Voting Equilibria in Multi-candidate Elections

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Working Paper No. 52
February 2008

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February 8, 2008

Abstract

We consider a general plurality voting game with multiple candidates, where voter preferences over candidates are exogenously given. In particular, we allow for arbitrary voter indifferences, as may arise in voting subgames of citizen-candidate or locational models of elections. We prove that the voting game admits pure strategy equilibria in undominated strategies. The proof is constructive: we exhibit an algorithm, the "best winning deviation" algorithm, that produces such an equilibrium in finite time. A byproduct of the algorithm is a simple story for how voters might learn to coordinate on such an equilibrium.

1 Introduction

The analysis of voting in elections arises not only in the pure theory of voting, but also in the study of candidate positioning, candidate entry, party formation, etc. In the latter studies, voting games appear as subgames of a larger game in which political platforms are determined before voting takes place. Since the outcomes of voting can theoretically affect prior political decisions, the analysis of voting has significance beyond the narrow focus on the ballots cast by voters. Of the issues arising in the theory of voting, the existence of minimally plausible equilibria in voting subgames, such as Nash equilibria in undominated strategies, is arguably of paramount importance. We consider a general plurality rule voting game with multiple candidates in which voter preferences over candidates are exogenously given, as in voting subgames of candidate competition or entry. In particular, we allow each voter to have arbitrary indifferences between candidates. We prove that the voting game admits pure strategy equilibria in undominated strategies. The proof is constructive: we exhibit an algorithm, the "best winning deviation" algorithm, that produces such an equilibrium in finite time. A byproduct of the algorithm is a simple story for how voters might learn to coordinate on such an equilibrium.

Existence of equilibria in undominated strategies is trivial when there are only two candidates or when there are multiple candidates but no voter is be indifferent between any two distinct candidates. In the latter case, we can reduce the multi-candidate election to a two-candidate one by choosing two candidates arbitrarily and specifying that each voter vote for his or her preferred of the two. If there are at least three voters, then this generates a Nash equilibrium in which no voter votes for his or her worst candidate, and so the equilibrium strategies are undominated. When indifferences are possible, however, this solution no longer works. Suppose, for example, that six voters rank four candidates, a, b, c, and d, as follows.

1	2	3	4	5	6
c	d	c	a	a	b
d	b	b	d	c	a
ab	ac	ad	bc	bd	cd

Then no mater which pair of candidates is pre-specified, there is some voter who ranks both of those candidates last, and for that voter, it is a dominated strategy to vote for either of those candidates. Thus, there are situations where at least three candidates must receive votes in a Nash equilibrium with undominated strategies. The same example shows that it need not be an equilibrium for each voter to simply vote for his or her favorite candidate: in the above example, that leads to tie between a and c, but voter 2 could profitably deviate by switching to candidate b.

The difficulty identified above clearly arises because of the possibility of voter indifferences. When voter preferences over candidates are modeled as exogenous, it is tempting to solve this problem by assumption, and in fact some work on voting games does explicitly this. (See, e.g., De Sinopoli and Turrini (2002) and Dhillon and Lockwood (2002).) But when voting games are viewed as subgames in a larger game in which candidate positions are determined endogenously, this is an undesirable limitation. If candidates compete by choosing policy platforms from a sufficiently rich policy space, for example, then the possibility of voter indifferences in some subgames cannot be avoided. The exclusion of indifferences is likewise a limitation if the goal is a general model of candidate entry, as in Besly and Coate (1997). In fact, the equilibrium existence claim of those authors, in their Proposition 1, relies on the existence of pure strategy equilibria in undominated strategies. Since Besley and Coate actually leave voter preferences over candidates unrestricted, their existence claim is compromised by the potential absence of equilibria in undominated strategies in some voting subgames. Our existence result fills that gap. A similar issue arises in Feddersen, Sened, and Wright (1990), who assume pure voting strategies and impose the restriction of undominated strategies. Those authors do not verify the existence of such equilibria in all subgames, but in contrast to Besley and Coate (1997), they examine Nash, rather than subgame perfect, equilibria, so the potential absence of such equilibria in "out of equilibrium" subgames does not affect their analysis.

Another simple solution to the problem of existence of voting equilibria in undominated strategies is to allow for mixing, or randomization, in voting behavior. We could simply eliminate the dominated votes for each voter, leaving a (smaller) finite voting game, which will then admit mixed strategy equilibria. Returning to the (larger) original voting game, the equilibrium mixed strategy of a voter will not put positive probability on a dominated vote. It is true that the strategic structure of some games leads players to behave in a manner that is not precisely predictable by their opponents, and so this approach is viable on a priori grounds. It leaves open the question, however, of whether the structure of voting games actually necessitates mixing by voters. (And since some voting games certainly have equilibria in pure strategies, it is not clear why the introduction of indifferences would fundamentally alter the competitiveness of voting games.) Furthermore, because the exact form of a voter's mixing probabilities will depend, in equilibrium, on indifference conditions involving other voters, the behavioral foundations for such equilibria may be less convincing than for equilibria in pure strategies.

A final alternative approach is to simply assume sincere voting on the part of voters, i.e., that each voter votes for his or her highest ranked candidate, neglecting strategic incentives to deviate (e.g., to avoid "wasting votes"). This approach is quite common in models with a continuum of voters (see, e.g., Osborne and Slivinski (1996) and Palfrey (1984)), because such voting games induce uninteresting incentives for individual voters. The assumption of sincere voting may or may not have empirical support in large elections, but it is of course less tenable in other settings that we would like to cover, such as small committees, and it does not address the status of the literature on strategic voting, where existence of equilibria in voting subgames is an issue.

The main result of this paper shows that it is not necessary to resort to the exclusion of voter indifferences or the possibility of mixed voting strategies. For completely arbitrary voter preferences over candidates, we establish the existence of pure strategy voting equilibria in which voters use undominated strategies. The proof is constructive: we give an algorithm, the "best winning deviation" algorithm, that will always generate an equilibrium of the sort desired. The algorithm is simple. We arbitrarily number voters and candidates, and we begin with a profile of votes in which each voter votes for his or her favorite candidate. Since voters may have more than one favorite candidate, we simply specify that a voter vote for the lowest indexed of his or her favorite candidates. If this is an equilibrium, then the algorithm stops. Otherwise, some voter has a profitable deviation, and we consider the lowest indexed such voter. We show, in fact, that this voter has a profitable deviation in which the voter switches to vote for a candidate who, as a result of the deviation, either wins outright or is tied for first place. We specify that the voter vote for the best candidate subject to this constraint, i.e., that the voter's new choice maximize his or her utility subject to the new candidate choice winning or tying after the switch. In case there is more than one such candidate, the voter switches to the lowest indexed of them. If this is an equilibrium, then the algorithm stops. Otherwise, we repeat the above procedure. The bulk of our analysis consists of demonstrating that this algorithm must terminate in finite time, whereupon it yields an equilibrium in undominated strategies.

