A Theory of Voting in Large Elections*

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Abstract

We explore an application of the *Quantal Response Equilibrium* (QRE) to spatial voting models. In this context, the QRE is a game theoretic formalization of probabilistic voting. We study candidate competition over a finite dimensional policy space between an arbitrary number of candidates. We assume that candidates maximize the margin of expected victory, and that voters' utility functions are uniformly bounded. In this setting we prove that for large enough electorates there exists a global equilibrium at the social optimum (the point which maximizes the expected sum of utilities of the voters). In two candidate contests, the equilibrium is unique.

1 Introduction

This paper investigates properties of the Quantal Response Equilibrium (see McKelvey and Palfrey [14, 15]) in spatial voting games. The Quantal Response Equilibrium (QRE) is a theory of behavior in games that assumes that individuals get privately observed random payoff disturbances for each action available to them. The QRE is then just the Bayesian equilibrium of this game of incomplete information. In a QRE, although voters adopt pure strategies, from the point of view of an outside observer who does not know the payoff disturbance, the players choose between strategies probabilistically, choosing actions that yield higher utility with higher probability than actions that yield lower utility. The probability that one action is chosen over another is based on the the utility difference between the alternatives.

Much other literature has studied probabilistic voting (see Coughlin [3] for a review of this literature). Hinich [9] showed that the median voter theorem does not always hold in a setting with probabilistic voting, and he constructed examples in a one dimensional space with equilibria at other locations. In particular, with quadratic utility functions, he obtained an equilibrium in two candidate elections at the mean (which is the social welfare optimum with those preferences). Coughlin and Nitzan [4, 5] (see also Coughlin [3], p. 96, Theorem 4.2) proved if voters have likelihood of voting functions satisfying the Luce axioms over subsets, there is a local equilibrium at a point maximizing the social log likelihood. While this work was not explicitly rooted in a utility maximization framework, subsequent work (see [3], p. 99-100, Corollaries 4.4 and 4.5, Theorem 4.2) shows how it can be so interpreted. Coughlin [3] also gives various conditions on voter likelihood functions or on preferences that result in a global equilibrium. If the likelihood functions are concave, there is a global equilibrium. In a re-distributional model where voters have logarithmic utility functions for income, and candidates use a logistic model to estimate the probability that voters vote for each candidate, there is a global equilibrium at the social utility maximum (p. 57, Theorem 3.7). All of the above results are for two candidate competition. Recently, Lin, Enelow and Dorussen [12] show that one can also obtain equilibrium for multi-candidate elections using probabilistic voting models. They assume preferences based on distance, with a random utility shock, and obtain local equilibria at the social utility maximum. Lin, et al. also find that if the utility shocks have high enough variance, then the expected vote function for each candidate becomes concave, implying the existence of a global equilibrium.

In all of the above cited probabilistic voting literature, game theoretic considerations for the voter are not modeled. Voters are assumed to vote based on their preferences for the candidate policy positions rather than based on the effect their vote will have on the outcome of the election. Ledyard [11] develops a Bayesian model of two candidate competition that does model the game theoretic considerations for the voter. In his model, voters vote deterministically (there is no random utility shock to preferences), but they can abstain as well as vote for one of the two candidates, and the cost of voting is a random variable. Voter types consist of preferences as well as a cost of voting. He shows that in large elections, if voting costs are non-negative, there is an equilibrium at the social welfare optimum, which under certain restrictive conditions on the distribution of costs, is a global equilibrium. Myerson [17] extends Ledyard's results in a model where the number of voters is a Poisson random variable, unknown to the voters. He shows that as long as the density function of the costs of voting is positive at zero, there is a global equilibrium in Ledyard's model as the number of voters becomes large. The Ledyard model, as well as Myerson's generalization of it require that no voters have negative costs of voting.

In this paper, we work in a Bayesian framework, as in Ledyard, and take into account the game theoretic considerations for the voters, but unlike Ledyard, we assume that voters have privately observed payoff distrubances associated with each action. Our only restrictions on preferences are that they are uniformly bounded. Further, we consider multi candidate contests. But our results basically extend those of the earlier literature. We find that for large enough electorates there is a convergent equilibrium at the alternative that maximizes social welfare. For two candidate contests, the equilibrium is unique. Our equilibrium is global, as in [12], but in our model, the conditions for a global equilibrium are satisfied by allowing the number of voters to grow large rather than by assuming the utility shock becomes large.

The main contribution of this paper over the previous work is to obtain a global candidate equilibrium in large electorates with very little in the way of assumptions about voter preferences. The main difference between our approach and previous work on probabilistic voting is the way in which we model the probabilistic voting. As in [11], by treating the voter decisions as a game, we explicitly include the pivot probability in the voters' expected utility calculations. In large electorates, because the probability of being pivotal goes to zero, the expected utility difference between any two candidates also goes to zero. Thus, under the QRE assumptions, the voter's choice is determined mainly by the candidate specific payoff disturbance. Hence, in aggregate, voters vote less based on policy, and more based on candidate attributes as the size of the electorate grows. However, even though individuals become less responsive to policy differences, in large electorates, since the total number of voters is also getting large, there is still enough policy voting at the aggregate level to force the candidates to the social optimum.

2 The Model

We assume the existence of a finite dimensional *policy space*, $X \subseteq \Re^m$, where X is bounded, and finite sets N and K of *voters* and *candidates*, respectively. Write $n = |N|$ and $k = |K|$ for the total number of each. We let 0 indicate *abstention*, and write $K_0 = K \cup \{0\}$ for the set of candidates plus abstention.

We assume that for each voter, $i \in N$, there is a space T_i of possible characteristics, or *types* of the voter. Write $T = \Pi_{i \in N} T_i$. We assume that $T_i = \mathcal{T} \times \left(\Re^{K_0}\right)^{\mathcal{T}}$ is partitioned into two parts, representing the policy and consumption based parts, respectively, and that $\mathcal{T}$ is a complete separable metric space. Voters' preferences over the policy space are described by a *utility function*, $u : X \times \mathcal{T} \to \Re$. Hence, the utility of voter $i \in N$, of type $t_i = (\tau_i, \eta_i(\tau_i)) \in T_i$ for the policy $x \in X$ is $u(x, \tau_i)$. We assume that u is *uniformly bounded* with respect to N, i. e., there exists a $D \in \Re$ such that for all $x \in X$ and $\tau \in \mathcal{T}$, $|u(x, \tau) - u(y, \tau)| < D$. For example, uniform boundedness would follow from continuity of u and compactness of X and $\mathcal{T}$. Assume that the distribution of the voter i's types is given by an atom-less probability measure of full support, ρ_i, over the Borel sets of T_i, and that the joint distribution is given by ρ. We assume that ρ is absolutely continuous with respect to the product measure $\prod_{i \in N} \rho_i$. Note we allow for possible correlation between the distribution of types for different voters.

