The Survival of the Welfare State

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Abstract

We construct a positive theory of income redistribution in a dynamic model with overlapping generations where agents act rationally, and private and collective decisions are mutually interlinked. We specify alternative political mechanisms such as majoritarian voting, probabilistic voting and lobbying, and provide analytical characterization of Markov perfect equilibria. The model shows that the long-run survival of the welfare state and the dynamics of redistribution depend qualitatively on the characterization of the political process, in particular the political influence of minorities. Multiple steady-states, with different levels of redistribution, exist under majority voting, whereas the equilibrium features a unique steady state without redistribution under probabilistic voting. If small groups are more effective at organizing lobbies, minorities have a stronger than proportional influence and the steady-state entails positive redistribution.

1 Introduction

The rise of the welfare state in industrialized countries in the 20th century has been an unprecedented change in the size and scope of governments. Moreover, the differences in the size of the welfare state across countries is remarkable. Recent trends, however, suggest a scaling back of the welfare state in both Western democracies and a number of developing countries. This development might turn out to be a historical blip or, alternatively, the start of a breakdown. The aim of this paper is to shed light on the scope for survival of the welfare state. In particular, we explore which aspects of the political process determine the make or break of the welfare state in a framework where there is no insurance role for a welfare state, and redistribution is driven entirely by political conflict. This is a complement to the

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standard explanation of the existence of a welfare state – that a government can deliver the insurance missing markets fail to provide.

We consider economies populated by two-period lived agents who are ex-ante identical, but ex-post heterogeneous. The overlapping generation structure is intended to capture the idea that, as life goes by, uncertainty about lifetime income is resolved. This resolution of uncertainty make young and old individuals have different evaluations of a welfare state that provides redistribution. In particular, in our model, young individuals are born identical and have common *ex-ante* preferences over redistribution, while the old have heterogenous preferences for redistribution, since the resolution of uncertainty has *ex-post* turned some of them into high income ("successful") and some of them into low-income ("unsuccessful") individuals.

A key assumption is that young individuals can affect the probability distribution of their lifetime income by taking a private (human capital) investment decision. The optimal decision is affected by the extent of redistribution, which is set at the beginning of each period in political elections. Voters are fully rational, and take into account the effects of policies on the current investments and on the future distribution of voters.[1]. Since agents are risk neutral and redistribution is distortionary, there would be no welfare state if the level of redistribution were set by a utilitarian planner caring equally about all agents currently alive. In this sense there is no intrinsic role for a welfare state. Similarly, there would be no welfare state in equilibrium if young agents could commit to vote in a particular way in the future. A central feature of democratic systems is, however, that such commitments are infeasible. We will show that, in the absence of commitment, a welfare state may arise and survive in the long run. The prospect of the welfare state surviving hinges on the way the *ex-post* conflict on redistribution is resolved by the political system. In particular, an important assumption for the long-run survival of the welfare state is that old individuals, for whom uncertainty has been resolved, have a higher degree of participation in (or, more generally, of influence on) the political process. Such an assumption is supported by empirical evidence that voting turnout is increasing with age. For example, Wolfinger and Rosenstone (1980) document that turnout in U.S. elections in 1972 was sharply increasing in age, rising from 45% for the 20-years old to 75% for the 65 years old (see also Mulligan and i Martin (1999) for theoretical justifications).

We consider two alternative political mechanisms of Downsian two-candidate electoral competition, each capturing important aspects of the real-world political systems. The first is a standard majority voting environment where redistribution is the single issue on which the candidates are evaluated, and the median voter theorem applies (Black (1948) and Downs (1957)). The second is an environment where even minorities carry some weight in the political process. In this case, the political equilibrium is determined by a probabilistic voting approach (Coughlin (1982) and Lindbeck and Weibull (1987)). Finally, we explore the implications of granting even more power to minorities by extending our setup to allow for lobbies to influence the political process, along the lines of Baron (1994).

One of our key findings is that the simple majority voting regime can sustain the welfare state. In particular, if the economy starts with a pro-redistribution majority, high levels of redistribution will be sustained over time, whereas there will never be a welfare state if

[1]Here, full rationality is defined up to the well-known "voting paradox". In other words, we assume that all agents vote, ignoring the standard problem that none of them has an incentive, individually, to cast its vote in the ballots.

the economy starts with an anti-redistribution majority. The equilibrium features multiple steady-states due to a self-reinforcing mechanism linking private and collective choices. Highly redistributive policies today decrease current investment, sustaining the constituency for the welfare state. In contrast, when there is low redistribution at the outset, no strong enough constituency for the welfare state ever arises. We refer to this self-reinforcing mechanism as *policy-behavior complementarity*. Thus, societies with identical preferences may, in equilibrium, feature large persistent differences in inequality, redistribution and incentives, due to self-sustained differences in the distributions of voters (see also Coate and Morris, 2000, Krusell and Ríos Rull (2000), and Hassler et al.1999 and 2000).

The simple majority voting regime can, however, also give rise to another equilibrium, featuring no welfare state in the long run. Thus, even in an economy starting with a welfare state, the pro-redistribution majority will vote strategically so as to induce a future anti-redistribution majority. The expectation that the next generation will terminate the welfare state has positive effects even for the current pro-redistribution majority, since the perspective of no redistribution in the future strengthens the incentives for the young to invest, increasing production and the current tax base. This opens up an interesting possibility of the termination of the welfare state: all old agents might be better off by convincing the young that the welfare state will disappear, thereby strengthening their incentives to inves. t and increasing the tax base. It important to note that this equilibrium can exist only because voters understand the impact of current redistribution on future election outcomes. Consequently, this equilibrium would not exist if voters were myopic.

When considering the second political mechanism, the probabilistic voting environment, where even minorities can influence the political outcome, we obtain very different results. In this case, the equilibrium path converges to a unique steady-state characterized by no welfare state. Under this political mechanism, it turns out that the winning platform maximizes a utilitarian welfare function giving an equal weight to every old agent. Like the utilitarian planner, the winning platform does not care about intra-generational redistribution. Moreover, since the welfare state is distortionary, the winning candidate will set redistribution to zero unless there is scope for intergenerational redistribution in favor of the age group with political rights. But, given a constant population, there is no role for such redistribution in steady-state. Thus, the steady-state must feature no welfare state. Along the transition path, however, there may be positive redistribution. The result changes, however, when the groups can organize lobbies to influence the political process. In particular, if small groups are more effective than large groups in lobbying (if the opportunity cost of lobbying is lower for low than for high income agents, say), the unique long-run equilibrium features a small welfare state.

In our model, collective choices are determined by full anticipation of the dynamic relationships between the current political choice, the evolution of income distribution, and its effects on future political choices. We restrict attention to Markov perfect equilibria. A number of papers has used this equilibrium concept to numerically compute equilibrium paths for a variety of applications (e.g., Krusell, Quadrini and Ríos Rull (1996) and Krusell and Ríos Rull (2000)). In contrast, our analysis results in closed form solutions, due to our focus on quadratic-linear preferences. To our knowledge, the only papers which work out analytical solutions to Markov perfect equilibria are Grossman and Helpman (1998) and Krusell, Kuruscu and Smith, Jr. (2000). Our paper differs substantially from both of these contributions. Grossman and Helpman (1998) analyze the political determination of redistribution in a growth model with overlapping generations and AK technology. Different from us, they focus on intergenerational redistribution only, while we focus on both intra and intergenerational redistribution. More importantly, in their model, agents make no private decisions. Thus, there is no feedback mechanism between public policy and individual behavior, which, in contrast, is a central feature of our analysis. Finally, their model exhibits equilibrium indeterminacy since the government, in equilibrium, is indifferent to the level of redistribution, due to linear preferences and technology. This limits the scope for deriving positive and normative implications. In contrast, our equilibria are determinate, and their interpretations are therefore sharper. Krusell, Kuruscu and Smith, Jr. (2000) is less directly related to our paper. They find an analytical time-consistent solution to an optimal fiscal policy problem in a context of quasi-geometric discounting. Formally related, but very different in the scope of the analysis, are Leininger (1986) and Bernheim and Ray (1989), who analyze properties of Markov perfect equilibria in models of growth with altruism and bequests.

The plan of the paper is the following. Section 2 describes the model. Section 3.1 describes the equilibrium concept which is used in the rest of the paper. Section 3 contains the main results of the paper, and characterize equilibrium under different political regimes (dictatorship and majority voting, subsection 3.2, probabilistic voting, subsection 3.3 and lobbying, subsection 3.4). Section 4 concludes. All proofs are in an appendix.

2 The model

The model economy consists of a continuum of risk-neutral two-period lived agents. Each generation has a unit mass. All agents are born identical, but the subsequent earnings are stochastic. "Successful" agents earn a high wage, normalized to unity, in both periods of their life, whereas "unsuccessful" agents earn a low wage, normalized to zero. At birth, each agent undertakes a costly investment, increasing the probability of subsequent success. The cost of investment, which can be interpreted as the disutility of educational effort, is e^2, and the probability of success is e. It is important for the analysis that agents earn income in both periods of life. However, the simplifying assumption that first and second period income are perfectly correlated is not essential; the qualitative results would be preserved provided earnings in the two periods are positively correlated.

The focal point of the paper is the political mechanism redistributing income from successful to unsuccessful agents. Each period, a transfer $b \in [0, 1]$ to each low-income agent is collectively set. The transfer is financed by collecting a lump-sum tax τ, and the government budget is assumed to balance every period. The transfer, and the associated tax rate, is determined before the young agents decide their investment. By assumption, we rule out age-dependent taxes and transfers.

The objective functions of the agents alive at time t are given as follows:

$$\begin{aligned}
\tilde{V}^{os}(b_t, b_{t+1}, \tau_t) &= 1 - \tau_t \\
\tilde{V}^{ou}(b_t, b_{t+1}, \tau_t) &= b_t - \tau_t \\
\tilde{V}^{y}(e_t, b_t, b_{t+1}, \tau_t) &= e_t(1+\beta) + (1-e_t)(b_t + \beta b_{t+1}) - e_t^2 - \tau_t - \beta\tau_{t+1}
\end{aligned}$$

where $\tilde{V}^{os}$, $\tilde{V}^{ou}$, and $\tilde{V}^{y}$ denotes the objective of the old successful, the old unsuccessful, and the young agents, respectively. $\tilde{V}^{y}$ is computed prior to individual success or faliure and $\beta \in [0,1]$ is the discount factor. It is straightforward to show that the solution to the optimal investment problem of the young, given b_t and b_{t+1}, is $e_t(b_t, b_{t+1}) = \frac{1+\beta-(b_t+\beta b_{t+1})}{2}$.

Since agents are *ex-ante* identical, agents of the same cohort choose the same investment. This implies that the proportion of old unsuccessful in period $t+1$ is given by

$$u_{t+1} = 1 - e_t = \frac{1-\beta+b_t+\beta b_{t+1}}{2}. \tag{1}$$

Thus, the future proportion of old unsuccessful depends on benefits in period t and $t+1$. To balance the budget, tax revenues must amount to $2\tau_t = (u_t + u_{t+1})\, b_t$, which yields:

$$\tau_t = \frac{1-\beta+b_t+\beta b_{t+1}+2u_t}{4} b_t. \tag{2}$$

Note that the marginal tax cost of redistribution, given by $\frac{1-\beta+b_t+\beta b_{t+1}+2u_t}{4}$, increases in u_t (because more old agents are benefit recipients) and in b_t and b_{t+1} (because more young agents become unsuccessful). Since the old in period t cannot enjoy benefits in period $t+1$, their utility is decreasing in b_{t+1}.

By subsituting for τ_t and e_t, the objective functions of all groups can be rewritten as:

$$\begin{aligned}
\hat{V}^{os}(b_t, b_{t+1}, u_t) &= 1 - \frac{(1-\beta)+(b_t+\beta b_{t+1})+2u_t}{4} b_t, \\
\hat{V}^{ou}(b_t, b_{t+1}, u_t) &= b_t - \frac{(1-\beta)+(b_t+\beta b_{t+1})+2u_t}{4} b_t, \\
\hat{V}^{y}(b_t, b_{t+1}, b_{t+2}, u_t) &= \frac{(1+\beta)^2}{4} + \frac{(1-\beta)-2u_t}{4} b_t - \frac{b_{t+1}+\beta b_{t+2}}{4} \beta b_{t+1}.
\end{aligned} \tag{3}$$

The old successful agents obviously prefer zero benefits, since redistribution implies positive taxes without providing them any benefits. The old unsuccessful agents, in contrast, are better off with some redistribution, even though their preferences for redistribution may be non-monotonic, as the marginal cost of redistribution is increasing. Concerning the preferences of the young, note that positive benefits lead to positive (negative) intergenerational redistribution from the old to the young if the number of old unsuccesful is sufficiently small (large). Holding future benefits constant, the young therefore prefer positive redistribution if and only if $u_t < (1-\beta)/2$.

3 Equilibria under alternative political mechanisms

In this section, we analyze the political determination of redistribution. We allow benefits in period t to distort the investment decisions in the same period by assuming the level of redistribution to be determined before the investment choice. We analyze equilibria where the outcome of the political mechanism can be represented as the maximization of a political objective function, given by a weighted sum of the objective functions of currently living individuals,

$$\begin{aligned}
\hat{V}(b_t, b_{t+1}, b_{t+2}, u_t) \equiv\; & W(u_t)\, \hat{V}^{ou}(b_t, b_{t+1}, u_t) + (1-W(u_t))\hat{V}^{os}(b_t, b_{t+1}, u_t) \\
& + W^y(u_t)\hat{V}^y(b_t, b_{t+1}, b_{t+2}, u_t),
\end{aligned}$$

where the weight on the unsuccessful is given by the function $W(u_t) \in [0,1]$ and $W^y(u_t) \in [0,1]$ is the weight on the young. We will show below that this representation covers all voting games discussed in the introduction, where the political weighting functions will be derived as equilibrium outcomes under different assumptions about the political environment.

A key assumtion for the model to generate non-trivial resuls is that young individuals have less political influence than old, i.e., that the weight on the old in the political objective function is larger than the weight on the young. We show in the appendix that with equal wheights, equilibrium redistribution is zero, except, possibly, in the initial period. For expositional convenience, we will first analyze the case when the young have no influence at all.

3.1 Definition of political equilibrium

We restrict attention to Markov perfect equilibria, where the state of the economy is summarized by the proportion of unsuccessful old agents (u_t). In the spirit of Krusell, Quadrini and Ríos Rull (1996), the political equilibrium is defined as follows.

Definition 1 *A (Markov perfect) political equilibrium is defined as a pair of functions* $\langle B, U\rangle$*, where* $B : [0,1] \rightarrow [0,1]$ *is a public policy rule,* $b_t = B(u_t)$ *, and* $U : [0,1] \rightarrow [0,1]$ *is a private decision rule,* $u_{t+1} = 1 - e_t = U(b_t)$ *, such that the following conditions hold:*

1. $B(u_t) = \arg\max_{b_t} \hat{V}(b_t, b_{t+1}, b_{t+2}, u_t)$ *subject to* $b_{t+2} = B(U(B(U(b_t))))$*,* $b_{t+1} = B(U(b_t))$ *, and* $b_t \in [0,1]$ *,*
2. $U(b_t) = (1 - \beta + b_t + \beta b_{t+1})/2$*, with* $b_{t+1} = B(U(b_t))$*.*

The first equilibrium condition implies that the political mechanism chooses b_t to maximize the political objective, taking into account that future redistribution depends on the current policy choice via the equilibrium private decision rule and future equilibrium public policy rules. The second equilibrium condition means that all young individuals choose their investment optimally, given b_t and b_{t+1}, and that agents hold rational expectations about future benefits. In general, U should be a function of both u_t and b_t. In our particular model, however, u_t has no direct effect on the investment choice of the young. Thus, we focus on equilibria where the equilibrium investment choice of the young is fully determined by the current benefit level.

For notational convenience, let us define equilibrium objective functions by replacing future benfits in (3) by their equilibrium counterparts ;

$$\begin{aligned}
V^j(b_t, u_t) &\equiv \hat{V}^j(b_t, B(U(b_t)), u_t), \text{ for } j \in \{os, ou\}, \\
V^y(b_t, u_t) &\equiv \hat{V}^y(b_t, B(U(b_t)), B(U(B(U(b_t)))), u_t) \\
V^p(b_t, u_t) &\equiv W(u_t) V^{ou}(b_t, u_t) + (1 - W(u_t)) V^{os}(b_t, u_t) + W^y(u_t) V^y(b_t, u_t), \\
&\qquad \text{for } p \in \{mv, pv, lo\},
\end{aligned}$$

In the following subsections, we will characterize equilibria under different assumptions about the political mechanism. In particular, the political mechanisms will differ in the extent to which minorities can influence the equilibrium political outcome. We will start with *majority voting (mv)*, where minorities have no weight at all. Thereafter, we will consider *probabilistic voting (pv)*, in which weights are assigned proportionally to the size of the group. Finally, we analyze the case of *lobbying (lo)*, in which minorities have more than proportional weight.

3.2 Majority voting

We consider, first, the case of majority voting. Since majority voting implies that the minority has no political influence, it is expositionally convenient to start by describing the equilibrium if the political power were permanently in the hands of one of the two groups of old agents in the society. We define as "plutocracy" (PL) and "dictatorship of proletariat" (DP), the regimes in which the successful and unsuccessful old agents, respectively, choose the level of redistribution. Formally, under DP, $W(u_t) = 1$ for all u_t, whereas, under PL, $W(u_t) = 0$ for all u_t. In both cases, we set $W^y = 0$.

