A Spatial Voting Model where Proportional Rule Leads to Two-Party Equilibria

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Abstract

In this paper we show that in a simple spatial model where the government is chosen under strict proportional rule, if the outcome function is a linear combination of parties' positions, with coefficient equal to their share of votes, essentially only a two-party equilibrium exists. The two parties taking a positive number of votes are the two extremist ones. Applications of this result include an extension of the well-known Alesina and Rosenthal model of divided government as well as a modified version of Besley and Coate's model of representative democracy. Different outcome functions are then analyzed.

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1 Introduction

If and how electoral rules affect the formation and the survival of political parties in mass elections are among the most analyzed questions in the voting literature. Duverger (1954) first observed a tendency to have just two serious candidates in plurality rule elections, whereas proportional systems are more likely to have several parties. Riker (1982) in a famous paper precisely defined *Duverger's Law* and *Duverger's Hypothesis*. *Duverger's Law* states that "the simple-majority single-ballot system [i.e. simple plurality rule] favors the two-party system" (Duverger 1954:217). *Duverger's Hypothesis* states that "proportional representation favors multipartyism" (Duverger 1954:239). *Duverger's Law* and *Hypothesis* have established themselves as two of the premier empirical regularities in political science.

The most common explanation of those regularities seems to be that strategic voting is present in a simple plurality system, acting to push down the number of parties, whereas it is absent in proportional representation, explaining multipartyism[1]. This is one of the reasons that strategic voting commonly means a reduction in the number of parties for which voters decide to vote. Duverger (1954:226) explained this fact in sociological terms: "in cases where there are three parties operating under the simple majority single-ballot system the electors soon realize that their votes are wasted if they continue to give them to the third party: hence their natural tendency to transfer their vote to the less evil of its two adversaries in order to prevent the success of the greater evil". So the sociological explanation given by Duverger has been translated into strategic voting by formal models.

Duverger's Law received a lot of attention in the political science literature. One reason may be found in Riker's words (1982:764): "The evidence renders it undeniable that a large amount of sophisticated voting occurs - mostly to the disadvantage of the third parties nationally so that the force of Duverger's psychological factor must be considerable".[2]

We want to cite here a few papers about *Duverger's Law* that have contributed to the success of this concept.

Palfrey (1989) cleverly proves that in an incomplete information framework, any symmetric Bayesian-Nash equilibrium is such that the share of votes that the third party gets in the election is small (except for what Palfrey conjectured to be knife-edge cases). Moreover, the share of votes of the third party goes to zero as soon as the number of voters goes to infinity.

Cox (1994) extends Palfrey's (1989) analysis to multimember districts (i.e., districts electing M members) operating under the single nontransferable rule. Cox proves that equilibria conform to the $(M + 1)$ rule; that is, strategic voting leads to $(M + 1)$ candidates who get votes in equilibrium.

Myerson and Weber (1993) develop a general model of a one-stage voting

[1] Leys (1959) and Sartori (1968) were the first scholars to claim that strategic voting, and its reduction of the number of parties, is also present under proportional representation.

[2] It is important to say that Riker here means by "sophisticated voting" just a sort of strategic voting, and not iterated dominance, as has become usual in the last decade.

game, deriving some results also for approval and Borda voting rules, where they prove the existence of what they define as *voting equilibria*[3] also for plurality rule. Furthermore, they show that non-Duvergerian and knife-edge *voting equilibria* exist under plurality rule, and so the derivation of Duverger's Law does not follow from the concept of *voting equilibrium* itself, requiring some additional assumptions.

Fey (1997) shows, in the same framework of Myerson and Weber (1993), that non-Duvergerian equilibria are unstable. More precisely, he shows that non-Duvergerian equilibria require extreme coordination, and any variation in beliefs leads voters away from them to one of the Duvergerian equilibria.

These results are obtained assuming incomplete information[4]. If players have complete information, Duvergerian equilibria cannot be justified (see De Sinopoli 2000) solely by strategic voting.

The literature does not give enough attention to *Duverger's Hypothesis*, in general assuming that strategic voting is absent under proportional representation. Riker (1982) himself understood that the relation between proportional representation and multi-partyism is weaker than the relation between plurality and bipartyism.

Cox (1997) shows that strategic voting can reduce the number of parties at equilibrium even in a model of proportional representation. More precisely, he investigates strategic voting equilibria in multimember districts operating under various largest-remainder methods of proportional representation. He shows that there are two kind of Bayesian equilibria. In one kind of equilibrium there is at most one vote-getting list that does not expect to win a seat; in the second kind of equilibrium more than one list does not expect to win a seat. The first kind of equilibria are the interesting ones: those where strategic voting leads to a bound on the number of the viable lists, and this bound is exactly $(M+1)$ (in districts with M seats). But, Cox himself concludes that "in case of large-magnitude PR the $M+1$ bound appears not to be binding, revealing that empirically observed effective numbers of lists are depressed below this upper bound by forces other than strategic voting" (1997:122).

Those results in mind, it is clear that there is a dearth of analysis of strategic voting in proportional representation. In this paper we investigate the strategic behavior of voters who face proportional representation. We have defined the simplest possible framework to fully analyze the effect of strategic voting on the number of parties at equilibrium.

In this paper, where we assume that the policy space is a closed interval of the real line, proportional representation is represented through a policy outcome defined as a linear combination of parties' positions weighted with the share

[3] An election result is a *voting equilibrium* if and only if there exists a vector of positive probabilities that justifies the election result and that satisfies a well-defined ordering condition. The ordering condition simply states that candidates expected to place third or lower in the poll are less likely to be tied for first than candidates expected to place first or second.

[4] The model developed by Myerson and Weber applies to both complete and incomplete information.

of votes that each party gets in the election. In this way we want to capture the spirit of proportional representation, i.e., any party that gets some votes is represented in the political process of policy determination, with a weight that is proportional to its share of votes. We realize that the policy outcome so defined is not "realistic"; nevertheless it deserves attention, because it can be seen as a polar case[5].

The main result of the paper is quite strong: in a large electorate strategic voters, regardless of the number of parties they can vote for, in any equilibrium will vote for only two parties. This result holds with a very weak assumption on voters' preferences, single peakedness. Moreover, we are able to identify which are those parties: they are the extremist ones. This result implies that strategic voting can have a devastating effect also under proportional representation.

Another message of this paper is the analysis of the game with a continuum of voters as the limit game of games with a finite number of voters. As a matter of fact, we think that if one wants to fully understand the strategic behavior of voters, it is necessary to start with a finite number of players. The main reason is that under the continuum assumption each player is negligible to the outcome. We start by analyzing the game with a finite number of voters, fully capturing the strategic incentive of the voters to vote for only two parties at equilibrium. More precisely we prove that essentially a unique Nash equilibrium exists, characterized by an outcome (defined cutpoint) such that any voter to its right votes for the rightmost party and any voter to its left votes for the leftmost party. The intuition of the result is clear: strategic voters misrepresent their preferences, voting for the extremist parties in order to drag the policy outcome toward their preferred policy point.[6]

The result above allows us to analyze the game with a continuum of voters as the limit game of games with a finite number of voters. In such a case each voter behaves as if he were decisive, and the "equilibrium" outcome is the policy obtained with every voter to its left voting for the leftmost party and every voter to its right for the rightmost party.

The general result can be useful in studying well-known voting models. First, we discuss the multi-party version of a divided government model, where in the spirit of Alesina and Rosenthal's (1996) analysis, we obtain a *moderation* result: we show that the more rightist the president is, more votes are taken by the leftist party. Second, we study, following Besley and Coate (1997), endogenous candidacy, finding that, when the cost of candidacy is small, only the two extremists will be candidates.