For notational simplicity, we will drop the argument of $\eta_i(\tau_i)$, and just write η_i when there is no confusion. Also, η_{ij} is used to represent the j^{th} component of $\eta_i(\tau_i)$. All of the η_{ij} for $i \in N$, $j \in K_0$, and $\tau_i \in \mathcal{T}$ are assumed to be independently distributed absolutely continuous random variables with full support, each with a cumulative density function that is twice continuously differentiable. We assume that the $\eta_{ij}(\tau_i)$ are identically distributed for all $i \in N$, $j \in K$, and $\tau_i \in \mathcal{T}$. However, we allow for η_{i0} to have a different distribution than η_{ij} to allow for costs or benefits of voting. Any joint distribution ρ on T satisfying all of the above conditions is said to be *admissible*. Let μ be the common mean of η_{ij} for $j \in K$, μ_0 be the mean of η_{i0}, and $c = \mu - \mu_0$. Then c is the expected cost of voting.

We now define a game, in which the candidates each simultaneously choose policy positions in X, and then after observing the candidate policy positions, the voters vote for a candidate. Thus, the strategy set Y_i for candidate $i \in K$ is $Y_i = X$, and the set of strategy profiles for the candidates is $Y = \Pi_{i\in K} Y_i$. The strategy set S_i for voter $i \in N$ is the set of functions $s_i : Y \times T_i \to K_0$, and the set of strategy profiles for the voters is $S = \Pi_{i\in N} S_i$. We will use the notation $S_{-i} = \Pi_{j\neq i} S_j$, and $s_{-i} \in S_{-i}$ to represent strategy profiles for all voters except voter i, with similar notation for candidates.

Given a strategy choice $y = (y_1, \dots, y_k) \in Y$ of the candidates, and $s = (s_1, \dots, s_n) \in S$ of the voters, define for any $j \in K_0$, and $t \in T^n$

$$V_j(y, s; t) = \frac{1}{n} |\{i \in N : s_i(y, t_i) = j\}| \tag{2.1}$$

to be the proportion of the vote for j, and

$$W(y, s; t) = \{j \in K : j \in \arg\max_{l\in K}(V_l(y, s; t))\} \tag{2.2}$$

to be the set of winners of the election. For any $J \subseteq K$, write

$$P_J(y, s; t_i) = \Pr[\{t_{-i} \in T_{-i} : W(y, s; t) = J\}]. \tag{2.3}$$

to be the probability of a first place tie among the candidates J. We assume that a fair lottery is used to select a winner when there is a tie, so that we can define voter utilities over subsets $J \subseteq K$ by

$$v_J(y, \tau_i) = \frac{1}{|J|} \sum_{j\in J} u(y_j, \tau_i). \tag{2.4}$$

The payoff to voter $i \in N$ of type $t_i = (\tau_i, \eta_i)$ from the strategy $(y, s) \in Y \times S$ is defined to be:

$$U(y, s, t_i) = \sum_{J\subseteq K} P_J(y, s; t_i) \cdot v_J(y, \tau_i) + \eta_{is_i(y,t_i)} \tag{2.5}$$

In other words, a voter voting for candidate $j = s_i(y, t_i)$ receives the expected utility of the policy of the winning candidate, plus a payoff disturbance η_{ij} that is associated with the vote, $j \in K_0$ that the voter makes. We write $U(j; y, s, t_i) = U(y, (j, s_{-i}); t_i)$ for the utility that voter i of type t_i gets from voting for strategy j, given y, and $s_{-i} \in S_{-i}$. Since $P_J(y, s; t_i)$ is a function of t_i only through s_i, it follows that $P_J(y, (j, s_{-i}); t_i)$ is independent of t_i . So we write $P_J(y, (j, s_{-i})) = P_J(y, (j, s_{-i}); t_i)$. Then, we can write for all $j \in K_0$,

$$U(j; y, s, t_i) \quad = \quad \bar{U}(j; y, s, \tau_i) + \eta_{ij} \tag{2.6}$$

where

$$\bar{U}(j; y, s, \tau_i) = \sum_{J\subseteq K} P_J(y, (j, s_{-i})) \cdot v_J(y, \tau_i) \tag{2.7}$$

is the expected utility to voter i of typeτ_i of voting for candidate j, unconditioned on the payoff disturbance, η_{ij}.

It follows from McKelvey and Ordeshook [13] that the difference in the expected utility of voting for j over abstaining can be written in the form:[1]

$$\bar{U}(j;y,s,\tau_i) - \bar{U}(0;y,s,\tau_i) = \sum_{k \neq j} \delta_i^{jk}(y,s) \cdot [u(y_j,\tau_i) - u(y_k,\tau_i)] \tag{2.8}$$

where $\delta_i^{jk}(y,s)$ is the *pivot probability* for j over k:

$$\delta_i^{jk}(y,s) = \sum_{j,k \in J \subseteq K} \frac{1}{|J|}\left(q_J^0 + \frac{q_J^j}{|J|-1}\right) \tag{2.9}$$

where we use the shorthand $q_J^k = P_J(y,(k,s_{-i}))$. The pivot probability is the probability that by voting for j rather than abstaining, voter i changes the outcome from a win for k to a win for j . It then follows from equation (2.8) that the difference in expected utility of voting for j over l is:

$$\begin{aligned} \bar{U}(j;y,s,\tau_i) - \bar{U}(l;y,s,\tau_i) &= \left(\delta_i^{jl}(y,s) + \delta_i^{lj}(y,s)\right) \cdot [u(y_j,\tau_i) - u(y_l,\tau_i)] \\ &+ \sum_{k \neq j,l} \left\{ \begin{array}{c} \delta_i^{jk}(y,s) \cdot [u(y_j,\tau_i) - u(y_k,\tau_i)] \\ +\delta_i^{lk}(y,s) \cdot [u(y_k,\tau_i) - u(y_l,\tau_i)] \end{array} \right\} \end{aligned} \tag{2.10}$$

which, for the case of two candidates, $K = \{j,l\}$, reduces to

$$\bar{U}(j;y,s,\tau_i) - \bar{U}(l;y,s,\tau_i) = \left(\delta_i^{jl}(y,s) + \delta_i^{lj}(y,s)\right) \cdot [u(y_j,\tau_i) - u(y_l,\tau_i)] \tag{2.11}$$

To define the candidate payoff functions, we first define $V_j(y,s)$ to to be the expected proportion of the votes for candidate j at the profile (y,s):

$$V_j(y,s) = E_t\left[V_j(y,s;t)\right] = \frac{1}{n} E_t\left[|\{i \in N : s_i(y,t_i) = j\}|\right] \tag{2.12}$$

Then we define the payoff to candidate j to be the *margin of expected victory* $\widehat{V}_j$, defined by:

$$\widehat{V}_j(y,s) = V_j(y,s) - \max_{l \in N - \{j\}} V_l(y,s) \tag{2.13}$$

Remark 1 Any voter with unbounded utility would be subject to the St. Petersburg paradox: If x_k is chosen to satisfy $u(x_k,\tau_i) > 2^k$, for $k = 1,2,\ldots$, the voter would not trade the lottery that gives prize x_k with probability $\frac{1}{2^k}$ for any x. Similarly, if the x_k satisfy $u(x_k,\tau_i) < -2^k$, they would not accept the lottery for any x. Thus, bounded utility for any one voter is implied if the voter is not subject to the St. Petersburg paradox. The uniform boundedness condition requires further that there be a common maximum and minimum bound across all voters.