The results concerning equilibrium under dictatorship are summarized by the following Proposition:

Proposition 1 *The PL equilibrium, $\langle B^{pl}, U^{pl} \rangle$, is characterized as follows;*

$$B^{pl}(u_t) = 0 \tag{4}$$

$$U^{pl}(b_t) = \frac{1 - \beta + b_t}{2} \tag{5}$$

Given any u_0, for all $t \geq 1$, $u_t = u^{pl} \equiv \frac{1-\beta}{2}$.
The DP equilibrium, $\langle B^{dp}, U^{dp} \rangle$, is characterized as follows;

$$B^{dp}(u_t) = \begin{cases} \frac{3}{2} - u_t & \text{if } u_t > \bar{u}(\beta) \\ \frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{2}{2-\beta} u_t & \text{if } u_t \in \left[\frac{3}{2} - \frac{2}{2+\beta}, \bar{u}(\beta)\right] \\ 1 & \text{if } u_t \in \left[0, \frac{3}{2} - \frac{2}{2+\beta}\right] \end{cases} \tag{6}$$

$$U^{dp}(b_t) = \begin{cases} \frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4} b_t & \text{if } b_t \in \left[\frac{2\beta}{2+\beta}, 1\right] \\ \frac{1+b_t}{2} & \text{if } b_t \in \left[0, \frac{2\beta}{2+\beta}\right) \end{cases} \tag{7}$$

where $\bar{u}(\beta) = \min\left\{\frac{\beta+6-\beta\sqrt{4-2\beta}}{2(2+\beta)}, 1\right\}$. The equilibrium law of motion is as follows;

$$u_{t+1} = \begin{cases} \frac{5}{4} - \frac{u_t}{2} & \text{if } u_t > \bar{u}(\beta) \\ \frac{1}{4}\left(5 + \frac{\beta^2}{2+\beta}\right) - \frac{u_t}{2} & \text{if } u_t \in \left[\frac{3}{2} - \frac{2}{(2+\beta)}, \bar{u}(\beta)\right] \\ \frac{\beta}{4} + \frac{2}{2+\beta} & \text{if } u_t \in \left[0, \frac{3}{2} - \frac{2}{(2+\beta)}\right] \end{cases} \tag{8}$$

Given any u_0, the economy converges (with an oscillatory pattern) to a unique steady-state, such that:

$$u = u^{dp} = \left(5 + \beta^2 / (2+\beta)\right)/6, \tag{9}$$

$$b = b^{dp} = \frac{4}{3}\frac{1+\beta}{2+\beta}. \tag{10}$$

Figure 1 represents the equilibrium policy function and law of motion for the PL and DP equilibrium. In the PL case, both functions are constant at $B(u_t) = 0$ and $u_{t+1} = (1-\beta)/2$, respectively. In the DP case, the equilibrium redistribution is always positive, and is 100% as long as $u_t \leq \frac{3}{2} - \frac{2}{(2+\beta)}$. Note that in steady state, benefits are smaller than 100%, irrespective of β. Figure 1 represents a parametric case where $\bar{u}(\beta) < 1$ (note that

$\bar{u}(\beta) = 1$ if $\beta \geq \left(\sqrt{17}-1\right)/4 \approx 0.78$). In this case, there is a range of high u_t ($u_t \in [\bar{u}, 1]$) such that, the unsuccessful in period t find it optimal to induce a u_{t+1} such that the constraint $b_{t+1} \leq 1$ is binding. When this constraint is not binding, equilibrium b_{t+1} is negatively related to b_t. This reduces the marginal distortion of current benefits. Thus, when $u_t = \bar{u}$ and the constraint on b_{t+1} starts to bind, optimal b_t fall discountinuously. Note, however, that $[\bar{u}, 1]$ is "non-recurrent", in the sense that no equilibrium path can lead to states belonging to this region, i.e., $u_t < \bar{u}$ $\forall t > 0$.

Let us now assume that political decisions are taken through majority voting. Agents vote on the single issue of redistribution, and the old only are entitled to vote. Under these assumptions, the political objective function is $V^{mv}(b_t, u_t)$, with $W(u_t) = 0$ if $u_t \leq 0.5$ and 1 otherwise, implying that, in case of a tie, the successful agents prevail.

As we shall see, majority voting can generate persistence in the equilibrium choice of redistribution: if the economy starts with a pro-welfare state majority (more than 50% of the agents are unsuccessful), there exists an equilibrium in which the welfare state survives over time. Conversely, if more than 50% of the agents are successful at time zero, the welfare state will never arise. This illustrates *policy-behavior complementarity;* high (low) benefits today implies a large (small) proportion of unsuccessful agents tomorrow, and thus a broad (narrow) future constituency for welfare state policies. Thus, majority voting can sustain multiple steady states for any rate of discounting.

There is, however, an important asymmetry concerning the robustness of the two steady-states. The welfare state will never arise when there is an initial majority of successful agents, irrespective of the discount factor. An initial majority of unsuccessful does, however, not guarantee the eternal survival of the welfare state. In particular, the survival of a welfare state is the unique equilibrium only if the discount factor is sufficiently low. For higher discount factors, and given an initial majority of unsuccessful, there exist both an equilibrium in which the welfare state survives forever, and an equilibrium in which any existing welfare state is dismantled. In this case, as we will see in detail, the survival of the welfare state depends on expectations. More specifically, if the young individuals believe that the welfare state will be dismantled, the ruling unsuccessful old agents find it optimal to induce a future political majority that will shut down redistribution to zero fulfilling the expectations of the young. On the other hand, if the young believe in the survival of the welfare state, the old agents will set redistribution high enough to sustain a constituency for the welfare state.

The following Proposition establishes the existence of an equilibrium with multiple steady-states; if the initial share of unsuccessful is higher than a half, the welfare state survives forever, otherwise it will never arise.

Proposition 2 *For any $\beta \in [0, 1]$, there exists a "multiple steady-state equilibrium" (MSSE), $\langle B^{mv}, U^{mv} \rangle$, with the following characteristics:*

1.

$$B^{mv}(u_t) = \begin{cases} B^{dp}(u_t) & \text{if } u_t \in \left(\frac{1}{2}, 1\right] \\ B^{pl}(u_t) & \text{if } u_t \in \left[0, \frac{1}{2}\right] \end{cases} \tag{11}$$

$$U^{mv}(b_t) = \begin{cases} U^{dp}(b_t) & \text{if } b_t \in (0, 1] \\ U^{pl}(b_t) & \text{if } b_t = 0 \end{cases} \tag{12}$$

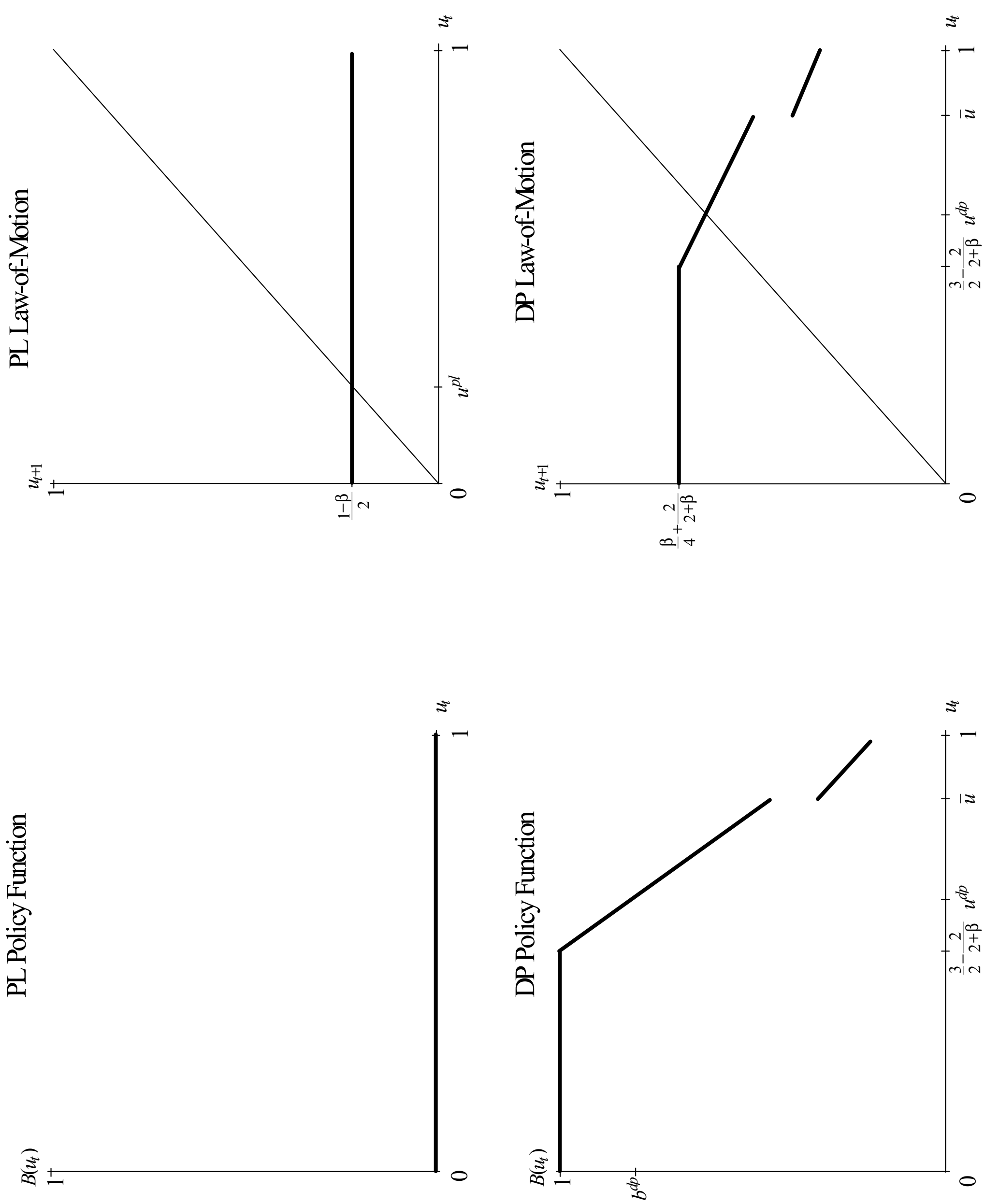
PL Law-of-Motion
DP Law-of-Motion
PL Policy Function
DP Policy Function

Figure 1:

where $B^{dp}(u_t), B^{pl}(u_t), U^{dp}(b_t)$ and $U^{pl}(b_t)$ are defined in Proposition 1. This implies the following equilibrium law of motion;

$$u_{t+1} = \begin{cases} \frac{5}{4} - \frac{u_t}{2} & \text{if } u_t > \bar{u}(\beta) \\ \frac{1}{4}\left(5 + \frac{\beta^2}{2+\beta}\right) - \frac{u_t}{2} & \text{if } u_t \in \left[\frac{3}{2} - \frac{2}{(2+\beta)}, \bar{u}(\beta)\right] \\ \frac{\beta}{4} + \frac{2}{2+\beta} & \text{if } u_t \in \left(\frac{1}{2}, \frac{3}{2} - \frac{2}{(2+\beta)}\right] \\ \frac{1-\beta}{2} & \text{if } u_t \in \left[0, \frac{1}{2}\right]. \end{cases} \tag{13}$$

2. *There are two locally stable steady-states. In particular,*

   (a) *if $u_0 \leq 0.5$, the economy converges in one period to a steady-state equilibrium with $b = b^{pl} = 0$ and $u = u^{pl} = \frac{1-\beta}{2}$.*

   (b) *if $0.5 < u_0 \leq 1$, the economy converges asymptotically to an equilibrium characterized by $b = b^{dp} \equiv \frac{4}{3}\frac{1+\beta}{2+\beta}$ and $u = u^{dp} \equiv \left(5 + \beta^2/(2+\beta)\right)/6$.*

Figure 2 represents geometrically the equilibrium policy function and law of motion for the case when $\bar{u}(\beta) < 1$ (i.e., $\beta > 0.78$). The left hand panel shows that, when $u_t \leq 1/2$, the equilibrium prescribes that $b_t = 0$, as the power is in the hands of the successful agents. At $u_t = 1/2$ the policy function increases discontinuously, as the unsuccessful become the decisive group. In fact, for $u_t \in \left(\frac{1}{2}, \frac{3}{2} - \frac{2}{(2+\beta)}\right]$, the equilibrium policy function prescribes 100% redistribution (the constraint that $b \leq 1$ binds).

For $u_t \geq \frac{3}{2} - \frac{2}{(2+\beta)}$, the equilibrium law of motion is downward sloping, reflecting the fact that redistribution becomes more and more expensive for the decisive unsuccessful as the current proportion of old successful agents becomes smaller. The right hand panel shows that the law of motion implies two fixed points for u, given by u^{pl} and u^{dp}. For all $u_t \leq 1/2$, the economy converges in one period to u^{pl}. For higher initial u_t, the relationship between u_{t+1} and u_t is downward sloping, implying that the economy converges to the steady-state with a welfare state, u^{dp}, following an oscillatory pattern.

Proposition 2 shows that our model provides an explanation for the historical persistence of the welfare state as generated by the ex-post conflict of interest between successful and unsuccessful. Our model does not explain why a pro-welfare majority first arose. However, suppose that for some reason, a pro-welfare political majority materializes ($u_0 > 0.5$), for example due to a depression or the introduction of democracy. Our model then predicts that the institutions that this majority promotes may survive, as they can fuel their own constituency over time. The model also predicts, however, that should these institutions happen to be dismantled, they would not resurrect.

So far, we have analyzed equilibria where an existing welfare state survives. As we will see, however, there may also exist an equilibrium where the welfare state breaks down. The simultaneous existence of these two qualitatively very different equilibria is directly related to the fact that since the public policy function B is non-linear and locally increasing, there may be multiple solutions to our equilibrium condition 2 in definition 1 ($u_{t+1} = (1 - \beta + b_t + \beta B(u_{t+1}))/2$). Intuitively, if agents believe that benefits next period will be high (low), there will be a large (small) share of unsuccessful next period, which, due to the increasing public policy function, leads to high (low) benefits next period. In particular,

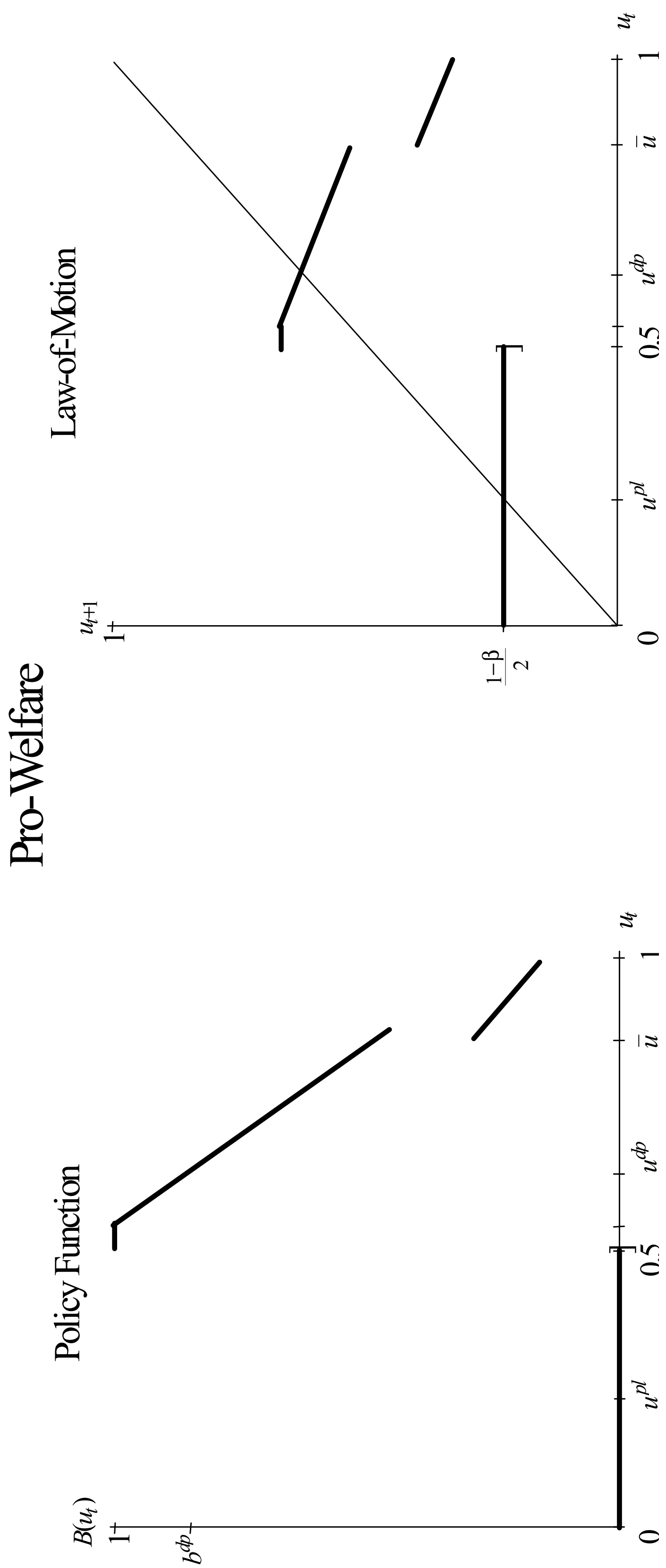

Figure 2: Equilibrium with majority voting; multiple steady-states.

any equilibrium policy function under majority voting features $B(u) = 0$ for $u \leq 1/2$ and $B(u) > 0$ for $u > 1/2$, and the upward discontinuity at $u = 1/2$ implies that there will generically exist two solutions to equilibrium condition 2. In the equilibrium described in proposition 2, the agents coordinate their beliefs on the *higher* of the two solutions to the rational expectation condition. Then, it is immediate to verify that a switch of majority ($u_{t+1} \leq 1/2$) can be attained only by setting $b_t = 0$ since there exists a solution to equilibrium condition 2 such that $u_{t+1} > 1/2$ for all $b_t > 0$. Since setting $b_t = 0$ can never be optimal for the ruling unsuccessful, the equilibrium will never feature a switch of majority. Thus, the welfare state necessarily survives when agents have these "pro-welfare" expectations.

