Strategy-Proofness and Single-Crossing

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Abstract

This paper analyzes collective choices in a society with strategic voters and single-crossing preferences. It shows that, in addition to single-peakedness, single-crossingness is another meaningful domain which guarantees the existence of non-manipulable social choice functions. A social choice function is shown to be anonymous, unanimous and strategy-proof on single-crossing domains if and only if it is an extended median rule with $n-1$ parameters distributed on the end points of the feasible set of alternatives. Such rules are known as *positional dictators*, and they include the median choice rule as a particular case. As a by-product, the paper also provides an strategic foundation for the so called "single-crossing version" of the Median Voter Theorem, by showing that the median ideal point can be implemented in dominant strategies through a simple mechanism in which each agent honestly reveals his preferences.

JEL codes: D70, D71.

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1 Introduction

It is well known in economic theory that majority rule and other voting rules may fail to produce acyclic social preferences if neither, the set of

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alternatives, nor individual preferences are suitably restricted. It is also known that any voting method defined for all rational preferences over a set of three or more alternatives may be subject to the misrepresentation of individual preferences (Gibbard [17] and Satterthwaite [32]).

To study the validity of these results in more specific economic and political environments, it is common in social choice theory to appropriately restrict the set of individual preferences. If alternatives can be placed over the real line, as for instance when different levels of a public good or different tax rates are the subject of a collective choice, a natural preference restriction is *single-crossingness* (SC). The other one is, of course, single-peakedness.

Single-crossingness makes sense in many political-economic settings. It is technically useful, because it accommodates non-convexities that arise in important applications of majority voting. And it has been extensively used in the literature on political economy in areas such as income taxation and redistribution (Roberts [26], Meltzer and Richard [21], Gans and Smart [16]), local public goods and stratification (Westhoof [34], Epple et al. [12], Epple and Platt [13], Epple et al. [14], Calabrese et al. [7]), coalition stability (Demange [9], Kung [19]) and, more recently, to study policies in the market for higher education (Epple et al. [15]) and the citizen candidate model under uncertainty (Eguia [10]).

In words, a society possesses single-crossing preferences if, given any two policies, one of them more to the right than the other, the more rightist the individual is with respect to the other agents, the more he will be willing to support the right-wing policy over the left-wing one. Thus, for example, if alternatives represent income tax rates, and individuals are *ordered* according to their incomes, this restriction simply means that, the richer the individual is, the lower the tax rate he will be willing to support.

Like other domain conditions, single-crossingness establishes restrictions across individual preferences, i.e. on the character of voters' heterogeneity. However, it does not impose any restriction on the shape of each individual preference relation. The main idea behind SC is that, in some cases, individual preferences can be *ordered* in such a way that, for every pair of alternatives, say x and y, whenever two preference orderings, say P' and P'', coincide in raking x above y, so do all preferences *in between*, so that the set of preference relations ranking one alternative above the other all lie to one side of those who have the opposite ranking.

Technically, SC not only guarantees the existence of majority voting equilibria, but it also provides a simple characterization of the core of the majority rule.[1] In effect, the core is simply the ideal point of the median agent, where the latter is defined over the ordering of individual preferences which makes the profile single-crossing.[2] Different versions of this result appeared first in the seminal works of Roberts [26] and Grandmont [18] and, more recently, in Rothstein [28], Gans and Smart [16] and Austen-Smith and Banks [1]. It is sometimes referred to as the *Representative Voter Theorem* (RVT) or, alternatively, as "the second version" of the *Median Voter Theorem* (MVT).

The problem with this result is that, unlike the MVT over single-peaked preferences, whose non-cooperative foundation was provided by Black [5], first, and then by Moulin [22], the RVT is based on the assumption that individuals honestly reveal their preferences. That is, it is derived assuming *sincere* voting. Hence, a natural question about its legitimacy arises when individual values are private information and voters can behave strategically.

This issue has been recently addressed by Saporiti and Tohmé [31]. In that paper, we showed that SC is sufficient to ensure the existence of non-manipulable social choice rules. In particular, this is true for the median choice rule, which is strategy-proof and group strategy-proof over the full set of alternatives and over every possible policy agenda.

Taking that work as the starting point, in this paper we characterize the family of *anonymous* (A), *unanimous* (U) and *strategy-proof* (SP) social choice functions on single-crossing domains. This family coincides with the class of *positional dictators*, which are extended median rules with $n-1$ parameters distributed on the end points of the feasible set of alternatives. It includes the median choice rule as a particular case.

Although the word "dictator" may initially generate a negative feeling toward our characterization, it is worth noting that the result is far from being a negative one. Anonymity and unanimity are very weak conditions, and strategy-proofness is a desirable incentive compatibility property that is frequently demanded in social choice. On the other hand, as will be clear in Section 2, a positional dictator is an *anonymous* social choice function that only considers the ordering of the announced most preferred alternatives, and always chooses one at a specified rank (e.g., the first ideal point, the second, the median, etc.). The preselected position is a "dictator". But,

[1] The core of a preference aggregation rule at any profile of individual preferences is the set of top ranked alternatives of the social preference relation (Austen-Smith and Banks [1], p. 99).

[2] Instead, under single-peakedness, the core of the majority rule is given by the median ideal point over the ordering of the alternatives that makes the profile single-peaked.

since in different profiles different individuals may locate at that position, there is no such a thing as a dictator, as it is understood in social choice.

In our model, positional dictators refer to the simple majority rule and other qualified majorities. Hence, the main message coming out from the analysis is that single-crossing is another simple example, besides single-peakedness, where majority voting works with "maximal" incentives properties. The article explains the root of this good property of single-crossing domains, and how far we can go in changing the majority rule.

To summarize the contribution of this article and to compare it with other important results over the real line, namely, with Moulin's [22] seminal work, we draw a diagram below that shows the family of A, U and SP social choice functions on single-peaked and single-crossing domains.[3] As the figure illustrates, since SC allows any shape in individual preferences, it leads to a smaller (but still large) family of strategy-proof social choice rules. Incidentally, the picture also points out that the class of non-manipulable rules in the intersection of these two domains (whenever nonempty) is still an open question. To the best of the author's knowledge, this subdomain, which contains preferences such as the Euclidean one, has not received enough attention, and a full characterization is still missing.

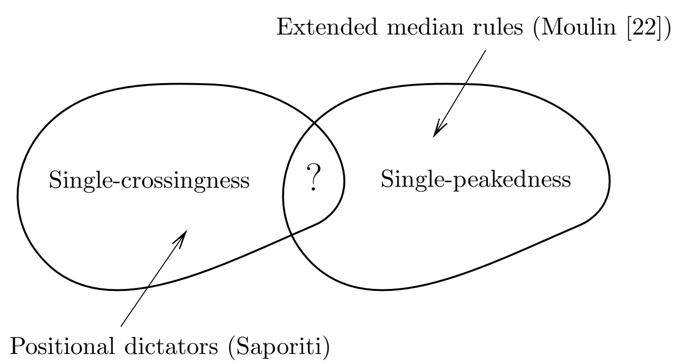


Figure 1:

The rest of the paper is organized as follows. Section 2 presents the

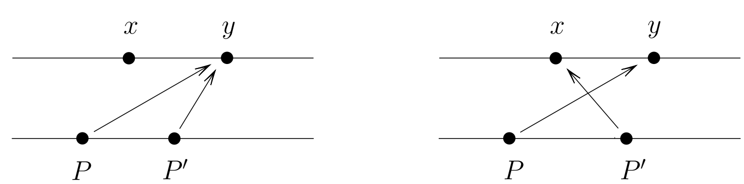
[3] Moulin's [22] original characterization on single-peaked preferences over the real line has been extended in several directions by many authors. Some important references within this literature are Border and Jordan [6], Zhou [36], Barberà et al. [2], Barberà and Jackson [3], Ching [8], Berga [4], Schummer and Vohra [33], and Ehlers et al. [11], but this list is by no means exhaustive.

model, the notation and the definitions. Section 3 contains all the results. We start by proving that every positional dictator is *group strategy-proof* (GSP) on single-crossingness (Proposition 1). Then, in Theorem 2, we state that, although single-crossing does not satisfy Weymark's [35] regularity, U and SP imply tops-onliness (TO). Finally, using anonymity and unanimity as auxiliary conditions, we prove that every strategy-proof social choice function is a positional dictator (Theorem 1), with the natural corollary that in our framework U, A and SP imply Pareto efficiency (Corollary 1). Final remarks appear in Section 4.