Though we avoid the use of mixed strategies, there is still the question of how voters might conceivably learn to coordinate on one among possibly many equilibria. One possible answer is provided by the best winning deviation algorithm described above. Taking the algorithm literally, it suggests a story about myopic learning, where individual voters adjust their votes over time by best responding to the current profile of voting strategies. This is termed an "introspective" learning dynamic by Bergin and Bernhardt (2004), who study learning in environments with economic structure not present in the voting games we consider. Nevertheless, the central problem addressed by those authors, namely, the termination of the learning algorithm, is similar in spirit to ours. Taking our algorithm less literally, it suggests that myopic adjustment of voters to public polls, as in McKelvey and Ordeshook (1985), may facilitate coordination in voting games.

2 The Model

We consider an election among a set $M = \{1, 2, \ldots, m\}$ of candidates to be decided by plurality voting by a set $N = \{1, 2, \ldots, n\}$ of voters. That is, each voter votes for a single candidate, and the candidate with the most votes wins, with ties being decided by a fair lottery. We take the positions of the candidates as fixed, and we assume that each voter i's preferences over candidates are given by a von Neumann-Morgenstern utility function $u_i\colon M \to \mathbb{R}$. Note that we impose no restrictions on voter utilities, and in particular we allow for arbitrary indifferences between the candidates.

We model voting as a strategic form game in which the pure strategy set of voter i is $C_i = M$, so that a pure strategy c_i for voter i is a vote for a single candidate. Let $C = M^n$ be the set of strategy profiles in the voting game, and let $c = (c_1, c_2, \ldots, c_n)$ be a profile of voting strategies.[1] Let $v(k|c) = \#\{i \in N \mid c_i = k\}$ be the number of votes that candidate k receives given strategy profile c. Let $w(c) = \max_{k \in M} v(k|c)$ be the maximum number of votes received by any candidate for profile c, and let $W(c) = \{k \in M \mid v(k|c) = w(c)\}$ be the set of candidates tied for first place. Each $k \in W(c)$ wins the election with the equal probability, $1/\#W(c)$. This determines a payoff function $U_i\colon C \to \mathbb{R}$ for voter i, defined as follows:

$$U_i(c) \;=\; \sum_{k \in W(c)} \frac{u_i(k)}{\#W(c)}.$$

A *pure strategy equilibrium* is a strategy profile c such that no voter can deviate to obtain a higher payoff, i.e., for all voters i and strategies c_i', $U_i(c) \geq U_i(c_i', c_{-i})$.

Given a voter i, we say that strategy c_i' *dominates* strategy c_i if (i) $U_i(c_i', c_{-i}) \geq U_i(c_i, c_{-i})$ for all $c_{-i} \in C_{-i}$, and (ii) $U_i(c_i', c_{-i}) > U_i(c_i, c_{-i})$ for some $c_{-i} \in C_{-i}$. Accordingly, c_i is *undominated* if there is no strategy c_i' that dominates c_i. The following proposition extends a well-known characterization of the undominated strategies to voting games with arbitrary indifferences: voting for a candidate is undominated if either the voter is completely indifferent or the candidate is not bottom-ranked. The proof is found in the appendix.

Proposition 1. *Strategy c_i is undominated for voter i if and only if either (i) for all $c_i', c_i'' \in C_i$, $u_i(c_i') = u_i(c_i'')$, or (ii) there exists $c_i''' \in C_i$ such that $u_i(c_i) > u_i(c_i''')$.*

As a consequence of Proposition 1, a strategy is undominated if it specifies that a voter vote for a utility maximizing candidate. Defining $c_i^0 = \min\{k \in M \mid \forall \ell \in M : u_i(k) \geq u_i(\ell)\}$ as the lowest indexed candidate who maximizes voter i's utility, the profile $c^0 = (c_1^0, \ldots, c_n^0)$ therefore consists of undominated strategies. It is not, however, necessarily a pure strategy equilibrium, as is easily checked.

Example 1. *A strategy profile in which every voter votes for his or her most preferred alternative is not a pure strategy Nash equilibrium in undominated strategies:*

Recall the example from the Introduction, where $n = 6$ and $M = 4$. Let each voter assign utility 1 to his or her favorite candidate, utility .8 to his or her second-favorite, and utility 0 to the last-ranked candidates. Then voter 2's expected payoff from the strategy profile, c^0, in which each voter votes for his or her favorite candidate is

[1] We use the notation c_{-i} to denote a profile of votes for voters other than i.

$U_2(c,d,c,a,a,b) = 0$. Though voting for c is a best response for voter 1, voter 2's expected payoff increases if the voter switches to b: $U_2(c,b,c,a,a,b) = .6 > 0$. $\square$

Note that in the previous example, voter 2 profitably deviates to candidate b, who then ties for first in the election. The next proposition establishes that, in general, if a voter has a profitable deviation, then the voter can deviate to vote for a candidate who, as a result of the deviation, either uniquely wins or at least ties for first in the election. The proof is found in the appendix.

Proposition 2. *For all strategy profiles c, if there is a deviation $c_i' \in C_i$ such that $U_i(c_i', c_{-i}) > U_i(c)$ for some $i \in N$, then there is a deviation c_i'' such that $U_i(c_i'', c_{-i}) > U_i(c)$ and $c_i'' \in W(c_i'', c_{-i})$.*

Note that in Example 1, the strategy profile (c,b,c,a,a,b) obtained after voter 2's deviation also is not an equilibrium. Nevertheless, the next example shows that it is a simple matter to generate an equilibrium in undominated strategies.

Example 2. *A pure strategy equilibrium in undominated strategies:*

Returning to Example 1, note that voters 1 and 2 are best responding in the profile (c,b,c,a,a,b), but now voter 3 can profitably deviate by switching to candidate b: $U_3(c,b,b,a,a,b) = .8 > .6 = U_3(c,b,c,a,a,b)$. Finally, the strategy profile (c,b,b,a,a,b) is indeed a pure strategy equilibrium in undominated strategies. $\square$

In the above examples, only two deviations were needed to achieve an equilibrium. In the next section, we extend the approach in Examples 1 and 2 to look for successive deviations of a certain form. The algorithm does not always terminate after two iterations, as it does above, but it always terminates in finite time and yields a pure strategy equilibrium in undominated strategies.