Alternatively, agents may coordinate their beliefs on the *lower* of the two solutions to the rational expectation condition. Then, it is straightforward to show that a switch of majority ($u_{t+1} \leq 1/2$) will be attained whenever $b_t \leq \beta$ since there is a solution to the rational expectation condition such that $u_{t+1} \leq 1/2$ and, consequently $B(u_{t+1}) = 0$, for all $b_t \leq \beta$. Under such "anti-welfare" expectations, there may exist an equilibrium such that the welfare state breaks down. The old unsuccessful know that if they keep today's demand for redistribution sufficiently low, they can induce the young to believe that no redistribution will be granted in future and that high investment is the optimal strategy. This implies lower taxes for everybody, including the old unsuccessful, and creates an incentive for lower benefits. But when is it optimal for the old to restrain their demand for redistribution and induce a switch of majority?

To answer this question, we first notice that it is never optimal for the old unsuccessful to set $b_t < \beta$ (proof omitted). Thus, the pay-off for the old unsuccessful to induce a switch of majority is $\hat{V}^{ou}(b_t, b_{t+1}, u_t) = \hat{V}^{ou}(\beta, 0, u_t)$. Moving from this observation, we can prove two results. First, if β is sufficiently low ($\beta < 1/4$), it is never optimal for the old unsuccessful to induce the breakdown of the WS (Proposition 1). In this case, the old unsuccessful will set $b_t > \beta$, the initial majority of unsuccessful will always regenerate itself and the welfare state will be sustained perpetually. Second, if β is sufficiently large ($\beta \geq \beta_M \simeq .555$), there exists an equilibrium featuring the termination of the WS in, at most, two periods (Proposition 4). Namely, along the equilibrium path, there will be a generation of old unsuccessful setting $b_t = \beta$ and inducing a majority of successful agents in the following period.[2]

We start by proving that when agents discount the future sufficiently much, there is no equilibrium featuring the dismantling of an existing welfare state.

Proposition 3

1. *Assume* $u_0 > 1/2$ *and* $\beta < 1/4$. *Then,* $\forall t \geq 0$, $u_t > 1/2$ *and* $b_t > 1/4$.

2. *If* $u_t > 1/2$, *then* $b_t \geq \left(\frac{1}{2}(3+\beta) - u_t - \frac{1}{2}\sqrt{\beta(6 - 4u_t + \beta)}\right) \geq 1 - \sqrt{3}/2$.

The first part of Proposition 1 states that, for $\beta < 1/4$, an initial majority of unsuccessful implies a pro-WS majority forever. It should be noted that this results holds for any optimal

[2]The analysis of the intermediate range of β is complicated by a technical problem. The problem originates from a discontinuity in the political policy functions causing nonexistence of a private decision rule, U, consistent with rational expectations. This case can be analyzed under a refinement of the equilibrium concept, allowing for some out-of-equilibrium beliefs to be inconsistent with part 2 of definition 1. Details of this analysis are available upon request.

public policy rule, also non-markovian and non-linear. The reason is that when $\beta < 1/4$ and for any $u_t > 1/2$,

$$\hat{V}^{ou}(\beta, 0, u_t) < \max_{b_t \in (\beta,1)} \hat{V}^{ou}(b_t, 1, u_t) \tag{14}$$

where the left-hand-side is an *upper* bound for attainable indirect utility of inducing a breakdown and the right-hand-side is a *lower* bound for the indirect utility of sustaining a pro-welfare majority. The second part establishes a lower bound for the choice of redistribution by a majority of unsuccessful. Note that the lower bound falls in u_t (since higher u_t implies that redistributing is more costly) and in β (since lower discounting means that taxes are more sensitive to current benefit levels).

Next, we turn to show that an equilibrium with the breakdown of the WS exists when β is sufficiently high.

Proposition 4 *Let $\beta_M \approx 0.555$ be the root in $[0,1]$ to the equation $0 = 2 - \beta_M - 2\beta_M^2 - 2\left(\sqrt{\beta_M}\right)^3$. For all $\beta \geq \beta_M$, there is a "anti-welfare equilibrium" (AWE) converging to zero redistribution in, at most, two periods.*

The characterization of this equilibrium is conceptually simple. The analytical description varies with β and is somewhat cumbersome. It is therefore deferred to the appendix (Proposition 9). Here, we provide a graphical description of two cases (see figure 3). The first case, described in the two upper panels, correspond to a range of high β's ($\beta > 0.618$). When $u_t \leq 1/2$, the equilibrium prescribes that $b_t = 0$, as usual, whereas, when $u_t > 1/2$, the equilibrium prescribes that $b_t = \beta$. This implies $u_{t+1} = 1/2$ (as the left hand panel shows), namely, the old unsuccessful in charge at t choose to strategically keep the current benefits low (i.e., set $b_t = \beta$), in order to induce a plutocratic majority setting zero benefits next period. Moderate redistribution at t together with the expectation of no redistribution at $t+1$ induce the young to exert high investment at t, and this implies low taxes today.

The second case, described in the two lower panels, corresponds to the range $\beta \in [0.570, 0.618]$. The main difference is that, in this case, there is a range of intermediate u's, larger than one half but smaller than u^a, where the proletarian majority choose 100% redistribution. As the lower right hand panel shows, the law of motion implies, in this case, a proletarian majority at $t+1$. The distribution at $t+1$ will be characterized, however, by $u_{t+1} > u^a$, and the equilibrium benefit rate will be set equal to β. This, in turn, implies that the switch of majority will occur at $t+2$. The intuition is the following. For this lower range of βs, the incentive for the old unsuccessful to vote for a benefit level inducing a plutocratic majority next period and $b_{t+1} = 0$, is weaker. In particular, setting $b_t = \beta$ is only optimal for large initial values of u ($u_t \geq u^a$). When $1/2 < u_t < u^a$, however, another strategic opportunity arises for the proletarian majority. That is, to vote for so high redistribution ($b_t > \beta$) as to induce $u_{t+1} > u^a$, implying that (i) a proletarian majority will hold power at $t+1$, and (ii) this majority will find it optimal to induce a switch into plutocracy at $t+2$. Thus the sequence of redistribution will be $b_t = 1$, $b_{t+1} = \beta$ and $b_{t+2} = 0$. Note that the anticipation of a switch of majority in two periods (implying that b_{t+1} is rather low, although larger than zero) induces the proletarian majority at t to insist on very high redistribution, so as to make sure that the next generation will find the welfare state too costly and act for its dismantling. Note that low discounting is crucial for this sequence of strategic voting to be optimal, since it implies the investment of the young to be very sensitive to future redistribution.

A third case, covered in proposition 9 in the appendix (no graph), arises when $\beta \in [0.555, 0.570]$. This case is qualitatively similar to that described in the lower panels of figure 3, with the only difference that, in this last case, there is a range of u's where the unsuccessful majority is not bounded by the constraint $b \leq 1$. In any case, they set benefits high enough to induce a shift of majority in two periods.

Propositions 2 and 4 imply, jointly, that multiple self-fulfilling equilibria exist when the economy starts with a pro-welfare majority, provided β is not too low. In one of them the welfare state survives, while in the other is terminated. As noted above, the source of this multiplicity is the fact that on the one hand the investment choice of the young depends on the expected future redistribution, while, on the other hand, the future political choice of redistribution depends on the current investment of the young. If the young believe that the welfare state will (will not) survive, they will choose low (high) investment, and many (few) of them will be unsuccessful forming a large (small) constituency for the survival of the welfare state.

3.2.1 Majority voting with young voters

In the previous section, we showed that a welfare state can arise and be sustained when only the old vote. We now extend the analysis and allow the young to participate in the political process, although we assume that, for exogenous reasons, the young have a lower turnout rate. We denote by $\varepsilon \in [0, 1]$ the share of the young individuals who participate in the voting process. It is straightforward to show that, if $\varepsilon = 1$ (equal turnout of young and old agents), then, for all $t > 0$, there will always be a strict majority in favor of zero benefits.[3] This is, however, not necessarily true when $\varepsilon < 1$, i.e., when the turnout is lower among the young. In this case, an equilibrium featuring the survival of the welfare state can still exist. We now analyze under which condition this is the case.

If a share ε of the young vote, the old unsuccessful are in majority among the voters if $u_t > (1 - u_t) + \varepsilon$, i.e., if $u_t > \frac{1+\varepsilon}{2}$, Similarly, if $1 - u_t \geq u_t + \varepsilon$, or $u_t < \frac{1+\varepsilon}{2}$, there is a majority of old successful. It is important to note that allowing the young to vote affects the political behavior for the old unsuccessful, including when these are in majority. In particular, when the young vote, it becomes more attractive for the old unsuccessful to vote strategically for low benefits so as to induce a switch of majority and the break-down of the welfare state. The reason is that, as argued above, the young will always vote for zero benefits. This means that it is no longer necessary to set $b_t = 0$ in order to induce a majority voting for zero benefits in the next period even if agents coordinate, in case of multiple self-fulfilling beliefs, on the *highest of the solutions to the rational expectation condition* (see discussion above). More precisely, at time t, no solution with $b_{t+1} > 0$ consistent with rational expectations can exist if

[3] To see this, first note from (**??**) that the indirect utility of the young is decreasing in b_t when u_t is larger than $\frac{1-\beta}{2}$. Second, whenever $b_{t+1} > 0$, $u_{t+1} > \frac{1-\beta}{2}$ implying that all young vote for zero benefits in period $t + 1$. Third, unless both b_t and b_{t+1} are unity, some old in period $t + 1$, will be successful. But if $b_{t+1} = 1$, all voters in period t prefer $b_t < 1$, making impossible both b_t and b_{t+1} being unity. Thus, under majoritarian voting with equal participation, there will always be a clear strict majority in favor of $b_{t+1} = 0$ $\forall t$.

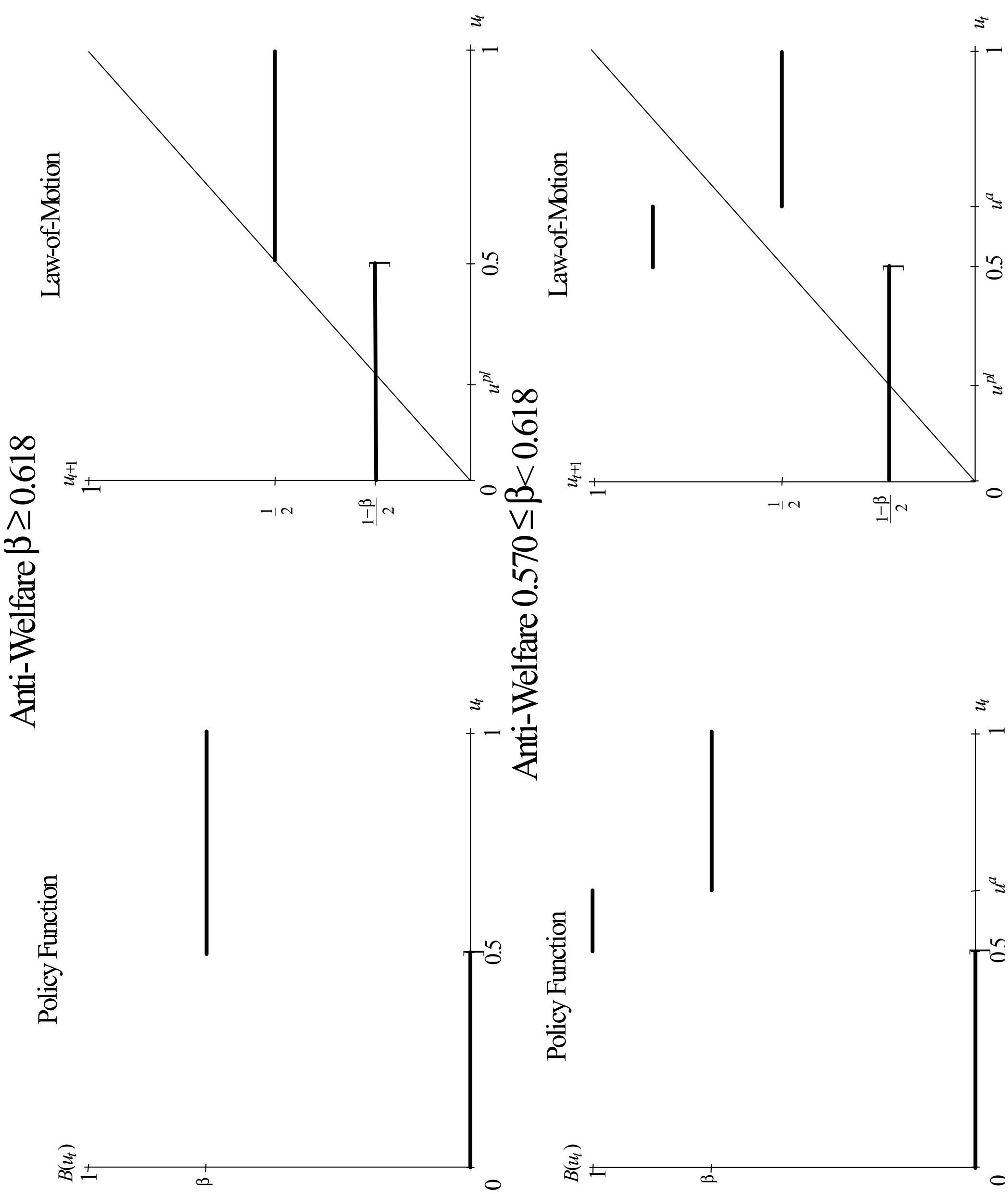

Figure 3: Equilibria under anti-welfare expectations and low rates of discounting

$$\frac{1-\beta+b_t+\beta B^{dp}\left(\frac{1+\varepsilon}{2}\right)}{2} \leq \frac{1+\varepsilon}{2},$$
$$\Rightarrow b_t \leq \varepsilon+\beta\left(1-B^{dp}\left(\frac{1+\varepsilon}{2}\right)\right) \equiv b^{bd}.$$

where $B^{dp}(.)$ is defined as in (6). When only the old vote, then $b^{bd}=0$, and the only choice consistent with a future majority against redistribution is to set benefits to zero. If $\varepsilon>0$, however, since also the young vote for zero benefits, a smaller share of old successful is needed to create a majority against redistribution. Then, there exists a positive range of choices of b_t which is consistent with a future anti-welfare majority, and the equilibrium featuring a welfare state becomes more fragile. Nevertheless, for a range of sufficiently low ε's, this equilibrium remains sustained. Under this equilibrium the economy converges to the same steady-state with positive benefits as in the DP-equilibrium. Formally,

Proposition 5 *Suppose that a share ε of the young are allowed to vote under majority voting. Then, if $u_0 \in \left(\frac{1+\varepsilon}{2}, \frac{\beta}{4}+\frac{2}{2+\beta}\right]$ and $\varepsilon \leq \bar{\varepsilon}(\beta)$, there exists an equilibrium such that the economy converges asymptotically to a steady state with $b=b^{dp}$ and $u=u^{dp}$ following the law-of-motion*

$$u_{t+1}=\begin{cases} \frac{1}{4}\left(5+\frac{\beta^2}{2+\beta}\right)-\frac{u_t}{2} & \textit{if } u_t>\frac{3}{2}-\frac{2}{(2+\beta)} \\ \frac{\beta}{4}+\frac{2}{2+\beta} & \textit{if } u_t \leq \frac{3}{2}-\frac{2}{(2+\beta)} \end{cases},$$

where

$$\bar{\varepsilon}(\beta)=\begin{cases} \frac{1}{4}\frac{\beta^2+8\beta+4-\sqrt{\left(\beta^4+34\beta^3+60\beta^2+24\beta\right)}}{\beta+2} & \textit{if } \beta>\beta^y \\ \frac{-\frac{1}{8}\beta^3+\frac{1}{4}\beta^2+\frac{3}{2}\beta+1-\frac{1}{8}\sqrt{\left(\beta^6+30\beta^5-72\beta^4-80\beta^3+144\beta^2+96\beta\right)}}{2+\beta} & \textit{if if } \beta \leq \beta^y, \end{cases}$$

and $\beta^y \equiv \frac{4}{51}\sqrt[3]{\left(586+102\sqrt{33}\right)}+\frac{16}{51\sqrt[3]{\left(586+102\sqrt{33}\right)}}-\frac{26}{51} \simeq .347.$

Note that $\bar{\varepsilon}(\beta)$ can take on values ranging between 0.174 and 0.5 and is strictly decreasing with β. This suggests that only if the participation of the young is rather limited (less than 50% of the participation of the old) the welfare state can survive. This feature hinges, however, on the assumption that the young are, ex-ante, perfectly homogenous, none of them being interested in distortionary redistribution. The result would change if we assumed, more realistically, that the young were heterogenous in ability, implying different preferences over redistribution. A low ability agent, for example, might want, *ex-ante*, high redistribution if he has an inherently low probability of becoming successful, or a high effort cost. In this case, a coalition of interests between old unsuccessful and low ability young would arise, and increase, ceteris paribus, the political support for redistribution.

