$\begin{aligned}\mathcal{G} &= \{i \in I : m_{i,i} \leq y'_j\} \\ \mathcal{L} &= \{k \in I : m_{j,j} \leq m_{k,k}\}.\end{aligned}$$

Note that, for all $i \in I \setminus (\mathcal{G} \cup \mathcal{L})$ such that $P(i|j) > 0$, we have $y'_j < m_{i,i} < m_{j,j}$. It follows that, for such i,

$$F_{i,j}((y'_j + m_{i,i})/2) - [1 - F_{i,j}((m_{i,i} + m_{j,j})/2)] \leq 0,$$

where we use (C5) to deduce $F_{i,j}((y'_j + m_{i,i})/2) \leq 1/2$ and $F_{i,j}((m_{i,i} + m_{j,j})/2) = 1/2$. Intuitively, B's gains from deviating when A receives signal s_i, with $i \in I \setminus (\mathcal{G} \cup \mathcal{L})$, are non-positive. Therefore, the change in B's interim expected payoff satisfies

$$\begin{aligned}&\Pi_B(X, Y'|j) - \Pi_B(X, Y|j) \\ &\leq \sum_{i \in \mathcal{G}} P(i|j)[1 - F_{i,j}((m_{i,i} + y'_j)/2) - 1 + F_{i,j}((m_{i,i} + m_{j,j})/2)] \\ &\quad + \sum_{k \in \mathcal{L}} P(k|j)[F_{j,k}((y'_j + m_{k,k})/2) - F_{j,k}((m_{j,j} + m_{k,k})/2)] \\ &= \sum_{i \in \mathcal{G}} P(i|j)[(1/2) - F_{i,j}((m_{i,i} + y'_j)/2)] \\ &\quad + \sum_{k \in \mathcal{L}} P(k|j)[F_{j,k}((y'_j + m_{k,k})/2) - (1/2)],\end{aligned}$$

which is non-positive if

$$\begin{aligned}&\sum_{i \in \mathcal{G}} P(i|j)[(1/2) - F_{i,j}((m_{i,i} + y'_j)/2)] \\ &\quad \leq \sum_{k \in \mathcal{L}} P(k|j)[(1/2) - F_{j,k}((y'_j + m_{k,k})/2)].\end{aligned}$$

Letting i^* minimize $F_{i,j}((m_{i,i} + y'_j)/2)$ over $\mathcal{G}$, and letting k^* maximize $F_{j,k}((m_{i,i} + y'_j)/2)$ over $\mathcal{L}$, the latter inequality holds if

$$\begin{aligned}&[(1/2) - F_{i^*,j}((m_{i^*,i^*} + y'_j)/2)] \sum_{i \in \mathcal{G}} P(i|j) \\ &\quad \leq [(1/2) - F_{j,k^*}((y'_j + m_{k^*,k^*})/2)] \sum_{k \in \mathcal{L}} P(k|j),\end{aligned}$$

which follows from conditions (C5)-(C7). Thus, deviating to $y'_j < m_{j,j}$ is not profitable for candidate B. A symmetric argument applies to the case $m_{j,j} < y'_j$. ∎

The following corollary for symmetric information models is immediate. Though it establishes a unique pure strategy Bayesian equilibrium, we will see that this uniqueness result extends to mixed strategies as well.

Corollary 3 *In the canonical model with symmetric information, the strategy pair in which candidates locate at m_i following signal $i \in I$ is the unique pure strategy Bayesian equilibrium.*

We have shown that our restriction, (C7), on the signals' conditional correlation, together with the regularity conditions (C5) and (C6), is sufficient for existence of pure strategy equilibrium. We now show that (C7), under (C5) and a stronger version of (C6), is actually necessary for equilibrium existence. In the next condition, we essentially strengthen (C6) by stating it with equality.

(C6′) For all $i, j, k \in I$ with $i \leq j \leq k$, we have

$$F_{i,j}((m_{i,i} + z)/2) = F_{j,k}((m_{k,k} + z)/2)$$

for all $z \in [m_{i,i}, m_{k,k}]$.

Theorem 4 *In the canonical model, under (C5) and (C6′), and (C7) is necessary for the existence of the unique pure strategy Bayesian equilibrium, where candidates locate at $m_{i,i}$, following signal $i \in I$.*

We need a real proof of this theorem.

It is immediate that, under (C5), condition (C6′) is satisfied in the shape-invariant conditional model. To see this, take any signals $i, j,$, and note that, by (C5), we have

$$\begin{aligned} \frac{m_{i,i} + z}{2} &= m_{i,j} + \frac{z - m_{j,j}}{2} \\ \frac{m_{k,k} + z}{2} &= m_{j,k} + \frac{z - m_{j,j}}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} F_{i,j}((m_{i,i} + z)/2) &= F_{i,j}(m_{i,j} + (z - m_{j,j})/2) \\ = \; F_{j,k}(m_{j,k} + (z - m_{j,j})/2) &= F_{j,k}((m_{k,k} + z)/2), \end{aligned}$$

where the second equality above uses shape-invariance. At the same time, under (C5), condition (C6′) holds also in the stacked uniform model when a sufficiently high, i.e., $a \geq b_c - b_1$. Take i, j, k as in the condition and $z \in [m_{i,i}, m_{k,k}]$. Then, because $F_{i,j}$ is linear with slope $1/2a$ over $[m_{i,i}, m_{k,k}]$, we have

$$\begin{aligned} F_{i,j}((m_{i,i}+z)/2) &= F_{i,j}(m_{i,j}+(z-m_{j,j})/2) \\ &= \frac{1}{2}+\frac{z-m_{j,j}}{4a}, \end{aligned}$$

and similarly for $F_{j,k}((m_{k,k}+z)/2)$. This establishes the claimed equality for the stacked uniform model.

4 Mixed Strategies

Our results for pure strategy equilibria suggest that, when there are many signals, existence of pure strategy equilibrium becomes difficult to maintain. We model mixed strategies in the electoral game as follows. We allow candidate A to randomize over campaign platforms following signal s according to a distribution G_s. A mixed strategy for A is then a vector $G = (G_s)$ of such distributions. Likewise, a mixed strategy for B is a vector $H = (H_t)$ of distributions conditional on signals. We may use the shorthand $G_s(z)^-$ or $H_t(z)^-$ for the lefthand limits of these distributions, e.g., $G_s(z)^- = \lim_{w \uparrow z} G_s(w)$. Accordingly, G_s has a mass point at x if $G_s(x) - G_s(x)^- > 0$.

To extend our definition of interim expected payoffs, define the probability that A wins using platform x following signal s, while B uses y following t, as

$$\pi_A(x,y|s,t) = \begin{cases} F_{s,t}((x+y)/2) & \text{if } x<y \\ 1-F_{s,t}((x+y)/2) & \text{if } y<x \\ 1/2 & \text{if } x=y, \end{cases}$$

and define $\pi_B(\cdot|s,t) = 1 - \pi_A(\cdot|s,t)$. Then, given mixed strategies (G, H), candidate A's interim expected payoff conditional on signal s is

$$\Pi_A(G,H|s) = \sum_{t \in T} P(t|s) \int \pi_A(x,y|s,t) G_s(dx) H_t(dy),$$

and B's interim payoff $\Pi_B(G, H|t)$ is defined similarly. Abusing notation slightly, we may write $\Pi_A(X, H|s)$ for the expected payoff when A uses the degenerate mixed strategy with $G_s(x_s)-G_s(x_s)^- = 1$ for all $s \in S$. Likewise for B. A *mixed strategy Bayesian equilibrium* is a pair (G, H) such that

$$\Pi_A(G, H|s) \quad \geq \quad \Pi_A(G', H|s)$$

for all $s \in S$ and all G', and

$$\Pi_B(G, H|t) \quad \geq \quad \Pi_B(G, H'|t)$$

for all $t \in T$ and all H'. Note that, in equilibrium, if x_i is a continuity point of $\Pi_A(X, H|i)$ in the support of G_i, then the expected payoff from x_i must equal $\Pi_A(G, H|i)$. Candidate A must therefore be indifferent over all such points.

As with pure strategies, we can define *ex ante* expected payoffs as

$$\begin{aligned} \Pi_A(G, H) &= \sum_{s \in S} P(s)\Pi_A(G, H|s) \\ \Pi_B(G, H) &= \sum_{t \in T} P(t)\Pi_B(G, H|t). \end{aligned}$$

Thus, we may again view mixed strategy Bayesian equilibria of the electoral game as equilibria of a two-player, constant-sum game. In the canonical model, the game is symmetric and we define a *symmetric* mixed strategy Bayesian equilibrium as a pair (G, H) of strategies such that $G = H$.

The next theorem establishes, for the general model, existence of mixed strategy equilibria in which candidates use mixed strategies with support bounded as follows. Define $\overline{m} = \max\{m_{s,t} : s \in S, t \in T\}$ and $\underline{m} = \min\{m_{s,t} : s \in S, t \in T\}$. Let $M = [\underline{m}, \overline{m}]$ denote the interval defined by the conditional medians. We say (G, H) *has support in* M if the candidates put probability one on this set following all signal realizations: for all $s \in S$, $G_s(\overline{m}) - G_s(\underline{m})^- = 1$, and likewise for all $t \in T$.

Theorem 5 *There exists a mixed strategy Bayesian equilibrium with support in M. Under (C1) and (C2), there exists a symmetric mixed strategy Bayesian equilibrium with support in M.*

Proof: We use the existence theorem of Dasgupta and Maskin (1986) for multi-player games with one-dimensional strategy spaces. To apply this result, we view the electoral game as a $|S| + |T|$-player game, in which each type (corresponding to different signal realizations) of each candidate is a separate player. Player s (or t) has strategy space $M \subseteq \Re$, a compact set, with pure strategies denoted x_s (or y_t). Then $(X, Y) = (x_s, y_t)_{s \in S, t \in T}$ is a pure strategy *profile,* one for each type. We use (X_{-s}, Y) to denote the result of deleting x_s from (X, Y). The payoff function of player $s \in S$ is

$$U_s(X, Y) \quad = \quad P(s)\Pi_A(X, Y|s),$$

and the payoff function of player $t \in T$ is

$$U_t(X, Y) \quad = \quad P(t)\Pi_B(X, Y|t).$$

The space of mixed strategies for each player type s (or t) is $\mathcal{M}$, the Borel probability measures on M, with mixed strategies denoted G_s (or H_t). Then (G, H) is a mixed strategy *profile,* one for each type. Note that

$$\sum_{s \in S} U_s(X, Y) + \sum_{t \in T} U_t(X, Y) \quad = \quad 1$$

for all X and Y, so the total payoff is trivially upper semi-continuous. Furthermore, payoffs are between zero and one, so they are bounded. Note that U_s is discontinuous at (X, Y) only if $x_s = y_t$ for some $t \in T$. Therefore, the discontinuity points of U_s lie in a set that can be written as $A^*(s)$, as in Dasgupta and Maskin's equation (2). Likewise for the discontinuity points of U_t. It remains to be shown that U_s (likewise U_t) is weakly lower semi-continuous in x_s, i.e., for all $x_s \in M$, there exists $\lambda \in [0, 1]$ such that, for all (X_{-s}, Y),

$$U_s(X, Y) \quad \leq \quad \lambda \liminf_{z \downarrow x_s} U_s(z, X_{-s}, Y) + (1 - \lambda) \liminf_{z \uparrow x_s} U_s(z, X_{-s}, Y).$$

In fact, it is straightforward to verify that this condition holds with equality for $\lambda = 1/2$. Let $T^+ = \{t \in T : x_s < y_t\}$, let $T^- = \{t \in T : y_t < x_s\}$, and let $T^0 = \{t \in T : x_s = y_t\}$. Since

$$\begin{aligned} U_s(X, Y) \quad = \quad & \sum_{t \in T^-} P(s, t)(1 - F_{s,t}((x_s + y_t)/2)) + \sum_{t \in T^0} P(s, t)(1/2) \\ & + \sum_{t \in T^+} P(s, t)F_{s,t}((x_s + y_t)/2), \end{aligned}$$

it follows that

$$\begin{aligned} & \liminf_{z \uparrow x_s} U_s(z, X_{-s}, Y) \\ & \quad = \quad \sum_{t \in T^-} P(s, t)(1 - F_{s,t}((x_s + y_t)/2)) + \sum_{t \in T^0 \cup T^+} P(s, t)F_{s,t}((x_s + y_t)/2) \end{aligned}$$

and

$$\liminf_{z \downarrow x_s} U_s(z, X_{-s}, Y)$$
$$= \sum_{t \in T^- \cup T^0} P(s,t)(1 - F_{s,t}((x_s + y_t)/2)) + \sum_{t \in T^+} P(s,t) F_{s,t}((x_s + y_t)/2).$$

The claim

$$U_s(X, Y) \;=\; (1/2) \liminf_{z \downarrow x_s} U_s(z, X_{-s}, Y) + (1/2) \liminf_{z \uparrow x_s} U_s(z, X_{-s}, Y)$$

then follows immediately. The condition is verified for U_t using exactly the same arguments. By Dasgupta and Maskin's (1986) Theorem 5, there exists a mixed strategy equilibrium of the multi-player game, and, therefore, of the electoral game when strategies are restricted to M. To see that, following a signal s, candidate A has no profitable deviations outside M, take any $x > \overline{m}$. Note that, for all $t \in T$ and all $y \in M$, we have $\pi_A(\overline{m}, y|s,t) \geq \pi_A(x, y|s,t)$. Letting G' be any deviation such that G'_s puts probability one on $x_s > \overline{m}$, and letting G'' put probability one on $\overline{m}$ instead, we have

$$\Pi_A(G', H|s) \leq \Pi_A(G'', H|s) \leq \Pi_A(G, H|s).$$

A similar argument applies when $x < \underline{m}$, yielding the claim. Adding (C1) and (C2), the electoral game can be viewed as a two-player, symmetric zero-sum game. Therefore, by existence of equilibrium and by interchangeability, there exists a symmetric mixed strategy equilibrium. ∎

We next study the continuity properties of the mixed strategy equilibrium correspondence as we vary the parameters of the model, namely, the candidates' marginal prior on $S \times T$ and the conditional distributions of μ. To state our continuity result, we index these parameters by a metric space Γ, with typical element γ representing a specification of the model. Let the marginal probability of (s,t) in game γ be $P^\gamma(s,t)$, and let the distribution of μ conditional on s and t in γ be $F^\gamma_{s,t}$. We assume that this indexing is continuous. That is, for each $s \in S$ and $t \in T$, if $\gamma_n \to \gamma$, then $P^{\gamma_n}(s,t) \to P^\gamma(s,t)$ and $F^{\gamma_n}_{s,t} \to F^\gamma_{s,t}$ weakly. Let $M(\gamma)$ denote the interval defined by the extreme conditional medians in game γ, and note that, by the assumption of continuous indexing, the correspondence $M\colon \Gamma \rightrightarrows \Re$ so-defined is continuous.