3 Best Winning Deviation Algorithm

We generalize the approach used in Examples 1 and 2 by defining an algorithm that proceeds as follows. We begin with the strategy profile c^0 in which every voter votes for his or her favorite candidate. Because we allow the possibility of indifferences, we arbitrarily specify that a voter with multiple utility-maximizing candidates initially vote for the lowest indexed of those candidates. We then consider whether any voter has a profitable deviation from c^0. If not, then using Proposition 1, the profile c^0 is a pure strategy equilibrium in undominated strategies. If there is a voter with a profitable deviation, then c^0 is not an equilibrium. Because there may be multiple voters with profitable deviations, we arbitrarily specify that the lowest indexed such voter deviate. Moreover, we require that the voter deviate to a candidate who, as a result of the deviation, either wins or ties for first in the election. By Proposition 2, there is always such a candidate. Since there may be multiple such candidates, we arbitrarily specify that the voter deviate to the lowest indexed of these candidates. We denote the resulting profile c^1. If there is no voter with a profitable deviation, then we have a pure strategy equilibrium in undominated strategies. If there is a voter with a profitable deviation, then we construct a new profile c^2 as above, and so on.

Formally, we define the *best winning deviation algorithm* by recursion. Initially, we define $c_i^0 = \min\{k \in M \mid \forall \ell \in M : u_i(k) \geq u_i(\ell)\}$ for all $i \in N$, and let $c^0 = (c_1^0, c_2^0, \ldots, c_n^0) \in C$. Now taking the profile c^t as given, we define profile c^{t+1} as follows. For all $i \in N$, let

$$D_i^t = \{k \in C_i \mid k \in W(k, c_{-i}^t), U_i(k, c_{-i}^t) > U_i(c^t)\}$$

denote the set of possible profitable winning deviations from c^t for voter i. As Proposition 2 shows, if c^t is not an equilibrium, then $D_i^t \neq \emptyset$ for some voter i. Let $N^t = \{i \in N \mid D_i^t \neq \emptyset\}$ denote the set of voters with a profitable deviation from c^t. If $N^t = \emptyset$, then the algorithm ends at c^t. Otherwise, the algorithm continues. Let $i(t) = \min N^t$ denote the lowest indexed voter with a profitable deviation, and define

$$c_{i(t)}^{t+1} = \min\{k \in D_{i(t)}^t \mid \forall \ell \in D_{i(t)}^t : U_{i(t)}(k, c_{-i(t)}^t) \geq U_{i(t)}(\ell, c_{-i(t)}^t)\}$$

to be the lowest indexed of the profitable winning deviations of the lowest indexed voter with such a deviation. We specify that the strategies of the other voters remain unchanged: $c_{-i(t)}^{t+1} = c_{-i(t)}^t$. Thus, $c^{t+1} = (c_{i(t)}^{t+1}, c_{-i(t)}^t)$. We write $c_{i(t)}^t \to c_{i(t)}^{t+1}$ to denote a best winning deviation from $c_{i(t)}^t$ to $c_{i(t)}^{t+1}$, and we say $i(t) \in N$ is the best winning deviator in round t.

4 Existence of Voting Equilibria in Undominated Strategies

As the goal of constructing the best winning deviation algorithm is to compute, and thereby establish the existence of, a pure strategy equilibrium in undominated strategies, it is of paramount importance that the algorithm terminate in finite time. Of course, it does so in Example 2. The next result establishes that the algorithm does indeed always terminates in finite time, despite the possibility of arbitrary voter indifferences.

Theorem 1. *The best winning deviation algorithm must stop at some round $T < \infty$. Thus, there exists a pure strategy equilibrium in undominated strategies.*

The remainder of this section consists in proving Theorem 1. Since the profile c^t in any round t is uniquely defined by the best winning deviation algorithm, we conserve notation by abbreviating $v(k|c^t)$ to $v^t(k)$, $w(c^t)$ to w^t, and $W(c^t)$ to W^t. Also, let

$W_q^t = \{k \in M \mid v^t(k) = w^t - q\}$, denote the set of candidates who are q votes out of first place in the election, and let $\widehat{W}^t = W_0^t \cup W_1^t$ denote the set of candidates who tie for first or lose by one vote. Note that $c_{i(t)}^{t+1}$ is an element of W^{t+1}, by construction of the algorithm.

Our analysis of the best winning deviation algorithm will make use of the following lemma, proved in the appendix. It shows that the candidate to whom the deviator switches his or her vote must maximize that voter's utility over the set of candidates who tie for first or lose by one vote.

Lemma 1. *For all candidates $k \in \widehat{W}^{t+1}$, we have $u_{i(t)}(c_{i(t)}^{t+1}) \geq u_{i(t)}(k)$.*

In the remainder of the analysis, we classify best winning deviations into eight possible cases, each defined by a condition on the status of the candidates voted for by the best winning deviator. The table below displays these cases, along with their defining conditions.

Case	Condition
1	$c_{i(t)}^t \in W^t$, $c_{i(t)}^t \in W^{t+1}$, $c_{i(t)}^{t+1} \in W^t$
2	$c_{i(t)}^t \in W^t$, $c_{i(t)}^t \in W^{t+1}$, $c_{i(t)}^{t+1} \notin W^t$
3	$c_{i(t)}^t \in W^t$, $c_{i(t)}^t \notin W^{t+1}$, $c_{i(t)}^{t+1} \in W^t$
4	$c_{i(t)}^t \in W^t$, $c_{i(t)}^t \notin W^{t+1}$, $c_{i(t)}^{t+1} \notin W^t$
5	$c_{i(t)}^t \notin W^t$, $c_{i(t)}^t \in W^{t+1}$, $c_{i(t)}^{t+1} \in W^t$
6	$c_{i(t)}^t \notin W^t$, $c_{i(t)}^t \in W^{t+1}$, $c_{i(t)}^{t+1} \notin W^t$
7	$c_{i(t)}^t \notin W^t$, $c_{i(t)}^t \notin W^{t+1}$, $c_{i(t)}^{t+1} \in W^t$
8	$c_{i(t)}^t \notin W^t$, $c_{i(t)}^t \notin W^{t+1}$, $c_{i(t)}^{t+1} \notin W^t$

The next lemma, proved in the appendix, establishes restrictions derived from the construction of the best winning deviation algorithm. For example, it turns out that Cases 1, 2, 5, and 6 cannot occur in the operation of this algorithm. A moment's consideration, and inspection of the proof of the lemma, reveal that the critical case is Case 2. Here, the deviator is currently voting for a winning candidate, then switches to vote for a candidate who is not currently winning, with the result that both candidates (and possibly others) are tied for the most votes after the switch. Whereas Cases 1, 5, and 6 can be precluded on a priori grounds, deviations in Case 2 are conceivable: such a deviation could occur if the deviator, say i, were voting for the uniquely winning candidate, say k, in round t; if candidate k were winning the election by two votes in round t; and if the deviator switches his or her vote to a candidate, say ℓ, who was losing by two votes. Such a deviation might fall into Case 2 if i prefers candidate ℓ to k, but we show that the construction of the best winning deviation algorithm precludes this possibility.