In the interest of space, the proposition does not cover initial conditions $u_0 \leq (1+\varepsilon)/2$ and $u_0>\beta/4+2/(2+\beta)$. These are, however, straightforward to analyze. It can be shown that if $u_0<\frac{1-\beta}{2}$, the young have a motive for increasing benefits above zero since this generates an intergenerational transfer from the old to the young. If β is sufficiently low, this motive may be strong enough to imply that the young vote for $b_t>b^{bd}$. If, in addition, the young are politically pivotal, they will induce a majority of unsuccessful old individuals in period 1. However, $u_t < \frac{1-\beta}{2}$ is not possible for $t > 0$, since, as we know, $u_{t+1} = \frac{1-\beta+b_t+\beta b_{t+1}}{2} \geq \frac{1-\beta}{2}$. Thus, the situation just described can only arise in the first period. Finally, if $u_0 > \frac{\beta}{4} + \frac{2}{2+\beta}$, the old unsuccessful are in majority and may, provided ε is large enough, vote for $b_0 = b^{bd}$, inducing a switch of majority and the breakdown of the welfare state. However, in equilibrium, $u_t < \frac{\beta}{4} + \frac{2}{2+\beta}$ for all $t > 0$. Thus, again, this scenario can only arise in the first period.

3.3 Probabilistic voting

In this subsection, we modify the political game in a way that implies that the preferences of all voters, including minorities, influence the political decisions. The political mechanism will be represented as a weighting function that gives a weight to the different groups which depend on their size. In appendix 5.1 we follow Lindbeck and Weibull (1987) and derive this weighting function as the equilibrium of a political game where voters cast their votes on one of two candidates, denoted A and B, who maximize the probability of becoming elected. As in the previous section, we use as a benchmark the case when only the old agents vote, and later extend the result to the case in which all agents vote, although the young have a lower influence on the political outcome.[4]

In general, the unique equilibrium under probabilistic voting has both candidates choosing the same platform, and

$$b_t^{pv} = \arg\max_{b_t} \left\{ V^{pv}(b_t, u_t) \right\},$$

where $V^{pv}(b_t, u_t) \equiv \omega V^y(b_t, u_t) + (1 - u_t)V^{os}(b_t, u_t) + u_t V^{ou}(b_t, u_t)$. Note, here, that the relative weight of the old unsuccessful relative to the successful is equal to the relative size of their group ($W(u_t) = u_t$). The size of the young is fixed to half of the population. Their political weight is assumed to be, however, less than proportional to their group size. In particular, in the benchmark case, $\omega = 0$, and the political mechanism maximizes the sum of the old individuals' utilities only. In the more general case, $\omega \leq 1/2$ and both the young and the old carry a weight in the political process.

We will now characterize the political equilibrium in the benchmark case.[5]

Proposition 6 *The PV political equilibrium,* $\langle B^{pv}, U^{pv} \rangle$ *, is characterized as follows;*

$$B^{pv}(u_t) = \begin{cases} -\frac{1-\beta}{2+\beta} + \frac{2}{2+\beta} u_t & \text{if } u_t \geq \frac{1-\beta}{2} \\ 0 & \text{if } u_t < \frac{1-\beta}{2} \end{cases} \tag{15}$$

$$U^{pv}(b_t) = \frac{1 - \beta + b_t \left(1 + \frac{\beta}{2}\right)}{2} \tag{16}$$

[4] In the case of probabilistic voting considered in this section, this lower influence needs not rely on the exogenous assumption that the young have a lower turnout in the polls. It can, alternatively, be derived from assuming that the political preferences of the young are less narrowly focused on the redistributional issue analyzed in this paper. This could be the case if, for instance, the young care more than the old about some exogenous traits of the candidates when they cast their ballot, according to preferences which are ex-ante unknown to the politicians (see Appendix 2 for a more formal discussion). Under this assumption, the young voters would be less attractive to power-seeking candidates and exert, endogenously, a lower influence on the equilibrium political outcome, irrespective of their turnout rate.

[5] This is, in general, a difficult problem, which previous literature has typically dealt with by resorting to numerical methods. Fortunately, our problem is sufficiently simple to allow analytical solution

implying the following equilibrium law of motion;

$$u_{t+1} = \begin{cases} \frac{1-\beta}{4} + \frac{u_t}{2} & \text{if } u_t \geq \frac{1-\beta}{2} \\ \frac{1-\beta}{2} & \text{if } u_t \leq \frac{1-\beta}{2} \end{cases} \tag{17}$$

Given any u_0, the economy converges (with a monotonic pattern) to a unique steady-state, such that $b = b^{pv} = 0$ and $u = u^{pv} = (1-\beta)/2$.

In the probabilistic voting equilibrium, redistribution only occurs along the transition path, if $u_0 > (1-\beta)/2$,. In the long-run, there is no welfare state. Figure 4 represents the equilibrium policy function and law of motion, with the steady-states levels, b^{pv} and u^{pv}. The left-hand panel shows that, when $u_t > (1-\beta)/2$, the equilibrium prescribes positive redistribution. Moreover, the equilibrium level of b_t increases rather than decreases with u_t.

Our result can be interpreted as follows;[6] under probabilistic voting, the political outcome is equivalent to maximizing average income of the old, regardless of the number of unsuccessful. This means that the goal of redistribution is not intra-generational income equalization. Thus, in the absence of intergenerational transfers, the political outcome would be zero redistribution, since it is distortionary and would reduce the average old agent's income. Therefore, a necessary condition for having $b_t > 0$ is that, in equilibrium, there is an intergenerational transfer from the young to the old, namely the proportion of old unsuccessful agents earning a zero wage is higher than the proportion of young unsuccessful. In our model, this implies that $u_t > u_{t+1}$. As the right-hand panel shows, the equilibrium law of motion implies a monotonic convergence to the steady-state. Thus, if $u_t > u^{dp}$, the planner would choose $b_t > 0$ in all period from t until the economy reaches its steady-state. At steady-state, however, there is no possibility to induce intergenerational redistribution in favor of the old, and the political equilibrium necessarily implies $b_t = 0$.

The qualitative results discussed so far generalize to the case in which also the young vote. To study this case formally, define

$$Z(\omega,\beta) \equiv 6^{-2/3} \frac{\left(\left(\left(9(1-\omega) + \left(3\left(27(1-\omega(2-\omega)) + \frac{16}{\omega\beta}\right)\right)^{1/2}\right)\beta^2\omega^2\right)^{1/3}\right)^2 - 6^{1/3}2\beta\omega}{\beta\omega\left(\left(9(1-\omega) + \left(3\left(27(1-\omega(2-\omega)) + \frac{16}{\omega\beta}\right)\right)^{1/2}\right)\beta^2\omega^2\right)^{1/3}},$$

where $Z_\omega(\omega,\beta) < 0$, $\lim_{\omega\to 0} Z(\omega,\beta) = 0.5$, and $Z(1,\beta) = 0$. Recall that $\omega \in [0,1]$ parameterizes the political influence of the young. The following proposition can be established.

Proposition 7 *The PV political equilibrium when both the young and the old vote is characterized as follows;*

$$B^{pv}(u_t) = \begin{cases} \frac{1-\omega}{1+\beta Z+\omega(1+\beta Z)\beta Z^2}\left(-\frac{1-\beta}{2} + u_t\right) & \text{if } u_t \geq \frac{1-\beta}{2} \\ 0 & \text{if } u_t < \frac{1-\beta}{2} \end{cases}$$

$$U^{pv}(b_t) = \frac{1-\beta}{2} + \frac{b_t}{2}\left(1 + \frac{\beta(1-\omega)}{2(1+\beta Z(1+\omega Z+\omega\beta Z^2)) - \beta(1-\omega)}\right),$$

[6] We thank Lars Ljungqvist for helping us with this intuition.

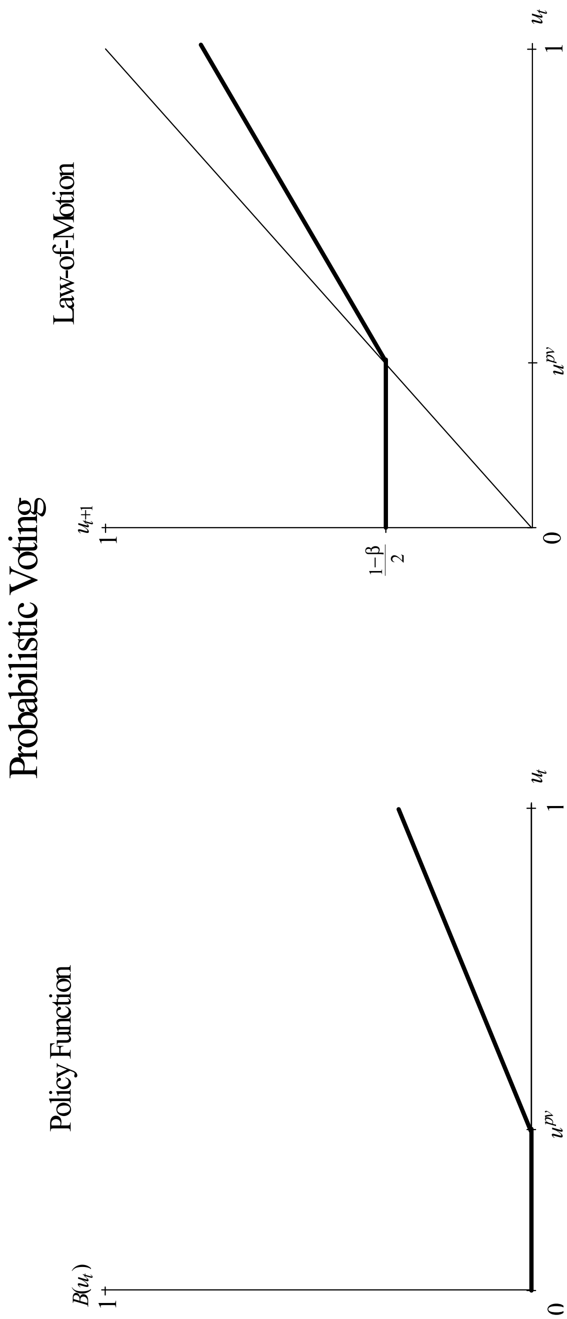

Figure 4: Equilibrium under probabilistic voting

implying the following equilibrium law of motion;

$$u_{t+1} = \begin{cases} \frac{1-\beta}{2} + \frac{1-\omega}{2(1+\beta Z+\omega(1+\beta Z)\beta Z^2)} \left(1 + \frac{\beta(1-\omega)}{2(1+\beta Z(1+\omega Z+\omega\beta Z^2))-\beta(1-\omega)}\right)\left(u_t - \frac{1-\beta}{2}\right) & \textit{if } u_t \geq \frac{1-\beta}{2} \\ \frac{1-\beta}{2} & \textit{if } u_t \leq \frac{1-\beta}{2} \end{cases}$$

Given any u_0, the economy converges (with a monotonic pattern) to a unique steady-state, such that $b = b^{pv} = 0$ and $u = u^{pv} = (1-\beta)/2$.

For $\omega < 1$, the equilibrium has the same qualitative features as in the benchmark case. In particular, redistribution only occurs along the transition path, and there is no welfare state the long-run. The higher is ω, the flatter the equilibrium policy function and law of motion in figure 4, with a kink at the steady-state whose level remains unchanged. This is due to the fact that the young exert political pressure to keep redistribution low. If $\omega = 1$ (full participation and influence of the young), there is no redistribution, irrespective of initial condition, and the economy converges to the steady-state in one period. This shows formally that a utilitarian social planner who cares equally about the old and the young would choose zero redistribution.

3.4 Lobbying

Let us now extend the analysis to the case when minorities have a political weight that is *larger* than their share of the electorate, i.e., $W(u_t) > (<)\ u_t$ if $u_t < (>) 1/2$. Such a weighting function can be derived as the equilibrium outcome in a probabilistic voting model where interest groups have the ability to influence the platforms of the competing candidates by lobbying as in Persson and Tabellini (2000). Here, we present a reduced-form version of a model where the average popularity of the parties depends on the amount of campaign contributions which they receive. In Appendix 5.2, we show that our weighting function can be derived as a first-order approximation of the equilibrium outcome in a model where small interest groups are more effective at organizing their lobbying activity than large groups. The weighting function is under probabilistic voting given by

$$W(u_t) = ((1-\gamma)(1-\alpha) + \gamma u_t),$$

and the equilibrium redistribution is characterized as follows;[7]

$$b_t^{lo} = B^{lo}(u_t) = \arg\max_{b_t} V^{lo}(b_t, u_t).$$

The parameter α captures the intrinsic ability of the successful relative to the unsuccessful group to lobby politicians. If $\alpha = 1/2$, the two groups have the same lobbying technology. If $\alpha = 1$ ($\alpha = 0$) only the successful (unsuccessful) have access to a lobbying technology. The parameter γ captures the extent of decreasing returns to lobbying activity. In particular, if $\gamma < 1$, the smaller group faces lower costs in organizing its contribution campaign,

[7] In this section, we will limit analysis to the benchmark case in which only the old have political influence. It is possible to characterize analytically equilibrium when also the young vote and lobby. We omitthe analysis of the more general case, since this does not add any major insight, and the expressions are somewhat more involved. Details are available upon request.

and agents belonging to the smaller group have a higher influence in the political process than agents belonging to the larger group. When $\gamma = 1$, we are back to the benchmark probabilistic voting environment.[8] As we will see, lobbying will imply that the welfare state may survive as a steady-state equilibrium.

For tractability, we will limit the formal characterization of the political equilibrium under lobbying (lo) to parameters such that along the equilibrium path redistribution is always strictly between zero and 100% (except for non-recurrent states when $\gamma \geq 1/2$, which is easy to deal with). These restrictions on the parameter space are formalized in the following set of assumptions.

Assumption 1 *We assume:*

1. $\gamma > \frac{1}{2}\frac{\beta}{2+\beta}$.
2. *If* $\gamma \in \left[\frac{1}{2}\frac{\beta}{2+\beta}, \frac{1}{2}\right)$, *then* $\frac{(2+3\beta(1-2\gamma))/8-\gamma}{1-\gamma} \leq \alpha \leq \frac{(2+\beta+2\beta\gamma)-\beta^2(1-2\gamma)}{8(1-\gamma)}$
3. *If* $\gamma \geq 1/2$, *then* $\alpha < (1+\beta)/2$.

Part 1 of the assumption ensures that the set of α's satisfying part 2 of the assumption is non-empty. Part 2 guarantees that redistribution is always strictly between 0 and 100% for $\gamma < 1/2$. Part 3, finally, guarantees that redistribution is strictly positive in steady-state when $\gamma \geq 1/2$. When these restrictions do not hold, the equilibrium path is characterized by non-linearities (see, for instance, Proposition 1 when $\bar{u} < 1$) which substantially complicate the analysis, without adding any major insight.

Proposition 8 *Let assumption 1 hold. The LO political equilibrium,* $\langle B^{lo}, U^{lo} \rangle$, *is then characterized as follows.*

1.

$$B^{lo}(u_t) = \begin{cases} c(\alpha,\beta,\gamma) + \frac{2\gamma-1}{1+\frac{\beta}{2}(2\gamma-1)}u_t & \text{if } \gamma < 1/2 \\ \begin{cases} c(\alpha,\beta,\gamma) + \frac{2\gamma-1}{1+\frac{\beta}{2}(2\gamma-1)}u_t & \text{if } u_t > u_y \\ 0 & \text{if } u_t \leq u_y \end{cases} & \text{if } \gamma \geq 1/2 \end{cases}$$

$$U^{lo}(b_t) = \left(\frac{1}{2} - \frac{2\beta^2(2\gamma-1) + 4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right) + \frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}B^{lo}(u_t),$$

where;

$$c(\alpha,\beta,\gamma) \equiv 2\frac{3+\beta-\beta(1-\beta)(\gamma-1/2)-4(\alpha+\gamma-\alpha\gamma)}{(2+\beta)(2+\beta(2\gamma-1))};$$

$$u_y(\alpha,\beta,\gamma) \equiv \frac{8(\alpha+\gamma-\alpha\gamma)-\beta^2(2\gamma-1)-3(2+\beta)+2\beta\gamma}{2(2+\beta)(2\gamma-1)}.$$

[8] This specification actually encompasses all models discussed earlier on. In particular, $\alpha = \gamma = 0$ corresponds to "dictatorship of proletariat", $\alpha = 1$ and $\gamma = 0$ corresponds to "plutocracy". Setting $\alpha = 1/2$ and letting γ approach infinity yields the majority voting model.

This implies the following equilibrium law of motion;

$$u_{t+1} = \begin{cases} \frac{5(2+\beta)-8(\gamma+\alpha(1+\beta)(1-\gamma))-2\beta\gamma(3+\beta)+\beta^2}{4(2+\beta)} + \left(\gamma - \frac{1}{2}\right) u_t & \text{if } \gamma < 1/2 \\ \frac{5(2+\beta)-8(\gamma+\alpha(1+\beta)(1-\gamma))-2\beta\gamma(3+\beta)+\beta^2}{4(2+\beta)} + \left(\gamma - \frac{1}{2}\right) u_t & \text{if } \gamma \geq 1/2,\ u_t \geq u_y \\ \frac{2-4\beta\alpha(1-\gamma)-\beta(1+\beta)(2\gamma-1)}{2(2+\beta)} & \text{if } \gamma < 1/2,\ u_t < u_y \end{cases}$$

2. *Given any u_0, the economy converges to a unique steady-state, such that*

$$b = b^{lo} = \frac{2(1-\gamma)(1+\beta-2\alpha)}{(2+\beta)(3/2-\gamma)};$$
$$u = u^{lo} = \left(1 - \beta + (1+\beta) b^{lo}\right)/2..$$

If $\gamma < 1/2$ the convergence path follows an oscillatory pattern. If $\gamma = 1/2$ convergence occurs instantaneously. If $\gamma \geq 1/2$ the convergence path is monotonic.

