Uncovered Sets

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UNCOVERED SETS

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ABSTRACT. This paper covers the theory of the uncovered set used in the literatures on tournaments and spatial voting. I discern three main extant definitions, and I introduce two new concepts that bound existing sets from above and below: the deep uncovered set and the shallow uncovered set. In a general topological setting, I provide relationships to other solutions and give results on existence and external stability for all of the covering concepts, and I establish continuity properties of the two new uncovered sets. Of note, I characterize each of the uncovered sets in terms of a decomposition into choices from externally stable sets; I define the minimal generalized covering solution, a nonempty refinement of the deep uncovered set that employs both of the new relations; and I define the acyclic Banks set, a nonempty generalization of the Banks set.

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In social choice theory, one must address the problem of constructing collective choice sets when majority voting (or some other method of aggregation) does not produce a maximal alternative. The abstract approach to the institution-free analysis of politics and collective decision-making takes as primitive a dominance relation, or social preference, over a set of alternatives that incorporates the structure of power and the distribution of preferences within a society. A maximal element of this relation is stable, in the sense that there is no group with the power and inclination to overturn it. If maximal elements do not exist, then this approach does not directly yield a plausible choice. One solution is to explicitly impose more institutional structure and to use non-cooperative game-theoretic analysis; another is to consider methods other than maximality for the construction of choice sets.

The notion of uncovered set is central to the latter approach. Given arbitrary social preferences, a covering relation is a subrelation of weak preference defined in terms of nested upper sections. For example, following Gillies' (1959) analysis of transferable utility cooperative games, an alternative x Gillies covers y if x is strictly preferred to y and every alternative strictly preferred to x is also strictly preferred to y. Regardless of the initial preferences, this covering relation is asymmetric and transitive, and its maximal set, which I term the Gillies uncovered set, will be nonempty under very general conditions. When the strict preference relation admits no indifferences, as in the literature on tournaments (see Moulin (1986)), there is no latitude in this definition: this and other notions of covering become equivalent in that context. But in the general setting, distinctions between different notions of covering arise, and the corresponding uncovered sets possess differing properties of interest.

In this paper, I attempt to present a systematic and general treatment of the theory of the uncovered set and to contribute to the theory by introducing two new covering relations — deep covering and shallow covering — that serve a useful role as benchmarks for the other covering relations in that the corresponding deep and shallow uncovered sets provide upper and lower bounds, respectively, on all other notions of the uncovered set. In addition to these new sets, I focus on three definitions of covering in the literature due to Gillies (1959), Bordes (1983), and McKelvey (1986). In contrast to Gillies, an alternative x Bordes covers y if it is strictly preferred to y and every alternative weakly preferred to x is also weakly preferred to y; McKelvey covering the the conjunction of Gillies and Bordes. I conduct the analysis at a general level, imposing topological conditions on preferences that generalize finite models of tournaments and weak tournaments and subsume the spatial voting model, in which the alternatives form a subset of multidimensional Euclidean space. Although nonemptiness of the shallow uncovered set is problematic, the other uncovered sets are nonempty and externally stable with respect to covering under very general conditions involving only compactness and continuity.

Other definitions of covering have appeared, notably in Fishburn (1977), Miller (1980), and Richelson (1981), but the five highlighted above are distinguished from the latter three by an interesting decomposition into choices from externally stable sets. A set of alternatives is externally stable with respect to strict preference if for every alternative outside the set, there is a strictly preferred alternative inside; external stability with respect to weak preference is defined analogously. For each such set, we can consider the subset (possibly empty) of alternatives maximal within the set. Then the Gillies uncovered set is the union of maximal subsets in all sets that are externally stable with respect to strict preference. The deep, shallow, and Bordes uncovered sets are characterized similarly by varying the type of external stability and the criterion for choice within a set. Such a characterization is also available for the McKelvey uncovered set, though it is more involved. An advantage of the deep and shallow uncovered sets is that if we view them as correspondences defined over preferences and feasible sets, the deep uncovered set is upper hemicontinuous generally, while the shallow uncovered set is lower hemicontinuous on a large domain. These continuity properties have bearing on computation of the uncovered set in spatial voting models, where an infinite set of alternatives must be approximated by finite grids.

Though not the main focus of this paper, I use the notions of deep and shallow covering to define a hybrid version of the minimal covering set satisfying internal stability with respect to deep covering and external stability with respect to shallow covering. Uniqueness of these minimal generalized covering sets does not hold, but the dual usage of the two covering relations delivers existence of a minimal generalized covering set under very general topological conditions. The union of these sets, the "minimal generalized covering solution," is therefore a well-defined choice set, it forms a subset of the deep uncovered set, and it in fact survives iteration of the deep uncovered set operation. Furthermore, as a byproduct of the analysis of the stability structures of the uncovered sets, I provide four versions of the Banks set (Banks (1985)) for general environments and relate them to the uncovered sets, and I establish general nonemptiness of the acyclic Banks set.

The uncovered set has appeared in different forms in a number of analyses of general tournaments. The Gillies uncovered set was studied in early work on the spatial voting model by Shepsle and Weingast (1984) and Cox (1987), and existence of Gillies uncovered alternatives was proved in general settings by Bordes, Le Breton, and Salles (1992) and Banks, Duggan, and Le Breton

(2002, 2006).[1] Although what I call "McKelvey covering" appeared in Bordes (1983), McKelvey (1986) was the first to develop the idea and apply it in the spatial setting. The McKelvey uncovered set was then used in later analyses of weak tournaments by Duggan and Le Breton (1999, 2001), Dutta and Laslier (1999), and Peris and Subiza (1999). A particular advantage of the McKelvey uncovered set is that it contains the minimal covering set and therefore the support of the maximal mixed strategy equilibrium, known as the essential set, in the canonical model of Downsian competition. I give a thorough account of the relationships between the various versions of the uncovered set and many of the solutions considered in the literature.

The analysis of this paper takes place in the general topological framework of Banks, Duggan, and Le Breton (2006), who focus on the McKelvey uncovered set. Here, I take an arbitrary preference relation as primitive, interpreted as a social preference generated by some unmodeled aggregation process, and I impose conditions directly on that relation. In contrast, Banks, Duggan, and Le Breton provide a general voting model (allowing for a finite or infinite set of voters) and give conditions on voter preferences and the voting rule sufficient to induce the properties desired of the social preference relation. Thus, I rely on that paper for foundations to generate the conditions needed for the analysis, submerging voters and preference aggregation except for a brief discussion of Pareto optimality in the discussion section. Many of the results I present on nonemptiness and external stability of the uncovered sets make use of the technical machinery developed in the earlier paper.

1. Formalities

Take as primitive a topological space X, an asymmetric relation P, and a complete relation R. Thus, for all $x, y \in X$, we cannot have xPy and yPx, and we must have xRy or yRx. View X as a set of alternatives, P as a strict preference relation, and R as a weak preference relation. Assume that these relations are dual, i.e., for all $x, y \in X$, xRy if and only if not yPx; equivalently, for all $x, y \in X$, xPy if and only if not yRx. Say (P, R) is a *tournament* if X is finite and for all distinct $x, y \in X$, either xPy or yPx; equivalently, for all $x, y \in X$, xRy and yRx implies $x = y$. The general finite setting, where P and R are only assumed asymmetric and complete, respectively, is sometimes referred to as a *weak tournament.* Whenever X is finite, it is endowed with the discrete topology.

Given an arbitrary relation Q on X, let $Q(x) = \{y \in X \mid yQx\}$ be the upper section of alternatives that bear Q to x, and let $Q^{-1}(x) = \{y \in X \mid$

[1] Epstein (1998) analyzes the Gillies uncovered set in the divide-the-dollar model of distributive politics, while Penn (2006a) also considers the Bordes and McKelvey uncovered sets.

$xQy\}$ be the lower section of alternatives to which x bears Q. Define the *dual relation of* Q, denoted Q^*, as follows: for all $x, y \in X$, xQ^*y if and only if not yQx. Obviously, the dual operation is idempotent, i.e., $Q^{**} = Q$, and so we may speak of two relations being "dual to each other." For example, the strict and weak preference relations P and R are dual to one another, and we refer to (P, R) as a dual pair. Say Q is *antisymmetric* if for all $x, y \in X$, xQy and yQx together imply $x = y$, weakening asymmetry. And Q is *total* if for all $x, y \in X$, $x \neq y$ implies either xQy or yQx, weakening completeness. Thus, (P, R) is a tournament if X is finite and P is total, or equivalently, R is antisymmetric.

Preferences (P, R) are *upper semicontinuous* if $R(x)$ is closed for all $x \in X$; equivalently, $P^{-1}(x)$ is open for all $x \in X$. Note that upper semicontinuity implies that the strict upper section correspondence $P \colon X \rightrightarrows X$, defined by $P(x) = \{y \in X \mid yPx\}$, is lower hemicontinuous. Preferences are *lower semicontinuous* if $P(x)$ is open for all $x \in X$; equivalently, $R^{-1}(x)$ is closed for all x. A dual pair is *continuous* if it is upper and lower semicontinuous. It is *uniformly continuous* if R is a closed subset of $X \times X$; equivalently, P is open. The dual pair (P, R) is *discriminating* if for all $x \in X$, we have $R(x) \subseteq \{x\} \cup \mathrm{clos} P(x)$, so that indifference curves are topologically thin. Preferences satisfy *full weak sections* if for all $x \in X$, we have $\{x\} \cup \mathrm{int} R(x) \subseteq \{x\} \cup P(x)$, or in words, the alternatives strictly preferred to x comprise the interior of the set of alternatives weakly preferred to x. When X is finite, the latter two conditions are separately equivalent to (P, R) being a tournament, but they hold more generally in infinite models.[2] Preferences are *rich* if for all distinct $x, y \in X$, we have $R(x) \neq R(y)$. This condition is satisfied when R is antisymmetric. In general, it is restrictive if R is transitive, for it then actually implies that R is antisymmetric, but the condition appears to be a reasonable restriction on social preferences in multidimensional voting models, where R is typically intransitive.

In addition to upper semicontinuity, I may assume that the weak upper section $R(x)$ is compact for some or all alternatives, and at times it is useful to assume the weak image of every compact set is compact: say preferences satisfy *compact weak images* if for all compact $Y \subseteq X$, the image $R(Y) = \bigcup\{R(x) \mid x \in Y\}$ is compact. When X is Hausdorff, this is a strengthening of compact weak upper sections, but it should be viewed as a weak condition.[3] The condition of compact weak images automatically holds, for example, if X is compact and (P, R) is uniformly continuous, as the image of a compact

[2] See Proposition 12 in Banks, Duggan, and Le Breton (2006) for general conditions on voter preferences under which $R(x)$ is closed for every alternative in the social choice model; see Proposition 17 for conditions under which social preferences are discriminating; and see Proposition 19 for conditions delivering full weak sections.

[3] See Propositions 14 and 15 in Banks, Duggan, and Le Breton (2006) for general conditions under which $R(x)$ is compact for all alternatives; see Proposition 16 for general conditions for compact weak images.

set under a closed correspondence with compact range is always compact (see Lemma 17.8 in Aliprantis and Border (2006)).

An additional condition used in the sequel is that the weak upper section correspondence $R\colon X \rightrightarrows X$, defined by $R(x) = \{y \in X \mid yRx\}$, is lower hemicontinuous.[4] A more than sufficient condition, as stated in the following proposition, is that (P, R) is upper semicontinuous and discriminating.

PROPOSITION 1: Assume (P, R) is upper semicontinuous and discriminating. Then $R(\cdot)$ is lower hemicontinuous.

PROOF: Discriminating preferences and upper semicontinuity imply, respectively, the inclusions $R(x) \subseteq \{x\} \cup \text{clos}P(x) \subseteq R(x)$. Therefore, $R(x) = \{x\} \cup \text{clos}P(x)$. The correspondence $x \to \{x\}$ is obviously lower hemicontinuous, and $x \to \text{clos}P(x)$ takes the closure of values of a lower hemicontinuous correspondence, so it is lower hemicontinuous as well. Then $R(\cdot)$, as the union of lower hemicontinuous correspondences, is also lower hemicontinuous. ∎

One can define, in the abstract, three choice sets given a relation Q on X. The *maximal set of* Q is the set $\mathfrak{M}(Q)$ of alternatives that return Q to any other alternatives, i.e.,

$$\mathfrak{M}(Q) \quad = \quad \{x \in X \mid \text{ for all } y \in X \setminus \{x\}, yQx \text{ implies } xQy\}.$$

The *undominated set of* Q is the set $\mathfrak{U}(Q)$ of alternatives to which no other bears Q, i.e.,

$$\mathfrak{U}(Q) \quad = \quad \{x \in X \mid \text{ for all } y \in X \setminus \{x\}, \text{ not } yQx\},$$

and the *dominant set of* Q is the set $\mathfrak{D}(Q)$ of alternatives that bear Q to all others, i.e.,

$$\mathfrak{D}(Q) \quad = \quad \{x \in X \mid \text{ for all } y \in X \setminus \{x\}, xQy\}.$$

Obviously, the undominated and dominant sets of Q are subsets of the maximal set, i.e., $\mathfrak{U}(Q) \cup \mathfrak{D}(Q) \subseteq \mathfrak{M}(Q)$. Moreover, maximality generalizes the ideas of undominated and dominant sets, in the sense that if Q is anti-asymmetric, then $\mathfrak{M}(Q) = \mathfrak{U}(Q)$; furthermore, if Q is total, then $\mathfrak{M}(Q) = \mathfrak{D}(Q)$. In particular, the maximal and undominated sets of P coincide, as do the maximal and dominant sets of R. At times I will consider choice sets from proper subsets of alternatives. Given $Y \subseteq X$, let

$$\mathfrak{U}(Q, Y) \quad = \quad \{x \in Y \mid \text{ for all } y \in Y, \text{ not } yQx \ \}$$

denote the set of alternatives undominated in the set Y, and define the choice sets $\mathfrak{D}(Q, Y)$ and $\mathcal{M}(Q, Y)$ similarly.

[4]See Proposition 13 in Banks, Duggan, and Le Breton (2006) for much weaker conditions on voter preferences under which $R(\cdot)$ is lower hemicontinuous.

It is straightforward to show that the undominated set of a relation is the dominant set of its dual, and visa versa.

PROPOSITION 2: Let Q be a relation on X. Then $\mathfrak{U}(Q) = \mathfrak{D}(Q^*)$ and $\mathfrak{D}(Q) = \mathfrak{U}(Q^*)$.

Thus, the undominated set of P, called the *core,* is equivalent to the dominant set of R. In fact, from the above discussion it follows that the choice determined by the dual pair (P, R) is unambiguous, i.e., $\mathfrak{U}(P) = \mathfrak{M}(P) = \mathfrak{M}(R) = \mathfrak{D}(R)$, as long as these sets are nonempty.

2. COVERING RELATIONS

The main situation of interest is that in which the core is empty, in which case we confront the problem of constructing a choice set consisting of reasonably plausible alternatives. One approach is to specify a "covering relation" — a subrelation of weak preference defined by inclusion relationships between upper sections — and to analyze the undominated elements of those relations. I discern three primary definitions of covering in the literature; all are subrelations of P and are, therefore, asymmetric.

Gillies covering	$x\ G\ y$	$\Leftrightarrow$	xPy and $P(x) \subseteq P(y)$
Bordes covering	$x\ B\ y$	$\Leftrightarrow$	xPy and $R(x) \subseteq R(y)$
McKelvey covering	$x\ M\ y$	$\Leftrightarrow$	$x\ G\ y$ and $x\ B\ y$.

The first of the above relations was introduced, in the context of TU cooperative games by Gillies (1959).[5] The second has often been attributed to Miller (1980),[6] but the formal definition used by the latter author is slightly different. The specification I give was introduced by Bordes (1983). In fact, that paper was also the first to present the McKelvey covering relation, but McKelvey was the first to develop the idea at length and in a spatial setting. The set of undominated elements of G, denoted $\mathfrak{U}(G)$, is the *Gillies uncovered set,* while $\mathfrak{U}(B)$ is the *Bordes uncovered set,* and $\mathfrak{U}(M)$ is the *McKelvey uncovered set.*

I introduce two new notions of covering, which play special roles in the analysis as benchmarks for the other notions. My initial definition of deep covering contains a redundancy to bring out the parallel structure common

[5]This was the definition of choice in some work on spatial voting models, such as Shepsle and Weingast (1984) and Cox (1987).

[6]For example, Bordes, Le Breton, and Salles (1992) and Banks, Duggan, and Le Breton (2002) refer to this as "Miller covering."

to it and shallow covering.

$$\textbf{deep covering} \qquad x\ D\ y \;\;\Leftrightarrow\;\; xPy \text{ and } R(x) \subseteq P(y)$$

$$\textbf{shallow covering} \qquad x\ S\ y \;\;\Leftrightarrow\;\; xRy \text{ and } P(x) \subseteq R(y)$$

Of course, since R is reflexive, the definition of deep covering can be simplified, and henceforth I rely on the more efficient formulation:

$$x\ D\ y \;\;\Leftrightarrow\;\; R(x) \subseteq P(y).$$

The set of undominated elements of D, denoted $\mathfrak{U}(D)$, is the *deep uncovered set,* while the set of undominated elements of S, denoted $\mathfrak{U}(S)$, is the *shallow uncovered set.* Though deep covering is a subrelation of P, shallow covering need not be, and indeed it is reflexive. In fact, shallow covering is not even assured of being antisymmetric, distinguishing it from the other definitions of covering above. Nevertheless, the connection between deep and shallow covering is deeper than may at first appear. In fact, the relations can be derived from a unified notion of covering. Given an arbitrary relation Q on X, we can define an abstract *covering relation of* Q*,* denoted $\mathcal{C}(Q)$, as follows: for all $x, y \in X$,

$$x\mathcal{C}(Q)y \;\;\Leftrightarrow\;\; xQy \text{ and } Q^*(x) \subseteq Q(y).$$

In these terms, it is easily seen that deep covering is the covering relation of P, and shallow covering is the covering relation of R.

Proposition 3:

(i) $D = \mathcal{C}(P)$
(ii) $S = \mathcal{C}(R)$.

The five concepts of covering defined above are the main focus of the analysis, but other formulations have been used. Given a relation Q, define the *umbra* of Q, denoted $Q^\bullet$, as follows: for all $x, y \in X$, $xQ^\bullet y$ if and only if $Q(x) \subseteq Q(Y)$. Three other definitions appearing in the literature are based on the concept of an umbra; I use the term "shading" relation, rather than "covering" relation, to distinguish relations of this type.[7]

$$\textbf{Fishburn shading} \qquad x\ G^F\ y \;\;\Leftrightarrow\;\; xRy \text{ and } P(x) \subseteq P(y)$$

$$\textbf{Miller shading} \qquad x\ B^M\ y \;\;\Leftrightarrow\;\; xRy \text{ and } R(x) \subseteq R(y)$$

$$\textbf{Richelson shading} \qquad x\ M^R\ y \;\;\Leftrightarrow\;\; x\ G^F\ y \text{ and } x\ B^M\ y.$$

Thus, Fishburn shading is the umbra of P, and Miller shading is the umbra of R, while Richelson shading is the intersection of umbras: $G^F = P^\bullet$, $B^M = R^\bullet$, and $M^R = P^\bullet \cap R^\bullet$. Each of the shading relations is a subrelation of R and

[7]Bordes (1983) provides yet another relation, which x bears to y if and only if $P(x) \subsetneq P(y)$ and $R(x) \subsetneq R(y)$.

can be converted to a covering relation by conjoining it with strict preference: evidently,

$$\begin{aligned} G &= P \cap G^F \\ B &= P \cap B^M \\ M &= P \cap M^R . \end{aligned}$$

The shading relations weaken the corresponding definitions of covering by no longer requiring the relation to be a subrelation of P, and each of the preceding three relations is reflexive and may fail to be antisymmetric.