Lemma 2. *The construction of the best winning deviation algorithm generates the following restrictions.*

Case 1 *Not possible.*

Case 2 *Not possible.*

Case 3 $w^{t+1} = w^t + 1$.

Case 4 $w^{t+1} = w^t$, $\#W^{t+1} = \#W^t$, $\widehat{W}^{t+1} = \widehat{W}^t$, *and* $W^{t+1} \setminus \{c_{i(t)}^{t+1}\} = W^t \setminus \{c_{i(t)}^t\}$.

Case 5 *Not possible.*

Case 6 *Not possible.*

Case 7 $w^{t+1} = w^t + 1$.

Case 8 $w^{t+1} = w^t$, $\#W^{t+1} = \#W^t + 1$.

In Cases 3, 4, 7, and 8, we see that after each iteration of the algorithm, the number of votes garnered by the winning candidate weakly increases. Since the number of voters is finite, this means that the number of votes accruing to the winner weakly increases until it reaches a certain level and then remains constant. This places us in Cases 4 and 8, where we see that the number of candidates tied for first place is constant in Case 4 and strictly increasing in Case 8. Because the set of candidates is finite, the number of candidates tied for first weakly increases until it reaches a certain level and then remains constant. This places us in Case 4. The end of the proof is to argue that we can only remain in Case 4 for a finite number of rounds, after which we conclude that the algorithm must terminate.

In the sequel, we make use of the following observations without reference. First, because the deviator $i(t)$ in round t switches from $c_{i(t)}^t$ to $c_{i(t)}^{t+1}$, we have

$$v^t(c_{i(t)}^t) = v^{t+1}(c_{i(t)}^t) + 1 \text{ and } v^t(c_{i(t)}^{t+1}) + 1 = v^{t+1}(c_{i(t)}^{t+1}) = w^{t+1},$$

where the last equality above holds because $c_{i(t)}^{t+1}$ is a winning deviation. Since the strategies of voters other than $i(t)$ are held fixed in round t, we also have $v^t(k) = v^{t+1}(k)$ for all $k \in M \setminus \{c_{i(t)}^t, c_{i(t)}^{t+1}\}$. Finally, since the deviator can only subtract or add one vote, at least one of the following holds: $w^{t+1} = w^t - 1$, $w^{t+1} = w^t$, or $w^{t+1} = w^t + 1$.

To prove Theorem 1, suppose that the best winning deviation algorithm does not stop, generating an infinite sequence of sets $\{W^t\}$ and profiles $\{c^t\}$. Recall that by Lemma 2, deviations in Cases 1, 2, 5, and 6 cannot occur at any round t. Thus, only deviations in Cases 3, 4, 7, and 8 are possible. As we have discussed, in these cases, we have $w^{t+1} \geq w^t$ for all t, and since N is finite, there exists T' such that $w^{t+1} = w^t$ for all $t \geq T'$. Then deviations in Cases 3 and 7 are not possible for $t \geq T'$, i.e., only deviations in Cases 4 and 8 are possible in rounds $t \geq T'$. In Cases 4 and 8, we have $\#W^{t+1} \geq \#W^t$ for all t, and since M is finite, there exists $T'' \geq T'$ such that

$\#W^{t+1} = \#W^t$ for all $t \geq T''$. Then deviations in Case 8 are not possible for $t \geq T''$, i.e., only deviations in Case 4 are possible in rounds $t \geq T''$.

By Lemma 2, we have $\widehat{W}^t = \widehat{W}^{t+1}$ in Case 4, and we conclude that $\widehat{W}^{t+1} = \widehat{W}^t$ for all $t \geq T''$. For simplicity, let $\widehat{W} = \widehat{W}^t$ for all $t \geq T''$. Since N is finite, there is at least one voter who is a deviator more than once after T''. Let i be the first such voter, and let s be the first round $s \geq T''$ in which i is deviator, i.e., $i = i(s)$, and let t be the second round such that $t \geq T''$ and $i = i(t)$. Since $c_i^t \to c_i^{t+1}$ is a best winning deviation, we have $U_i(c^{t+1}) > U_i(c^t)$. Using $\#W^t = \#W^{t+1}$, from Lemma 2, this is equivalent to

$$u_i(c_i^{t+1}) + \sum_{k \in W^{t+1} \setminus \{c_i^{t+1}\}} u_i(k) \quad > \quad u_i(c_i^t) + \sum_{k \in W^t \setminus \{c_i^t\}} u_i(k),$$

and since $W^{t+1} \setminus \{c_i^{t+1}\} = W^t \setminus \{c_i^t\}$, by Lemma 2, this implies $u_i(c_i^{t+1}) > u_i(c_i^t)$. Note that i does not deviate between rounds $s+1$ and t, so we have $c_i^t = c_i^{s+1}$, with the implication that $u_i(c_i^{t+1}) > u_i(c_i^{s+1})$. Note also that $c_i^{t+1} \in W^{t+1} \subseteq \widehat{W}$. But Lemma 1 implies that $u_i(c_i^{s+1}) \geq u_i(k)$ for all $k \in \widehat{W}$, and then $c_i^{t+1} \in \widehat{W}$ implies $u_i(c_i^{s+1}) \geq u_i(c_i^{t+1})$, a contradiction. Therefore, the best winning deviation algorithm must stop at some $T < \infty$.

By Proposition 2, the resulting strategy profile c^T is a pure strategy equilibrium. That each voter's strategy in c^T is undominated follows by application of Proposition 1. In the initial profile c^0, each voter begins by voting for a utility maximizing candidate. If voter i never deviates in the operation of the algorithm, then by Proposition 1, the strategy $c_i^T = c_i^0$ is undominated. If voter i does deviate to strategy c_i^{t+1} in some round t, then because $c_i^{t+1} \in D_i^t$, we have

$$\frac{1}{\#W^{t+1}} \sum_{k \in W^{t+1}} u_i(k) \quad > \quad \frac{1}{\#W^t} \sum_{k \in W^t} u_i(k),$$

and so there exist $k \in W^{t+1}$ and $\ell \in W^t$ such that $u_i(k) > u_i(\ell)$. Since $c^t \to c^{t+1}$ is a best winning deviation, we have $c_i^{t+1} \in W^{t+1}$, and Lemma 1 then implies $u_i(c_i^{t+1}) \geq u_i(k) > u_i(\ell)$. By Proposition 1, c_i^{t+1} is undominated, and since this holds for arbitrary deviations, we conclude that c^T is a pure strategy equilibrium in undominated strategies.