The introduction of lobbies makes some degree of welfare state sustainable in steady-state as long as the unsuccessful agents are not too much at disadvantage in organizing their lobbying activity relatively to the successful agents (there is no welfare state in the long-run if $\alpha > (1+\beta)/2$). In particular, if the two groups have access to the same lobbying technology ($\alpha = 1/2$) redistribution is sustained in the long-run provided that there are decreasing returns to the group size in the lobbying technology. This result contrasts with the benchmark case of probabilistic voting, as well as to identical case in which the lobbying technology has constant returns to the group size featuring zero long-run redistribution. The intuition is that, when $\gamma = 1$, as the size of the unsuccessful group falls, its political influence diminishes proportionally and is too small, in steady-state, to induce any redistribution. When $\gamma < 1$, however, as the unsuccessful turn less numerous, they become more efficient in organizing their lobbying activity, and the political influence of each of its member, therefore, increases.

Figure 5 represents the equilibrium policy function and law of motion with lobbying behavior, with the steady-states levels, b^{lo} and u^{lo}. When $\gamma < 1/2$ (upper panels) there is positive redistribution throughout. The figure is qualitative identical to figure 1 (dictatorship of proletariat, in section 3.2), except that the parameter restrictions imposed ensure, here, that the policy function is continuous. In particular, the policy function has a non-increasing shape. The law of motion is downward sloping, implying an oscillatory pattern of convergence. In this case, the political process is rather insensitive to the group size, as small groups can form more powerful lobbies (recall that $\gamma = 0$ would mean that each group has an exogenous power in the political mechanism, α, irrespective of group sizes). In this case, the predominant force is the fact that redistribution is more costly when u_t is high than when u_t is low. Thus, if the economy starts off with u_t above steady-state, the political process will set b_t relatively low, and this will induce relatively high investment from the young, causing u_{t+1} to be lower than steady-state. This in turn will justify a high

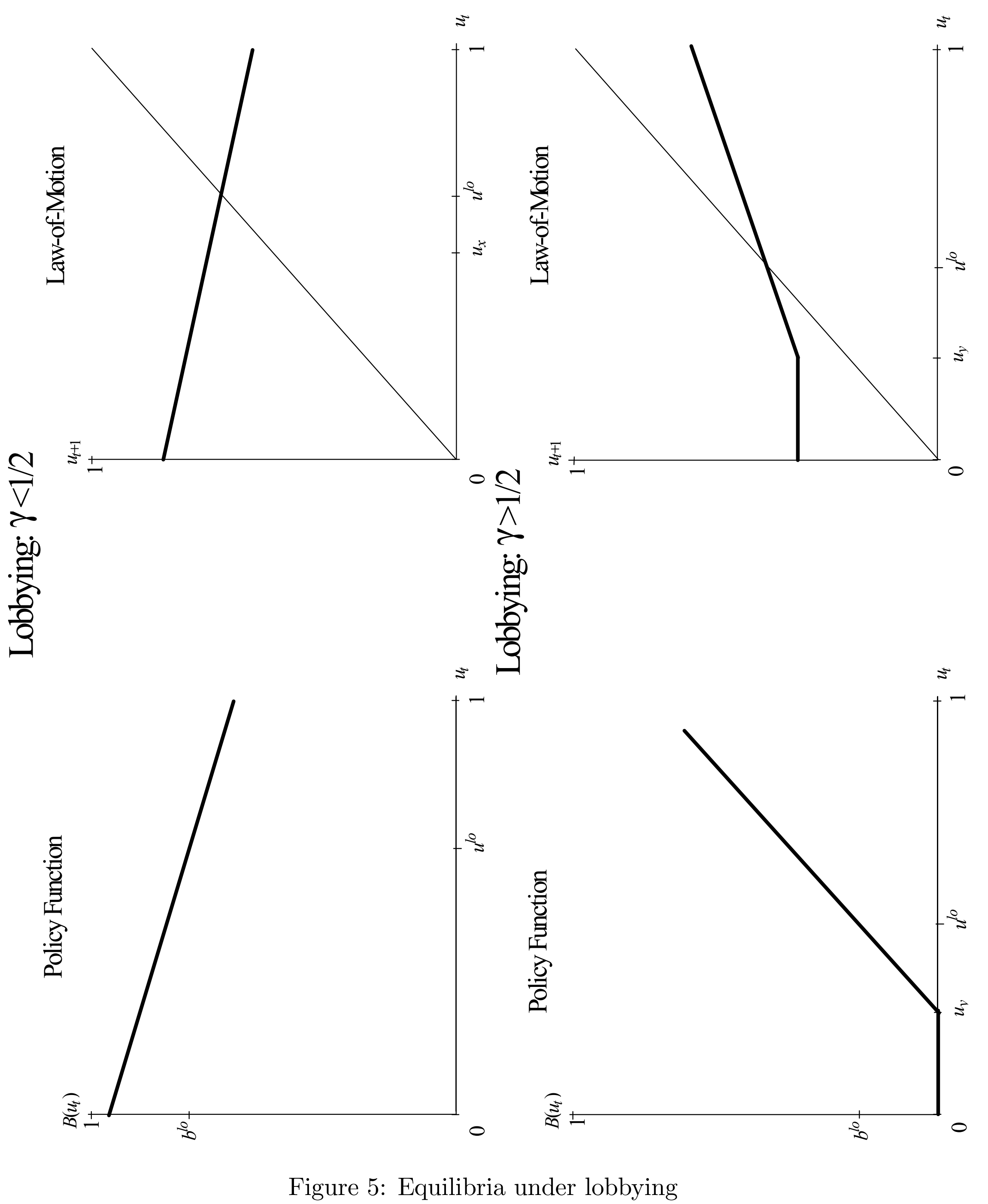

Figure 5: Equilibria under lobbying

b_{t+1} leading to high u_{t+2} and so on. When $\gamma < 1/2$ (lower panels) there is positive redistribution except, possibly, for low levels of u_t. Furthermore, the policy function is (weakly) upward sloping. Since the political process is in this case more sensitive to the group size, higher redistribution is chosen when the unsuccessful are more numerous. Accordingly, the law of motion is (weakly) upward sloping. If the economy starts off with u_t above steady-state, the political process will set b_t relatively high, and this will induce relatively low investment from the young, causing u_{t+1} to be, again, higher than steady-state. This in turn will justify a high b_{t+1} leading to high u_{t+2} and so on. The process is not explosive, and converges to the steady-state (namely, $u_{t+2} < u_{t+1} < u_t$), but convergence follows a monotonic pattern. When $\gamma = 1/2$ (no graph), the policy function is flat, i.e., $B(u_t) = b^{lo}$, and so is the law of motion. Thus, the economy converges in one period to the steady-state.

4 Conclusions

We have analyzed the dynamics of redistribution under different political systems, maintaining the assumption that agents are fully rational. Following Krusell, Quadrini and Ríos Rull (1996), among others, we have restricted attention to Markov perfect equilibria. Differently from most previous papers, we have achieved analytical characterization of equilibria.

We have considered political mechanisms which all can be represented as the maximization of a political objective function, being a weighted average of the utility of the different groups of agents who are entitled to vote. In each political mechanisms, the weight given to the voting groups is a different function of the group's size: Under majority voting, the larger group has a unit weight. Under the probabilistic voting, each group's weight is proportional to its size. Under lobbying, the interests of small groups tend to be overrepresented, the reason being that we assume that they are more effective in organizing their lobbying activities.

We have shown that the equilibrium outcome and the dynamics of redistribution crucially depends on the way the political objective function aggregates the conflicting interests on redistribution. In particular, if the political process is (at least locally) very sensitive to changes in the relative group size, as under majority voting, multiple steady-states may exist. Then, initial conditions determine the long-run level of redistribution since the existence (non-existence) of a welfare state, created by an initial majority of unsuccessful (successful), leads to private investment decisions that regenerates the political support for redistribution.

We have also shown that it may be tempting for agents to engage in strategic voting, because current tax-rates are affected by the expectations about future levels of redistribution. When small changes in composition of the future electorate has large effects on the chosen redistribution policy, such strategic considerations become very important for current voters, since small changes in current policy may have large consequences for future redistribution levels and thus on current taxes. In this case, multiple self-fulfilling equilibria, some with the eternal survival and some with the termination of the welfare state, may co-exist. If, instead, political power is insensitive to the relative size of the different groups of voters, as under lobbying, the level of transfers converges over time, following a cyclical pattern (periods of high redistribution are followed by period of low redistribution), to a unique positive steady-state level. Finally, if the political process gives weights that are proportional to group size, the economy converges monotonically to a steady state with zero redistribution.

In our model, redistribution from rich to poor agents has a distortionary effect and agents are risk neutral, attaching no *ex-ante* value to redistribution. Nevertheless, some agents want redistribution *ex-post* , and as we have seen, this may, under some circumstances, be sufficient to sustain welfare state institutions. In particular, we have found that the welfare state can survive under two qualitatively different cases.

First, if minorities have no political influence, it is possible that the welfare state survives due to its ability to regenerate its own political support. We have called this mechanism *policy behavior complementarity.* For this mechanism to be able to make the welfare state viable, it is necessary that young individuals have little political influence so that the *ex-post* interests over redistribution are strong. However, if young individuals do not believe in the survival of the welfare state, it will not prevail.

Second, the welfare state may survive if the interest of minorities is overrepresented in the political process. This is true even if there is no intrinsic difference between successful and non-successful in the ability to influence political decisions. In such a case, the welfare state may survive despite the existence of a majority against it, due to the over-representation of the interest of a small group of unsuccessful.

Our stylized model and mechanisms can be interpreted as representing investment in human capital via education. Assume that education is costly to achieve, but increases the expected permanent income. The agent's decision about his human capital investment depends on both current and future redistribution. In a society with high redistribution, agents will invest less. The individuals who are anyway successful and earn a high income will be subject to high taxation in order to finance the transfer system. Thus, multiple steady-state can arise. **a)** One in which the economy starts with high redistribution and a political majority supporting high levels of redistribution. The high redistribution leads to low levels of private investment in education, guaranteeing the continuation of a constituency for the welfare state. **b)** Another steady state, in which the economy starts with low redistribution and no constituency for the welfare state is ever formed.

The existence of multiple self-fulfilling equilibria under majority voting (given parameters and initial distribution), one where the welfare state survives, and the other where it collapses, can provide insights to the debate on the future of the pay-as-you-go pension system. A conceivable scenario is that the system will be kept alive, possibly by increasing the contribution of the working agents, or by marginally reducing the benefits. If young agents believe in this perspective, there will be no major change in their saving behavior, and today's young will become an active constituency for the pay-as-you-go system when old. Thus, the system will survive. Another scenario, however, is that agents come to expect that the system is destined to collapse. Under this expectation, the young agents will work and save more. This change in the private behavior of the young will dry up – or at least reduce the activism of – the future constituency for the survival system. Once the system is abandoned, there is no constituency for its restoration. Thus, supporters and detractors of the system have an incentive to make people believe that the system is or is not sustainable, respectively, since what people believe is decisive in order to determine the size of the future constituency of the system. This is arguably a feature of the current policy debate.

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5 Appendix A.

5.1 Probabilistic voting

Here we show that the case in which all agents have a proportional weight in the political decision adopted ($W(u_t) = u_t$) can be interpreted as the outcome of a standard political process where voters electoral decisions depend not only in the platforms that different candidates hold about redistribution, but also about their platforms about other issues, orthogonal to redistribution and that are left, mostly, unmodelled. We adopt a simplified version of Lindbeck and Weibull (1987).

The voters cast their votes on one of two candidates, denoted A and B. There are two candidates who maximize the probability of becoming elected and can commit to any transfer policy, $b_t \in [0,1]$. The two candidates have exogenous traits which implies that voters generically are not indifferent between them, even if they propose the same level of transfers. More specifically, a individual voter i of type $j \in \{y, os, ou\}$, prefers candidate A, proposing b_A, over candidate B proposing b_B if

$$V^j(b_{A,t}, u_t) > V^j(b_{B,t}, u_t) + \sigma_{i,j} + \delta_t$$

where $\sigma_{i,j}$ is drawn from a rectangular distribution with support over $\left[-\frac{1}{2\varepsilon_j}, \frac{1}{2\varepsilon_j}\right]$.[9] We maintain, throughout, that $\varepsilon_j = 1$ for $j \in \{os, ou\}$, implying that all old voters are equally attractive to power-seeking candidates, whereas $\varepsilon_y \leq 1$, implying that the preferences of the young voters are, possibly, more spread out than those of the old. The benchmark case in which the young have no influence on the level of redistribution corresponds to the limit as $\varepsilon_y \to 0$. For simplicity, we limit discussion here to this limit case in which the young exert no influence, although the extension is straightforward. δ_t is a common shock to preferences over the candidates and is drawn i.i.d. over time from a rectangular distribution with support over $\left[-\frac{1}{2\psi}, \frac{1}{2\psi}\right]$. Individuals know their preferences when they cast their votes, while candidates only know the distributions when they fix their proposals. Thus, given b_A and b_B and δ_t, all individuals of type j with a $\sigma_{i,j}$ lower than $V_t^j(b_A) - V_t^j(b_B) - \delta_t$ will vote for candidate A. The number of votes for candidate A,denoted p_A, is, thus,

$$\begin{aligned} &\frac{1}{2} + (1 - u_t)\left(V^{os}(b_{A,t}, u_t) - V^{os}(b_{B,t}, u_t)\right) \\ &+ u_t\left(V^{ou}(b_{A,t}, u_t) - V^{ou}(b_{B,t}, u_t)\right) - \delta_t \end{aligned} \tag{18}$$

and the probability that candidate A wins the election is

$$\begin{aligned} &\Pr\left[p_A > \frac{1}{2}\right] = \\ &\frac{1}{2} + \psi\left[(1 - u_t)\left(V^{os}(b_{A,t}, u_t) - V^{os}(b_{B,t}, u_t)\right) + u_t\left(V^{ou}(b_{A,t}, u_t) - V^{ou}(b_{B,t}, u_t)\right)\right] \end{aligned}$$

[9] In Persson and Tabellini (2000), the support of the distribution from which $\sigma_{i,j}$ is drawn is allowed to vary with j, whereas here we assume it to be the same for all relevant groups. This means that, in our model, voters belonging to each group are equally attractive to power-seeking candidates. Dealing with the more general case in the context of our model complicates the analysis substantially. In particular, the first order condition for the problem of setting b_t optimally for the winning candidate becomes non-linear, and we cannot use the guess-and-verify technique adopted in section 3.2.

Denoting $V^{pv}(b_t, u_t) \equiv (1-u_t)V^{os}(b_t, u_t) + u_t V^{ou}(b_t, u_t)$, the unique equilibrium has both candidates choosing the same platform, and

$$b_t^{pv} = \arg\max_{b_t} \{V^{pv}(b_t, u_t)\}$$

is the equilibrium redistribution, irrespective of which candidate wins the election.

5.2 A voting model with lobbies.

The model assumes that the average popularity of the parties depends on the amount of campaign contributions which they receive. Let C_n^j denote the campaign contribution *per person* to party n from type j individuals, and recall that δ_t stands for the popularity of party B. Then, we assume that $\delta = \tilde{\delta} + (C_B - C_A)$, where $C_n = (1-u)C_n^{os} + uC_n^{ou}$. The number of voters for candidate A is, thus modified to;

$$\begin{aligned} & \frac{1}{2} + (1-u_t)\left(V^{os}(b_{A,t}, u_t) - V^{os}(b_{B,t}, u_t)\right) + \\ & u_t\left(V^{ou}(b_{A,t}, u_t) - V^{ou}(b_{B,t}, u_t)\right) - \delta_t - (C_B - C_A) \\ \equiv\ & p_A^{lo}. \end{aligned} \tag{19}$$

The expected utility of an individual in group $j \in [os, ou]$ is assumed to be;

$$p_A^{lo} V^j(b_{A,t}, u_t) + \left(1 - p_A^{lo}\right) V^j(b_{B,t}, u_t) - \kappa\left(C_A^j, C_B^j, u_t\right), \tag{20}$$

where

$$\kappa\left(C_A^j, C_B^j, u_t\right) = \frac{\left(C_A^j + C_B^j\right)^2}{2\theta^j(u_t)}$$

where $\kappa\left(C_A^j, C_B^j, u_t\right)$ is the cost suffered by each member in lobby j in order for this lobby to make a contribution per member equal to C_A^j to candidate A and equal to C_B^j to candidate B. We assume that $\theta^{ou\prime}(u_t) \leq 0$ and $\theta^{os\prime}(u_t) \geq 0$, implying that organizing lobbies becomes more difficult as the group size increases (coordination and cohesion is easier to rich in smaller groups). Furthermore, we assume, although with a different interpretation) that each lobby, given u_t, is subject to increasing costs with respect to the size of their contribution to all candidates, and parameterize the cost function to be quadratic in the size of the contribution.[10] Given these assumptions, the optimal choice of contributions yields;

$$\begin{aligned} C_A^j &= \theta^j(u_t) \cdot Max\left[0, V^j(b_{A,t}, u_t) - V^j(b_{B,t}, u_t)\right] \\ C_B^j &= \theta^j(u_t) \cdot Max\left[0, V^j(b_{A,t}, u_t) - V^j(b_{B,t}, u_t)\right]. \end{aligned} \tag{21}$$

[10] The assumption of convex-quadratic costs follows Persson and Tabellini (2000), although our interpretation differs from the one they provide. In our model, agents have linear utilities, implying that, the individual disutility to make contributions should be linear with the size of the contribution. We therefore motivate our assumption as X-inefficiencies at the level of the lobby management. Note that, with slight abuse of notation, V^j denotes, here, the utility of an agent in group j gross of the cost of contributing to the lobby.

