Strategic Candidacy and Voting Procedures

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Draft: December 19, 1998

Abstract

We study the impact of considering the incentives of candidates to strategically affect the outcome of a voting procedure. First we show that *every* non-dictatorial voting procedure that satisfies unanimity, is open to strategic entry or exit by candidates: there necessarily exists some candidate who can affect the outcome by entering or exiting the election, even when she does not win the election. Given that strategic candidacy always matters, we analyze the impact of strategic candidacy effects. We show that the equilibrium set of outcomes of the well-known voting by successive elimination procedure expands in a well-defined way when strategic candidacy is accounted for.

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1 Introduction

The decision of a candidate to enter an election can affect the outcome of the election even in situations where the candidate is not the winner of the election. For instance, consider a scenario in which three national parties A, B and C can contest an election in which the winner is decided by plurality rule. Although party A may have the highest number of first-preference votes, it may still fail to win the election if, for instance, B drops out of the race in order to let C win. If the voting process is viewed as a mapping from preferences to outcomes, the strategic behavior in the first stage is just as important as strategic voting in the second stage. As we shall show, this phenomenon is important to *all* voting procedures, and thus spans applications ranging from political elections to committee decisions. Despite this important form of sensitivity of voting procedures to strategic candidacy, there is no theoretical analysis of how the outcomes of various voting procedures are affected by strategic choices by candidates of whether to enter. Such an analysis is the subject of our paper.

More precisely, we consider a framework in which there is a finite set of voters and potential candidates. We allow for the possibility that some or all of the candidates may also be voters, and consider situations where each individual (including candidates) has preferences over the set of all candidates. We examine a two-stage procedure where in a first stage candidates decide on whether or not they will enter the election, and then in a second stage a voting procedure is implemented to select from the candidates who enter.

Before outlining our analysis, let us describe in more detail the way in which we model voting procedures. We model a voting procedure as specifying the winning candidate as a function of the set of entering candidates and voters' preferences over the entering candidates. The only restriction that we place on such a voting procedure is that it satisfy unanimity. Unanimity requires that if all voters find the same candidate most preferred out of the entering candidates, then that candidate is selected.

We focus on two main questions.

The first question is whether it is possible to find a voting procedure that satisfies the following condition: a candidate who is *not* winning the election cannot affect the outcome of the election by choosing not to enter the election. We call this condition "candidate stability," and show that the only voting procedures that satisfy candidate stability are dictatorial procedures.

We also show that if the set of voters and candidates are distinct, then imposing a weaker form of candidate stability which requires only that it be a Nash equilibrium for all candidates to enter, also implies that the voting procedure must be dictatorial. We spend some time exploring variations on these results: they continue to hold (under some modifications) with a natural preference restriction where candidates find themselves most preferred, and with overlaps in the sets of candidates and voters.

We should mention that these results are not simple extensions of an Arrow-type impossibility theorem, even though we invoke Arrow's theorem at one point in the proofs. The bulk of the proofs develop the joint implications of candidate stability and unanimity. We discuss this in more detail in what follows.

Since these results show that it is impossible to avoid strategic candidacy, a second question naturally arises: "What are the outcomes of voting procedures when one allows for strategic candidacy and accounts for its implications?" We answer this question for a specific, but important, class of voting procedures: namely voting by successive amendments, which is more generally known as voting by successive elimination.

Voting by successive elimination has been well-studied in the context of a fixed set of candidates, and thus provides a nice benchmark. In such voting, candidates are ordered and then compared pairwise. So, for instance, the first and the second candidates are put to a vote. The losing candidate is eliminated, and the winning candidate is then matched against the third candidate for a vote, and so on. For a fixed set of candidates, a fixed profile of voters' preferences, and a fixed ordering of candidates, a single winner emerges. This has been nicely characterized via an algorithm due to Shepsle and Weingast (1984). Moreover, Banks (1985) characterized the set of candidates as the ordering of candidates is varied to admit all possible orderings. This set, which we refer to as the Banks set, has nice properties and in particular is a subset of the uncovered set.

The problem that we consider is as follows: there is a fixed set of potential candidates, a fixed profile of voters' preferences, and a fixed ordering of candidates. Candidates anticipating the outcome of the voting by successive elimination procedure, simultaneously choose whether or not to enter. The voting procedure then takes place only over the candidates who entered. Examining an equilibrium of this overall game leads to a prediction of who will enter in the first stage, and ultimately who will win in the procedure.

Most importantly, it can lead to a different winning candidate than in the case where the set of candidates is fixed exogenously. Now, as we vary the ordering over candidates we can perform an exercise similar to that of Banks (1985), but where we have allowed for strategic entry by the candidates. We end up with a well-identified set that we call the "candidate stable set" and that we fully characterize.

There are several things that we note about the candidate stable set. First, the candidate stable set is larger than the Banks set. Thus, for this class of voting procedures, accounting for strategic candidacy enlarges the set of winning candidates. Second, the candidate stable set has an intuitive relationship to the Banks set that we outline in detail in the characterization. Third, the candidate stable set is a superset of the uncovered set, but is only "slightly" larger than the uncovered set, in a way that we are able to make precise.

Before proceeding to the body of our analysis, let us briefly discuss the relationship of this work to the most closely related literature.

The Related Literature

The most closely related papers to this one are papers by Dutta and Pattanaik (1978), Osborne and Slivinsky (1996), and Besley and Coate (1997). Dutta and Pattanaik (1978) analyze a setting where in a first stage (before voting) individuals sponsor or propose alternatives out of a set. Next, in a second stage, voting takes place over the set of proposed alternatives. Their main result is to show that there are circumstances under which sponsors indulge in strategic behavior by not proposing their most preferred alternatives.[1] More recently, Besley and Coate (1997) and Osborne and Slivinsky (1996) analyzed strategic candidacy in the context of representative democracy. In their models citizen-candidates can contest an election in which the winner is decided by plurality rule. The main focus of these papers is to determine the number of candidates who will enter the election as well as the pattern of entry. They exhibit situations where candidates who have no chance of winning may enter the fray simply in order to influence the outcome of the election, thus noting strategic candidacy.

Although each of the three above-mentioned papers analyzes issues related to strategic candidacy, the focus of our paper is quite different. First,

[1] See Majumdar (1956) for an earlier analysis of sponsoring behavior.

we are interested in understanding whether strategic candidacy can be overcome by a careful choice of the voting procedure. We conclude that it cannot - and so this is an issue that is endemic. Second, we are interested in understanding how the set of elected candidates is affected by strategic candidacy, relative to the fixed candidates setting. We show that for the case of voting by successive elimination, the set of winning candidates is expanded in a well identifiable and intuitive way.

There is also a rich theoretical literature on the strategic effects of agenda manipulation, which is not as closely related to our work, but still should be mentioned.[2] The agenda manipulation literature takes seriously the strategic ordering of alternatives, or more generally the effects of varying the game form used for voting. However, the agenda manipulation literature considers the set of alternatives as fixed, and instead focuses on effects of changes in the voting procedure.

Finally, the underlying issue that we are considering here is of importance to the general literature on mechanism design and implementation. This literature usually takes the set of feasible alternatives to be exogenous, and focuses on the difficulties raised by the presence of incentives of players regarding information that they might privately hold. There are notable exceptions, however. Papers by Postlewaite (1979), Hurwicz, Maskin and Postlewaite (1995), and Hong (1996, 1998), have considered strategic withholding of endowments in exchange settings,[3] and thus are similarly motivated to understand the strategic effects of control of the feasible set of alternatives. As we examine a very different setting, our work is complementary to these papers with regards to the broader goal of developing an understanding of collective decision making when the set of alternatives is endogenous.

The remainder of the paper is structured as follows. In section 2, we outline the setting and provide definitions. In section 3, we consider the question of candidate stability showing that the only candidate stable and unanimous voting procedures are dictatorial, when the domain of preferences is unrestricted. In section 4, we follow up on this question when the set of

[2] A nice discussion of some of the main contributions to the agenda manipulation literature appears in Ordeshook (1986).

[3] Hong (1998) addresses private information of an agent about the feasible set of alternatives.

candidate preferences are restricted to finding themselves most-preferred. In section 5, we examine the implications of strategic candidacy on voting by successive elimination. In section 6, we provide a preliminary examination of a number of important issues for further analysis, such as the existence of pure strategy equilibrium in the entry game.

2 Preliminaries

Candidates and Voters

Let $\mathcal{N} = \{1, \ldots, n\}$ be a finite set of individuals.

$\mathcal{C} \subset \mathcal{N}$ is the set of potential *candidates*. Generic candidates are denoted a, b, c, d, e. We consider the case where $\#\mathcal{C} \geq 3$, as the case with just two potential candidates is easily handled with a majority vote and the possibilities for strategic candidacy are trivial.

$\mathcal{V} \subset \mathcal{N}$ is the set of *voters*. Generic voters are denoted i, j, k.

Without loss of generality, assume that $\mathcal{N} = \mathcal{C} \cup \mathcal{V}$. In different situations it may be that $\mathcal{C} = \mathcal{V}$, $\mathcal{C} \subset \mathcal{V}$, $\mathcal{C} \cap \mathcal{V} \neq \emptyset$ or $\mathcal{C} \cap \mathcal{V} = \emptyset$. We will discuss how the overlap between candidates and voters matters, as it applies. Unless otherwise stated, there is no fixed assumption.