The outcome function used in the general model is, as we acknowledged, a polar case. Motivated by this consideration, we study two very different outcome functions. The result that only the two extremist parties get votes, holds even

[5]For a similar policy outcome see, for example, Alesina and Rosenthal (2000) and Ortuño-Ortin (1997).

[6]The incentive to vote for an extreme is given by the maximal effect that such a vote has on the outcome. Vice versa, if the policy outcome is the median, voting sincerely is a dominant strategy because the effect of each vote, but the median one, is solely "directional" (i.e., any vote to the left of the median has the same effect as well as any vote to the right).

when the policy outcome is a weighted average of the platforms of the members of a predetermined winning coalition, or if the policy is defined as the weighted average of the platforms of the top two vote getters.

The primary motivation to write this paper was a desire to understand the relation between strategic voting and the number of parties resulting at equilibrium. Nevertheless, we present here some considerations that may reconcile our theoretical predictions with reality. First, we try to determine whether proportional representation always implies multi-partyism; second, we present some support for its application to Alesina and Rosenthal's model.

Proportional representation and a two-party system. Riker himself, after the analysis of four counterexamples[7] to *Duverger's Hypothesis*, concluded that "we can therefore abandon Duverger's Hypothesis in its deterministic form" (1982:760). We have shown that Cox (1997) proved that strategic voting can reduce the number of parties at equilibrium even under proportional representation.

We present here two cases where proportional representation did not imply multi-partyism.

The first case is Austria, defined by Riker as a "true counterexample" (1982:758) that experienced a stable two-party system under proportional representation[8]. The two major parties, the Christian Socialist (OVP) and the Social Democrats (SPO), were essentially duopolists with eighty to ninety per cent (or more) of the vote from 1945 to 1987 (see Engelmann, 1988:87).

Ireland was defined by Riker (1982:758) as "a devastating counterexample" to *Duverger's Hypothesis*. The reason is that proportional representation[9] favored a decrease in the number of parties: since the elections of 1927, when there were seven parties and fourteen independents, the number of parties decreased, and from the election of 1969 three parties were on the scene together with a few independent parties. From the elections of 1932 a stable "two-party and half" system (Carty, 1988:224) was founded.

Moderation in the multiparty version of Alesina and Rosenthal's model. The empirical evidence for the *moderation* result that our model predicts for the multiparty version of the model of Alesina and Rosenthal (1996) may have some support. Formally, we define a two-stage game where first there is an election of the president with plurality rule, and then an election of a legislature

[7] Riker analyzed four cases: Australia, Austria, Germany, and Ireland.

[8] More precisely, the electoral system is as follows. Each list receives as many seats as its vote contains full Hare quotas, and those seats are then allocated to the list's candidates, in accordance with the list order. Seats unallocated in the first step are aggregated in accordance with each secondary list's vote and then reallocated to the list's candidate (see Cox 1997).

[9] The Irish system is proportional representation by means of the single transferable vote (STV). Under STV the voter has the opportunity to indicate a range of preferences by placing numbers in correspondence with candidates' names on the ballot paper. A vote can be transferred from one candidate to another if it is not required by the prior choice to make up that candidate's quota (or if, as a result of poor support, that candidate is eliminated from the contest).

with proportional rule. The main finding is that the share of votes taken by the leftmost party in the legislative election is increasing in the position of the president, i.e., the more rightist is the president, the more votes will be taken by the leftmost party. Shugart (1995), analyzing some presidential countries, observes that "as elections are held later in a president's term, the share of seats won by the president's party tends to decline"[10]. The empirical findings offered by Shugart (1995) are coherent with our theoretical prediction.

The paper is organized as follows. In sections 2 and 3 we present the general model and we characterize the equilibrium. In section 4 we present two applications, one to the Alesina and Rosenthal (1996) model of divided government in subsection 4.1, and one to the Besley and Coate (1997) model in subsection 4.2. In section 5 we analyze the extension to two other outcome functions, and section 6 concludes.

2 The basic model

We define the simplest framework to analyze an election called with proportional rule:

Policy Space. The policy space $\mathbb{X}$ is a closed interval of the real line, and without loss of generality we assume $\mathbb{X} = [0, 1]$.

Parties. Parties are fixed both in number[11] and in their positions, in that there is no strategic role for them: there is an exogenously given set of parties $M = \{1, ..., k, ...m\}$ $(m \geq 2)$, indexed by k. Each party k is characterized by a policy $\zeta_k \in [0, 1]$.

Strategy. Given the set of parties M, each voter can cast his vote for a party[12]. The pure strategy space of each player i is $S_i = \{1, ..., k, ..., m\}$ where each $k \in S_i$ is a vector of m components with all zeros except for a one in position k, which represents the vote for party k.

A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^1, ...\sigma_i^k, ..., \sigma_i^m)$ where each σ_i^k represents the probability that player i votes for party k.

Policy outcome. The position of the government, i.e., the policy outcome, is a linear combination of parties' policies, each coefficient being equal to the corresponding share of votes. Given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, $v(s) = \sum_{i \in N} \frac{s_i}{n}$ is the vector representing for each party its share of votes, hence the policy outcome can be written as:

$$X(s) = \sum_{k=1}^{m} \zeta_k v_k(s). \tag{1}$$

[10] Shugart considers eleven countries, among which are France, Chile, and El Salvador.

[11] We will relax this assumption in the application to Besley and Coate's model of represenative democracy (1997), when the number of parties will be endogenous.

[12] In this paper we do not allow for abstention. We cannot claim that this assumption is neutral. In our proof we use the fact that, as the number of players goes to infinity, the weight of each player goes to zero, and this result does not hold if a large number of voters abstain.

Voters. Each voter is characterized by his bliss point $\theta \in \Theta = [0,1]$. Voters' preferences are single peaked. We stress that this is the only assumption needed to reach the result for pure strategy equilibria. To analyze mixed strategy equilibria, we assume that a fundamental utility function $u : \Re^2 \rightarrow \Re$ exists, continuously differentiable with respect to the first argument[13], which represents the preferences, that is, $u_i(X) = u(X, \theta_i)$.

Given the set of parties and the utility function u, a finite game Γ is characterized by a set of players $N = \{1, ..., i, ..., n\}$ and their bliss points. Given $\Gamma = \left\{N, \{\theta_i\}_{i \in N}\right\}$ we denote by $H^\Gamma(\theta)$ the distribution of players' bliss points[14], i.e., $H^\Gamma(\theta^*)$ is the proportion of players with a bliss point less than or equal to θ^*.

The utility that player i gets under the strategy combination s is:

$$U_i(s) = u(X(s), \theta_i)$$

Given a mixed strategy combination $\sigma = (\sigma_1, ..., \sigma_n)$, because players make their choice independently of each other, the probability that $s = (s_1, s_2, ..., s_n)$ occurs is:

$$\sigma(s) = \prod_{i \in N} \sigma_i^{s_i}.$$

The expected utility that player i gets under the mixed strategy combination σ is:

$$U_i(\sigma) = \sum \sigma(s) U_i(s).$$

In the following, as usual, we shall write $\sigma = (\sigma_{-i}, \sigma_i)$, where $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$ denotes the $(n-1)$ −tuple of strategies of the players other than i. Furthermore s_i will denote the mixed strategy σ_i that gives probability one to the pure strategy s_i.