Theorem 5 establishes the existence of a mixed strategy equilibrium for all $\gamma \in \Gamma$. Therefore, since the electoral game is constant sum, the *ex ante* expected payoff of a candidate in game γ is the same for in all mixed strategy equilibria. These payoffs, or "values," are denoted $v_A(\gamma)$ and $v_B(\gamma)$. Furthermore, each candidate has a mixed strategy, called an "optimal strategy," that guarantees the candidate's value, no matter which strategies are used by the opponent. If (C1) and (C2) hold for the game γ, then the game is symmetric, so these values are identical and constant at one half. The next theorem establishes that the values of the candidates vary continuously in the parameters of the game, even when asymmetries are allowed.

Theorem 6 *The mapping $v_A \colon \Gamma \to \Re$ is continuous.*

Proof: We prove lower semi-continuity of v_A. A symmetric argument proves lower semi-continuity of $v_B = 1 - v_A$, which gives us upper semi-continuity of v_A, as well. Let $\gamma_n \to \gamma$, and suppose $v_A(\gamma) > \liminf v_A(\gamma_n)$. Let Π_A^n denote A's *ex ante* expected payoff function corresponding to γ_n, and let Π_A denote the *ex ante* payoffs corresponding to γ. Let M^n denote the interval $M(\gamma^n)$, let $M = M(\gamma)$, and let $\hat{M}$ be any compact set containing M in its interior. By continuity, therefore, we have $M^n \subseteq \hat{M}$ for high enough n. For each n, let (G^n, H^n) be an equilibrium with support in M^n for the electoral game indexed by γ_n, so $\Pi_A^n(G^n, H^n) = v_A(\gamma_n)$ and $\Pi_B^n(G^n, H^n) = v_B(\gamma_n)$. By compactness of $\hat{M}$, we may go to a weakly convergent subsequence of $\{(G^n, H^n)\}$, also indexed by n, with limit (G, H). Going to a further subsequence if necessary, we may assume $\{v_A(\gamma_n)\}$ converges to limit $v < v_A(\gamma)$. Let (G^*, H^*) be an equilibrium of the electoral game indexed by γ, so G^* is an optimal strategy for A, which guarantees a payoff of at least $v_A(\gamma)$ in game γ. Thus, $\Pi_A(G^*, H) \geq v_A(\gamma)$. In particular, there exists X^* such that

$$\Pi_A(X^*, H) \;\geq\; v_A(\gamma) \;>\; v.$$

We claim that, as a consequence, there exists X' such that

$$\Pi_A^n(X', H^n) \quad > \quad \frac{\Pi_A(X^*, H) + v}{2}$$

for high enough n. But this, with $v(\gamma_n) \to v$, contradicts the assumption that G^n is a best response to H^n for candidate A. We establish the claim in three steps.

Step 1. By Lemma 1 (in the appendix), for every $s \in S$, we have either

$$\begin{aligned}\sum_{t\in T} & P(t|s)[H_t(x_s^*) - H_t(x_s^*)^-][F_{s,t}(x_s^*)] \\ & \geq \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(x_s^*) - H_t(x_s^*)^-] \end{aligned} \tag{1}$$

or

$$\begin{aligned}\sum_{t\in T} & P(t|s)[H_t(x_s^*) - H_t(x_s^*)^-][1 - F_{s,t}(x_s^*)] \\ & \geq \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(x_s^*) - H_t(x_s^*)^-]. \end{aligned} \tag{2}$$

Let S^- be the set of $s \in S$ such that (1) holds, and let S^+ be the set of $s \in S \setminus S^-$ such that (2) holds. For $s \in S^-$, let $\{x_s^k\}$ be a sequence increasing to x_s^*, and for $s \in S^+$, let $\{x_s^k\}$ be a sequence decreasing to x_s^*. In addition, we choose each x_s^k to be a continuity point of H_t for all $t \in T$. (This is possible because T is finite and each H_t has a countable number of discontinuity points.) Thus, $H_t(x_s^k) - H_t(x_s^k)^- = 0$ for all $t \in T$. For each m, define the strategy $X^k = (x_s^k)$ for candidate A.

Step 2. We now argue that X^k satisfies $\liminf \Pi_A(X^k, H) \geq \Pi_A(X^*, H)$. For each $t \in T$, let λ_t denote the measure generated by the distribution H_t, let μ_t denote the degenerate measure with probability $H_t(x_s^*) - H_t(x_s^*)^-$ on each x_s^*, and let $\nu_t = \lambda_t - \mu_t$. Let

$$\pi_{s,t}^*(z) \quad = \quad \pi_A(x_s^*, z|s, t)$$

denote A's probability of winning using x_s^*, conditional on signal s, when B receives signal t and chooses platform z, and let

$$\pi_{s,t}^k(z) \quad = \quad \pi_A(x_s^k, z|s, t)$$

denote A's probability of winning using x_s^k, conditional on signal s, when B

receives signal t and chooses platform z. Note that

$$\begin{aligned}
&\Pi_A(X^k,H)-\Pi_A(X^*,H)\\
&\quad = \sum_{s\in S} P(s)\sum_{t\in T} P(t|s)\int [\pi^k_{s,t}(z)-\pi^*_{s,t}(z)]\,\lambda_t(dz)\\
&\quad = \sum_{s\in S^-}\sum_{t\in T} P(s,t)[H_t(x^*_s)-H_t(x^*_s)^-][F_{s,t}((x^*_s+x^k_s)/2)-(1/2)]\\
&\qquad + \sum_{s\in S^+}\sum_{t\in T} P(s,t)[H_t(x^*_s)-H_t(x^*_s)^-][1-F_{s,t}((x^*_s+x^k_s)/2)-(1/2)]\\
&\qquad + \sum_{s\in S}\sum_{t\in T} P(s,t)\int [\pi^k_{s,t}(z)-\pi^*_{s,t}(z)]\,\nu_t(dz).
\end{aligned}$$

Since $\pi^k_{s,t}-\pi^*_{s,t}\to 0$ almost everywhere (ν_t), the corresponding integral terms above converge to zero. Thus, by our construction of X^k,

$$\lim\inf_{m\to\infty} \Pi_A(X^k,H)\ \geq\ \Pi_A(X^*,H)\ >\ v,$$

as desired.

Step 3. Choose m such that $\Pi_A(X^k,H)>(\Pi_A(X^*,H)+v)/2$ and set $X'=X^k$. To prove our claim that $\Pi^n_A(X',H^n)>v$ for high enough n, define the functions

$$\begin{aligned}
\phi'_{s,t}(z) &= \pi_A(x'_s,z|s,t)\\
\phi^n_{s,t}(z) &= \pi^{\gamma_n}_A(x'_s,z|s,t).
\end{aligned}$$

Note that

$$\Pi_A(X',H) \;=\; \sum_{s\in S}\sum_{t\in T} P(s,t)\int \phi'_{s,t}(z)\,H_t(dz),$$

and, letting $P^n=P^{\gamma_n}$,

$$\Pi^n_A(X',H^n) \;=\; \sum_{s\in S}\sum_{t\in T} P^n(s,t)\int \phi^n_{s,t}(z)\,H^n_t(dz).$$

Thus, it suffices to show that

$$\int \phi^n_{s,t}(z)\,dH^n_t\to\int \phi'_{s,t}(z)\,dH_t$$

for each $s\in S$ and $t\in T$. To prove this, fix $\epsilon>0$. Because $x'_s=x^k_s$ is not a mass point of H_t, we may specify an interval $Z=[\underline{z},\overline{z}]$ with $x'_s\in(\underline{z},\overline{z})$ such that $H_t(\overline{z}) - H_t(\underline{z})^- < \epsilon/4$. By weak convergence, we have $H_t^n(\overline{z}) - H_t^n(\underline{z})^- < \epsilon/2$ for sufficiently high n. Furthermore, $\{\phi_{s,t}^n\}$ is a sequence of functions that are non-decreasing on $[\underline{m}, \underline{z}]$ and converge pointwise to $\phi'_{s,t}$ on this interval, so they converge uniformly to $\phi'_{s,t}$ on the interval. Similarly, each $\phi_{s,t}^n$ is non-increasing on $[\overline{z}, \overline{m}]$, so the functions converge uniformly to $\phi'_{s,t}$ on this interval. Choosing n high enough that $|\phi_{s,t}^n(z) - \phi'_{s,t}(z)| < \epsilon/2$ for all $z \in [\underline{m}, \underline{z}] \cup [\overline{z}, \overline{m}]$, we have

$$\left| \int \phi_{s,t}^n(z)\, dH_t^n - \int \phi'_{s,t}(z)\, dH_t \right| \quad < \quad \epsilon,$$

as required. ∎

Letting $\mathcal{B}$ denote the Borel probability measures over X, define the mixed strategy equilibrium correspondence $E \colon \Gamma \rightrightarrows \mathcal{B}^{S \cup T}$ so that $E(\gamma)$ consists of all mixed strategy equilibrium pairs (G, H). We have seen that this correspondence has non-empty values. The next result establishes an important continuity property of the equilibrium correspondence.

Theorem 7 *The correspondence $E \colon \Gamma \rightrightarrows \mathcal{B}^{S \cup T}$ has closed graph.*

Proof: Let $\gamma_n \to \gamma$, let $(G^n, H^n) \in E(\gamma_n)$ for each n, and suppose that $(G^n, H^n) \to (G, H)$. If $(G, H) \notin E(\gamma)$, then, using the notation from the proof of Theorem 6, one candidate, say A, has a pure strategy X such that

$$\Pi_A(X, H) \quad > \quad v_A(\gamma). \tag{3}$$

But then, as in the proof of Theorem 6, we can find X' satisfying (3) such that no x'_s is a mass point of any H_t, and then we can show that

$$\Pi_A^n(X', H^n) \quad > \quad \frac{\Pi_A(X, H) + v_A(\gamma)}{2}$$

for high enough n. But $v_A(\gamma_n) \to v_A(\gamma)$ by Theorem 6, so it follows that

$$\Pi_A^n(X', H^n) \quad > \quad v_A(\gamma_n)$$

for high enough n, contradicting the assumption that G^n is a best response to H^n for A in the electoral game indexed by γ_n. ∎

To this point, our results have established existence of mixed strategy equilibria with support in M, but we have not provided a necessary condition to bound the supports of equilibrium mixed strategies. Our next result does just that for the canonical model, showing that all mixed strategy equilibria have support in M.

Theorem 8 *In the canonical model, if (G,H) is a mixed strategy Bayesian equilibrium, then it has support on M.*

Proof: Let (G,H) be a mixed strategy Bayesian equilibrium, let $\underline{x}_i = \sup\{x \in \Re : G_i(x) = 0\}$ be the lower bound of the support of G_i for each $i \in I$, and let $\underline{x} = \min_{i \in I} \overline{x}_i$ be the minimum of these lower bounds. Suppose that $\underline{x} < \underline{m}$, and take i such that $\underline{x}_i = \underline{x}$. By symmetry and interchangeability, (G,G) is also an equilibrium, so we may assume that $H = G$. Define the pure strategy X^n as follows. In case G_i puts positive probability on $\underline{x}$, i.e., $G_i(\underline{x}) - G_i(\underline{x})^- > 0$, then let $x_i^n = \underline{x}$ for all n. Otherwise, let $\{x^n\}$ be a sequence decreasing to $\underline{x}$ such that each x^n is in the support of G_i. Furthermore, choose x^n so that $\Pi_A(X^n, H|i) = \Pi_A(G,H|i)$ for all n. To see that this is possible, set x^1 arbitrarily and note that we can choose any continuity point of A's expected payoff function in the support of G_i and in the interval $[\underline{x}, (\underline{x} + x^{n-1})/2]$ to satisfy the desired condition. Since there is at most a countable number of discontinuity points of A's payoff function, such a point can be found unless the support of G_i in $[\underline{x}, (\underline{x} + x^{n-1})/2]$ is countable. In that case, however, any point in the support of G_i in this interval will satisfy the desired condition, and there will be at least one such point by the assumption that $\underline{x}_i = \underline{x}$. In any case, we have $\Pi_A(X^n, H|i) = \Pi_A(G,H|i)$ for all n and $\lim_{n\to\infty} G_i(x_i^n)^- = 0$. Now consider a pure strategy X' satisfying $x_i' = \underline{m}$, and note that

$$
\begin{aligned}
&\Pi_A(X', H|i) - \pi_A(X^n, H|i) \\
&= \sum_{j \in I} P(j|i) \Bigg[\int_{[\underline{x}, x_i^n)} [F_{i,j}((x_i^n + z)/2) - F_{i,j}((x_i'/ + z)/2)]\, H_j(dz) \\
&\quad + \int_{[x_i^n, x_i')} [1 - F_{i,j}((x_i' + z)/2) - F_{i,j}((x_i^n + z)/2)]\, H_j(dz) \\
&\quad + (H_j(x_i') - H_j(x_i')^-)[(1/2) - F_{i,j}((x_i^n + x_i')/2)] \\
&\quad + \int_{(x_i', \infty)} [F_{i,j}((x_i' + z)/2) - F_{i,j}((x_i^n + z)/2)]\, H_j(dz) \Bigg].
\end{aligned}
$$

For each $j \in I$, the first integral goes to zero, because $\lim_{n\to\infty} H_j(x_i^n)^- = \lim_{n\to\infty} G_j(x_i^n)^- = 0$. The third integral is clearly non-negative, and the integrand of the second integral is strictly positive, because $F_{i,j}((w+z)/2) < 1/2$ for all $w \leq \underline{m}$ and all $z < \underline{m}$. The remaining term is similarly non-negative. This establishes that, for all $j \in I$, the limit of the expression in brackets is non-negative. It is actually strictly positive for $j = i$, for in that case

$$\lim_{n\to\infty} H_j(x_i')^- - H_j(x_i^n)^- \;=\; G_i(\underline{m})^- \;>\; 0,$$

and the second integral converges to

$$\int_{[\underline{x},x_i')} [1 - F_{i,j}((x_i' + z)/2) - F_{i,j}((\underline{x} + z)/2)]\, H_j(dz),$$

which is positive. Since $P(i|i) > 0$, we conclude that

$$\Pi_A(X', H|i) \;>\; \pi_A(X^n, H|i) \;=\; \pi_A(G, H|i)$$

for high enough n, contradicting the assumption that (G, H) is a mixed strategy Bayesian equilibrium. ∎

Theorem 8 can be applied to symmetric information models to substantially strengthen our existence and uniqueness results for that class of model. The next corollary follows from Theorem 8 by decomposing any symmetric information game into its component games. If F is the distribution corresponding to one component, then M, defined for the component game, is just the singleton consisting of the median of F, and Theorem 8 establishes that the unique *mixed* strategy equilibrium of the component game is the strategy pair such that each type chooses the median with probability one.

Corollary 4 *In the canonical model with symmetric information, the strategy pair in which candidates locate at m_i with probability one following signal $i \in I$ is the unique mixed strategy Bayesian equilibrium.*

With Theorem 7, this tells us that, in games close to symmetric information, mixed strategy equilibria must be close to the pure strategy equilibrium, in the sense of weak convergence. This applies to our above example of non-existence of pure strategy equilibria: though pure strategy equilibria do not exist, for every open set around m and for high enough k, mixed strategy equilibria exist and all put probability one on that open set. Thus, while the example shows the fragility of pure strategy equilibria, we obtain a robustness result in mixed strategies.