Fishburn shading was introduced by Fishburn (1977); Miller shading was defined by Miller (1980); and Bordes (1983) attributes Richelson shading to unpublished work by Richelson (Richelson, 1981). The latter is essentially the weak dominance relation defined by McKelvey (1986),[8] and the maximal elements of Richelson shading comprise the weakly undominated set, as defined by McKelvey.

Obviously, the undominated set of Fishburn shading is a subset of the Gillies uncovered set, and similarly for Miller shading and Bordes covering and for Richelson shading and McKelvey covering. The next proposition establishes that the inclusions hold even when we consider the maximal elements of the shading relations.

PROPOSITION 4:

(i) $\mathfrak{M}(G^F) \subseteq \mathfrak{U}(G)$
(ii) $\mathfrak{M}(B^M) \subseteq \mathfrak{U}(B)$
(iii) $\mathfrak{M}(M^R) \subseteq \mathfrak{U}(M)$

PROOF: For (i), take any $x \in \mathfrak{M}(G^F)$, and suppose there exists $y \in X$ such that $y\ G\ x$, i.e., yPx and $P(y) \subseteq P(x)$. In particular, $y\ G^F\ x$, so by maximality we have $x\ G^F\ y$, which implies $P(y) = P(x)$. Then we have $y \in P(x) = P(y)$, contradicting irreflexivity of P. Therefore, $x \in \mathfrak{U}(G)$. For (ii), take any $x \in \mathfrak{M}(B^M)$, and suppose there exists $y \in X$ such that $y\ B\ x$, i.e., yPx and $R(y) \subseteq R(x)$. In particular, $y\ B^M\ x$, so by maximality we have $x\ B^M\ y$, which implies $R(y) = R(x)$. Then we have $x \in R(x) = R(y)$, contradicting yPx. Therefore, $x \in \mathfrak{U}(B)$. For (iii), take any $x \in \mathfrak{M}(M^R)$, and suppose there exists $y \in X$ such that $y\ M\ x$, i.e., yPx, $P(y) \subseteq P(x)$, and $R(y) \subseteq R(x)$. In particular, $y\ M^R\ x$, so by maximality, we have $x\ M^R\ y$, which implies $P(x) = P(y)$ and $R(x) = R(y)$, which similarly leads to a contradiction. Therefore, $x \in \mathfrak{U}(M)$. ∎

It may seem arbitrary to focus on five covering relations while neglecting the three shading relations, but in Section 8, I give foundations for the first

[8]McKelvey (1986) actually defines weak dominance as the asymmetric part of the Richelson shading relation, but this does not affect the maximal set.

five uncovered sets (the McKelvey uncovered set to a lesser extent) in terms of a decomposition into stable sets. While the choice sets generated by the shading relations may also have characterizations in terms of stability properties, those characterizations would lack the simple form I give for the Gillies, Bordes, deep, and shallow (and McKelvey to a lesser degree) uncovered sets.

The next proposition establishes several obvious relationships between the above notions of covering and shading and the uncovered sets. Accordingly, the new variants of the uncovered set provide upper and lower bounds on all of those previously considered: the deep uncovered set is the largest choice set of those defined, while the shallow uncovered set is the smallest. Each of the covering relations, save shallow covering, is asymmetric and transitive, i.e., a strict partial order, with the important implication that when X is finite, the Gillies and Bordes uncovered sets are non-empty, as are the McKelvey and deep uncovered sets. That the shallow covering relation can violate transitivity is seen in the following example. Suppose $X = \{a, b, c, d\}$ and $P = \{(a, b), (b, c), (c, d), (d, a)\}$. Then $S = P$ is actually asymmetric, but it is nevertheless intransitive. Indeed, the shallow uncovered set in this example is empty: $\mathfrak{U}(S) = \emptyset$. Because shallow covering is not always antisymmetric, the shallow uncovered set may not coincide with its maximal set, so I include the latter in the comparisons for now.

PROPOSITION 5:

(i) The covering relations G, B, M, and D are asymmetric and transitive, and the shading relations G^F, B^M, and M^R are reflexive and transitive.

(ii) The covering relations are nested as follows:

$$D \subseteq M = (G \cap B) \ \subseteq \ (G \cup B) \ \subseteq S\ .$$

(iii) The shading and shallow covering relations are nested as follows:

$$(G^F \cap B^M) \ = M^R \subseteq \ (G^F \cup B^M) \ \subseteq S\ .$$

(iv) The uncovered sets are nested as follows:

$$\mathfrak{U}(S) \ \subseteq \ \mathfrak{M}(S) \ \subseteq \ (\mathfrak{U}(G) \cap \mathfrak{U}(B)) \ \subseteq \ (\mathfrak{U}(G) \cup \mathfrak{U}(B)) \ \subseteq \ \mathfrak{U}(M) \ \subseteq \ \mathfrak{U}(D).$$

PROOF: Part (i) is fairly clear, but I prove transitivity of B to illustrate. Take any $x, y, z \in X$ such that $x\ B\ y\ B\ z$. If not xPz, then zRx. But then $z \in R(x) \subseteq R(y)$, contradicting yPz. Thus, xPz. Then $R(x) \subseteq R(y) \subseteq R(z)$ implies $R(x) \subseteq R(z)$, and we conclude that $x\ B\ z$. Parts (ii) and (iii) follow directly from definitions. Part (iv) is implied directly by definition of undominated set and part (ii), except the second inclusion. To verify it, take $x \in \mathfrak{M}(S)$ and suppose $y\ G\ x$ or $y\ B\ x$ for some $y \in X$. Then yPx and $y\ S\ x$. By maximality, we have $x\ S\ y$, which implies xRy, contradicting yPx, as required. ∎

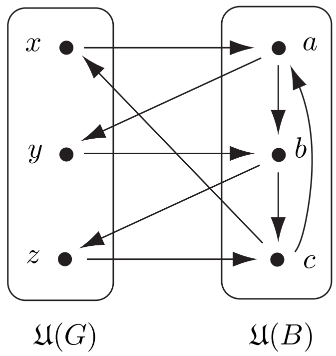

FIGURE 1. Inconsistency of Gillies and Bordes

Of the covering relations defined, Proposition 5 delineates clear logical relationships between all of the uncovered sets but Gillies and Bordes. It turns out that not only are the two uncovered sets logically nonnested, but they can actually be inconsistent: it is possible that their intersection is empty. For an example, consider Figure 1, which also appears in Figure 1 of Brandt and Fischer (2008) and Figure 3 of Bordes (1983). Here, $x\ G\ a$, $y\ G\ b$, and $z\ G\ c$, while $a\ B\ y$, $b\ B\ z$, and $c\ B\ x$, and the uncovered sets are $\mathfrak{U}(G) = \{x, y, z\}$ and $\mathfrak{U}(B) = \{a, b, c\}$, which are disjoint.

Under the assumptions of upper semicontinuity and discriminating preferences, we lose some diversity of the covering relations: Gillies covering and McKelvey covering typically coincide, as do the corresponding shading relations, and shallow covering is equivalent to Bordes shading. Then Gillies covering implies Bordes covering, and the two uncovered sets are nested with Bordes contained in Gillies, precluding the inconsistency demonstrated in Figure 1.

PROPOSITION 6: Assume (P, R) is upper semicontinuous and discriminating. Then $G^F = M^R \subseteq B^M = S$, and $G = M \subseteq B$.

PROOF: Equivalence of Gillies and McKelvey covering under upper semicontinuity and discriminating preferences is established in Proposition 5 of Banks, Duggan, and Le Breton (2006). To see equivalence of Fishburn and Richelson shading, note that $M^R \subseteq G^F$, and the opposite inclusion follows from Lemma 1 of Banks, Duggan, and Le Breton (2006). Then equivalence of Gillies and McKelvey covering follows from $G = P \cap G^F = P \cap M^R = M$. For equivalence of shallow covering and Miller shading, note that $B^M \subseteq S$. Now take any $x, y \in Y$ such that $x\ S\ y$, so xRy and $P(x) \subseteq R(y)$. Then by discriminating preferences and upper semicontinuity, we have $R(x) \subseteq \{x\} \cup \text{clos}P(x) \subseteq \{y\} \cup \text{clos}R(y) \subseteq \{y\} \cup R(y)$. With yRy, this implies $R(x) \subseteq R(y)$, and we conclude that $x\ B^M\ y$, so $S \subseteq B^M$, as required. ∎

It is not the case that Bordes and McKelvey covering coincide under the same conditions because of technicalities illustrated in Figure 1 of Banks, Duggan, and Le Breton (2006), where two alternatives satisfy $R(c) \subseteq R(a)$ but $P(c) \nsubseteq P(a)$. The problem is that an alternative b is indifferent to a, but is it isolated from the alternatives to which a is strictly preferred. Replacing discriminating preferences with full weak sections in Proposition 6 precludes this problem and delivers corresponding conditions under which Bordes and McKelvey covering coincide.

Proposition 7: Assume (P, R) is lower semicontinuous and satisfies full weak sections. Then $B^M = M^R \subseteq G^F = S$, and $B = M \subseteq G$.

Proof: Equivalence of Bordes and McKelvey covering under full weak sections follows directly from Lemma 1 of Banks, Duggan, and Le Breton (2006). To see equivalence of Bordes and Richelson shading, note that $M^R \subseteq B^M$, and the opposite inclusion follows from Lemma 1 of Banks, Duggan, and Le Breton (2006). Then equivalence of Bordes and McKelvey covering follows from $B = P \cap B^M = P \cap M^R = M$. For equivalence of shallow covering and Gillies shading, note that $G^M \subseteq S$. Now take any $x, y \in Y$ such that $x \; S \; y$, so xRy and $P(x) \subseteq R(y)$. Then by lower semicontinuity and full weak sections, we have $P(x) \subseteq \mathrm{int}R(y) \subseteq \{y\} \cup \mathrm{int}R(y) \subseteq \{y\} \cup P(y)$. With xRy, this implies $P(x) \subseteq P(y)$, and we conclude that $x \; B^M \; y$, so $S \subseteq B^M$, as required. ∎

Combining the previous propositions, we have the following corollary on the equivalence of Gillies and Miller covering.

Corollary 8: Assume (P, R) is continuous, discriminating, and satisfies full weak sections. Then $G^F = B^M = M^R = S$, and $G = B = M$.

The latter result gives conditions under which the three extant covering relations — Gillies, Bordes, and McKelvey — coincide, so that the five initially defined covering relations reduce to three. Moreover, shallow covering coincides with the shading relations, giving us equivalence of their maximal sets. But the shallow covering relation can violate antisymmetry, so as noted above, its maximal set need not coincide with its undominated set. The next result uses richness, with the assumption of discriminating preferences, to derive antisymmetry of shallow covering and, therefore, equivalence of the shallow uncovered set with its maximal set.

Proposition 9: Assume (P, R) is discriminating and rich. Then S is antisymmetric, and therefore $\mathfrak{M}(S) = \mathfrak{U}(S)$.

Proof: Take any $x, y \in X$ such that $x \; S \; y$ and $y \; S \; x$. By discriminating preferences, we have $R(y) \subseteq \{y\} \cup P(y) \subseteq R(x)$ and $R(x) \subseteq \{x\} \cup P(x) \subseteq R(y)$, and we conclude that $R(x) = R(y)$. Then richness implies $x = y$. ∎

Combining all of the assumptions of this section — upper semicontinuity, discriminating preferences, and richness — the distinctions between the maximal and undominated sets of shallow covering and the shading relations disappear. Thus, although I technically focus on just the five covering relations, there appear to be reasonable conditions under which the choice sets based on shading relations are captured via the shallow uncovered set.

Corollary 10: Assume (P, R) is upper semicontinuous, discriminating, rich, and satisfies full weak sections. Then $\mathfrak{U}(S) = \mathfrak{M}(M^R)$, and therefore the shallow uncovered set is equal to the maximal and undominated sets of all shading relations.

Finally, when preferences admit no indifferences, as in the literature on tournaments when X is finite, all distinctions between the uncovered sets and maximal sets of shading relations disappear, and we can essentially talk of a single "covering relation." Note that shallow covering is not technically equivalent to the other covering relations because it is reflexive, but this does not affect the equivalence of the undominated sets of the covering relations. In the context of tournaments, I can therefore refer to simply the "uncovered set" without ambiguity.

Proposition 11: Assume P is total.

(i) All of the covering relations, save shallow covering, are equivalent:

$$D = M = G = B \ .$$

(ii) All of the shading relations, with shallow covering, are equivalent:

$$M^R = G^F = B^M = S \ .$$

(iii) All of the uncovered sets and maximal and undominated sets of shading relations are equivalent.

3. Two-Step Principles

Corresponding to each notion of covering from the previous section is a dual relation, which is easily characterized. The straightforward exercise of calculating duals of the shading relations is omitted.

shallow covering*	$x\ S^*\ y$	$\Leftrightarrow$	xPy or $\exists z \in X : xPzPy$
Gillies covering*	$x\ G^*\ y$	$\Leftrightarrow$	xRy or $\exists z \in X : xRzPy$
Bordes covering*	$x\ B^*\ y$	$\Leftrightarrow$	xRy or $\exists z \in X : xPzRy$
McKelvey covering*	$x\ M^*\ y$	$\Leftrightarrow$	xRy or $\exists z \in X : xPzRy$ or $\exists z \in X : xRzPy$
deep covering*	$x\ D^*\ y$	$\Leftrightarrow$	$\exists z \in X : xRzRy.$

The duals of shallow and deep covering appear to have the simplest forms, as each step involves only strict (for shallow) or weak (for deep) preference.

With these dual covering relations defined, alternative characterizations of the corresponding uncovered sets are easily generated. Recall the duality between undominated and dominant sets: $\mathfrak{U}(Q) = \mathfrak{D}(Q^*)$. In particular, $\mathfrak{U}(D) = \mathfrak{D}(D^*)$, so we see that an alternative is in the deep uncovered set if and only if for all other alternatives $y \in X$, there exists $z \in X$ such that $xRzRy$. For another example, $\mathfrak{U}(S) = \mathfrak{D}(S^*)$ implies that an alternative is in the shallow uncovered set if and only if for all other alternatives $y \in X$, either xPy or there exists $z \in X$ such that $xPzPy$. These variations on what is known in the literature on tournaments as the "two step principle" are gathered together in the next proposition.[9]

Proposition 12: For all $x \in X$,

(i) $x \in \mathfrak{U}(S) \quad \Leftrightarrow \quad \forall y \in X \setminus \{x\} : xPy$ or $\exists z \in X : xPzPy$

(ii) $x \in \mathfrak{U}(G) \quad \Leftrightarrow \quad \forall y \in X \setminus \{x\} : xRy$ or $\exists z \in X : xRzPy$

(iii) $x \in \mathfrak{U}(B) \quad \Leftrightarrow \quad \forall y \in X \setminus \{x\} : xRy$ or $\exists z \in X : xPzRy$

(iv) $x \in \mathfrak{U}(M) \quad \Leftrightarrow \quad \forall y \in X \setminus \{x\} : xRy$ or $\exists z \in X : xRzPy$ or $\exists z \in X : xPzRy$

(v) $x \in \mathfrak{U}(D) \quad \Leftrightarrow \quad \forall y \in X \setminus \{x\} : \exists z \in X : xRzRy.$

Furthermore, the duals of the deep and shallow covering relations can be derived from a unified notion of "dual covering." In general, define the *dual covering relation of* Q, denoted $\mathcal{C}^*(Q)$, as follows: for all $x, y \in X$,

$$x\mathcal{C}^*(Q)y \quad \Leftrightarrow \quad xQ^*y \text{ or } Q^*(y) \nsubseteq Q(x).$$

[9]Viewed from this dual perspective, the deep and shallow uncovered sets also appear in McKelvey's Proposition 4.1, though the intersection representing the shallow uncovered set should be over sets $P(y) \cup \{y\}$.

Clearly, dual covering is the dual of covering: $\mathcal{C}^*(Q) = \mathcal{C}(Q)^*$. Given Proposition 3, this immediately yields the following result.

Proposition 13:

(i) $D^* = \mathcal{C}^*(P)$
(ii) $S^* = \mathcal{C}^*(R)$.

4. Related Choice Sets

The choice set of chief interest is of course the core, $\mathfrak{U}(P)$, but this set is often empty in the social choice framework. Many solutions have been proposed to address that problem, and this section (and the appendix) traces the logical relationships between the uncovered sets and a number of those solutions commonly applied to weak tournaments: the essential set, the minimal covering set, the ultimate uncovered set, the strong top cycle, and the weak top cycle. Of note, examples are provided in which the Gillies and Bordes uncovered sets are disjoint from the essential set. In Appendix A, I include several other solutions — the mixed saddle, the uncaptured set, and the untrapped set — in the analysis and provide a broad overview of the interconnections among these theories.

Before proceeding, the next proposition records the simple facts that the core forms a subset of all of the uncovered sets except the shallow uncovered set, and if preferences are also discriminating, then all of our solutions agree with the core when it is nonempty. For trivial example in which the core contains an alternative that does not belong to the shallow uncovered set, let $X = \{x, y\}$ and $P = \emptyset$, and note that both alternatives belong to the core, but they bear shallow covering to each other, so $\mathfrak{U}(S) = \emptyset$. The core is, however, always contained in the dominant set, and therefore the maximal set, of shallow covering. Interestingly, the core is always a subset of the maximal set of Fishburn shading, i.e., $\mathfrak{U}(P) \subseteq \mathfrak{M}(G^F)$, but it need not be contained in the maximal set of Miller shading, letting $X = \{x, y, z\}$ and $P = \{(y, z)\}$, we have $\mathfrak{U}(P) = \{x, y\}$, $y \ B^M \ x$, but not $x \ B^M \ y$, so the core alternative x is not maximal with respect to Miller shading.

Proposition 14:

(i) The core is contained in the dominant set of shallow covering, and therefore in all versions of the uncovered set except the shallow uncovered set:

$$\mathfrak{U}(P) \subseteq \mathfrak{D}(S) \subseteq \mathfrak{M}(S) \subseteq (\mathfrak{U}(G) \cap \mathfrak{U}(B)) \subseteq \mathfrak{U}(M) \subseteq \mathfrak{U}(D).$$

(ii) Assume (P, R) is discriminating. If the core is non-empty, then it is a singleton and it coincides with all of the uncovered sets and the dominant set of P:

$$\mathfrak{U}(P) = \mathfrak{U}(D) = \mathfrak{U}(M) = \mathfrak{U}(G) = \mathfrak{U}(B) = \mathfrak{U}(S) = \mathfrak{D}(P).$$

PROOF: I prove only that when preferences are discriminating, the core is the dominant set of P. Take any $x \in \mathfrak{U}(P)$ and any $y \in X$. If not xPy, then $y \in R(x) = \{x\} \cup \mathrm{clos}P(x)$. But $P(x) = \emptyset$, so $x = y$, as required. ∎

Next, I establish some simple inclusion relationships with two versions of the top cycle. For an arbitrary relation Q, let $Q^1 = Q$, and recursively define Q^k as follows: for all $x, y \in X$, xQ^ky if and only if there exists $z \in X$ such that $xQzQ^{k-1}y$. Then $Q^\infty = \bigcup_{k=1}^{\infty} Q^k$ is the *transitive closure* of Q. Then define the *weak top cycle* as the maximal elements of the transitive closure of weak preference: $WTC = \mathfrak{M}(R^\infty)$. The *strong top cycle* consists of the maximal elements of the transitive closure of strict preference: $STC = \mathfrak{M}(P^\infty)$. Of course, $STC \subseteq WTC$, and since R^∞ is complete, we can write $WTC = \mathfrak{M}(R^\infty) = \mathfrak{D}(R^\infty)$.[10] If P is total, then it is easily seen that $STC = WTC$, and I can simply refer to the "top cycle" without ambiguity, but in general the inclusion may be proper. An obvious property of the strong top cycle, and one used in the sequel, is that if xPy for some $y \in STC$, then $x \in STC$.