Preferences

Individuals have strict preferences over the set of candidates represented by a complete, transitive, and asymmetric binary relation, P_i (often referred to as an order or a linear order). Let $\mathcal{P}$ denote the set of all profiles of such preference relations. The notation $P \in \mathcal{P}$ denotes a generic profile $P = (P_1, \ldots, P_n)$.

Let aR_ib denote the situation where either aP_ib or $a = b$.

Given any $A \subset \mathcal{C}$, let $P_i|_A$ denote the binary relation on A induced by P_i, and $P|_A$ the profile of induced relations.

Given any nonempty $B \subset \mathcal{C}$, let $\text{top}(B, P_i)$ denote the candidate $a \in B$ such that aP_ib for all $b \in B$, $b \neq a$.

Voting Procedures

A *voting procedure* is a function $V : 2^{\mathcal{C}} \setminus \emptyset \times \mathcal{P} \rightarrow \mathcal{C}$ such that for all $A \subset \mathcal{C}$ and $P \in \mathcal{P}$

(i) $V(A,P) \in A$,

(ii) $V(A,P) = V(A,P')$ for all P' such that $P_i = P_i'$ for all $i \in \mathcal{V}$, and

(iii) $V(A,P) = V(A,P')$ for all P' such that $P|_A = P'|_A$.

Item (i) says that a voting procedure chooses from the set of available candidates A.

Item (ii) says that a voting procedure is determined only by voters' preferences. In our setting the profile P includes a specification of candidates' preferences, who in some cases may not be voters. Restriction (ii) is essentially without loss of generality, as we could simply define $\mathcal{V}$ to be the set of individuals whose preferences matter for V.

Item (iii) says that the voting procedure depends only on preferences over the set of feasible (i.e., entering) candidates. This condition is similar to Arrow's independence of irrelevant alternatives condition, except defined over voting procedures instead of social welfare orderings.[4] We emphasize that this independence condition is very weak in the context of voting procedures, as there are many voting procedures that are non-dictatorial and satisfy unanimity, as illustrated in Example 1, below. We make this statement formal in the following claim. The proof is obvious and thus omitted.

Unanimity

V satisfies *unanimity* if $V(B,P) = b$ for any $B \subset \mathcal{C}$, $P \in \mathcal{P}$, and $b \in B$ such that $top(B,P_i) = b$ for each $i \in \mathcal{V}$.

Claim *For each set of candidates $A \subset \mathcal{C}$ consider any finite extensive game form of perfect information[5] G_A with range A. Define $V(A,P)$ to be the subgame perfect equilibrium[6] outcome of G_A given the preference profile P. Then V is a voting procedure that satisfies unanimity.*

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[4]This condition is sometimes called "independence of infeasible alternatives" in the choice setting.

[5]As this definition is quite standard we omit it. We refer the interested reader to Dutta and Sen (1993, Def. 6) for details. The claim can actually be significantly strengthened to simply require that for each A, $V(A,\cdot)$ be a social choice function that is implementable via some standard solution concept such as Nash equilibrium, subgame perfect equilibrium, undominated Nash equilibrium, or the iterative elimination of weakly dominated strategies, via some game form with range A (not necessarily finite or of perfect information).

[6]For such game forms, subgame perfect equilibrium, backwards induction, and the iterative elimination of weakly dominated strategies all coincide.

Thus, if for each A the voting rule that society uses can be modeled (implemented) by some extensive game form, then that will result in a unanimous voting procedure. This, of course, includes most common voting methods.

If the voters' preference profile is fixed, then a voting procedure becomes a choice rule. If the set of entering candidates is fixed, then the voting procedure becomes a social choice function. The fact that a voting procedure is more general than what are standardly defined as either choice rules or social choice functions, is why we use the new name voting procedure.[7]

Candidate Stability

Suppose all members of $\mathcal{C}$ know that the voting procedure V describes the choice out of any given set of candidates as a function of the profile of voters' preferences. This gives rise to a normal form game where $\mathcal{C}$ is the set of players, with each player having two possible strategies of either entering the election or not entering the election. The profile of entry decisions, together with the choice function specifies the eventual outcome. Since players in $\mathcal{C}$ have preferences over $\mathcal{C}$, this is a well-defined game. Our focus here is on the strategic behavior of the members of $\mathcal{C}$ in this game. The following conditions describe properties of this game.

A voting procedure V is *candidate stable* if for each $a \in \mathcal{C}$ and $P \in \mathcal{P}$ such that $a \neq V(\mathcal{C}, P)$, $V(\mathcal{C}, P) = V(\mathcal{C} \setminus \{a\}, P)$.

Candidate stability implies that a candidate *who is not being elected*, cannot exit and affect the outcome.

This is a very weak way of capturing the idea that candidates cannot strategically affect the outcome of a voting procedure by withdrawing from a contest. First, the condition only applies to candidates who are not being elected. Thus, it is only addressing situations where the candidate entering or exiting only affects the outcome in terms of another candidate being elected. Second, candidate stability only requires that this be true when all candidates enter. That is, the condition only compares $V(\mathcal{C}, P)$ and $V(\mathcal{C} \setminus \{a\}, P)$, but makes no statement about the relationship between $V(A, P)$ and $V(A \setminus \{a\}, P)$ for $A \subset \mathcal{C}$.

[7] The term "social choice function" has been used in the non-binary choice literature to describe the same functions that we are calling "voting procedures". We chose not to use the name social choice function, as that term is now commonly used to indicate a function for which the set of candidates is fixed, and we want to explicitly focus on the importance of candidate entry.

The above condition might be thought of as requiring that entry by all candidates be a Nash equilibrium. Such a condition is slightly weaker than candidate stability and may be described as follows.

A voting procedure V is *weakly candidate stable* if $V(\mathcal{C}, P) R_a V(\mathcal{C} \setminus \{a\}, P)$ for all $a \in \mathcal{C}$ and $P \in \mathcal{P}$.

Note that these conditions are weaker than requiring that it be a dominant strategy for candidates to enter, or requiring that the Nash equilibrium be unique.

3 The Implications of Candidate Stability

A voting procedure V is *dictatorial* if there exists a voter $i \in \mathcal{V}$ such that $V(\mathcal{C}, P) = top(\mathcal{C}, P_i)$ and $V(\mathcal{C} \setminus \{a\}, P) = top(\mathcal{C} \setminus \{a\}, P_i)$ for all $P \in \mathcal{P}$ and $a \in \mathcal{C}$.

THEOREM 1 *If a voting procedure is candidate stable and unanimous, then it is dictatorial.*[8]

Theorem 1 says that any voting procedure that satisfies unanimity and candidate stability must be dictatorial. The implication is that *every* non-dictatorial method by which a society elects a candidate will be open to strategic manipulation on the part of the candidates.

Remark that our conclusion of the voting procedure being dictatorial only holds on the sets $\mathcal{C}$ and $\mathcal{C} \setminus \{a\}$. If we impose candidate stability more broadly, then the conclusion of a dictatorial voting procedure would hold more broadly too.[9] This version of candidate stability seems more natural given that we are looking for all candidates entering to be a Nash equilibrium, as we explore in detail in Corollary 1 below.

The proof of Theorem 1 appears in the appendix.[10]

[8] A limited converse to this theorem holds as dictatorship only has implications on $\mathcal{C}$ and $\mathcal{C} \setminus \{a\}$ for each $a \in \mathcal{C}$, and so unanimity only holds for that part of the domain.

[9] Also, it is easy to show that under the much stronger condition that it is a dominant strategy for each candidate to enter, then V is dictatorial on all sets.

[10] We remark that if we changed unanimity to Pareto optimality, then we could use a theorem of Grether and Plott (1982) from the non-binary choice literature to prove this result. Although the Grether and Plott theorem is not concerned with candidate stability,

To see some of the logic behind the proof of Theorem 1, note that if such a voting procedure were rationalizable by some social welfare ordering (i.e., represented the choices consistent with some social welfare ordering), then one could apply Arrow's (1951) impossibility theorem to deduce the result. However, there are voting procedures that are unanimous and candidate stable, but are *not* rationalizable. Thus, the logic of Arrow's theorem cannot be directly applied. The way in which we prove Theorem 1 is to show that candidate stability and unanimity imply that the voting procedure is rationalizable on very restricted domains. In particular, the voting procedure is rationalizable on a domain where all voters find the same three alternatives most preferred, and agree with some fixed preference profile on other alternatives. We thus conclude that the voting procedure is dictatorial on such a restricted domain. We then tie such domains together through repeated application of candidate stability to show that the same voter must dictate on each such restricted domain. Finally, we use these conditions again to conclude that the same voter must dictate on the rest of the domain.

The important role of candidate stability in the proof of Theorem 1 can be seen by noting that there are many non-dictatorial voting procedures that satisfy unanimity (and even Pareto optimality), but not candidate stability. This follows from our earlier claim and the theorem above. It is easily illustrated, since voting by successive elimination defines a unanimous and non-dictatorial voting procedure. A specific example of voting by successive elimination as follows. (More complete and general definitions appear in Section 5.)

Example 1

Let $\mathcal{C} = \{a, b, c\}$, $\#\mathcal{V}$ be odd, and consider $A \subset \mathcal{C}$.

- If $\#A = 1$, then $V(A, P) = A$.
- If $\#A = 2$, then $V(A, P)$ is determined by majority rule.