We have not yet considered whether the distributions used by candidates in equilibrium may contain atoms or not. Our next result shows that, with some reasonable structure on the electoral game, the candidates may have atoms only at the conditional medians, $m_{i,i}$. This is true for pure strategy equilibria by Theorem 1, but it actually holds for mixed strategy equilibria quite generally. In addition to (C1)-(C4), we will impose the following "monotone likelihood ratio" on the candidates' beliefs.

(C8) For all $i, j, i', j' \in I$, if $i < i'$ and $j < j'$, then

$$P(j|i')P(j'|i) \quad \leq \quad P(j|i)P(j'|i').$$

Assuming $P(j|i)$ and $P(j|i')$ are positive, this can be written more intuitively as

$$\frac{P(j'|i)}{P(j|i)} \quad \leq \quad \frac{P(j'|i')}{P(j|i')},$$

which gives the condition its name. The condition is clearly satisfied in the case of perfect correlation and in the case of independent signals.

A plausible interpretation of signals in our model is that they indicate the ideological leanings of the electorate, i.e., whether the cut point μ is likely to be located more to the left or more to the right. Under that interpretation, the following stochastic dominance condition, which also presumes (C1)-(C4), is natural.

(C9) For all $i, i' \in I$ with $i < i'$, for all $j \in I$, and for all $z \in M$ with $0 < F_{i',j}(z) < 1$, we have $F_{i',j}(z) < F_{i,j}(z)$.

A consequence of (C9) is that $F_{i',j}(z) \leq F_{i,j}(z)$. This claim is certainly true if $F_{i',j}(z) = 0$ or $0 < F_{i',j}(z) < 1$. If $F_{i',j}(z) = 1$ but $F_{i,j}(z) < 1$, then, using continuity, decrease z to z' such that $F_{i,j}(z') < F_{i',j}(z') < 1$, contradicting (C9). Thus the claim follows. Note that (C9) is implied by (C4) in the stacked uniform model when $a \geq b_c - b_1$, for in that case $m_{i',j} > m_{i,j}$, and

$$F_{i',j}(z) \;=\; \frac{1}{2} + \frac{z - m_{i',j}}{2a} \;<<\; \frac{1}{2} + \frac{z - m_{i,j}}{2a} \;=\; F_{i,j}(z),$$

yields the condition.

The next lemma, which is proved in the appendix, establishes an important consequence of these conditions. Setting $\alpha_j = 1$ for all $j \in H$ and $\alpha_j = 0$ otherwise, we see that this consequence actually strengthens (C4).

Lemma 4 *In the canonical model, assume (C8) and (C9). For each $j \in I$, let $\alpha_j \in [0,1]$. For all $i, i' \in I$ with $i < i'$ and for all $z \in M$ with*

$$0 \;<\; \alpha_j P(j|i)P(j|i')F_{i',j}(z) \;<\; \alpha_j P(j|i)P(j|i')$$

for at least one j, we have

$$\frac{\sum_{j\in I}\alpha_j P(j|i)F_{i,j}(z)}{\sum_{j\in I}\alpha_j P(j|i)} \;\;>\;\; \frac{\sum_{j\in I}\alpha_j P(j|i')F_{i',j}(z)}{\sum_{j\in I}\alpha_j P(j|i')}.$$

This allows us to prove (in the appendix) a final lemma on the location of mass points of equilibrium mixed strategies. It parallels, under the extra conditions of (C8) and (C9), the earlier Lemma 3.

Lemma 5 *In the canonical model, assume (C8) and (C9). Let (G, H) be a mixed strategy Bayesian equilibrium. For all $z \in M$, if $G_i(z) - G_i(z)^- > 0$ for some $i \in I$ and $H_j(z) - H_j(z)^- > 0$ for some $j \in j$ with $P(i,j) > 0$, then $z = m_{i,j}$.*

For the intuition behind this lemma, suppose that candidates A and B both put positive mass on the same point $z \in M$ following two signal realizations, i and j. Lemma 4 allows us to assume that i and j are the only signal realizations after which the candidates put positive mass at this point. At this point, the argument proceeds as for Lemma 3. Conditional on realizations i and j, each candidate expects to choose z with positive probability, and, if z is not equal to $m_{i,j}$, then either candidate, say A, can transfer probability mass from z and move it toward $m_{i,j}$ an arbitrarily small amount. This increases A's expected payoff discretely in case B chooses z, and it affects A's expected payoff continuously otherwise. Therefore, a small enough deviation will increase A's expected payoff.

We now derive our restriction on atoms of mixed strategy equilibria: in the symmetric model with (C8) and (C9), the only possible atom of an equilibrium distribution, G_i or H_i, is $m_{i,i}$. Note that, save for the added assumptions of (C8) and (C9), the result generalizes Theorem 1.

Theorem 9 *In the canonical model, assume (C8) and (C9). Let* (G, H) *be a mixed strategy Bayesian equilibrium. If* $G_i(z) - G_i(z)^- > 0$ *for some* $i \in I$*, then* $z = m_{i,i}$*. If* $H_j(z) - H_j(z)^- > 0$ *for some* $j \in I$*, then* $z = m_{j,j}$*.*

Proof: Let (G, H) be a mixed strategy Bayesian equilibrium, and suppose $G_i(z) - G_i(z)^- > 0$ for some $i \in I$ but $z \neq m_{i,i}$. By symmetry and interchangeability, (G, G) is an equilibrium, and we have$P(i, i) > 0$, by (C3). By Theorem 8, we must have $z \in M$, and then Lemma 5 implies that $z = m_{i,i}$, a contradiction. ∎

Theorem 9 does not quite allow us to easily use differentiable methods in the analysis of mixed strategy equilibria. While the result limits the potential discontinuities of equilibrium mixed strategies to a finite set, there may be other points at which an equilibrium distribution G_i is non-differentiable, though continuous. In fact, the Cantor-Lebesgue function (see Wheeden and Zygmund, 1977) is an example of a continuous distribution that puts probability one on its points of non-differentiability, so this technical problem is a potentially significant one. In our analysis of equilibrium uniqueness in the three-signal stacked uniform model, we therefore restrict attention to a subset of mixed strategy equilibria: we say a strategy pair (G, H) is *regular* if, for all $i \in I$ and all $z \in \Re$, either G_i is differentiable at z or it is discontinuous at z, and similarly for H_i. This restriction eliminates the problem anticipated above, at the cost of omitting some pathological equilibria from the analysis.

Suppose (G, H) is a regular mixed strategy Bayesian equilibrium. Under the conditions of Theorem 9, we can decompose the probability measure generated by H_j into a degenerate measure with mass $H_j(m_{j,j}) - H_j(m_{j,j})^-$ on $m_{j,j}$ and an absolutely continuous measure with density h_j. The expected payoff of candidate A from pure strategy X against H, conditional on signal

i, is then

$$
\begin{aligned}
&\Pi_A(X, H|i) \\
&\quad = \sum_{j:m_{j,j}<x_i} P(j|i)(1 - F_{i,j}((x_i + m_{j,j})/2))(H_j(m_{j,j}) - H_j(m_{j,j})^-) \\
&\qquad + \sum_{j:m_{j,j}=x_i} P(j|i)(1/2)(H_j(m_{j,j}) - H_j(m_{j,j})^-) \\
&\qquad + \sum_{j:x_i<m_{j,j}} P(j|i)F_{i,j}((x_i + m_{j,j})/2)(H_j(m_{j,j}) - H_j(m_{j,j})^-) \\
&\qquad + \sum_{j\in I} P(j|i)\left[\int_{-\infty}^{x_i} (1 - F_{i,j}((x_i + z)/2))h_j(z)\,dz\right. \\
&\qquad \left. + \int_{x_i}^{\infty} F_{i,j}((x_i + z)/2)h_j(z)\,dz\right].
\end{aligned}
$$

Note that this expected payoff is differentiable at x_i if there is no j such that $x_i = m_{j,j}$. Indeed, it is enough that, if $x_i = m_{j,j}$, then H_j does not put positive mass on $m_{j,j}$. In this case, the usual first order condition must be satisfied at x_i, i.e.,

$$
\begin{aligned}
0 \;=\; & \sum_{j:m_{j,j}<x_i} P(j|i)\left[-\frac{1}{2}f_{i,j}((x_i + m_{j,j})/2)(H_j(m_{j,j}) - H_j(m_{j,j})^-)\right] \\
& \sum_{j:x_i<m_{j,j}} P(j|i)\left[\frac{1}{2}f_{i,j}((x_i + m_{j,j})/2)(H_j(m_{j,j}) - H_j(m_{j,j})^-)\right) \\
& + \sum_{j\in I} P(j|i)\left[\int_{-\infty}^{x_i} -\frac{1}{2}f_{i,j}((x_i + z)/2)h_j(z)\,dz + (1 - F_{i,j}(x_i))h_j(x_i)\right. \\
& \left. + \int_{x_i}^{\infty} \frac{1}{2}f_{i,j}((x_i + z)/2)h_j(z)\,dz - F_{i,j}(x_i)h_j(x_i)\right].
\end{aligned}
$$

We will use these observations next to construct a regular mixed strategy equilibrium in the stacked uniform model.

5 The Stacked Uniform Model

In this section, we construct an example of a mixed strategy equilibrium in the stacked uniform model and examine some properties of that equilibrium. The construction works by conjecturing the general form of the equilibrium, then deducing strong necessary conditions for the conjectured strategies to be an equilibrium, and finally checking that the strategies so-characterized are indeed mutual best responses. We consider the canonical stacked uniform model assuming sufficiently large support of the disturbance term, i.e., $a \geq b_c - b_1$, and imposing several restrictions on the marginal prior on signal pairs.

(C10) For all $i, i' \in I$, if $i < i'$, then

$$0 \;\leq\; \sum_{j:j<i} P(j|i) - \sum_{j:j<i} P(j|i') \;\leq\; \sum_{j:i\leq j<i'} P(j|i').$$

The first inequality is a stochastic dominance condition, formalizing the obvious intuition: the higher a candidate's signal, the more likely is the other candidate to also receive a higher signal. The second inequality, which can be rewritten as

$$\sum_{j:j<i} P(j|i') \quad\leq\quad \sum_{j:j<i'} P(j|i'),$$

limits the extent of this dominance: if we move from i to i', the signals below i will lose probability, but not too much.

(C11) For all $i, j \in I$, $P(i|i) \geq P(i|j)$.

That is, the likelihood a candidate receives a signal is greatest when the other candidate receives the same signal.

An important consequence of (C10) is that the subset of signals i such that

$$\begin{aligned} \sum_{j\in I:j\leq i} P(j|i) \quad&\geq\quad \sum_{j\in I:j>i} P(j|i) \\ \sum_{j\in I:j<i} P(j|i) \quad&\leq\quad \sum_{j\in I:j\geq i} P(j|i) \end{aligned}$$

is an interval. That is, letting C denote this subset, with $\underline{c} = \min C$ and $\overline{c} = \max C$, if $\underline{c} \leq i \leq \overline{c}$, then $i \in C$. To see this, suppose

$$\sum_{j\in I:j<i} P(j|i) \quad>\quad \sum_{j\in I:j\geq i} P(j|i),$$

and take $i' > i$. Then

$$\frac{1}{2} < \sum_{j \in I: j < i} P(j|i) \leq \sum_{j \in I: j < i'} P(j|i'),$$

where the first inequality follows by supposition and the second by (C10). The following condition is self-explanatory.

(C12) $C \neq \emptyset$.

The last condition we impose is a technical one needed for the proof of equilibrium existence in the multi-signal stacked uniform model.

(C13) For all $i, j \in I$, if $\bar{c} < j < i$, then

$$P(j|i)(m_{i,j} - m_{j,j}) \quad \geq \quad (P(j|j) - P(j|i))(m_{i,j} - m_{\bar{c},\bar{c}}),$$

and, if $i < j < \underline{c}$, then

$$P(j|i)(m_{j,j} - m_{i,j}) \quad \geq \quad (P(j|j) - P(j|i))(m_{\underline{c},\underline{c}} - m_{i,j}).$$

The condition limits (C11), saying roughly that $P(j|j)$ cannot exceed $P(j|i)$ by too much, where the stated bound depends on distances between conditional medians. Note that the condition is satisfied if signals are uninformative, when we would have $P(j|j) = P(j|i)$. It is satisfied if signals are perfectly correlated, for then $C = I$, and the condition is vacuously satisfied. Finally, note that it is also vacuously satisfied in the three signal model, with $I = \{-1, 0, 1\}$ and $C = \{0\}$, for in this case there do not exist distinct signals to the right of $\bar{c} = 0$ or the the left of $\underline{c} = 0$.

We conjecture a symmetric equilibrium in which candidates who receive a central signal $i \in C$ choose the conditional median $m_{i,i}$ with probability one. Candidates who receive other signals use mixed strategies that are non-atomic with differentiable densities and convex supports, say $[\underline{x}_i, \overline{x}_i]$. We conjecture supports that are non-overlapping, adjacent, and ordered identically to signals, so $\underline{x}_{\bar{c}+1} = m_{\bar{c},\bar{c}}$ and, for all $i > \bar{c}$, $\underline{x}_i = \overline{x}_{i-1}$. Our analysis will mainly be concerned with the conditions that need to be fulfilled by the distribution, say G_i, used by the candidates after signal realizations $i \notin C$. Under the assumptions of this section, we can analyze the cases $i < \underline{c}$ and $i > \bar{c}$ independently, and we will therefore focus on the latter. Once we have fully characterized what such a strategy pair, say (G, G), would have to look like *if it were* an equilibrium, we will check that *it actually is* an equilibrium.

Because we assume G satisfies non-overlapping supports, we see that, for every differentiability point x in the support of G_i, $g_j(x) > 0$ implies $j = i$. This allows us to simplify the first order condition for $i > \overline{c}$ from the previous section to: for all $x \in [\underline{x}_i, \overline{x}_i]$,

$$\begin{aligned}
&P(i|i)g_i(x)(2F_{i,i}(x) - 1) \\
&\quad= -\sum_{j\in C} \frac{1}{2}P(j|i)f_{i,j}((x + m_{j,j})/2) \\
&\qquad - \sum_{j\in I\setminus C} P(j|i)\left[\int_{-\infty}^{x} \frac{1}{2}f_{i,j}((x+z)/2)g_j(z)\,dz\right. \\
&\qquad \left. + \int_{x}^{\infty} \frac{1}{2}f_{i,j}((x+z)/2)g_j(z)\,dz\right],
\end{aligned}$$

unless $i = \overline{c}+1$, in which case the condition holds on the half open interval $(m_{\overline{c},\overline{c}}, \overline{x}_{\overline{c}+1}]$. Since the candidate's expected payoff is constant over the relevant interval, it must, in particular be linear over this interval, so the second order condition must be satisfied with equality. Using the assumption of uniform distributions, the second order condition reduces to the following: for all $x \in [\underline{x}_i, \overline{x}_i]$,

$$3g_i(x)f_{i,i}(x) + g_i'(x)(2F_{i,i}(x) - 1) = 0,$$

with the same qualification if $i = \overline{c}+1$. Since the platform $m_{\overline{c},\overline{c}}$ will have no mass when the candidate receives signal $\overline{c}+1$, we include it in the interval as well, yielding a differential equation in g_i that is easily solved: we find that, for all $x \in [\underline{x}_i, \overline{x}_i]$,

$$g_i(x) = g_i(\underline{x}_i)\left(\frac{1 - 2F_{i,i}(\underline{x}_i)}{1 - 2F_{i,i}(x)}\right)^{\frac{3}{2}},$$

which yields the distribution

$$G_i(x) = g_s(\underline{x}_i)((1/2) - F_{i,i}(\underline{x}_i))^{3/2}\int_{\underline{x}_i}^{x} \frac{1}{((1/2) - F_{i,i}(z))^{3/2}}\,dz.$$

Thus, the second order condition pins down the density g_i up to the location of $\underline{x}_i$ and the initial condition $g_i(\underline{x}_i)$.