A Proofs of Auxiliary Results

Proposition 1. *Strategy c_i is undominated for voter i if and only if either (i) for all $c_i', c_i'' \in C_i$, $u_i(c_i') = u_i(c_i'')$, or (ii) there exists $c_i''' \in C_i$ such that $u_i(c_i) > u_i(c_i''')$.*

First, assume c_i is undominated. Suppose that u_i is not constant on M, violating (i). To show that (ii) holds, suppose that candidate c_i minimizes voter i's utility, i.e., $u_i(\tilde{c}_i) \geq u_i(c_i)$ for all $\tilde{c}_i \in C_i$. Choose strategy $\hat{c}_i$ to maximize i's utility, which implies $u_i(\hat{c}_i) > u_i(c_i)$. We claim that for all c_{-i}, we have $U_i(\hat{c}_i, c_{-i}) \geq U_i(c_i, c_{-i})$. To see this, note that either (a) $w(\hat{c}_i, c_{-i}) = w(c_i, c_{-i}) + 1$, or (b) $w(\hat{c}_i, c_{-i}) = w(c_i, c_{-i})$, or (c) $w(\hat{c}_i, c_{-i}) = w(c_i, c_{-i}) - 1$. In cases (a) and (c), we have $W(\hat{c}_i, c_{-i}) = \{\hat{c}_i\}$ and $W(c_i, c_{-i}) = \{c_i\}$, respectively, both of which imply $U_i(\hat{c}_i, c_{-i}) \geq U_i(c_i, c_{-i})$. In case (b), we have either $W(\hat{c}_i, c_{-i}) = W(c_i, c_{-i}) \setminus \{c_i\}$ or $W(\hat{c}_i, c_{-i}) = \{\hat{c}_i\} \cup (W(c_i, c_{-i}) \setminus \{c_i\})$, and again $U_i(\hat{c}_i, c_{-i}) \geq U_i(c_i, c_{-i})$, fulfilling the claim. Now, if n is odd, then choose c'_{-i} such that $\hat{c}_i$ and c_i both receive $\frac{n-1}{2}$ votes. Then $U_i(\hat{c}_i, c'_{-i}) = u_i(\hat{c}_i) > u_i(c_i) = U_i(c_i, c'_{-i})$. If n is even, then choose c''_{-i} such that $\hat{c}_i$ receives $\frac{n}{2} - 1$ votes and c_i receives $\frac{n}{2}$ votes. Then $U_i(\hat{c}_i, c''_{-i}) = \frac{u_i(\hat{c}_i) + u_i(c_i)}{2} > u_i(c_i) = U_i(c_i, c''_{-i})$. By Proposition 1, it follows that $\hat{c}_i$ dominates c_i, contradicting the assumption that c_i is undominated.

Second, assume that (i) or (ii) holds. If (i) holds, then it is clear that all strategies are undominated for i. So suppose (ii) holds, i.e., there is a strategy $c'''_i \in M$ such that $u_i(c_i) > u_i(c'''_i)$. If n is odd, then choose any c'_{-i} such that c_i and c'''_i both receive $\frac{n-1}{2}$ votes. Then $U_i(c_i, c'_{-i}) > U_i(\hat{c}_i, c'_{-i})$ for all $\hat{c}_i \in C_i \backslash \{c_i\}$, since $U_i(c_i, c'_{-i}) = u_i(c_i)$, $U_i(c'''_i, c'_{-i}) = u_i(c'''_i)$, and $U_i(\hat{c}_i, c'_{-i}) = \frac{u_i(c_i) + u_i(c'''_i)}{2}$ for all $\hat{c}_i \in C_i \backslash \{c_i, c'''_i\}$. If n is even, then choose any c''_{-i} such that c_i receives $\frac{n}{2} - 1$ votes and c'''_i receives $\frac{n}{2}$ votes. Then $U_i(c_i, c''_{-i}) > U_i(\hat{c}_i, c''_{-i})$ for all $\hat{c}_i \in C_i \backslash \{c_i\}$, since $U_i(c_i, c''_{-i}) = \frac{u_i(c_i) + u_i(c'''_i)}{2} > u_i(c'''_i) = U_i(\hat{c}_i, c''_{-i})$ for all $\hat{c}_i \in C_i \backslash \{c_i\}$. Therefore, c_i is dominated by no $\hat{c}_i$, as desired.

Proposition 2. *For all strategy profiles c, if there is a deviation $c'_i \in C_i$ such that $U_i(c'_i, c_{-i}) > U_i(c)$ for some $i \in N$, then there is a deviation c''_i such that $c''_i \in W(c''_i, c_{-i})$ and $U_i(c''_i, c_{-i}) > U_i(c)$.*

Suppose there exists $c'_i \in C_i$ such that $U_i(c'_i, c_{-i}) > U_i(c)$. Then,

$$U_i(c'_i, c_{-i}) = \sum_{k \in W(c'_i, c_{-i})} \frac{u_i(k)}{\#W(c'_i, c_{-i})}.$$

Take any $\ell \in W(c'_i, c_{-i})$ such that $u_i(\ell) \geq u_i(k)$ for all $k \in W(c'_i, c_{-i})$. If $\ell = c'_i$, then $U_i(\ell, c_{-i}) = U_i(c'_i, c_{-i}) > U_i(c)$. If $\ell \neq c'_i$, then $W(\ell, c_{-i}) = \{\ell\}$, so $U_i(\ell, c_{-i}) = u_i(\ell) \geq U_i(c'_i, c_{-i}) > U_i(c)$. Then, $U_i(\ell, c_{-i}) > U_i(c)$. Setting $c''_i = \ell$, we are done.