The next proposition, a trivial implication of the two-step principle, establishes that the weak top cycle encompasses the deep uncovered set and, therefore, all of the other choice sets defined above. The two choice sets are indeed distinct, for it is known in that in tournaments, the uncovered set may be a proper subset of the top cycle.

PROPOSITION 15: $\mathfrak{U}(D) \subseteq WTC$.

PROOF: Take any $x \in \mathfrak{U}(D)$ and any $y \in X \setminus \{x\}$. By Proposition 12, either xRy or there exists $z \in X$ such that $xRzRy$, both implying $xR^\infty y$. Therefore, $x \in WTC$. ∎

Before considering the strong top cycle, the next proposition generalizes Proposition 1 of Duggan (2007a) by weakening compactness of X to compactness of a single weak upper section, and in fact the proof given here relies on that earlier result.[11] The result establishes that the strong top cycle, and with it the weak top cycle, is nonempty under very general topological conditions. Moreover, we can locate elements of the strong top cycle in any compact weak upper section.

[10] Schwartz (1986) refers to the weak and strong top cycles, respectively, as GETCHA and GOTCHA.

[11] Although Duggan (2007a) assumes that X is a metric space, the proof of Proposition 1 in that paper does not rely on this assumption and goes through unchanged in the general topological setting.

PROPOSITION 16: Assume $R(x)$ is compact for some $x \in X$ and (P,R) is upper semicontinuous. Then $STC \cap R(x) \neq \emptyset$.

PROOF: Assume $R(x)$ is compact. If $x \in STC$, then we are done, so assume there exists $y \in X$ such that $yP^{\infty}x$ and not $xP^{\infty}y$. Note that $P^{\infty}(y) \subseteq R(x)$, for otherwise we would have $z \in X$ such that $xPzP^{\infty}y$, implying $xP^{\infty}y$, a contradiction. Let $Y = \text{clos}P^{\infty}(y)$, a compact subset of $R(x)$, let $P|_Y = P \cap (Y \times Y)$ denote the restriction of P to Y, let $R|_Y = R \cap (Y \times Y)$ denote the restriction of R to Y, and note that $(P|_Y, R|_Y)$ is upper semicontinuous in the relative topology on $R(x)$. Let $P|_Y^{\infty}$ denote the transitive closure of $P|_Y$. By Proposition 1 of Duggan (2007a), there exists $z \in \mathfrak{M}(P|_Y^{\infty}, Y)$. I claim that $z \in STC$. Consider any $w \in X$ such that $wP^{\infty}z$, so there exist $w_1, \ldots, w_m \in X$ such that $w = w_1 P \cdots w_m P z$. In particular, $z \in P^{-1}(w_m)$. By upper semicontinuity, $P^{-1}(w_m)$ is open, and therefore there exists $v \in P^{\infty}(y)$ such that $v \in P^{-1}(w_m)$, i.e., $w_m P v$, and we conclude that $w_m \in P^{\infty}(y)$. An induction argument based on these observations yields $w_i \in P^{\infty}(y)$ for each $i = 1, \ldots, m$, which implies $wP|_Y^{\infty}z$. Since $z \in \mathfrak{M}(P|_Y^{\infty}, Y)$, we then have $zP|_Y^{\infty}w$, which implies $zP^{\infty}w$, as required. ∎

Another trivial consequence of the two-step principle is the inclusion of the shallow uncovered set within the strong top cycle. In fact, the proof shows that the shallow uncovered set is included in the potentially smaller dominant set of P^{∞}.

PROPOSITION 17: $\mathfrak{U}(S) \subseteq STC$.

PROOF: Take $x \in \mathfrak{U}(S)$. Then Proposition 12 implies that for all $y \in X \setminus \{x\}$, we have xPy or there exists $z \in X$ such that $xPzPy$, both implying $xP^{\infty}y$. Thus, $x \in STC$. ∎

The logical nesting stated in the preceding proposition does not hold generally for the larger uncovered sets. When preferences form a tournament, the strong and weak top cycles are equivalent, and it is well-known that the uncovered set can be a proper subset of the top cycle of a tournament. When preferences admit nontrivial indifferences, however, the opposite inclusion can hold strictly, as when $X = \{x, y, z, w\}$ and $P = \{(x,y), (y,z), (z,w)\}$. Here, x bears P^{∞} to all other alternatives, but none bears the relation to x, so $STC = \{x\}$. But we have $zRxPy$, so not $y \; G \; z$, and we have $zPwRy$, so not $y \; B \; z$, and in fact, $\mathfrak{U}(G) = \{x, w, z\}$ and $\mathfrak{U}(B) = \{x, y, z\}$. Thus, the strong top cycle is not nested with the uncovered sets, but the analysis of external stability of these sets, in Section 6, will show that these uncovered sets do have nonempty intersection with the strong top cycle under quite general conditions.

The logic underlying the uncovered set can be used to locate smaller sets of some interest. An initial reduction of the uncovered set can be achieved by iterating the removal of covered alternatives, a procedure that necessarily terminates with a nonempty set when X is finite. Following Dutta and Laslier (1999), I define this choice set, called the *ultimate uncovered set,* using the notion of McKelvey covering, though other concepts of covering would in general produce different versions of the ultimate uncovered set. Given a subset $Y \subseteq X$ and alternatives $x, y \in Y$, say *x M-covers y in Y*, written $xM_Y y$, if xPy, $P(x) \cap Y \subseteq P(y) \cap Y$, and $R(x) \cap Y \subseteq R(y) \cap Y$. Letting $Y^1 = \mathfrak{U}(M)$ and letting Y^k be the kth power of the McKelvey uncovered set, define $Y^{k+1} = \mathfrak{U}(M_{Y^k}, Y^k)$. Then the ultimate uncovered set is $UC^\infty = \bigcap_{k=1}^\infty Y^k$. Propositions 18 and 19, below, imply that the Gillies and Bordes uncovered sets have nonempty intersection with the ultimate uncovered set. It is known in the tournament setting that the ultimate uncovered set may be a proper subset of the uncovered set, and the discussion prior to Proposition 18 shows that the opposite inclusion may hold, so there is no general logical relationship between the two uncovered sets and the ultimate uncovered set. The shallow uncovered set can be empty and can therefore have empty intersection with the ultimate uncovered set. In fact, an example in the appendix shows that the two sets can be disjoint, even when the shallow uncovered set is nonempty.

A further refinement of the uncovered set is the *minimal covering set,* denoted MC. To define this solution, say Y is a *M-covering set* if (i) no $x \in Y$ is M-covered in Y, and (ii) for all $y \in X \setminus Y$, there is some $z \in Y$ that M-covers y in $Y \cup \{y\}$. (It follows that an M-covering set is nonempty.) Then a minimal covering set is an M-covering set that includes no other M-covering sets.[12] Dutta (1988) proves that in a tournament, there is exactly one minimal covering set, and Dutta and Laslier (1999) and Peris and Subiza (1999) prove that uniqueness of the minimal covering set carries over to the general finite case. It is straightforward to verify (and it follows from Proposition 31 on external stability) that when X is finite, $\mathfrak{U}(M)$ is itself is a covering set, so the minimal covering set that it is contained in the McKelvey uncovered set.

A more subtle question is the relationship between the minimal covering set and the Gillies and Bordes uncovered sets. It is well-known that when X is finite and preferences form a tournament, the minimal covering set may be a proper subset of the Gillies and Bordes uncovered sets, which coincide. In fact, the opposite inclusions may hold: in Example 7 of Duggan and Le Breton (1999), the minimal covering set is the entire set X of alternatives, yet $x_6\ G\ x_2$, so $\mathfrak{U}(G) \subsetneq MC$; and in the same example, $x_5\ B\ x_7$, so we have $\mathfrak{U}(B) \subsetneq MC$. Thus, the solutions are logically nonnested. Of course, the

[12]See Brandt and Fischer (2008) for minimal covering sets defined in terms of Gillies and Bordes covering. The authors show that a weak tournament can admit multiple minimal Gillies covering sets and multiple Bordes covering sets, and the latter can actually fail to exist.

same example shows that the two uncovered sets may be proper subsets of the ultimate uncovered set. The next proposition establishes, however, that the Gillies uncovered set always has nonempty intersection with the minimal covering set, and therefore with the ultimate uncovered set.

Proposition 18: Assume X is finite. Then $\mathfrak{U}(G) \cap MC \neq \emptyset$.

Proof: Let $Y = MC$, let $P|_Y = P \cap (Y \times Y)$ be the restriction of P, and define Gillies covering in Y, denoted G_Y, as follows: for all $x, y \in Y$, $x\, G_Y\, y$ if and only if xPy and $P(x) \cap Y \subseteq P(y) \cap Y$. Let x be any element of the Gillies uncovered set in Y, i.e., $x \in \mathfrak{U}(G_Y, Y)$. Suppose that $y\, G\, x$ for some $y \in X$. If $y \in Y$, then this implies $y\, G_Y\, x$, a contradiction. Thus, $y \in X \setminus Y$, so there exists $z \in Y$ such that z M-covers y in Y, implying zPy and $P(z) \cap Y \subseteq P(y) \cap Y \subseteq P(x) \cap Y$. But then $z\, G_Y\, x$, contradicting $x \in \mathfrak{U}(G_Y, Y)$. We conclude that $x \in \mathfrak{U}(G) \cap MC$, as required. ∎

Essentially the same argument (couched in terms of Bordes covering rather than Gillies covering) establishes nonempty intersection of the Bordes uncovered set and the minimal covering set. Because the shallow uncovered set, even when nonempty, can be disjoint from the ultimate uncovered set, as shown in the appendix, the same holds true for the shallow uncovered set in relation to the minimal covering set.

Proposition 19: Assume X is finite. Then $\mathfrak{U}(B) \cap MC \neq \emptyset$.

A related class of solutions is based on equilibrium concepts applied to a particular two-player, symmetric, zero-sum noncooperative game derived from the preferences (P, R). Let the pure strategy sets of the players be X, finite for the remainder of this section, and let Π be the payoff function for player 1 defined by

$$\Pi(x,y) \quad = \quad \begin{cases} 1 & \text{if } xPy, \\ -1 & \text{if } yPx, \\ 0 & \text{else.} \end{cases}$$

Let σ and σ' denote probability distributions on X representing mixed strategies of the players, and let $\Pi(\sigma, \sigma')$ denote the expected payoff to player 1 from σ when player 2 uses strategy σ'. For a mixed strategy degenerate on a single alternative, I simply insert that alternative in the argument of Π.

Laffond, Laslier, and Le Breton (1993) prove that in a tournament, there is a unique mixed strategy equilibrium (necessarily symmetric) of the above game, and they refer to the support of the unique equilibrium mixed strategy as the *bipartisan set.* In general, uniqueness is lost when P fails to be total, but Dutta and Laslier (1999) establish that there is a unique mixed strategy equilibrium (necessarily symmetric) with maximal support, and they refer to this support set as the *essential set,* denoted ES. The authors show

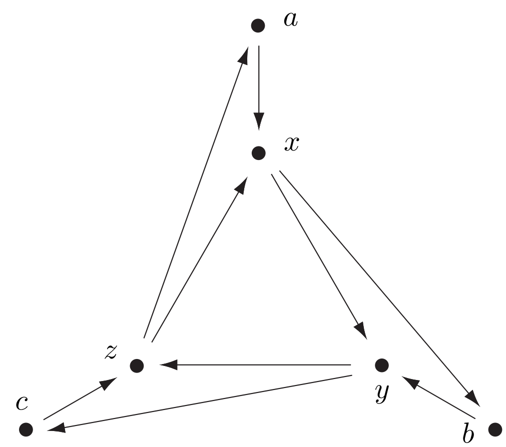

FIGURE 2. Gillies contained in essential set

the essential set is contained in the minimal covering set, and therefore in the McKelvey uncovered set. In the tournament setting, we may have $ES \subsetneq \mathfrak{U}(G) = \mathfrak{U}(B)$. For the opposite inclusion, consider the example in Figure 2. Here, $a \; G \; x$, $b \; G \; y$, and $c \; G \; z$, but letting σ be the uniform distribution on X, the pair (σ, σ) is a mixed strategy equilibrium. Therefore, $\mathfrak{U}(G) = \{a, b, c\} \subsetneq X = ES$. To obtain a corresponding example for the Bordes uncovered set, replace the (z, a) arc with (a, y), replace (x, b) with (b, z), and replace (y, c) with (c, x).

In fact, the relationship between these uncovered sets and the essential set is more tenuous than suggested so far: the Gillies and Bordes uncovered sets can actually have *empty* intersection with the essential set. To illustrate this, I extend the previous example in Figure 3. Note that $a \; G \; x$, $b \; G \; y$, $c \; G \; z$, $d \; G \; w$, and $e \; G \; v$, and in particular, $\mathfrak{U}(G) = \{a, b, c, d, e\}$. I claim, however, that there is no mixed strategy equilibrium that places positive probability on this set. Suppose, for example, that there is an equilibrium mixed strategy σ^{+0} such that

$$\begin{aligned}(\sigma^{+0}(a), \sigma^{+0}(b), \sigma^{+0}(c), \sigma^{+0}(d), \sigma^{+0}(e)) &= (p_1, p_2, p_3, p_4, p_5)\\ (\sigma^{+0}(x), \sigma^{+0}(y), \sigma^{+0}(z), \sigma^{+0}(w), \sigma^{+0}(v)) &= (q_1, q_2, q_3, q_4, q_5)\end{aligned}$$

with $p_1 > 0$. The graph in Figure 3 is rotationally symmetric, so it follows that the mixed strategy σ^{+1} defined by

$$\begin{aligned}(\sigma^{+1}(a), \sigma^{+1}(b), \sigma^{+1}(c), \sigma^{+1}(d), \sigma^{+1}(e)) &= (p_5, p_1, p_2, p_3, p_4)\\ (\sigma^{+1}(x), \sigma^{+1}(y), \sigma^{+1}(z), \sigma^{+1}(w), \sigma^{+1}(v)) &= (q_5, q_1, q_2, q_3, q_4)\end{aligned}$$

is also played in equilibrium. By rotational symmetry, we can define an equilibrium mixed strategy σ^{+2} by applying this permutation again, as well

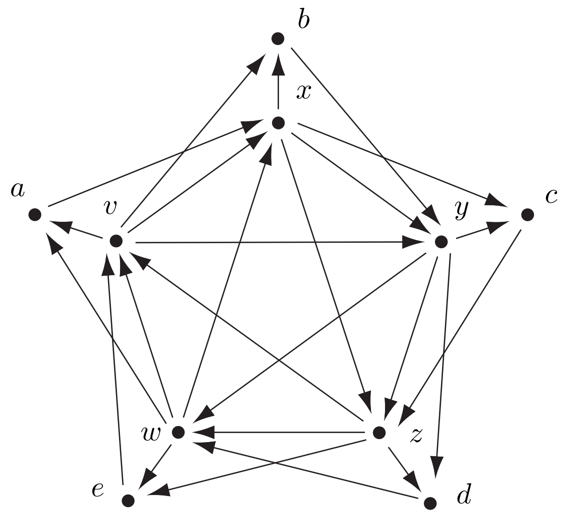

FIGURE 3. Gillies disjoint from essential set

as σ^{+3} and σ^{+4}. Because the set of equilibrium mixed strategies of a two-player, zero-sum game is convex, it follows that the mixed strategy

$$\sigma \quad = \quad \frac{1}{5}(\sigma^{+0} + \sigma^{+1} + \sigma^{+2} + \sigma^{+3} + \sigma^{+4})$$

is also played in equilibrium. Furthermore, note that

$$\sigma(a) = \sigma(b) = \sigma(c) = \sigma(d) = \sigma(e) = \frac{p_1 + p_2 + p_3 + p_4 + p_5}{5} = p \quad > \quad 0$$
$$\sigma(x) = \sigma(y) = \sigma(z) = \sigma(w) = \sigma(v) = \frac{q_1 + q_2 + q_3 + q_4 + q_5}{5} = q.$$

But then

$$\Pi(x,\sigma) \quad = \quad p + p + q + q - p - q - q \; = \; p \; > \; 0,$$

a contradiction. We conclude that the probability of alternative a in the initial equilibrium must be zero, as claimed, and by symmetry it follows that no equilibrium strategy puts positive probability on $\{a,b,c,d,e\}$. In fact, $ES = \{x,y,z,w,v\}$, and we have $ES \cap \mathfrak{U}(G) = \emptyset$.

A related example, in Figure 4, illustrates the possibility that the Bordes uncovered set and essential set have empty intersection. Now $a \; B \; x$, $b \; B \; y$, $c \; B \; z$, $d \; B \; w$, and $e \; B \; v$, so again $\mathfrak{U}(B) = \{a,b,c,d,e\}$. By a similar argument using rotational symmetry and convexity of the equilibrium set, if a belongs to the essential set, then there is an equilibrium mixed strategy σ that puts probability $p > 0$ on a, b, c, d, and e and probability q on x, y, z, w, and v. But then

$$\Pi(x,\sigma) \quad = \quad p + p - p \; = \; p \; > \; 0,$$

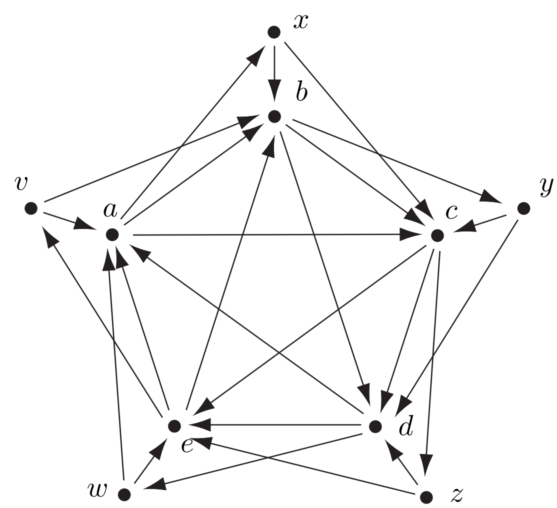

FIGURE 4. Bordes disjoint from essential set

a contradiction. Thus, by symmetry, no equilibrium strategy puts positive probability on the set $\{a, b, c, d, e\}$, and in fact, $ES = \{x, y, z, w, v\}$, so we have $ES \cap \mathfrak{U}(B) = \emptyset$.

5. Existence Results

A central question is whether the various uncovered sets are generally non-empty. We have noted that all of the uncovered sets, except for the shallow uncovered set, are non-empty whenever the set of alternatives is finite. When X is infinite, additional structure is required to ensure nonemptiness of these sets, and in fact it is sufficient to impose minimal compactness and continuity conditions to obtain nonemptiness of the Gillies uncovered set and the larger deep uncovered set. Further reasonable structure is imposed to obtain a result for the Bordes set, but stronger conditions are needed for the shallow uncovered set. Before proceeding to the analysis, next result, which follows follows directly from Proposition A4 of Banks, Duggan, and Le Breton (2006), provides a useful technical tool by showing that upper semicontinuity of (P, R) carries over to Fishburn shading.

PROPOSITION 20: Assume (P, R) is upper semicontinuous. Then G^F is upper semicontinuous.

For deep covering, the best technically behaved covering relation among those considered, we can actually obtain not only nonemptiness but compactness of the uncovered set quite generally, as the next proposition shows. Compactness of the set of alternatives and uniformly continuous preferences are sufficient for this result, but compactness can be relaxed substantially to local compactness, as long as we have compact weak images.

Proposition 21: Assume X is locally compact and (P, R) is uniformly continuous and satisfies compact weak images. Then D is open and $\mathfrak{U}(D)$ is nonempty and compact.