These parameters are determined by the first order condition for each signal realization and by our assumption of non-overlapping adjacent supports. In the stacked uniform model, which we consider here, the first order condition reduces to

$$g_i(x) \quad = \quad \frac{-\sum_{j\in I:j<i} P(j|i) + P(i|i)(1-2G_i(x)) + \sum_{j\in I:i<j} P(j|i)}{4aP(i|i)(2F_{i,i}(x)-1)}.$$

Evaluated as $\underline{x}_i$, this becomes

$$g_i(\underline{x}_i) \quad = \quad \frac{-\sum_{j\in I:j<i} P(j|i) + \sum_{j\in I:j\geq i} P(j|i)}{4aP(i|i)(2F_{i,i}(\underline{x}_i)-1)},$$

which is positive for $i > \overline{c}$, as is required. For signal $\overline{c}+1$, we have $\underline{x}_{\overline{c}+1} = m_{\overline{c},\overline{c}}$ by construction. This pins down $g_{\overline{c}+1}(\underline{x}_{\overline{c}+1})$ through the first order condition. We then find $\underline{x}_{\overline{c}+2}$ as the solution to

$$G_{\overline{c}+1}(x) \quad = \quad 1,$$

if such a solution exists. This pins down $g_{\overline{c}+2}(\underline{x}_{\overline{c}+2})$ through the first order condition, and so on. It is straightforward to verify that, given $\underline{x}_i < m_{i,i}$, a solution to $G_i(x) = 1$ exists for all $i > \overline{c}$. Using $F_{i,i}(x) = F_{i,i}(\underline{x}_i) + (x - \underline{x}_i)/2a$ for $x \in [\underline{x}_i, m_{i,i,}]$, note that

$$\int_{\underline{x}_i}^{x} \frac{1}{((1/2)-F_{i,i}(z))^{3/2}}\, dz \quad = \quad \frac{4a}{\sqrt{\frac{1}{2} - F_{i,i}(\underline{x}_i) + \frac{\underline{x}_i - x}{2a}}} - \frac{4a}{\sqrt{\frac{1}{2} - F_{i,i}(\underline{x}_i)}}.$$

Solving $F_{i,i}(\underline{x}_{i,i}) + (m_{i,i} - \underline{x}_i)/2a$ for $m_{i,i}$, we see that

$$\lim_{x\uparrow m_{i,i}} \int_{\underline{x}_i}^{x} \frac{1}{((1/2)-F_{i,i}(z))^{3/2}}\, dz \quad = \quad \infty,$$

which yields the desired solution.

It is prohibitively difficult to solve for these parameters in the general case, but the next result establishes several properties of our conjectured equilibrium. We first show that the strategies defined above do, indeed, form an equilibrium. We then note that the density used following signal realizations $i > \overline{c}$ is increasing, so more extreme positions are more likely than more moderate ones, and that the support of this density is bounded above by the conditional median $m_{i,i}$. Thus, a candidate never chooses positions more extreme than what might be chosen in the pure strategy equilibrium.

In the third part of the theorem, we index the marginal prior on $I \times I$ by the elements of a metric space Γ, as in $P^{\gamma}(i,j)$. We consider a sequence $\{\gamma_n\}$ of games, where, for each n, we let G^n denote the strategy profile defined above, g_i^n the corresponding density used after signal realization i, and $\underline{x}_i^n$ and $\overline{x}_i$ the lower and upper bounds of the support of g_i^n. We show that, as the net probability that his/her opponent has a more moderate signal goes to zero, the mixed strategy used by a candidate approaches the point mass on the candidate's conditional median.

Theorem 10 *In the canonical stacked uniform model with $a \geq b_c - b_1$, assume (C10)-(C13).*

1. *The above strategies form a mixed strategy Bayesian equilibrium.*
2. *For $i > \overline{c}$, g_i is increasing and convex and $\overline{x}_i < m_{i,i}$.*
3. *Let $\{\gamma_n\}$ be a sequence with $C^{\gamma_n} = C$ for all n and, for all $i \in I$, $\liminf_{n\to\infty} P^{\gamma_n}(i|i) > 0$. For $i > \overline{c}$, if*
$$\sum_{j\in I:j<i} P^{\gamma_n}(j|i) - \sum_{j\in I:j\geq i} P^{\gamma_n}(j|i) \to 0,$$
then $\overline{x}_i^n \to m_{i,i}$ and, for all $x < m_{i,i}$ $g_i^n(x) \to 0$.

Proof: 1. Take any $i,k \in I$ with $\overline{c} < k \leq i$. Let X be a pure strategy such that $x_i = \overline{x}_k$, and let X' be a pure strategy such that $x_i' = x' \in (\underline{x}_k, \overline{x}_k]$. Define
$$\psi_k(x) = \frac{\overline{x}_k - x}{4a},$$
and note that this quantity is positive. The change in candidate A's expected payoff, conditional on signal i, upon moving from $\overline{x}_k$ to x' is
$$\begin{aligned}
&\Pi_A(X',G|i) - \Pi_A(X,G|i) \\
&= \sum_{j\in C:j<k} P(j|i)\psi_k(x') - \sum_{j\in C:j>k} P(j|i)\psi_k(x') \\
&\quad + \sum_{j\notin C:j<k} P(j|i)\int_{-\infty}^{\underline{x}_k} \psi_k(x')g_j(z)\,dz - \sum_{j\notin C:k<j} P(j|i)\int_{\overline{x}_k}^{\infty} \psi_k(x')g_k(z)\,dz \\
&\quad +P(k|i)\left[\int_{\underline{x}_k}^{x'} \psi_k(x')g_k(z)\,dz \right.\\
&\quad \left. + \int_{x'}^{\overline{x}_k} (F_{i,k}((x+z)/2) + F_{i,k}((\overline{x}_k+z)/2) - 1)g_k(z)\,dz\right],
\end{aligned}$$

where the second term on the righthand side is zero, by construction. The last terms on the righthand side, corresponding to signal k, simplify to $P(k|i)$ times

$$\begin{aligned}
&\psi_k(x')G_k(x') + \int_{x'}^{\overline{x}_k} \frac{\overline{x}_k + x' + 2z}{4a} g_k(z)\,dz - \frac{1-G_k(x')}{a}\sum_{h=1}^{c} Q(h|i,k)b_h \\
&= \psi_k(x')G_k(x') + \int_{x'}^{\overline{x}_k} \frac{\overline{x}_k + x' + 2z - 4\underline{x}_k}{4a} g_k(z)\,dz \\
&\quad + \frac{1-G_k(x')}{a}\sum_{h=1}^{c} Q(h|i,k)(\underline{x}_k - b_h).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Pi_A(X', G|i) - \Pi_A(X, G|i) \\
&= \psi_k(x')\left[\sum_{j:j<k} P(j|i) - \sum_{j:j>k} P(j|i)\right] + P(k|i)\left[\psi_k(x')G_k(x')\right. \\
&\quad \left. + \int_{x'}^{\overline{x}_k} \frac{\overline{x}_k + x' + 2z - 4\underline{x}_k}{4a} g_k(z)\,dz + \frac{1-G_k(x')}{a}\sum_{h=1}^{c} Q(h|i,k)(\underline{x}_k - b_h)\right].
\end{aligned}$$

Note that this change is equal to zero, by construction, if $i = k$. If $k < i$, then

$$\sum_{j:j<k} P(j|i) - \sum_{j:j>k} P(j|i) \quad \le \quad \sum_{j:j<k} P(j|k) - \sum_{j:j>k} P(j|k)$$

by (C10). Furthermore,

$$\begin{aligned}
&P(k|i)\left[\psi_k(x')G_k(x') + \int_{x'}^{\overline{x}_k} \frac{\overline{x}_k + x' + 2z - 4\underline{x}_k}{4a} g_k(z)\,dz\right. \\
&\quad \left. + \frac{1-G_k(x')}{a}\sum_{h=1}^{c} Q(h|i,k)(\underline{x}_k - b_h)\right] \\
&\le P(k|k)\left[\psi_k(x')G_k(x') + \int_{x'}^{\overline{x}_k} \frac{\overline{x}_k + x' + 2z - 4\underline{x}_k}{4a} g_k(z)\,dz\right. \\
&\quad \left. + \frac{1-G_k(x')}{a}\sum_{h=1}^{c} Q(h|k,k)(\underline{x}_k - b_h)\right]
\end{aligned}$$

if and only if

$$\begin{aligned}
&P(k|i)\left(\sum_{h=1}^{c}Q(h|k,k)b_h-\sum_{h=1}^{c}Q(h|i,k)b_h\right)\\
&\leq\ (P(k|k)-P(k|i))\left[\psi_k(x')G_k(x')+\int_{x'}^{\overline{x}_k}\frac{\overline{x}_k+x'+2z-4\underline{x}_k}{4a}g_k(z)\,dz\right.\\
&\qquad\left.+\frac{1-G_k(x')}{a}\sum_{h=1}^{c}Q(h|i,k)(\underline{x}_k-b_h)\right].
\end{aligned}$$

This is implied by

$$\begin{aligned}
&P(k|i)\left(\sum_{h=1}^{c}Q(h|k,k)b_h-\sum_{h=1}^{c}Q(h|i,k)b_h\right)\\
&\leq\ (P(k|k)-P(k|i))\left(\sum_{h=1}^{c}Q(h|i,k)(m_{\overline{c},\overline{c}}-b_h)\right),
\end{aligned}$$

which follows from (C13). Therefore,

$$\Pi_A(X',G|i)-\Pi_A(X,G|i)\ \leq\ \Pi_A(X',G|k)-\Pi_A(X,G|k)\ =\ 0,$$

and we conclude that

$$\Pi_A(X',G|i)\quad\leq\quad\Pi_A(X,G|i)$$

whenever $x'\in(\underline{x}_k,\overline{x}_k]$. Indeed, if $\overline{c}+1<k$, then $\underline{x}_k$ is a continuity point of A's expected payoff function, so the inequality also holds at $\underline{x}_k$.

If $k=\overline{c}+1$, then $\underline{x}_k=m_{\overline{c},\overline{c}}$ is not a continuity point, and this argument no longer holds. In this case, let $x'=m_{\overline{c},\overline{c}}$, let $\{x^n\}$ be a decreasing sequence converging to $m_{\overline{c},\overline{c}}$, and let X^n be defined as X' but with $x_i^n=x^n$. Then we have

$$\begin{aligned}
&[\Pi_A(X',G|i)-\Pi_A(X,G|i)]-\lim_{n\to\infty}[\Pi_A(X^n,G|i)-\Pi_A(X,G|i)]\\
&=\ P(\overline{c}|i)\left[\frac{1}{2}-(1-F_{i,\overline{c}}((\overline{x}_{\overline{c}+1}+m_{\overline{c},\overline{c}})/2))-\psi_{\overline{c}+1}(m_{\overline{c},\overline{c}})\right]\\
&\quad+P(\overline{c}+1|i)\left[\int_{m_{\overline{c},\overline{c}}}^{\overline{x}_{\overline{c}+1}}(2F_{i,\overline{c}+1}((z+m_{\overline{c},\overline{c}})/2)-1)g_{\overline{c}+1}(z)\,dz\right.\\
&\quad\left.-\lim_{n\to\infty}\int_{x^n}^{\overline{x}_{\overline{c}+1}}(2F_{i,\overline{c}+1}((z+m_{\overline{c},\overline{c}})/2)-1)g_{\overline{c}+1}(z)\,dz\right]\\
&=\ P(\overline{c}|i)\left[\frac{1}{2}-(1-F_{i,\overline{c}}((\overline{x}_k+m_{\overline{c},\overline{c}})/2))-\psi_{\overline{c}+1}(m_{\overline{c},\overline{c}})\right].
\end{aligned}$$

Note that

$$\psi_{\overline{c}+1}(m_{\overline{c},\overline{c}}) \;\; = \;\; F_{i,\overline{c}+1}((\overline{x}_{\overline{c}+1} + m_{\overline{c},\overline{c}})/2) - F_{i,\overline{c}+1}(m_{\overline{c},\overline{c}})),$$

so we have

$$\begin{aligned}
&[\Pi_A(X', G|i) - \Pi_A(X, G|i)] - \lim_{n\to\infty}[\Pi_A(X^n, G|i) - \Pi_A(X, G|i)] \\
&= \;\; P(\overline{c}|i)\left[F_{i,\overline{c}}(m_{\overline{c},\overline{c}}) - \frac{1}{2}\right] \\
&\leq \;\; 0,
\end{aligned}$$

where the inequality follows from $\overline{c} < i$ and (C4), which implies $m_{\overline{c},\overline{c}} < m_{i,\overline{c}}$. Therefore,

$$[\Pi_A(X', G|i) - \Pi_A(X, G|i)] \;\leq\; \lim_{n\to\infty}[\Pi_A(X^n, G|i) - \Pi_A(X, G|i)] \;\leq\; 0,$$

and we conclude that

$$\Pi_A(X', G|i) \;\; \leq \;\; \Pi_A(X, G|i),$$

as in the earlier case.

Now take any $i, k \in I$ with $k \leq i$ and $\underline{c} < k \leq \overline{c}$. Let X be a pure strategy such that $x_i = m_{k,k}$, and let X' be a pure strategy such that $x_i = x' \in (m_{k-1,k-1}, m_{k,k}]$. Then

$$\begin{aligned}
&\Pi_A(X', G|i) - \Pi_A(X, G|i) \\
&= \;\; \psi_k(x')\left[\sum_{j:j<k} P(j|i) - \sum_{j:k<j} P(j|k) + P(k|i)(F_{i,k}((x' + m_{k,k})/2) - 1/2)\right].
\end{aligned}$$

Note that

$$F_{i,k}((x' + m_{k,k})/2) - 1/2 \;\; \leq \;\; -\psi_k(x')$$

follows from

$$\psi_k(x') \;\; = \;\; F_{i,k}(m_{k,k}) - F_{i,k}((x' + m_{k,k})/2)$$

and $m_{k,k} \leq m_{i,k}$. Therefore,

$$\begin{aligned}
&\Pi_A(X', G|i) - \Pi_A(X, G|i) \\
&\leq \;\; \psi_k(x')\left[\sum_{j:j<k} P(j|i) - \sum_{j:k\leq j} P(j|k)\right],
\end{aligned}$$

which is non-positive if $i = k$ by definition of $k \in C$. If $i > k$, then the inequality follows from (C10), and we conclude that

$$\Pi_A(X', G|i) \quad \le \quad \Pi_A(X, G|i),$$

whenever $x' \in (m_{k-1,k-1}, m_{k,k}]$. That the inequality actually holds for $x' = m_{k-1,k-1}$ as well follows from (C4) and an argument similar to that used in the previous paragraph.