2 Preliminaries

Consider a society $I = \{1, 2, \ldots, n\}$ with a finite number $n \geq 2$ of agents, who must choose an alternative from a finite set $X = \{x, y, \ldots\}$, $|X| > 2$.[4]

A preference relation P over X is a complete, transitive and antisymmetric binary relation on X. We say that a set $\mathcal{SC}$ of preference relations has the ***single-crossing property*** if there exists a linear order $\succeq$ of the elements of $\mathcal{SC}$, and a linear order $\geq$ over the set of social alternatives X such that, for all $x, y \in X$, and all $P, P' \in \mathcal{SC}$, $[y > x,\ P' \succ P,\ \&\ y\,P\,x] \ \Rightarrow\ y\,P'\,x$, and $[y > x,\ P' \succ P,\ \&\ x\,P'\,y] \ \Rightarrow\ x\,P\,y$.[5] Figure 3 below illustrates the concept.



Single-crossing Non single-crossing

Figure 2:

A preference profile $P = (P_i)_{i \in I}$ is ***single-crossing* (SC)** over X if for all $i \in I$, $P_i \in \mathcal{SC}$.[6] We call $\mathcal{SC}^n$ the set of all single-

[4] For every set A, $|A|$ stands for the cardinality of the set.

[5] As usual, $>$ is the strict part of $\geq$, and $\succ$ the strict part of $\succeq$.

[6] Other domain restrictions related with single-crossing are *hierarchical adherence*, *intermediateness*, *order-restriction* and *unidimensional alignment*. For more details, see Roberts [26], Grandmont [18], Rothstein [27] and [28], Gans and Smart [16], Myerson

crossing preference profiles. As usual, for any profile $P = (P_1, \ldots, P_n) \in \mathcal{SC}^n$, $P_{-i} = (P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$; for each $\hat{P}_i \in \mathcal{SC}$, $(\hat{P}_i, P_{-i}) = (P_1, \ldots, P_{i-1}, \hat{P}_i, P_{i+1}, \ldots, P_n)$; and, for every set $S \subseteq I$, $(P_S, P_{\bar{S}}) = (\{P_i\}_{i \in S}, \{P_j\}_{j \in \bar{S}})$, where $\bar{S} = I \backslash S$ is the complement of S.

The next example, taken from Persson and Tabellini [25], illustrates how our abstract setup may naturally emerge in political economy.

Example 1 *Consider Roberts' [26] model on redistributive linear tax schemes. Suppose each agent i has preferences $u(c_i, l_i) = c_i + v(l_i)$, where c_i denotes private consumption, l_i leisure time, and $v(l_i)$ a continuous and concave function. Let $c_i \leq (1-t)h_i + f$ be the individual budget constraint, where $t \in (0,1)$ is an income tax rate, h_i the individual labor supply, and $f = (\sum_{i \in I} t\, h_i)/n$ a lump-sum transfer.*[7] *Assume each agent is endowed with productivity $\theta_i \in \Re$, and let $l_i + h_i \leq 1 - \theta_i$ be his effective time constraint. If we solve the constrained maximization problem of each individual and substitute the solution into his utility function, then the indirect utility associated with a tax rate t is given by $w(t, \theta_i) = u(c_i^*(t, \theta_i), l_i^*(t, \theta_i)) = h(t) + v[1 - h(t) - \bar{\theta}] - (1-t)(\theta_i - \bar{\theta})$, where $h(t) = 1 - \bar{\theta} - v_l^{-1}(1-t)$ is the average labor supply, v_l the first derivative of $v(l_i)$, and $\bar{\theta}$ the mean productivity. Hence, the profile of induced preferences is single-crossing on the interval $(0,1)$, because for any two policies t', $t'' \in (0,1)$, such that $t' > t''$, the difference $w(t', \theta) - w(t'', \theta)$ is strictly increasing in θ.* $\square$

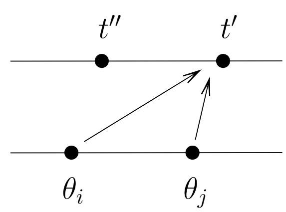


Figure 3:

The recent interest in single-crossingness is due to the fact that, like single-peakedness, this domain restriction is sufficient to guarantee the existence of majority voting equilibria. However, apart from this, it should

[24], Austen-Smith and Banks [1], List [20] and Saporiti and Tohmé [31].

[7] The real wage is exogenous and normalized at 1.

be clear that both conditions are totally independent, in the sense that neither property is logically implied by the other.[8] Examples 2 and 3 below illustrate this point.[9]

Example 2 *Assume individual preferences are as in Table 1. This profile is single-crossing on* $X = \{x, y, z\}$ *with respect to* $z > y > x$ *and* $P_3 \succ P_2 \succ P_1$*. However, for any ordering of the alternatives, it violates single-peakedness, because every alternative is ranked bottom in one preference relation.* □

Example 3 *Consider the profile displayed in Table 2. These preferences are single-peaked with respect to* $z > y > x > w$*. On the contrary, for every ordering of the binary relations, they violate single-crossing. Moreover, they violate SC not only for* $z > y > x > w$*, but for every ordering of them.* □

Table 1: Single-crossingness

$P_1 : xyz$
$P_2 : xzy$
$P_3 : zyx$

Table 2: Single-peakedness

$P_1 : xyzw$
$P_2 : zyxw$
$P_3 : yxwz$

Since we are interested in social choice functions that are not manipulable over $\mathcal{SC}^n$, in what follows we restrict our attention to *maximal* domains of single-crossing preferences, in the sense that it would be impossible to add another preference relation in $\mathcal{SC}$ such that every profile of the enlarged domain $\mathcal{SC}^n$ still satisfies SC. These domains contain the largest number of possible deviations. Therefore, they are the appropriate framework to analyze incentive compatibility.

In order to make social choices, individual preferences must be aggregated. The aggregation process is represented by a social choice function. A social choice function is a single-valued mapping $f : \mathcal{SC}^n \to X$ that associates to each profile $P \in \mathcal{SC}^n$ a unique outcome $f(P) \in X$. Denote by $r_f = \{x \in X : \exists\, P \in \mathcal{SC}^n$ such that $f(P) = x\}$ the range of f. We are interested in social choice functions that satisfy the following properties on

[8] As Gans and Smart [16] showed, single-crossingness is equivalent to Rothstein's [27] and [28] order-restriction (OR), and OR (on triples) is strictly weaker than single-peakedness and single-cavedness, but strictly stronger than Sen's value-restriction, (see Theorems 2 and 3 in Rothstein [27]).

[9] The interesting difference between single-crossing and single-peakness is that the latter is a unique domain once alternatives are ordered, whereas there are still many different SC domains compatible with a given ordering of X. On the other hand, unlike single-peaked preferences, their union covers all preferences on X.