Proof: Take any net $\{(x_\alpha, y_\alpha)\}$ in the dual D^* of D, i.e., for each α there exists $z_\alpha \in X$ such that $x_\alpha R z_\alpha R y_\alpha$, and suppose this net converges to (x, y). Since X is locally compact, there is an open set V containing y and with compact closure, $\overline{V} = \text{clos}V$. By compact weak images, $R(\overline{V})$ is compact, and there exists β such that for all $\alpha \geq \beta$, $y_\alpha \in V$. Then for such α, we have $z_\alpha \in R(\overline{V})$, and we may therefore consider a subnet of $\{z_\alpha\}$, still indexed by α for simplicity, that converges to some $z \in R(\overline{V})$. Since R is closed, we have $xRzRy$, i.e., $(x, y) \in D^*$. Thus, the dual D^* is closed, and it follows that D is open. For existence of an undominated element of D, consider any $w \in X$ and any open set V containing w with compact closure $\overline{V}$, so $R(\overline{V})$ is compact. By Proposition 20, G^F is upper semicontinuous. Furthermore, $G^F(w) \subseteq R(w) \subseteq R(\overline{V})$, so $G^F(w)$ is compact, and Proposition A1 of Banks, Duggan, and Le Breton (2006) implies $\mathfrak{M}(G^F) \neq \emptyset$, and therefore, by Propositions 4 and 5, the deep uncovered set is nonempty. Closedness is evident from

$$X \setminus \mathfrak{U}(D) \;\; = \;\; \bigcup_{y \in X} D^{-1}(y),$$

which shows that complement of the deep uncovered set is the union of open sets and is, therefore, open. To prove compactness, note that by the two-step principle, $\mathfrak{U}(D) \subseteq R^2(w) \subseteq R(R(\overline{V}))$, which is compact. Thus, as a closed subset of a compact set, the deep uncovered set is compact. ∎

An immediate implication is that the deep uncovered set can be iterated. Given a subset $Y \subseteq X$ and alternatives $x, y \in Y$, say *x D-covers y in Y*, written $xD_Y y$, if $R(x) \cap Y \subseteq P(y) \cap Y$. Proposition 21 implies that when Y is compact and P is open, the relation D_Y is open in the relative topology on $Y \times Y$, and therefore the deep uncovered set relative to Y, $\mathfrak{U}(D_Y, Y)$, is nonempty and compact. Let $Y^1 = \mathfrak{U}(D)$, let Y^k be the kth power of the deep uncovered set, and define $Y^{k+1} = \mathfrak{U}(D_{Y^k}, Y^k)$. Of course, if X is compact and (P, R) is uniformly continuous, then it follows that Y^k is nonempty and compact for all k. Vartiainen (2011) uses these observations to define the *ultimate deep uncovered set*, $UD^\infty = \bigcap_{k=1}^\infty Y^k$. The next corollary, which follows directly from Proposition 21, establishes that the ultimate undominated set is nonempty and compact in very general topological settings.

Corollary 22: Assume X is locally compact and (P, R) is uniformly continuous and satisfies compact weak images. Then UD^∞ is nonempty and compact.

The next result establishes that under even weaker continuity and compactness conditions, the maximal set of Fishburn shading is nonempty: it is sufficient that preferences are upper semicontinuous and that at least one weak upper section is compact. Moreover, we can locate a maximal element within any compact section. With Proposition 20, it follows immediately from Proposition A1 of Banks, Duggan, and Le Breton (2006).[13]

PROPOSITION 23: Assume (P, R) is upper semicontinuous and $R(x)$ is compact for some $x \in X$. Then $\mathfrak{M}(G^F) \cap G^F(x) \neq \emptyset$.

Recall that Proposition 4 shows that any alternative x maximal with respect to G^F belongs to the Gillies uncovered set. Thus, a corollary is non-emptiness of the Gillies uncovered set, as proved in Banks, Duggan, and Le Breton (2006).[14] Obviously, nonemptiness of the McKelvey follows, and we now also have nonemptiness of the deep uncovered set without the assumptions of local compactness, lower semicontinuity, or the full force of compact weak images. Note, however, that we no longer obtain compactness of the deep uncovered set.

COROLLARY 24: Assume (P, R) is upper semicontinuous and $R(x)$ is compact for some $x \in X$. Then $\mathfrak{U}(G) \neq \emptyset$, and therefore $\mathfrak{U}(M) \neq \emptyset$ and $\mathfrak{U}(D) \neq \emptyset$.

For the corresponding result for Bordes shading, I impose the condition that the weak upper section correspondence $R(\cdot)$ is lower hemicontinuous. The next result follows from Proposition A3 of Banks, Duggan, and Le Breton (2006).

PROPOSITION 25: Assume (P, R) is upper semicontinuous and $R(\cdot)$ is lower hemicontinuous. Then B^M is upper semicontinuous.

With lower hemicontinuity of weak upper sections, we obtain corresponding conditions under which the maximal set of B^M is non-empty. With Proposition 25, the proof again follows from Proposition A1 of Banks, Duggan, and Le Breton (2006).

PROPOSITION 26: Assume that (P, R) is upper semicontinuous, that $R(x)$ is compact for some $x \in X$, and that $R(\cdot)$ is lower hemicontinuous. Then $\mathfrak{M}(B^M) \cap B^M(x) \neq \emptyset$.

[13] A proof of this result assuming compactness of the set of alternatives can also found in the proof of Theorem 3 of Banks, Duggan, and Le Breton (2002).

[14] This is also in Theorem 2 of Bordes, Le Breton, and Salles (1992) and Theorem 3 of Banks, Duggan, and Le Breton (2002). The results of those paper are less general in that the set X of alternatives is assumed compact, rather than the weaker compactness condition used here; but the setting of Banks, Duggan, and Le Breton is a more general model of two-player, zero-sum games.

With Proposition 4, a corollary is non-emptiness of the Bordes uncovered set.

Corollary 27: Assume that (P, R) is upper semicontinuous, that $R(x)$ is compact for some $x \in X$, and that $R(\cdot)$ is lower hemicontinuous. Then $\mathfrak{U}(B) \neq \emptyset$.

As an aside, Proposition 3 of Banks, Duggan, and Le Breton (2006) establishes non-emptiness of the maximal set of M^R, i.e., the weakly undominated set, under the same conditions.

Proposition 28: Assume that (P, R) is upper semicontinuous, that $R(x)$ is compact for some $x \in X$, and that $R(\cdot)$ is lower hemicontinuous. Then $\mathfrak{M}(M^R) \cap B^M(x) \neq \emptyset$.

Nonemptiness of the shallow uncovered set is a more difficult issue. It is trivial to construct examples in which the shallow uncovered set is empty, as when $X = \{a, b\}$ and $P = \emptyset$, for then both alternatives are in the core, yet each bears S to the other. In such cases, we should instead consider the maximal set of shallow covering, but the example preceding Proposition 5 shows that the shallow uncovered set may be empty even when S is antisymmetric. Note that preferences are not discriminating in that example. The next result verifies that upper semicontinuity of (P, R) carries over to shallow covering without any additional assumptions, an important — but not sufficient — condition for existence of an uncovered alternative.

Proposition 29: Assume (P, R) is upper semicontinuous. Then S is upper semicontinuous.

Proof: Take any $x \in X$ and any net $\{y_\alpha\}$ in $S(x)$ converging to y. For each α, we have $y_\alpha \in R(x)$ and $P(y_\alpha) \subseteq R(x)$. By upper semicontinuity, $R(x)$ is closed, and we conclude that $y \in R(x)$. If not $P(y) \subseteq R(x)$, then there exists $w \in X$ such that $xPwPy$. By upper semicontinuity, $P^{-1}(w)$ is open, so we can choose high enough α such that $y_\alpha \in P^{-1}(w)$, but then $w \in P(y_\alpha) \subseteq R(x)$, contradicting xPw. Therefore, $y \; S \; z$, as required. ∎

To obtain existence, examples such as that preceding Proposition 5 must be precluded, and to do so I add the assumption that preferences are discriminating. Then Proposition 26 yields maximal elements of Miller shading, and by Proposition 6, Miller shading is equivalent to shallow covering, easily delivering maximal elements of shallow covering. To strengthen this to nonemptiness of the shallow uncovered set, I impose the assumption that preferences are rich. This implies that Miller shading is antisymmetric, and therefore the maximal set is equivalent to the undominated set.

Corollary 30: Assume that $R(x)$ is compact for some $x \in X$, and that (P, R) is upper semicontinuous, discriminating, and rich. Then $\mathfrak{U}(S) \neq \emptyset$.

6. External Stability

A subset is Q-*externally stable* if for all $y \in X \setminus Y$, there exists $x \in Y$ such that xQy. Let $\mathcal{E}(Q)$ denote the collection of all Q-externally stable subsets of alternatives. Propositions 23 and 26 directly imply external stability of the maximal sets of Fishburn and Miller shading, while Proposition 4 of Banks, Duggan, and Le Breton (2006) provides conditions under which the McKelvey uncovered set and the maximal set of Richelson shading are externally stable. The next proposition adds external stability of the Bordes and deep uncovered sets.

Proposition 31: Assume $R(x)$ is compact for all $x \in X$ and $R(\cdot)$ is lower hemicontinuous.

(i) $\mathfrak{U}(B)$ is B-externally stable.
(ii) $\mathfrak{U}(M)$ is M-externally stable.
(iii) $\mathfrak{U}(D)$ is D-externally stable.

Proof: For part (i), suppose $x \notin \mathfrak{U}(B)$, so there exists $y \in X$ such that $y\ B\ x$, i.e., yPx and $R(y) \subseteq R(x)$. Propositon 26, with Proposition 4, yields $z \in \mathfrak{U}(B) \cap B^M(y)$. Clearly, $z\ B^M\ y$ implies $R(z) \subseteq R(y) \subseteq R(x)$. Furthermore, zPx, for otherwise we have $x \in R(z) \subseteq R(y)$, contradicting yPx. Therefore, $z \in \mathfrak{U}(B)$ and $z\ B\ x$, as required. Part (ii) is given by Proposition 4 of Banks, Duggan, and Le Breton (2006). To prove part (iii), suppose $x \notin \mathfrak{U}(D)$, so there exists $y \in X$ such that $y\ D\ x$, i.e., $R(y) \subseteq P(x)$. If $y \in \mathfrak{U}(B)$, then because $\mathfrak{U}(B) \subseteq \mathfrak{U}(D)$, we are done. Otherwise, by part (i), there exists $z \in \mathfrak{U}(B)$ such that $z\ B\ y$, which implies $R(z) \subseteq R(y) \subseteq P(x)$. Then we have $z \in \mathfrak{U}(D)$ and $z\ D\ x$, as required. ∎

External stability of the Gillies uncovered set is more difficult to obtain. It holds, of course, if the set of alternatives is finite: since G is a strict partial order, standard results in the theory of binary relations deliver the result. In the infinite setting, however, it seems that stronger conditions must be imposed. The approach in Proposition 31 does not appear to be effective. Indeed, suppose $x \notin \mathfrak{U}(G)$, so there exists $y \in X$ such that $y\ G\ x$. Proposition 23, with Proposition 4, yields $z \in \mathfrak{U}(G) \cap G^F(y)$, and clearly $P(z) \subseteq P(x)$. This implies zRx, but the problem is in deducing that zPx. This is illustrated in Figure 5, where z is Gillies uncovered, $I(z) = R(z) \cap R^{-1}(z)$ is the set of alternatives indifferent to z (shaded), and $x \in I(z)$.

To preclude this possibility, I add the assumption of discriminating preferences. Then Proposition 6 implies that Gillies covering coincides with McKelvey covering, i.e., $G = K$, and external stability of the Gillies uncovered set follows from Proposition 31. This is to some extent unsatisfactory, because discriminating preferences rule out indifferences when X is finite, an unneeded restriction in that case. Thus, the following result does not state

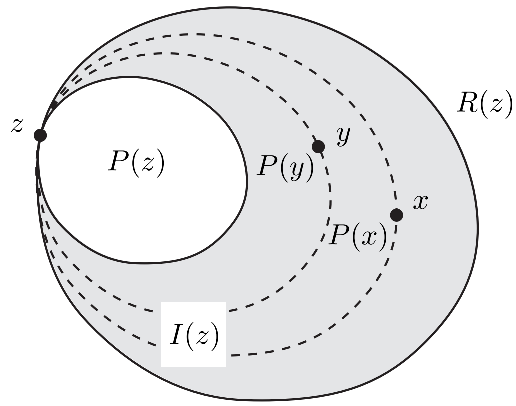

FIGURE 5. External stability difficulties for Gillies

a simple sufficient condition that generalizes the finite case to the infinite one. Note that upper semicontinuity and discriminating preferences imply lower hemicontinuity of weak upper sections, by Proposition 1, so I omit the explicit statement of that condition.

COROLLARY 32: Assume $R(x)$ is compact for all $x \in X$. If either X is finite or (P, R) is discriminating, then $\mathfrak{U}(G)$ is G-externally stable.

Finally, augmenting the above conditions with the assumption that preferences are rich, the external stability of the shallow uncovered set follows.

PROPOSITION 33: Assume that $R(x)$ is compact for all $x \in X$ and that (P, R) is discriminating and rich. Then $\mathfrak{U}(S)$ is S-externally stable.

PROOF: Lower hemicontinuity of $R(\cdot)$ follows from Proposition 1, so Proposition 26 implies that $\mathfrak{M}(B^M)$ is B^M-externally stable. Corollary 8 implies $B^M{=}S$, and Proposition 9 implies $\mathfrak{M}(S) = \mathfrak{U}(S)$. Combining these observations, $\mathfrak{U}(S)$ is S-externally stable. ∎

The above external stability results reveal general connections to the strong top cycle for the uncovered sets other than the shallow uncovered set. The next result, a corollary of Proposition 31, relies on external stability of the Bordes uncovered set to establish nonempty intersection with the strong top cycle. An obvious implication is that the strong top cycle has nonempty intersection with the McKelvey and deep uncovered sets.

COROLLARY 34: Assume that $R(x)$ is compact for all $x \in X$ and $R(\cdot)$ is lower hemicontinuous. Then $\mathfrak{U}(B) \cap STC \neq \emptyset$, and therefore $\mathfrak{U}(M) \cap STC \neq \emptyset$ and $\mathfrak{U}(D) \cap STC \neq \emptyset$.

PROOF: By Proposition 16, there exists $x \in STC$. If $x \in \mathfrak{U}(B)$, then we are done. Otherwise, Proposition 31 yields $y \in \mathfrak{U}(B)$ such that $y\ B\ x$. In particular, yPx, and therefore $y \in STC \cap \mathfrak{U}(B)$. ∎

A corollary of Corollary 32, by a similar argument, is that the strong top cycle has nonempty intersection with the Gillies uncovered set when preferences are upper hemicontinuous and discriminating.

COROLLARY 35: Assume that $R(x)$ is compact for all $x \in X$. If either X is finite or (P, R) is upper semicontinuous and discriminating, then we have $\mathfrak{U}(G) \cap STC \neq \emptyset$.

7. Implications for Minimal Covering Sets

The definition given above for the minimal covering set follows standard lines in using McKelvey covering, providing a well-defined solution for weak tournaments, but one that is not apparently tractable in infinite settings. Because of the continuity properties of deep and shallow covering, it is of interest to consider a formulation of the minimal covering set that takes advantage of those properties to deliver existence of a form of minimal covering set. Care must be taken in the definition of covering set in the abstract setting, as it is important that the notion of covering used for internal stability of a covering set have open lower sections and that the notion used for external stability has closed upper sections. This leads to a hybrid concept of covering set in which no element of the set deeply covers another, and each alternative outside the set is shallow covered by an element of the set. Moreover, an issue that is trivial in weak tournaments, but important here, is that a covering set be required to be compact. I show that there is very generally at least one generalized minimal covering set, so-defined. There may in fact be multiple such sets, but I show that their union, the "minimal generalized covering solution," is a subset of the ultimate deep uncovered set.

Following above conventions, given a set Y of alternatives and $x, y \in Y$, say x *S-covers y in Y*, written xS_Yy if xRy and $P(x) \cap Y \subseteq R(y) \cap Y$. Given a subset $Y \subseteq X$, say Y is a *generalized covering set* if (i) no $x \in Y$ is D-covered in Y, (ii) for all $y \in X \setminus Y$, there is some $x \in Y$ that S-covers y in $Y \cup \{y\}$, and (iii) Y is compact. (It follows that a generalized covering set is nonempty.) Then a *minimal generalized covering set* is a generalized covering set that includes no other generalized covering set. The next proposition provides general conditions under which there is at least one minimal generalized covering set. In fact, the proposition proves somewhat more than that: within any compact set satisfying external stability with respect to shallow covering, there is a minimal generalized covering set. Formally, a set Y of alternatives is an *outer S-covering set* if Y is compact and for all $x \in X \setminus Y$, there exists $y \in Y$ such that y S-covers x in $Y \cup \{x\}$.

This generality is useful in the following analysis of the location of minimal generalized covering sets in relation to the ultimate deep uncovered set. Of course, when X is compact, it is itself an outer S-covering set, and more generally, if the deep uncovered set is compact and externally stable, then it is an outer S-covering set. Note, as well, that if there is an alternative x such that the closure of $P^{\infty}(x)$ is compact, then $\mathrm{clos}P^{\infty}(x)$.

Proposition 36: Assume (P, R) is uniformly continuous, and let W be an outer S-covering set. Then there exists a minimal generalized covering set $Y \subseteq W$.

Proof: Let $\mathcal{Y}$ denote the collection of all outer S-covering sets contained in W, and note $W \in \mathcal{Y}$, so the collection is nonempty. Let $\mathcal{C}$ be any chain in $\mathcal{Y}$, so that any two sets belonging to the chain are related by set inclusion. Index the chain as $\mathcal{C} = \{Y_\alpha\}$, and define the direction $\geq$ on indices so that $\alpha \geq \beta$ if and only if $Y_\alpha \subseteq Y_\beta$. The collection $\mathcal{C}$ has the finite intersection property, so compactness of W implies that the intersection $Y = \bigcap \mathcal{C}$ is a nonempty, compact subset of W. The first step in the proof is to show that $Y \in \mathcal{Y}$. To this end, consider any $y \in X \setminus Y$. Then there is some β such that for all $\alpha \geq \beta$, $y \notin Y_\alpha$. For each such α, there exists $x_\alpha \in Y_\alpha$ such that x_α S-covers y in $Y_\alpha \cup \{y\}$. By compactness, the net $\{x_\alpha\}$ has a subnet, still indexed by α, that converges to some $x \in Y$. For each $\alpha \geq \beta$, we have $x_\alpha \in R(y)$ and $P(x_\alpha) \cap Y_\alpha \subseteq R(y) \cap Y_\alpha$. By continuity, $R(y)$ is closed, and therefore $x \in R(y)$, so xRy. If not $P(x) \cap (Y \cup \{y\}) \subseteq R(y) \cap (Y \cup \{y\})$, then there exists $z \in Y \cup \{y\}$ such that $yPzPx$. Then xRy implies $z \in Y$. Since $x \in P^{-1}(z)$, an open set by uniform continuity, we can choose high enough α such that $x_\alpha \in P^{-1}(z)$, but then $z \in P(x_\alpha) \cap Y_\alpha \subseteq R(y) \cap Y_\alpha$, contradicting yPz. Therefore, x S-covers y in $Y \cup \{y\}$. It follows that $Y \in \mathcal{Y}$, and since the chain $\mathcal{C}$ was arbitrary, Zorn's lemma implies that $\mathcal{Y}$ contains a minimal element, say Z.