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it uses a revealed preference axiom that is mathematically similar. Despite the similarity, in the absence the Pareto condition we have to pursue a very different route of proof, so that their theorem and even their methods turn out not to be useful in our proof. The difference between unanimity and the Pareto condition is very important to our approach. In fact, it is critical in the next section when we consider overlap in the set of candidates and voters, and a restricted domain for which Theorem 1 can be extended, but for which the Grether and Plott type of reasoning cannot.

- If $\#A = 3$, then $V(A, P)$ is determined by first holding a majority vote over a versus b, then matching the winner in a majority vote against c.

When $\#A = 3$, $V(A, P)$ can be selected either assuming sincere voting or sophisticated (i.e., strategic) voting and will still serve the purposes of this example. To keep things consistent, let us analyze what happens with sophisticated voting. V is clearly a voting procedure, satisfies unanimity, and is non-dictatorial. However, V is not candidate stable. This is seen, for instance when voters have the following preferences: aP_1bP_1c, bP_2cP_2a, and cP_3aP_3b. Then, b will be the outcome with sophisticated voting (1 and 2 vote for b in the first stage, since 1 knows that a would lose in the second stage). However, if c were to exit, then the outcome would be a since a majority prefers a to b, and so candidate stability is violated. Note that if c had the preferences of voter 3, then c would choose to exit.

4 A Restricted Domain of Candidate Preferences

In the previous section, there were no restrictions placed on the preferences of candidates. Nor were there any requirements concerning the overlap of voters and candidates. In some situations it is natural to assume that a candidate finds him or herself most preferred.[11]

So, we now consider the restricted domain

$$\mathcal{P}^r = \{P \in \mathcal{P} | a \in \mathcal{C} \Rightarrow aP_ab \ \forall b \in \mathcal{C}, b \neq a\}.$$

We first show that for the case where there is no overlap in candidates and voters, the results of the previous section carry over directly. In fact, the result holds even with weak candidate stability. This is a corollary to Theorem 1, as we now show.

COROLLARY 1 *If $\mathcal{C} \cap \mathcal{V} = \emptyset$ and V is weakly candidate stable and unanimous on $\mathcal{P}^r$, then V is dictatorial on $\mathcal{P}^r$.*

This follows from Theorem 1 and the following lemma.

[11]For instance, this will be true in the framework of Besley and Coate(1997), where each candidate is identified with her most preferred alternative in some policy space.

LEMMA 1 *If $\mathcal{C} \cap \mathcal{V} = \emptyset$ and V is weakly candidate stable on $\mathcal{P}^r$, then V is candidate stable.*

Proof of Lemma 1: Consider $a \in \mathcal{C}$ and $P \in \mathcal{P}^r$ such that $a \neq V(\mathcal{C}, P)$. We need to show that $V(\mathcal{C}, P) = V(\mathcal{C} \setminus \{a\}, P)$. Suppose to the contrary that $V(\mathcal{C}, P) \neq V(\mathcal{C} \setminus \{a\}, P)$. Since, $V(\mathcal{C}, P) R_a V(\mathcal{C} \setminus \{a\}, P)$, it must be that $V(\mathcal{C}, P) P_a V(\mathcal{C} \setminus \{a\}, P)$. Since $V(\mathcal{C}, P) \neq a$, we can find P'_a such that $P_{-a}, P'_a \in \mathcal{P}^r$ and $V(\mathcal{C} \setminus \{a\}, P) P'_a V(\mathcal{C}, P)$. Since V depends only on voters' preferences, it follows that $V(\mathcal{C} \setminus \{a\}, P_{-a}, P'_a) P'_a V(\mathcal{C}, P_{-a}, P'_a)$. This contradicts the fact that $\mathcal{C}$ is a candidate entry equilibrium, and so our supposition was incorrect. ∎

Corollary 1 deals with the case where $\mathcal{C} \cap \mathcal{V} = \emptyset$. With restricted candidate preferences, the case where $\mathcal{C} \cap \mathcal{V} \neq \emptyset$ can also be addressed, but with added complications. For instance, when the set of candidates is a subset of the set of voters, and the domain of candidates' preferences is restricted to $\mathcal{P}^r$, then the unanimity condition is vacuous! To understand why, note that if $P \in \mathcal{P}^r$, then each candidate must have a different most preferred candidate (him or herself), and thus there are no unanimous profiles.

In order to address this problem, we need to strengthen the unanimity condition if the domain is restricted to $\mathcal{P}^r$ when $\mathcal{C} \cap \mathcal{V} \neq \emptyset$.

Strong Unanimity: V satisfies *strong unanimity* if for all $B \subset \mathcal{C}$ and $P \in \mathcal{P}^r$, if $b \in B$ and $b = top(B, P_i)$ for each $i \in \mathcal{V} \setminus \mathcal{C}$ and $b = top(B \setminus \{a\}, P_a)$ for every $a \in \mathcal{C} \cap \mathcal{V}$, $a \neq b$, then $V(B, P) = b$.

This condition coincides with unanimity when $\mathcal{C} \cap \mathcal{V} = \emptyset$. More generally, it is a unanimity condition which ignores candidates' preferences for themselves.

Theorem 2 states that this strengthening of unanimity is enough to extend Theorem 1 to the restricted domain where candidates find themselves most preferred.

THEOREM 2 *On the domain $\mathcal{P}^r$, if a voting procedure is candidate stable and satisfies strong unanimity, then it is dictatorial and the dictator is in $\mathcal{V} \setminus \mathcal{C}$.*

Theorem 2 does not have any requirements regarding $\mathcal{C} \cap \mathcal{V}$, but requires the strong unanimity condition in order to address the restriction of candidates' preferences when $\mathcal{C} \cap \mathcal{V} \neq \emptyset$.

Note that Theorem 2 explicitly states that the dictator must be in $\mathcal{V} \setminus \mathcal{C}$. There is an obvious explanation for this requirement. If an individual $i \in \mathcal{V} \cap \mathcal{C}$ was the dictator, then i would always be the chosen outcome given that i prefers herself to all other candidates. But this would violate strong unanimity. Hence, one implication of the theorem is that if $\mathcal{V} = \mathcal{C}$, then there is no voting procedure that can satisfy the stated conditions.

The proof of Theorem 2 is very similar to the proof of Theorem 1, using strong unanimity in place of unanimity to draw implications on the restricted domain, $\mathcal{P}^r$.[12]

Theorem 2 requires candidate stability, while Corollary 1 only required weak candidate stability. For the case where $\mathcal{V} \cap \mathcal{C} \neq \emptyset$, this difference is important, as illustrated in the following voting procedure.

Example 2.

Let $\mathcal{C} = \{a, b, c\}$ and $\mathcal{V} = \{a, b, c, 1\}$. The voting procedure is described as follows. If only *two* candidates enter, then Voter 1 chooses the winner. If all the candidates enter, then Voter 1 determines an ordering of the three candidates $\{a, b, c\}$. Then, the candidates are voted on by successive elimination (as in example 1), but according to the ordering of the candidates suggested by voter 1. In particular, only the candidates a, b, and c get to vote in the successive elimination procedure. Thus, voter 1 only sets the ordering of the agenda. It is easily checked that if, for instance, candidate a is second most preferred by candidates b and c, then a is a Condorcet winner (considering only candidate preferences) and so regardless of voter 1's ordering of the agenda candidate a will be selected. Thus, a can be selected even when it is 1's worst alternative. Moreover, each candidate's least preferred alternative can be the outcome for some preference profile. It can be checked by direct calculation that this procedure is weakly candidate stable and strongly unanimous.

Thus, Example 2 shows that for the case where some candidates are voters (i.e., $\mathcal{V} \cap \mathcal{C} \neq \emptyset$), the difference between candidate stability and weak candidate stability allows for the existence of non-dictatorial and strongly

[12]The only exception is in the proof of Lemma 3, where we need to replace the appeal to Arrow's theorem, with an extension of his theorem. This extension essentially shows that in the presence of strong unanimity, the dictatorship result can still be derived despite the domain restriction.

unanimous voting procedures. Further work needs to be done in order to characterize the set of weakly candidate stable voting procedures.

5 Accounting for Strategic Candidacy

The results in the previous sections imply that strategic action on the part of candidates is unavoidable. This means that voting theory which treats the set of candidates as fixed and exogenous needs to be reconsidered. What are the implications of strategic decisions on the part of candidates? How greatly will their actions affect voting outcomes?

We begin answering these questions by examining a class of voting procedures that are often used in practice and have been well-studied. In particular we examine the well-known procedure called *voting by successive elimination* (also known as voting by successive amendments in some situations) when individuals vote sophisticatedly.[13]

Voting by Successive Elimination

We first describe the voting by successive elimination procedure. Let $\sigma : \mathcal{C} \rightarrow \{1, \ldots, \#\mathcal{C}\}$ be an ordering (where σ is one-to-one) of candidates. Let us refer to the candidates by $a_1, \ldots, a_{\#\mathcal{C}}$, where $\sigma(a_k) = k$. In the successive elimination procedure, a vote is first taken to eliminate either a_1 or a_2. The 'winning' candidate from the first round, denoted w_1, is compared to a_3, and a vote is taken to eliminate either w_1 or a_3, and so on. After $(K \Leftrightarrow 1)$ comparisons, the surviving candidate is declared to be the voting outcome.