Finally, take any $i, k \in I$ with $k \le i$ and $k < \underline{c}$. Let X be a pure strategy such that $x_i = \underline{x}_k$, and let X' be a pure strategy such that $x'_i = x' \in [\underline{x}_k, \overline{x}_k)$. Then, we have

$$\begin{aligned}
&\Pi_A(X, G|i) - \Pi_A(X', G|i) \\
&= \quad \psi_k(x') \left[\sum_{j:j<k} P(j|i) - \sum_{j:k<j} P(j|i) \right] + P(k|i) \left[-\psi_k(x')(1 - G_k(x')) \right. \\
&\qquad \left. + \int_{\underline{x}_k}^{x'} (F_{i,k}((\underline{x}_k + z)/2) + F_{i,k}((x' + z)/2) - 1) g_k(z)\, dz \right].
\end{aligned}$$

If $i = k$, then, by definition of $k < \underline{c}$, we know that the first term on the righthand side is negative. Since $\Pi_A(X', G|k) - \Pi_A(X, G|k) = 0$ by construction, the second term is positive. If $k < i$, then the first term decreases, by (C10). Furthermore,

$$\begin{aligned}
& P(k|i) \left[-\psi_k(x')(1 - G_k(x')) \right. \\
&\qquad \left. + \int_{x'}^{\overline{x}_k} (F_{i,k}((x' + z)/2) + F_{i,k}((\overline{x}_k + z)/2) - 1) g_k(z)\, dz \right] \\
&\le \quad P(k|k) \left[-\psi_k(x')(1 - G_k(x')) \right. \\
&\qquad \left. + \int_{x'}^{\overline{x}_k} (F_{k,k}((x' + z)/2) + F_{k,k}((\overline{x}_k + z)/2) - 1) g_k(z)\, dz \right],
\end{aligned}$$

where the inequality follows from (C11) and from the fact that $F_{i,k}$ stochastically dominates $F_{k,k}$, by stochastic dominance. Thus, the second term decreases as well. We conclude that

$$\Pi_A(X, G|i) \quad \le \quad \Pi_A(X', G|i),$$

whenever $x' \in [\underline{x}_k, \overline{x}_k)$. Indeed, if $k < \underline{c} - 1$, then $\overline{x}_k$ is a continuity point, so the inequality also holds at $\overline{x}_k$. If $k = \underline{c} - 1$, then $\overline{x}_k = m_{\underline{c},\underline{c}}$, which is not a continuity point of A's expected payoffs. In this case, we may argue as above that the inequality remains true for $x' = m_{\underline{c},\underline{c}}$.

We now argue that the candidate has no profitable deviation, conditional on signal i, from the mixed strategy G. First, consider the case $i > \overline{c}$. Let X satisfy $x_i = \underline{x}_i$, and let X' satisfy $x'_i = x$. Suppose $x < \underline{x}_i$ and that x does not lie below the supports of all distributions. (That case is easily checked and is omitted.) Then either $x \in [\underline{x}_k, \overline{x}_k]$ for some $k \notin C$, or $x \in [m_{k-1}, m_{k-1}]$ for some $k \in C$. Suppose the former. In fact suppose $x_k < \underline{c}$, and define the pure strategies X^j as follows: for $j \notin C$, let $x_i^j = \underline{x}_j$, and for $j \in C$, let $x_i^j = m_{j,j}$. Our above arguments show that

$$
\begin{aligned}
&\Pi_A(X', G|i) - \Pi_A(X, G|i) \\
&= [\Pi_A(X', G|i) - \Pi_A(X^{k+1}, G|i)] \\
&\quad + \sum_{j:k<j<\underline{c}} [\Pi_A(X^j, G|i) - \Pi_A(X^{j+1}, G|i)] \\
&\quad + \sum_{\underline{c}\leq j<\overline{c}} [\Pi_A(X^j, G|i) - \Pi_A(X^{j+1}, G|i)] \\
&\quad + \sum_{j:\overline{c}<j<i} [\Pi_A(X^j, G|i) - \Pi_A(X^{j+1}, G|i)] \\
&\leq 0.
\end{aligned}
$$

Other cases are proved the same way, by decomposing a deviation to the left after signal i into a finite number of moves across the supports for other signal realizations. This proves that A has no deviation to the left after signal i, and a symmetric argument shows that there are no deviations to the right. Finally, the same argument for candidate B establishes that the strategy pair (G, G) is a mixed strategy equilibrium.

2. That g_i is increasing and convex is apparent from the functional form of the density. That $\overline{x}_i < m_{i,i}$ follows from the discussion above, where it is shown that, given $\underline{x}_i < m_{i,i}$, we have

$$
\lim_{x\uparrow m_{i,i}} \int_{\underline{x}_i}^{x} \frac{1}{((1/2) - F_{i,i}(z))^{3/2}}\, dz \quad = \quad \infty.
$$

This, with an induction argument starting with $\underline{x}_{\overline{c}+1} = m_{\overline{c},\overline{c}}$ yields the desired conclusion.

3. Since $C^{\gamma_n} = C$ for all n, $\underline{x}_{\overline{c}+1}$ is constant along the sequence. By part 2, we therefore have $\underline{x}_i^n < m_{i-1,i-1}$ for all $i > \overline{c}$. This implies that

$1 - 2F_{i,i}(\underline{x}_i^n) \geq 1 - 2F_{i,i}(m_{i-1,i-1}) > 0$ for all n, so the denominated in the above expression for $g_i(\underline{x}_i)$ does not go to zero. We therefore have $g_i^n(\underline{x}_i^n) \to 0$. Then, fixing $x < m_{i,i}$, we see that $g_i^n(x) \to 0$. And, from the above expression for $G_i(x)$, we see that $G_i^n(\overline{x}_i^n) = 1$ for all n implies $\overline{x}_i^n \to m_{i,i}$. ∎

6 Three Signal Realizations

We now turn to the special case of three signal realizations in the canonical stacked uniform model. Thus, $I = \{-1, 0, 1\}$. We again assume sufficiently large support of the noise term, i.e., $a \geq b_c - b_1$, without imposing any restriction on the number of disturbances. In contrast to the previous section, we impose symmetry in the model.

(C14) For all $z \in X$ and all $i, j \in I$, $P(i|0) = P(-i,|0)$, $P(-i|i) = P(i|-i)$, $P(0|i) = P(0|-i)$, and $F_{i,j}(z) = 1 - F_{-i,-j}(-z)$.

Note the consequence of the first part of the condition that $P(1|1) = P(-1|-1)$, while the last part of the condition implies that $m_{0,0} = 0$ and $m_{-1,-1} = -m_{1,1}$. With only three signals, condition (C10) only binds when $i = 0$ and $i' = 1$, or when $i = -1$ and $i' = 0$. It turns out that, with the symmetry imposed in this section, this will not be needed for existence here. Condition (C11) is also simplified, but we will only use two inequalities from this condition: $P(1|1) \geq P(1|-1)$ (and its symmetric counterpart). Condition (C12) is automatically satisfied, because $0 \in C$ under our symmetry assumption. In fact, C contains only zero unless $P(1|1) \geq 1/2$, in which case $C = I$. Finally, condition (C13) is vacuously satisfied.

The consideration of this special case allows us to strengthen the results of the previous section. We prove that the equilibrium characterized there is actually unique among all "fully symmetric" equilibria, and we give more precise results on the possibility that candidates choose extremal positions. Finally, we examine two special information structures. In this model, we say a mixed strategy G is *symmetric about zero* if, for all $i \in I$ and all $z \in X$, $G_i(z) = 1 - G_{-i}(-z)$. A mixed strategy Bayesian equilibrium (G, H) is *fully symmetric* if G and H are symmetric about zero

Theorem 11 *In the canonical stacked uniform model with $a \geq b_c - b_1$, assume $I = \{1, 0, -1\}$, (C8), (C9), (C14), and $P(i|i) \geq P(i|-i)$ for all $i \in I$. There is a unique fully symmetric equilibrium. In it, each candidate plays $x_0 = 0$ upon receiving the signal $i = 0$. Upon receiving signal $i \neq 0$, each candidate plays $x = m_{i,i}$ if $P(1|1) \geq 1/2$. Otherwise, if $P(1|1) < 1/2$, then the mixed strategy played upon receiving signal $i = 1$ is given by*

$$G_1(z) \quad = \quad \frac{(1 - 2P(1|1))(\sqrt{m_{1,1}} - \sqrt{m_{1,1} - z})}{2P(1|1)\sqrt{m_{1,1} - z}},$$

for all $z \in [0, 4m_{1,1}P(1|1)(1 - P(1|1))]$.

Proof: Note that, by Theorem 8, in any equilibrium (G, H), we must have $G_i(m_{1,1}) - G_i(m_{-1,-1})^- = H_i(m_{1,1}) - H_i(m_{-1,-1})^-$ for all $i \in I$. The first step of the proof is to show that, given any potential fully symmetric equilibrium strategy for one candidate, a best response for the other, conditional on the zero signal, is to locate at zero. The second step is to prove, using the construction of the previous section, the existence of a fully symmetric equilibrium with support in M. The third step is to show that, in every fully symmetric equilibrium, the candidates locate at zero following the zero signal. The final step is to pin down the strategies of the candidates following extreme signal realizations.

Step 1: Now, let H be any strategy symmetric about zero for candidate B with support in M and with mass points (if any) only at conditional medians, i.e., $H_i(y) - H_i(y)^- > 0$ implies $y = m_{i,i}$. Let X be a pure strategy with $x_0 = 0$, let X' be a pure strategy with $x'_0 = x' \neq 0$, and note that

$$\begin{aligned}
&\Pi_A(X, H|0) - \Pi_A(X', H|0) \\
&= \sum_{j=-1}^{1} P(j|0) \left[\int_{x'}^{m_{1,1}} \left[F_{0,j}\left(z/2\right) - F_{0,j}\left((x' + z)/2\right) \right] H_j(dz) \right. \\
&\quad + \int_0^{x'} [F_{0,j}(z/2) - (1 - F_{0,j}((x' + z)/2))] H_j(dz) \\
&\quad \left. + \int_{-m_{1,1}}^{0} [(1 - F_{0,j}(z/2)) - (1 - F_{0,j}((x' + z)/2))] H_j(dz) \right].
\end{aligned}$$

Recall that, for all $j \in I$ and all $w, z \in [b_c - a, b_1 + a]$,

$$F_{0,j}\left(w\right) - F_{0,j}\left(z\right) \quad = \quad \frac{w - z}{2a}.$$

Therefore, we have

$$F_{0,j}(z/2) - F_{0,j}((x'+z)/2) \quad = \quad (1 - F_{0,j}((x'+z)/2)) - (1 - F_{0,j}(z/2)),$$

and the above expressions corresponding to $j = 1, -1$ can be simplified to

$$\int_0^{x'} [F_{0,j}(z/2) - (1 - F_{0,j}((x'+z)/2))] H_j(dz)$$
$$+ \int_{-x'}^{0} [(1 - F_{0,j}(z/2)) - (1 - F_{0,j}((x'+z)/2))] H_j(dz).$$

By assumption, we have $F_{0,-1}(-z) = 1 - F_{0,1}(z)$ for all $z \in \Re$. Using $P(1|0) = P(-1|0)$, we then see that

$$\sum_{j=-1,1} P(j|0) \left[\int_0^{x'} [F_{0,j}(z/2) - (1 - F_{0,j}((x'+z)/2))] H_j(dz) \right.$$
$$\left. + \int_{-x'}^{0} [(1 - F_{0,j}(z/2)) - (1 - F_{0,j}((x'+z)/2))] H_j(dz) \right]$$

is equal to $P(1|0)$ times

$$\int_0^{x'} [1 - F_{0,1}(-z/2) - F_{0,1}(-(x'+z)/2) + F_{0,1}(z/2)$$
$$-1 + F_{0,1}((x'+z)/2)] H_j(dz) + \int_{-x'}^{0} [F_{0,1}(-z/2) - F_{0,1}(-(x'+z)/2)$$
$$-F_{0,1}(z/2) + F_{0,1}((x'+z)/2)] H_j(dz).$$

To see that each of these integrals is positive, note that the first integrand is equal to $x'/4a$, and the second is equal to $(x'/2a) + (z/a)$, which is positive for all $z \in [0, x']$. Therefore, we have

$$\begin{aligned}
&\Pi_A(X, H|0) - \Pi_A(X', H|0) \\
&\geq \ P(0|0) \left[\int_{x'}^{m_{1,1}} \left[F_{0,0}\left(z/2\right) - F_{0,0}\left((x'+z)/2\right) \right] H_0(dz) \right. \\
&\quad + \int_0^{x'} [F_{0,0}(z/2) - (1 - F_{0,0}((x'+z)/2))] H_0(dz) \\
&\quad \left. + \int_{-m_{1,1}}^{0} [(1 - F_{0,0}(z/2)) - (1 - F_{0,0}((x'+z)/2))] H_0(dz) \right] \\
&= \ P(0|0) \left[\int_{x'}^{m_{1,1}} [F_{0,0}(z/2) - F_{0,0}((x'+z)/2) \right. \\
&\quad + F_{0,0}(z/2) - F_{0,0}((x'+z)/2)] H_0(dz) \\
&\quad + \int_0^{x'} [F_{0,0}(z/2) - 1 + F_{0,0}((x'+z)/2) \\
&\quad \left. + F_{0,0}(z/2) - F_{0,0}((-x'+z)/2)] H_0(dz) \right] \\
&= \ \frac{P(0|0)}{2a} \left[\int_{x'}^{m_{1,1}} -x' H_0(dz) + \int_0^{x'} (x'+z) H_0(dz) \right] \\
&= \ \frac{P(0|0)}{2a} \left[x'(2H_0(x') - 1) + \int_0^{x'} z H_0(dz) \right] \\
&> \ 0,
\end{aligned}$$

where the first inequality follows from the preceding remarks; the first equality follows from symmetry of $F_{0,0}$ and $H_{0,0}$; the second equality uses $F_{0,0}(z/2) - 1 = -F_{0,0}(-z/2)$; and the last inequality relies on the assumption that the median of H_0 is zero. A symmetric argument holds for $x' < 0$, and we conclude that $x_0 = 0$ is a best response to H.