[1] Equation (2.8) follows by reversing the order of summation in the expression for $(E^j - E^0)$ of the Theorem on p. 49 of [13].

Remark 2 Note that our assumptions do not preclude atoms in the marginal distribution of ρ over $\mathcal{T}$. The requirement that ρ be atomless is automatically satisfied via the assumptions that are imposed by admissibility on the distribution of the η_i's. Thus, our assumption of admissibility of ρ encompasses on the one hand the classical framework, in which all voter ideal points are known and common knowledge, and on the other hand, models such as that of Ledyard, in which all voter types are independent and drawn $i.i.d$ from a common distribution on voter types. The classical framework arises if we let the marginal distribution of ρ on $\mathcal{T}$ be discrete.

Remark 3 The assumption that the distribution of the η_{ij} are $i.i.d$ with respect to voters is an implicit normalization of utility functions. This is important in interpreting the main theorem, since the weights that individuals are given in the social utility function is determined by this normalization.

Remark 4 While we work with the objective function of "margin of expected victory" in this paper, the mathematical trick we use is to show that as the size of the electorate grows, the local equilibria that are guaranteed under this objective function expand to be global equilibria. Recent work by Patty [19] shows an equivalence between local equilibria under the objective function of expected vote and probability of winning. Thus, although we do not formally prove it here, we conjecture that our results would also hold when the candidate objective function is probability of winning.

3 Voter Equilibrium

In this section, we consider the voter equilibrium to the game defined by equation (2.6) conditional on fixed candidate positions, $y \in Y$. Since the candidate positions are fixed, the strategy space for the voter reduces from S_i (the set of functions $s_i : Y \times T_i \to K_0$) to the set of functions of the form $s_i(y, \cdot) : T_i \to K_0$. We write $S_i(y)$ to designate this conditional strategy space, and $S(y)$ to designate the set of profiles of conditional strategies.

For any fixed $y \in Y$, we define a *voter equilibrium for* y to be a pure strategy Bayesian Nash equilibrium (BNE) to the voter game defined by (2.6) over the strategy space $S(y)$. This is any profile, $s \in S(y)$, in which voters always choose an action that maximizes expected utility conditional on their type. Thus, s is a voter equilibrium for y if for all $i \in N$, $t_i \in T_i$, and $j \in K_0$,

$$\begin{aligned} s_i(y, t_i) = j \quad &\Leftrightarrow \quad U(j; y, s, t_i) = \max_{l \in K_0} U(l; y, s, t_i) \\ &\Leftrightarrow \quad \bar{U}(j; y, s, \tau_i) + \eta_{ij} = \max_{l \in K_0} \left[\bar{U}(l; y, s, \tau_i) + \eta_{il}\right] \end{aligned} \tag{3.1}$$

Note the structure of the payoffs is exactly the same as used in McKelvey and Palfrey [15] in defining the agent quantal response equilibrium (AQRE) for extensive form games. So as long as the distribution of the errors, η_{ij} is admissible, a Bayes Nash equilibrium to the voter game is exactly the same as an AQRE to the game. Note further that in any voter equilibrium for y, except on a set of measure zero, the strategy $s_i(y, t_i)$ depends on i only through t_i. So we can drop the subscript on s without loss of generality.

Proposition 1 *For any $y \in Y$, there exists a voter equilibrium for y.*

Proof: This is a game of incomplete information, with action spaces $A_i = K_0$ and type space T_i for each $i \in N$. The action spaces are finite, and the distribution of types is independent across individuals. Thus, we can apply Theorem 1 of Milgrom and Weber [16] to conclude that there exists an equilibrium in distributional strategies. Further, since the distribution of player i's types, ρ_i, is assumed atomless, it follows from Theorem 4 in the same paper that the equilibrium can be purified to be in pure strategies. ∎

Of particular interest is the average behavior of a voter i of type t_i, after integrating out η_i. For any $s_i(y, \cdot) \in S_i(y)$, define $\bar{s}_i(y, \cdot) : \mathcal{T} \to \Delta^{K_0}$, as the marginal distribution of s_i with respect to η_i: for any $\tau_i \in \mathcal{T}$ and $j \in K_0$,

$$\bar{s}_i(y, \tau_i)(j) = \Pr[\eta_i : s_i(y, (\tau_i, \eta_i)) = j]. \tag{3.2}$$

We have assumed that the η_{ij} are independently distributed, for all i, j and τ_i, and identically distributed for all $j \in K$. Let $H(\cdot)$ be the cumulative distribution function of η_i, i. e., $H(w) = Pr[\eta_{ij} \leq w_j \text{ for all } j \in K_0]$ for $w \in \Re^{K_0}$. And let $G_j(\cdot)$ be the cumulative distribution function of $\zeta \in \Re^K$, where $\zeta_l = \eta_{il} - \eta_{ij}$ for $l \in K - \{j\}$, and $z_j = \eta_{i0} - \eta_{ij}$. Thus,

$$G_j(z) = \Pr[\eta_{i0} - \eta_{ij} \leq z_j \text{ and } \eta_{il} - \eta_{ij} \leq z_l \text{ for all } l \neq j] \tag{3.3}$$

for any $z \in \Re^K$. Under the assumptions we have made on the η_{ij}, for all $j \in K$, both $H(w)$ and $G_j(z)$ are twice continuously differentiable and strictly increasing in all arguments, and everywhere positive. Thus, if s is a Bayes Nash equilibrium, applying equation (3.1), for $j \in K$,

$$\begin{aligned} \bar{s}_i(y, \tau_i)(j) &= \Pr[\bar{U}(j; y, s, \tau_i) + \eta_{ij} = \max_{l \in K_0} [\bar{U}(l; y, s, \tau_i) + \eta_{il}]] \\ &= \Pr[\eta_{il} - \eta_{ij} \leq \bar{U}(j; y, s, \tau_i) - \bar{U}(l; y, s, \tau_i) \text{ for all } l \in K_0 - \{j\}] \\ &= G_j(\bar{\mathbf{U}}^j(y, s, \tau_i)). \end{aligned} \tag{3.4}$$

where $\bar{\mathbf{U}}^j(y, s, \tau_i)$ is a vector in $\Re^K$ with components $\bar{\mathbf{U}}^j_l(y, s, \tau_i) = \bar{U}(j; y, s, \tau_i) - \bar{U}(l; y, s, \tau_i)$ for $l \neq j$, and $\bar{\mathbf{U}}^j_j(y, s, \tau_i) = \bar{U}(j; y, s, \tau_i) - \bar{U}(0; y, s, \tau_i)$.