At each stage, the elimination of one candidate is on the basis of majority voting. However, in order to determine the sophisticated voting outcome, it is also necessary to describe how voters act. If they vote strategically, then their votes at any stage is influenced by which candidates they believe the winner from the current stage will face in later stages. Clearly, at the last stage, when the comparison is between the two surviving candidates, each individual has a dominant strategy - to vote for his or her preferred candidate. Put differently, it is a dominated strategy to vote for the less preferred candidate in the last stage. Using this forecast, the voters can move back one stage and

[13]Sophisticated voting as defined by Farquharson (1969) is found by the iterative elimination of weakly dominated strategies, as analyzed in Moulin (1986). See Moulin (1988) for an overview of some of the literature on voting by successive elimination.

then essentially choose between two determinate outcomes in the penultimate stage. This can be rolled back, in effect successively eliminating weakly dominated strategies at each stage. Only one voting strategy profile survives this iteration on the elimination of dominated strategies, and the resulting winning candidate is the 'sophisticated' outcome.

Before we describe the algorithm for determining the sophisticated outcome of the successive amendments procedure, let us describe a generalization of the idea where one need not consider individual preferences, but instead only consider the *tournament* that they induce.[14]

Tournaments

A *tournament*, T, on $\mathcal{C}$ is a complete and asymmetric binary relation on $\mathcal{C}$.

Note that majority voting naturally induces a tournament when $\#\mathcal{V}$ is odd or a deterministic tie-breaking procedure is in place. The tournament $T(P)$ associated with the preference profile $P \in \mathcal{P}^r$ is the majority relation corresponding to P. That is, $T(P)$, is defined by

$$aT(P)b \Leftrightarrow \#\{i \in \mathcal{V} | aP_i b\} > \#\{i \in \mathcal{V} | bP_i a\}.$$

Let us make an important remark at this point. In the remainder of this section, we will treat the tournament and *candidate* preferences as independent objects. For this to be a valid exercise, we need one of two things to hold: either the voters who induce the tournament do not include the candidates (e.g., $\mathcal{V} \cap \mathcal{C} = \emptyset$), or other conditions are satisfied which assure that the candidates are a small enough minority so that changes in their preferences do not result in changes in the tournament. We identify such conditions in the next section. For now, the reader may just keep in mind the case where voters and candidates do not overlap.

Sophisticated Voting by Successive Elimination

Shepsle and Weingast (1984) defined an algorithm for determining the sophisticated outcome of the voting by successive elimination (or amendments) procedure for an arbitrary tournament T and ordering of candidates σ on a set $A \subset \mathcal{C}$. This is described as follows.

[14]See Laslier (1997) for an illuminating account of the principal results in the vast literature on tournaments.

The sophisticated voting procedure corresponding to sophisticated voting by a tournament T on a set of candidates A that is ordered by σ, denoted $S(A,T,\sigma)$, is w_1, where

$$\begin{array}{rcl} w_\ell &=& a_\ell, \text{ and} \\ \forall k < l, \quad w_k(A,T,\sigma) &=& \left\{ \begin{array}{lll} a_k & \text{if} & a_k T w_{k'} \forall k' > k \\ w_{k+1} & \text{otherwise} & \end{array} \right. , \end{array}$$

and where $a_1, \ldots, a_\ell$ is the ordering on A consistent with σ (and $\ell = \#A$).

In the case where $\mathcal{V} \cap \mathcal{C} = \emptyset$, then fixing the ordering of candidates σ results in $S(A, T(P), \sigma)$ being a voting procedure as A and P are varied.

Banks (1985) provides a characterization of the set of outcomes which can emerge as sophisticated outcomes of the voting by successive elimination (or amendments) procedure when every possible ordering of a given feasible set $\mathcal{C}$ of candidates is taken into account.

The Banks set

The *Banks set* associated with a tournament T, denoted $BS(T)$, is defined by

$$BS(T) = \{a | \exists \sigma \text{ s.t. } a = S(\mathcal{C}, T, \sigma)\}.$$

In order to show the characterization of this set given by Banks (1985), we need additional definitions.

A *chain* of T is a set $H \subset \mathcal{C}$ such that T is a transitive relation when restricted to H.

Thus, a chain is a collection of candidates that can be ordered so that each candidate in the order beats all the following candidates in the order. Note that a tournament T is generally not transitive on all of $\mathcal{C}$. Thus, chains are generally strict subsets of $\mathcal{C}$.

Given a candidate $a \in \mathcal{C}$, an *a-chain* of T is a chain H with $a \in H$ such that aTb for all $b \in H$. The set of all a-chains is denoted $H(a,T)$.

Thus, an a-chain is a chain where a beats all the other candidates in the chain

The result reported by Banks (1985) can now be stated as follows:

Theorem (Banks 1985)

$$BS(T) = \{a | \exists H \in H(a,T) \text{ s.t. } \forall b \notin H \exists c \in H \text{ s.t. } cTb\}.$$

Thus, Banks showed that the outcomes that could be elected by varying the ordering (for a fixed tournament) when voting by successive elimination correspond to the endpoints of chains, where the chains are such that any candidate not included in the chain is beaten by something in the chain. The intuition behind the characterization is that the candidates in the chain represent the candidates who temporarily "win" at some stage in the voting by successive elimination, and the remaining candidates are those who are eliminated at their stages.

What we do here is to re-examine the characterization of outcomes that can be elected under voting by successive elimination, when one accounts for the strategic choices of candidates.

Equilibrium of Candidate Entry

We now define a candidate entry equilibrium for the case where a tournament is fixed and independent of the candidates' preferences.

Given a candidate preference profile P, a tournament T, and ordering σ, a candidate c is an *entry equilibrium outcome* if there exists $A \subset \mathcal{C}$ such that

- $c = S(A, T, \sigma)$,
- $S(A, T, \sigma) R_a S(A \setminus \{a\}, T, \sigma)$ for all $a \in A$, and
- $S(A, T, \sigma) R_b S(A \cup \{b\}, T, \sigma)$ for all $b \in \mathcal{C} \setminus A$.

Thus, an entry equilibrium outcome is an outcome where the set of candidates entering is an entry (Nash) equilibrium, given the resulting sophisticated choice made by the voting by successive amendments procedure.

We denote the set of entry equilibrium outcomes by $E(P, T, \sigma)$.

We now define the analog of the Banks set accounting for the possibility of strategic choices of candidates. We call the following CS, the *candidate stable set*.

The Candidate Stable Set

The *candidate stable set* associated with a tournament T, denoted $CS(T)$, is defined by

$$CS(T) = \{a \in \mathcal{C} | \exists \sigma, P \in \mathcal{P}^r \text{ s.t. } a \in E(P, T, \sigma)\}.$$

Thus, the candidate stable set is found by not only varying the ordering on candidates, but also accounting for strategic choice on the part of candidates for the preferences that they might have, holding the tournament fixed. (Again, see the next section for a discussion of conditions under which the tournament and candidate preferences can be treated as independent.)

We can now state a characterization of the candidate stable set, which has a close intuitive relationship to the Bank's set.

Given a tournament T, and $\{a, b\} \subset \mathcal{C}$, b *covers* a if bTa and bTc for all $c \in \mathcal{C}$ such that aTc.

THEOREM **3** *The candidate stable set may be characterized as follows:*

$$CS(T) = \{a \in \mathcal{C} |\ \exists H \in H(a, T) \text{ s.t. } \forall b \notin H\ \exists c \in H \text{ s.t. } b \text{ does not cover } c\}.$$

Theorem 3 shows that a simple characterization may be found for the candidate stable set, and thus we can account for the strategic impact of candidates' choices and preferences. The characterization bears an intuitive relationship to the characterization of the Banks set. The only change in the characterization is that b not cover c replaces cTb (which may be thought of as "b not Tc"). This enlarges the set of candidates that may be realized, as it follows directly that $BS(T) \subset CS(T)$, since cTb implies b not cover c.

The proof of Theorem 3 appears in the appendix. The intuition for the theorem is as follows. First, consider a in the candidate stable set. Let H be the chain corresponding to the candidates that a beats in the elimination procedure. There cannot exist b that covers every c in H, otherwise b would be the outcome of the procedure if b entered. To see the converse, order the elimination procedure according to H, with elements not in H but beaten by an element appearing before H in the ordering, and the remaining elements appearing after H in the ordering. Then it is an equilibrium for only candidates in H and lower in the ordering to enter, and for the remaining candidates not to enter as given their position in the ordering, they can only win if they beat all candidates in H and those beaten by a candidate in H.

We can say much more about the relationship between the two sets, and well-known sets such as the uncovered set and the top-cycle set. To give precise definitions, consider the following.

Let aT^kb for some $k \in \{1, \ldots, \#\mathcal{C} \Leftrightarrow 1\}$ denote the situation where there exists a sequence of candidates $a = a_1, \ldots, a_{k'} = b$ with $k' \leq k + 1$ and a_hTa_{h+1} for each $h \in \{1, \ldots, k' \Leftrightarrow 1\}$.

Thus, aT^kb if one can find a string of candidates of length no more than $k+1$ linking a to b, such that each candidate in the string beats the next candidate in the string.

Let $A^k(T) = \{a | aT^kb \; \forall b \neq a\}$.