$\mathcal{SC}^n$. The main one is that agents, acting individually or in groups, never have incentives to misrepresent their preferences.

Definition 1 (SP) *A social choice function f is* ***strategy-proof*** *on $\mathcal{SC}^n$ if $\forall\, i \in I$, and $\forall\, (P_i, P_{-i}) \in \mathcal{SC}^n$, $\nexists \hat{P}_i \in \mathcal{SC}$ such that $f(\hat{P}_i, P_{-i})\, P_i\, f(P_i, P_{-i})$.*

In words, a social choice function f is SP on $\mathcal{SC}^n$ if for *any* possible report $P_{-i} \in \mathcal{SC}^{n-1}$ that the rest of the agents could make, no individual $i \in I$ would find profitable to make a declaration $\hat{P}_i \in \mathcal{SC}$ different from his own ordering P_i. On the contrary, if f is not strategy-proof, then there must exist at least one agent who would be strictly better off misrepresenting his preferences. Therefore, we say that f is ***manipulable*** by this individual.

Proceeding in a similar way, we can also define *group strategy-proofness*, to study the possibility of group deviations.

Definition 2 (GSP) *A social choice function f is* ***group strategy-proof*** *on $\mathcal{SC}^n$ if $\forall\, S \subseteq I$, and $\forall\, (P_S, P_{\bar{S}}) \in \mathcal{SC}^n$, $\nexists\, \hat{P}_S \in \mathcal{SC}^{|S|}$ such that $\forall i \in S$, $f(\hat{P}_S, P_{\bar{S}})\, P_i\, f(P_S, P_{\bar{S}})$.*

Another property that we may seek in a social choice function is *unanimity.* This property ensures that, if all agents have the same most preferred alternative, then that alternative is socially selected. For any $P \in \mathcal{SC}$, let $\tau(P) \equiv \arg\max_X\ P$.

Definition 3 (U) *A social choice function f is* ***unanimous*** *on $\mathcal{SC}^n$ if $\forall\, x \in X$, and $\forall\, P \in \mathcal{SC}^n$ such that $\tau(P_i) = x$ $\forall i \in I$, $f(P) = x$.*

Let $\sigma : I \to I$ be a permutation of the set of individuals. A profile $P \in \mathcal{SC}^n$ is a σ-permutation of another profile $P^* \in \mathcal{SC}^n$ if for every individual $i \in I$, $P_i = P^*_{\sigma(i)}$. That is, P is a σ-permutation of P^* if the lists of preferences under P and P^* are identical up to a renaming of agents. We refer to such a pair (P, P^*) as a σ-permutation.

Definition 4 (A) *A social choice function f is* ***anonymous*** *on $\mathcal{SC}^n$ if for each σ-permutation (P, P^*), $f(P) = f(P^*)$.*

In words, a social choice function is anonymous if the names of the individuals holding particular preferences are immaterial in deriving social choices.

One last property that a social choice function may satisfy is *tops-onliness.* We say that f is *tops-only* on $\mathcal{SC}^n$ if for any preference profile, the social choice is exclusively determined by individuals' most preferred alternatives on the range of the social choice function.

Definition 5 (TO) *A social choice function f is **tops-only** on $\mathcal{SC}^n$ if, $\forall P, \hat{P} \in \mathcal{SC}^n$ such that $\tau|_{r_f}(P_i) = \tau|_{r_f}(\hat{P}_i)\ \forall i \in I$, $f(P) = f(\hat{P})$.*

Tops-onliness dramatically constrains the scope for manipulation. No agent can expect to be able to affect the social outcome without modifying the peak on r_f of his reported ordering. However, as we show later in Theorem 2, this condition is closely related to SP, in the sense that every U and SP social choice function on single-crossing domains is also TO.

Now we define a class of social choice functions that plays a crucial role in the characterization given in Section 3. To do that we introduce the following notation. For any odd positive integer k, we say that $m^k : X^k \to X$ is the *k-median* function on X^k if for each $x = (x_1, \ldots, x_k) \in X^k$, $|\{x_i : m^k(x) \geq x_i\}| \geq \frac{(k+1)}{2}$, and $|\{x_j : x_j \geq m^k(x)\}| \geq \frac{(k+1)}{2}$. Since k is odd, $m^k(x)$ is always well defined.

Definition 6 (EMR) *A social choice function f is an **extended median rule** on $\mathcal{SC}^n$ if there exist $n+1$ parameters $\alpha_i \in X$, $i = 1, 2, \ldots, n+1$, also called fixed ballots or phantom voters, such that $\forall P \in \mathcal{SC}^n$, $f(P) = m^{2n+1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n+1})$.*

We denote by f^e a social choice function that satisfies Definition 6, and by $EMR = \{f^e : (\alpha_1, \ldots, \alpha_{n+1}) \in X^{n+1}\}$ the family of all such functions, obtained by reallocating the parameters $\alpha_1, \ldots, \alpha_{n+1}$ in X^{n+1}. A particular case of interest within this family is the well known ***median choice rule***, noted f^m, which is obtained from f^e by assigning $(n+1)/2$ fixed ballots at $\underline{X} \equiv \min X$ and the rest at $\overline{X} \equiv \max X$, if n is odd, and $n/2$ at $\underline{X}$ and $n/2+1$ at $\overline{X}$ if n is even.

Proceeding in a similar way, we can derive other rules from EMR, by restricting each α_i to a particular value of X. For example, if $\alpha_i = \alpha$ for all $i = 1, 2, \ldots, n+1$, then f^e is completely insensitive to the preferences reported by the individuals. We might want to exclude such undesirable rules and, in particular, require Pareto efficiency.[10] To do that, we eliminate the possibility of inefficiency by setting $\alpha_n = \underline{X}$ and $\alpha_{n+1} = \overline{X}$. Then, we obtain a social choice rule, noted f^*, with the property that for all $P \in \mathcal{SC}^n$, $f^*(P) = m^{2n-1}(\tau(P_1), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n-1})$. This rule is called the ***efficient extended median rule***, and it is characterized by $n-1$ parameters distributed on X^{n-1}. The set of all such rules is denoted $EMR^* = \{f^* : (\alpha_1, \ldots, \alpha_{n-1}) \in X^{n-1}\}$.

[10] A social choice function f is ***Pareto efficient*** on $\mathcal{SC}^n$ if for all $P \in \mathcal{SC}^n$, $\nexists y \in X$ such that $y\, P_i\, f(P)$ for all $i \in I$.

Finally, we can also restrict each α_i to take its value at either $\underline{X}$ or $\overline{X}$, so that each phantom voter is either a *leftist* or a *rightist*. The family of social choice functions obtained in that way was first introduced by Moulin [23], and it is known as ***positional dictators***.

These rules select the j-th peak among the tops of the reported preference orderings, for some $j \in \{1, \ldots, n\}$. For example, if $j = 1$, we have the *leftist rule*, which always chooses the smallest reported peak. The median choice rule f^m is also a particular case. We denote by f^j the positional dictator that selects, for all $P \in \mathcal{SC}^n$, the alternative of the sequence $\tau(P_1), \ldots, \tau(P_n)$ placed at the j-th position according with the order of X. This rule is obtained from f^* by distributing $n-j$ fixed ballots at $\underline{X}$ and the remaining $j-1$ at $\overline{X}$. The family of all such rules is denoted $PD = \{f^j; j = 1, \ldots, n\}$.