The second, and last, part of the proof is to show that Z is a minimal generalized covering set. To this end, consider any $x, y \in Z$, and suppose that xD_Zy. Note that preferences restricted to Z, namely $(P|_Z, R|_Z)$, are uniformly continuous in the relative topology on Z, and therefore, by Proposition 21, the relation D_Z is open in the relative topology on $Z \times Z$. It follows that there is an open set $V \subseteq X$ such that $D_Z^{-1}(x) = Z \cap V$. Then the set $Y = Z \setminus D_Z^{-1}(x) = Z \setminus V$ is compact and contains x, and I claim it is an outer S-covering set. Indeed, consider any $w \in X \setminus Y$. If $w \in Z$, then $w \in V$, so x D-covers w in Z, and therefore it S-covers w in $Y \cup \{w\}$. If $w \notin Z$, then there exists $s \in Z$ such that s S-covers w in $Z \cup \{w\}$, and in particular sRw. In case $s \in Y$, then we have $s \in Y$ such that s S-covers w in $Y \cup \{w\}$. Otherwise, $s \in V$, so xD_Zs. To see that x S-covers w in $Y \cup \{w\}$, first note that since x D-covers s in Z and s S-covers w in $Y \cup \{w\}$, we have $x \in P(s) \cap (Y \cup \{w\}) \subseteq R(w) \cap (Y \cup \{w\})$, so xRw. Now consider any $t \in Y \cup \{w\}$ such that tPx, which implies $t \neq w$. Then $t \in Y$, so xD_Zs implies tPs, and then the fact that s S-covers w in $Z \cup \{w\}$ implies tRw. Thus, Y is an outer S-covering set such that $Y \subsetneq Z$, contradicting minimality of Y. Therefore, Z is a generalized covering set, and since any smaller generalized covering set would be an outer S-covering set, Z is indeed a minimal generalized covering set. ∎

The preceding result permits the definition of the *minimal generalized covering solution,* denoted MGC, as the union of minimal generalized covering sets. Recall that when X is compact, it is itself an outer S-covering set, and Proposition 36 can be applied. More generally, however, the proposition only requires an outer S-covering set, which exist much more generally. The next proposition exploits the Fishburn shading relation to obtain such a set. It uses the very weak assumption that X is Hausdorff, an assumption that can be replaced by upper semicontinuity if desired. This result is not only useful in establishing nonemptiness of the minimal generalized covering solution, but also, later, in connecting it to the Gillies uncovered set.

PROPOSITION 37: Assume that X is Hausdorff and that (P, R) satisfies compact weak images. Then $\mathrm{clos}\mathfrak{M}(G^F)$ is an outer S-covering set.

PROOF: To see that $\mathfrak{M}(G^F)$ is S-externally stable, consider any $y \notin \mathfrak{M}(G^F)$. Since X is Hausdorff, compact weak images implies that (P, R) is upper semicontinuous and that $R(y)$ is compact, so Proposition 23 yields $x \in \mathfrak{M}(G^F) \cap G^F(y)$. In particular, xG^Fy implies $x\ S\ y$, as required. The next step is to bound $\mathfrak{M}(G^F)$ within a compact set. Consider any $y \in X$, and note that by compact weak images, $R^2(y) = R(R(y))$ is compact. Take any $x \in \mathfrak{M}(G^F)$. If yG^Fx, then xG^Fy, which implies xRy, so $x \in R^2(y)$. Otherwise, if not yG^Fx, there exists $z \in X$ such that $xRzPy$, and again $x \in R^2(y)$. Then $\mathfrak{M}(G^F) \subseteq R^2(y)$, and it follows that $\mathrm{clos}\mathfrak{M}(G^F)$ is compact, and therefore, it is an outer S-covering set. ∎

A direct implication of the preceding proposition, with Proposition 36, is a general result on the existence of minimal generalized covering sets.

COROLLARY 38: Assume that X is Hausdorff and that (P, R) satisfies compact weak images. Then $MGC \neq \emptyset$.

Note that the framework allows for the possibility that a minimal generalized covering set contains a proper subset satisfying conditions (i) and (ii) in the definition of generalized covering set. If there is one, however, it cannot be compact. It is easy to see that there can be multiple minimal generalized covering sets, even in weak tournaments: letting $X = \{a, b, c, x, y, z\}$, and letting P be the union of a cycle through $\{a, b, c\}$ and a cycle through $\{x, y, z\}$, both sets are minimal generalized covering sets. In tournaments, of course, deep and shallow covering coincide with the usual notion, so there is a unique minimal generalized covering set, which coincides with the standard minimal covering set. Despite losing uniqueness, the minimal generalized covering solution is relatively small, being a refinement of the deep uncovered set and even the ultimate deep uncovered set.

PROPOSITION 39: Assume X is Hausdorff and (P, R) is uniformly continuous. Then $MGC \subseteq UD^{\infty}$.

PROOF: Let $Y^0 = X$, and let Y^k be the kth power of the deep uncovered set in the definition of the ultimate deep uncovered set. Consider any minimal generalized covering set Z. Clearly, $Z \subseteq Y^0$, so it suffices to assume $Z \subseteq Y^k$ and show $Z \subseteq Y^{k+1}$. To deduce a contradiction, suppose otherwise, so there exists $y \in Z$ and $x \in Y^k$ such that $xD_{Y^k}y$, i.e., $R(x) \cap Y^k \subseteq P(y) \cap Y^k$. Note that $x \notin Z$, for otherwise x would S-cover y in Z, contradicting (i) in the definition of generalized covering set. Therefore, there exists $z \in Z$ that S-covers x in $Z \cup \{x\}$, and in particular zRx. To ease notation, define $Z' = Z \cup \{x\}$, and note that since X is Hausdorff, Z' is compact. Moreover, restricted preferences $(P|_{Z'}, R|_{Z'})$ are uniformly continuous in the relative topology on Z', so Proposition 21 implies that $D_{Z'}$ is relatively open in $Z' \times Z'$, so there is an open set $V \subseteq X$ such that $y \in D_{Z'}^{-1}(x) = Z' \cap V$. Then $W = Z \setminus D_{Z'}^{-1}(x) = Z \setminus V$ is compact, and moreover $z \in W$, for otherwise we have $z \in V$, but then $xD_{Z'}z$ implies xPz, which contradicts zRx.

I claim W is an outer S-covering set. Consider any $v \in X \setminus W$. If $v \in Z$, then $v \in V$, so $xD_{Z'}v$. To see that z S-covers v in $W \cup \{v\}$, first note that zRx, with $xD_{Z'}v$, implies zRv. Now consider any $t \in W \cup \{v\}$ such that tPz, which implies $t \in W$. Then since z S-covers x in Z', we have $t \in P(z) \cap Z' \subseteq R(x) \cap Z'$, so tRx. With $xD_{Z'}v$, this implies tRv. Thus, z S-covers v in $W \cup \{v\}$. If $v \notin Z$, then there exists $w \in Z$ such that w S-covers v in $Z \cup \{v\}$ and therefore in $W \cup \{v\}$. In case $w \in W$, then we have an alternative $w \in W$ that S-covers v in $W \cup \{v\}$. In case $w \in Z \setminus W$, then $w \in V$, so $xD_{Z'}w$. To see that z S-covers v in $W \cup \{v\}$, first note that zRx, with $xD_{Z'}w$, implies zPw, and since w S-covers v in $W \cup \{v\}$, we have $z \in P(w) \cap (W \cup \{v\}) \subseteq R(v) \cap (W \cup \{v\})$, so zRv. Now consider any $t \in W \cup \{v\}$ such that tPz, which implies $t \in W$. Then since z S-covers x in Z', we have $t \in P(z) \cap Z' \subseteq R(x) \cap Z'$, so tRx. With $xD_{Z'}w$, this implies tPw, and since w S-covers v in $W \cup \{v\}$, this implies tRv. Thus, z indeed S-covers v in $W \cup \{v\}$. We conclude that W is an outer S-covering set, as claimed. But then Proposition 36 implies that there is a minimal generalized covering set that is contained in W and that is, therefore, a proper subset of Z, contradicting minimality of Z. Finally, we conclude that $Z \subseteq Y^{k+1}$, as required. ∎

The analysis of the external stability of the uncovered sets gives a partial result on their juxtaposition relative to the minimal generalized covering solution. In short, if either the Gillies or Bordes uncovered sets is externally stable and has compact closure, then it will contain a minimal generalized covering set in its closure. For the Bordes uncovered set, connections to the minimal generalized covering solution hold under conditions slightly stronger than Proposition 31, strengthening compact weak upper sections to the requirements that X is Hausdorff and preferences satisfy compact weak images.

PROPOSITION 40: Assume that X is Hausdorff, that (P, R) satisfies compact weak images, and that $R(\cdot)$ is lower hemicontinuous. Then there exists a minimal generalized covering set $Y \subseteq \text{clos}\mathfrak{U}(B)$.

PROOF: Consider any $z \in X$. Since X is Hausdorff, compact weak images implies $R^2(z) = R(R(z))$ is compact. Note that $\mathfrak{U}(B) \subseteq R^2(z)$, for otherwise there exists $y \in \mathfrak{U}(B) \setminus R^2(z)$, but then $z\ D\ y$, which implies $z\ B\ y$, a contradiction. Furthermore, Proposition 31 implies that $\mathfrak{U}(B)$ is S-externally stable. Therefore, $\text{clos}\mathfrak{U}(B)$ is an outer S-covering set, and Proposition 36 implies there is a minimal generalized covering set $Y \subseteq \text{clos}\mathfrak{U}(B)$. ∎

In fact, because it is enough that the closure of the Gillies uncovered set be an outer S-covering set, we obtain connections with the minimal generalized covering solution uncovered set under weaker conditions than in Corollary 32, dropping the condition of discriminating preferences. Compared to the Bordes uncovered set, we omit the lower hemicontinuity condition. Of course, these observations hold also for the larger McKelvey uncovered set.

PROPOSITION 41: Assume that X is Hausdorff and that (P, R) satisfies compact weak images. Then there exists a minimal generalized covering set $Y \subseteq \text{clos}\mathfrak{U}(G)$.

PROOF: Consider any $z \in X$. Since X is Hausdorff, compact weak images implies $R^2(z) = R(R(z))$ is compact. Note that $\mathfrak{U}(G) \subseteq R^2(z)$, for otherwise there exists $y \in \mathfrak{U}(G) \setminus R^2(z)$, but then $z\ D\ y$, which implies $z\ G\ y$, a contradiction. Now consider any $y \in X \setminus \mathfrak{U}(G)$. Proposition 23, with Proposition 4, yields $x \in \mathfrak{U}(G) \cap G^F(y)$. Since xG^Fy, it follows that xRy and $P(x) \subseteq P(y) \subseteq R(y)$, so $x\ S\ y$. Then $\text{clos}\mathfrak{U}(G)$ is an outer S-covering set, and Proposition 36 implies there is a minimal generalized covering set $Y \subseteq \text{clos}\mathfrak{U}(G)$. ∎

Similar observations hold for the shallow uncovered set, the proof following the same lines but relying on Proposition 33.

PROPOSITION 42: Assume that X is Hausdorff, that (P, R) satisfies compact weak images and is discriminating and rich. Then there exists a minimal generalized covering set $Y \subseteq \text{clos}\mathfrak{U}(S)$.

As a final remark on the location of the minimal generalized covering solution, the next result establishes that it intersects the closure of the strong top cycle under the general conditions of Proposition 41.

PROPOSITION 43: Assume that X is Hausdorff and that (P, R) satisfies compact weak images. Then $MGC \cap \text{clos}STC \neq \emptyset$.

PROOF: Consider any $y \in X$. Since X is Hausdorff, compact weak images implies that (P, R) is upper semicontinuous and $Y = R(R(y))$ is compact. By Proposition 16, $STC \neq \emptyset$, and note that STC is S-externally stable. If $STC \subseteq Y$, then closSTC is an outer S-covering set, and Proposition 36 implies there is a minimal covering set $Z \subseteq \text{clos}STC$, so $MGC \cap STC \neq \emptyset$. Assume, then, that $STC \setminus Y \neq \emptyset$. Note that the restricted preferences $(P|_Y, R|_Y)$ are upper semicontinuous, and Proposition 16 implies that the restricted strong top cycle, $W = \mathfrak{M}(P|_Y^\infty, Y)$, is nonempty. I claim that clos$W$ is an outer S-covering set. Indeed, take any $z \in X \setminus W$. If $z \in Y$, then xRz for all $x \in W$, and therefore x S-covers z in $W \cup \{z\}$ for all $x \in W$. If $z \in X \setminus Y$, then $y\ D\ z$. In case $y \in W$, this implies y S-covers z in $W \cup \{z\}$. Otherwise, there exists $w \in W$ such that w S-covers y in $W \cup \{y\}$. Then wRy, and with $y\ D\ z$, this implies wRz. Now consider any $v \in W \cup \{z\}$ such that vPw. This implies $v \neq z$, so $v \in W$, and $v \in P(w) \cap (W \cup \{y\}) \subseteq R(y) \cap (W \cup \{y\})$. Then vRy, which with $y\ D\ z$ implies vRz. Therefore, w S-covers z in $W \cup \{z\}$, and W is S-externally stable. Since Y is compact, it follows that clos$W$ is an outer S-covering set. By Proposition 36, there exists a minimal generalized covering set $Z \subseteq \text{clos}W$. To see that $Z \cap STC \neq \emptyset$, take any $x \in STC \setminus Y$, assumed nonempty, so that $y\ D\ x$. If $y \in Z$, then yPx and $x \in STC$ imply $y \in STC$. Otherwise, there is some $z \in Z$ that S-covers y in Z. In particular, zRy, which with $y\ D\ x$ implies zPx, and $z \in STC$, as required. ∎

An unresolved question is whether condition (ii) in the definition of generalized covering set can be modified to require external stability with respect to D-covering. The corresponding solution would be a "minimal deep covering" set. It seems that the technical issue confronting such a solution is the possibility that external stability of the deep uncovered set is lost when applied to a subset of alternatives. That is, although Proposition 31 establishes external stability of $\mathfrak{U}(D)$ when $R(\cdot)$ is lower hemicontinuous, this property is not inherited by the weak upper sections of the restriction of R to an arbitrary compact Y, so $R|_Y(\cdot)$ need not be lower hemicontinuous, and $\mathfrak{U}(D_Y, Y)$ may violate external stability. These technical issues are moot in the setting of weak tournaments, where minimal deep covering sets obviously exist and can be shown to be unique by standard arguments as in, e.g., Dutta and Laslier (1999).

8. Stability Structure

In this section, I provide exact characterizations of the five main versions of the uncovered set in terms of choices from externally stable sets, each being obtained by an appropriate specification of a choice criterion and form of external stability. Beginning with the deep uncovered set, the next result establishes that it is the union of undominated sets of P in R-externally stable sets.

Proposition 44:

$$\mathfrak{U}(D) \quad = \bigcup_{Y \in \mathcal{E}(R)} \mathfrak{U}(P, Y).$$

Proof: Take any $x \in \mathfrak{U}(D)$, and note that $x \in \mathfrak{U}(P, R^{-1}(x))$. To see that $R^{-1}(x)$ is R-externally stable, take any $y \in X \setminus R^{-1}(x)$. By the two-step principle, there exists $z \in X$ such that $xRzRy$. Thus, $z \in R^{-1}(x)$ and zRy, as required. Therefore, $x \in \bigcup_{Y \in \mathcal{E}(R)} \mathfrak{U}(P, Y)$. Now take any $Y \in \mathcal{E}(R)$ and any $x \in \mathfrak{U}(P, Y)$. Suppose there exists $y \in X \setminus \{x\}$ such that $y \; D \; x$, i.e., $R(y) \subseteq P(x)$. Since yPx, it follows that $y \in X \setminus Y$, and R-external stability yields $z \in Y$ such that zRy. But then $z \in R(y) \subseteq P(x)$, implying zPx, which contradicts $x \in \mathfrak{U}(P, Y)$. Therefore, $x \in \mathfrak{U}(D)$, as required. ∎

The shallow uncovered set is the union of dominant sets of P in P-externally stable sets. With Corollary 30, an immediate implication is that, under standard compactness and continuity conditions and the restriction that preferences are rich and discriminating, there exists at least one P-externally stable set that admits dominant alternatives.

Proposition 45:

$$\mathfrak{U}(S) \quad = \bigcup_{Y \in \mathcal{E}(P)} \mathfrak{D}(P, Y).$$

Proof: Take any $x \in \mathfrak{U}(S)$, define $Y = \{x\} \cup P^{-1}(x)$, and note that $x \in \mathfrak{D}(P, Z)$. To see that Z is P-externally stable, take any $y \in X \setminus Z$, so yRx. By the two-step principle, either xPy or there exists $z \in X$ such that $xPzPy$. The former is precluded by yRx, so we have $z \in P^{-1}(x) \subseteq Y$ with zPy, as required. Therefore, $x \in \bigcup_{Y \in \mathcal{E}(P)} \mathfrak{D}(P, Y)$. Now take any $Y \in \mathcal{E}(P)$ and any $x \in \mathfrak{D}(P, Y)$. Suppose there exists $y \in X \setminus \{x\}$ such that $y \; S \; x$, i.e., yRx and $P(y) \subseteq R(x)$. Since $y \neq x$ and yRx, it follows that $y \in X \setminus Y$, and P-external stability yields $z \in Y$ such that zPy, which further implies $z \neq x$. But then $z \in P(y) \subseteq R(x)$, implying zRx, which contradicts $x \in \mathfrak{D}(P, Y)$. Therefore, $x \in \mathfrak{U}(S)$, as required. ∎

The Gillies uncovered set is the union of undominated sets of P in P-externally stable sets.[15] Compared to Proposition 44, here the union is over the sets satisfying the stronger requirement of P-external stability rather than R-external stability, producing a potentially smaller choice set. Compared to Proposition 45, we take the union of undominated sets of P, rather than dominant sets of P, producing a potentially larger choice set. With Corollary 24, an immediate implication is that, under standard compactness and continuity conditions, there exists at least one P-externally stable set that admits undominated alternatives.

PROPOSITION 46:

$$\mathfrak{U}(G) \quad = \bigcup_{Y \in \mathcal{E}(P)} \mathfrak{U}(P, Y).$$

PROOF: Take any $x \in \mathfrak{U}(G)$, and note that $x \in \mathfrak{U}(P, R^{-1}(x))$. To see that $R^{-1}(x)$ is P-externally stable, take any $y \in X \setminus R^{-1}(x)$, so yPx. By the two-step principle, either xRy or there exists $z \in X$ such that $xRzPy$. The former is precluded by yPx, so we have $z \in R^{-1}(x)$ with zPy, as required. Therefore, $x \in \bigcup_{Y \in \mathcal{E}(P)} \mathfrak{U}(P, Y)$. Now take any $Y \in \mathcal{E}(P)$ and any $x \in \mathfrak{U}(P, Y)$. Suppose there exists $y \in X \setminus \{x\}$ such that $y \; G \; x$, i.e., yPx and $P(y) \subseteq P(x)$. Since yPx, it follows that $y \in X \setminus Y$, and P-external stability yields $z \in Y$ such that zPy. But then $z \in P(y) \subseteq P(x)$, implying zPx, which contradicts $x \in \mathfrak{U}(P, Y)$. Therefore, $x \in \mathfrak{U}(G)$, as required. ∎

A set Y is a *von Neumann-Morgenstern stable set* if it is P-externally stable and $\mathfrak{U}(P, Y) = Y$. An easy corollary is that every stable set is contained in the Gillies uncovered set, generalizing McKelvey's (1986) result that every stable set is contained in the McKelvey uncovered set. In the framework of Brandt and Fischer (2008), a von Neumann-Morgenstern stable set is actually a minimal Gillies covering set, so the result extends their Theorem 2 to the general setting.

COROLLARY 47: If Y is a von Neumann-Morgenstern stable set, then $Y \subseteq \mathfrak{U}(G)$.

The Bordes uncovered set is the union of dominant sets of P in R-externally stable sets. Compared to Proposition 44, we take the union of dominant sets of P rather than undominated sets, producing a potentially smaller choice set. Compared to Proposition 45, the union is over sets satisfying R-external stability rather than P-external stability, producing a potentially larger choice set. With Corollary 27, an immediate implication is that, under general conditions involving lower hemicontinuity of $R(\cdot)$, there exists at least one R-externally stable set that admits dominant alternatives.

[15]This result is cited in Lemma 1 of Penn (2006b) from an earlier version of this paper.