Also, each group will contribute to at most one candidate and, in particular, to that proposing the most preferred platform. The candidates rationally anticipate the lobbying activity. However, the symmetry of the problem implies that they will choose, in equilibrium, the same platform. Thus, no contributions will be made in equilibrium, in accordance with (21). Yet, the capability of each group to influence the political choice depends positively on the effectiveness of their lobbying technology. In particular, the two candidates will set $b_t = b_t^{lo}$ where;[11]

$$b_t^{loe} = \arg\max_{b_t} V^{lo}(b_t, u_t) \equiv (1 - W^{ou}(u_t))\, V^{os}(b_t, u_t) + W^{ou}(u_t) V^{ou}(b_t, u_t), \tag{22}$$

where

$$\hat{W}^{ou}(u_t) = \frac{u_t\,(1 + \theta^{ou}(u_t))}{1 + (1 - u_t)\,\theta^{os}\,(u_t) + u_t \theta^{ou}\,(u_t)}.$$

With an eye to keep the formulation tractable, we parameterize the θ^j functions as follows;

$$\theta^{ou}(u) \equiv (\hat{\alpha}\hat{\gamma}u)^{\hat{\gamma}-1},$$
$$\theta^{os}(u) \equiv ((1 - \hat{\alpha})\hat{\gamma}(1 - u))^{\hat{\gamma}-1}\,.$$

where $\hat{\gamma} \in (0, 1]$ and $\hat{\alpha} \in [0, 1]$ can be given the following interpretations. α denotes the comparative advantage in lobbying of the unsuccessful. If $\alpha = 1/2$ both groups are equally effective at organizing their lobbying activity, whereas if $\hat{\alpha} > (<) 1/2$, the successful (unsuccessful) are more effective. The parameter γ captures the sensitivity of the lobbying technology to the group size. The smaller $\hat{\gamma}$ the more decreasing the returns to group size (namely, smaller groups can lobby more effectively). Note that $\hat{\gamma}$ is assumed to affect also the productivity of lobbying for both groups. This is just a technical simplification which implies the convenient property that $W^{ou}\,(1 - \hat{\alpha}) = 1 - \hat{\alpha}$.

Non-linearities in the $\hat{W}^{ou}\,(u_t)$ function prevent an analytical characterization of the equilibrium. The problem simplifies substantially, however, if we approximate the $\hat{W}^{ou}\,(u_t)$ function by a first order linear approximation, such that $W^{ou}\,(u_t) \simeq z_0 + z_1 u_t \equiv (1 - \gamma)\,(1 - \alpha) + \gamma u_t \equiv \hat{U}^{ou}\,(u_t)$, where we set $\alpha = \hat{\alpha}$. It is straightforward to show that $1 - \alpha = W^{ou}\,(1 - \alpha) = \hat{W}^{ou}\,(1 - \hat{\alpha}) = 1 - \hat{\alpha}$. With this in mind, we calculate the slope coefficient of the approximating function, γ, by equating it to the derivative of $\hat{W}^{ou}\,(u_t)$ evaluated at $u_t = 1 - \hat{\alpha}$. This yields;

$$\frac{d\hat{W}^{ou}(1 - \hat{\alpha})}{du_t} = \hat{\gamma}\frac{\left[(\hat{\alpha}\hat{\gamma}(1 - \hat{\alpha}))^{\hat{\gamma}} - \hat{\alpha}(1 - \hat{\alpha})\right]}{\left[(\hat{\alpha}\hat{\gamma}(1 - \hat{\alpha}))^{\hat{\gamma}} - \hat{\alpha}\hat{\gamma}(1 - \hat{\alpha})\right]} \equiv \gamma$$

where γ is a monotonically increasing function of $\hat{\gamma}$ (note that $\gamma \to \hat{\gamma}$ when $\hat{\alpha} \to 0$ or $\hat{\alpha} \to 1$).

[11] (22) is obtained by substituting (21) into (19), recalling that candidate A aims at maximizing p_A taking the choice of candidate B as given and that, by the symmetry of the problem, candidate B faces an analogous problem.

We can therefore express the (approximate) political equilibrium in terms of the following maximization;

$$b_t^{lo} = \arg\max_{b_t} V^{lo}(b_t, u_t),$$

where

$$\begin{aligned} V^{lo}(b_t, u_t) \quad \equiv \quad & ((1-\gamma)\alpha + \gamma(1-u_t))\, V^{os}(b_t, u_t) \\ & + ((1-\gamma)(1-\alpha) + \gamma u_t)\, V^{ou}(b_t, u_t), \end{aligned}$$

6 Appendix B. Characterization of the Anti-Welfare Equilibrium.

Proposition 9 *An AWE, $\langle B^{aw}, U^{aw} \rangle$, has the following characteristics:*

1. *For $\beta \geq \frac{\sqrt{5}-1}{2} \simeq 0.618$;*

$$B^{aw}(u_t) = \begin{cases} \beta & \text{if } u_t > 1/2 \\ 0 & \text{if } u_t \in \left[0, \frac{1}{2}\right] \end{cases} \tag{23}$$

$$U^{aw}(b_t) = \begin{cases} \frac{1-\beta-\beta^2+b_t}{2} & \text{if } b_t > \beta \\ U^{pl}(b_t) & \text{if } b_t \leq \beta \end{cases} \tag{24}$$

where $U^{pl}(b_t)$ is defined in proposition 1. This implies,

$$u_{t+1} = \begin{cases} \frac{1}{2} & \text{if } u_t \in (0.5, 1] \\ \frac{1-\beta}{2} & \text{if } u_t \in [0, 0.5] \end{cases} \tag{25}$$

2. *For $\beta \in \left[\beta_H, \frac{\sqrt{5}-1}{2}\right)$;*

$$B^{aw}(u_t) = \begin{cases} \beta & \text{if } u_t \geq u^a(\beta) \\ 1 & \text{if } u_t \in (0.5, u^a(\beta)) \\ B^{pl}(u_t) & \text{if } u_t \in \left[0, \frac{1}{2}\right] \end{cases} \tag{26}$$

$$U^{aw}(b_t) = \begin{cases} \frac{1-\beta+b_t+\beta^2}{2} & \text{if } b_t > \beta \\ U^{pl}(b_t) & \text{if } b_t \leq \beta \end{cases} \tag{27}$$

$$u_{t+1} = \begin{cases} \frac{1}{2} & \text{if } u_t \in [u^a(\beta), 1] \\ 1 - \frac{\beta(1-\beta)}{2} & \text{if } u_t \in (0.5, u^a(\beta)) \\ \frac{1-\beta}{2} & \text{if } u_t \in [0, 0.5] \end{cases} \tag{28}$$

where $u^a(\beta) \equiv 1 - \frac{\beta^2}{2(1-\beta)}$, and $\beta_H \simeq .570$ is the solution in $[0,1]$ to

$$\beta^3 - \beta^2 + 2\beta - 1 = 0.$$

3. For $\beta \in [\beta_M, \beta_H]$;

$$B^{aw}(u_t) = \begin{cases} \beta & if \quad u_t \in \left[u^d(\beta), 1\right] \\ \frac{3}{2} + \frac{\beta(1-\beta)}{2} - u_t & if \quad u_t \in \left[u^c(\beta), u^d(\beta)\right) \\ 1 & if \quad u_t \in (0.5, u^c(\beta)) \\ 0 & if \quad u_t \in [0, 0.5] \end{cases} \tag{29}$$

$$U^{aw}(b_t) = \begin{cases} \frac{1-\beta+b_t+\beta^2}{2} & if \quad b_t > \beta \\ U^{pl}(b_t) & if \quad b_t \in [0, \beta] \end{cases}, \tag{30}$$

$$u_{t+1} = \begin{cases} \frac{1}{2} & if \quad u_t \in \left[u^d(\beta), 1\right] \\ \frac{5-\beta(1-\beta)}{4} - \frac{1}{2}u_t & if \quad u_t \in \left[u^c(\beta), u^d(\beta)\right) \\ 1 - \frac{\beta(1-\beta)}{2} & if \quad u_t \in (0.5, u^c(\beta)) \\ \frac{1-\beta}{2} & if \quad u_t \in [0, 0.5] \end{cases}$$

where $u^c(\beta) \equiv \frac{1}{2} + \frac{\beta(1-\beta)}{2}$, $u^d(\beta) \equiv \frac{3}{2} - \frac{1}{2}\beta(1+\beta) - \left(\sqrt{\beta}\right)^3$, $\beta_M \approx 0.555$ is the root in $[0,1]$ to the equation $0 = 2 - \beta_M - 2\beta_M^2 - 2\left(\sqrt{\beta_M}\right)^3$, and β_H defined as above.[12]

7 Appendix C. Proofs.

This section is highly preliminary and incomplete.

7.1 Proof of proposition 1

Proof. A) $B^{dp}(u_t) = \arg\max_{b_t}\{V^{ou}(b_t, u_t)\}$, subject to $u_{t+1} = U^{dp}(b_t)$, $b_t \in [0,1]$ and $b_{t+1} = B^{dp}(u_{t+1})$;

B) $U^{dp}(b_t) = \left(1 - \beta + b_t + \beta B^{dp}\left(U^{dp}(b_t)\right)\right)/2$.

Given $u_{t+1} = U^{dp}(b_t)$, $b_t \in [0,1]$ and $b_{t+1} = B^{dp}(u_{t+1}) = B^{dp}\left(U^{dp}(b_t)\right) = \left(\frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{2}{2-\beta}\left(\frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t\right)\right)$, then V_t^{ou} can be expressed as;

$$V_t^{ou}(b_t, u_t) = b_t - \frac{(1-\beta) + \left(b_t + \beta B^{dp}\left(U^{dp}(b_t)\right)\right) + 2u_t}{4} b_t$$

$$= \begin{cases} b_t - \frac{1}{4}\left(1 - \beta + b_t + \beta\left(\frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{2}{2-\beta}\left(\frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t\right)\right) + 2u_t\right) b_t & if \quad b_t \geq \frac{2\beta}{2+\beta} \\ b_t - \frac{1}{4}(1 + b_t + 2u_t) b_t & if \quad b_t < \frac{2\beta}{2+\beta}. \end{cases}$$

(note that if (and only if) $b_t \leq 2\beta/(2+\beta)$ then $b_{t+1} = 1$). Maximizing V_t^{ou} over b_t yields;

$$b_t = \begin{cases} \frac{3}{2} - u_t & \text{if } u_t > \bar{u}(\beta) \\ \frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{2}{2-\beta}u_t & \text{if } u_t \in \left[\frac{3}{2} - \frac{2}{(2+\beta)}, \bar{u}(\beta)\right] \\ 1 & \text{if } u_t \in \left[0, \frac{3}{2} - \frac{2}{(2+\beta)}\right] \end{cases} = B^{dp}(u_t), \tag{31}$$

[12] The equation $0 = 2 - \beta_M - 2\beta_M^2 - 2\left(\sqrt{\beta_M}\right)^3$ originates from setting $\beta_M = \bar{b}(\beta_M)$, where $\bar{b}(\beta)$ is defined below as the infimum of benefits b_t which will generate $u_{t+1} \geq u^d(\beta)$ and hence $b_{t+1} = \beta$.

where $\bar{u}(\beta) = \frac{\beta+6-\beta\sqrt{4-2\beta}}{2(2+\beta)}$. This proves part (A) of the proposition. To see the steps of this maximization in more details, define $V^a(u_t)$ and $V^b(u_t)$ as follows;

$$V^a(u_t) \equiv \max\left\{V_t^{ou}(b_t, u_t)\,|_{b_t\in\left[0,\frac{2\beta}{2+\beta}\right]}\right\} \tag{32}$$
$$= \begin{cases} \frac{9}{16} - \frac{3}{4}u_t + \frac{1}{4}u_t^2 \equiv V^{a,int}(u_t) & if \quad u_t > \frac{6-\beta}{2(2+\beta)} \\ \beta\frac{6+\beta-2u_t(2+\beta)}{2(2+\beta)^2} \equiv V^{a,cor}(u_t) & if \quad u_t \leq \frac{6-\beta}{2(2+\beta)} \end{cases}$$
$$V^b(u_t) \equiv \max\left\{V_t^{ou}(b_t, u_t)\,|_{b_t\in\left[\frac{2\beta}{2+\beta},1\right]}\right\} \tag{33}$$
$$= \begin{cases} \frac{1}{8}\frac{\left(\beta^2-3\beta+2u_t(2+\beta)-6\right)^2}{(2-\beta)(2+\beta)^2} \equiv V^{b,int}(u_t) & if \quad u_t \geq \frac{3}{2} - \frac{2}{(2+\beta)} \\ \frac{1}{8}\frac{8+\beta(6-\beta)}{2+\beta} - \frac{u_t}{2} \equiv V^{b,cor}(u_t) & if \quad u_t \leq \frac{3}{2} - \frac{2}{(2+\beta)} \end{cases}$$

where $V^{a,c}(u_t)$ and $V^{b,c}(u_t)$ result from corner solutions in the respective ranges (the corners being, respectively, equal to $b_t = \frac{2\beta}{2+\beta}$ and $b_t = 1$). First, standard algebra establishes that $V^{b,int}(u_t) - V^{a,cor}(u_t) = \frac{1}{8}\frac{\left(\beta^2-2\beta u_t-\beta+6-4u_t\right)^2}{(2-\beta)(2+\beta)^2} > 0$ and that, in the range where $u_t \leq \frac{3}{2} - \frac{2}{(2+\beta)}$, $V^{b,cor}(u_t) - V^{a,cor}(u_t) > \frac{1}{8}(2-\beta)\frac{4(1-\beta)+\beta^2}{(2+\beta)^2} > 0$. Thus, whenever $V^a(u_t) = V^{a,cor}(u_t)$, then $V^b(u_t) > V^a(u_t)$. Second, if $\beta < \frac{2}{3}$, then $\frac{6-\beta}{2(2+\beta)} > 1$ and $V^a(u_t) = V^{a,cor}(u_t)$ for all u_t. Thus, $V^b(u_t) > V^a(u_t)$ if $\beta < 2/3$. Third, note that, if $\beta \geq 2/3$, then there exists a range of u_t, where $V^a(u_t) = V^{a,int}(u_t)$. In this range, standard algebra establishes that $V^{b,int}(u_t) > V^{a,int}(u_t)$ for all u_t provided that $\beta < \left(\sqrt{17}-1\right)/4$. Thus, $\beta < \left(\sqrt{17}-1\right)/4$ implies that $V^b(u_t) > V^a(u_t)$ for all $u_t \in [0,1]$. Consider now the range of parameters such that $\beta \geq \left(\sqrt{17}-1\right)/4$. In this case, for all $u_t > \bar{u}(\beta) = \frac{\beta+6-\beta\sqrt{4-2\beta}}{2(2+\beta)}$, $V^b(u_t) < V^a(u_t)$. Thus, the choice of b_t maximizing V_t^{ou} is in the range $b_t \in \left[0, \frac{2\beta}{2+\beta}\right]$ and, in particular, it will be $argmax_{b_t}\left\{b_t - \frac{1}{4}(1+b_t+2u_t)\,b_t\right\} = 3/2 - u_t$.

To prove part (B), i.e., that $U^{dp}(b_t) = \left(1-\beta+b_t+\beta B^{dp}\left(U^{dp}(b_t)\right)\right)/2$, observe that, from (7);

$$\left(1-\beta+b_t+\beta B^{dp}\left(U^{dp}(b_t)\right)\right)/2 = \begin{cases} \frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t & \text{if } b_t \in \left[\frac{2\beta}{2+\beta}, 1\right] \\ \frac{1+b_t}{2} & \text{if } b_t \in \left[0, \frac{2\beta}{2+\beta}\right) \end{cases} \tag{34}$$

Take, first, the range $b_t \in \left[\frac{2\beta}{2+\beta}, 1\right]$. Then;

$$\left(1-\beta+b_t+\beta B^{dp}\left(\frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t\right)\right)/2 = \frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t$$

Next, note that $b_t \in \left[\frac{2\beta}{2+\beta}, 1\right] \Rightarrow \left(\frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t\right) \in \left[\frac{3}{2} - \frac{2}{(2+\beta)}, \bar{u}(\beta)\right]$. Hence, using (7);

$$\left(1-\beta+b_t+\beta\left(\frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{2}{2-\beta}\left(\frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t\right)\right)\right)/2$$
$$= \frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t$$

which is true.

Take, next, the range $b_t \in \left[0, \frac{2\beta}{2+\beta}\right)$. In this case, $u_{t+1} = (1+b_t)/2$, and, necessarily, $u_{t+1} \in \left[0, \frac{3}{2} - \frac{2}{(2+\beta)}\right]$. Thus, $B^{dp}(u_{t+1}) = 1$, which can be substituted into the left hand-side of (34) to yield $(1+b_t)/2$ which verifies the second part of (34) This concludes part (B) of the proof.