3 Characterization

In this section, we prove that positional dictators is the only family of social choice functions that satisfies U, A and SP on single-crossing domains. At the end, we also show that this is a tight characterization, in the sense that relaxing any of the previous axioms enlarges the family of social choice functions.

We start by proving that every positional dictator is GSP.

Proposition 1 *Each positional dictator f^j is group strategy-proof on $\mathcal{SC}^n$.*

Proof: Fix $f^j \in PD$. Suppose, by contradiction, there exist a coalition $S \subseteq I$, a profile $(P_S, P_{\bar{S}}) \in \mathcal{SC}^n$, and a joint deviation $\hat{P}_S \in \mathcal{SC}^{|S|}$ for S such that $f^j(\hat{P}_S, P_{\bar{S}})\, P_i\, f^j(P_S, P_{\bar{S}})$ for all $i \in S$. To simplify, denote $f^j(P_S, P_{\bar{S}}) \equiv \tau$ and $f^j(\hat{P}_S, P_{\bar{S}}) \equiv \hat{\tau}$, and let $\hat{\tau} > \tau$.

Note that $f^j \in PD \Rightarrow \alpha_i \in \{\underline{X}, \overline{X}\}$ for all $i = 1, 2, \ldots, n-1$. Hence, τ and $\hat{\tau}$ must coincide with the tops reported by two *real* voters. Denote these agents k and k', and their preferences P_k and $P_{k'}$, respectively. Then, for all $i \in S$, $\tau(P_i) > \tau$. Suppose not. That is, assume $\tau \geq \tau(P_i)$ for some agent $i \in S$. If $\tau(P_i) = \tau$, then $\tau\, P_i\, \hat{\tau}$, which contradicts our initial hypothesis. Instead, suppose $\tau > \tau(P_i)$. Since $\hat{\tau}\, P_i\, \tau$ and $(P_S, P_{\bar{S}}) \in \mathcal{SC}^n$, we have that $\hat{\tau}\, P\, \tau$ for all $P \succ P_i$. Then, $P_i \succ P_k$. Otherwise, $\hat{\tau} > \tau$, $P_k \succ P_i$ and $\hat{\tau}\, P_i\, \tau$ would imply $\hat{\tau}\, P_k\, \tau$, contradicting that $\tau = \tau(P_k)$. And, again, since $(P_S, P_{\bar{S}}) \in \mathcal{SC}^n$, $\tau\, P_k\, \tau(P_i)$ implies $\tau\, P_i\, \tau(P_i)$: contradiction. Hence, $\tau(P_i) > \tau$ for all $i \in S$.

By definition, $\tau = m^{2n-1}(\{\tau(P_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$ and $\hat{\tau} = m^{2n-1}(\{\tau(\hat{P}_i)\}_{i \in S}, \{\tau(P_j)\}_{j \in \bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$. Thus, there must exist $i \in S$

such that $\tau > \tau(\hat{P}_i)$. Otherwise, if $\tau(\hat{P}_i) \geq \tau$ for all $i \in S$, we would have that $\hat{\tau} = \tau$. Therefore, if we rename $(\{\tau(\hat{P}_i)\}_{i\in S}, \{\tau(P_j)\}_{j\in \bar{S}}, \alpha_1, \ldots, \alpha_{n-1})$ as $(y_1, \ldots, y_{2n-1})$, it follows that $|\{j \in \{1, \ldots, (2n-1)\} : \tau \geq y_j\}| \geq n$. But then $\tau \geq m^{2n-1}(y_1, \ldots, y_{2n-1})$. That is, $f^j(P_S, P_{\bar{S}}) \geq f^j(\hat{P}_S, P_{\bar{S}})$, contradicting that $\hat{\tau} > \tau$. Hence, f^j is GSP on $\mathcal{SC}^n$. $\square$

Falling short of Moulin's [22] results, Proposition 1 shows that every extended median rule is GSP (and, consequently, SP) on single-crossing domains, provided that each fixed ballot is placed at the end points of X, (i.e., at either $\underline{X}$ or $\overline{X}$). Instead, all other extended median rules, which allow the collective outcome to be the top of a *fictitious* voter, are not guaranteed to be SP on $\mathcal{SC}^n$.

To see this, consider the profile of Table 1, and a rule $f \in EMR^*$, such that $\alpha_1 = y$ and $\alpha_2 = z$. Note that α_1 coincides with neither voters' most preferred alternatives nor the end points of $X = \{x, y, z\}$, (recall that $\underline{X} = x$ and $\overline{X} = z$). Furthermore, $f(P) = m^5(x, x, z, \alpha_1, \alpha_2) = y$. But, since y is agent 2's worst outcome on X, he could report $\hat{P}_2 : zyx$, and generate the outcome $m^5(x, z, z, \alpha_1, \alpha_2) = z$. Agent 2's deviation would be profitable, because $z\, P_2\, y$. Hence, individual manipulation cannot be excluded.[11]

As the example illustrates, SP is not ensured for extended median rules other than positional dictators because the latter are the only one within the class of *anonymous* social choice functions which guarantee that the social choice always coincides with a voter's most preferred alternative. However, as we showed in the proof of Proposition 1, without this information manipulation on single-crossing domains cannot be ruled out, because the argument exploits precisely the correlation among individual preferences together with the fact that the outcome is the ideal point reported by a real voter.

The point is that SC does not restrict the shape of individual preferences. Instead, it allows orderings that do not decrease monotonically to both sides of the ideal point. In fact, this is one of the main reasons why SC is an attractive restriction in certain problems of political economy (such as majority voting over income taxation). The price for this flexibility, however, is that in general it is impossible to ensure that no agent could be better off misrepresenting his values.

In Figure 4, for instance, $f(\hat{P}_i, P_{-i})\, P_i\, f(P_i, P_{-i})$, so that in principle agent i would like to manipulate f at (P_i, P_{-i}) via $\hat{P}_i$. However, this is not possible if f is a positional dictator. In that case, SC is sufficient to rule out

[11]Interestingly, in the example, agent 2 would prefer to misrepresent his ordering even if the other agents report their true preferences. That means extended median rules other than positional dictators not only fail to be SP over $\mathcal{SC}^n$, but also Nash implementable.

any attempt of individual and group manipulation. For example, suppose that $f(P_i, P_{-i})$ is j's most preferred alternative. If $f(\hat{P}_i, P_{-i})\, P_i\, f(P_i, P_{-i})$, like in Figure 4, SC would imply $f(\hat{P}_i, P_{-i})\, P_k\, f(P_i, P_{-i})$ for all $P_k \succ P_i$. Thus, $f(P_i, P_{-i}) = \tau(P_j) \Rightarrow P_i \succ P_j$. But then agent i's preferences cannot be like in the figure. Otherwise, $(P_i, P_{-i}) \in \mathcal{SC}^n$, $f(P_i, P_{-i})\, P_j\, \tau(P_i)$ and $P_i \succ P_j$ would imply $f(P_i, P_{-i})\, P_i\, \tau(P_i)$, contradicting that $\tau(P_i)$ is agent i's ideal point.