Lemma 1. *For all candidates $k \in \widehat{W}^{t+1}$, we have $u_{i(t)}(c^{t+1}_{i(t)}) \geq u_{i(t)}(k)$.*

Let $k \in \widehat{W}^{t+1}$ maximize voter i's utility over $\widehat{W}^{t+1}$, and suppose that $u_{i(t)}(k) > u_i(c^{t+1}_{i(t)})$. If $k \in W^{t+1}_0$, then $U_{i(t)}(k, c^{t+1}_{-i(t)}) = u_{i(t)}(k) > U_{i(t)}(c^{t+1})$, where the strict inequality follows since $c^{t+1}_{i(t)} \in W^{t+1}$. If $k \in W^{t+1}_1$, then $W(k, c^{t+1}_{-i(t)}) \setminus \{k\} = W(c^{t+1}) \setminus \{c^{t+1}_{i(t)}\}$, and this implies $U_{i(t)}(k, c^{t}_{-i(t)}) > U_{i(t)}(c^{t+1})$. In both cases, we contradict the fact that $c^t_{i(t)} \to c^{t+1}_{i(t)}$ is a best winning deviation.

Lemma 2. *The construction of the best winning deviation algorithm generates the following restrictions.*

Case 1 *Not possible.*

Case 2 *Not possible.*

Case 3 $w^{t+1} = w^t + 1$.

Case 4 $w^{t+1} = w^t$, $\#W^{t+1} = \#W^t$, $\widehat{W}^{t+1} = \widehat{W}^t$, *and* $W^{t+1} \setminus \{c^{t+1}_{i(t)}\} = W^t \setminus \{c^t_{i(t)}\}$.

Case 5 *Not possible.*

Case 6 *Not possible.*

Case 7 $w^{t+1} = w^t + 1$.

Case 8 $w^{t+1} = w^t$, $\#W^{t+1} = \#W^t + 1$.

The proof proceeds by establishing a number of claims. The task of proving the impossibility of Case 2 is reserved for the end of the proof.

Claim 1. *Case 1 is not possible.*

Suppose $c^t_{i(t)} \in W^t$, $c^t_{i(t)} \in W^{t+1}$, and $c^{t+1}_{i(t)} \in W^t$. Then $v^{t+1}(c^t_{i(t)}) = v^t(c^t_{i(t)}) - 1 = w^t - 1 = v^t(c^{t+1}_{i(t)}) - 1 = v^{t+1}(c^{t+1}_{i(t)}) - 2$, but then $c^t_{i(t)} \notin W^{t+1}$, a contradiction.

Claim 2. *In Case 2,* $w^{t+1} = w^t - 1$, $c^{t+1}_{i(t)} \in W^t_2$, *and* $W^t = \{c^t_{i(t)}\}$.

Suppose $c^t_{i(t)} \in W^t$, $c^t_{i(t)} \in W^{t+1}$, and $c^{t+1}_{i(t)} \notin W^t$. Then $w^t = v^t(c^t_{i(t)}) = v^{t+1}(c^t_{i(t)}) + 1 = w^{t+1} + 1$. Thus, $w^{t+1} = w^t - 1$. Then $v^t(c^{t+1}_{i(t)}) + 1 = v^{t+1}(c^{t+1}_{i(t)})) = w^{t+1} = w^t - 1$, which implies $c^{t+1}_{i(t)} \in W^t_2$. Of course, $c^t_{i(t)} \in W^t$. To show that $W^t \setminus \{c^t_{i(t)}\} = \emptyset$, suppose there exists $k \in W^t \setminus \{c^t_{i(t)}\}$. Note that $k \neq c^t_{i(t)}, c^{t+1}_{i(t)}$, so $v^{t+1}(k) = v^t(k) = w^t = v^t(c^{t+1}_{i(t)}) + 2 = v^{t+1}(c^{t+1}_{i(t)}) + 1$. Then $v^{t+1}(k) > v^{t+1}(c^{t+1}_{i(t)})$, so $c^{t+1}_{i(t)} \notin W^{t+1}$, a contradiction. Therefore, $W^t = \{c^t_{i(t)}\}$.

Claim 3. *In Cases 3 and 7,* $w^{t+1} = w^t + 1$.

Suppose $c^{t+1}_{i(t)} \in W^t$. Then $v^t(c^{t+1}_{i(t)}) = w^t$, so $w^{t+1} = v^{t+1}(c^{t+1}_{i(t)}) = v^t(c^{t+1}_{i(t)}) + 1 = w^t + 1$, as required.

Claim 4. *In Case 4,* $w^{t+1} = w^t$ *and* $W^{t+1} \setminus \{c^{t+1}_{i(t)}\} = W^t \setminus \{c^t_{i(t)}\}$.

Suppose $c^t_{i(t)} \in W^t$, $c^t_{i(t)} \notin W^{t+1}$, and $c^{t+1}_{i(t)} \notin W^t$. To see that $w^{t+1} = w^t$, first note that $v^t(c^{t+1}_{i(t)}) \geq w^t - 1$, for otherwise we would have $w^{t+1} = v^{t+1}(c^{t+1}_{i(t)}) = v^t(c^{t+1}_{i(t)}) + 1 \leq w^t - 1 = v^t(c^t_{i(t)}) - 1 = v^{t+1}(c^t_{i(t)})$, which implies $c^t_{i(t)} \in W^{t+1}$, a contradiction. And since $c^{t+1}_{i(t)} \notin W^t$, this implies $v^t(c^{t+1}_{i(t)}) = w^t - 1$. Therefore, $w^{t+1} = v^{t+1}(c^{t+1}_{i(t)}) = v^t(c^{t+1}_{i(t)}) + 1 = w^t$. To see that $W^{t+1} \setminus \{c^{t+1}_{i(t)}\} = W^t \setminus \{c^t_{i(t)}\}$, take any $k \in W^t \setminus \{c^t_{i(t)}\}$. Note that $k \neq c^t_{i(t)}, c^{t+1}_{i(t)}$. Then $v^t(k) = v^{t+1}(k) = w^t = w^{t+1}$, so $k \in W^{t+1} \setminus \{c^{t+1}_{i(t)}\}$. Now take any $k \in W^{t+1} \setminus \{c^{t+1}_{i(t)}\}$. Note that $k \neq c^t_{i(t)}, c^{t+1}_{i(t)}$. Then $v^t(k) = v^{t+1}(k) = w^t = w^{t+1}$, so $k \in W^t \setminus \{c^t_{i(t)}\}$, as required.

Claim 5. *Cases 5 and 6 are not possible.*

Suppose $c^t_{i(t)} \notin W^t$ and $c^t_{i(t)} \in W^{t+1}$. Then $w^t > v^t(c^t_{i(t)}) = v^{t+1}(c^t_{i(t)}) + 1 = w^{t+1} + 1$, a contradiction.