3 The equilibrium

In this section we analyze the equilibrium of the game defined above. First, we analyze voters' behavior when only pure strategies are allowed. We show that in any pure strategy Nash equilibrium of the game, voters vote only for the extreme parties, except for a neighborhood inversely related to the number of players. We define then the cutpoint outcome, i.e., the outcome obtained such that any voter strictly on its right votes for the rightmost party and any voter strictly on its left votes for the leftmost party. Such a strategy combination is a pure strategy Nash equilibrium of the game, if it does not coincide with a voter's bliss point.

[13] Hence, by single-peakedness, $\forall \bar{x}_2 \in [0,1]$, $\frac{\partial u(x_1, \bar{x}_2)}{\partial x_1} \gtreqless 0$ for $x_1 \lesseqgtr \bar{x}_2$ and $x_1 \in [0,1]$.

[14] Sometimes we will identify a player with his bliss point.

As nothing assures us that this sufficient condition for the existence of a pure strategy equilibrium is satisfied, or that mixed strategy equilibria behave completely differently, we extend the analysis to the case when voters are allowed to play mixed strategies. We prove the main result of this paper: in any equilibrium any player on the right of the cutpoint outcome votes for the rightmost party, and any player on the left of the cutpoint outcome votes for the leftmost party, except for a neighborhood inversely related to the number of voters.

We then study the game with a continuum of voters as the limit game of games with a finite number of voters, i.e., each voter behaves as if he could be decisive. The previous analysis, carried for games with a finite number of players, lets us consider the cutpoint outcome as the "right" solution of the game with a continuum of voters.

In order to simplify the notation, in the following we will denote L the leftmost party and R the rightmost (i.e., $L = \arg\min_{k\in M} \zeta_k$, $R = \arg\max_{k\in M} \zeta_k$)[15].

3.1 Pure strategy equilibria

We start by analyzing the pure strategy equilibria in order to stress the intuition behind the result, that is, strategic voters have an incentive to vote for the extremist parties in order to drag the policy outcome toward their bliss policy. First, we underline that only the assumption of single peakedness of voters' preferences is needed to get the result. We prove that every pure strategy equilibrium is such that (except for a neighborhood whose length is inversely proportional to the number of players) everybody votes for one of the two extremist parties.

Proposition 1 *Let s be a pure strategy equilibrium of a game Γ with n voters:*
(α) if $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$ then $s_i = L$,
(β) if $\theta_i \geq X(s) + \frac{1}{n}(\zeta_R - \zeta_L)$ then s_i=R.

Proof. (α) Notice that if $X(s_{-i}, L) \geq \theta_i$ then, by single-peakedness, L is the only best reply, for player i, to s_{-i} (i.e., $\forall k \neq L$, $X(s_{-i}, k) > X(s_{-i}, L)$). Because $X(s_{-i}, L) = X(s) - \frac{1}{n}(\zeta_{s_i} - \zeta_L) \geq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, the assumption $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$ implies that L is the unique best reply, for player i, to s_{-i}. (β) A symmetric argument holds. ■

The proposition above implies that in every pure strategy Nash equilibrium of a game, the proportion of votes taken by the less extreme parties goes to zero as n goes to infinity.[16]

At this point, it is natural to analyze the case when anybody strictly on the left of the policy outcome votes for L, and anybody strictly on the right of the policy outcome votes for R.

Given a game Γ and its distribution of bliss points $H^\Gamma(\theta)$, let $\tilde{\theta}^\Gamma$, defined as cutpoint policy, be the unique policy outcome obtained with voters strictly on

[15] We assume that there is only one party at ζ_L as well as at ζ_R. This assumption simplifies the notation, but it does not affect the result. Without this assumption, if we denote L and R the set of extremist parties, everything still holds.

[16] At least if voters' bliss points are sufficiently spread.

its left voting for L and voters strictly on its right voting for R, i.e., let $\tilde{\theta}^{\Gamma}$ be implicitly defined by:

$$\tilde{\theta}^{\Gamma} \in \zeta_L \bar{H}^{\Gamma}\left(\tilde{\theta}^{\Gamma}\right) + \zeta_R(1 - \bar{H}^{\Gamma}\left(\tilde{\theta}^{\Gamma}\right))$$

where $\bar{H}^{\Gamma}$ is the correspondence defined by $\bar{H}^{\Gamma}(\theta) = \left[\lim_{y \to \theta^-} H^{\Gamma}(y), H^{\Gamma}(\theta)\right]$.

Let us assume that no player's preferred policy coincides with the cutpoint outcome. No player on the left of the cutpoint outcome has an incentive to vote for any party different from L, because doing so would push the policy outcome further away from his preferred policy. The same argument holds for any player on the right of the policy outcome. We can, then, state the following proposition:

Proposition 2 *If $\theta_i \neq \tilde{\theta}^{\Gamma}$ $\forall i \in N$, then the strategy combination given by $\forall \theta_i < \tilde{\theta}^{\Gamma}$ $s_i = L$ and $\forall \theta_i > \tilde{\theta}^{\Gamma}$ $s_i = R$ is a pure strategy Nash equilibrium of the game Γ.*

It is clear that nothing assures us that pure strategy equilibria exist; moreover we have to check if mixed strategy equilibria prescribe a dramatically different behavior for individual voters.

3.2 Mixed strategy equilibria

We analyze the case when players are allowed to play mixed strategies. In order to undertake this analysis we have to assume also that the utility function u is continuously differentiable with respect to the first argument.[17]

We recall that, given the set of candidates M and the utility function u, a game Γ is characterized by the set of players and their bliss points. Let $\sigma = (\sigma_1, ..., \sigma_n)$ and $\bar{\mu}^{\sigma} = \sum_{i \in N} \frac{\sigma_i}{n}$. With abuse of notation, let $X(\bar{\mu}^{\sigma}) = \sum_{k=1}^{m} \zeta_k \bar{\mu}_k^{\sigma}$.

We can state the following proposition:

Proposition 3 *$\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ if σ is a Nash equilibrium of a game Γ with n voters, then:*
(α) if $\theta_i \leq X(\bar{\mu}^{\sigma}) - \varepsilon$ then $\sigma_i = L$
(β) if $\theta_j \geq X(\bar{\mu}^{\sigma}) + \varepsilon$ then $\sigma_j = R$.

Proof. See Appendix. ■

In the appendix we will show that $\bar{\mu}^{\sigma}$ is the expected vote shares for the parties. The proposition above says that in any Nash equilibrium, except for a

[17] To study mixed strategies equilibria some cardinal assumptions on the utility function are needed. Because we use the mean value theorem the cardinal assumption we have made is the differentiability one, which seems to be the weakest one to get the results. Furthermore, the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ in X guarantees the existence, for each player, of a lower bound on the number of players for which the results hold. The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ in θ assures that a bound can be found independently of the set of players.

neighborhood whose length decreases as the number of players increases, everybody to the left of $X(\bar{\mu}^{\sigma})$ votes for L, while everybody to the right votes for R.

Using the definition of cutpoint policy outcome, we can state the main result of this paper: essentially an unique Nash equilibrium of the game exists:

Corollary 4 *$\forall \eta > 0$, $\exists n_1$ such that $\forall n \geq n_1$ if σ is a Nash equilibrium of a game Γ with n voters, then:*
(α) if $\theta_i \leq \tilde{\theta}^{\Gamma} - \eta$ then $\sigma_i = L$
(β) if $\theta_j \geq \tilde{\theta}^{\Gamma} + \eta$ then $\sigma_j = R$.