Thus, $A^1(T)$ is the Condorcet winner, which is often non-existent. $A^2(T)$ is the *uncovered set*, as it is easily seen to be precisely the set of alternatives that are not covered. and $A^{\#\mathcal{C}-1}$ is the *top-cycle* set.

The following relationship then holds. Let UC denote the uncovered set and TC denote the top cycle set.

THEOREM 4

$$BS(T) \subset UC(T) \subset CS(T) \subset TC(T).$$

More specifically:

$$A^1(T) \subset BS(T) \subset A^2(T) = UC(T) \subset CS(T) \subset A^3(T) \subset A^{\#\mathcal{C}-1}(T) = TC(T).$$

There are examples where each relationship is strict.

Proof of Theorem 4: We show that $A^2(T) \subset CS(T) \subset A^3(T)$, and that these inclusions can be strict. The other relationships are known (e.g., see Banks (1985)).

First, it is clear that $A^2(T) = UC(T) \subset CS(T)$, directly from the characterization given in Theorem 3, as any uncovered a is in $CS(T)$ using $H = \{a\}$. Second, we show that $CS(T) \subset A^3(T)$. Consider any $a \in CS(T)$. We need to show that for every $b \neq a$, aT^3b. According to Theorem 3, consider $H \in H(a,T)$ such that for each $b \notin H$ there exists $c \in H$ with b not covering c. Note that for each $b \neq a$, $b \in H$, aTb, since $H \in H(a,T)$. So, consider $b \notin H$. Since there exists $c \in H$ with b not covering c, it follows that there exists $c \in H$ with cT^2b. Since $c \in H$, it follows that either $c = a$, or aTc. Thus, aT^3b.

The fact that these inclusions can be strict is illustrated in Examples 3 and 4. Example 4 (see below) shows that there exists T for which the uncovered set and candidate stable set are distinct. The next example shows a T for which $CS(T) \neq A^3(T)$.

Example 3 Let $\mathcal{C} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$, and T be given by

- $a_1 T a_2,\ a_1 T a_3,$
- $a_2 T a_4,$
- $a_3 T a_5,$
- $a_4 T a_1,\ a_4 T a_6,$
- $a_5 T a_1,\ a_5 T a_7,$
- $a_6 T a_1,\ a_6 T a_3,\ a_6 T a_4,\ a_6 T a_5,$
- $a_7 T a_1,\ a_7 T a_2,\ a_7 T a_3,\ a_7 T a_4,$

The remaining comparisons are inconsequential to the point. Note that $a_1 \in A^3(T)$ since $a_1 T a_2 T a_4 T a_6$ and $a_1 T a_3 T a_5 T a_7$. However, $a_1 \notin CS(T)$. Note that $H(a_1, T) = \{\{a_1\}, \{a_1, a_2\}, \{a_1, a_3\}\}$. (For instance, $\{a_1, a_2, a_4\} \notin H(a_1, T)$ since T is not transitive on $\{a_1, a_2, a_4\}$.) Consider a_7. Note that a_7 covers a_1 and a_2. Thus, we need to set $H = \{a_1, a_3\}$ so that a_7 does not cover all candidates in H. However, note that a_6 covers a_1 and a_3. Thus, the conditions for a_1 to be in $CS(T)$ cannot be satisfied. ∎

The consideration of strategic candidacy has expanded the set of potential elected candidates, although in a very well-defined way.

The following example[15] shows that the expansion due to accounting for strategic candidacy has some negative implications. In particular, since the uncovered set contains only Pareto optimal outcomes (with respect to voters' preferences for the corresponding tournament), the Bank's Set also only contains Pareto optimal outcomes. However, it turns out that the candidate stable set can include candidates that are not Pareto optimal. Thus, the strategic actions on the part of candidates may result in Pareto inferior candidates being chosen, from the *voters'* perspective.

Notice, however, that the restriction on candidates' preferences implies *every* candidate is Pareto optimal when candidates' preferences are taken into account. Pareto optimality needs only to ignore this self-preference on the part of candidates, in order for this example to hold.

Example 4. Consider the set of candidates $\mathcal{C} = \{a, b, c, d\}$ and the set of voters $\mathcal{V} = \{1, 2, 3\}$. Suppose that voters' preferences are given by $c P_1 d P_1 a P_1 b$,

[15]We thank Jeff Banks for pointing this example out to us.

$dP_2aP_2bP_2c$, and $bP_3cP_3dP_3a$. Majority voting results in the tournament that has $aT(P)bT(P)cT(P)dT(P)a$, $dT(P)b$ and $cT(P)a$. It is easily checked from Theorem 3 that $a \in CS(T(P))$, since for the a-chain $H = \{a, b\}$: $bT(P)c$ and so c cannot cover b, and d does not cover b since $bT(P)cT(P)d$. However, a is Pareto dominated by d.

6 Concluding Remarks

The results that we have presented here suggest that strategic candidacy is always important, and in at least some cases of interest can be nicely accounted for. Our analysis here is thus a first step in a much larger analysis of endogenous candidacy in voting rules. While concluding remarks are often short, we break with that tradition and present here a series of observations and results that are important both for interpreting the previous analysis, and engaging in a broader analysis of endogenous and strategic candidacy.

Tournaments and Preferences

In our characterization of the candidate stable set, we treated T as fixed, while we varied the preferences of candidates. If we are thinking of T as being derived from the preferences of voters, then there are implicit conditions required for this to be a valid exercise.

If T is generated by the preferences of voters, then this exercise is clearly valid if $\mathcal{C} \cap \mathcal{V} = \emptyset$. It is also valid if $P_{-\mathcal{C}}$ is such that $T(P)$ turns out to be independent of $P_{\mathcal{C}}$. This situation applies for certain preferences of the voters in $\mathcal{N} \setminus \mathcal{C}$, but generally requires that the number of candidates be small relative to the number of voters.

Note that there are certain profiles of preferences of voters outside of the set of candidates, for which the tournament will always be affected by changes in the candidates' preferences. For example, suppose that there is a very large number of voters relative to the three candidates a, b, c. There are six possible preferences over candidates, and suppose that the population of voters is exactly evenly divided in terms of the preferences they have (one sixth of the population having each preference). Then regardless of how large the population is, the candidates will always be pivotal.

The following proposition (in the spirit of McGarvey (1953)) provides a bound on the number of voters required to ensure that any arbitrary tournament can be obtained as the majority relation corresponding to *some* profile of preferences of the voters not in $\mathcal{C}$, and independent of the preferences of individuals in $\mathcal{C} \cap \mathcal{V}$.

PROPOSITION **1** *Let $\#\mathcal{C} = m$ and $\#(\mathcal{C} \cap \mathcal{V}) = k$. If k is even (respectively odd), and if $\#\mathcal{V} \geq \frac{(k+2)(m-1)m}{2} + k$ (resp. $\#\mathcal{V} \geq \frac{(k+3)(m-1)m}{2} + k$), then for any arbitrary tournament T over $\mathcal{C}$ and $P_{\mathcal{V}\cap\mathcal{C}} = (P_i)_{i\in\mathcal{V}\cap\mathcal{C}}$, there exists $\bar{P}_{\mathcal{V}\setminus\mathcal{C}}$ such that $T = T(\bar{P}_{\mathcal{V}\setminus\mathcal{C}}, P_{\mathcal{V}\cap\mathcal{C}})$.*

Proof of Proposition 1: Assume that k is even, and take $a, b \in \mathcal{C}$ with aTb. Let $\{a_1, \ldots, a_{m-2}\} = \mathcal{C} \setminus \{a, b\}$, and consider the two linear orderings P_{ab} and P'_{ab} defined as follows:

- P_{ab} : $a\ b\ a_1\ a_2 \ldots a_{m-2}$
- P'_{ab} : $a_{m-2}\ a_{m-3} \ldots a_1\ a\ b$

Consider $\frac{k+2}{2}$ individuals in $\mathcal{V} \setminus \mathcal{C}$ with the preference P_{ab} and $\frac{k+2}{2}$ individuals in $\mathcal{V} \setminus \mathcal{C}$ with the preference P'_{ab}.

Repeat the operation for all pairs of candidates. We have assigned a preference to $\frac{(k+2)(m-1)m}{2}$ voters in $\mathcal{V} \setminus \mathcal{C}$. This is possible since $\#\mathcal{V} \geq \frac{(k+2)(m-1)m}{2} + k$. Suppose $(\#\mathcal{V} \Leftrightarrow k) \Leftrightarrow \frac{(k+2)(m-1)m}{2} = r > 0$. Then, if r is even, endow each extra pair of voters with opposite preferences. If r is odd, then endow each extra pair of voters with opposite preferences, and the remaining with an arbitrary preference.

The reader can check that this construction yields the desired majority relation. ∎

Existence of pure strategy equilibria in candidate entry

In our analysis of candidate entry, we have focused on pure strategies on the part of candidates. However, there may be voting procedures for which there are no pure strategy entry equilibria. Example 5 below shows that there may even be a voting procedure induced by voting on a tree for which there are no pure strategy entry equilibria.

Example 5.

Consider $\mathcal{C} = \{a, b, c, d\}$.[16] Consider the voting procedure described by the extensive game form pictured in Figure 1. If a candidate does not enter, then just trim the tree to eliminate all terminal nodes that result in that candidate.