PROPOSITION 48:

$$\mathfrak{U}(B) \quad = \bigcup_{Y \in \mathcal{E}(R)} \mathfrak{D}(P, Y).$$

PROOF: Take any $x \in \mathfrak{U}(B)$, define $Y = \{x\} \cup P^{-1}(x)$, and note that $x \in \mathfrak{D}(P,Y)$. To see that Y is R-externally stable, take any $y \in X \setminus Y$, so yRx. By the two-step principle, either xRy or there exists $z \in X$ such that $xPzRy$. In the former case, we are done, and otherwise we have $z \in P^{-1}(x) \subseteq Y$ with zRy, as required. Therefore, $x \in \bigcup_{Y \in \mathcal{E}(R)} \mathfrak{D}(P,Y)$. Now take any $Y \in \mathcal{E}(R)$ and any $x \in \mathfrak{D}(P,Y)$. Suppose there exists $y \in X \setminus \{x\}$ such that $y\ B\ x$, i.e., yPx and $R(y) \subseteq R(x)$. Since yPx, it follows that $y \in X \setminus Y$, and R-external stability yields $z \in Y$ such that zRy, which further implies $z \neq x$. But then $z \in R(y) \subseteq R(x)$, implying zRx, which contracts $x \in \mathfrak{D}(P,Y)$. Therefore, $x \in \mathfrak{U}(B)$, as required. ∎

A form of stability result for the McKelvey uncovered set is possible, but it takes a less elegant form, as we have to consider stability with respect to pairs of alternatives. Define the relation PR on the product set $X \times X$ so that $(x,y)\ PR\ (w,z)$ if and only if either $x \in P(w) \cup P(z)$ or $y \in R(w) \cup R(z)$. That is, $(x,y)\ PR\ (w,z)$ either if x is strictly preferred to w or z, or if y is weakly preferred to at least one of those two alternatives.

PROPOSITION 49:

$$\mathfrak{U}(M) \quad = \bigcup_{(A \times B) \in \mathcal{E}(PR)} (\mathfrak{U}(P,A) \cap \mathfrak{D}(P,B)).$$

PROOF: Take any $x \in \mathfrak{U}(M)$, define $A = R^{-1}(x)$ and $B = \{x\} \cup P^{-1}(x)$. Clearly, we have $x \in \mathfrak{U}(P,A) \cap \mathfrak{D}(P,B)$. To see that $A \times B$ is PR-externally stable, take any $(y,z) \in (X \times X) \setminus (A \times B)$, so either $y \notin A$ or $z \notin B$. In the first case, we have yPx. By the two-step principle, either (i) xRy or (ii) there exists $w \in X$ such that $xRwPy$ or (iii) there exists $v \in X$ such that $xPvRy$. Of course, (i) is precluded by yPx. If (ii) holds, then $(w,x) \in A \times B$, and we have $(w,x)\ PR\ (y,z)$. And if (iii) holds, then $(x,v) \in A \times B$, and we have $(x,v)\ PR\ (y,z)$. In the second case, we have zRx, and the two-step principle yields (i) xRz or (ii) there exists $w \in X$ such that $xRwPz$, or (iii) there exists $v \in X$ such that $xPvRz$. If (i) holds, then we have $(x,x)\ PR\ (y,z)$. If (ii) holds, then $(w,x) \in A \times B$, and we have $(w,x)\ PR\ (y,z)$. And if (iii) holds, then $(x,v) \in A \times B$, and we have $(x,v)\ PR\ (y,z)$. Now take any $A \times B \in \mathcal{E}(PR)$ and any $x \in \mathfrak{U}(P,A) \cap \mathfrak{D}(P,B)$. Suppose there exists $y \in X \setminus \{x\}$ such that $y\ M\ x$, i.e., yPx and $P(y) \subseteq P(x)$ and $R(y) \subseteq R(x)$. Note that yPx implies $y \notin A$, so PR-external stability yields $(w,z) \in A \times B$ such that $(w,z)\ PR\ (y,y)$, so either wPy or zRy. In the latter case, $z \in R(y) \subseteq R(x)$ implies zRx, and with yPx this implies $z \neq x$, but then we contradict $x \in \mathfrak{D}(B,P)$; and in the former case, $w \in P(y) \subseteq P(x)$ implies wPx, which contradicts $x \in \mathfrak{U}(P, A)$. Therefore, $x \in \mathfrak{U}(M)$, as required. ∎

9. Implications for Banks Sets

The stability structure of the Gillies and Bordes uncovered sets has interesting implications for the Banks set, which was first presented in Banks (1985) by way of characterizing the sophisticated voting outcomes of amendment agendas. Defined in the context of a tournament, the Banks set consists of the maximal elements of maximal chains of P. Recall that a *chain* of P is a subset Y such that the restriction $P|_Y$ is transitive and total, i.e., a linear order. When preferences form a tournament, $P|_Y$ is a linear order if and only if it is acyclic, so there is little latitude in defining the Banks set. In general, however, there are at least four possible approaches, depending on the transitivity properties imposed on subsets of alternatives.

Define four collections of subsets, which differ with respect to the transitivity properties of the restriction of the strict preference relation P:

$$\begin{aligned} \mathcal{Y}_1 &= \{Y \subseteq X \mid P|_Y \text{ is total and transitive }\} \\ \mathcal{Y}_2 &= \{Y \subseteq X \mid P|_Y \text{ is negatively transitive }\} \\ \mathcal{Y}_3 &= \{Y \subseteq X \mid P|_Y \text{ is transitive }\} \\ \mathcal{Y}_4 &= \{Y \subseteq X \mid P|_Y \text{ is acyclic }\}. \end{aligned}$$

Let $\mathcal{Y}_i^*$ consist of the sets in $\mathcal{Y}_i$ that are maximal with respect to set-inclusion, i.e., $Y \in \mathcal{Y}_i^*$ if and only if there is no $Z \in \mathcal{Y}_i$ such that $Y \subsetneq Z$. To each transitivity condition corresponds a Banks set defined as

$$BS_i = \bigcup_{Y \in \mathcal{Y}_i^*} \mathfrak{U}(P, Y),$$

$i = 1, 2, 3, 4$, or in words defined to consist of the undominated alternatives from all maximal subsets in $\mathcal{Y}_i$.[16] In Figure 1, for example, alternative a is undominated in the set $\{a, y, b\}$, P restricted to this set is total and transitive, and the set is maximal with respect to this property. Similarly, b is maximal in $\{b, z, c\}$ and c is maximal in $\{c, x, a\}$, both sets being maximal in $\mathcal{Y}_1$, and in fact, $BS_1 = \{a, b, c\}$. Indeed, x does not belong to this set, because it is strictly preferred only to a, but c is preferred to both x and a. On the other hand, x is maximal in $\{x, a, y, b, z\}$, which is maximal in $\mathcal{Y}_4$, and in fact, $BS_4 = \{x, y, z\}$. Indeed, a does not belong to this set, because it is maximal in $\{a, y, b, z\}$, but $\{x, a, y, b, z\}$ is a strictly larger set on which P is acyclic and in which a is not maximal.

The different versions of the Banks set can be connected to the uncovered sets via their stability structure. It cannot always be the case that BS_1 coincides with the Bordes uncovered set or that BS_4 coincides with the

[16]Penn (2006b) uses BS_1, which is strictly closest to the definition given by Banks.

Gillies uncovered set: in tournaments, BS_1 and BS_4 coincide with the Banks set, which can be a proper subset of the uncovered set. But one inclusion follows easily from Propositions 46 and 48. Evidently, every element $Y \in \mathcal{Y}_1^*$ is R-externally stable: if there exists $x \in X \setminus Y$ such that xPy for all $y \in Y$, then $Y \cup \{x\}$ is a strictly larger set on which P is total and transitive. And since $P|_Y$ is total, we have $\mathfrak{U}(P,Y) = \mathfrak{D}(P,Y)$, establishing the inclusion of BS_1 within the Bordes uncovered set. The following result, which states this inclusion, is related to Theorem 3 of Brandt and Fischer (2008), which establishes in the setting of weak tournaments that every Bordes covering set has nonempty intersection with BS_1.

Proposition 50: $BS_1 \subseteq \mathfrak{U}(B)$.

As well, every element $Y \in \mathcal{Y}_4^*$ is R-externally stable: for each $x \in X \setminus Y$, P is acyclic on Y but not on $Y \cup \{x\}$, so there is a cycle including x. Thus, BS_4 is a subset of the Gillies uncovered set.

Proposition 51: $BS_4 \subseteq \mathfrak{U}(G)$.

Finally, every element $Y \in \mathcal{Y}_2^* \cup \mathcal{Y}_3^*$ is R-externally stable, so BS_2 and BS_3 are contained in the deep uncovered set.

Proposition 52: $BS_2 \cup BS_3 \subseteq \mathfrak{U}(D)$.

The set BS_4, the *acyclic Banks set,* has the advantage that nonemptiness is relatively straightforward to establish under general topological conditions. The next proposition establishes, specifically, that under standard continuity conditions, there is a maximal set on which the restriction of P is acyclic, and that every such set is closed. When X is compact, it then follows from standard results that P has maximal elements in every maximal acyclic set, and therefore the acyclic Banks set is nonempty. This implication is left unstated here, as a more powerful result is given in the subsequent proposition. The proof approach for the result at hand does not appear to apply to the other versions of the Banks set, as they all exclude weak preferences in some way, creating difficulties for establishing compactness.

Proposition 53: Assume (P,R) is uniformly continuous. Then $\mathcal{Y}_4^* \neq \emptyset$, and for all $Y \in \mathcal{Y}_4^*$, Y is closed.

Proof: Of course, $\{x\}$ acyclic for all x, so the collection $\mathcal{Y}_4$ is nonempty. Consider any chain $\mathcal{C}$ in $\mathcal{Y}_4$ such that all pairs of sets in $\mathcal{C}$ are related by set inclusion, and let $Y = \bigcup \mathcal{C}$. Since the condition of acyclicity is closed upward (see Duggan (1999)), it follows that $P|_Y = \bigcup_{Z \in \mathcal{C}} P|_Z$. Then by Zorn's lemma, there exists $Y \in \mathcal{Y}_4$ that is maximal with respect to set inclusion. I claim that Y is closed, and for this it suffices to show that $\text{clos}Y \in \mathcal{Y}_4$. Suppose otherwise, in order to deduce a contradiction, so that there exist $y_1, \ldots, y_k \in \mathrm{clos}Y$ such that $y_1 P y_2 P \cdots y_k P y_1$, i.e., $(y_k, y_1) \in P$ and $(y_{i-1}, y_i) \in P$ for all $i = 2, \ldots, k$. Since P is open, by uniform continuity, there exist $x_1, \ldots, x_k \in Y$ such that $(x_k, x_1) \in P$ and $(x_{i-1}, x_i) \in P$ for all $i = 2, \ldots, k$, i.e., $x_1 P x_2 P \cdots x_k P x_1$, contradicting $Y \in \mathcal{Y}_4$. Therefore, Y is closed, as required. ∎

A last advantage of the acyclic Banks set is that it intersects the closure of the strong top cycle under the continuity condition of the previous proposition and the additional assumption of compact weak upper sections. That it may not be a subset of the strong top cycle is easily demonstrated by a finite example, where STC is trivially closed. Indeed, letting $X = \{x, y, z, w\}$ and $P = \{(y, z), (z, w), (w, y), (w, x)\}$, the set $\{x, y, z\}$ is a maximal acyclic set, and $x \in \mathfrak{U}(P, \{x, y, z\})$, so $x \in B_4$; nevertheless, $STC = \{w, y, z\}$. None of the other Banks sets are generally subsets of the strong top cycle either: consider $X = \{x, y, z\}$ and $P = \{(z, y), (y, x)\}$, and note that $\{x, y\}$ is maximal among the sets on which P is transitive or negatively transitive or both total and transitive, so $y \in BS_1 \cap BS_2 \cap BS_3$; nevertheless, the strong top cycle consists of z alone. Clearly, the next result implies nonemptiness of the acyclic Banks set, and when X is finite, it implies that $BS_4 \cap STC \neq \emptyset$.

Proposition 54: Assume that $R(x)$ is compact for all $x \in X$ and that (P, R) is uniformly continuous. Then $BS_4 \cap \mathrm{clos}STC \neq \emptyset$.

Proof: By Proposition 16, there exists $x \in STC$. The result clearly holds if $P(x) = \emptyset$, so assume $P(x) \neq \emptyset$. Note that $\hat{X} = R(x)$ is compact, and preferences restricted to $\hat{X}$, denoted $(\hat{P}, \hat{R}) = (P|_{\hat{X}}, R|_{\hat{X}})$, are uniformly continuous in the relative topology on $\hat{X}$. Let $\hat{\mathcal{Y}}_4$ consist of subsets $Y \subseteq \hat{X}$ such that $\hat{P}|_Y$ is acyclic, and let $\hat{\mathcal{Y}}_4^*$ be the sets in $\hat{\mathcal{Y}}_4$ that are maximal with respect to set inclusion. By Proposition 53, there exists $Y \in \hat{\mathcal{Y}}_4^*$, and Y is relatively closed in $\hat{X}$ and, therefore, compact. Since $\hat{X} = R(x)$, it must be that $x \in Y$, for otherwise we have $Y \cup \{x\} \in \hat{\mathcal{Y}}_4$ and $Y \subsetneq Y \cup \{x\}$, contradicting maximality of Y.

Next, I identify a particular element, denoted z^*, of $\mathfrak{U}(\hat{P}, Y)$. If $\hat{P}(x) \cap Y = \emptyset$, so that $x \in \mathfrak{U}(\hat{P}, Y)$, then specify $z^* = x$. Otherwise, define the collection $\mathcal{Z}$ to consist of every set $Z \subseteq Y$ such that (i) $Z \subseteq \hat{P}|_Y^\infty(x)$, (ii) $\hat{P}|_Y^\infty \cap (Z \times Z)$ is total, and (iii) $\mathfrak{D}(\hat{P}|_Y^\infty, Z) \neq \emptyset$. Note that the transitive closure of $\hat{P}|_Y^\infty$ restricted to Z is a necessarily asymmetric, by acyclicity of $\hat{P}|_Y$, and is of course transitive, so the transitive closure of $\hat{P}|_Y^\infty$ restricted to Z is a linear order. Furthermore, $\mathcal{Z} \neq \emptyset$ since $z\hat{P}x$ for some $z \in Y$, and then $\{z\} \in \mathcal{Z}$. Giving $\mathcal{Z}$ the partial order of set inclusion, the Hausdorff maximality principle implies that the collection $\mathcal{Z}$ possesses a maximal chain; I select one such chain, denoted $\hat{\mathcal{C}}$, and I index it as $\hat{\mathcal{C}} = \{Z_\alpha\}$. For each α, let $\{z_\alpha\} = \mathfrak{D}(\hat{P}|_Y^\infty, Z_\alpha)$, and view $\{z_\alpha\}$ as a net in Y with direction $\geq$ defined so that $\alpha \geq \beta$ if and only if $Z_\alpha \supseteq Z_\beta$. Note that for each α, we have $z_\alpha \hat{P}|_Y^\infty x$, which implies $z_\alpha \in STC$. Since Y is compact, the net possesses a convergent subnet; for later use, I select one such subnet and continue to index it by α, for simplicity. Let $z_\alpha \to z \in Y$. Since $z_\alpha \in STC$ for all α, it follows that $z \in \mathrm{clos}STC$. I claim that $z \in \mathfrak{U}(\hat{P}, Y)$. If not, then there exists $w \in Y$ such that $w\hat{P}z$, i.e., $z \in \hat{P}^{-1}(w)$. Since $\hat{P}^{-1}(w)$ is relatively open in $\hat{X}$, it follows that $z_\alpha \in \hat{P}^{-1}(w)$ for sufficiently high α. I consider two cases. First, $w \in \bigcup \hat{\mathcal{C}}$, which implies $w \in Z_\beta$ for some β. Then for high enough $\alpha > \beta$, transitivity of $\hat{P}|_Y^\infty$ implies $z_\alpha \hat{P}|_Y^\infty z_\beta = w\hat{P}z_\alpha$, contradicting acyclicity of $\hat{P}|_Y$. Second, $w \notin \bigcup \hat{\mathcal{C}}$. Define $W = \{w\} \cup \bigcup \hat{\mathcal{C}}$. Then transitivity of $\hat{P}|_Y^\infty$ implies that $\mathfrak{D}(\hat{P}, W) = \{w\}$, which in turn implies $W \in \mathcal{Z}$. But then $\mathcal{C}' = \hat{\mathcal{C}} \cup \{W\}$ is a chain in $\mathcal{Z}$ such that $\hat{\mathcal{C}} \subsetneq \mathcal{C}'$, contradicting maximality of $\hat{\mathcal{C}}$. Therefore, $z \in \mathfrak{U}(\hat{P}, Y)$, as claimed, and I specify $z^* = z$. Note that whichever case in the definition of z^* holds, we have $z^* \in \mathrm{clos}STC$.

Finally, I claim that $z^* \in BS_4$. Let $\mathcal{W} \subseteq \mathcal{Y}_4$ denote the collection consisting of every set W such that $Y \subseteq W$ and $P|_W$ is acyclic, and note that $\{Y\} \in \mathcal{W}$, so the collection is nonempty. Let $\mathcal{C}$ be any chain in $\mathcal{W}$, and note that $\bigcup \mathcal{C} \in \mathcal{W}$, and therefore Zorn's lemma yields a set $W \in \mathcal{W}$ that is maximal with respect to set inclusion. Accordingly, $W \in \mathcal{Y}_4^*$, and it then suffices to show that $z^* \in \mathfrak{U}(P, W)$. Suppose not, in order to deduce a contradiction. Then there exists $w \in W$ such that wPz^*. It follows that $w \notin Y$, since z^* is undominated in Y, and that $w \notin \hat{X} \setminus Y$, by maximality of Y in $\hat{\mathcal{Y}}_4$. Thus, $w \in X \setminus \hat{X} = P^{-1}(x)$, so xPw, so $z^* = z$, where z is defined above. Since $z \in P^{-1}(w)$, an open set, it follows that for the net $\{z_\alpha\}$ converging to z, above, we have wPz_α for high enough α. Since $Y \subseteq W$, we have $\{w, x, z_\alpha\} \subseteq W$, but then $wPz_\alpha P^\infty xPw$ for high enough α, contradicting $W \in \mathcal{Y}_4$. Thus, $z^* \in \mathfrak{U}(P, W)$, and we conclude that $z^* \in BS_4 \cap \mathrm{clos}STC$. ∎

Of course, Propositions 50–52, with Proposition 15, immediately imply that all of the Banks sets are contained in the weak top cycle.

Corollary 55: $BS_1 \cup BS_2 \cup BS_3 \cup BS_4 \subseteq WTC$.

10. Continuity Properties

In this section, I analyze the continuity properties of the two benchmark sets, the deep and shallow uncovered sets. In contrast to the Gillies and Bordes uncovered sets, which suffer from discontinuity problems discussed by Bordes, Le Breton, and Salles (1992), these two correspondences possess nice continuity properties: they are upper and lower hemicontinuous, respectively, as a function of preferences and feasible sets.[17] Thus, they provide

[17]Although Bordes, Le Breton, and Salles (1992) do not give examples for the McKelvey uncovered set, the same difficulties are expected. In fact, Banks, Duggan, and Le

robust upper and lower bounds on the other uncovered sets. Because I allow for the approximation of the set of alternatives by a sequence of finite grids, the result has particular bearing on efforts to compute the uncovered set in spatial voting models using numerical methods, such as Bianco, Jeliazkov, and Sened (2004). A straightforward further implication of upper hemicontinuity is that when preferences are discriminating and the core is nonempty, the deep uncovered set, along with the smaller sets, is continuous.