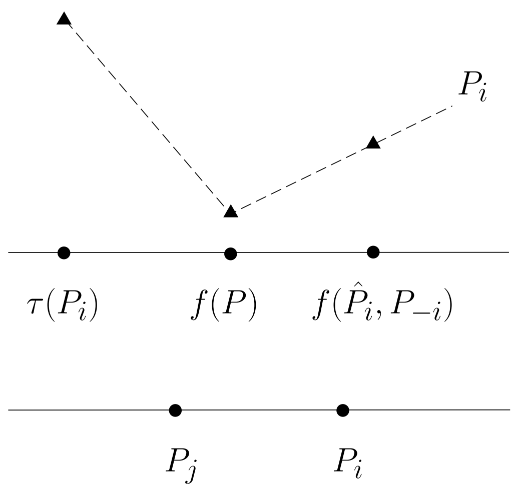


Figure 4:

Thus, when the choice rule associates to each preference profile an individual's peak, like in the case of positional dictators, the ordering of that agent together with the relation among preferences in single-crossing domains is sufficient to reject any incentive for manipulation. Remarkably, *no additional information about the shape of each preference relation is needed.*

Instead, if social choices are not individual tops, we might think that individuals' preferences can still be inferred from the correlation with other agents' rankings. However, there are profiles on single-crossingness where the way in which one agent orders alternatives bears no relation with other orderings. In those cases, it is impossible to guarantee that all individuals will have the *right* incentives, (i.e., no one will hold an ordering like Figure 4). So, manipulation cannot be excluded.

This conjecture stands in sharp contrast with the main result on single-peaked domains, where extended median rules have been shown to be strategy-proof without any restriction on the distribution of phantom voters.

Moreover, it suggests that the family of SP social choice functions on $\mathcal{SC}^n$ is strictly smaller than the corresponding class on single-peakedness. This is now formally stated in Theorem 1 and proved in the rest of this section.

Theorem 1 *A social choice function f is unanimous, anonymous, and strategy-proof on $\mathcal{SC}^n$ if and only if f is a positional dictator.*

Corollary 1 *If a social choice function f is unanimous, anonymous and strategy-proof on $\mathcal{SC}^n$, then it is Pareto efficient.*

Proof: Suppose, by contradiction, that there exists a social choice function f that satisfies all the hypotheses of Corollary 1, but that f is not Pareto efficient on $\mathcal{SC}^n$. Then, there must exist $P \in \mathcal{SC}^n$, and a pair $x, y \in X$, $x \neq y$, such that $f(P) = x$, while $y\, P_i\, x$ for all $i \in I$. Thus, for all $i = 1, \ldots n$, $f(P) \neq \tau(P_i)$, contradicting that, by Theorem 1, $f \in PD$. $\square$

The proof of Theorem 1 rests on three main results. The first one, summarized in Theorem 2 below, shows that on single-crossing domains tops-onlyness is implied by strategy-proofness and unanimity. This result is consistent with other results in the literature on strategy-proofness, and captures the intuitive idea that social choice functions that use too much information from society are easier to manipulate.

Theorem 2 *A social choice function f is unanimous and strategy-proof on $\mathcal{SC}^n$ only if f is tops-only on $\mathcal{SC}^n$.*

Proof: See Saporiti [30]. $\square$

Apart from Theorem 2, the proof of Theorem 1 also invokes two additional results, which are summarized in Lemma 1 and 2, respectively. The first of these lemmas points out that, if a social choice function is SP and U (and therefore TO), then no individual must be able to profit by reporting extreme ideal points, unless such extreme preferences constitute the individual's true ordering. This "median property" at the individual level must simultaneously hold for *every* agent.

To present this more formally, in the sequel we use $\underline{P}_i$ (respectively, $\overline{P}_i$) to denote agent i's most leftist (respectively, rightist) preference relation on X according with $\succeq$ and $\geq$, so that for all $x, y \in X$, $x\, \underline{P}_i\, y$ (respectively, $y\, \overline{P}_i\, x$) if and only if $y > x$. Clearly, $\tau(\underline{P}_i) = \underline{X}$ and $\tau(\overline{P}_i) = \overline{X}$. Moreover, it is easy to check that these rankings always belong to $\mathcal{SC}$.

Lemma 1 *A social choice function f is unanimous and strategy-proof on $\mathcal{SC}^n$ only if, for all $i \in I$, and all $P \in \mathcal{SC}^n$,*

$$f(P_i, P_{-i}) = m^3(\tau(P_i), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})).$$

Proof: Let f be U and SP on $\mathcal{SC}$.[12] By Theorem 2, f is TO on $\mathcal{SC}^n$. Fix a profile $P \in \mathcal{SC}^n$ and an agent $i \in I$. If $f(\underline{P}_i, P_{-i}) > f(\overline{P}_i, P_{-i})$, then $f(\underline{P}_i, P_{-i})\, \overline{P}_i\, f(\overline{P}_i, P_{-i})$. Thus, i would like to manipulate f at $(\overline{P}_i, P_{-i})$ via $\underline{P}_i$: contradiction. Hence, $f(\overline{P}_i, P_{-i}) \geq f(\underline{P}_i, P_{-i})$. Two cases are possible:

<u>Case 1</u>: $f(\overline{P}_i, P_{-i}) > \tau(P_i) > f(\underline{P}_i, P_{-i})$. Then, $m^3(\tau(P_i), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) = \tau(P_i)$. Assume, by contradiction, $f(P) \neq \tau(P_i)$. Without loss of generality, suppose $\tau(P_i) > f(P)$ $\Rightarrow f(\overline{P}_i, P_{-i}) > f(P)$. By SP, $f(P_i, P_{-i})\, P_i\, f(\overline{P}_i, P_{-i})$ and $f(\overline{P}_i, P_{-i})\, \overline{P}_i\, f(P_i, P_{-i})$. Define a preference relation P_i' such that (i) $\tau(P_i') = \tau(P_i)$, and (ii) $f(\overline{P}_i, P_{-i})\, P_i'\, f(P_i, P_{-i})$ (see Figure 5 below). Since P_i' is *between* P_i and $\overline{P}_i$, $P_i' \in \mathcal{SC}$ and $\overline{P}_i \succ P_i' \succ P_i$. By TO, $f(P_i', P_{-i}) = f(P_i, P_{-i})$. Thus, $f(\overline{P}_i, P_{-i})\, P_i'\, f(P_i', P_{-i})$: contradiction.

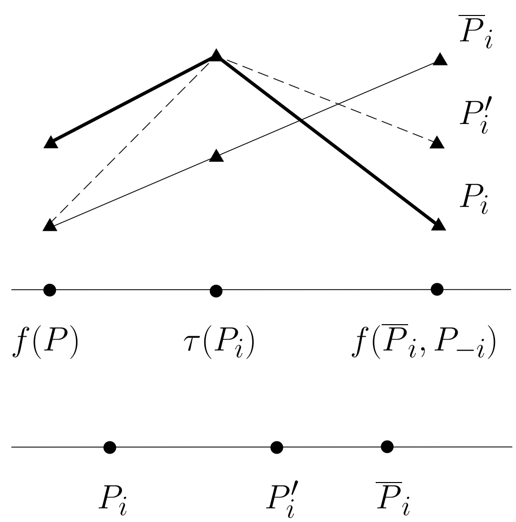


Figure 5:

<u>Case 2</u>: $f(\underline{P}_i, P_{-i}) \geq \tau(P_i)$.[13] Then, $m^3(\tau(P_i), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) = f(\underline{P}_i, P_{-i})$. Assume, by contradiction, $f(P) \neq f(\underline{P}_i, P_{-i})$. First, suppose

[12] Note that U implies $r_f = X$; hence, for all $i \in I$, and all $P_i \in \mathcal{SC}$, $\tau(P_i) = \tau|_{r_f}(P_i)$.