Step 2: All of the conditions of Theorem 10 are fulfilled, except for part of (C11), namely, $P(1|1) \geq P(1|0)$ and $P(0|0) \geq P(0|1)$. For the special case of three signals, the latter inequality is not used, and the former is only used to prove that $x_0 = 0$ is a best response following $i = 0$, which is proved in Step 1. We conclude that the strategies defined in the previous section form a fully symmetric equilibrium.

Step 3: Let (G, H) be a fully symmetric equilibrium. To see that G_0 and H_0 both put probability one on $m_{0,0} = 0$, suppose that $G_0(0) < 1$. By symmetry and interchangeability, we may also suppose $H = G$. Since G_0 has no other mass points, by Theorem 9, we can choose a strictly increasing sequence $\{x^n\}$ in the support of G_0 such that $G_0(x^n) \to 1$. Since each x^n is a continuity point of A's expected payoff, conditional on $i = 0$, we have

$$\Pi_A(X^n, H|0) \quad = \quad \Pi_A(G, H|0)$$

for all n, where X^n is a pure strategy with $x_0^n = x^n$. Take any n such that $G_0(x^n) = H_0(x^n) > 1/2$, and recall that

$$\Pi_A(X, H|0) - \Pi_A(X^n, H|0) \quad = \quad \frac{P(0|0)}{2a}\left[x^n(2H_0(x^n) - 1) + \int_0^{x^n} zH_0(dz)\right],$$

as shown in Step 2. This quantity is positive, however, a contradiction. Therefore, $G_0(0) = 1$, and a symmetric argument shows that $G_0(0)^- = 0$ as well. We conclude that G_0 puts probability one on zero.

Step 4: To pin down G_1, again let (G, H) be a fully symmetric equilibrium, and suppose $H = G$. To see that $G_1(0) = 0$, let X be such that $x_0 = x \in (m_{-1,-1}, 0)$, and note that

$$\begin{aligned}
&\Pi_A(X, H|1) \\
&= \quad P(0|1)F_{0,1}(x/2) + \sum_{j=-1,1} P(j|1)\left[\int_{m_{-1,-1}}^{x} (1 - F_{j,1}((x+z)/2)H_j(dz)\right. \\
&\qquad \left. + \int_x^{m_{1,1}} F_{j,1}((x+z)/2)H_j(dz)\right].
\end{aligned}$$

Using Theorem 9, each H_j is differentiable at x, and candidate A's payoff is differentiable at x, so we have

$$\begin{aligned}
&\frac{\partial}{\partial x_1}\Pi_A(X, H|1) \\
&= \quad \frac{P(0,1)}{2}f_{0,1}(x/2) + \sum_{j=-1,1} P(j|1)\left[\int_{m_{-1,-1}}^{x} -\frac{1}{2}f_{j,1}((x+z)/2)H_j(dz)\right. \\
&\qquad \left. +(1 - F_{j,1}(x))h_j(x) + \int_x^{m_{1,1}} \frac{1}{2}f_{j,1}((x+z)/2)H_j(dz) - F_{j,1}(x)h_j(x)\right] \\
&= \quad \frac{P(0|1)}{4a} + \sum_{j=-1,1} P(j|1)[(1 - 2F_{j,1}(x))h_j(x) + \frac{1}{4a}(1 - 2H_j(x))].
\end{aligned}$$

Since the median of $F_{1,1}$ is positive and the median of $F_{-1,1}$ is zero, by (C14), the only term above that may be positive is $1 - 2H_j(x)$. If $G_1(0) > 0$, then, by Theorem 9, we may take a strictly decreasing sequence $\{x^n\}$ in the support of G_1 such that $G_1(x^n) \to 0$. Because each x^n is a differentiable point of Π_A, we must have $\frac{\partial}{\partial x_1}\Pi_A(x^n, H|1) = 0$ for all n. Note, however, that $1 - H_1(x^n) = 1 - G_1(x^n) \to 1$, while $1 - 2H_{-1}(x^n)$ is bounded below by -1. Therefore, by the assumption that $P(1|1) \geq P(-1|1)$, we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{\partial}{\partial x_1}\Pi_A(X^n, H|1) &> \frac{P(1|1) - P(-1|1)}{4a} \\ &\geq 0 \end{aligned}$$

a contradiction. We conclude that $G_1(0) = 0$, and a symmetric argument shows $G_{-1}(0) = 1$.

Now suppose $P(1|1) < 1/2$ and the support of G_1 is not an interval with lefthand endpoint zero, i.e., for some $c, d \in \Re$ such that $0 < c < d$, we have $G_1(c) = G_1(d) < 1$. Without loss of generality, assume d is in the support of G_1, i.e., $G_1(d) < G_1(e)$ for all $e > d$. Note that, by Theorem 9, d is a continuity point of A's expected payoff conditional on $i = 1$. Let X be a pure strategy with $x_1 = d$, so

$$\Pi_A(X, H|1) \quad = \quad \Pi_A(G, H|1),$$

and let X' be a pure strategy with $x_1' = d' = (c + d)/2$. We will argue that a deviation to X' is, conditional on $i = 1$, profitable for candidate A. By Theorem 9, the distribution G_1 is differentiable on $[0, m_{1,1})$, and we let g_1

denote this density. Using symmetry of G about zero, we have

$$\begin{aligned}
&\Pi_A(X', H|1) - \Pi_A(X, H|1) \\
= &\ P(-1|1)\left[\int_0^{m_{1,1}} [F_{1,-1}((d-z)/2) - F_{1,-1}((d'-z)/2)]g_1(z)\,dz\right. \\
&\left. + (1 - G_1(m_{1,1})^-)[F_{1,-1}((d - m_{1,1})/2) - F_{1,-1}((d' - m_{1,1})/2)]\right] \\
&+ P(0|1)[F_{1,0}(d/2) - F_{1,0}(d'/2)] \\
&+ P(1|1)\left[\int_0^{c} [F_{1,1}((d+z)/2) - F_{1,1}((d'+z)/2)]g_1(z)\,dz\right. \\
&+ \int_d^{m_{1,1}} [F_{1,1}((d'+z)/2) - F_{1,1}((d+z)/2)]g_1(z)\,dz \\
&\left. + (1 - G_1(m_{1,1})^-)[F_{1,1}((d' + m_{1,1})/2) - F_{1,1}((d + m_{1,1})/2)]\right] \\
= &\ P(-1|1)\left[G_1(m_{1,1})^-\left(\frac{d-d'}{4a}\right) + (1 - G_1(m_{1,1})^-)\left(\frac{d-d'}{4a}\right)\right] \\
&+ P(1|1)\left[[G_1(m_{1,1})^-\left(\frac{d'-d}{4a}\right) + (1 - G_1(m_{1,1})^-)\left(\frac{d'-d}{4a}\right)\right] \\
&+ P(0|1)\left(\frac{d-d'}{4a}\right) \\
= &\ \left(\frac{d-d'}{4a}\right)[P(-1|1) + P(0|1) - P(1|1)] \\
> &\ 0,
\end{aligned}$$

where the last inequality follows from $P(1|1) < 1/2$, a contradiction. We conclude that the support of G_1 is an interval with lefthand endpoint zero.

Suppose $P(1|1) > 1/2$ but the support of G_1 is not an interval with righthand endpoint $m_{1,1}$, i.e., for some $c, d \in \Re$ such that $c < d < m_{1,1}$, we have $0 < G_1(c) = G_1(d)$. Without loss of generality, assume c is in the support of G_1. Then, moving from c to $c' = (c + d)/2$, the previous inequality again yields a contradiction.

From Theorem 9, the support of G_1 must be, in fact, a non-degenerate interval if $P(1|1) < 1/2$. Because candidate A must be indifferent over all platforms in this interval, the first and second order conditions from the previous section must be satisfied, and, as shown in the discussion before Theorem 10, it follows that

$$G_1(z) \quad = \quad g_s(\underline{x}_1)((1/2) - F_{1,1}(\underline{x}_1))^{3/2} \int_{\underline{x}_1}^{x} \frac{1}{((1/2) - F_{1,1}(z))^{3/2}}\,dz$$

for all z in the support of G_1, where

$$g_1(\underline{x}_1) \quad = \quad \frac{-1+2P(1|1)}{4aP(1|1)(2F_{1,1}(\underline{x}_1)-1)}$$

and $\underline{x}_1 = 0$. Using

$$F_{1,1}(z) \quad = \quad \frac{1}{2} + \frac{z - m_{1,1}}{2a},$$

this simplifies to

$$G_1(z) \quad = \quad \frac{(1-2P(1|1))(\sqrt{m_{1,1}} - \sqrt{m_{1,1} - z})}{2P(1|1)\sqrt{m_{1,1} - z}},$$

for all z in the support of G_1. Solving $G_1(z) = 1$ gives us the upper bound of the support, $4m_{1,1}P(1|1)(1 - P(1|1))]$. If $P(1|1) > 1/2$, then the support of G_1 must be degenerate: otherwise, the same first and second order conditions must hold, and the density of G_1 is given above, but this is negative. From our above argument, it follows that G_1 is degenerate on $m_{1,1}$, as required. ∎

The closed form of G_1 in Proposition 11 immediately yields results about the likelihood that the candidates locate more extremely than their information suggests. Let Γ^3 index parameterizations of the canonical three signal model as considered in this section, where $P^\gamma(i,j)$ denotes the marginal probability of signals i and j and $F^\gamma_{i,j}$ the distribution of μ conditional on i and j in the game γ. As usual, we assume these parameterizations are continuous. Let $m^\gamma_{i,j}$ denote the median of $F^\gamma_{i,j}$ and let m^γ_i denote the median of F^γ_i, which are also continuous in γ. The first corollary establishes that, when the probability the other candidate receives the same signal is close to one half, a candidate who receives a left or right signal almost always chooses a platform more extreme than suggested by that signal alone.

Corollary 5 *Let $\{\gamma_n\}$ be a sequence converging to γ such that $F^{\gamma_n}_{i,j} = F^\gamma_{i,j}$ for all $i,j \in I$. Let $\{(G^n, H^n)\}$ be the corresponding sequence of mixed strategy Bayesian equilibria, and let*

$$\overline{z}_n \quad = \quad 4m^\gamma_{1,1}P^{\gamma_n}(1|1)(1 - P^{\gamma_n}(1|1))$$

denote the upper bound of the support of G^n_1. If $P^{\gamma_n}(1|1) \uparrow 1/2$, then $\overline{z}_n \to m^\gamma_{1,1}$ and $G^n_1(m^\gamma_1) \to 0$. If $P^{\gamma_n}(1|1)) \downarrow 0$, then $\overline{z}_n \to 0$ and $G^n_1(m^\gamma_1) \to 0$.

The corollary also shows that, when the probability that the other candidate receives the same signal is very small, the candidates take very moderate positions. In fact, upon inspecting the closed form of G_1 in Proposition 11, we see that, fixing z in the support of G_1, the derivative of G_1 at z is

$$-\frac{\sqrt{m_{1,1}} - \sqrt{m_{1,1} - z}}{2P(1|1)^2\sqrt{m_{1,1} - z}},$$

which is negative, so $G_1(z)$ is decreasing in $P(1|1)$. This observation yields the next corollary.

Corollary 6 *Let $\gamma, \gamma' \in \Gamma^3$ be parameterizations such that $P^{\gamma}(1|1) < P^{\gamma'}(1|1)$ and $m^{\gamma}_{1,1} = m^{\gamma'}_{1,1}$, and let (G^{γ}, H^{γ}) and (G^{γ}, H^{γ}) be the corresponding mixed strategy Bayesian equilibria. Then $G^{\gamma'}_1$ first order stochastically dominates $G^{\gamma'}_1$.*

This result implies that there is a unique threshold in $P(1|1)$ that determines whether a candidate is likely to choose a location more extreme than m_1, the "non-strategic" choice induced by the candidate's private information only. We end this section by surveying three examples of the three-signal stacked uniform model, in which we consider the probability of extremal electoral platforms as a function of different parameters of the model. The key feature of the first example is that, because $P(1|1)$ is naturally bounded strictly above zero, the candidates receiving left or right signals will always be much more likely to take extreme positions.

Example: Three Spatial Signals with Correlation. Consider the three-signal stacked uniform model in which there are three equally likely preliminary cut points, β, may take three values: -1, 0, or 1. Three possible signal realizations, -1, 0, or 1, correspond to these as follows. With probability q, the candidates receive conditionally independent signals: the probability that the signal is correct is p, while the probability that the signal is "off by one" is r.

$$\begin{aligned} p &= \Pr(s_i = 0|\beta = 0) \ = \ \Pr(s_i = 1|\beta = 1) \ = \ \Pr(s_i = -1|\beta = -1) \\ r &= \Pr(s_i = 1|\beta = 0) \ = \ \Pr(s_i = -1|\beta = 0) \ = \ \Pr(s_i = 0|\beta = 1) \\ &= \Pr(s_i = 0|\beta = -1) \end{aligned}$$

We assume that $p > r > 1 - p - r$, so we require $r < 1/3$ as well. With probability $1 - q$, the candidates receive identical signals, with the same distribution. Given $\beta = k$, the joint probability of signals i and j is

$$Q(i,j|k) \quad = \quad q\Pr(i|k)\Pr(j|k) + (1-q)\Pr(i|k)$$

if $i = j$, and $Q(i,j|k) = q\Pr(i|k)\Pr(j|k)$ otherwise. It is easily shown that

$$\begin{aligned} P(1|1) &= q\frac{p^2+r^2+(1-p-r)^2}{p+r+(1-p-r)} + (1-q) \\ m_{1,1} &= q\frac{p^2-(1-p-r)^2}{p^2+r^2+(1-p-r)^2} + (1-q)\frac{p-(1-p-r)}{p+r+(1-p-r)} \\ m_1 &= \frac{p-(1-p-r)}{p+r+(1-p-r)}. \end{aligned}$$

From these expressions, we can obtain an expression for $G_1(m_1)$ as a function of p, q, and r, which we omit because not very insightful. Calculations executed with Maple found that $G(m_1)$ is maximal for $q = 1$, p approaching 1/3 (from above), and r approaching 1/3 (from below). This allows us to calculate an upper bound for the mass that the candidates place on moderate positions after left and right signals:[3]

$$\lim_{p\downarrow\frac{1}{3},r\uparrow\frac{1}{3},q\to 1} G_1(m_1) \quad = \quad \frac{1}{2}(\sqrt{2}-1) \approx 0.207.$$

Figure 3 illustrates these comparative statics.

[Figure 3 here.]

In Figure 3a, we fix $q = 1$ and view $G_1(m_1)$ as a function of p and r alone; in 3b, we fix $p = .35$ and view $G_1(m_1)$ as a function of q and r; and in 3c, we fix $r = .3$ and view $G_1(m_1)$ as a function of p and q. In each, negative values correspond to pairs (p,r) such that G_1 is degenerate on $m_{1,1}$.

The key characteristic of the second example is that the moderate signal $s_i = 0$ is completely uninformative. In this case, we show that the probability that the candidates receiving the informative signal extremize the informational content of the signal is inversely related with the likelihood of the uninformative signal.