To conduct the analysis, let $\mathbf{X}$ denote the collection of all closed subsets of X, and let $\mathbf{R}$ denote the space of all complete, closed relations on X. Given such a relation R, we may define the dual R^*, which is asymmetric, and we may then consider the various notions of covering generated by these relations. Define the corresponding deep covering relation on Y, denoted $\mathcal{D}(R,Y)$, as follows: for all $x, y \in Y$,

$$x \; \mathcal{D}(R,Y) \; y \quad \Leftrightarrow \quad R(x) \cap Y \subseteq R^*(y) \cap Y.$$

This is just the standard definition of deep covering, applied to the restriction $R|_Y$. Denote the corresponding deep uncovered set by $\mathfrak{UD}(R,Y) = \mathfrak{U}(\mathcal{D}(R,Y),Y)$. The shallow covering relation, denoted $\mathcal{S}(R,Y)$, is defined as follows: for all $x, y \in Y$,

$$x \; \mathcal{S}(R,Y) \; y \quad \Leftrightarrow \quad xRy \text{ and } R^*(x) \cap Y \subseteq R(y) \cap Y,$$

and the corresponding shallow uncovered set is $\mathfrak{US}(R,Y) = \mathfrak{U}(\mathcal{S}(R,Y),Y)$.

Assume X is a compact metric space, with metric d, and endow the space $\mathbf{X}$ with the Hausdorff metric, ξ, which is defined by

$$\xi(Y,Z) \quad = \quad \max\left\{\max_{y\in Y}\min_{z\in Z} d(y,z), \max_{z\in Z}\min_{y\in Y} d(y,z)\right\}.$$

It is well-known that $Y_m \to Y$ in the Hausdorff metric if and only if both of the following conditions hold:

(i) for every sequence $\{y_m\}$ converging to limit y with $y_m \in Y_m$ for all m, we have $y \in Y$,
(ii) for every $y \in Y$, there is a subsequence $\{Y_{m_k}\}$ and a sequence $\{y_k\}$ such that $y_k \in Y_{m_k}$ for all k and $y_k \to y$.

Give the space $X \times X$ the product metric, still denoted d and defined by $d((x,y),(w,z)) = d(x,w) + d(y,z)$, and endow the space $\mathbf{R}$ with the Hausdorff metric, ρ, which is defined by

$$\begin{aligned}\rho(R,R') \quad = \quad \max\Bigg\{ & \max_{(x,y)\in R}\min_{(w,z)\in R'} d((x,y),(w,z)), \\ & \max_{(w,z)\in R'}\min_{(x,y)\in R} d((x,y),(w,z))\Bigg\}.\end{aligned}$$

Breton (2002) prove analyticity of the McKelvey uncovered set, but even measurability is unknown.

If $\{R_m\}$ is a sequence in $\mathbf{R}$ and $R \in \mathbf{R}$, then the following conditions in conjunction are necessary and sufficient for $R_m \to R$ in the Hausdorff metric:

(iii) for every sequence $\{(x_m, y_m)\}$ converging to some (x, y) with $x_m R_m y_m$ for all m, we have xRy,
(iv) for all $x, y \in X$ with xRy, there is a subsequence $\{R_{m_k}\}$ and a sequence $\{(x_k, y_k)\}$ such that $x_k R_{m_k} y_k$ for all k and $(x_k, y_k) \to (x, y)$.

With these metrics, $\mathbf{X}$ and $\mathbf{R}$ are compact metric spaces.

It is now a simple matter to establish upper hemicontinuity of the deep uncovered set. Note that the proof of the next proposition only uses properties (i) and (iii) of Hausdorff convergence, so that an even stronger continuity property could be stated.

Proposition 56: Assume X is a compact metric space. Then $\mathfrak{UD}\colon \mathbf{R}\times\mathbf{X} \rightrightarrows X$ is upper hemicontinuous with closed values.

Proof: It suffices to show that the correspondence has closed graph. Take any sequence $\{(R_m, Y_m, x_m)\}$ in $\mathbf{R}\times\mathbf{X}\times X$ and any triple (R, Y, x) such that $x_m \in \mathfrak{UD}(R_m, Y_m)$ for all m and $(R_m, Y_m, x_m) \to (R, Y, x)$. Given arbitrary $y \in Y$, Hausdorff convergence of $\{Y_m\}$ yields a sequence $\{y_m\}$ such that $y_m \in Y_m$ for all m and $y_m \to y$. The two-step principle yields for each m an alternative $z_m \in Y_m$ such that $x_m R_m z_m R_m y_m$. Since X is compact, there is a subsequence $\{z_m\}$, still indexed by m, that converges to some alternative $z \in X$. By Hausdorff convergence of $\{Y_m\}$, we have $z \in Y$. And by property (i) of Hausdorff convergence of $\{R_m\}$, we have $xRzRy$. Since y was an arbitrary alternative belonging to Y, the two-step principle implies $x \in \mathfrak{UD}(R, Y)$, as required. ∎

To draw implications for computation, fix the weak preference R. A numerical approach to calculating the uncovered set will take a sequence $\{Y_m\}$ of finite approximations of X and for each m, compute a version of the uncovered set, say Z_m, for the preferences restricted to Y_m. As the grid is refined, we take the limit $Z_m \to Z$ (assuming one exists) to obtain an approximation of the uncovered set in X. For each m, the deep uncovered set encompasses the computed set, i.e., $Z_m \subseteq \mathfrak{UD}(R, Y_m)$, regardless of the definition of covering used. By upper hemicontinuity, from Proposition 56, we then have $Z \subseteq \mathfrak{UD}(R, X)$. It is theoretically possible that the approximation of the uncovered set obtained by numerical methods may omit some elements of the actual uncovered set, but it is bounded above by the deep uncovered set: the approximation will not contain any deeply covered alternatives.

A further consequence of Proposition 56 is that when preferences are discriminating and the core is nonempty, the deep uncovered set (and all of the other uncovered sets) is fully continuous. This observation is a simple implication of Proposition 14, which states that the core is a singleton and coincides with the deep uncovered set, and the fact that an upper hemicontinuous correspondence is continuous whenever it is single-valued. Given weak preference R and feasible set Y, we say (R, R^*) is *discriminating in* Y if the restricted preference relations are discriminating in the relative topology on Y: for all $x \in Y$, $R(x) \cap Y \subseteq \{x\} \cup \mathrm{clos}(R^*(x) \cap Y)$.

Corollary 57: Assume X is a compact metric space, and let $(R, Y) \in \mathbf{R} \times \mathbf{X}$ be such that (R, R^*) is discriminating in Y. Then $\mathfrak{UD}$ is continuous at (R, Y).

In contrast, I prove the lower hemicontinuity of the shallow uncovered set, when restricted to a subdomain of weak preference relations and feasible sets: let

$$\mathbf{\Theta} \;=\; \left\{ (R,Y) \in \mathbf{R} \times \mathbf{X} \mid \begin{array}{c} \text{either } Y = X \text{ and } (R^*, R) \text{ is rich} \\ \text{and discriminating or} \\ Y \text{ is finite and } R|_Y \text{ is antisymmetric} \end{array} \right\}$$

denote the domain of pairs (R, Y) such that either the feasible set is the entire set of alternatives and preferences are discriminating and rich, or the feasible set is finite and there are no ties between feasible alternatives. We give Θ the relative topology in $\mathbf{R} \times \mathbf{X}$, essentially allowing us to approximate the original set of alternatives with a sequence of tournaments.

Proposition 58: Assume X is a compact metric space. Then $\mathfrak{US} \colon \mathbf{\Theta} \rightrightarrows X$ is lower hemicontinuous.

Proof: Take any $(R, Y) \in \mathbf{\Theta}$, and let $Z \subseteq X$ be an open set such that $Z \cap \mathfrak{US}(R) \neq \emptyset$. I must establish an open set around (R, Y) such that $\mathfrak{US}(R', Y') \cap Z \neq \emptyset$ for every element (R', Y') of that open set. If there is no such open set, then there is a sequence $\{(R_m, Y_m)\}$ converging to (R, Y) such that for all m, $\mathfrak{US}(R_m, Y_m) \cap Z = \emptyset$. Choose any $x \in Z \cap \mathfrak{US}(R, Y)$. Since $Y_m \to Y$, there is a sequence $\{x_m\}$ such that $x_m \in Y_m$ for all m and $x_m \to x$. For sufficiently high m, we have $x_m \in Z$, and therefore $x_m \notin \mathfrak{US}(R_m, Y_m)$. Then there exists $y_m \in Y_m$ such that $y_m \,\mathcal{S}(R_m, Y_m)\, x_m$ for all m, i.e., $y_m R_m x_m$ and $R_m^*(y_m) \cap Y_m \subseteq R_m(x_m) \cap Y_m$. Note that discriminating preferences and upper semicontinuity imply that, in fact, $R_m(y_m) \cap Y \subseteq R_m(x_m) \cap Y$. Then external stability, from Proposition 33, implies we may further specify that $y_m \in \mathfrak{US}(R_m, Y_m)$ for all m. I claim that $y_m \to x$. Otherwise, by compactness of X, we may consider a subsequence of $\{y_m\}$, still indexed by m for simplicity, that converges to an alternative $y \neq x$. Since $R_m(y_m) \cap Y_m \subseteq R_m(x_m) \cap Y_m$ for all m, property (i) of Hausdorff convergence implies yRx and $R(y) \cap Y \subseteq R(x) \cap Y$, which implies $y \,\mathcal{S}(R, Y)\, x$, contradicting $x \in \mathfrak{US}(R, Y)$. ∎

The requirements for lower hemicontinuity of the shallow uncovered set are more stringent the Proposition 56. Returning to the problem of computing the uncovered set, the finite sets Y_m now must be chosen so that the restriction of R to the set is antisymmetric — assuming majority rule and an odd number of voters with continuous, strictly convex preferences, this is satisfied by almost all finite subsets. Since preferences restricted to Y_m form a tournament, the definition of the uncovered set used to construct Z_m is immaterial. By lower hemicontinuity, from Proposition 58, we conclude $\mathfrak{US}(R, X) \subseteq Z$. It is possible that the approximation of the uncovered set contains alternatives outside the actual shallow uncovered set, but the latter provides a lower bound of the calculated set: the approximation will include the shallow uncovered set. If we know, through other means, that the actual deep and shallow uncovered sets coincide — or are very close to one another — then the computed set can approximate the true uncovered set to any desired degree of precision.

11. Discussion

I have attempted in this paper to systematically develop the theory of the uncovered set and to motivate two new solutions, the deep and shallow uncovered sets, that arguably deserve attention in their role as benchmarks for the other uncovered sets. I discern three existing definitions of covering, in addition to the deep and shallow covering relations, as primary. Underlying this distinction is an analysis of the stability structure of the main uncovered sets: the deep, shallow, Gillies, and Bordes (and McKelvey to a lesser extent) uncovered sets can be characterized as the union of choice from externally stable sets. These four (or five) uncovered sets are achieved by varying the type of external stability and the choice criterion, and the characterizations have some interesting implications for the Banks set in general environments. I provide general results on nonemptiness of the uncovered sets, the most stringent for the shallow uncovered set, and on their external stability. I compare the uncovered sets to a number of other solutions considered in the literature, and in particular I define the minimal generalized covering solution and the acyclic Banks set and examine connections between them and the uncovered sets. Finally, I establish that the deep and shallow uncovered sets vary upper and lower hemicontinuously, respectively, as a function of parameters. Thus, the bounds they provide are robust to perturbations of the underlying model.

Several topics deserve further mention.

Downsian Competition: Laffond, Laslier, and Le Breton (1993) prove in the context of tournaments that the support of the unique mixed strategy equilibrium of the canonical Downsian model, called the bipartisan set, is contained in the uncovered set. Dutta and Laslier (1999), in their analysis of the essential set of a weak tournament, extend this result by showing that the equilibrium outcomes of Downsian competition are contained in the McKelvey uncovered set. A conjecture by McKelvey (1986) is that the uncovered set, under his definition, provides an upper bound on the equilibrium outcomes of Downsian competition in the multidimensional spatial voting model. Banks, Duggan, and Le Breton (2002) confirm this conjecture by showing that each equilibrium mixed strategy puts probability one on a measurable subset of the McKelvey uncovered set in a general topological setting.[18] It is not known, however, whether mixed strategy equilibria generally exist in the canonical Downsian model, which presents discontinuities in the parties' payoffs of a particularly difficult form.[19]

Duggan and Jackson (2006) move from the canonical setting, in which voters' behavior is predetermined (as a function of their preferences), to a game-theoretic setting, in which voters are strategic and assumed to play undominated Nash equilibrium strategies. By specifying the behavior of indifferent voters appropriately, the authors prove that the Downsian game does admit a mixed strategy equilibrium, but they give an example in which the parties put positive probability on platforms outside the McKelvey uncovered set. Thus, the McKelvey uncovered set no longer provides a bound on equilibrium behavior of parties. An advantage of the deep uncovered set, however, is that it *does* provide an upper bound on equilibrium platforms. The deep uncovered set can, moreover, be shown to bound outcomes of a class of equilibria in amendment agenda games, providing further support for this solution.

Pareto Optimality: The analysis to this point has been abstract, taking social preferences as primitive rather than explicitly modeling voter preferences and an aggregation mechanism. It is worth, nonetheless, briefly touching on the efficiency properties of the uncovered sets. We now take as given a set $N = \{1, \ldots, n\}$ of voters, a profile $((P_1, R_1), \ldots, (P_n, R_n))$ of voter preference relations, a collection $\mathcal{W}$ of winning coalitions, and a collection $\mathcal{B}$ of blocking coalitions. Assume:

- each (P_i, R_i) is a weak order, so P_i is asymmetric and negatively transitive and R_i is complete and transitive
- $\mathcal{W}$ is nonempty *proper,* i.e., $C \in \mathcal{W}$ implies $N \setminus C \notin \mathcal{W}$
- $\mathcal{W}$ is *monotonic,* i.e., $C \in \mathcal{W}$ and $C \subseteq C'$ imply $C' \in \mathcal{W}$
- $\mathcal{W}$ and $\mathcal{B}$ are dual in the sense that $C \in \mathcal{W}$ if and only if $N \setminus C \notin \mathcal{B}$.

[18] The measurable subset may depend on the equilibrium; the necessity of this technicality is that the McKelvey uncovered set is potentially non-measurable. Please note the typo in Theorem 4 of Banks, Duggan, and Le Breton (2002), in which the inclusion $\hat{U} \supseteq U$ should be reversed to $\hat{U} \subseteq U$!

[19] See Duggan (2007b) for discussion and results on this point.

We then define social preferences as follows:

$$P = \bigcup_{C \in \mathcal{W}} \bigcap_{i \in C} P_i \quad \text{and} \quad R = \bigcup_{C \in \mathcal{B}} \bigcap_{i \in C} R_i.$$

It can be checked that $\mathcal{W} \subseteq \mathcal{B}$, that $\mathcal{B}$ is *strong,* i.e., $C \notin \mathcal{W}$ implies $N \setminus C \in \mathcal{W}$, and that (P, R) is a dual pair. Furthermore, if $\mathcal{W}$ is itself strong, then $\mathcal{W} = \mathcal{B}$.[20]

Given voter preferences, we define two notions of Pareto optimality. The narrower concept is the set of *strongly Pareto optimal* alternatives, denoted $\mathfrak{PO}^s(P_1, \ldots, P_n)$, which consists of any alternative $x \in X$ such that there does not exist $y \in X$ satisfying (i) for all $i \in N$, yR_ix and (ii) for some $j \in N$, yP_jx. The more encompassing concept is the set of *weakly Pareto optimal* alternatives, denoted $\mathfrak{PO}^w(P_1, \ldots, P_n)$, which consists of any alternative $x \in X$ such that there does not exist $y \in X$ with yP_ix for all $i \in N$. It is well-known, and formalized for example in Proposition 24 of Banks, Duggan, and Le Breton (2006), that the McKelvey uncovered set is a subset of the weak Pareto optimals; of course, this result carries over to all of the smaller uncovered sets.

Proposition 59: $\mathfrak{U}(M) \subseteq \mathfrak{PO}^w(P_1, \ldots, P_n)$.

Although the deep uncovered set has a comparative advantage in encompassing supports of equilibrium strategies in the game-theoretic Downsian model, a drawback of this solution is that it may contain alternatives that are not even weakly Pareto optimal. Consider the following profile of linear orders for $n = 4$ voters.

1	2	3	4
x	x	z	z
y	y	x	x
z	z	y	y

When voting is by majority rule, so $\mathcal{W}$ consists of all groups of three or more voters, $y \in \mathfrak{U}(D) \setminus \mathfrak{PO}^w(P_1, \ldots, P_n)$. In particular, each voter strictly prefers x to y, and so y is not weakly Pareto optimal, but y is weakly majority preferred to z, and so belongs to the deep uncovered set.

The collection $\mathcal{W}$ of winning coalitions is not strong in the above example, and this is no coincidence. If $\mathcal{W}$ is strong, as is the case with majority rule and n odd, then the deep uncovered set is contained in the weakly Pareto optimals.

Proposition 60: Assume $\mathcal{W}$ is strong. Then $\mathfrak{U}(D) \subseteq \mathfrak{PO}^w(P_1, \ldots, P_n)$.

[20] See Banks, Duggan, and Le Breton (2006) for a more general formulation of simple voting games with a measure space of voters.

PROOF: Take any $x, y \in X$ such that yP_ix for all $i \in N$. It suffices to show that $y\ D\ x$. Take any $z \in X$ such that zRy, which implies $C = \{i \in N \mid zR_iy\} \in \mathcal{B}$. Since $\mathcal{W}$ is strong, this implies $C \in \mathcal{W}$. For each $i \in C$, we then have zR_iyP_ix, which implies zP_ix. Thus, $\{i \in N \mid zP_ix\} \in \mathcal{W}$, and we conclude that zPx, as required. ∎

The more stringent requirement of strong Pareto optimality is satisfied by the shallow uncovered set but is problematic for the larger uncovered sets. For example, suppose $X = \{x, y\}$ and consider any profile such that voters $1, \ldots, n-1$ are indifferent between the two alternatives and voter $n \geq 2$ strictly prefers x to y; under majority voting, we have yRx, so y belongs to the Gillies, Bordes, McKelvey, and deep uncovered sets. Of course, the latter example holds when n is odd, so $\mathcal{W}$ being strong is no recourse.

PROPOSITION 61: $\mathfrak{U}(S) \subseteq \mathfrak{PO}^s(P_1, \ldots, P_n)$.

PROOF: Take any $x, y \in X$ such that (i) for all $i \in N$, yR_ix and (ii) for some $j \in N$, yP_jx. To see that $y\ S\ x$, first note that $N = \{i \in N \mid yR_ix\}$, so yRx. Next, take any $z \in X$ such that zRy. Then $C = \{i \in N \mid zR_iy\} \in \mathcal{B}$, and for all $i \in C$, we have zR_iyR_ix, which implies zR_ix, and therefore zRx. Taking any $z \in X$ such that zPy, a similar argument in terms of winning coalitions shows zPx, as required. ∎

EVALUATION: It is interesting to consider the strength and weaknesses of the various uncovered sets defined in this paper. The most significant dimensions of comparison seem to be: existence, external stability, bounds for equilibria in Downsian competition, and Pareto optimality. Consider the four uncovered sets with the strongest stability structure.

- The deep uncovered set fares best in terms of existence and is even compact quite generally; it is the only solution that provides an upper bound on equilibria in the game-theoretic Downsian model; and it is among the best in terms of external stability. The main drawback to this solution is that it admits Pareto inefficient choices, at least in voting games that are not strong.
- The shallow uncovered set fares best with respect to Pareto optimality and is the only strongly Pareto optimal solution, but it can be disjoint from outcomes of electoral competition and even the ultimate uncovered set. Moreover, it is the weakest of the solutions with respect to existence and external stability.
- The Gillies uncovered set has excellent existence properties and fares well with respect to Pareto optimality, but it can be disjoint from equilibrium outcomes of electoral competition, and its external stability properties appear weaker than the deep and Bordes uncovered sets.

- The Bordes uncovered set is among the best with respect to external stability, it fares well with respect to Pareto optimality and existence, but it too can be disjoint from equilibrium outcomes of electoral competition.