Example: One example of the above is the logit AQRE, where the density functions of $w_0 = \eta_{i0} + c$ and $w_j = \eta_{ij}$ for $j \in K$ follow a type one extreme value distribution, $H_j(w_j) = \exp[-\exp[-\lambda w_j]]$. Thus, with independence, we have $H(w) = \prod_j H_j(w_j)$.

This leads to the logistic formula $G_j(z) = \frac{1}{1+\exp\lambda(c+z_j)+\sum_{l\neq j}\exp(\lambda z_l)}$. In this case, for fixed λ, we get:

$$\begin{aligned}\bar{s}_i(y,\tau_i)(j) &= G_j(\bar{\mathbf{U}}^j(y,s,\tau_i)) \\ &= \frac{1}{\begin{array}{c}1+\exp\left[\lambda\cdot\left(c+\bar{U}(0;y,s,\tau_i)-\bar{U}_i(j;y,s,\tau_i)\right)\right] \\ +\sum_{l\neq j}\left(\exp\left[\lambda\cdot\left(\bar{U}(l;y,s,\tau_i)-\bar{U}(j;y,s,\tau_i)\right)\right]\right)\end{array}},\end{aligned}$$

and in the case of two candidates, where $K=\{j,l\}$,

$$\bar{s}_i(y,\,\tau_i)(j) = \frac{1}{\begin{array}{c}1+\exp\left(\lambda\cdot\left(c+\delta^{jl}(y,s)\cdot[u(y_j,\tau_i)-u(y_l,\tau_i)]\right)\right) \\ +\exp\left(\lambda\cdot(\delta^{jl}(y,s)+\delta^{lj}(y,s))\cdot[u(y_j,\tau_i)-u(y_l,\tau_i)]\right)\end{array}}.$$

We now show that for fixed candidate positions at $y\in Y$, and for any voter equilibrium, that all pivot probabilities go to zero and the probability of voting for any two candidates in K becomes equal as $n\to\infty$. The reason for this result is simple: one's vote only matters when it is pivotal.[2] Thus, one's vote only matters when the other voters are either evenly split between the two top candidates or when the vote difference between the two top candidates differs by one vote. As n grows large, this becomes a very low probability event. Thus, in general, one's vote doesn't make a difference very often. This implies that voters effectively become indifferent with respect to which candidate they vote for as $n\to\infty$. We formalize the above in the following proposition:

Proposition 2 *Assume u is uniformly bounded. Fix $y\in Y$, and for each integer n, let ρ^n be any admissible joint distribution over $\Pi_{i=1}^n T_i$, and let s^n be any AQRE for the voters. Then for any $j,l\in K$ and $i,k>0$,*

(a) $\lim_{n\to\infty}\delta_i^{jl}(y,s^n)=0$ *and*

(b) $\lim_{n\to\infty}\delta_i^{jl}(y,s^n)/\delta_k^{jl}(y,s^n)=1$

(c) $\lim_{n\to\infty}\delta_i^{jl}(y,s^n)/\delta_i^{lj}(y,s^n)=1$

(d) $\lim_{n\to\infty}[\bar{s}_i^n(y,\tau_i)(j)-\bar{s}_i^n(y,\tau_i)(l)]=0.$

Further, in all cases, the convergence is uniform. I. e., for any $\varepsilon>0$, there is an n_ε such that for all i,k,j,l,y,ρ^n,s^n if $n>n_\varepsilon$, $\delta_i^{jl}(y,s^n)<\varepsilon$, $\left|\delta_i^{jl}(y,s^n)/\delta_k^{jl}(y,s^n)-1\right|<\varepsilon$, $\left|\delta_i^{jl}(y,s^n)/\delta_i^{lj}(y,s^n)-1\right|<\varepsilon$ and $|\bar{s}_i^n(y,\tau_i)(j)-\bar{s}_i^n(y,\tau_i)(l)|<\varepsilon$.

To prove the proposition, we need a Lemma.

[2]The logic of pivotal voting is explained in the voting literature. See eg. Myerson and Weber [18]

Lemma 1 *Fix $\varepsilon^* > 0$, and let $\mathcal{Z}^n$ be the set of sequences $Z = (Z_1, \ldots, Z_n)$ of independent random vectors $Z_i \in \Re^K$ of the form*

$$Z_i = \alpha_j \text{ w. p. } p_{ij}$$

where α_j is the j^{th} unit basis vector in $\Re^K$, and $p \in \left(\Delta^{K_0}\right)^n$ satisfies $p_{ij} \geq \varepsilon^*$ for all i, j. For any $J \subseteq K$, define

$$B_J = \{z \in \Delta^K : z_j = z_k > z_l \text{ for all } j, k \in J,\ l \notin J\}.$$

Write $\bar{Z} = \sum_i Z_i$, and define

$$\delta_J^{n*} = \max_{Z \in \mathcal{Z}^n} \Pr[\bar{Z} \in B_J] \tag{3.5}$$

Then for any $J \subseteq K$ with $|J| \geq 2$

(a) $\lim_{n\to\infty} \delta_J^{n*} = 0$

(b) $\lim_{n\to\infty} \delta_{J'}^{n*}/\delta_J^{n*} = 0$ *for any $J \subsetneq J'$*

Proof: An element $Z = (Z_1, \ldots, Z_n) \in \mathcal{Z}^n$ consists of independent, but not identically distributed random vectors, and is characterized by a vector $p = (p_1, \ldots, p_n)$, where $p_i = (p_{i0}, p_{i1}, \ldots, p_{iK}) \in \Delta^{K_0}$. The mean of Z_i is $\mu_i = (p_{i1}, \ldots, p_{iK})$ which consists of all but the first component of p. Pick $Z^n = (Z_1^n, \ldots, Z_n^n) \in \mathcal{Z}^n$ to attain the maximum in equation (3.5). Since $\Pr[\bar{Z} \in B_J]$ is continuous as a function of p, which ranges over a compact set, it follows that such a δ_J^{n*} and Z^n exist. Define V_{ni} to be the variance covariance matrix of Z_i^n, and $X_i^n = Z_i^n - \mu_i$. Set $V_n = \frac{1}{n}\sum_i V_{ni}$ and $T_n^2 = V_n^{-1}$. From our assumption that $p_{ij} > \varepsilon^*$ for all $j \in K_0$, it follows that V_n is strictly positive definite and hence invertible. Then