Consider the following preference profile. Preferences of the voters $i \in \mathcal{V}$ are described by

- $bP_1dP_1aP_1c$
- $dP_2cP_2bP_2a$
- $aP_3dP_3cP_3b$
- $aP_4bP_4cP_4d$

The preferences of the candidates in $\mathcal{C}$ are described by

- $aP_abP_adP_ac$
- $bP_baP_bcP_bd$
- $cP_cbP_caP_cd$
- $dP_daP_dbP_dc$

Here, $\mathcal{V} \cap \mathcal{C} = \emptyset$.

Direct calculation shows that there is no $A \subset \mathcal{C}$ for which having exactly the set of candidates in A enter is a candidate entry equilibrium. For example, if all candidates enter, the outcome is a while if c exits then the outcome is b and so c would prefer not to enter. If $\{a, b, d\}$ enter the outcome is b, while if only $\{a, b\}$ enter, then the outcome is $\{a\}$. Thus d would choose not to enter and so $\{a, b, d\}$ is not an equilibrium. Similar calculations show that there is no A that is a pure strategy entry equilibrium.

Example 5 is not representative of all voting procedures, as there are some for which there always exist pure strategy entry equilibria. The candidate entry game associated with voting by successive elimination procedure

[16] It is necessary to this example to have 4 candidates, as with three candidates and any voting procedure an equilibrium can be shown to always exist.

has a pure strategy entry equilibrium for *all* orders of candidates, for *all* touraments, and for *all* preferences of candidates.

We make a further observation on this subject. While voting by successive elimination always has a pure strategy entry equilibrium, Example 6 shows that existence of *undominated* Nash equilibria in pure strategies is not assured. First, we state the following without proof.

Claim. Fix $\sigma : \mathcal{C} \to \{1, 2, \ldots, \#\mathcal{C}\}$, and define $a_1, \ldots, a_{\#\mathcal{C}}$ to be such that $\sigma(a_i) = i$. Then, the "no entry" strategy is weakly dominated for candidates a_1 and a_2.

The claim is true because candidates a_1 and a_2 can change the outcome of the election by dropping out *only* when they win the election by contesting.

Example 6. Let $\#\mathcal{C} = 5$. As in the claim above, let σ be such that candidate a_i comes before candidate a_{i+1} for all $i = 1, 2, 3, 4$. So, Claim 1 establishes that a_1 and a_2 always enter the contest in an undominated Nash equilibrium.

Let T denote the tournament, which is described (partially) below. Here, we define $W(a_i) = \{a_j | a_i T a_j\}$. That is, $W(a_i)$ is the set of candidates which are "worse" than a_i according to the tournament T.

- $a_4 \in W(a_1)$
- $a_3, a_1 \in W(a_2)$
- $a_1, a_4 \in W(a_3)$
- $a_2, a_5 \in W(a_4)$
- $a_1, a_2, a_3 \in W(a_5)$

For any $B \subset \mathcal{C}$, let $w(B)$ denote the winner of the contest under the amendment procedure given this tournament. Then, the following statements are true.

(1) $w(\mathcal{C}) = a_4$

(2) $w(\{a_1, a_2, a_3, a_4\}) = a_3$

(3) $w(\{a_1, a_2, a_4, a_5\}) = a_4$

(4) $w(\{a_1, a_2, a_3, a_5\}) = a_5$

(5) $w(\{a_1, a_2, a_3\}) = a_2$

(6) $w(\{a_1, a_2, a_4\}) = a_1$

(7) $w(\{a_1, a_2, a_5\}) = a_5$

(8) $w(\{a_1, a_2\}) = a_2$

Assuming that $a_3 P_{a_5} a_4$, (1) cannot be an equilibrium because a_5 can ensure a_3's victory by dropping out of the contest. Similarly, by assuming that $a_2 P_{a_4} a_3$, (2) cannot be an equilibrium. Also, assuming that $a_1 P_{a_5} a_4$, we can conclude that (3) is not an equilibrium. Finally, note that cases (4)-(8) cannot be equilibria because in each some candidate can enter and win the election.

Since a_1 and a_2 must enter the contest at any undominated Nash equilibrium, we have therefore shown that there is no pure strategy undominated Nash equilibrium in this example.

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Appendix

The following choice axiom is a single-valued version of axioms in Chernoff (1954) and Arrow (1959).[17]

Choice Axiom (CA): V satisfies the *choice axiom* if $\forall P \in \mathcal{P}$ and $A, B \subset \mathcal{C}$, if $B \subset A$ and $V(A, P) \in B$, then $V(A, P) = V(B, P)$.

Two other definitions will be useful in the proofs that follow.

$B \subset \mathcal{C}$ is a *top*$\Leftrightarrow$*set* for $P \in \mathcal{P}$ if for every $i \in \mathcal{V}$: $bP_i a$ for every $b \in B$ and $a \notin B$.

Consider any three distinct candidates $a, b, c \in \mathcal{C}$ and any preference profile $\overline{P} \in \mathcal{P}$. Let $\mathcal{P}_{\{a,b,c\}}(\overline{P})$ be the set of $P \in \mathcal{P}$ such that for each $i \in \mathcal{V}$

(i) $P_i|_{\mathcal{C}\setminus\{a,b,c\}} = \overline{P}_i|_{\mathcal{C}\setminus\{a,b,c\}}$, and

(ii) $\{a, b, c\}$ is a top-set for P.

Proof of Theorem 1: We establish the theorem from the following sequence of lemmas.

Lemma 2 *Consider any three distinct candidates $a, b, c \in \mathcal{C}$ and $\overline{P} \in \mathcal{P}$. If V satisfies candidate stability and unanimity, then $V(\mathcal{C}, P) \in \{a, b, c\}$ for any $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$.*

Proof of Lemma 2: Pick any $a, b, c \in \mathcal{C}$ and $\overline{P} \in \mathcal{P}$. Consider $P' \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$ such that $\{a, b\}$ is a top-set. Either $V(\mathcal{C}, P') \neq b$ or $V(\mathcal{C}, P') \neq a$. Without loss of generality, suppose that $V(\mathcal{C}, P') \neq b$. By candidate stability $V(\mathcal{C}\setminus\{b\}, P') = V(\mathcal{C}, P')$. By unanimity $V(\mathcal{C}\setminus\{b\}, P') = a$. Combining these two equalities implies that $V(\mathcal{C}, P') = a$. Since the choice of a and b was arbitrary, it follows that if $\{d, e\} \subset \{a, b, c\}$ is a top-set of $P' \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$, then $V(\mathcal{C}, P') \in \{d, e\}$.

So, consider any $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$. Suppose to the contrary of the lemma that $V(\mathcal{C}, P) = f \notin \{a, b, c\}$. By candidate stability it follows that $V(\mathcal{C} \setminus \{c\}, P) = f$. Consider $P' \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$ such that $P'|_{\mathcal{C}\setminus\{c\}} = P|_{\mathcal{C}\setminus\{c\}}$, and $\{a, b\}$

[17] This choice axiom is also equivalent to what Nash (1950) called Independence of Irrelevant Alternatives.

is a top-set for P' (so we have found P' from P by moving c to third position in each preference ranking and leaving the other relative rankings unchanged). It follows from (iii) in the definition of voting procedure that $V(\mathcal{C}\setminus\{c\}, P') = f$. From our previous argument we also know that $V(\mathcal{C}, P') \in \{a, b\}$. This contradicts candidate stability, since $f \notin \{a, b\}$. Thus, our supposition was wrong. ▮

LEMMA **3** *Consider any three distinct candidates $a, b, c \in \mathcal{C}$ and $\overline{P} \in \mathcal{P}$. If V satisfies candidate stability and unanimity, then V is dictatorial on $\mathcal{P}_{\{a,b,c\}}(\overline{P})$.*[18]

Proof of Lemma 3: Consider any three distinct candidates $a, b, c \in \mathcal{C}$ and $\overline{P} \in \mathcal{P}$.

First, note that unanimity implies that $V(\mathcal{C} \setminus \{a, b\}, P) = c$, $V(\mathcal{C} \setminus \{a, c\}, P) = b$, and $V(\mathcal{C} \setminus \{b, c\}, P) = a$, for any $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$. Next, it follows from Lemma 2 that $V(\mathcal{C}, P) \in \{a, b, c\}$ for all $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$. Candidate stability implies that $V(\mathcal{C}, P) = a \Rightarrow V(\mathcal{C}\setminus\{b\}, P) = V(\mathcal{C}\setminus\{c\}, P) = a$, and likewise that $V(\mathcal{C}, P) = b \Rightarrow V(\mathcal{C} \setminus \{a\}, P) = V(\mathcal{C} \setminus \{c\}, P) = b$, and $V(\mathcal{C}, P) = c \Rightarrow V(\mathcal{C} \setminus \{a\}, P) = V(\mathcal{C} \setminus \{b\}, P) = c$. Thus, it follows that $V(\mathcal{C} \setminus \{d\}, P) \in \{a, b, c\}$ for all $d \in \{a, b, c\}$ and $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$.