Proof. Fix η and, in Proposition 3, take $\varepsilon = \frac{\eta}{2}$. For the corresponding n_0 it is easy to see that if $n \geq n_0$ and σ is a Nash equilibrium of Γ, $\tilde{\theta}^{\Gamma} - \frac{\eta}{2} \leq X(\bar{\mu}^{\sigma}) \leq \tilde{\theta}^{\Gamma} + \frac{\eta}{2}$. In fact, suppose by contradiction that $\tilde{\theta}^{\Gamma} - \frac{\eta}{2} > X(\bar{\mu}^{\sigma})$ then Proposition 3 implies that all voters to the right of $\tilde{\theta}^{\Gamma}$ vote for the rightmost party and hence $\tilde{\theta}^{\Gamma} \leq X(\bar{\mu}^{\sigma})$, contradicting $\tilde{\theta}^{\Gamma} - \frac{\eta}{2} > X(\bar{\mu}^{\sigma})$. Analogously for the second inequality. Hence $\tilde{\theta}^{\Gamma} - \eta \leq X(\bar{\mu}^{\sigma}) - \frac{\eta}{2}$ and $\tilde{\theta}^{\Gamma} - \eta \geq X(\bar{\mu}^{\sigma}) + \frac{\eta}{2}$, which, with Proposition 3, complete the proof. ■

Every equilibrium conforms to such a cutpoint, and hence, for n large enough, only the two extremist parties take a significant amount of votes.

3.3 Game with a continuum of voters

We now analyze analogous games with a continuum of voters. In such games every strategy combination is a Nash equilibrium, because each player's vote does not affect the outcome. The results obtained in the previous pages lead us to analyze the game with a continuum of players as the limit game of games with a finite number of players. In such a case each voter behaves as if he could be decisive, and the "equilibrium" outcome is the policy obtained with every voter to its left voting for the leftmost party and every voter to its right for the rightmost party.

Let the bliss point distribution function characterizing the game with a continuum of voters $H(\theta)$ be continuous and strictly increasing, and let $\tilde{\theta}$ be the unique policy outcome obtained with voters on the left of $\tilde{\theta}$ voting for L and voters on the right voting for R, i.e., $\tilde{\theta}$ is the unique solution of

$$\tilde{\theta} = \zeta_L H\left(\tilde{\theta}\right) + \zeta_R(1 - H\left(\tilde{\theta}\right)).$$

The previous analysis implies that $\tilde{\theta}$ is the "equilibrium" of the game characterized by $H(\theta)$ when this game is seen as a limit of finite games.

Moreover, considering the game with a continuum of voters, but where every player acts as if his vote would be no-negligible, the cutpoint strategy can be obtained through a process of iterated elimination of dominated strategies, if the distribution function $H(\theta)$ is not too steep. This is the content of the next proposition.

Proposition 5 *Let $H(\theta)$ be differentiable. If*

$$(\zeta_R - \zeta_L)H'(\theta) < 1,$$

then $\tilde{\theta}$ is the only strategy combination that survives the iterated elimination of dominated strategies.

Proof. Given ζ_L, every player between 0 and ζ_L has voting for L as dominant strategy (whatever the others do, the outcome will be to his right). Hence, eliminating all the other strategies, every player between $\zeta_L H(\zeta_L) + \zeta_R(1 - H(\zeta_L))$ and 1 has voting for R as a dominant strategy. We can iterate this process[18]. The iterations obey the following dynamic:

$$\theta_{t+1} = \zeta_R - (\zeta_R - \zeta_L)H(\theta_t)$$

which clearly converges to $\tilde{\theta}$ if

$$(\zeta_R - \zeta_L)H'(\theta) < 1.$$

■

4 Two Applications

We have fully exploited the incentive for individual voters to vote for only two parties when an election is called under proportional representation and there are m parties. We want to apply our general result to two well-known models of political economy.

The first model we consider is Alesina and Rosenthal (1996). In such a model the policy outcome is described through a compromise between the executive, elected by plurality rule, and the legislature, elected by proportional rule. Considering the two-stage game in which first the president and then the legislature is elected, backward induction implies that in the second stage only the two extremists will obtain votes. In the spirit of Alesina and Rosenthal's analysis, we obtain a *moderation* result: we show that further right the president is, the more votes are taken by the leftmost party in the legislative election.

We have pointed out in the introduction the empirical support for the *moderation* result we obtain (Shugart 1995). Division of government (that is, when moderation shows its maximum effect giving the majority in the legislature to the loser in the presidential election) is explained by Alesina and Rosenthal in the following way: "division of power balances polarized parties" (1995:244). Clearly, in our model strategic voting explains this feature.

The second model we consider is that of Besley and Coate (1997), where the set of candidates is endogenous. Each citizen decides whether to become

[18] The same conclusion can be obtained eliminating simultaneously the dominated strategies of the leftist and the rightist voters (cf. the analogous procedure for a two-party model in Iannantuoni (1999)). With such a procedure, however, we would have to analyze a system. For this reason we have preferred a simpler way to proceed.

a candidate, incurring a cost, or not. Our result implies that as the cost of candidacy goes to zero, only the two extremist citizens will be candidates.

We analyze both models assuming a continuum of voters and under the assumption that the distribution of bliss points $H(\theta)$ is strictly increasing and continuously differentiable, because in such a case we have uniqueness and differentiability of the "equilibrium". We underline that the game with a continuum of voters has to be interpreted as an approximation of a finite game with a large number of voters.

4.1 Divided Government

In a recent paper, Alesina and Rosenthal (1996) describe the formation of national policies as the result of institutional complexity that is captured by the existence of two decision branches of the government: the executive (i.e., the president), elected under plurality rule, and the legislature, elected under proportional rule. In their model, two parties announce their policies and then voters vote. The main implication of this model is that "divided government" can be explained through the behavior of voters with intermediate (that is, situated between parties' announced positions) preferences, who take advantage of the institutional structure to balance the plurality of the winning party in the executive by voting for the opposite party in the legislative election. The main result of Alesina and Rosenthal can be expressed as: *a party receives more votes in the legislative election if it has lost the executive election.*

In this section, we limit the analysis to a two-stage game in which first the president and then the legislature is elected, and we show that analogous results hold for any finite number of parties. More precisely, the results presented in the previous section imply that, in the proportional stage, only the two extremists take votes, and we show that the further to the right the president is, the more votes are taken by the leftmost party. As shown above, our solution rests on *purely individual behavior*, viewing the game with a continuum of players as a limit of finite games. Alesina and Rosenthal's solution is instead based on coalitions, to circumvent the difficulties arising from the fact that with a continuum of voters everyone is negligible to the outcome.

In the first stage players vote for the president, elected with plurality rule, then in the second stage they vote for the legislature, elected with proportional rule.

Given the result of the elections, let the position of the legislature be given by

$$X^{leg} = \sum_{k=1}^{m} \zeta_k v_k$$

where v_k denotes the share of votes taken by party k, and let the policy outcome be a convex combination of presidential and legislative positions:

$$X = (1 - \alpha)\zeta_P + \alpha X^{leg}$$

where ζ_P denotes the position of the party winning the presidential election and $0 < \alpha < 1$.