Example: A Conditionally Independent Uninformative Signal.

[3] The case for $p = 1/3, r = 1/3$ is meaningless because $m_1 = 0$, and $G(m_1)$ is indeterminate.

Assume that $\beta \in B = \{-1, 1\}$, with $\Pr(\beta = 1) = 1/2$. Each candidate i independently receives one of two possible signals, s_{i1} and s_{i2}, with realizations in B. The probability that i receives signal $s_{i1}$is set to be r. Signal s_{i2} is completely uninformative, so that $\Pr(s_{i2} = \beta|\beta) = 1/2$. Signal s_{i1} is informative and we parametrize $\Pr(s_{i1} = \beta|\beta) = q + (1-q)/2$, so that s_{i1} is completely uninformative if $q = 0$, and s_{i1} is fully informative if $q = 1$. This formulation falls into our analysis if we let $s_i = 0$ when i receives signal s_{i2}, and $s_i = s_{i1}$ if i receives signal s_{i1}. Note that $s_i|\beta$ are i.i.d.

Our Theorem implies that the pure-strategy equilibrium exists if and only if $r \geq \frac{1}{1+q^2}$, i.e. fixing how informative it is, if the informative signal is likely enough.

If $r < \frac{1}{1+q^2}$ there is a unique mixed strategy equilibrium such that each player i plays $m_{0,0} = 0$, upon receiving the uninformative signal $s_i = 0$, and that if she receives the signal $s_i = 1$, she plays the mixed strategy distribution:

$$G(z) = \frac{(1 - r(1+q^2))\left(\sqrt{2rq} - \sqrt{2qr - r(1+q^2)z}\right)}{r(1+q^2)\sqrt{2qr - r(1+q^2)z}} \text{ for } z \in (0, 2rq(2-r(1+q^2)).$$

The median of the distribution of the random median voter's bliss point is obtained by solving

$$\Pr(\beta = 1|s_i = 1)\int_{-a-1}^{m_{1,1}} \frac{1}{2a}dx + \Pr(\beta = -1|s_i = 1)\int_{-a+1}^{m_{1,1}} \frac{1}{2a}dx = 1/2, \text{ hence } m_1 = q.$$

By plotting the level curves of $G(m_1)$ in the space of parameters (q, r), we can have a feeling for how likely the candidate receiving an informative signal will extremize its informational content in equilibrium. In the figure below, the higher is the level curve, the lower is $G(m_1)$ (i.e. the more likely is the candidate to choose a location that extremizes the informational content of her signal). The top level stands for $G(m_1) = 0$, and the bottom level curve for $G(m_1) = 1$.

We conclude this section by constructing an example of a polling technology that induces the candidates never to play a location that is more extreme than what is suggested by their private information. The key assumption is that the candidates share the same informational sources. Thus if an agent observes an informative signal, the signal obtained by the opponent does not contain any additional information. As a result the median conditional on the agent's private information coincides with the median conditional on both agents having the same signal realization. Since this statistics is the upper bound of the agents' mixed strategies, the agents will never extremize their choices.

Example: A Correlated Uninformative Signal. Assume that $\beta \in \{-1, 1\}$, with $\Pr(\beta = 1) = 1/2$. There is a single informative signal $s \in \{-1, 1\}$, with $\Pr(s = \beta|\beta) = q > 1/2$. Each candidate independently observes s with some probability. This environment is captured by our framework by saying that $s_i = s$ with probability r and $s_i = 0$ with probability $1 - r$.

Straightforward calculations give:

$$\Pr(\beta = b|s_i = s) = \Pr(\beta = b|s_1 = s, s_2 = s), \quad \text{for } b = -1, 1.$$

It follows that $m_{s,s} = m_s$, and that $P(1|1) = r$. Thus if $r < 1/2$, then the candidate receiving the informative signal $s = 1$ will randomize on locations that belong to $(0, m_1)$. If instead $r \geq 1/2$, she will choose the location m_1 with probability one.

7 Welfare

In this section we explore how voter welfare is affected by the polling technology. We seek to glean insights into what determines whether voters prefer that candidates have access to noisier or less noisy polling technologies; and how preferences over polling technology varies with ideology. We will distinguish both the direct welfare impact of the polling technology quality, and the indirect impact of the polling technology due to the strategic responses of candidates. For ease of simplicity we will investigate welfare within the environment of 2-signal realization uniform stacked models, parametrized as follows.

The median voter's position is given by $\mu = \alpha + \beta$, *where* α *is independently and uniformly distributed on* $[-a, a]$, *and* $\beta \in \{-1, 1\}$ *with* $\Pr(\beta = 1) = 1/2$. *Candidate* j *receives signal* $s_j \in \{-1, 1\}$, $j = A, B$. *With probability* $q \geq 0$, *candidates receive the same signal, where* $Pr(s_A = s_B = \mu|\mu) = p > 0.5$; *while with probability* $1 - q$, *candidates receive conditionally independent signals of the same precision, so that* $Pr(s_j = \mu|\mu) = p$, $j = A, B$. *A voter with ideological location* θ, *derives quadratic disutility* *of* $u(\theta, y) = -(\theta - y)^2$ *from a winning candidate who adopts platform* y.[4]

In equilibrium, each candidate j adopts pure strategies, locating at

$$m_{1,1} = (2p-1)[q + (1-q)\frac{1}{p^2 + (1-p)^2}], \quad \text{if } s_j = 1;$$

and locating at

$$m_{-1,-1} = -(2p-1)[q + (1-q)\frac{1}{p^2 + (1-p)^2}], \quad \text{if } s_j = -1.$$

Since $\frac{1}{p^2+(1-p)^2} > 1$, it follows that $m_{1,1} - m_1$ rises as the correlation, q, between signals falls. Platform separation is zero if signals are perfectly correlated so that $m_1 = m_{1,1} = 2p - 1$, and, and as correlation is reduced, candidates choose to strategically separate their platforms by more and more.

With probability $qp + (1-q)p^2$, both candidates receive the signal corresponding to β; with probability $2(1-q)p(1-p)$, candidates receive distinct signals; and with probability $q(1-p) + (1-q)(1-p)^2$, both candidates receive the signal $-\beta$. The welfare of a voter with ideology $\mu + z$ is thus given by

$$\begin{aligned} W(z;p,q,a) \quad = \quad & \frac{1}{2}\left[\int_{-a}^{a}\left[\begin{array}{c} \min\{L(\alpha+1, m_{1,1}), L(\alpha+1, m_{-1,-1})\} \Pr(s_A \neq s_B | \beta = 1) \\ +L(\alpha+1, m_{1,1}) \Pr(s_A = s_B = 1 | \beta = 1) \\ +L(\alpha+1, m_{-1,-1}) \Pr(s_A = s_B = -1 | \beta = 1) \end{array}\right] \frac{d\alpha}{2a}\right] \\ & +\frac{1}{2}\left[\int_{-a}^{a}\left[\begin{array}{c} \min\{L(\alpha-1, m_{1,1}), L(\alpha, m_{-1,-1})\} \Pr(s_A \neq s_B | \beta = -1) \\ +L(\alpha-1, m_{1,1}) \Pr(s_A = s_B = 1 | \beta = -1) \\ +L(\alpha-1, m_{-1,-1}) \Pr(s_A = s_B = -1 | \beta = -1) \end{array}\right] \frac{d\alpha}{2a}\right], \end{aligned}$$

[4]One interpretation of this environment is that at date 1, candidates observe signals about the median voter's current location $\beta \in \{-1, 1\}$, and then choose their platforms. The election is at date 2. Between dates 1 and 2, the median voter's preferred platform may change. For example, if the economy declines between dates 1 and 2, the median voter may become more predisposed toward a platform that features a more aggressive economic stimulus package. The support of α captures the degree to which the median voter's most preferred platform can change over time. While p captures the precision that candidates receive about the median voter's initial location, and q captures the initial signal correlation, as the support of α is increased, the effective precision in a candidate's signal falls, while the effective correlation between signals rises (if $\alpha > 0$, candidate signals will tend to be below μ, while if $\alpha < 0$, candidate signals will tend to exceed μ).

after some calculations, we obtain

$$W(z;p,q,a) = -\frac{1}{2}(qp+(1-q)p^2)(\int_{-a}^{a}\frac{(1+\alpha+z-m_{1,1})^2}{2a}d\alpha + \int_{-a}^{a}\frac{(-1+\alpha+z+m_{1,1})^2}{2a}d\alpha)$$
$$-\frac{1}{2}(q(1-p)+(1-q)(1-p)^2)(\int_{-a}^{a}\frac{(1+\alpha+z+m_{1,1})^2}{2a}d\alpha + \int_{-a}^{a}\frac{(-1+\alpha+z-m_{1,1})^2}{2a}d\alpha)$$
$$-(1-q)p(1-p)(\int_{-a}^{-1}\frac{(1+\alpha+z+m_{1,1})^2}{2a}d\alpha + \int_{-1}^{a}\frac{(1+\alpha+z-m_{1,1})^2}{2a}d\alpha)$$
$$-(1-q)p(1-p)(\int_{-a}^{1}\frac{(-1+\alpha+z+m_{1,1})^2}{2a}d\alpha + \int_{1}^{a}\frac{(-1+\alpha+z-m_{1,1})^2}{2a}d\alpha)$$

Here, we have substituted for $m_{-1,-1} = -m_{1,1}$, to write welfare solely as a function of $m_{1,1}$. With quadratic dis-utility, expected utility depends only on the mean payoff, and the variance. Note that ideology, z, does not affect the (symmetric) bounds of integration, reflecting that the median voter is decisive, and enters only separably and symmetrically elsewhere; the difference between the median voter's welfare and that of voter z is z^2, independent of the polling technology, (p,q). It follows that all voters share a common appraisal of the attractiveness of a particular polling technology, and thus we can set $z=0$ without loss of generality. By exploiting the symmetry of the induced expression, we obtain:

$$W(p,q,a) = -(1-q)p(1-p)\left[\int_{-a}^{0}(\alpha+1+m_{1,1})^2\frac{d\alpha}{2a} + \int_{0}^{a}(\alpha+1-m_{1,1})^2\frac{d\alpha}{2a}\right]$$
$$-\frac{1}{2}\int_{-a}^{a}\left[(\alpha+1-m_{1,1})^2(qp+(1-q)p^2)+(\alpha+1+m_{1,1})^2\left(q(1-p)+(1-q)(1-p)^2\right)\right]\frac{d\alpha}{2a}.$$

When exploring how the voters' welfare changes as a function of the underlying characteristics of the model, our results identify three major forces that determine ex-ante welfare. Two of these forces are statistical properties polling technology, they are signals precision and signals correlation. The third one is the strategic effect on candidates' location, which compares the equilibrium candidates choice, with the location of "non-fully-strategic" candidates that condition their choice only on their private information. Since we would like to isolate the statistical effects from the strategic effect, we first consider a simplified model, where the strategic effect is shut down, and also signal precision and correlation are uniquely pinned down by our parameters in the model.

Specifically, we suppose that $a = 0$, in which case the median is just β. Then the precision of the signal that candidates receive about the median voter's location is p, and the signal correlation is q. In this benchmark case, strategic considerations do not arise because $m_1 = m_{1,1} = 1$ and $m_{-1} = m_{-1,-1} = -1$, so that each candidate locates as if she were non-strategically locating at the median given only his own information.

The median voter's welfare is simply given by

$$W(p, q) = -Pr(\sigma_A = \sigma_B = -\mu)4 = -4(q(1-p) + (1-q)(1-p)^2).$$

Differentiating with respect to p and q, it follows immediately that increased precision raises welfare, but increased correlation reduces welfare:

$$\frac{\partial W(p,q)}{\partial q} = -4p(1-p) < 0; \quad \frac{\partial W(p,q)}{\partial p} = 4[(2-q)(1-p) + qp] > 0.$$

While the parameter p has a positive effect on welfare, it is also the case that according to Blackwell theory, a more informative signal structure may reduce welfare. If correlation is high, in fact, then a "Blackwell garble" of the signals can raise welfare. For example, suppose that $q = 1$ so that signals are perfectly correlated. Introduce i.i.d. garbles to this signal such that $\Pr(\sigma_j = \sigma|\sigma) = z \geq 0.5,\ j = A, B$. Then median voter welfare becomes $W(z) = -4[(1-z)^2 p + z^2(1-p)]$. Differentiating with respect to z we see that reducing the garble lowers welfare if $p < z$:

$$\frac{dW(z)}{dz} = 8(p - z) < 0 \text{ if } p < z.$$

We can use this result to understand the puzzle of why voters may dislike improved polling technologies, that more accurately identifies his preferred position. On first pass, it appears that "information is bad". However, it is really the correlation that is bad. While the Blackwell garble reduces the precision of the signal, it also reduces the correlation between signals. In turn, this reduce correlation raises the probability that the median voter will have a choice between platforms. The garble would always hurt voters if they had to select a *given* candidate, because the garble causes any given candidate to target the median less accurately; but voters can choose between candidates when they offer distinct platforms, and this option to choose has significant value.

Turning to our full-fledged 2-signal uniform stacked model, we will first analyze the strategic effect. We compare the equilibrium candidates choice, with the location of "non-fully-strategic" candidates that condition their choice only on their private information, and with optimal location on the stand point of voters. By combining previous analysis with this section results, it turns out that these three quantities are ordered. We have seen that the strategic interaction between candidates causes them to locate more extremely than their private information suggests is the likely median. If the median voter could choose the amount by which candidates 'biased' their location following a signal realization, they would choose to increase the bias in their location beyond what candidates strategically choose, as voter's value the increased choice.

Proposition 1 *The optimal location for the voters m_1^* is increasing in p, and decreasing q. For any p, q, a,*

$$m_1^* > m_{1,1} > m_1.$$

The difference $m^* - m_{1,1}$ is decreasing in q, and increasing in p if a is sufficiently small, but decreasing in p if a is sufficiently large.

Proof: Solving the first order condition of (**??**) with respect to "m", we obtain

$$m^* = 2p - 1 + (1-q)p(1-p)(a + \frac{1}{a}).$$

Comparing the optimal m^* with the actual $m_{1,1}$ chosen by the candidates we obtain

$$m^* - m_{1,1} = (1-q)\left[(2p-1) + p(1-p)(a + \frac{1}{a}) - \frac{1}{1-2p(1-p)}\right] > 0.$$

∎

Turning our consideration to the statistical properties of the polling technology, it is easy to show that signal correlation is always bad in terms of welfare, as it reduces the likelihood that candidates take different locations. For any p, and a, an increase in q is to be understood as in increase of the correlation among signals, conditional on the actual median realization μ.