Bringing the McKelvey uncovered set into the comparisons, it is nonempty if the Gillies set is, it shares the external stability and Pareto optimality of the Bordes set, and it bounds the outcomes of electoral competition in the canonical setting. But the latter property does not persist when voters are modeled game-theoretically, and its stability structure is weaker than the other solutions.

Open Questions: The analysis in this paper raises (at least) five open questions. The first regards general conditions on voter preferences under which social preferences (P, R) are rich. I conjecture that in a multidimensional spatial setting, assuming continuity and convexity of voter preferences and $\mathcal{W}$ is strong, richness is generic in some meaningful sense. The second question is whether external stability of the Gillies uncovered set can be obtained under topological conditions weaker than discriminating preferences — conditions that generalize the assumption of finite X, which delivers external stability even if preferences are not discriminating. The third is whether minimal deep covering sets, defined using deep covering for both internal and external stability, can be shown to exist and even be unique. The fourth question is the status of the Banks sets, and in particular the acyclic Banks set, relative to other solutions for weak tournaments. The fifth evolves from McKelvey's (1986) Proposition 4.1, where he states, essentially, that the deep uncovered set is contained in the closure of the shallow uncovered set. His proof contains an error,[21] however, so the question remains whether the relationship holds under reasonable conditions on primitives. I conjecture that McKelvey's claim holds under reasonable conditions. If so, in line with the discussion at the end of Section 10, the result would greatly facilitate computational approaches to the uncovered set.

Appendix A. Further Connections

The first task of this appendix is to provide an example, in Figure 6, showing that the shallow uncovered set can be disjoint from the ultimate uncovered set, even when the former is nonempty. Using the two-step principle, it can be checked that the shallow uncovered set consists of v alone: every alternative can be reached from v in either one or two strict preference steps, while that is true of no other alternative; in particular, x cannot reach w in one or two steps, y cannot reach x, and so on. On the other hand, the McKelvey uncovered set is $\{x, y, z, w, v\}$; in particular, a is McKelvey covered by x, b is McKelvey covered by y, and so on. But then x McKelvey

[21]This was originally pointed out by Banks, Duggan, and Le Breton (2006).

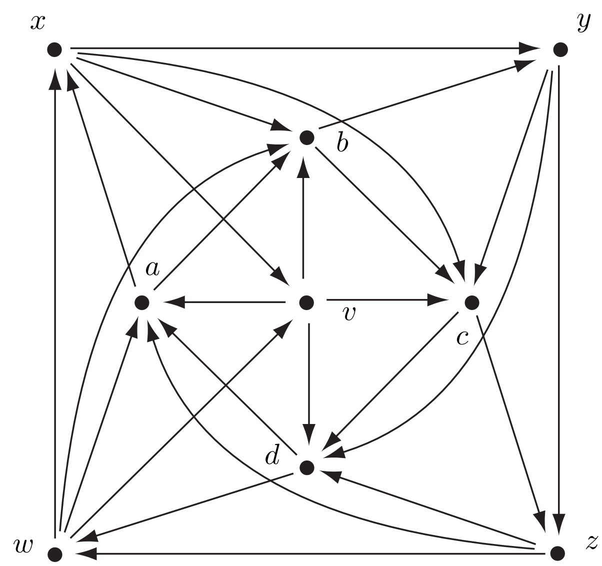

FIGURE 6. Shallow uncovered set disjoint from ultimate uncovered set

covers v in $\{x, y, z, w, v\}$, and the ultimate uncovered set is $\{x, y, z, w\}$, which is disjoint from the shallow uncovered set.

Now return to the two-player, zero-sum game induced by the preferences (P, R), assuming X is finite. A subset $Y \subseteq X$ is a *mixed generalized saddle point* (MGSP) if for all $x \in X \setminus Y$, there exists a probability distribution σ with support in Y such that for all $y \in Y$, $\Pi(\sigma, y) > \Pi(x, y)$. That is, Y is a MGSP if every alternative outside the set is strictly dominated by a mixed strategy on Y for player 1, if we vary player 2's strategies only within Y. We say a MGSP is a *mixed saddle* if it is minimal among MGSP's with respect to set-inclusion. Duggan and Le Breton (1997, 1999, 2001) show that there is a unique mixed saddle, denoted MS. Moreover, the mixed saddle coincides with the unique minimal rationalizable set, meaning that it contains all best responses to mixed strategies with support within the set, and it is contained in every other set with this property, with the implication that it is a superset of the strong top cycle and the essential set.

Those authors show that the mixed saddle is nested between the strong and weak top cycles, and so in the tournament context, it is equivalent to the top cycle and is known to be a superset of the uncovered set. For the opposite inclusion, Duggan and Le Breton's (1999) Example 8 provides a weak tournament in which the mixed saddle is $MS = \{x_1, x_2, x_3\}$, while the Gillies and Bordes uncovered sets contain the entire set X of six alternatives. That the Gillies and Bordes uncovered sets generally have nonempty intersection with the mixed saddle follows from nonempty intersection with the strong top cycle, from Corollaries 34 and 35, and $STC \subseteq MS$. Thus, unlike the essential set, the mixed saddle always intersects the Gillies and Bordes uncovered sets.

Allowing for a general metric space of alternatives, Duggan (2007a) defines the *uncaptured set,* denoted UC^+, as the union of maximal sets of transitive subrelations of P and shows that an alternative x belongs to UC^+ if and only if for all $y \in X$, at least one of the following four conditions holds:

(i) xRy
(ii) there exists $w \in X$ such that $xPwRy$
(iii) there exists $z \in X$ such that $xRzPy$
(iv) there exist $s, t \in X$ such that $xPsRtPy$.

Clearly, by the two-step principle, the McKelvey uncovered set, along with the smaller Gillies, Bordes, and shallow uncovered sets, are contained in the uncaptured set. Thus, as long as the Gillies or Bordes sets are nonempty, we have $\mathfrak{U}(D) \cap UC^+ \neq \emptyset$. In general, however, there is no logical nesting between the deep uncovered set and the uncaptured set. To see that the deep uncovered set may be a proper subset of the uncaptured set, let $X = \{x, y, z, w\}$ and $P = \{(x, y), (x, z), (y, z), (y, w), (z, w)\}$, and note that $y\ M\ z$ yet $xPwRxPy$. In particular, $UC^+ = X$ and $\mathfrak{U}(D) = \{x, y, w\}$. For the opposite inclusion, let $X = \{x, y, z, w\}$ and $P = \{(y, z)\}$. Then z is not deeply covered, but there is no chain of preferences as in (i)–(iv) from z to y. In particular, $UC^+ = \{x, y, w\}$ and $\mathfrak{U}(D) = X$.

Duggan (2007a) also defines the *untrapped set,* denoted UT, as the union of maximal elements of acyclic subrelations of P. This set is characterized as the maximal elements of the *trapping* relation, T, defined as follows: for all $x, y \in X$, $x\ T\ y$ if and only if xPy and not $yP^\infty x$. The untrapped set is nested between the strong top cycle and the mixed saddle, so it coincides with the top cycle in a tournament, and it is well-known that the uncovered set is a subset of the top cycle in tournaments. Since the shallow uncovered set is generally a subset of the strong top cycle, it follows that $\mathfrak{U}(S) \subseteq UT$ always holds, but this inclusion does not hold for the larger uncovered sets. To see this, let $X = \{x, y, z\}$ and $P = \{(x, y), (y, z)\}$, and note that $UT = \{x\}$, but $\mathfrak{U}(G) = \{x, z\}$ and $\mathfrak{U}(B) = \{x, y\}$. That the Gillies and Bordes uncovered sets generally have nonempty intersection with the untrapped set follows from nonempty intersection with the strong top cycle, from Corollaries 34 and 35, and $STC \subseteq UT$.

Table 1 summarizes the relationships studied in Section 4 and in this appendix. Here, $\emptyset$ indicates that two solutions can be disjoint, even when both are nonempty, and that each solution can be a proper subset of the other; $\cap$ indicates that two solutions have nonempty intersection when X is finite (or perhaps under more general topological conditions) and that each solution can be a proper subset of the other; and $\subset$ means that the row solution is

	$\mathfrak{U}(S)$	$\mathfrak{U}(G)$	$\mathfrak{U}(B)$	MC	UC^∞	STC	UT	MS	$\mathfrak{U}(M)$	UC^+	$\mathfrak{U}(D)$	WTC
ES	$\emptyset^{1}$	$\emptyset^{2}$	$\emptyset^{3}$	$\subset$	$\subset^{4}$	$\subset$	$\subset^{5}$	$\subset$	$\subset$	$\subset^{6}$	$\subset^{7}$	$\subset$
$\mathfrak{U}(S)$		$\subset^{8}$	$\subset^{9}$	$\emptyset^{10}$	$\emptyset^{11}$	$\subset^{12}$	$\subset^{13}$	$\subset^{14}$	$\subset^{15}$	$\subset^{16}$	$\subset^{17}$	$\subset^{18}$
$\mathfrak{U}(G)$			$\emptyset^{19}$	$\cap^{20}$	$\cap^{21}$	$\cap^{22}$	$\cap^{23}$	$\cap^{24}$	$\subset^{25}$	$\subset^{26}$	$\subset^{27}$	$\subset^{28}$
$\mathfrak{U}(B)$				$\cap^{29}$	$\cap^{30}$	$\cap^{31}$	$\cap^{32}$	$\cap^{33}$	$\subset^{34}$	$\subset^{35}$	$\subset^{36}$	$\subset^{37}$
MC					$\subset^{38}$	$\cap$	$\cap^{39}$	$\subset$	$\subset$	$\subset^{40}$	$\subset^{41}$	$\subset$
UC^∞						$\cap^{42}$	$\cap^{43}$	$\cap^{44}$	$\subset^{45}$	$\subset^{46}$	$\subset^{47}$	$\subset^{48}$
STC							$\subset^{49}$	$\subset$	$\cap$	$\cap^{50}$	$\cap^{51}$	$\subset$
UT								$\subset^{52}$	$\cap^{53}$	$\cap^{54}$	$\cap^{55}$	$\subset^{56}$
MS									$\cap$	$\cap^{57}$	$\cap^{58}$	$\subset$
$\mathfrak{U}(M)$										$\subset^{59}$	$\subset^{60}$	$\subset$
UC^+											$\cap^{61}$	$\subset^{62}$
$\mathfrak{U}(D)$												$\subset^{63}$

TABLE 1. Logical connections

generally a subset of column. Superscripts point to brief explanations, below, of the entries in Table 1, where entries without a superscript are found Table 1 of Duggan and Le Breton (1999).

In the remainder of the appendix, I verify the numbered entries of Table 1.

1. The possibility of empty intersection follows from $ES \subseteq UC^\infty$ and the example in Figure 6. It is well-known that in tournaments, the essential set can be a proper subset of the uncovered set. For the opposite inclusion, recall from the example preceding Proposition 5 that the shallow uncovered set can be empty, while the essential set is always nonempty.
2. The possibility of empty intersection follows from the example in Figure 3. It is well-known that in tournaments, the essential set can be a proper subset of the uncovered set. For the opposite inclusion, see the example in Figure 2.
3. The possibility of empty intersection follows from the example in Figure 4. It is well-known that in tournaments, the essential set can be a proper subset of the uncovered set. The opposite inclusion follows from a simple modification of the example in Figure 2.
4. The inclusion is established in Theorem 4.3 of Dutta and Laslier (1999). It is well-known that in tournaments, the essential set (or bipartisan set) may be a proper subset of the minimal covering set.
5. From 49 and $ES \subseteq STC$.
6. From 59 and $ES \subseteq \mathfrak{U}(M)$.
7. From 60 and $ES \subseteq \mathfrak{U}(M)$.
8. From Proposition 5.
9. From Proposition 5.
10. The possibility of empty intersection follows from 4 and the example in Figure 6. It is well-known that in tournaments, the minimal covering set can be a proper subset of the uncovered set. For the

opposite inclusion, recall from the example preceding Proposition 5 that the shallow uncovered set can be empty, while the minimal covering set is always nonempty.

11. The possibility of empty intersection follows from the example in Figure 6. It is well-known that in tournaments, the ultimate uncovered set can be a proper subset of the uncovered set. For the opposite inclusion, recall from the example preceding Proposition 5 that the shallow uncovered set can be empty, while the ultimate uncovered set is always nonempty.
12. From Proposition 17.
13. From 12 and 49.
14. From 13 and 52.
15. From Proposition 5.
16. From 15 and 59.
17. From 15 and 60.
18. From 15 and $\mathfrak{U}(M) \subseteq WTC$.
19. The possibility of empty intersection follows from the example in Figure 1. For $\mathfrak{U}(G) \subsetneq \mathfrak{U}(B)$, let $X = \{x, y, z, w\}$ and $P = \{(x, y), (y, z), (z, w), (w, y)\}$, so that $\mathfrak{U}(B) = X$ but $x\ G\ y$. For $\mathfrak{U}(B) \subsetneq \mathfrak{U}(G)$, redefine $P = \{(x, y), (x, z), (z, w), (w, x)\}$, so that $\mathfrak{U}(G) = X$ but $x\ B\ y$.
20. Nonempty intersection follows from Proposition 18. It is well-known that in tournaments, the minimal covering set may be a proper subset of the uncovered set. For the opposite inclusion, see Example 7 in Duggan and Le Breton (1999), where $MC = X$ but $x_6\ G\ x_2$.
21. Nonempty intersection follows from 20 and 38. For nonnestedness, it is well-known that in tournaments, the ultimate uncovered set may be a proper subset of the uncovered set. For the opposite inclusion, see Example 7 in Duggan and Le Breton (1999), where $UD^\infty = X$ but $x_6\ G\ x_2$.
22. Nonempty intersection follows from Corollary 35. For nonnestedness, see the examples following Proposition 17.
23. Nonempty intersection follows from 22 and 49. For nonnestedness, it is well-known that in tournaments, the uncovered set may be a proper subset of the top cycle, which coincides with the untrapped set. For the opposite inclusion, see the discussion after the definition of the untrapped set.
24. Nonempty intersection follows from 22 and $STC \subseteq MS$. For nonnestedness, it is well-known that in tournaments, the uncovered set may be a proper subset of the top cycle. For the opposite inclusion, see Example 8 of Duggan and Le Breton (1999), where $MS = \{x_1, x_2, x_3\}$, while $\mathfrak{U}(G) = X \supsetneq MS$.
25. From Proposition 5.
26. From 25 and 59.
27. From 25 and 60.

28. From 25 and $\mathfrak{U}(M) \subseteq WTC$.
29. Nonempty intersection follows from Proposition 19. It is well-known that in tournaments, the minimal covering set may be a proper subset of the uncovered set. For the opposite inclusion, see Example 7 in Duggan and Le Breton (1999), where $MC = X$ but $x_5 \; B \; x_7$.
30. Nonempty intersection follows from 29 and 38. For nonnestedness, it is well-known that in tournaments, the uncovered set may be a proper subset of the ultimate uncovered set. For the opposite inclusion, see Example 7 in Duggan and Le Breton (1999), where $UD^{\infty} = X$ but $x_5 \; B \; x_7$.
31. Nonempty intersection follows from 29 and $MC \subseteq STC$. For nonnestedness, see the examples following Proposition 17.
32. Nonempty intersection follows from 31 and 49. For nonnestedness, see the discussion after the definition of the untrapped set.
33. Nonempty intersection follows from 31 and $STC \subseteq MS$. For nonnestedness, it is well-known that in tournaments, the uncovered set may be a proper subset of the top cycle, which coincides with the mixed saddle. For the opposite inclusion, see Example 8 of Duggan and Le Breton (1999), where $MS = \{x_1, x_2, x_3\}$, while $\mathfrak{U}(B) = X \supsetneq MS$.
34. From Proposition 5.
35. From 34 and 59.
36. From 34 and 60.
37. From 34 and $\mathfrak{U}(M) \subseteq WTC$.
38. The inclusion is proved in Theorem 2 of Peris and Subiza (1999), and it is well-known that in tournaments, the minimal covering set may be a proper subset of the ultimate uncovered set.
39. Nonempty intersection follows from 49 and $STC \cap MC \neq \emptyset$. For nonnestedness, it is well-known that in tournaments, the minimal covering may be a proper subset of the top cycle. For the opposite inclusion, see the example following Corollary 2 of Duggan (2007a), where $MC = X = \{a, b, c\}$ yet $STC = \{a\}$. From $MC \subseteq STC$ and 49.
40. From $MC \subseteq \mathfrak{U}(M)$ and 59.
41. From $MC \subseteq \mathfrak{U}(M)$ and 60.
42. Nonempty intersection follows from 4 and $ES \subseteq STC$. For nonnestedness, it is well-known that in tournaments, the ultimate uncovered set may be a proper subset of the top cycle. For the opposite inclusion, see Example 8 of Duggan and Le Breton (1999), where $STC = \{x_1, x_2, x_3$ and $UC^{\infty} = X \supsetneq STC$.
43. Nonempty intersection follows from 4 and 5. For nonnestedness, it is well-known that in tournaments, the ultimate uncovered set may be a proper subset of the top cycle, which coincides with the untrapped set. For the opposite inclusion, see Example 8 of Duggan and Le Breton (1999), where $UT = \{x_1, x_2, x_3$ and $UC^{\infty} = X \supsetneq UT$.

44. Nonempty intersection follows from 4 and $ES \subseteq MS$. For nonnestedness, it is well-known that in tournaments, the ultimate uncovered set may be a proper subset of the top cycle, which coincides with the mixed saddle. For the opposite inclusion, see Example 8 of Duggan and Le Breton (1999), where $MS = \{x_1, x_2, x_3$ and $UC^{\infty} = X \supsetneq MS$.
45. The inclusion follows from definition of the ultimate uncovered set, and the possibility of strict inclusion is well-known in tournaments.
46. From 45 and 59.
47. From 45 and 60.
48. From 45 and $\mathfrak{U}(M) \subseteq WTC$.
49. From Theorem 3 of Duggan (2007a) and surrounding discussion.
50. Nonempty intersection follows from $STC \cap \mathfrak{U}(M) \neq \emptyset$ and 59. For nonnestedness, see the examples after Theorem 1 of Duggan (2007a).
51. Nonempty intersection follows from $STC \cap \mathfrak{U}(M) \neq \emptyset$ and 60. For nonnestedness, see the discussion after the definition of the uncaptured set.
52. From Theorem 5 of Duggan (2007a) and surrounding discussion.
53. Nonempty intersection follows from 49 and $STC \cap \mathfrak{U}(M) \neq \emptyset$. For nonnestedness, see the examples following Corollary 2 of Duggan (2007a).
54. Nonempty intersection follows from 53 and 59. For nonnestedness, see the examples following Corollary 2 of Duggan (2007a).
55. Nonempty intersection follows from 53 and 60. For nonnestedness, see the discussion after the definition of the untrapped set.
56. From Theorem 3 of Duggan (2007a) and surrounding discussion.
57. Nonempty intersection follows from $MS \cap \mathfrak{U}(M) \neq \emptyset$ and 59. An example after Theorem 1 of Duggan (2007a) shows that the uncaptured set may be a proper subset of the strong top cycle, and therefore of the mixed saddle. For the other inclusion, see Example 8 of Duggan and Le Breton (1999), where $MS \subsetneq X = \mathfrak{U}(M) = UC^{+}$.
58. Nonempty intersection follows from $MS \cap \mathfrak{U}(M) \neq \emptyset$ and 60. It is well-known that the uncovered set may be a proper subset of the top cycle, which coincides with the mixed saddle. For the other inclusion, see Example 8 of Duggan and Le Breton (1999), where $MS \subsetneq X = \mathfrak{U}(M) = \mathfrak{U}(D)$.
59. From Theorem 1 of Duggan (2007a) and surrounding discussion.
60. From Proposition 5.
61. Nonempty intersection follows from 59 and 60. For nonnestedness, see the discussion after the definition of the uncaptured set.
62. From Theorem 1 of Duggan (2007a) and surrounding discussion.
63. From Proposition 15. It is well-known that in tournaments, the uncovered set may be a proper subset of the top cycle.

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