Finally, the characterization of the equilibrium law of motion of u_t, (8), and of the steady-state, (9)-(10), is straightforward. ■

7.2 Proof of proposition 2

Proof. B) $U^{mv}(b_t) = (1 - \beta + b_t + \beta B^{mv}(U^{mv}(b_t)))/2$.
As to part (A), consider first the range where $u_t > 1/2$:

$$V_t^{mv}(b_t, u_t) = V_t^{ou}(b_t, u_t) =$$
$$\begin{cases} b_t - \frac{1}{4}\left(1 - \beta + b_t + \beta\left(\frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{2}{2-\beta}\left(\frac{\beta(1+\beta)+2}{2(2+\beta)} + \frac{2-\beta}{4}b_t\right)\right) + 2u_t\right)b_t & if \quad b_t \geq \frac{2\beta}{2+\beta} \\ b_t - \frac{1}{4}(1 + b_t + 2u_t)b_t & if \quad b_t < \frac{2\beta}{2+\beta} \end{cases}.$$

Note that $V_t^{mv}(0, u_t) = 0$, which shows that setting $b_t = 0$ can never be optimal for the unsuccessful (recall that both (32) and (33) are strictly positive)

Next, if $u_t \leq 1/2$, then $V_t^{mv}(b_t, u_t) = V_t^{os}(b_t, u_t)$, and this is maximized by setting $b_t = 0$.

To prove part (B), i.e., that $U^{mv}(b_t) = (1 - \beta + b_t + \beta B^{mv}(U^{mv}(b_t)))/2$, observe that, from (11) and (12);

$$b_t = B^{mv}(u_t) = \begin{cases} B^{dp}(u_t) & \text{if } u_t \in \left(\frac{1}{2}, 1\right] \\ B^{pl}(u_t) & \text{if } u_t \in \left[0, \frac{1}{2}\right] \end{cases}$$
$$U^{mv}(b_t) = \begin{cases} U^{dp}(b_t) & \text{if } b_t \in (0,1] \\ U^{pl}(b_t) & \text{if } b_t = 0 \end{cases}$$

while;

$$\frac{(1 - \beta + b_t + \beta B^{mv}(U^{mv}(b_t)))}{2}$$
$$= \begin{cases} \frac{\left(1-\beta+b_t+\beta B^{mv}\left(U^{dp}(b_t)\right)\right)}{2} & \text{if } b_t \in (0,1] \\ \frac{\left(1-\beta+b_t+\beta B^{mv}\left(U^{pl}(b_t)\right)\right)}{2} & \text{if } b_t = 0 \end{cases}$$
$$= \begin{cases} \frac{\left(1-\beta+b_t+\beta B^{dp}\left(U^{dp}(b_t)\right)\right)}{2} & \text{if } b_t \in (0,1] \\ \frac{1-\beta}{2} & \text{if } b_t = 0 \end{cases}$$

The first equality follows directly from ((12). The latter follows from (5)-(7)-(12) and, in particular, from the following two observations. First, for all $b_t \in (0,1]$, $U^{dp}(b_t) > 1/2$. Hence, in this range $B^{mv}\left(U^{dp}(b_t)\right) = B^{dp}\left(U^{dp}(b_t)\right)$. Second, if $b_t = 0$, then $U^{pl}(b_t) = U^{pl}(b_t) = \frac{1-\beta}{2} < 1/2$. Hence, in this range, $B^{mv}\left(U^{pl}(b_t)\right) = B^{pl}\left(U^{pl}(b_t)\right) = 0$, and, consequently, $\left(1 - \beta + b_t + \beta B^{pl}\left(U^{pl}(b_t)\right)\right)/2 = (1-\beta)/2$. Thus, in order to prove (B), we need to show that:

$$\begin{cases} U^{dp}(b_t) = \frac{\left(1-\beta+b_t+\beta B^{dp}\left(U^{dp}(b_t)\right)\right)}{2} & \text{if } b_t \in (0,1] \\ U^{pl}(b_t) = \frac{1-\beta}{2} & \text{if } b_t = 0 \end{cases}$$

But this follows immediately from Proposition 1, so part (B) is also established.

Finally, the characterization of the equilibrium law of motion of u_t, (13), and of the steady-state is straightforward. ■

7.3 Proof of proposition 9

Proof. B) $U^{aw}(b_t) = (1 - \beta + b_t + \beta B^{aw}(U^{aw}(b_t)))/2$.

1. Consider first the case when $\beta \geq \frac{\sqrt{5}-1}{2}$. As to part (A), consider first the range where $u_t > 1/2$.

$$V_t^{aw}(b_t, u_t) = V_t^{ou}(b_t, u_t) =$$
$$\begin{cases} b_t - \frac{1}{4}\left(1-\beta+b_t+\beta\left(B^{aw}\left(U^{aw}(b_t)\right)\right)+2u_t\right)b_t & \text{if} \quad b_t > \beta \\ b_t - \frac{1}{4}\left(1-\beta+b_t+\beta\left(B^{pl}\left(U^{pl}(b_t)\right)\right)+2u_t\right)b_t & \text{if} \quad b_t \leq \beta \end{cases}.$$
$$= \begin{cases} b_t - \frac{1}{4}\left(1-\beta+b_t+\beta^2+2u_t\right)b_t & \text{if} \quad b_t > \beta \\ b_t - \frac{1}{4}\left(1-\beta+b_t+2u_t\right)b_t & \text{if} \quad b_t \leq \beta \end{cases}.$$

Simple algebra shows that $V_t^{aw}(b_t, u_t)$ is increasing in b_t for $b_t \leq \beta$. Furthermore, $V_t^{aw}(\beta, u_t) \geq V_t^{aw}(b_t, u_t)$ for all $b_t > \beta, u_t > 1/2$ and $\beta \geq \frac{\sqrt{5}-1}{2}$. So $B^{aw}(u_t) = \beta$ for $u_t > 1/2$.
If $u_t \leq 1/2$, $V_t^{aw}(b_t, u_t) = V_t^{os}(b_t, u_t)$ which is decreasing in b_t so $B^{aw}(u_t) = 0$, for $u_t \leq 1/2$.
To prove part (B), i.e., that $U^{aw}(b_t) = \left(1-\beta+b_t+\beta B^{aw}\left(U^{aw}(b_t)\right)\right)/2$, observe that

$$\frac{\left(1-\beta+b_t+\beta B^{aw}\left(U^{aw}(b_t)\right)\right)}{2}$$
$$= \begin{cases} \frac{\left(1-\beta+b_t+\beta B^{aw}\left(U^{dp}(b_t)\right)\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{\left(1-\beta+b_t+\beta B^{aw}\left(U^{pl}(b_t)\right)\right)}{2} & \text{if} \quad b_t \leq \beta \end{cases}$$
$$= \begin{cases} \frac{\left(1-\beta+b_t+\beta B^{dp}\left(\frac{\beta(1+\beta)+2}{2(2+\beta)}+\frac{2-\beta}{4}b_t\right)\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{\left(1-\beta+b_t+\beta B^{pl}\left(\frac{1-\beta+b_t}{2}\right)\right)}{2} & \text{if} \quad b_t \leq \beta \end{cases}$$
$$= \begin{cases} \frac{\left(1-\beta+b_t+\beta^2\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{\left(1-\beta+b_t\right)}{2} & \text{if} \quad b_t \leq \beta \end{cases} = U^{aw}(b_t)$$

where the second equality follows from the facts that $\left[\beta(1+\beta)+2\right]/\left[2(2+\beta)\right]+(2-\beta)b_t/4 > \frac{1}{2}$ for all $b_t \in [\beta, 1]$ and $(1-\beta+b_t)/2 < 1/2$ for all $b_t < \beta$.

2. Consider now the case when $\beta \in \left[\beta_H, \frac{\sqrt{5}-1}{2}\right)$. As to part (A), consider first the range where $u_t > 1/2$.

$$V^{aw}(b_t, u_t) = V^{ou}(b_t, u_t)$$
$$= b_t - \frac{1}{4}\left(1-\beta+b_t+\beta\left(B^{aw}\left(U^{aw}(b_t)\right)\right)+2u_t\right)b_t$$
$$= \begin{cases} b_t - \frac{1}{4}\left(1-\beta+b_t+\beta^2+2u_t\right)b_t & \text{if} \quad b_t > \beta \\ b_t - \frac{1}{4}\left(1-\beta+b_t+2u_t\right)b_t & \text{if} \quad b_t \leq \beta \end{cases}.$$

Simple algebra shows that $V^{aw}(b_t, u_t)$ is increasing in b_t in both the region $b_t \leq \beta$ and $b_t > \beta$, and that at $b_t = \beta$ the value function has a discontinuous fall. Furthermore, $V^{aw}(1, u_t) > V^{aw}(\beta, u_t)$ when $u_t \in (0.5, u^a(\beta))$, $V^{aw}(1, u_t) < V^{aw}(\beta, u_t)$ when $u_t \in (u^a(\beta), 1]$, and $V^{aw}(1, u^a(\beta)) = V^{aw}(\beta, u^a(\beta))$, where $u^a(\beta)$ is defined in the text. Thus, $B^{aw}(u_t) = \beta$ for $u_t > u^a(\beta)$ and $B^{aw}(u_t) = 1$ for $u_t \in (0.5, u^a(\beta))$. If $u_t \leq 1/2$, $V_t^{aw}(b_t, u_t) = V_t^{os}(b_t, u_t)$ which is decreasing in b_t so $B^{aw}(u_t) = 0$, for $u_t \leq 1/2$.

To prove part (B), i.e., that $U^{aw}(b_t) = (1-\beta+b_t+\beta B^{aw}(U^{aw}(b_t)))/2$, observe that

$$\begin{aligned}
&\frac{(1-\beta+b_t+\beta B^{aw}(U^{aw}(b_t)))}{2} \\
&= \begin{cases} \frac{1-\beta+b_t+\beta B^{aw}\left(\left(1-\beta+b_t+\beta^2\right)/2\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{1-\beta+b_t+\beta B^{aw}\left(\frac{1-\beta+b_t}{2}\right)}{2} & \text{if} \quad b_t \leq \beta \end{cases} \\
&= \begin{cases} \frac{\left(1-\beta+b_t+\beta^2\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{(1-\beta+b_t)}{2} & \text{if} \quad b_t \leq \beta \end{cases} = (U^{aw}(b_t))
\end{aligned}$$

where the second equality follows from the facts that $\left(1-\beta+b_t+\beta^2\right)/2 \geq u^a(\beta)$ for all $b_t \in (\beta, 1]$ (since $\beta \geq \beta_H$), and that $(1-\beta+b_t)/2 < 1/2$ for all $b_t < \beta$. QED

3. Consider now the case when $\beta \in [\beta_M, \beta_H)$. As to part (A), consider first the range where $u_t > 1/2$. Applying the equilibrium objects B^{aw} and U^{aw}, the indirect utility function can be written as

$$\begin{aligned}
V^{aw}(b_t, u_t) &= V^{ou}(b_t, u_t) \\
&= b_t - \frac{1}{4}(1-\beta+b_t+\beta(B^{aw}(U^{aw}(b_t))) + 2u_t)\, b_t \\
&= \begin{cases} b_t - \frac{1}{4}\left(1-\beta+b_t+\beta\left(B^{aw}\left(\left(1-\beta+b_t+\beta^2\right)/2\right)\right)+2u_t\right) b_t & \text{if} \quad b_t > \beta \\ b_t - \frac{1}{4}\left(1-\beta+b_t+\beta\left(B^{aw}\left(U^{pl}(b_t)\right)\right)+2u_t\right) b_t & \text{if} \quad b_t \leq \beta \end{cases} \\
&= \begin{cases} b_t - \frac{1}{4}\left(1-\beta+b_t+\beta^2+2u_t\right) b_t & \text{if} \quad b_t > \beta \\ b_t - \frac{1}{4}(1-\beta+b_t+2u_t)\, b_t & \text{if} \quad b_t \leq \beta \end{cases}
\end{aligned}$$

Simple algebra shows that $V^{aw}(b_t, u_t)$ is increasing in b_t in the region $b_t \leq \beta$. Moreover, conditional on $b_t \in (\beta, 1]$ and $\beta \in [\beta_M, \beta_H)$, the (constrained) optimal benefit level $\widetilde{b}$ would be

$$\widetilde{b}(u_t) = \begin{cases} 1 & \text{if} \quad u_t \in (0.5, u^c(\beta)] \\ \frac{3}{2} + \frac{\beta(1-\beta)}{2} - u_t & \text{if} \quad u_t \in (u^c(\beta), 1] \end{cases}.$$

where $u^c(\beta)$ is defined in the text. Hence, $V^{aw}(\beta, u_t) \leq V^{aw}\left(\widetilde{b}(u_t), u_t\right)$ if and only if $u_t \in \left(\frac{1}{2}, u^d(\beta)\right]$, with equality for $u_t = u^d(\beta)$, where, again, $u^d(\beta)$ is defined in the text. Finally, if $u_t \leq 1/2$, $V_t^{aw}(b_t, u_t) = V_t^{os}(b_t, u_t)$ which is decreasing in b_t so $B^{aw}(u_t) = 0$, for $u_t \leq 1/2$.

To prove part (B), i.e., that $U^{aw}(b_t) = (1-\beta+b_t+\beta B^{aw}(U^{aw}(b_t)))/2$, for all $b_t \in [0,1]$, observe that

$$\begin{aligned}
u_{t+1} &= \frac{(1-\beta+b_t+\beta B^{aw}(U^{aw}(b_t)))}{2} \\
&= \begin{cases} \frac{1-\beta+b_t+\beta B^{aw}\left(\left(1-\beta+b_t+\beta^2\right)/2\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{1-\beta+b_t+\beta B^{aw}\left(\frac{1-\beta+b_t}{2}\right)}{2} & \text{if} \quad b_t \leq \beta \end{cases} \\
&= \begin{cases} \frac{\left(1-\beta+b_t+\beta^2\right)}{2} & \text{if} \quad b_t > \beta \\ \frac{(1-\beta+b_t)}{2} & \text{if} \quad b_t \leq \beta \end{cases} \\
&= U^{aw}(b_t) \;\; \forall\, b_t \in [0,1],
\end{aligned}$$

where the second equality follows from the fact that if $\beta \geq \beta_M$, then $\left(1-\beta+b_t+\beta^2\right)/2 \geq u^d(\beta)$, which implies that $B^{aw}\left(\left(1-\beta+b_t+\beta^2\right)/2\right) = \beta$ if $b_t > \beta$. This concludes the proof for the case when $\beta \in [\beta_M, \beta_H)$.

■

7.4 Proof of proposition 1

Proof. First, note from equilibrium definition 2 that $b_t \leq \beta$ is a necessary condition for $u_{t+1} \leq 1/2$. Thus, since $u_{t+1} \leq$ implies $b_{t+1} = 0$, the value of inducing a welfare state breakdown is bounded from above by

$$\hat{V}^{ou}(\beta, 0, u_t) = \beta - \frac{1+2u_t}{4}\beta, \tag{35}$$

since $\hat{V}^{ou}(b_t, 0, u_t)$ is increasing in b_t.

On the other hand, sustaining the welfare state implies $b_{t+1} > 0$, which has a negative impact on the indirect utility of the current old. However, since $b_{t+1} \leq 1$, the indirect utility of sustaining the welfare state is bounded from below by

$$\max_{b_t} \hat{V}^{ou}(b_t, 1, u_t) \text{ s.t. } b_t > \beta, \tag{36}$$

where the constraint ensures that $u_{t+1} > 1/2$. Simple algebra shows that for $\beta < 1/2$ the previous expression is solved by $b_t = 3/2 - u_t$ yielding

$$\hat{V}^{ou}(3/2 - u_t, 1, u_t) = \frac{9}{16} - \frac{3}{4}u_t + \frac{1}{4}u_t^2. \tag{37}$$

Finally, we note that

$$\frac{9}{16} - \frac{3}{4}u_t + \frac{1}{4}u_t^2 - \left(\beta - \frac{1+2u_t}{4}\beta\right) > 0 \tag{38}$$

for all u_t when $\beta < 1/4$. Thus, inducing a welfare state break-down necessarily implies lower indirect utility for a majority of unsuccessful than any alternative.

Now, consider the second statement in the proposition, where we find a lower bound for the choice of b_t of a majority of unsuccessful might choose. To find this, we do the following thought experiment. Suppose that we want to find a (possibly sub-optimal) policy function in period in period $t+1$ that, given u_t, induces the unsuccessful in period t to choose some particular $b_t = b$. Clearly, for this to be the case, b must be incentive compatible, i.e., the indirect utility of the unsuccessful in period t by choosing $b_t = b$ must be higher than any alternative where, of course, the unsuccessful in period t understands that their choice affects b_{t+1} indirectly via the policy function in period $t+1$. Finally, we find a lower bound for b consistent with incentive compatibility. Since this bound holds for any policy function in period $t+1$, it certainly holds for the smaller class of equilibrium policy functions.