Proposition 2 *For any p, q, a, an increase in the correlation q decreases welfare $W(p, q, a)$.*

Proof: First we show that by substituting $m_{1,1}$ in $W(p,q,a)$, and differentiating with respect to q, we obtain an expression which is decreasing in a. In fact

$$\begin{aligned} \frac{\partial^2 W}{\partial q \partial a} &\propto \left(-20p^3 + 8p^4 - 4p + 16p^2\right) q + 1 - p - 4p^4 - 6p^2 + 10p^3 \\ &< \left(-20p^3 + 8p^4 - 4p + 16p^2\right) + 1 - p - 4p^4 - 6p^2 + 10p^3 < 0, \end{aligned}$$

where the last inequality has been checked with Maple for the relevant range, $p \in [1/2, 1]$, and the first one follows from $-20p^3 + 8p^4 - 4p + 16p^2 > 0$, which again has been checked with Maple.

Secondly, we see that

$$\frac{\partial}{\partial q}\left(\left.\frac{\partial W}{\partial q}\right|_{a=1}\right) \propto -8p - 112p^3 + 48p^2 + 128p^4 - 72p^5 + 16p^6 > 0$$

in the relevant range, as checked with Maple.

Finally, we see that

$$\left.\frac{\partial W}{\partial q}\right|_{\substack{a=1\\q=1}} \propto 22p^2 - 7p + 1 - 40p^3 + 44p^4 - 28p^5 + 8p^6 < 0,$$

as Maple concludes for the relevant range. ∎

With respect to the effect of the “precision” parameter p, the picture gets blurrier. For general values of a, it turns out that the interaction between the strategic effect and the precision of the signals upsets the monotonic relation of welfare with respect to precision. I turns out that welfare is increasing in p over a certain range, and then decreasing. The following indifference curves (derived for the case that $q = 0$, perfectly independent signals, and for the case $a = 2$) illustrate the point.

We can plot the optimal $p^*(a,q)$ for welfare, in the following three-dimensional picture, in which the optimal p^* is decreasing in a, and increasing in q.

In essence, the reason why welfare is not monotonic in p is analogous to the reason of why it is possible to obtain a welfare-increasing Blackwell garble in the simpler model where $a = 0$. Even if $q = 0$, the signals s_A and s_B are not independent given μ, but are only independent given β. Hence, when $a > 0$, any realization $\alpha \neq 0$ induces spurious correlation between the signals s_A and s_B. Moreover, the extent of this spurious correlation depends itself on p (it is increasing in p). As a result, an increase in p does not correspond only to an increase of signals' precisions, but also to an increase in their correlation. These two quantities have opposite effect on welfare, as a result welfare admits an optimal p strictly smaller than 1.

A Appendix

Lemma 1 *Let (G, H) be any pair of mixed strategies. For all $s \in S$ and all $w, z \in \Re$, one of three possibilities obtains: either*

$$\begin{aligned}
&\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][F_{s,t}(w)] \\
&\quad = \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-] \\
&\quad = \sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][1 - F_{s,t}(w)],
\end{aligned}$$

or

$$\begin{aligned}
&\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][F_{s,t}(w)] \\
&\quad < \frac{1}{2}\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-] \\
&\quad < \sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-][1 - F_{s,t}(w)],
\end{aligned}$$

or the reverse inequalities hold. Likewise for G and all $t \in T$ and all $w, z \in \Re$.

Proof: Given $s \in S$ and $z \in \Re$, the first possibility clearly obtains if $P(t|s)[H_t(z) - H_t(z)^-] = 0$ for all $t \in T$. Suppose $P(t|s)[H_t(z) - H_t(z)^-] > 0$ for some $t \in T$, and define the distribution function F^* as follows:

$$F_s^*(w) = \frac{\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-]F_{s,t}(w)}{\sum_{t\in T} P(t|s)[H_t(z) - H_t(z)^-]}.$$

Then the three possibilities above correspond to the three possibilities: $F_s^*(w) = 1/2$, $F_s^*(w) < 1/2$, and $F_s^*(w) > 1/2$. ∎

Lemma 2 *Let (G,H) be a mixed strategy Bayesian equilibrium. For all $z \in \Re$, if $G_{s'}(z) - G_{s'}(z)^- > 0$ for some $s' \in S$ and $H_{t'}(z) - H_{t'}(z)^- > 0$ for some $t' \in T$ with $P(s',t') > 0$, then*

$$\begin{aligned}
&\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][F_{s',t}(z)] \\
&\quad = \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-] \\
&\quad = \sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][1 - F_{s',t}(z)].
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s\in S} P(s|t')[G_s(z) - G_s(z)^-][F_{s,t'}(z)] \\
&\quad = \frac{1}{2}\sum_{s\in S} P(t'|s)[G_s(z) - G_s(z)^-] \\
&\quad = \sum_{s\in S} P(s|t')[G_s(z) - G_s(z)^-][1 - F_{s,t'}(z)].
\end{aligned}$$

Proof: We prove the first equalities. If they do not hold for some z and some s' and t' with $P(s',t') > 0$, then, by Lemma 1, we may assume that

$$\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][1 - F_{s',t}(z)] \quad > \quad \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-]$$

or

$$\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][F_{s',t}(z)] \quad > \quad \frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-].$$

We focus on the first inequality, as a symmetric proof addresses the second. For each $t \in T$, let λ_t denote the measure generated by the distribution H_t, let μ_t denote the degenerate measure with probability $H_t(z) - H_t(z)^-$ on z, and let $\nu_t = \lambda_t - \mu_t$. Let $\{x^n\}$ be a sequence decreasing to z, and let G^n be the mixed strategy defined by replacing G_s in G with point mass on x^n. Let

$$\pi_{s',t}(w) \quad = \quad \pi_A(z, w|s', t)$$

denote A's probability of winning using z when B receives signal t and chooses platform w, and

$$\pi^n_{s',t}(w) \quad = \quad \pi_A(x^n, w|s', t)$$

denote A's probability of winning using x^n when B receives signal t and chooses platform w. Note that

$$\begin{aligned}
&\Pi_A(G^n, H|s') - \Pi_A(G, H|s') \\
&= \sum_{t\in T} P(t|s') \int [\pi_t^n(w) - \pi_t(w)]\, \lambda_t(dw) \\
&= \sum_{t\in T'} P(t|s')[H_t(z) - H_t(z)^-][1 - F_{s',t}((z+x^n)/2) - (1/2)] \\
&\quad + \sum_{t\in T} P(t|s') \int [\pi_t^n(w) - \pi_t(w)]\, \nu_t(dw).
\end{aligned}$$

Since $\pi^n_{s',t} - \pi_{s',t} \to 0$ almost everywhere (ν_t), the corresponding integral terms above converge to zero. Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \Pi_A(G^n, H|s') - \Pi_A(G, H|s') \\
&= \sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-][F_{s',t}(z)] \\
&\quad -\frac{1}{2}\sum_{t\in T} P(t|s')[H_t(z) - H_t(z)^-] \\
&> 0,
\end{aligned}$$

and it follows that $\Pi_A(G^n, H|s) > \Pi_A(G, H|s)$ for high enough n, a contradiction. ∎

We now define a condition that imposes distinct medians, conditional on sets of signal realizations. It weakens (C4) and, in particular, is stated without the symmetry conditions (C1)-(C3).

(C4′) For all $s, s' \in S$, not conditionally equivalent, and all $T' \subseteq T$ with $P(T'|s) > 0$ and $P(T'|s') > 0$, we have $m_{s,T'} \neq m_{s',T'}$. For all $t, t' \in T$, not conditionally equivalent, and all $S' \subseteq S$ with $P(S'|t) > 0$ and $P(S'|t') > 0$, we have $m_{S',t} \neq m_{S',t'}$.

We argue that (C4′) should be thought of as a very weak restriction on the candidate's beliefs. Fixing the conditional distributions $\{F_{s,t}\}$ and viewing the marginal P on $S \times T$ as an element p of the unit simplex $\Delta \subseteq \Re^{|S|+|T|}$, the next proposition establishes that (C4′) is satisfied generically. Let $m_{s,T'}(p)$ and $m_{S',t}(p)$ be the conditional medians determined by p and the conditional distributions.

Proposition 3 *Assume that, for all $s \in S$ and all $z \in \Re$, there exist $t, t' \in T$ such that $F_{s,t}(z) \neq F_{s,t'}(z)$. And assume that, for all $t \in T$ and all $z \in \Re$, there exist $s, s' \in S$ such that $F_{s,t}(z) \neq F_{s',t}(z)$. Then the set*

$$\mathcal{P} \;=\; \{p \in \Delta : \textit{ (C4') is satisfied } \}$$

is open and dense in Δ.

Proof: Fix $s, s' \in S$, not conditionally equivalent, and $T' \subseteq T$. It suffices to show that the set of p's such that $m_{s,T'}(p) = m_{s',T'}(p)$ is closed and nowhere dense. That it is closed is obvious, following from continuity of the conditional distributions. Take any $p \in \Delta$, and note that, by assumption, there exist $t, t' \in T$ such that $F_{s',t}(m_{s',T'}(p)) < F_{s',t'}(m_{s',T'}(p))$. Now perturb p so that $P(\cdot|s)$ is unchanged, but $P(\cdot|s')$ moves an arbitrarily small amount of probability from t to t'. This will lead to a decrease in the median conditional on s' and T', as required. ∎

Lemma 3 *Assume (C4'). Let (X, Y) be a pure strategy Bayesian equilibrium. If $x_s = y_t$ for some $s \in S$ and some $t \in T$ with $P(s,t) > 0$, then $x_s = y_t = m_{s,t}$.*

Proof: Let $S' = \{s \in S : x_s = z\}$, and let $T' = \{t \in T : y_t = z\}$. Take any $s' \in S'$ and any $t' \in T'$ with $P(s', t') > 0$. Lemma 2 implies

$$\sum_{t \in T'} \frac{P(t|s')}{P(T'|s')}[F_{s',t}(z)] \;=\; \frac{1}{2}, \tag{4}$$

where we use $H_t(z) - H_t(z)^- = 1$. Thus, $z = m_{s',T'}$. If there exists $s \in S'$ with $s' \neq s$ and $P(s,t') > 0$, then (4) must hold for s as well, implying $z = m_{s,T'}$. Since $P(T'|s') > 0$ and $P(T'|s) > 0$, (C4') implies that s and s' are conditionally equivalent. The symmetric argument for candidate B establishes that $P(s',t) > 0$ implies that t and t' are conditionally equivalent. Now take any $t \in T$ such that $P(t|s') > 0$. This implies $P(s',t) > 0$, so $F_{s',t} = F_{s',t'}$. Therefore, (4) reduces to $F_{s',t'}(z) = 1/2$, i.e., $z = m_{s',t'}$. ∎

Lemma 4 *In the canonical model, assume (C8) and (C9). For each $j \in I$, let $\alpha_j \in [0,1]$. For all $i, i' \in I$ with $i < i'$ and for all $z \in M$ with*

$$0 \;<\; \alpha_j P(j|i)P(j|i')F_{i',j}(z) \;<\; \alpha_j P(j|i)P(j|i')$$

for at least one j, we have

$$\frac{\sum_{j\in I}\alpha_j P(j|i)F_{i,j}(z)}{\sum_{j\in I}\alpha_j P(j|i)} \quad > \quad \frac{\sum_{j\in I}\alpha_j P(j|i')F_{i',j}(z)}{\sum_{j\in I}\alpha_j P(j|i')}.$$

Proof: Take $i, i' \in I$, $z \in \Re$, and α_j's as in the statement of the lemma. By assumption, $\alpha_j P(j|i) > 0$ and $\alpha_j P(j|i') > 0$ for some j, so cross multiply and rewrite the desired inequality as

$$\sum_{j,j'\in I}\alpha_j\alpha_{j'}P(j|i)P(j'|i')F_{i,j}(z) \quad > \quad \sum_{j,j'\in I}\alpha_j\alpha_{j'}P(j|i')P(j'|i)F_{i',j}(z).$$

We compare the two sides of the inequality one pair $\{j, j'\}$ at a time. For $j = j'$, we have

$$\alpha_j^2 P(j|i)P(j|i')F_{i,j}(z) \quad \geq \quad \alpha_j^2 P(j|i)P(j|i')F_{i',j}(z)$$

from (C9). Moreover, there is at least one j such that $\alpha_j^2 P(j|i)P(j|i) > 0$ and $F_{i',j}(z) \in (0,1)$, which implies $F_{i,j}(z) > F_{i',j}(z)$ and gives us a strict inequality. For distinct j and j', say $j < j'$, we want to show that

$$\alpha_j\alpha_{j'}[P(j|i)P(j'|i')F_{i,j}(z) + P(j'|i)P(j|i')F_{i,j'}(z)]$$

is greater than or equal to

$$\alpha_j\alpha_{j'}[P(j|i')P(j'|i)F_{i',j}(z) + P(j'|i')P(j|i)F_{i',j'}(z)].$$

Note that, by (C9), we have

$$F_{i,j}(z) \;\geq\; \max\{F_{i,j'}(z), F_{i',j}(z)\} \;\geq\; \min\{F_{i,j'}(z), F_{i',j}(z)\} \;\geq\; F_{i',j'}(z),$$

and therefore

$$F_{i,j}(z) - F_{i',j'}(z) \quad > \quad F_{i',j}(z) - F_{i,j'}(z).$$

Then (C8) implies

$$P(j|i)P(j'|i')(F_{i,j}(z) - F_{i',j'}(z)) \quad \geq \quad P(j|i')P(j'|i)(F_{i',j}(z) - F_{i,j'}(z)),$$

which yields the desired inequality. ∎

Lemma 5 *In the canonical model, assume (C8) and (C9). Let* (G, H) *be a mixed strategy Bayesian equilibrium. For all* $z \in M$*, if* $G_i(z) - G_i(z)^- > 0$ *for some* $i \in I$ *and* $H_j(z) - H_j(z)^- > 0$ *for some* $j \in I$ *with* $P(i,j) > 0$*, then* $z = m_{i,j}$*.*

Proof: Define the sets

$$\begin{aligned} S' &= \{i \in I : G_i(z) - G_i(z)^- > 0\} \\ T' &= \{j \in I : H_j(z) - H_j(z)^- > 0\}. \end{aligned}$$

Take any $i' \in S'$ and $j' \in T'$ such that $P(i', j') > 0$. Lemma 2 implies

$$\sum_{j \in I} P(j|i')[H_j(z) - H_j(z)^-][F_{i',j}(w)] = \frac{1}{2} \sum_{j \in I} P(j|i')[H_j(z) - H_j(z)^-].$$

If $P(i, j') > 0$ for some $i \in S'$ with $i \neq i'$, then the above equality must hold for i as well. Setting $\alpha_j = H_j(z) - H_j(z)^-$, we see that (C8), (C9), and Lemma 4 imply that i and i' are conditionally equivalent. The symmetric argument for candidate B establishes that $P(i', j) > 0$ implies that j and j' are conditionally equivalent. Now take any $j \in I$ such that $P(j|i') > 0$. This implies $P(i', j) > 0$, so $F_{i',j} = F_{i',j'}$. Therefore, the above implication of Lemma 2 reduces to $F_{i',j'}(z) = 1/2$, i.e., $z = m_{i',j'}$. ∎

References

[1] Alesina

[2] Aragones and Postlewaite

[3] Ball

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[6] Downs

[7] Groseclose

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