$$\begin{aligned}
\delta^{n*} &= \Pr[\bar{Z}^n \in B_J] \\
&= \Pr\left[\begin{array}{c} \sum_i Z_{ij}^n - \sum_i Z_{ik}^n = 0 \text{ for } j, k \in J, \text{ and} \\ \sum_i Z_{ij}^n - \sum_i Z_{il}^n > 0 \text{ for } j \in J, l \notin J \end{array}\right] \\
&= \Pr\left[\begin{array}{c} \sum_i \left(X_{ij}^n - X_{ik}^n\right) = \sum_i (p_{ik} - p_{ij}) \text{ for } j, k \in J, \text{ and} \\ \sum_i \left(X_{ij}^n - X_{il}^n\right) > \sum_i (p_{il} - p_{ij}) \text{ for } j \in J, l \notin J \end{array}\right] \\
&= \Pr\left[\begin{array}{c} \frac{1}{\sqrt{n}} T_n \sum_i \left(X_{ij}^n - X_{ik}^n\right) = \frac{1}{\sqrt{n}} T_n \sum_i (p_{ik} - p_{ij}) \text{ for } j, k \in J, \text{ and} \\ \frac{1}{\sqrt{n}} T_n \sum_i \left(X_{ij}^n - X_{il}^n\right) > \frac{1}{\sqrt{n}} T_n \sum_i (p_{il} - p_{ij}) \text{ for } j \in J, l \notin J \end{array}\right]
\end{aligned} \tag{3.6}$$

But now the X_i^n form a triangular array where each random variable X_i^n has zero mean, and for each n, the X_i^n are independent. Further, writing Q_i^n for the cumulative density function of X_i^n, the random vectors satisfy the following multivariate Lindeberg condition: For every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_i \int_{\|T_n X_i\| > \epsilon\sqrt{n}} \|T_n X_i\|^2 \, dQ_i^n(X) = 0 \tag{3.7}$$

To see this, note that Z_i^n is in the simplex Δ^K. Hence, $\|X_i^n\| \leq 2$. The probability that $Z_{ij}^n = 1$ is $p_{ij} \geq \varepsilon^*$. Further, the variances and covariance of V_{ni} are all uniformly bounded away from zero and one, since $p_{ij} \geq \varepsilon^*$ for all i, j. Thus, the same will be true of V_n. So V_n will be invertible, and for any ϵ, we can pick large enough n so that $\|T_n X_i\| < \epsilon\sqrt{n}$. So each term in the summation of equation (3.7) goes to zero with n, which establishes (3.7). It follows by the multivariate version of the central limit theorem for triangular arrays (see Bhattacharya and Rao, [2], Corollary 18.2, p 183) that the distribution of $\frac{1}{\sqrt{n}} T_n \sum_i X_i^n$ converges weakly to a multivariate unit normal distribution. Hence the probability it falls in a subset of any lower dimensional subspace goes to zero. Thus, when $|J| \geq 2$, the right hand side of equation (3.6) converges to 0 with n. I. e., $\lim_{n\to\infty} \delta^{n*} = 0$, proving (a). To prove (b), we note that $B_{J'}$ describes a lower dimensional subspace than B_J. Hence, an argument similar to above shows that for all sequences, the $\Pr[\bar{Z} \in B_{J'}]$ goes to zero faster than $\Pr[\bar{Z} \in B_J]$ establishing the result. ∎

We now proceed to a proof of the proposition.

Proof: To prove (a), define $D = 2 \cdot (|K| - 1) \cdot \sup_{x,y,\tau} [u(x, \tau) - u(y, \tau)]$, and $\varepsilon^* = \min_{j \in K} G_j(-\mathbf{1} \cdot D)$, where $\mathbf{1} = (1, \ldots, 1)$ is the unit vector of length $|K|$. By the assumptions we have made on the η_{ij}, $\varepsilon^* > 0$. Then from equation (2.8), using the fact that $\delta_i^{jl} \leq 1$ for all i, j, k,we have $-D \leq \bar{U}(j; y, s, \tau_i) - \bar{U}(l; y, s, \tau_i) \leq D$ for all $j, l \in K$, which implies that $\bar{s}_i(y, \tau_i)(j) = G_j(\bar{\mathbf{U}}^j(y, s, \tau_i)) \geq G_j(-\mathbf{1} \cdot D) \geq \varepsilon^*$.

Now, given any sequence $\tau = (\tau_1, \ldots, \tau_n)$ with $\tau_i \in \mathcal{T}$ for all $i > 0$, define the random variable

$$Z_{ni}(\tau_i) = \alpha_j \text{ if } s_i^n(y, (\tau_i, \eta_i)) = j$$

So $Z_{ni}(\tau_i) \in \mathcal{Z}^n$, with $p_{ij} = \bar{s}_\alpha^n(y, \tau_\alpha)(j)$.

Then, letting $(0, s_{-i}^n)$ be the profile where the voter i abstains, and (j, s_{-i}^n) be the profile where voter i votes for candidate j, we have, from equation (2.9):

$$\delta_i^{jl}(y, s^n) = \sum_{j,k \in J \subseteq K} \frac{1}{|J|} \left(q_J^0 + \frac{q_J^j}{|J| - 1} \right) \tag{3.8}$$

But, from equation (2.3), for any $J \subseteq K$,

$$\begin{aligned} q_J^0 &= P_J(y, (0, s_{-i})) = \Pr[\{t_{-i} \in T_{-i} : W(y, s; t) = J\}] &(3.9) \\ &= E_{t_{-i}} \left[W(y, (0, s_{-i}^n); t_{-i}) = J \right] = E_{t_{-i}} \left[\sum_{l \neq i} Z_{nl}(\tau_l) \in B_J \right] &(3.10) \\ &= E_{\tau_{-i}} E_{\eta_{-i}} \left[\sum_{l \neq i} Z_{nl}(\tau_l) \in B_J \right] \leq E_{\tau_{-i}}[\delta_J^{n*}] = \delta_J^{n*}, &(3.11) \end{aligned}$$

where the inequality follows from the definition of δ_J^{n*} in Lemma 1. A similar argument shows the second term in equation (3.8) is less than or equal to δ_J^{n*}. Thus, $\delta_i^{jl}(y, s^n) \leq$

$\sum_{j,k\in J\subseteq K}\left(\frac{1}{|J|-1}\right)\delta_J^{n*} \leq \left(\sum_{j,k\in J\subseteq K}\frac{1}{|J|-1}\right)\delta^{n*}$, where $\delta^{n*} = \max_{J\subseteq K}\delta_J^{n*}$ By Lemma 1, $\lim_{n\to\infty}\delta^{n*} = 0$, which proves *(a)*. Since δ^{n*} is independent of i, j, l, y, the convergence is uniform in all arguments.