Now, note that the highest possible "penalty" associated with choosing $b_t \neq b$ is that $b_{t+1} = 1$. Conversely, the "reward " for choosing $b_t = b$ is bounded from above by $b_{t+1} = 0$. Thus, a necessary condition for b to be incentive compatible, is that

$$\begin{aligned} \hat{V}^{ou}(b, 0, u_t) &\geq \max_{b_t} \hat{V}^{ou}(b_t, 1, u_t) \\ b - \frac{1-\beta+b+2u_t}{4}b &\geq \frac{9}{16} - \frac{3}{4}u_t + \frac{1}{4}u_t^2. \\ -\frac{1}{4}b^2 + \left(\frac{3}{4} + \frac{1}{4}\beta - \frac{1}{2}u_t\right)b + \frac{3}{4}u_t - \frac{1}{4}u_t^2 - \frac{9}{16} &\geq 0, \end{aligned} \tag{39}$$

from which we get the second condition in the proposition. ■

7.5 Proof of proposition 5

Proof. ε, the equilibrium in proposition 2 remains unchanged, expect for the fact that now $u_t > \frac{1+\varepsilon}{2}$ is required for the old unsuccessful to be in majority. Above, we have shown that if the old unsuccessful are in power at period t, $b_t = B^{dp}(u_t)$ maximize their indirect utility if $b_{t+1} = B^{dp}(u_{t+1})$ and expectations are of the "pro-welfare" type. When $\varepsilon = 0$, $b_{t+1} = B^{dp}(u_{t+1})$ for all $b_t > 0$. For $\varepsilon > 0$, this is no longer true since we will show that the young will join forces with the old successful and vote for zero benefits. This will imply that for $b_t \in \left[0, b^{bd}\right]$, where b^{bd} depends on ε and β, $u_{t+1} \leq \frac{1+\varepsilon}{2}$, implying $b_{t+1} = 0$. Nevertheless, it is straightforward to show that $\varepsilon \leq \bar{\varepsilon}(\beta)$ implies that $U^{dp}\left(B^{dp}(u_t)\right) \in \left(\frac{1+\varepsilon}{2}, \frac{\beta}{4} + \frac{2}{2+\beta}\right]$ for all $u_t \in \left(\frac{1+\varepsilon}{2}, \frac{\beta}{4} + \frac{2}{2+\beta}\right]$. Thus, choosing $b_t = B^{dp}(u_t)$ under the expectation that there will be a majority of unsuccessful in period $t+1$ who choose $b_{t+1} = B^{dp}(u_{t+1})$is consistent with rational expectations and optimal choices of e_t. Furthermore, we will show that $B^{dp}(u_t) \geq b^{dp}$. Thus, $B^{dp}(u_t)$ remains the best choice whenever b_t is chosen in a range such that the majority remains in the hands of the unsuccessful (i.e., $b_t \in \left(b^{bd}, 1\right]$). Thus, the welfare state remains if it is better for a majority of old unsuccessful to choose $b_t = B^{dp}(u_t)$ than any $b_t \in \left[0, b^{bd}\right]$.

We start by finding the range $\left[0, b^{bd}\right]$, which under pro-welfare expectations is the range of b_t such that it is inconsistent with rational expectations to believe that $u_{t+1} > \frac{1+\varepsilon}{2}$. For this purpose, we note that if $b_t = b^{bd}$ and $b_{t+1} = B^{dp}(u_{t+1})$,

$$\frac{1-\beta+b^{bd}+\beta B^{dp}\left(\frac{1+\varepsilon}{2}\right)}{2} = \frac{1+\varepsilon}{2}, \tag{40}$$

and

$$\frac{d}{db_t}\frac{1-\beta+b^{bd}+\beta B^{dp}\left(u_{t+1}\right)}{2} = \frac{1}{2-\frac{dB^{dp}(u_{t+1})}{du_{t+1}}} > 0. \tag{41}$$

Thus, if $b_t > b^{bd}$, there is a solution to the second condition in the equilibrium definition such that $u_{t+1} > \frac{1+\varepsilon}{2}$ and $b_{t+1} = B^{dp}(u_{t+1})$, while for $b_t \leq b^{bd}$, there is no such solution and agents instead believe $u_{t+1} \leq \frac{1+\varepsilon}{2}$. It follows from (40) that

$$b^{bd} = \varepsilon + \beta\left(1 - B^{dp}\left(\frac{1+\varepsilon}{2}\right)\right) \tag{42}$$

$$. = \begin{cases} \varepsilon & \text{if } \varepsilon \leq \frac{2\beta}{2+\beta} \\ 2\frac{\varepsilon(2+\beta)-\beta^2}{(2-\beta)(2+\beta)} & \text{if } \varepsilon > \frac{2\beta}{2+\beta} \end{cases}, \tag{43}$$

since $B^{dp}\left(\frac{1+\varepsilon}{2}\right) = 1$ if $\varepsilon \leq \frac{2\beta}{2+\beta}$, and $B^{dp}\left(\frac{1+\varepsilon}{2}\right) = \frac{3(2+\beta)-\beta^2}{4-\beta^2} - \frac{1+\varepsilon}{2-\beta}$ if $\varepsilon > \frac{2\beta}{2+\beta}$.

We conclude that

$$u_{t+1} = \begin{cases} U^{dp}\left(b_t\right) & \text{if } b_t \in \left(b^{bd}, 1\right] \\ U^{pl}\left(b_t\right) & \text{if } b_t = \left[0, b^{bd}\right] \end{cases} \tag{44}$$

satisfies the second condition in the equilibrium definition. Note also that $b^{bd}$is smaller than $1/2$ if $\varepsilon \leq \bar{\varepsilon}(\beta)$. Then, since $B^{dp}(u_t) \geq 1/2$for any u_t, $B^{dp}(u_t) \geq b^{bd}$.

Now, we need to establish that the equilibrium in the proposition satisfies the first condition in the equilibrium definition. Thus, we need to show that

a) for any $t \geq 0$ and $u_t \in \left(\frac{1+\varepsilon}{2}, \frac{\beta}{4}+\frac{2}{2+\beta}\right]$ and $\varepsilon \leq \bar{\varepsilon}(\beta)$, the old unsuccessful optimally chooses $b_t = B^{dp}(u_t)$ if they expect $b_{t+1} = 0$ for $u_{t+1} \leq \frac{1+\varepsilon}{2}$ and $b_{t+1} = B^{dp}(u_{t+1})$ for $u_{t+1} \in \left(\frac{1+\varepsilon}{2}, \frac{\beta}{4}+\frac{2}{2+\beta}\right]$, and

b) for any $t > 0$ and $u_t \leq \frac{1+\varepsilon}{2}$, b_t is optimally chosen to 0 if it is expected that $b_{t+1} = 0$ for $u_{t+1} \leq \frac{1+\varepsilon}{2}$and $b_{t+1} = B^{dp}(u_{t+1})$ for $u_{t+1} \in \left(\frac{1+\varepsilon}{2}, \frac{\beta}{4}+\frac{2}{2+\beta}\right]$.

Note that we have not specified the policy action for $u_t > \frac{\beta}{4}+\frac{2}{2+\beta}$. This, however, turns out not to be necessary to prove the proposition. To see this, first note that $U^{dp}(b_t) = \frac{1-\beta+b_t+B^{dp}(u_{t+1})}{2} \leq \frac{\beta}{4}+\frac{2}{2+\beta}$. Second, only if b_{t+1} is chosen higher than $B^{dp}(u_{t+1})$ for some $u_{t+1} > \frac{\beta}{4}+\frac{2}{2+\beta}$, it would be possible to reach $u_{t+1} > \frac{\beta}{4}+\frac{2}{2+\beta}$. However, $b_{t+1} = B^{dp}(u_{t+1})$ in this region (which is optimal if political power is to remain in the hands of the unsuccessful) or, $b_{t+1} = b^{bd}$, and we have already shown that $B^{dp}(u_{t+1}) \geq b^{bd}\,\forall\, u_{t+1}$.

To check condition a), we start by computing the indirect utility of choosing $b_t \in \left[0, b^{bd}\right]$. In fact, it is straightforward to show that for any ε covered in the proposition (i.e., $\varepsilon \leq 1/2$), $b_t = b^{bd}$ solves $\max_{b\in[0,b^{bd}]} V^{ou}(b_t, 0, u_t)$ for any $u_t > \frac{1+\varepsilon}{2}$, yielding a indirect utility

$$\frac{1}{4}\left((3+\beta-2u_t)\, b^{bd} - \left(b^{bd}\right)^2\right) \equiv V^{ou}_{bd}(\varepsilon, u_t). \tag{45}$$

Now, we calculate the indirect utility of the old unsuccessful if the choose $b_t = B^{dp}(u_t)$. Let us define $V^{ou}\left(B^{dp}(u_t), B^{dp}(U^{dp}(B^{dp}(u_t)), u_t\right) \equiv V^{ou}_c(u_t)$. It is intuitive (and straightforward to show) that $V^{ou}_c(u_t) - V^{ou}_{bd}(\varepsilon, u_t)$ decreases in u_t. Thus, for the welfare state to remain, we require

$$V^{ou}_c\left(\frac{\beta}{4}+\frac{2}{2+\beta}\right) - V^{ou}_{bd}\left(\varepsilon, \frac{\beta}{4}+\frac{2}{2+\beta}\right) \geq 0 \Leftrightarrow \varepsilon \leq \bar{\varepsilon}(\beta), \tag{46}$$

where $\bar{\varepsilon}(\beta)$ is defined in the proposition. This shows that condition a) above is satisfied. Now consider condition b). First, we note that the old successful always prefer zero redistribution, implying $b_t = 0$ if $u_t \leq \frac{1-\varepsilon}{2}$. Then, consider the young, who have an indirect utility of

$$V^y(b_t, b_{t+1}, b_{t+2}, u_t) = \frac{(1+\beta)^2}{4} + \frac{1-\beta-2u_t}{4} b_t - \frac{b_{t+1}+\beta b_{t+2}}{4}\beta b_{t+1}. \tag{47}$$

Clearly, the young prefer zero benefits whenever $u_t \geq \frac{1-\beta}{2}$. Furthermore, since $u_{t+1} = \frac{1-\beta+b_t+\beta b_{t+1}}{2} \geq \frac{1-\beta}{2}$, the young prefer zero benefits at all $t > 0$ and by assumption, $u_0 \geq \frac{1+\varepsilon}{2} > \frac{1-\beta}{2}$. Thus, b_t is optimally chosen to zero whenever $u_t \leq \frac{1+\varepsilon}{2}$, which proves condition b), showing that the first condition in the equilibrium definition is satisfied and concluding the proof. ■

7.6 Proof of proposition 6

Proof. B) $U^{pv}(b_t) = (1-\beta+b_t+\beta B^{pv}(U^{pv}(b_t)))/2$, for all $b_t \in [0,1]$

Given $u_{t+1} = U^{pv}(b_t)$, $b_t \in [0,1]$ and $b_{t+1} = B^{pv}(u_{t+1})$, $V^{pv}(b, u_t)$ can be expressed as;

$$V^{pv}_t(b_t, u_t) = 1 - u_t + u_t b_t$$
$$-\left[1-\beta+b_t+\beta\left(-\frac{1-\beta}{2+\beta}+\frac{2}{2+\beta}\left(\frac{1-\beta+b_t\left(1+\frac{\beta}{2}\right)}{2}\right)\right)+2u_t\right]\frac{b_t}{4}.$$

Maximizing V_t^{pv} over b_t yields;

$$b_t = \begin{cases} -\frac{1-\beta}{2+\beta} + \frac{2}{2+\beta} u_t & \text{if } u_t \geq \frac{1-\beta}{2} \\ 0 & \text{if } u_t < \frac{1-\beta}{2} \end{cases} = B^{pv}(u_t),$$

which proves (A).

To prove (B), observe that

$$\begin{aligned} & \frac{1-\beta+b_t+\beta B^{pv}\left(U^{pv}(b_t)\right)}{2} \\ &= \frac{1-\beta+b_t+\beta B^{pv}\left(\frac{1-\beta+b_t\left(1+\frac{\beta}{2}\right)}{2}\right)}{2} \\ &= \frac{1-\beta+b_t+\beta\left(-\frac{1-\beta}{2+\beta}+\frac{2}{2+\beta}\left(\frac{1-\beta+b_t\left(1+\frac{\beta}{2}\right)}{2}\right)\right)}{2} \\ &= \frac{1-\beta+b_t\left(1+\frac{\beta}{2}\right)}{2} = U^{pv}(b_t) \end{aligned}$$

Finally, having proved that $b_t = B^{pv}(u_t)$ and $u_{t+1} = U(b_t)$, the characterization of the equilibrium law of motion of u_t, (16), and of the steady-state is straightforward. ■

7.7 Proof of proposition 7

To be written.

7.8 Proof of proposition 8

Proof. $B^{lo}(u_t)$=arg $\max_{b_t} V^{lo}(b_t, u_t)$, where

$$\begin{aligned} V^{lo}(b_t,u_t) &\equiv \left((1-\hat{\gamma})\alpha+\hat{\gamma}(1-u_t)\right)V^{os}(b_t,u_t)+\left((1-\hat{\gamma})(1-\alpha)+\hat{\gamma}u_t\right)V^{ou}(b_t,u_t) \\ &= \left((1-\hat{\gamma})\alpha+\hat{\gamma}(1-u_t)\right)+\left((1-\hat{\gamma})(1-\alpha)+\hat{\gamma}u_t\right)b_t-(1+b_t+\beta b_{t+1}+2u_t)\,b_t/4 \end{aligned} \tag{48}$$

subject to $u_{t+1} = U^{lo}(b_t)$, $b_t \in [0,1]$ and $b_{t+1} = B^{lo}(u_{t+1})$;

B) $U^{lo}(b_t) = \left(1-\beta+b_t+\beta B^{lo}\left(U^{lo}(b_t)\right)\right)/2$, for all $b_t \in [0,1]$

Consider, first, the case in which $\gamma < 1/2$. Given $u_{t+1} = U^{lo}(b_t)$, $b_t \in [0,1]$ and $b_{t+1} = B^{lo}(u_{t+1})$, V_t^{ou} can be expressed as;

$$\begin{aligned} V_t^{lo}(b_t,u_t) &= \left((1-\hat{\gamma})(1-\alpha)+\hat{\gamma}u_t\right)b_t \\ -\frac{1}{4}\left(1-\beta+b_t+\beta\cdot\right.&\left(c_1(\alpha,\beta,\gamma)+\frac{2\gamma-1}{1+\frac{\beta}{2}(2\gamma-1)}\left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4\left(2\beta\alpha(1-\gamma)+\beta\gamma\right)}{4(2+\beta)}\right)\right. \\ &\left.+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t+2u_t\right)b_t, \end{aligned}$$

Maximizing V_t^{lo} over b_t yields;

$$b_t = c(\alpha,\beta,\gamma)+\frac{2\gamma-1}{1+\frac{\beta}{2}(2\gamma-1)}u_t = B^{lo}(u_t), \tag{49}$$

where assumption 1 (parts 1 and 2), together with the definition of $c(\alpha,\beta,\gamma)$, ensure that the constraint $b_t \in [0,1]$ is never binding. This proves part (A) of the proposition when $\gamma < 1/2$.

Next, consider the case where $\gamma \geq 1/2$. First, it is easy to check that for any $u_t \in [0,1]$, $u_{t+1} > u_y(\alpha,\beta,\gamma)$. In particular, if $u_t < u_y(\alpha,\beta,\gamma)$, then $u_{t+1} = \frac{2-4\beta\alpha(1-\gamma)-\beta(1+\beta)(2\gamma-1)}{2(2+\beta)} > u_y(\alpha,\beta,\gamma)$, provided that $\alpha < (1+\beta)/2$ (note that $\frac{2-4\beta\alpha(1-\gamma)-\beta(1+\beta)(2\gamma-1)}{2(2+\beta)} = u_y(\alpha,\beta,\gamma)$, if $\alpha = (1+\beta)/2$). Thus, V_t^{lo} can be expressed as (48) for any choice of b. Hence, the maximization yields (49), and this concludes part A.

For part B, we start, again with the case in which $\gamma < 1/2$. Then,

$$\begin{aligned}
&\frac{\left(1-\beta+b_t+\beta B^{lo}\left(U^{lo}(b_t)\right)\right)}{2}\\
&= \frac{1-\beta+b_t}{2}\\
&+\frac{\beta}{2}B^{lo}\left(\left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right)+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t\right)\\
&= \frac{1-\beta+b_t}{2}+\frac{\beta}{2}\left(c(\alpha,\beta,\gamma)+\frac{2\gamma-1}{1+\frac{\beta}{2}(2\gamma-1)}\cdot\right.\\
&\left.\cdot\left(\left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right)+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t\right)\right)\\
&= \left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right)+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t = U^{lo}(b_t).
\end{aligned}$$

Next, consider the case where $\gamma \geq 1/2$.

$$\begin{aligned}
&\frac{\left(1-\beta+b_t+\beta B^{lo}\left(U^{lo}(b_t)\right)\right)}{2}\\
&= \frac{1-\beta+b_t}{2}+\frac{\beta}{2}B^{lo}\left(\left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right)+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t\right)\\
&= \begin{cases}
\begin{array}{c}\frac{1-\beta+b_t}{2}+\frac{\beta}{2}\left(c(\alpha,\beta,\gamma)+\frac{2\gamma-1}{1+\frac{\beta}{2}(2\gamma-1)}\cdot\right.\\ \left.\left(\left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right)+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t\right)\right)\end{array} & \text{if } u_t > u_y(\alpha,\beta,\gamma)\\
0 & \text{if } u_t \leq u_y(\alpha,\beta,\gamma)
\end{cases}\\
&= \begin{cases}
\left(\frac{1}{2}-\frac{2\beta^2(2\gamma-1)+4(2\beta\alpha(1-\gamma)+\beta\gamma)}{4(2+\beta)}\right)+\frac{4(1+\beta\gamma)+\beta^2(2\gamma-1)}{4(2+\beta)}b_t & \text{if } u_t > u_y(\alpha,\beta,\gamma)\\
0 & \text{if } u_t \leq u_y(\alpha,\beta,\gamma)
\end{cases} = U^{lo}(b_t)
\end{aligned}$$

which concludes the prove of part B.

Finally, having proved that $b_t = B^{lo}(u_t)$ and $u_{t+1} = U^{lo}(b_t)$, the characterization of the equilibrium law of motion of u_t, and of the steady-state is straightforward. Note that Assumption 1 guarantees that $\alpha \leq (1+\beta)/2$, for any γ and β, thus $b^{lo} \geq 0$. ■