Solving this game by backward induction, it is evident that, given the election of the president P, the proportional stage is equivalent to the "proportional game" studied in the previous sections, with translated positions of the parties. In other words, given P, we have to analyze the "proportional game" with the set of parties M^P where each party k is characterized by the policy

$$\zeta_k^P = (1-\alpha)\zeta_P + \alpha\zeta_k.$$

The results of the previous sections imply that the equilibrium is such that only the two extremist parties[19] L and R take votes. Moreover, the cutpoint strategy $\tilde{\theta}^P$ is given by the unique solution to:

$$\tilde{\theta}^P = \zeta_L^P H(\tilde{\theta}^P) + \zeta_R^P (1 - H(\tilde{\theta}^P),$$

which can be re-written as:

$$\tilde{\theta}^P = (1-\alpha)\zeta_P + \alpha\zeta_R - \alpha(\zeta_R - \zeta_L) H\left(\tilde{\theta}^P\right). \tag{2}$$

Hence we have

$$\frac{\partial \tilde{\theta}^P}{\partial \zeta_P} = \frac{1-\alpha}{1 + \alpha H'\left(\tilde{\theta}^P\right)(\zeta_R - \zeta_L)} > 0. \tag{3}$$

Because $H(\tilde{\theta}^P)$ represents the share of votes taken in the legislative election by the leftmost party and $H(\theta)$ is strictly increasing, (3) implies that such a share is increasing in the position of the president. Hence also in multi-party systems, we have a *moderation* result.

The main difficulties in analyzing such a model arise in the presidential stage, because multi-candidate election with plurality rule in a complete information framework leads to a multiplicity of equilibria and, to have sensible solution, a strong refinement (as Mertens' stability one) seems to be needed.[20]

Nevertheless, for some specification of the parameters of the model, the plurality stage can be solved by iterated elimination of dominated strategies. The following example shows a case where, given the equilibrium outcome for each subgame, the plurality stage is dominance solvable and the center wins the presidential election, while the two extremists win the legislative one.

EXAMPLE 2

There are three parties L, C, and R with $\zeta_L = 0, \zeta_C = \frac{1}{2}$, and $\zeta_R = \frac{3}{5}$. Suppose the voters' bliss points are distributed uniformly on $[0, 1]$, with symmetric utility functions and $\alpha = \frac{1}{6}$.

[19] Obviously we have $L = \arg\min_k \zeta_k = \arg\min_k \zeta_k^P$ and $R = \arg\max_k \zeta_k = \arg\max_k \zeta_k^P$.
[20] We refer to De Sinopoli (2000) for a discussion on this point.

If we solve the game backward, equation (2) gives us the equilibrium outcome for each possible president. It is not difficult to compute that

$$\tilde{\theta}^L = \frac{1}{11}, \tilde{\theta}^C = \frac{31}{66}, \text{ and } \tilde{\theta}^R = \frac{6}{11}.$$

In the first stage, hence, citizens choose with plurality among $\tilde{\theta}^L, \tilde{\theta}^C$, and $\tilde{\theta}^R$. Obviously we have the following preference orders on the election of L, C, and R as president:

$0 \leq \theta_i < \frac{37}{132}$	$L \succ_i C \succ_i R$
$\theta_i = \frac{37}{132}$	$L =_i C \succ_i R$
$\frac{37}{132} < \theta_i < \frac{7}{22}$	$C \succ_i L \succ_i R$
$\theta_i = \frac{7}{22}$	$C \succ_i L =_i R$
$\frac{7}{22} < \theta_i < \frac{67}{132}$	$C \succ_i R \succ_i L$
$\theta_i = \frac{67}{132}$	$C =_i R \succ_i L$
$\frac{67}{132} < \theta_i \leq 1$	$R \succ_i C \succ_i L$

In a plurality election, the strategy of voting for the least preferred candidate is dominated (by voting for the most preferred). In the reduced game obtained by eliminating such strategies, the players have the following strategies:

$\theta_i < \frac{7}{22}$	L, C
$\theta_i = \frac{7}{22}$	C
$\theta_i > \frac{7}{22}$	$R, C.$

In this reduced game there is no chance of candidate L being elected president, because he takes at most $\frac{7}{22}$ of the total number of votes. Hence voting for him is dominated, as are voting for R if $\frac{7}{22} < \theta_i < \frac{67}{132}$ and voting for C if $\frac{67}{132} < \theta_i \leq 1$. As a result, candidate C wins the plurality election, and in the proportional stage L and R take, respectively, $\frac{31}{66}$ and $\frac{35}{66}$ of the votes.

4.2 Representative Democracy

In this section we analyze what can happen when the set of candidates is not exogenous. To this end, we adopt a model analogous to Besley and Coate (1997). We consider a community consisting of a set of citizens N that, in order to implement a policy X, must elect some representatives among themselves.

The selection of the community representatives requires an election. Each citizen is allowed to run for election, acting as a candidate. All citizens choosing to be a candidate face a utility cost δ.

The political process consists of a three-stage game. In the first stage, each citizen decides whether to become a candidate or not. In the second stage, the election occurs. In the third stage, the policy is implemented. In Besley and Coate's (1997) model the election is run with plurality rule and, because there is no commitment, each elected candidate implements his preferred policy.

Let us consider what happens when the election is run with proportional rule and the policy is given by:

$$X = \sum_{k=1}^{m} \zeta_k v_k$$

where v_k denotes the share of votes taken by citizen-candidate k.[21]

If we let the number of citizens go to infinity, we know that for a given set of candidates only the two extremists will take votes. Hence in every pure strategy subgame perfect equilibrium we will have only two candidates. Moreover[22] we have:

$$\frac{\partial \tilde{\theta}}{\partial \zeta_L} = \frac{H(\tilde{\theta})}{1 + (\zeta_R - \zeta_L) H'(\tilde{\theta})} > 0$$

$$\frac{\partial \tilde{\theta}}{\partial \zeta_R} = \frac{1 - H(\tilde{\theta})}{1 + (\zeta_R - \zeta_L) H'(\tilde{\theta})} > 0.$$

This implies that a more extreme citizen, if he decides to be a candidate, will move the outcome toward him. Hence, for a given cost of candidacy, if the leftmost candidate is sufficiently far from the extremist citizen, the latter will prefer to become a candidate. As a result, in every pure strategy equilibrium, as the cost of candidacy goes to zero, only the two extremists decide to become candidates.

Remark: The fact that $\frac{\partial \tilde{\theta}}{\partial \zeta_L} > 0$ and $\frac{\partial \tilde{\theta}}{\partial \zeta_R} > 0$ has an interesting implication in a model where there are two policy-oriented parties that can commit to a policy before the election is called. The equilibrium choices of the parties do not converge toward centrist policy, but either both parties are "radical" (i.e., the policies they commit to will be respectively 0 and 1) or one is "radical" and the outcome coincides with the preferred policy of the other, the choice of the latter being, however, more extremist than its preferred policy. A similar result has been proved with sincere voting and further assumptions on the distribution of voters by Ortuño-Ortin (1997), while we obtain it with strategic voting.[23] Furthermore, Alesina and Rosenthal (2000) prove that parties offer divergent platforms, in an incomplete information setting and when parties care both about winning and about the policy, the latter being a compromise between the executive and the legislature.

[21] To avoid confusion we still denote ζ_i as the preferred policy of candidate i. We have proved the basic results for a finite number of parties, hence we have to assume that the number of candidates is finite. This is NOT an assumption when we consider the game with a continuum of citizens as an "approximation" of the game with a finite number of players.

[22] Assuming $\tilde{\theta} \notin \{0, 1\}$.

[23] A move toward a more extreme position produces two effects: on one hand the number of votes decreases, on the other hand the votes are on a more extreme position. With sincere voting the net effect can be either positive or negative, depending upon the distribution of the voters, whereas with strategic voting the second effect always dominates the first one.

5 Some extensions

In this section we analyze two different outcome functions.