To show (b), for each $J \subseteq K$, we can write $P_J(y,(0,s_{-i})) = E_{t_{-i}}\left[\sum_{l\neq i} Z_{nl}(\tau_l) \in B_J\right]$ the corresponding expression for voter j is $P_J(y,(0,s_{-j})) = E_{t_{-j}}\left[\sum_{l\neq j} Z_{nl}(\tau_l) \in B_J\right]$. But the RHS of these two expressions differ only by the i and j^{th} terms, and hence, by Lemma 1, both converge weakly to the same multivariate normal distribution. Hence, in the limit, the ratio of the two must approach one. The same argument applies to all terms in the sum in (3.8). Thus, the result follows. A similar argument suffices to establish (c).

To show (d), we have from equation (3.2) that

$$\bar{s}_i^n(y,\tau_i)(j) = \Pr[\max_{l\neq j}\bar{U}(l;y,s^n,\tau_i) + \eta_{il} \leq \bar{U}(j;y,s^n,\tau_i) + \eta_{ij}].$$

Now, in the first part of the proposition we showed all pivot probabilities go to zero uniformly as n gets large. Hence, using equation (2.10) we get that as $n\to\infty$, for $j,l\in K$, $\bar{U}(l;y,s^n,\tau_i) - \bar{U}(j;y,s^n,\tau_i)\to 0$ uniformly in i,j,l,y,τ. But then we get

$$\begin{aligned}\lim_{n\to\infty}[\bar{s}_i^n(y,\tau_i)(j) - \bar{s}_i^n(y,\tau_i)(l)] &= \Pr[\max_{\alpha\neq j}\eta_{ia} - \eta_{ij} \leq 0] - \Pr[\max_{a\neq l}\eta_{ia} - \eta_{il} \leq 0] \\ &= G_j(0) - G_l(0) = 0. \end{aligned} \tag{3.12}$$

Since the convergence of $\bar{U}(l;y,s^n,\tau_i) - \bar{U}(j;y,s^n,\tau_i)$ is uniform in all arguments, it follows that the convergence in equation (3.12) is also. ∎

Based on Proposition 2 (b), it follows that for large n, we can ignore the voter subscript on δ, and write $\delta_i^{jl}(y,s^n) = \delta^{jl}(y,s^n) = \delta^{lj}(y,s^n)$. Further, from Lemma 1, it follows that in any voter equilibrium, all ties involving three or more candidates will be small in relation to the two candidate ties. Recall the notation $q_J^k = P_J(y,(k,s_{-i}))$. Then for $J \subsetneq J'$,

$$\lim_{n\to\infty} q_J^k/q_{J'}^k = \lim_{n\to\infty} P_J(y,(k,s_{-i}))/P_{J'}(y,(k,s_{-i})) = 0$$

Hence, for large electorates, formula (2.9) for the pivot probability has the following approximation:

$$\delta_i^{jk}(y,s) = \sum_{j,k\in J\subseteq K}\frac{1}{|J|}\left(q_J^0 + \frac{q_J^j}{|J|-1}\right) \cong \frac{1}{2}\left(q_{\{j,k\}}^0 + q_{\{j,k\}}^j\right)$$

Remark 5 Note that the requirement that voters adopt a Bayesian equilibrium means that voters vote strategically in multi candidate elections, Thus, a voter may rank $u(y_j,\tau) > u(y_l,\tau)$, and yet (even if the realization of the payoff disturbances is zero) vote for their second ranked alternative l over their first ranked alternative j if the pivot probability for the first ranked alternative is sufficiently low in relation to that for the second ranked alternative so that we have $\bar{U}(l;y,s^n,\tau_i) - \bar{U}(j;y,s^n,\tau_i) > 0$.

4 Candidate Equilibrium

This section examines the incentives of candidates competing for votes in a world populated by voters who play quantal response equilibrium strategies. We establish that for a large enough electorate, N, all candidates adopting the social optimum constitutes a global equilibrium. In the case of two candidates, the global equilibrium is unique. Our results hold regardless of the how the measure ρ changes as the size of the electorate increases, as long as the admissibility condition is met. More specifically, recall that admissibility required that the η_{ij} are *i.i.d.* with full support. We also assume that the distribution of the η_{ij} is independent of the size n of N.

For a fixed electorate, N, and measure ρ on $T = \prod_{i \in N} T_i$, let s be any strategy profile for the voters[3] such that for any candidate positions, $y \in Y$, $s(y, \tau)$ is a quantal response equilibrium for the voters, as described in the previous section. We use the notation

$$V_j(y) = V_j(y, s(y, \cdot)) = E_t\left[V_j(y, s(y, t); t)\right] \tag{4.1}$$

to represent the expected vote for the candidates j, assuming that the voters follow the strategy s in response. Then,

$$\begin{aligned} V_j(y) &= \frac{1}{n} E_t\left[\left|\{i \in N : s_i(y, t_i) = j\}\right|\right] \\ &= \frac{1}{n} E_\tau\left[E_\eta\left[\left|\{i \in N : s_i(y, \tau_i, \eta_i) = j\}\right|\right]\right] \\ &= \frac{1}{n} E_\tau\left[\sum_{i \in N} \bar{s}_i(y, \tau_i)(j)\right] = \frac{1}{n} \sum_{i \in N} E_{\tau_i}\left[\bar{s}_i(y, \tau_i)(j)\right]. \end{aligned} \tag{4.2}$$

We assume that candidates seek to maximize the margin of expected victory. So the payoff of candidate $j \in K$ at the profile (y, s) is given by:

$$\widehat{V}_j(y) = \widehat{V}_j(y, s) = V_j(y) - \max_{l \in N - \{j\}} V_l(y). \tag{4.3}$$

Let

$$x^*_\rho = \arg\max_{x \in X} \sum_{i \in N} E_{\tau_i}[u(x, \tau_i)] \tag{4.4}$$

denote the *expected social optimum*. We assume for each N and ρ, that such a point exists and is unique.

Theorem 1 *Let u be uniformly bounded. There exists an integer n^* such that for any set of voters N with $|N| = n > n^*$, and any admissible ρ on $T = \prod_{i \in N} T_i$, $y^* = (x^*_\rho, \ldots, x^*_\rho)$ constitutes a global equilibrium under the margin of expected victory: for any $j \in K$ and $y_j \in X$, $\hat{V}_j(y) = \hat{V}_j(y_j, y^*_{-j}) \leq \hat{V}_j(y^*)$, with the weak inequality becoming strict whenever $y_j \neq x^*$.*

[3] To be technically correct, since we are considering N and ρ to be variables, we should subscript voter and candidate strategies on these variables. To simplify notation, we leave off these parameters.

Proof: For any set of voters N, and admissible ρ, let $y = (y_j, y^*_{-j})$, where $y^*_l = x^*_\rho$ for all $l \neq j$ and $y_j \neq x^*_\rho$. We first show that for large enough n, $V_j(y) = V_j(y_j, y^*_{-j}) \leq V_j(y^*)$.