[13] The remaining case where $\tau(P_i) \geq f(\overline{P}_i, P_{-i})$ is similar.

that $f(\underline{P}_i, P_{-i}) > f(P)$. Then, $\underline{P}_i \succ P_i$. Otherwise, if $P_i \succ \underline{P}_i$, SC would imply that $f(\underline{P}_i, P_{-i})\, P_i\, f(P_i, P_{-i})$, which contradicts SP. However, since $\underline{P}_i$ is agent i's most leftist preference relation, $\underline{P}_i \succ P_i$ implies $\tau(P_i) = \tau(\underline{P}_i) = \underline{X}$. Hence, by TO, $f(P_i, P_{-i}) = f(\underline{P}_i, P_{-i})$: contradiction. Thus, $f(P) > f(\underline{P}_i, P_{-i}) \Rightarrow f(P_i, P_{-i}) > \tau(P_i)$. Note that $\tau(P_i) \neq f(\underline{P}_i, P_{-i})$. Otherwise, if $\tau(P_i) = f(\underline{P}_i, P_{-i})$, then $f(P_i, P_{-i}) \neq f(\underline{P}_i, P_{-i})$ would imply that i would like to manipulate f at (P_i, P_{-i}) via $\underline{P}_i$. On the other hand, SP $\Rightarrow f(P_i, P_{-i})\, P_i\, f(\underline{P}_i, P_{-i})$. And, $f(\underline{P}_i, P_{-i}) \neq \tau(\underline{P}_i)$, because $f(\underline{P}_i, P_{-i}) > \tau(P_i) \geq \tau(\underline{P}_i) = \underline{X}$.

In fact, as it can be inferred from Figure 6 below, $f(\underline{P}_i, P_{-i}) \neq \tau(P_j)$ for all $j \neq i$. Otherwise, if $f(\underline{P}_i, P_{-i}) = \tau(P_j)$ for some $j \in I$, $j \neq i$, then $P_j \succ P_i$, because $f(\underline{P}_i, P_{-i}) > \tau(P_i)$. However, by SC, $P_j \succ P_i$, $f(P_i, P_{-i}) > f(\underline{P}_i, P_{-i})$, and $f(\underline{P}_i, P_{-i})\, P_j\, f(P_i, P_{-i})$ would imply $f(\underline{P}_i, P_{-i})\, P_i\, f(P_i, P_{-i})$: contradiction. Hence, there exists an ordering $P_i' \in \mathcal{SC}$ such that (i) $\tau(P_i') = \tau(P_i)$, and (ii) $f(\underline{P}_i, P_{-i})\, P_i'\, f(P_i, P_{-i})$. By TO, $f(P_i', P_{-i}) = f(P_i, P_{-i})$. Therefore, $f(\underline{P}_i, P_{-i})\, P_i'\, f(P_i', P_{-i})$: contradiction. Thus, since $P \in \mathcal{SC}^n$ and $i \in I$ were arbitrarily chosen, Cases 1 and 2 prove the claim. $\square$

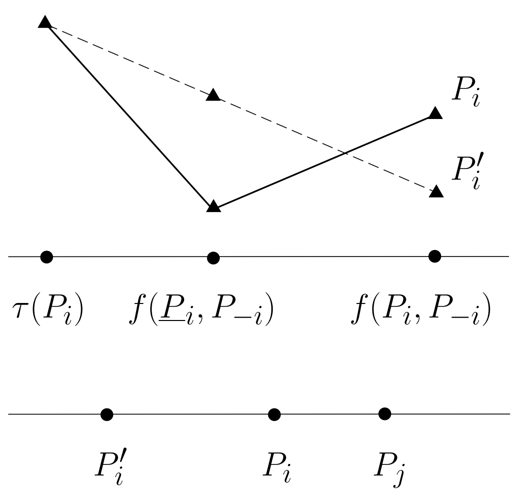


Figure 6:

Finally, before proving Theorem 1, we show below in Lemma 2 that a U and SP social choice function must also satisfy *top-monotonicity* on

$\mathcal{SC}^n$. Roughly speaking, this property ensures that collective choices do not respond perversely to changes in individuals' ideal points.

Definition 7 (TM) *A social choice function f is **top-monotonic** on $\mathcal{SC}^n$ if for all $i \in I$, all $(P_i, P_{-i}) \in \mathcal{SC}^n$, and all $P_i' \in \mathcal{SC}$ such that $\tau|_{r_f}(P_i') \geq \tau|_{r_f}(P_i)$, $f(P_i', P_{-i}) \geq f(P_i, P_{-i})$.*

Like before, let us assume that $\underline{P}_i$ (respectively, $\overline{P}_i$) denote agent i's most leftist (respectively, rightist) preference relation on X.

Lemma 2 *If a social choice function f is unanimous and strategy-proof on $\mathcal{SC}^n$, then f is top-monotonic.*

Proof: Let f be U and SP on $\mathcal{SC}^n$. Consider any individual $i \in I$, any profile $(P_i, P_{-i}) \in \mathcal{SC}^n$ and any admissible deviation $P_i' \in \mathcal{SC}$, such that $\tau(P_i') \geq \tau(P_i)$. We want to show that $f(P_i', P_{-i}) \geq f(P_i, P_{-i})$. Three cases are possible:

Case 1: If $\tau(P_i) \geq f(\overline{P}_i, P_{-i}) \Rightarrow m^3(\tau(P_i), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) = m^3(\tau(P_i'), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i}))$, because SP implies that $f(\overline{P}_i, P_{-i}) \geq f(\underline{P}_i, P_{-i})$, and $\tau(P_i') \geq \tau(P_i)$ by hypothesis. Therefore, by Lemma 1, $f(P_i', P_{-i}) = f(P_i, P_{-i})$.

Case 2: If $f(\overline{P}_i, P_{-i}) > \tau(P_i) > f(\underline{P}_i, P_{-i})$, then $m^3(\tau(P_i), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) = \tau(P_i)$ and, given that $\tau(P_i') \geq \tau(P_i)$, $m^3(\tau(P_i'), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) \geq \tau(P_i)$. Therefore, by Lemma 1, $f(P_i', P_{-i}) \geq f(P_i, P_{-i})$.

Case 3: Finally, if $f(\underline{P}_i, P_{-i}) \geq \tau(P_i)$, then $m^3(\tau(P_i'), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) \geq m^3(\tau(P_i), f(\underline{P}_i, P_{-i}), f(\overline{P}_i, P_{-i})) = f(\underline{P}_i, P_{-i})$. Hence, by Lemma 1, $f(P_i', P_{-i}) \geq f(P_i, P_{-i})$. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1: (**Sufficiency**) Immediate from Proposition 1 and the definition of positional dictators.

(Necessity) Suppose f is U, A and SP on $\mathcal{SC}^n$. We want to show that $f \in PD$. By Theorem 2, f is TO on $\mathcal{SC}^n$. Consider first the case where $|I| = 2$. Fix a profile $P \in \mathcal{SC}^n$. Without loss of generality, assume $\tau(P_2) \geq \tau(P_1)$. By Lemma 1, $f(P_1, P_2) = m^3(\tau(P_1), f(\underline{P}_1, P_2), f(\overline{P}_1, P_2))$. Applying Lemma 1 once again, $f(\underline{P}_1, P_2) = m^3(\tau(P_2), f(\underline{P}_1, \underline{P}_2), f(\underline{P}_1, \overline{P}_2))$, and $f(\overline{P}_1, P_2) = m^3(\tau(P_2), f(\overline{P}_1, \underline{P}_2), f(\overline{P}_1, \overline{P}_2))$. By unanimity, $f(\underline{P}_1, \underline{P}_2) = \underline{X}$

and $f(\overline{P}_1, \overline{P}_2) = \overline{X}$. By anonymity, $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2)$. Furthermore, by SP, $f(\underline{P}_1, \overline{P}_2), f(\overline{P}_1, \underline{P}_2) \in \{\underline{X}, \overline{X}\}$. Suppose not. That is, assume for instance that $f(\underline{P}_1, \overline{P}_2) = z \in X \backslash \{\underline{X}, \overline{X}\}$.