The first institutional context we consider is the following. There are two coalitions of parties and the outcome function is a linear combination, with coefficients equal to the relative share of seats, of the parties' positions in the coalition that takes more votes. Such a model can be an approximation of the *apparentement* system used, at district level, in the French legislative election in 1951 and 1956 (see Rosenthal, 1975). The *apparentements* were preelectoral coalitions of parties, and, even if each party had its own list of candidates, seats were allocated by treating the *apparentement* as single bloc. Moreover, if any *apparentement* had more than half of the votes, it won all the seats of the district.

We obtain, in pure strategies and under the assumption that there is not a pivotal voter (i.e., a voter whose vote can affect the winning coalition), that only two-party equilibria can emerge, where the two parties taking a significant amount of votes are the extremes of the winning coalition.

The second outcome function we will consider is a linear combination, with coefficients equal to the relative share of votes, of the positions of the two first-ranked parties. In such a case, we will show that, in pure strategies, only two-party equilibria can emerge.

5.1 Coalitions

Suppose that the set of parties M is divided into two coalitions A and B, and the outcome function is a linear combination of the winning coalition's parties, each coefficient being equal to the relative share of votes taken by the corresponding party. Formally, given a pure strategy combination $s = (s_1, s_2, ..., s_n)$, let $\eta(s) = \sum\limits_{i \in N} s_i$ be the vector representing the number of votes taken by each party and let $\eta_A = \sum\limits_{k \in A} \eta_k$ and $\eta_B = n - \eta_A$. The outcome function is given by:

$$X(s) = \begin{cases} \frac{1}{\eta_A} \sum\limits_{k \in A} \zeta_k \eta_k(s) & \text{if } \eta_A \geq \frac{n}{2} \\ \frac{1}{\eta_b} \sum\limits_{k \in B} \zeta_k \eta_k(s) & \text{if } \eta_A < \frac{n}{2} \end{cases} .$$

If no pivotal voter exists, it is straightforward to see that only one coalition takes a significant amount of votes, and, within the winning coalition, only the two extremist parties share it.

Let L_A denote the leftmost party and R_A the rightmost party in coalition A; formally $L_A = \arg\min_{k \in A} \zeta_k$, and $R_A = \arg\max_{k \in A} \zeta_k$. Analogously, we define L_B and R_B as the two extremist parties for the coalition B. It is not difficult to see that every pure strategy equilibrium where no pivotal voter exists is such that (except for a neighborhood whose length is inversely proportional to the number of players) everybody votes for one of the two extremist parties of the winning coalition:

Proposition 6 *Let s be a pure strategy equilibrium of a game with n voters, and assume that in s at least two parties in the winning coalition take some votes.*

(a) If $\eta_A \geq \frac{n+2}{2}$, $\theta_i \leq X(s) - \frac{2}{n}(\zeta_{R_A} - \zeta_{L_A})$ implies $s_i = L_A$ and $\theta_i \geq X(s) + \frac{2}{n}(\zeta_{R_A} - \zeta_{L_A})$ implies $s_i = R_A$.

(b) If $\eta_A < \frac{n-2}{2}$, $\theta_i \leq X(s) - \frac{2}{n}(\zeta_{R_B} - \zeta_{L_B})$ implies $s_i = L_B$ and $\theta_i \geq X(s) + \frac{2}{n}(\zeta_{R_B} - \zeta_{L_B})$ implies $s_i = R_B$.

Proof. The conditions $\eta_A \geq \frac{n+2}{2}$ and $\eta_A < \frac{n-2}{2}$ imply that there is not a pivotal voter. We show the first part of (a), the other cases being symmetric. Suppose $s_i \neq L_A$. We have to analyze two cases:

(i) $s_i \in A \backslash L_A$

$$X(s) > X\left(s_{-i}, L_A\right) = X\left(s\right) - \frac{1}{\eta_A}\left(\zeta_{s_i} - \zeta_{L_A}\right) > X(s) - \frac{2}{n}(\zeta_{R_A} - \zeta_{L_A}) \geq \theta_i$$

Hence $s_i \in A \backslash L_A$ cannot be a best reply to s_{-i}

(ii) $s_i \in B$

$$X(s) > X\left(s_{-i}, L_A\right) = \frac{\eta_A}{\eta_A+1} X\left(s\right) + \frac{1}{\eta_A+1}\zeta_{L_A} = X\left(s\right) - \frac{1}{\eta_A+1}\left[X\left(s\right) - \zeta_{L_A}\right] > \theta_i$$

Hence $s_i \neq L_A$ is not a best reply.[24] ■

We stress that, in order to obtain the result, only the hypothesis on single peakedness of voters' preferences is needed and an analogue to Proposition 2 can be easily proved for each coalition. Moreover the result could be extended to mixed strategy equilibria, with the condition that, however, no player is pivotal among the winning coalition, and to the case where there are more than two coalitions.

EXAMPLE 3

(a) There are 101 voters equidistant on the $[0, 1]$ interval; i.e., one voter is in 0, one voter is in 0.01 and so on until the last one, who is in 1. There are six parties, positioned at $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$ and two coalitions, A and B. Coalition A is formed among the first three parties, and coalition B is formed among the other three parties.

It is easy to see that at least two equilibria conform to proposition 6.

The first one, which leads to the victory of coalition A, is the following. Any voter in $[0, 0.28]$ votes for the leftmost party in coalition A, i.e., for the party situated in 0; any voter in $[0.29, 1]$ votes for the rightmost party in coalition A, i.e., for the party situated in 0.4. The resulting policy outcome is $\frac{288}{1010} \simeq 0.285$. Clearly any player in $[0, 0.28]$ has no incentive to vote for any other party of coalition A or for any party of coalition B because by doing so, he would shift the policy outcome more to the right, i.e., further away from his bliss policy point. At the same time any voter in $[0.29, 1]$ has no incentive to vote for any other party of the coalition, or for any party of coalition B, because by doing so, he would shift the policy outcome toward the left, i.e., further away from his bliss policy.

[24] The condition that at least two parties in the winning coalition take some votes is necessary to have, in this case, $X(s) \neq X(s_{-i}, L_A)$.

The other equilibrium occurs when coalition B is the winning one, and any voter in $[0, 0.71]$ votes for the leftmost party in the coalition, i.e. for the party positioned in 0.6; while any voter in $[0.72, 1]$ votes for the rightmost party in the coalition, i.e., for the party positioned in 1, and the policy outcome is equal to $\left(\frac{72}{101}0.6 + \frac{29}{101}\right) \simeq 0.714$.

(b) Let us consider an other case. Given the same set of voters and parties, let coalition A be formed between parties located in 0 and in 1. Coalition B is formed among the other four parties.

An equilibrium exists equilibrium such that any voter in $[0, 0.49]$ votes for coalition A and, within the coalition, for the party situated in 0; any voter in $[0.51, 1]$ votes for coalition A and, within the coalition, for the party situated in 1. The voter situated in 0.5 votes in such a way as not to affect the outcome, i.e., he casts his vote for any party in the losing coalition. The outcome resulting from the strategy combination above is 0.5, and it is simple to verify that this is an equilibrium[25].

5.2 Two Leading Parties

Up to now, we have analyzed the extreme situation in which the parties move the outcome toward them with strength exactly proportional to the numbers of votes they take. At another extreme, we can consider a multi-party system where only the two leading parties determine the political outcome. If the outcome is a linear combination, with coefficients equal to the relative share of seats, of the position of the two leading parties, we prove that, in pure strategies, only two-party equilibria can emerge.