Claim 6. *In Case 8, $w^{t+1} = w^t$ and $W^{t+1} \setminus \{c^{t+1}_{i(t)}\} = W^t$.*

Suppose $c^t_{i(t)} \notin W^t$, $c^t_{i(t)} \notin W^{t+1}$, and $c^{t+1}_{i(t)} \notin W^t$. The proof that $w^{t+1} = w^t$ is exactly as in the proof of Claim 4, as it only uses the last two of these conditions. Take any $k \in W^{t+1} \setminus \{c^{t+1}_{i(t)}\}$, so that $k \neq c^t_{i(t)}, c^{t+1}_{i(t)}$. Then $v^t(k) = v^{t+1}(k) = w^{t+1} = w^t$, so $k \in W^t$. Now take any $k \in W^t$, and note that $c^t_{i(t)}, c^{t+1}_{i(t)} \notin W^t$. Then $v^{t+1}(k) = v^t(k) = w^t = w^{t+1}$, so $k \in W^{t+1}$, as required.

Claim 7. *In Cases 3, 4, 7, and 8, $\widehat{W}^{t+1} \subseteq \widehat{W}^t$.*

Recall that $c^t_{i(t)} \in \widehat{W}^{t+1}$ in these cases by Claims 3, 4, and 6. Also, $c^t_{i(t)} \in W^t$. Note that $c^{t+1}_{i(t)} \in W^{t+1}$. Also, $c^{t+1}_{i(t)} \in W^t \subseteq \widehat{W}^t$ in Cases 3 and 7. In Cases 4 and 8, we have $w^t = w^{t+1} = v^{t+1}(c^{t+1}_{i(t)}) = v^t(c^{t+1}_{i(t)}) + 1$, so $c^{t+1}_{i(t)} \in W^t_1 \subseteq \widehat{W}^t$. Recall that $w^{t+1} \geq w^t$ in these cases by Claims 3, 4, and 6. Now take any $k \in \widehat{W}^{t+1} \setminus \{c^t_{i(t)}, c^{t+1}_{i(t)}\}$. Then $v^t(k) = v^{t+1}(k) \geq w^{t+1} - 1 \geq w^t - 1$, so $k \in \widehat{W}^t$. Thus, $\widehat{W}^{t+1} \subseteq \widehat{W}^t$.

Claim 8. *In Case 4, $\widehat{W}^t = \widehat{W}^{t+1}$.*

Of course, $c^t_{i(t)} \in W^t \subseteq \widehat{W}^t$. Note that $c^{t+1}_{i(t)} \in W^t \subseteq \widehat{W}^t$ in Cases 3 and 7. In Cases 4 and 8, we have $w^t = w^{t+1} = v^{t+1}(c^{t+1}_{i(t)}) = v^t(c^{t+1}_{i(t)}) + 1$, so $c^{t+1}_{i(t)} \in W^t_1 \subseteq \widehat{W}^t$. Now take any $k \in \widehat{W}^{t+1} \setminus \{c^t_{i(t)}, c^{t+1}_{i(t)}\}$. Then $v^t(k) = v^{t+1}(k) \geq w^{t+1} - 1 \geq w^t - 1$, so $k \in \widehat{W}^t$. Thus, $\widehat{W}^{t+1} \subseteq \widehat{W}^t$.

Claim 9. *Case 2 is not possible.*

We prove the claim in a number of steps. Suppose that $c^t_{i(t)} \to c^{t+1}_{i(t)}$ is in Case 2 for some round t, and without loss of generality let t be the first round belonging to Case 2. Let $i = i(t)$.

Step 1. $c_i^t \neq c_i^0$.

Proof. By Claim 2, $W^t = \{c_i^t\}$, so that $U_i(c^t) = u_i(c_i^t)$. If $c_i^t = c_i^0$, then,

$$\frac{1}{\#W^{t+1}} \sum_{k \in W^{t+1}} u_i(k) = U_i(c^{t+1}) > U_i(c^t) = u_i(c_i^t) = u_i(c_{i(t)}^0).$$

So there is a candidate ℓ such that $u_i(\ell) > u_i(c_i^0)$, a contradiction. $\square$

That is, i must have deviated at least once before round t. Let round s be the last round in which i deviated before t. That is, choose s so that $i = i(s)$, $s < t$, and there is no round r such that $s < r < t$ and $i = i(r)$. Then $c_i^{s+1} = c_i^t \in W_0^t$. By Claims 1 and 5, Cases 1, 5, and 6 are not possible, so all deviations before $c_i^t \to c_i^{t+1}$ belong to Cases 3, 4, 7, and 8.

Step 2. $u_i(c_i^{t+1}) > u_i(c_i^t)$.

Proof. Suppose $u_i(c_i^{t+1}) \leq u_i(c_i^t)$. Note that $c_i^{s+1} = c_i^t$ and, by Claim 7, $\widehat{W}^t \subseteq \widehat{W}^s$. Therefore, Lemma 1 implies that $u_i(c_i^t) \geq u_i(k)$ for all $k \in \widehat{W}^t$. By Claim 2, we have $w^{t+1} = w^t - 1$, and therefore $W^{t+1} = \widehat{W}^t \cup \{c_i^{t+1}\}$. Then $u_i(c_i^t) \geq u_i(k)$ for all $k \in W^{t+1}$, which yields

$$U_i(c^t) = u_i(c_i^t) \geq \frac{1}{\#W^{t+1}} \sum_{k \in W^{t+1}} u_i(k) = U_i(c^{t+1}),$$

where the first equality follows from $W^t = \{c_i^t\}$ in Claim 2. Then $c_i^t \to c_i^{t+1}$ is not a best winning deviation, a contradiction. $\square$

Step 3. $v^{s+1}(c_i^{t+1}) \geq v^{s+2}(c_i^{t+1}) \geq \cdots \geq v^{t-1}(c_i^{t+1}) = v^t(c_i^{t+1})$.

Proof. By Step 2, we have $u_i(c_i^{t+1}) > u_i(c_i^t)$. Since $c_i^t = c_i^{s+1}$, this implies, with Lemma 1, that $c_i^{t+1} \notin \widehat{W}^s$. By Claim 7, this implies $c_i^{t+1} \notin \widehat{W}^{s+1}$, i.e., $v^{s+1}(c_i^{t+1}) < w^{s+1} - 1$. Since all deviations before $c_i^t \to c_i^{t+1}$ belong to Cases 3, 4, 7, and 8, we then have $v^{s+1}(c_i^{t+1}) + 1 < w^{s+1} \leq w^{s+2} \leq \cdots \leq w^{t-1} \leq w^t$. Since $v^{s+2}(c_i^{t+1}) \leq v^{s+1}(c_i^{t+1}) + 1 < w^{s+2}$ and $c_{i(s+1)}^{s+2} \in W^{s+2}$, it follows that $c_{i(s+1)}^{s+2} \neq c_i^{t+1}$. Therefore, $v^{s+1}(c_i^{t+1}) \geq v^{s+2}(c_i^{t+1})$. Similarly, it must be that $c_{i(s+2)}^{s+3} \neq c_i^{t+1}$, and so on. An induction argument based on these observations then yields $v^{s+1}(c_i^{t+1}) \geq v^{s+2}(c_i^{t+1}) \geq \cdots \geq v^{t-1}(c_i^{t+1}) = v^t(c_i^{t+1})$. $\square$

Step 4. $c_i^{t+1} \in W_2^{s+1}$.