Let $\mathcal{P}^{\mathcal{V}}_{\{a,b,c\}}$ denote the set of all strict preferences (linear orders) of voters over the set $\{a, b, c\}$. Define $\widehat{V} : 2^{\{a,b,c\}} \setminus \emptyset \times \mathcal{P}^{\mathcal{V}}_{\{a,b,c\}} \to \{a, b, c\}$, by

$$\widehat{V}(B, P|^{\mathcal{V}}_{\{a,b,c\}}) = V(B \cup (\mathcal{C} \setminus \{a, b, c\}), P)$$

for each $B \subset \{a, b, c\}$ ($B \neq \emptyset$), and $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$, where $P|^{\mathcal{V}}_{\{a,b,c\}}$ denotes the profile of preferences for $i \in \mathcal{V}$ restricted to $\{a, b, c\}$, induced by P. This is well defined, since $V(\mathcal{C}, P) \in \{a, b, c\}$, $V(\mathcal{C} \setminus \{d\}, P) \in \{a, b, c\}$, and $V(\mathcal{C} \setminus \{d, e\}, P) \in \{a, b, c\}$ for all $\{d, e\} \subset \{a, b, c\}$ and $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$. Moreover, by the arguments above, the choice axiom is satisfied by $\widehat{V}$ relative to $\{a, b, c\}$ on $\mathcal{P}^{\mathcal{V}}_{\{a,b,c\}}$. This implies, by a theorem of Sen (1971), that $\widehat{V}(\cdot, P)$ is rationalizable by a linear order (see Moulin (1988), page 308) for any P. Also, since V satisfies unanimity and (iii) in the definition of voting procedure, it

[18] There exists a voter $i \in \mathcal{V}$ such that $V(\mathcal{C}, P) = top(\mathcal{C}, P_i)$, $V(\mathcal{C} \setminus \{d\}, P) = top(\mathcal{C} \setminus \{d\}, P_i)$, for all $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$ and $d \in \mathcal{C}$.

follows that $\widehat{V}$ satisfies unanimity and Arrow's independence of irrelevant alternatives relative to $\{a, b, c\}$ on $\mathcal{P}^{\mathcal{V}}_{\{a,b,c\}}$. It then follows from Arrow's theorem (noting that on this domain unanimity and the choice axiom imply Pareto efficiency) that $\widehat{V}$ is dictatorial on $\mathcal{P}^{\mathcal{V}}_{\{a,b,c\}}$. Thus, from the definition of $\widehat{V}$ it follows that there exists $i \in \mathcal{V}$ such that $V(\mathcal{C}, P) = top(\mathcal{C}, P_i)$ and $V(\mathcal{C} \setminus \{d\}, P) = top(\mathcal{C} \setminus \{d\}, P_i)$ for all $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$ and $d \in \{a, b, c\}$. Then from candidate stability it follows that $V(\mathcal{C} \setminus \{d\}, P) = top(\mathcal{C} \setminus \{d\}, P_i)$ for all $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$ and $d \notin \{a, b, c\}$. ∎

Let
$$\mathcal{P}_{\{a,b,c\}} = \{P \in \mathcal{P} \mid \{a, b, c\} \text{ is a top} \Leftrightarrow \text{set for } P\}.$$

LEMMA 4 *Consider any three distinct candidates $a, b, c \in \mathcal{C}$. If V satisfies candidate stability and unanimity, then V is dictatorial on $\mathcal{P}_{\{a,b,c\}}$.*

Lemma 4 is stronger than Lemma 3 in that it applies to a larger set of preferences ($\mathcal{P}_{\{a,b,c\}}$ instead of $\mathcal{P}_{\{a,b,c\}}(\overline{P})$).

Proof of Lemma 4. Consider $\{a, b, c\}$, $\overline{P}$ and $d \notin \{a, b, c\}$. Let $\tilde{P}$ be such that $\overline{P}|_{\mathcal{C}-d} = \tilde{P}|_{\mathcal{C}-d}$. By lemma 3 there exists a dictator $i \in \mathcal{V}$ on $\mathcal{P}_{\{a,b,c\}}(\overline{P})$. Similarly, there exists a dictator $j \in \mathcal{V}$ on $\mathcal{P}_{\{a,b,c\}}(\tilde{P})$. To establish the lemma, we need only show that $i = j$, since $\overline{P}$ and d are arbitrary (and this logic can be applied iteratively).

Suppose to the contrary that $i \neq j$. Consider $P \in \mathcal{P}_{\{a,b,c\}}(\overline{P})$ with $a = top(\mathcal{C}, P_i)$ and $b = top(\mathcal{C}, P_j)$, and take any $P' \in \mathcal{P}_{\{a,b,c\}}(\tilde{P})$ such that $P|_{\mathcal{C}-d} = P'|_{\mathcal{C}-d}$. It follows that $V(\mathcal{C}, P) = a$ and $V(\mathcal{C}, P') = b$, and so candidate stability implies that $V(\mathcal{C} \setminus \{d\}, P) = a$ and $V(\mathcal{C} \setminus \{d\}, P') = b$. However, this contradicts the fact that V satisfies (iii) in the definition of voting procedure, since $P|_{\mathcal{C}-d} = P'|_{\mathcal{C}-d}$. ∎

LEMMA 5 *If V satisfies candidate stability and unanimity, then V is dictatorial on $\mathcal{P}_{\{a,b,c\}}$ for every $\{a, b, c\} \subset \mathcal{C}$ (with the same dictator on each of these domains).*

Lemma 5 is stronger than lemma 4, since it implies that the same i is dictator on $\mathcal{P}_{\{a,b,c\}}$ for every $\{a, b, c\} \subset \mathcal{C}$.

Proof of Lemma 5. This follows directly if $\#\mathcal{C} = 3$. So suppose that $\#\mathcal{C} \geq 4$. It is enough to consider any $\{a, b, d\}$ distinct from $\{a, b, c\}$, and show that the same voter dictates on $\mathcal{P}_{\{a,b,c\}}$ and $\mathcal{P}_{\{a,b,d\}}$. By lemma 4, there exists a dictator j on $\mathcal{P}_{\{a,b,d\}}$ and a dictator i on $\mathcal{P}_{\{a,b,c\}}$. Suppose to the contrary that $i \neq j$. Consider $P \in \mathcal{P}_{\{a,b,c\}}$ with $\{a,b,c,d\}$ a top-set for P and with $aP_ibP_icP_id$, $bP_jaP_jcP_jd$, and $aP_kbP_kcP_kd$ for any $k \in \mathcal{V} \setminus \{i,j\}$. Consider $P' \in \mathcal{P}_{\{a,b,d\}}$ such that $P|_{\mathcal{C}\setminus d} = P'|_{\mathcal{C}\setminus d}$, and $P|_{\mathcal{C}\setminus c} = P'|_{\mathcal{C}\setminus c}$. Thus, we have only reversed the place of c and d in the rankings. It follows that $V(\mathcal{C},P) = a$ and $V(\mathcal{C},P') = b$, and so candidate stability implies that $V(\mathcal{C}\setminus\{d\},P) = a$ and $V(\mathcal{C}\setminus\{d\},P') = b$. However, this contradicts the fact that V satisfies (iii) in the definition of voting procedure, since $P|_{\mathcal{C}-d} = P'|_{\mathcal{C}-d}$. ∎

We now complete the proof of Theorem 1. Find i from Lemma 5, and consider any $P \in \mathcal{P}$. Without loss of generality suppose that $\{a,b,c\}$ is top relative to P_i and that $top(\mathcal{C},P_i) = a$. We need to show that $V(\mathcal{C},P) = top(\mathcal{C},P_i) = a$, and $V(\mathcal{C}\setminus\{d\},P) = top(\mathcal{C}\setminus\{d\},P_i)$, for any $d \in \mathcal{C}$.

Consider P'_{-i} such that $P'_{-i}|_{\{a,b,c\}} = P_{-i}|_{\{a,b,c\}}$, $P'_{-i}|_{\mathcal{C}-\{a,b,c\}} = P_{-i}|_{\mathcal{C}-\{a,b,c\}}$, and $P_i, P'_{-i} \in \mathcal{P}_{\{a,b,c\}}$ (so $\{a,b,c\}$ is a top set for P'_{-i}). It follows from Lemma 5 that $V(\mathcal{C},P_i,P'_{-i}) = top(\mathcal{C},P_i) = a$, and $V(\mathcal{C}\setminus\{d\},P_i,P'_{-i}) = top(\mathcal{C}\setminus\{d\},P_i)$, for any $d \in \mathcal{C}$. Find a voter j and alternatives $f = bottom(\{a,b,c\},P_j)$ and $e = top(\mathcal{C}\setminus\{a,b,c\},P_j)$ such that eP_jf and fP'_je. (Clearly $j \neq i$.) Consider P''_j which agrees with P'_j on $\mathcal{C}\setminus\{e\}$ and $\mathcal{C}\setminus\{f\}$, and agrees with P_j on $\{e,f\}$. (Thus, we have only switched e and f in the ranking of j.) Here $V(\mathcal{C},P_i,P'_{-i}) = top(\mathcal{C},P_i) = a = V(\mathcal{C}\setminus\{e\},P_i,P'_{-i})$. Since P''_j agrees with P'_j on $\mathcal{C}\setminus\{e\}$ it follows from (iii) in the definition of voting procedure that $a = V(\mathcal{C}\setminus\{e\},P_i,P'_{-i,j},P''_j)$. Thus, from candidate stability it follows that $V(\mathcal{C},P_i,P'_{-i,j},P''_j) \in \{a,e\}$. Consider two cases:

Case 1. $f = a$

Since P''_j agrees with P'_j on $\mathcal{C}\setminus\{f\}$, (iii) in the definition of voting procedure implies that $V(\mathcal{C}\setminus\{f\},P_i,P'_{-i,j},P''_j) = V(\mathcal{C}\setminus\{f\},P_i,P'_{-i}) = b$. Then, since V is candidate stable it must be that $a = V(\mathcal{C},P_i,P'_{-i,j},P''_j)$. In this case, candidate stability also implies that $a = V(\mathcal{C}\setminus\{d\},P_i,P'_{-i,j},P''_j)$, for any $d \neq a$, and since we know that $f = a$, it follows that $V(\mathcal{C}\setminus\{a\},P_i,P'_{-i,j},P''_j) = b$. The last two sentences imply that in this case i dictates at $P_i,P'_{-i,j},P''_j$.