More formally, fix a strategy combination s. Define W^1 as the set of parties that receive more votes under the strategy combination s. If this set contains only one party, define W^2 as the set of parties that receives more votes, except W^1 (i.e., $W^1 = \{k : \nexists k' s.t.\ v_{k'} > v_k\}$, $W^2 = \{k : \exists! k' s.t.\ v_{k'} > v_k\}$). If $\#W^1 \geq 2$, let $I = \arg\min_{k \in W^1} \zeta_k$ and $II = \arg\max_{k \in W^1} \zeta_k$. If $\#W^1 = 1$, call I its element and II the leftist element[26] of W^2. In a multi-party system where the outcome is a linear combination, with coefficients equal to the relative share of seats, of the position of the two parties that take more votes[27], we have:

$$X(s) = \frac{\zeta_I v_I + \zeta_{II} v_{II}}{v_I + v_{II}} \tag{4}$$

In such a case we cannot obtain a "uniqueness" result as in section 3. Nevertheless, the following proposition implies that, in pure strategy equilibria, only two parties take a significant amount of votes. Given a pure strat-

[25] Of course, other equilibria exist. For example, any voter in $[0, 0.49]$ votes for coalition B, and within the coalition for the party positioned in 0.2, and any voter in $[0.51, 1]$ votes for coalition B, and within the coalition for the party situated in 0.8, while the player in 0.50 votes for any party in the losing coalition.

[26] Or any other predefined element of W^2, as well as if $\#W^1 \geq 3$, we can choose the two winning parties in any deterministic way. For our analysis it is necessary that every tie is deterministically broken.

[27] If the ties are broken as we have done.

egy combination s, the set $\{I, II\}$ of winning parties is deterministic. Let $l(s) = \arg\min_{k\in\{I,II\}} \zeta_k$ and $r(s) = \arg\max_{k\in\{I,II\}} \zeta_k$. In the following, for simplicity, we indicate $l(s)$ as l and $r(s)$ as r.

Proposition 7 *Let the outcome function be as in (4), and let s be a pure strategy equilibrium of a game Γ with n voters. Then:*
(α) if $\theta_i \leq X(s) - \frac{m(\zeta_R - \zeta_L)}{2n}$ then $s_i = l$
(β) if $\theta_i \geq X(s) + \frac{m(\zeta_R - \zeta_L)}{2n}$ then $s_i = r$.

Proof. (α) Suppose, by contradiction, that player i does not vote for party l. Let n^l (resp.n^r) be the number of players who vote for l (resp. r), according to the pure strategy combination s. Clearly, $n^l + n^r \geq \frac{2n}{m}$. We distinguish the case when player i votes for party r from the case when player i votes for any other party.

Suppose $s_i = r$. We have that:

$$X(s) > X(s) + \frac{1}{n^l + n^r}(\zeta_l - \zeta_r) = X(s_{-i}, l) \geq X(s) - \frac{m(\zeta_R - \zeta_L)}{2n} \geq \theta_i,$$

which contradicts s being an equilibrium.

Suppose $s_i \neq r$, hence $X(s) > \zeta_l$. We have that:

$$X(s) > (\tfrac{n^l+n^r}{n^l+n^r+1})X(s) + \tfrac{1}{n^l+n^r+1}\theta_l = X(s) - \tfrac{1}{n^l+n^r+1}(X(s) - \zeta_l) =$$
$$= X(s_{-i}, l) \geq X(s) - \tfrac{1}{n^l+n^r+1}(\zeta_R - \zeta_L) > \theta_i,$$

which again contradicts s being an equilibrium.

(β) A symmetric argument holds. ■

Again, this proposition is based only on the single peakedness assumption on voters' preferences and an analogous of Proposition 2 can be easily proved for each couple of parties. Also in this case the proof could be extended to mixed strategy equilibria, with the condition that, however, no player is pivotal among the set of winning parties. Furthermore, we could get a two-party result even if the outcome function is a linear combination, with coefficients equal to the relative share of seats, of the positions of the m' first ranked parties ($2 \leq m' \leq m$). Unfortunately, we cannot analyze a mixed strategy equilibrium without the no-pivotal assumption, because a different behavior by a single player could imply a dramatically different outcome. Hence our proof cannot be extended for lack of continuity.

EXAMPLE 4

Let's take exactly the same set of parties and players as in example 3. It is easy to see that the same argument developed for example 3 shows us that the strategy combination where any voter in $[0, 0.28]$ votes for the party situated in 0 while any voter in $[0.29, 1]$ votes for the party situated in 0.4 is an equilibrium, as is the strategy combination where any voter in $[0, 0.71]$ votes for the party positioned in 0.6 while any voter in $[0.72, 1]$ votes for the party positioned in 1. Moreover, the analogous argument of the case (b) in example 3 clarifies the equilibrium represented by the strategy combination where any voter in $[0, 0.49]$ votes for the party situated in 0, and any voter in $[0.51, 1]$ votes for the party situated in 1, while the player situated in 0.5 votes for any loser party in order not to affect the outcome.[28]

6 Conclusion

This paper is a first step in understanding the effect of strategic voting in proportional rule elections. The insight is quite "obvious": under proportional representation strategic voters have an incentive to vote for the extremist parties in order to drag the policy outcome toward their ideal point. The main consequence is that *Duverger's hypothesis*, that proportional representation favors multi-partyism, may be incorrect under strategic voting. If the policy is a weighted average of parties' platforms, with weights equal to the share of votes, or if the policy is the weighted average of the platforms of the members of the winning preelectoral coalition, or if the policy is a weighted average of the top two vote-getters, only two parties get votes.

Future work will take into account many possible extensions that we only briefly cite here.

Strategic parties. Readers familiar with spatial models of elections where parties are the strategic players of the game may feel uncomfortable with their absence in this paper. For this reason, introducing parties as active players in this game is the first extension we will undertake.

Incomplete information. We assumed throughout the model that voters possess complete information. In this respect, a natural extension is to consider changes in this model when we add uncertainty.

Multidimensional policy space. Another assumption that we think would be interesting to relax is the unidimensionality of the policy space.

Legislative bargaining. Finally, a very interesting research project is to understand how the policy outcome could be interpreted as a "reduced form" of legislative bargaining.

It is clear that this paper represents a first step toward better understanding the effect of strategic voting in proportional representation elections, and we hope that it will spur interest in this research agenda.

7 Appendix

Proof of proposition 3:

(α) Given a mixed strategy σ_j, the player j's vote is a random vector[29] $\tilde{s}_j$ with $Pr(\tilde{s}_j = k) = \sigma_j^k$. Given $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$, let $\bar{\tilde{s}}^{-i} =$

[28] Of course other equilibria exist, for example, one for every possible pairs of parties.

[29] We remind readers that a vote is a vector with m components.

$\frac{1}{n-1}\sum_{j\in N/i}\tilde{s}_j$ and $\bar{\mu}^{\sigma_{-i}} = \frac{1}{n-1}\sum_{j\in N/i}\sigma_j$. The first step of the proof consists in proving the following lemma:

Lemma 8 *$\forall\phi > 0$ and $\forall\delta > 0$, if $n > \frac{m}{4\phi^2\delta} + 1$, then $\forall\sigma, \forall i$*

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

Proof. To prove the lemma we can use Chebychev's inequality component by component. Given σ_{-i}, it is easy to verify that $E(\tilde{s}_j^k) = \sigma_j^k$ and $Var(\tilde{s}_j^k) = \sigma_j^k(1-\sigma_j^k) \leq \frac{1}{4}$, hence $E(\bar{\tilde{s}}_k^{-i}) = \bar{\mu}_k^{\sigma_{-i}}$ and $Var(\bar{\tilde{s}}_k^{-i}) \leq \frac{1}{4(n-1)}$. By Chebychev's inequality we know that $\forall k, \forall\phi$:

$$\Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \leq \frac{1}{4(n-1)\phi^2}.$$

Hence

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) \geq 1 - \sum_k \Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \geq 1 - \frac{m}{4(n-1)\phi^2},$$

which is strictly greater than $1-\delta$ for $n > \frac{m}{4\phi^2\delta} + 1$. ■

Now we show that $\forall\varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$, if $\theta_i \leq X(\bar{\mu}^\sigma) - \varepsilon$, then L is the only best reply for player i to σ^{-i}.