For $z \in \Re^K$, write $Q(z) = G_j(z)$, where G_j is as defined in equation (3.3). Given an individual $i \in N$, and using equations (2.8) and (2.10), the probability of a vote for candidate j is given by

$$\begin{aligned} s_i(y, \tau_i)(j) &= \Pr\left[\max_{l \in K_0 - \{j\}} [U(l; y, s, t_i) - U(j; y, s, t_i)] \leq 0\right] \\ &= \Pr\left[\begin{array}{c} \eta_{ik} - \eta_{ij} \leq \Delta_i^k(y, s) \cdot [u(y_j, \tau_i) - u(x^*_\rho, \tau_i)] \text{for } k \in K - \{j\} \\ \text{and } \eta_{i0} - \eta_{ij} \leq \Delta_i^j(y, s) \cdot [u(y_j, \tau_i) - u(x^*_\rho, \tau_i)] \end{array}\right] \\ &= Q(\Delta_i(y, s) \cdot [u(y_j, \tau_i) - u(x^*_\rho, \tau_i)]) \end{aligned} \tag{4.5}$$

where $\Delta_i(y, s) = (\Delta_i^1(y, s), \ldots, \Delta_i^k(y, s))$, $\Delta_i^l(y, s) = 2\delta_i^{lj}(y, s) + \sum_{\alpha \neq j, l} \delta_i^{\alpha j}(y, s)$, for all $l \in K - \{j\}$, and $\Delta_i^j(y, s) = \sum_{\alpha \neq j} \delta_i^{j\alpha}(y, s)$.

Using equation (4.2) we can express the vote for candidate j as

$$V_j(y) = \frac{1}{n} \sum_{i \in N} E_{\tau_i} [\bar{s}_i(y, \tau_i)(j)] \tag{4.6}$$

Then, from equation (4.5), we have that

$$V_j(y) = \frac{1}{n} \sum_{i \in N} E_{\tau_i} \left[Q(\Delta_i(y, s) \cdot [u(y_j, \tau_i) - u(x^*_\rho, \tau_i)])\right]$$

Without loss of generality, we can assume utility functions are normalized with $u(x^*, \tau_i) = 0$ for all $i \in N$ and $\tau_i \in T$. Write $u_i = u(y_j, \tau_i)$, and $\Delta_i = \Delta_i(y, s)$. Then, the above can be written as:

$$V_j(y) = \frac{1}{n} \sum_{i \in N} E_{\tau_i} [Q(\Delta_i(y, s) \cdot u(y_j, \tau_i))] = \frac{1}{n} \sum_{i \in N} E_{\tau_i} [Q(\Delta_i \cdot u_i)] \tag{4.7}$$

Using parts (b) and (c) of Proposition 2, normalize the Δ_i by Δ_1 in the following manner. For $i \in N$, let

$$\chi_i = \left(\frac{\Delta_i^1}{\Delta_1^1}, \ldots, \frac{\Delta_i^k}{\Delta_1^k}\right)^T,$$

and

$$D = \begin{bmatrix} \Delta_1^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Delta_1^k \end{bmatrix}.$$

It is easily shown that $\Delta_i^j > 0$ for all $i \in N$ and $j \in K$, so that χ_i is well defined. By parts (b) and (c) of Proposition 2, and using the fact that all terms are positive, it can be shown that $\lim_{n\to\infty} \chi_i^j = 1$ for all $i \in N$, $j \in K$, where the convergence is uniform with respect to y, ρ,and s. So for any ϵ, we can find a large enough n^* such that if $n > n^*$, then $|\chi_i^j - 1| \le \epsilon$, and simultaneously, $\Delta_i^j \le \epsilon$ Then, we can write

$$\begin{aligned}
V_j(y) - V_j(y^*) &= \frac{1}{n}\sum_{i\in N}\left\{E_{\tau_i}\left[Q\left(D\cdot\chi_i\cdot u_i\right)\right] - E_{\tau_i}\left[Q\left(\mathbf{0}\right)\right]\right\} \\
&\approx \frac{1}{n}\sum_{i\in N} E_{\tau_i}\left[Q'\left(\mathbf{0}\right)D\cdot\chi_i\cdot u_i + \frac{1}{2}(\Delta_i\cdot u_i)^T Q''\left(\mathbf{0}\right)\left(\Delta_i\cdot u_i\right)\right] \\
&= \frac{1}{n}Q'\left(\mathbf{0}\right)\cdot D\cdot\sum_{i\in N} E_{\tau_i}\left[\mathbf{1}\cdot u_i + (\chi_i - \mathbf{1})\cdot u_i\right] \\
&\quad + \frac{1}{2n}\sum_{i\in N} E_{\tau_i}\left[\Delta_i^T\cdot Q''\left(\mathbf{0}\right)\cdot\Delta_i\cdot u_i^2\right] \\
&\le \frac{k}{n}(1-\epsilon)\epsilon Q^*\sum_{i\in N} E_{\tau_i}\left[u_i\right] + \frac{k^2}{2n}\epsilon^2 Q^{**}\sum_{i\in N} E_{\tau_i}\left[u_i^2\right]
\end{aligned} \tag{4.8}$$

where $Q'(\mathbf{0})$ is a k dimensional vector consisting of the gradient of Q evaluated at $\mathbf{0}$, $Q''(\mathbf{0})$ is a $k \times k$ symmetric matrix of second partial derivatives of Q evaluated at $\mathbf{0}$, Q^* is the smallest element of $Q'(\mathbf{0})$, Q^{**} is the greatest element of $Q''(\mathbf{0})$, and $\mathbf{0}$ and $\mathbf{1}$ represent k dimensional vectors of zeros and ones, respectively. So for small enough ϵ, the first term in the above summation is negative, since $Q^* > 0$ and $\sum_{i\in N} E_{\tau_i}[u_i] < 0$, and the second term is small in absolute value in relation to the first if ϵ is small enough. It follows that for large enough n, the above expression is negative. Thus, for any $y_j \in Y_j$, $V_j(y) = V_j(y_j, y_{-j}^*) \le V_j(y^*)$ with strict inequality whenever $y_j \ne x_\rho^*$.