Case 2. $f \neq a$.

Since P''_j agrees with P'_j on $\mathcal{C}\setminus\{f\}$, (iii) in the definition of voting procedure implies that $a = V(\mathcal{C}\setminus\{f\},P_i,P'_{-i,j},P''_j)$. So, from candidate stability it follows that $V(\mathcal{C},P_i,P'_{-i,j},P''_j) \in \{a,f\}$. Thus, since we also know that $V(\mathcal{C},P_i,P'_{-i,j},P''_j) \in \{a,e\}$, it follows that $V(\mathcal{C},P_i,P'_{-i,j},P''_j) = a$. Candidate stability then implies that $a = V(\mathcal{C} \setminus \{d\}, P_i, P'_{-i,j}, P''_j)$, for any $d \neq a$, for any $d \neq a$. To show that in this case i dictates at $P_i, P'_{-i,j}, P''_j$, it is only left to show that $b = V(\mathcal{C} \setminus \{a\}, P_i, P'_{-i,j}, P''_j)$. Notice that by the same reasoning that we have applied up to this point, we can conclude that $b = V(\mathcal{C} \setminus \{a\}, P'''_i, P'_{-i,j}, P''_j)$, where P'''_i differs from P_i only in switching the ranking of a and b. Thus, it follows from AIIA that $b = V(\mathcal{C} \setminus \{a\}, P_i, P'_{-i,j}, P''_j)$.

This argument can be repeated, with one such change at each stage for some j between an $f \in \{a, b, c\}$ and $e \notin \{a, b, c\}$, to complete the transition from P'_{-i} to P_{-i}. ∎

The proof of Theorem 2 is exactly analogous to the proof of Theorem 1, except that we cannot invoke Arrow's theorem as we did in the last paragraph of the proof of Lemma 3, because of the restricted domain. However, we show in the following lemma that Arrow's theorem extends to this domain under the strong unanimity condition.

Let $\mathcal{P}^{rV}_{\{a,b,c\}}$ denote the set of preferences of voters over the set $\{a, b, c\}$, where preference profiles are restricted to be in $\mathcal{P}^r$.

LEMMA **6** *Suppose V is a choice function that satisfies strong unanimity and is such that $V(\cdot, P)$ is rationalizable by a linear order on the set $\{a, b, c\} \subset \mathcal{C}$ for each $P \in \mathcal{P}^{rV}_{\{a,b,c\}}$. Then, V is dictatorial on $\mathcal{P}^{rV}_{\{a,b,c\}}$, with the dictator being some $i \notin \{a, b, c\}$.*

Proof of lemma 6 : We first show that V depends only on the preferences of voters in $\mathcal{V} \setminus \{a, b, c\}$. Since V satisfies strong unanimity, there exists $P \in \mathcal{P}^{rV}_{\{a,b,c\}}$ such that $V(\{a, b\}, P) = a$. Similarly, there exists $P' \in \mathcal{P}^{rV}_{\{a,b,c\}}$ such that $V(\{b, c\}, P') = b$. Construct $P'' \in \mathcal{P}^{rV}_{\{a,b,c\}}$ such that

(i) For all $i \in \mathcal{V} \setminus \{a, b\}$, $P''_i|_{\{a,b\}} = P_i|_{\{a,b\}}$.

(ii) For all $i \in \mathcal{V} \setminus \{b, c\}$, $P''_i|_{\{b,c\}} = P'_i|_{\{b,c\}}$.

Then, by (iii) in the definition of voting procedure $V(\{a, b\}, P'') = a$ and $V(\{b, c\}, P'') = b$. Since V is rationalizable by a linear order, we must have $V(\{a, c\}, P'') = c$. But notice that only the preferences of individuals in $\mathcal{V} \setminus \{a, b, c\}$ have been specified over $\{a, c\}$. An obvious repetition of this argument establishes that the preferences of individuals in $\mathcal{V} \setminus \{a, b, c\}$ determine the choice out of all pairs in $\{a, b, c\}$. Since the preferences of individuals in $\mathcal{V} \setminus \{a, b, c\}$ over $\{a, b, c\}$ are unrestricted, Arrow's theorem applies. ∎

Proof of Theorem 2: The proof of Theorem 2 is almost identical to that of Theorem 1. Essentially, strong unanimity is used instead of unanimity to derive corresponding versions of Lemmas 2-5. Also, suitable restricted top sets (instead of top sets) are to be used in these lemmas. Also, Lemma 6 can be used to prove the corresponding version of Lemma 3. We omit these details. ∎

Proof of Theorem 3: First we show that

$$CS(T) \subset \{a | \exists H \in H(a, T) \text{ s.t. } \forall b \notin H \exists c \in H \text{ s.t. } b \text{ not cover } c\}$$

Consider $P \in \mathcal{P}^r$, T and σ, and $a = E(P, T, \sigma)$. We show that there exists $H \in H(a, T)$ such that for any $b \notin H$ there exists $c \in H$ which is not covered by b. Find A which is an equilibrium at P, T, σ with $a = S(A, T, \sigma)$. Thus, $a = w_1$ in the Shepsle-Weingast algorithm described previously. Let $H = \{w_1, \ldots, w_\ell\}$. By the Shepsle-Weingast algorithm it follows that $H \in H(a, T)$. First, consider any $b \in A$, such that $b \notin H$. It follows from the Shepsle-Weingast algorithm that there exists some $c \in H$ with cTb, otherwise, we would have $w_k = b$ for some k, which contradicts the fact that $b \notin H$. Next, consider any $b \notin A$. It must be that $b \neq S(A \cup \{b\}, T, \sigma)$, otherwise A would not be an equilibrium given that $bP_b a$, since $P \in \mathcal{P}^r$. Thus, it follows from the Shepsle-Weingast algorithm that there exists some $d \in A$ with dTb. If $d \in H$, then b does not cover d. If $d \notin H$, the we know from the previous argument that there exists $c \in H$ with cTd. Thus, b does not cover c.

We now prove that

$$\{a | \exists H \in H(a, T) \text{ s.t. } \forall b \notin H \exists c \in H \text{ s.t. } b \text{ not cover } c\} \subset CS(T)$$

Consider a and $H \in H(a, T)$ such that for any $b \notin H$ there exists $c \in H$ which is not covered by b. We need to show that there exist P and σ such that there is an A which is an equilibrium at P, T, σ with $a = SV(A, T, \sigma)$.

Let $P \in P^r$ be such that for all $c \in \mathcal{C} \setminus \{a\}$, $cP_j aP_j b$ for all $b \in \mathcal{C} \setminus \{a, c\}$. Let $Z = \{b \notin H | \exists c \in H, cTb\}$. Let $A = H \cup Z$. Let σ be such that

(i) $\sigma(b) < \sigma(c)$ for any $b \in Z$ and $c \in H$,

(ii) $\sigma(c) < \sigma(d)$ for any $c \in H$ and $d \notin A$, and

(iii) $\sigma(c) < \sigma(d)$ implies cTd for $c \in H$ and $d \in H$.

Note that (iii) is possible by the transitivity of T on H, since $H \in H(a,T)$. Thus, candidates in Z come first under σ, then candidates in H, and then the remaining candidates. Let us verify that A is an equilibrium and that $a = S(A,T,\sigma)$. First, we check that $a = S(A,T,\sigma)$. Let $\ell = \#A$ and $\ell' = \#H$. Ordering the elements in A according to σ results in the sequence $a_1, \ldots, a_{\ell-\ell'} = a, \ldots, a_\ell$, where $H = \{a_{\ell-\ell'} = a, \ldots, a_\ell\}$. First, note that since $H \in H(a,T)$, and by the ordering under (iii), it follows that $w_k = a_k$ for each $k \geq \ell \Leftrightarrow \ell'$. Then, by the definition of Z, it follows that $w_1 = w_{\ell-\ell'} = a$. Thus, $a = S(A,T,\sigma)$. Now let us check that A is an equilibrium, given P and σ. No candidate in A can benefit from exiting, since each prefers a to any other candidate besides him or herself. Consider a candidate $b \notin A$. Given the preference P_b (a is b's second most preferred candidate), it suffices to show that $S(A \cup \{b\}, T, \sigma) \neq b$. We know from our original choice of a and H that there exists $c \in H$ which is not covered by b. Thus, either cTb, or there exists d such that $cTdTb$. In the second case, note that from the definition of Z it follows that either $d \in Z$ or $d \in H$. Note that by the ordering σ, it follows from the Shepsle Weingast formula, for $b = S(A \cup \{b\}, T, \sigma)$ it must be that bTe for all $e \in A$. However, this cannot be due to the existence of c or d as just described. ∎

Figure 1.