Fix $\varepsilon > 0$. Define $\forall\theta \in \left[0, 1-\frac{\varepsilon}{2}\right]$

$$M_\varepsilon(\theta) = \max_{X\in\left[\theta+\frac{\varepsilon}{2},1\right]} \frac{\partial u(X,\theta)}{\partial X}.$$

By single-peakedness we know that $M_\varepsilon(\theta) < 0$. Moreover, given the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ we can apply the theorem of the maximum[30] to deduce that the function $M_\varepsilon(\theta)$ is continuous, hence it has a maximum on $\left[0, 1-\frac{\varepsilon}{2}\right]$, which is strictly negative. Let

$$M_\varepsilon^* = \max_{\theta\in\left[0,1-\frac{\varepsilon}{2}\right]} M_\varepsilon(\theta).$$

Let M denote the upper bound[31] of $\left|\frac{\partial u(X,\theta)}{\partial X}\right|$ on $[0,1]^2$, and let $\delta_\varepsilon^* = \frac{-M_\varepsilon^*}{M-M_\varepsilon^*} > 0$ and $\phi^* = \frac{(-2+\sqrt{6})\varepsilon}{m}$. We prove that if $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, then every strategy

[30] Because there are various versions of the theorem of the maximum, we prefer to state explicitly the version we are using (cf. Th.3.6 in Stokey and Lucas, 1989). Let $f : \Psi \times \Phi \rightarrow \Re$ be a continuous function and $g : \Phi \rightarrow P(\Psi)$ be a compact-valued, continuous correspondence, then $f^*(\phi) := \max\{f(\psi,\phi) \mid \psi \in g(\phi)\}$ is continuous on Φ.

[31] The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ assures that such a bound exists.

other than L cannot be a best reply for player i, which, setting n_0 equal to the smallest integer strictly greater than $\frac{m}{4\phi^{*2}\delta^*_\varepsilon}+1$, directly implies the claim.

Take a party $c \neq L$. By definition $c \in BR_i(\sigma) \Longrightarrow$

$$\sum_{s_{-i}\in S_{-i}} \sigma(s_{-i})\left[u\left(X\left(s_{-i},c\right),\theta_i\right)-u\left(X\left(s_{-i},L\right),\theta_i\right)\right]\geq 0, \tag{5}$$

which can be written as:

$$\sum_{s_{-i}\in S_{-i}} \sigma(s_{-i})\left[u\left(X\left(s_{-i},c\right),\theta_i\right)-u\left(X\left(s_{-i},c\right)-\frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]\geq 0. \tag{6}$$

Because the outcome function $X(s)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters, (6) can be written as:

$$\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i}=v_n^{-i})\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right)-u\left(X\left(v_n^{-i},c\right)-\frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]\geq 0 \tag{7}$$

where, with abuse of notation, $X\left(v_n^{-i},c\right)=\frac{\zeta_c}{n}+\frac{n-1}{n}\sum_{k=1}^{m}\zeta_k v_{n(k)}^{-i}$. Multiplying both sides of (7) by $\frac{n}{\zeta_c-\zeta_L}>0$ we have:

$$\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i}=v_n^{-i})\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right)-u\left(X\left(v_n^{-i},c\right)-\frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]}{\frac{1}{n}(\zeta_c-\zeta_L)}\geq 0. \tag{8}$$

By the *mean value theorem* we know that $\forall v_n^{-i}$,
$\exists X^*\in\left[X\left(v_n^{-i},c\right)-\frac{1}{n}(\zeta_c-\zeta_L),X\left(v_n^{-i},c\right)\right]$ such that

$$\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right)-u\left(X\left(v_n^{-i},c\right)-\frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]}{\frac{1}{n}(\zeta_c-\zeta_L)}=\left.\frac{\partial u(X,\theta_i)}{\partial X}\right|_{X=X^*}.$$

Hence we have:

$$\sum_{v_n^{-i}\in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i}=v_n^{-i})\frac{\left[u\left(X\left(v_n^{-i},c\right),\theta_i\right)-u\left(X\left(v_n^{-i},c\right)-\frac{1}{n}(\zeta_c-\zeta_L),\theta_i\right)\right]}{\frac{1}{n}(\zeta_c-\zeta_L)}\leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i}-\bar{\mu}^{\sigma_{-i}}\right|\leq\vec{\phi}^*)M_n^*(\vec{\phi}^*,\theta_i)+(1-\Pr(\left|\bar{\tilde{s}}^{-i}-\bar{\mu}^{\sigma_{-i}}\right|\leq\vec{\phi}^*))M$$

where

$$M_n^*(\vec{\phi}^*, \theta_i) = \max_{X \in \left[X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L), 1\right]} \frac{\partial u(X, \theta_i)}{\partial X}.$$

Now we prove that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*}+1$, $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon^*$. From the definition of M_ε^*, it suffices to prove that $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon(\theta_i)$, which is true if $X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L)$ is greater than $\theta_i + \frac{\varepsilon}{2}$.

$$X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L) = \frac{n-1}{n}\sum_k \bar{\mu}_k^{\sigma_{-i}}\zeta_k - \frac{n-1}{n}\sum_k \phi^*\zeta_k + \frac{1}{n}\zeta_L =$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n}\sum_k \sigma_i^k \zeta_k + \frac{1}{n}\zeta_L - \frac{n-1}{n}\sum_k \phi^*\zeta_k >$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n}(\zeta_R - \zeta_L) - m\phi^*\zeta_R \geq \theta_i + \varepsilon - \frac{1}{n} - m\phi^*.$$

Hence this step of the proof is concluded by noticing that δ_ε^* is by definition less than $\frac{1}{2}$, hence[32]

$$\theta_i + \varepsilon - \frac{1}{n} - m\phi^* > \theta_i + \varepsilon - \frac{2\phi^{*2}}{m} - m\phi^* =$$

$$\theta_i + \varepsilon - \frac{(20 - 8\sqrt{6})\varepsilon^2}{m^3} - \varepsilon\left(-2 + \sqrt{6}\right) \geq \theta_i + \varepsilon(1 - \frac{(20-8\sqrt{6})}{8} + 2 - \sqrt{6}) =$$

$$\theta_i + \frac{1}{2}\varepsilon.$$

By Lemma 8, we know that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$,

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))M <$$

$$(1 - \delta_\varepsilon^*)M_\varepsilon^* + \delta_\varepsilon^* M = (1 - \frac{-M_\varepsilon^*}{M - M_\varepsilon^*})M_\varepsilon^* + \frac{-M_\varepsilon^*}{M - M_\varepsilon^*}M = 0.$$

[32]In the following we assume that $\varepsilon \leq 1$, since otherwise the proposition is trivially true.

Summarizing, we have proved that for $n > \frac{m}{4\phi^{*2}\delta^*_\varepsilon} + 1$, for every strategy $c \neq L$

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*)) M <$$

$$(1 - \delta^*_\varepsilon) M^*_\varepsilon + \delta^*_\varepsilon M = 0,$$

which implies that c is not a best reply for player i.

(β) A symmetric argument holds.

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