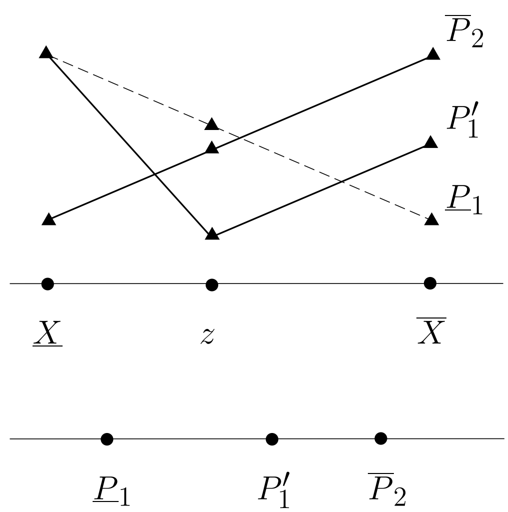


Figure 7:

Then, as we show in Figure 7 above, there must exist an ordering $P_1' \in \mathcal{SC}$ such that $\tau(P_1') = \tau(\underline{P}_1)$, and $\overline{X}\, p(P_1')\, z$. By TO, $f(P_1', \overline{P}_2) = f(\underline{P}_1, \overline{P}_2) = z \Rightarrow$ agent 1 would manipulate f at $(P_1', \overline{P}_2)$ via $\overline{P}_1$: contradiction. Thus, $f(\underline{P}_1, \overline{P}_2), f(\overline{P}_1, \underline{P}_2) \in \{\underline{X}, \overline{X}\}$. Furthermore, if $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2) = \underline{X}$, $f(\underline{P}_1, P_2) = m^3(\tau(P_2), \underline{X}, \underline{X}) = \underline{X}$, and $f(\overline{P}_1, P_2) = m^3(\tau(P_2), \underline{X}, \overline{X}) = \tau(P_2)$. Thus, $f(P_1, P_2) = m^3(\tau(P_1), \underline{X}, \tau(P_2)) = \tau(P_1)$. Instead, if $f(\underline{P}_1, \overline{P}_2) = f(\overline{P}_1, \underline{P}_2) = \overline{X}$, then a similar argument shows that $f(P_1, P_2) = m^3(\tau(P_1), \tau(P_2), \overline{X}) = \tau(P_2)$.

Thus, if $|I| = 2$ and f satisfies the hypotheses of Theorem 1, (i.e. f is U, A and SP), the previous paragraphs show that there exists a parameter (or fixed ballot) $\alpha \in \{\underline{X}, \overline{X}\}$ such that, for all $P \in \mathcal{SC}^n$, $f(P) = m^3(\tau(P_1), \tau(P_2), \alpha)$. Hence, $f \in PD$.

Now, suppose $|I| = 3$. Take any profile $P \in \mathcal{SC}^n$. Without loss of generality, relabel I if necessary so that $\tau(P_3) \geq \tau(P_2) \geq \tau(P_1)$. Using Lemma 1, it is easy to see that,

$$f(P) = m^3 \Big[\tau(P_1),\, m^3 \Big(\tau(P_2),\, m^3(\tau(P_3), a_3, a_2),\, m^3(\tau(P_3), a_2, a_1)\Big),\, m^3 \Big(\tau(P_2),\, m^3(\tau(P_3), a_2, a_1),\, m^3(\tau(P_3), a_1, a_0)\Big)\Big], \tag{2}$$

where $a_3 = f(\underline{P}_1, \underline{P}_2, \underline{P}_3)$, $a_0 = f(\overline{P}_1, \overline{P}_2, \overline{P}_3)$, and

$$a_2 = f(\underline{P}_1, \underline{P}_2, \overline{P}_3) = f(\underline{P}_1, \overline{P}_2, \underline{P}_3) = f(\overline{P}_1, \underline{P}_2, \underline{P}_3), \tag{3}$$

and

$$a_1 = f(\underline{P}_1, \overline{P}_2, \overline{P}_3) = f(\overline{P}_1, \overline{P}_2, \underline{P}_3) = f(\overline{P}_1, \underline{P}_2, \overline{P}_3), \tag{4}$$

where the equalities in (3) and in (4), respectively, follow from the fact that f is A on $\mathcal{SC}^n$. By U and TM, we have that $\overline{X} = a_0 \geq a_1 \geq a_2 \geq a_3 = \underline{X}$. By SP, a_1, $a_2 \in \{\underline{X}, \overline{X}\}$. Otherwise, if for example $f(\underline{P}_1, \overline{P}_2, \overline{P}_3) = z \in X \backslash \{\underline{X}, \overline{X}\}$, we can find an ordering $P_1' \in \mathcal{SC}$ such that $\tau(P_1') = \tau(\underline{P}) = \underline{X}$ and $\overline{X}\, p(P_1')\, z$. By TO, $f(P_1', \overline{P}_2, \overline{P}_3) = f(\underline{P}_1, \overline{P}_2, \overline{P}_3) \Rightarrow$ agent 1 would like to manipulate f at $(P_1', \overline{P}_2, \overline{P}_3)$ via $\overline{P}_1$. Then,

(i) If $\tau(P_1) \geq a_0$, then $\forall\, i = 1, 2, 3,\ \tau(P_i) = \overline{X}$. Thus, independently of the distribution of a_1 and a_2, it follows from (2) that $f(P) = \overline{X}$;

(ii) Similarly, if $a_3 \geq \tau(P_3)$, then $\forall\, i = 1, 2, 3,\ \tau(P_i) = \underline{X}$, and $f(P) = \underline{X}$;

(iii) If $a_1 = \underline{X}$, then $a_2 = \underline{X}$, because, by TM, $a_1 \geq a_2$. Therefore, (2) can be rewritten as $f(P) = m^3(\tau(P_1), \underline{X}, \tau(P_2)) = \tau(P_1)$;

(iv) Similarly, if $a_2 = \overline{X}$, then $a_1 = \overline{X}$, and $f(P) = m^3(\tau(P_1), \tau(P_3), \overline{X}) = \tau(P_3)$;

(v) Finally, if $a_1 = \overline{X}$ and $a_2 = \underline{X}$, then (2) can be rewritten as $f(P) = m^3(\tau(P_1), \tau(P_2), \tau(P_3)) = \tau(P_2)$.

Thus, since P was arbitrarily chosen, (i)-(v) imply that, if $|I| = 3$ and f is A, U and SP, then there exists $\alpha_1, \alpha_2 \in \{\underline{X}, \overline{X}\}$ such that, for all $P \in \mathcal{SC}^n$, $f(P) = m^5(\tau(P_1), \tau(P_2), \tau(P_3), \alpha_1, \alpha_2)$. Hence, $f \in PD$.