Next, we show that for some $l \ne j$, $V_l(y_j, y_{-j}^*) \ge V_l(y^*)$. We pick $l \in K - \{j\}$ for which $\delta^{jl}(y, s)$ is maximized. For $z \in \Re^K$, write $Q(z) = G_l(z)$, where G_l is as defined in equation (3.3). Then we have

$$\begin{aligned}
s_i(y,\tau_i)(l) &= \Pr\left[\begin{array}{c} U\left(0;y,s,\tau_i\right) - U(l;y,s,\tau_i) \le 0, \text{and} \\ U(j;y,s,\tau_i) - U\left(l;y,s,\tau_i\right) \le 0, \text{and} \\ \max_{k\in K-\{l,j\}}\left[U(k;y,s,\tau_i) - U(l;y,s,\tau_i)\right] \le 0, \end{array}\right] \\
&= \Pr\left[\begin{array}{l} \eta_{i0} - \eta_{il} \le \Delta_i^l(y,s)\cdot\left[u(x_\rho^*,\tau_i) - u(y_\alpha,\tau_i)\right], \text{and} \\ \eta_{ij} - \eta_{il} \le \Delta_i^j(y,s)\cdot\left[u(x_\rho^*,\tau_i) - u(y_j,\tau_i)\right], \text{and} \\ \eta_{ik} - \eta_{il} \le \max_{k\in K-\{l,j\}}\left(\Delta_i^k(y,s)\cdot\left[u(x_\rho^*,\tau_i) - u(y_j,\tau_i)\right)\right] \end{array}\right] \\
&= Q(\Delta_i(y,s)\cdot\left[u(x_\rho^*,\tau_i) - u(y_j,\tau_i)\right])
\end{aligned}$$

where $\Delta_i(y, s) = (\Delta_i^1(y, s), \ldots, \Delta_i^k(y, s))$, with $\Delta_i^l(y, s) = \sum_{\alpha\ne l} \delta_i^{l\alpha}(y, s)$, $\Delta_i^j(y, s) = 2\delta_i^{lj}(y, s) + \sum_{\alpha\ne j,l} \delta_i^{j\alpha}(y, s)$, and $\Delta_i^k(y, s) = \delta_i^{jl}(y, s) - \delta_i^{kj}(y, s)$ for all $k \in K - \{l, j\}$.

Using equation (4.2) we can express the vote for candidate l as

$$V_l(y) = \frac{1}{n}\sum_{i\in N} E_{\tau_i}\left[s_i(y,\tau_i)(l)\right] = \frac{1}{n}\sum_{i\in N} E_{\tau_i}\left[Q(\Delta_i(y,s)\cdot\left[u(x_\rho^*,\tau_i) - u(y_j,\tau_i)\right])\right] \tag{4.9}$$

As above, we can assume utility functions are normalized with $u(x^*, \tau_i) = 0$ for all $i \in N$ and $\tau_i \in T$. As before, write $u_i = u(y_j, \tau_i)$, and $\Delta_i = \Delta_i(y, s)$. Then, the above can be written as:

$$V_l(y) = \frac{1}{n} \sum_{i \in N} E_{\tau_i} \left[Q(-\Delta_i(y,s) \cdot u(y_j, \tau_i)) \right] = \frac{1}{n} \sum_{i \in N} E_{\tau_i} \left[Q(-\Delta_i u_i) \right] \tag{4.10}$$

Note that the above takes exactly the same form as equation (4.6) above, with the exception of the negative sign. Consequently, an analagous argument to that in (4.8) establishes that we can find large enough n so that $V_l(y) - V_l(y^*)$ is positive. Thus, for any $y_j \in Y_j$, $V_l(y) = V_l(y_j, y^*_{-j}) \geq V_l(y^*)$ with strict inequality whenever $y_j \neq x^*_\rho$. We have shown that $V_j(y_j, y^*_{-j}) \leq V_l(y^*)$ and $V_l(y_j, y^*_{-j}) \geq V_l(y^*)$. So $\hat{V}_j(y_j, y^*_{-j}) \leq \hat{V}_j(y^*)$. So y^* is a global equilibrium for the objective function $\hat{V}$. ∎

For the case of two candidates, the above theorem can be strengthened:

Corollary 1 *If $k = 2$, then the equilibrium found in Theorem 1 is unique.*

Proof: Suppose there is another equilibrium, y. Then for at least one candidate j, $y_j \neq x^*$. Assume W.L.O.G. that $j = 2$. By Theorem 1, $\hat{V}_1(y_1, y_2) \geq \hat{V}_1(x^*, y_2) > 0$. Hence, $\hat{V}_2(y_1, y_2) < 0$. But this cannot be an equilibrium for candidate 2, Since $\hat{V}_2(y_1, x^*) \geq 0 > \hat{V}_2(y_1, y_2)$. This yields a contradiction. Hence the equilibrium is unique. ∎

Note that in the equilibrium defined by Theorem 1, that $y^*_j = y^*_l = x^*_\rho$ for all $j, l \in K$. Hence, we have $u(y^*_j, \tau_i) = u(y^*_l, \tau_i)$ for all $j, l \in K$. Thus, the level of abstention in equilibrium is determined by $V_0(y^*, s) = \frac{1}{n} \sum_{i \in N} E_{\tau_i} [s_i(y^*, \tau_i)(0)]$. But

$$\begin{aligned}
\bar{s}_i(y^*, \tau_i)(0) &= \Pr\left[\max_{l \in K} \left[\bar{U}(l; y^*, s, t_i) - \bar{U}(0; y^*, s, t_i) + \eta_{il} - \eta_{i0} \right] \leq 0 \right] \\
&= \Pr\left[\max_{l \in K} \left[\sum_{\alpha \neq l} \delta^{l\alpha}(y, s) \cdot [u(y^*_l, \tau_i) - u(y^*_\alpha, \tau_i)] + \eta_{il} - \eta_{i0} \right] \leq 0 \right] \\
&= \Pr\left[\eta_{i0} \geq \max_{l \in K} [\eta_{il}] \right].
\end{aligned}$$

For example, if $c_i = 0$ for all $i \in N$, then under the assumptions we have made, all of the η_{il} for $l \in K_0$ are *i. i. d.* Hence the above evaluates to $\frac{1}{K+1}$. It follows that

$$V_0(y^*) = \frac{1}{n} \sum_{i \in N} E_{\tau_i} [s_i(y^*, \tau_i)(0)] = \frac{1}{K+1}.$$

So that in a two candidate election, one would obtain equilibrium turnout of about two thirds of the electorate. Of course, the above calculation would be very sensitive to the assumed distribution of costs of voting.

Thus, asymptotically we find that the social optimum is a global equilibrium so long as preferences are uniformly bounded. Note that this result does not give us any indication as to how big n^* must be.

5 Conclusions and Extensions

In this paper we have provided a general framework for probabilistic spatial voting models in large electorates. In particular, we have extended equilibrium results of Coughlin, Ledyard, and other researchers to spaces of arbitrary finite dimensionality and elections with both abstention and arbitrary numbers of candidates. In addition, our model allows for strategic behavior by the voters.

As an aside, our model is agnostic as to the cause of probabilistic choice. The probabilistic choice in a QRE model can be assumed to arise either as the result of rational behavior under payoff disturbances (as we have modeled it here), or as the result of boundedly rational behavior.

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