Proof. The first part of the proof of Step 3 shows that $c_i^{t+1} \notin \widehat{W}^{s+1}$. Suppose $c_i^{t+1} \notin W_2^{s+1}$. Then $v^{s+1}(c_i^{t+1}) < w^{s+1} - 2$. Since all deviations before $c_i^t \to c_i^{t+1}$ belong to Cases 3, 4, 7, and 8, we have $w^{s+1} \leq w^{s+2} \leq \cdots \leq w^{t-1} \leq w^t$. By Claim 2, this implies $w^{s+1} \leq w^{t+1} + 1$. Using Step 3, we then have $v^t(c_i^{t+1}) \leq v^{s+1}(c_i^{t+1}) < w^{s+1} - 2 \leq w^{t+1} - 1$, and $c_i^{t+1} \notin W_2^t$, contradicting Claim 2. Therefore, $c_i^{t+1} \in W_2^{s+1}$. $\square$

Step 5. $w^{s+1} = w^{s+2} = \cdots = w^{t-1} = w^t$.

Proof. By Claim 2 and Step 4, we have $c_i^{t+1} \in W_2^{s+1} \cap W_2^t$. Then, using Step 3, we have $w^{s+1} - 2 = v^{s+1}(c_i^{t+1}) \geq v^{s+2}(c_i^{t+1}) \geq \cdots \geq v^{t-1}(c_i^{t+1}) = v^t(c_i^{t+1}) = w^t - 2$, which implies $w^{s+1} \geq w^t$. With $w^{s+1} \leq w^{s+2} \leq \cdots \leq w^{t-1} \leq w^t$, this yields $w^{s+1} = w^{s+2} = \cdots = w^{t-1} = w^t$. □

Step 6. *For all rounds r with $s \leq r \leq t$, $c_{i(r)}^r \to c_{i(r)}^{r+1}$ belongs to Cases 4 or 8.*

Proof. By Step 5, we have $w^{s+1} = w^{s+2} = \cdots = w^{t-1} = w^t$, and by Claim 3, no deviation $c_{i(r)}^r \to c_{i(r)}^{r+1}$ can belong to Cases 3 or 7. This leaves only Cases 4 and 8. □

Step 7. $\#W^{s+1} = \#W^{s+2} = \cdots = \#W^{t-1} = \#W^t = 1$.

Proof. Since the relevant cases are Cases 4 and 8, we have $\#W^{s+1} \leq \#W^{s+2} \leq \cdots \leq \#W^{t-1} \leq \#W^t$. By Claim 2, we have $\#W^t = 1$, which then implies $\#W^{s+1} = \#W^{s+2} = \cdots = \#W^{t-1} = \#W^t = 1$. □

Step 8. *For all rounds r with $s \leq r \leq t$, $c_{i(r)}^r \to c_{i(r)}^{r+1}$ belongs to Case 4.*

Proof. By Step 7, we have $\#W^{s+1} = \#W^{s+2} = \cdots = \#W^{t-1} = \#W^t = 1$, and by Claim 6, no $c_{i(r)}^r \to c_{i(r)}^{r+1}$ can belong to Case 8. With Step 6, this leaves only Case 4. □

Step 9. $W^{s+1} = W^t$ *and* $W_1^{s+1} = W_1^t$.

Proof. Note that $c_i^{s+1} \in W^{s+1}$, $c_i^t \in W^t$, and by Step 7, $\#W^{s+1} = \#W^t = 1$. Then we have $W^{s+1} = \{c_i^{s+1}\} = \{c_i^t\} = W^t$. Then $W_1^{s+1} = W_1^t$ follows from Step 8 and Claim 8. □

Step 10. $W(c_i^{t+1}, c_{-i}^{s+1}) = W(c_i^{t+1}, c_{-i}^t)$.

Proof. By Claim 2 and Step 9, $W(c_i^{s+1}, c_{-i}^{s+1}) = W^{s+1} = \{c_i^t\}$, and by Step 4, $c_i^{t+1} \in W_2^{s+1}$. Therefore, $v(c_i^t | c_i^{t+1}, c_{-i}^{s+1}) = w^{s+1} - 1$ and $v(c_i^{t+1} | c_i^{t+1}, c_{-i}^{s+1}) = w^{s+1} - 1$, and we deduce that

$$W(c_i^{t+1}, c_{-i}^{s+1}) \quad = \quad \{c_i^t, c_i^{t+1}\} \cup W_1^{s+1}.$$

By Claim 2, we also have

$$W(c_i^{t+1}, c_{-i}^t) \quad = \quad \{c_i^t, c_i^{t+1}\} \cup W_1^t,$$

and then Step 9 implies $W(c_i^{t+1}, c_{-i}^{s+1}) = W(c_i^{t+1}, c_{-i}^t)$. □

Finally, we use Step 10 to deduce that

$$W(c_i^{t+1}, c_{-i}^s) = W(c_i^{t+1}, c_{-i}^{s+1}) = W(c_i^{t+1}, c_{-i}^t) = W(c^{t+1}) = W^{t+1}.$$

Then, using $W^{s+1} = W^t = \{c_i^t\}$ from Step 9 and Claim 2, we have

$$\begin{aligned} U_i(c_i^{t+1}, c_{-i}^s) &= U_i(c^{t+1}) = U_i(c_i^{t+1}, c_{-i}^t) > U_i(c^t) \\ &= u_i(c_i^t) = u_i(c_i^{s+1}) = U_i(c_i^{s+1}, c_{-i}^s). \end{aligned}$$

So $U_i(c_i^{t+1}, c_{-i}^s) > U_i(c_i^{s+1}, c_{-i}^s)$, but this contradicts the assumption that $c_i^s \to c_i^{s+1}$ is a best winning deviation, and we conclude that $c_i^t \to c_i^{t+1}$ belongs to Case 2 for no round t.

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