Now let us extend the proof to $|I| = n > 3$. For all $K \subseteq I$, let $a_{|K|} = f(\underline{P}_K, \overline{P}_{\bar{K}})$, where $\bar{K} = I \backslash K$. By unanimity, $K = \emptyset$ implies $a_0 = f(\overline{P}_1, \ldots, \overline{P}_n) = \overline{X}$. Similarly, if $K = I$, then $a_n = f(\underline{P}_1, \ldots, \underline{P}_n) = \underline{X}$. By anonymity,

$$\begin{array}{rcl} a_1 & = & f(\underline{P}_i, \overline{P}_{-i}),\ \forall \{i\} \subset I, \\ a_2 & = & f(\underline{P}_{\{i,j\}}, \overline{P}_{-\{i,j\}}),\ \forall \{i, j\} \subseteq I, \\ \vdots & & \vdots \\ a_{n-1} & = & f(\underline{P}_{-j}, \overline{P}_j),\ \forall \{j\} \subset I. \end{array}$$

Thus, by top-monotonicity, $a_0 \geq a_1 \geq a_2 \geq \ldots \geq a_{n-1} \geq a_n$. Moreover, for all $k = 0, 1, \ldots, n$, $a_k \in \{\underline{X}, \overline{X}\}$. In effect, if either $k = 0$

or $k = n$, then the result follows immediately from U. So, assume that $a_k \in \{\underline{X}, \overline{X}\}$ for some $k = 0, 1, \ldots, n-2$, and let us prove the claim for a_{k+1}. On the contrary, suppose $a_{k+1} \notin \{\underline{X}, \overline{X}\}$. Specifically, assume $a_{k+1} = f(\underline{P}_1, \ldots, \underline{P}_{k+1}, \overline{P}_{k+2}, \ldots, \overline{P}_n) = z \in X \backslash \{\underline{X}, \overline{X}\}$. Without loss of generality, let $a_k = f(\underline{P}_1, \ldots, \underline{P}_k, \overline{P}_{k+1}, \ldots, \overline{P}_n) = \overline{X}$. Consider $P'_{k+1} \in \mathcal{SC}$ such that $\tau(P'_{k+1}) = \tau(\underline{P}_{k+1})$ and $\overline{X}\, P'_{k+1}\, z$ (recall Figure 7 above). By TO, $f(\underline{P}_1, \ldots, \underline{P}_k, P'_{k+1}, \overline{P}_{k+2}, \ldots, \overline{P}_n) = z \Rightarrow$ agent $k+1$ would like to manipulate f at $(\underline{P}_1, \ldots, \underline{P}_k, P'_{k+1}, \overline{P}_{k+2}, \ldots, \overline{P}_n)$ via $\overline{P}_{k+1}$: contradiction.

Now, fix any profile $P \in \mathcal{SC}$, and relabel I if necessary, so that $\tau(P_n) \geq \tau(P_{n-1}) \geq \ldots \geq \tau(P_1)$. By repeated application of Lemma 1, it follows that:

(i) If $\tau(P_1) \geq a_0$, then $\forall\, i = 1, \ldots, n$, $\tau(P_i) = \overline{X}$, and $f(P) = m^3(\tau(P_1), a_1, a_0) = \overline{X}$;

(ii) If $a_n \geq \tau(P_n)$, then $\forall\, i = 1, \ldots, n$, $\tau(P_i) = \underline{X}$, and we have that $f(P) = m^3(\tau(P_1), a_n, a_{n-1}) = \underline{X}$;

(iii) If $a_k = \overline{X}$ for all $k = 1, \ldots, n-1$, then $f(P) = m^3(\tau(P_1), \tau(P_n), \overline{X}) = \tau(P_n)$;

(iv) If $a_k = \underline{X}$ for all $k = 1, \ldots, n-1$, then $f(P) = m^3(\tau(P_1), \underline{X}, \tau(P_2)) = \tau(P_1)$;

(v) Finally, if for some $k = 1, 2, \ldots, n-2$, $a_1 = \ldots = a_k = \overline{X}$ and $a_{k+1} = \ldots = a_{n-1} = \underline{X}$, then $f(P) = m^3(\tau(P_1), \tau(P_{k+1}), \tau(P_{k+2})) = \tau(P_{k+1})$.

Therefore, since $P \in \mathcal{SC}^n$ was arbitrarily chosen and, for every $k = 0, 1 \ldots, n$, a_k is independent of P, if f is A, U and SP, then items (i)-(v) imply that there exist $n-1$ parameters $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ on $\{\underline{X}, \overline{X}\}$ such that, for all $P \in \mathcal{SC}^n$, $f(P) = m^{2n-1}(\tau(P_1), \tau(P_2), \ldots, \tau(P_n), \alpha_1, \ldots, \alpha_{n-1})$. Hence, $f \in PD$. □

We close this section showing the independence of the axioms used in Theorem 1. First, consider the consequence of relaxing SP. As we explained before, any efficient extended median rule that it is not a positional dictator may be subject to individual manipulation on single-crossing domains. However, all of them are U and A. Thus, the family that satisfies U and A on $\mathcal{SC}^n$ is larger than PD.

Next consider the consequences of relaxing U. Define a social choice function f in such a way that, for each $P \in \mathcal{SC}^n$, $f(P) = a \in X$. It is clear that f is A and SP; however, f violates U, since $r_f = \{a\}$. Hence, $f \notin PD$.

Finally, regarding A, for any coalition $S \subset I$, define a social choice function f in such a way that, for all $P \in \mathcal{SC}^n$, $f(P) = m^{2|S|-1}(\{\tau(P_i)\}_{i \in S}, \alpha_1, \ldots, \alpha_{|S|-1})$. It is immediate to see that f is U. Moreover, following a reasoning analogous to the proof of Proposition 1, it is also easy to prove that f is SP on $\mathcal{SC}^n$, provided that for all $i = 1, \ldots, |S| - 1$, $\alpha_i \in \{\underline{X}, \overline{X}\}$. However, f violates A, since the preferences of all agents in the set $\bar{S} = I \backslash S$ are ignored to make social choices.

4 Final remarks

This paper analyzes collective choices in a society with strategic voters and single-crossing preferences. While this preference domain ensures that the core of the majority rule is nonempty, this result has been derived assuming sincere voting. This naturally raises the issue of potential individual and group manipulation, motivating the current research.

The main contributions of the paper are the following. First of all, it shows that, in addition to single-peakedness, single-crossingness is another meaningful domain which guarantees the existence of strategy-proof social choice functions. More precisely, it proves that each *positional dictator* is group strategy-proof on single-crossing domains. Conversely, every social choice function that satisfies anonymity, unanimity and strategy-proofness is shown to be a member of this family, with the natural consequence that A, U and SP imply Pareto efficiency and tops-onliness.

As we argue in the text, strategy-proofness over single-crossing preferences requires that the social choice be always an individual's most preferred alternative. This is necessary to rule out orderings that might produce incentives for manipulation, because the argument exploits (i) that the outcome is an individual's ideal point, (ii) the ordering of that agent, and (iii) the *correlation* among individual preferences in single-crossing domains. Remarkably, *no additional information about the shape of each preference relation is necessary to guarantee strategy-proofness.*

To put it in other terms, the results of this paper show that, in the case of public goods, convexity of individual preferences is not necessary to prevent manipulation, provided that a "certain amount of correlation" among preferences is simultaneously imposed. Unfortunately, this is no longer true when the collective choice problem refers to the allocation of a private good among a finite number of agents. In that case, Saporiti [29] have shown that *intermediateness*, a preference restriction essentially equivalent to single-crossingness, is not sufficient to ensure the existence of Pareto efficient, anonymous and strategy-proof allocation rules.

Furthermore, even in the case of public goods relaxing convexity is costly, because any extended median rule is A, U and SP on single-peaked preferences, without any restriction on the distribution of fixed ballots. However, in our framework, the family characterized by A, U and SP coincides with the class of positional dictators, which is a subset of extended median rules.

Finally, the paper also shows that the Representative Voter Theorem, i.e. "the single-crossing version" of the Median Voter Theorem, has a well defined strategic foundation, in the sense that its prediction can be implemented in dominant strategies. However, this result only holds on a subdomain of single-crossing preferences, the rectangular one. So, relaxing sincere voting is not free either.

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