Repeated Downsian Electoral Competition

John Duggan
Department of Political Science
and Department of Economics
University of Rochester

Mark Fey
Department of Political Science
University of Rochester

August 22, 2002

Abstract

We analyze an infinitely repeated version of the Downsian model of elections. The folk theorem suggests that a wide range of policy paths can be supported by subgame perfect equilibria when parties and voters are sufficiently patient. We go beyond this result by giving separate weak conditions on the patience of voters and the patience of parties under which every policy path can be supported. We also show that this indeterminacy holds, regardless of the patience of parties and voters, when the policy space and the voters' utilities are unbounded. On the other hand, we show that when parties and voters are sufficiently impatient, all supportable policy paths lie in the core, bringing us back to the median voter result in one dimension and generic non-existence of equilibria in multiple dimensions. Two distinctive features of these results are that they apply over a wide range of discount factors and they employ a refinement of subgame perfect equilibrium that restricts the types of punishments that can be used.

1 Introduction

The Median Voter Theorem of Black (1958) establishes that if voters have single-peaked preferences over a one-dimensional set of alternatives, then the median of the distribution of voter ideal points is majority-preferred to all other alternatives. In multidimensional policy spaces, however, such points, called "core points," typically do not exist (Plott, 1967; Rubinstein, 1979; Schofield, 1983; Cox, 1984; Le Breton, 1987; Banks, 1995). The standard game-theoretic model of two-party spatial competition, originally examined by Downs (1957), assumes that parties are office-motivated, they can commit to campaign platforms, and that voters eliminate weakly dominated strategies. In this setting, it is easy to show that a choice of policy by the parties is a (pure strategy) Nash equilibrium if and only if both parties locate at a core point. Thus, the median is the unique equilibrium in the one-dimensional model, but electoral equilibria typically fail to exist in multiple dimensions. These conclusions rely, however, on the often unnoticed assumption, implicit in the standard model, that parties and voters care only about the current election.

In this paper, we investigate the consequences of repeating this Downsian game and allowing parties and voters to anticipate the effects of their actions on future elections. Drawing on the theory of infinitely repeated games, it is a folk theorem that if players have sufficiently high discount factors, the set of subgame perfect equilibrium outcomes of a repeated game can be sizeable (Fudenberg and Maskin, 1986). Motivated by this result, we consider the following questions concerning infinitely repeated two-party competition. Is the median the unique equilibrium in one dimension? Do equilibria exist in multiple dimensions when the core is empty? How large does the set of equilibrium outcomes become? How do the equilibrium outcomes depend on the patience of the parties and voters?

Our first set of results show that under modest conditions, not only do electoral equilibria exist (in a single or in multiple dimensions), but *every* possible sequence of policies is supportable by a subgame perfect equilibrium. We find three separate sets of conditions under which this "indeterminacy" conclusion holds. The first requires only that *voters* place more weight on the future than the present (discount factors greater than one half) and imposes no restriction on the parties. The second assumes that the core is non-empty, or that it is "close" to non-empty, and requires only that *parties* place more weight on the future than present. The third examines the case in which the policy space and the voters' utility functions are unbounded and imposes *no* conditions on patience.[1] These results go beyond the standard folk theorem in two ways. First, they do not rely on arbitrary patience, but rather impose either modest requirements on discount factors or none at all. Second, we use a refinement of subgame perfect equilibrium that restricts the kinds of punishments available to parties and voters. In particular, we exclude equilibria in which voters or parties condition on how particular voters voted in the past, or even on total vote tallies in previous elections. Furthermore, we suppose each voter acts as though pivotal in every election, essentially eliminating weakly dominated strategies in every period in the spirit of the one-shot Downsian model. Finally, we restrict ourselves to equilibria in which any voter, when indifferent concerning which party wins, flips a fair coin to decide his/her vote, treating the parties symmetrically.

With these results in hand, we next investigate the conditions under which parties must choose core points (the median, in one dimension) in equilibrium, a phenomenon we refer to as "core equivalence." We first note that if voters and parties use stationary (that is, history-independent) strategies, then we obtain core equivalence. Therefore, we are back to the median voter theorem in one dimension and generic non-existence of equilibria in multiple dimensions. If we allow for history-dependent equilibria, then we know from our above results that the voters and parties must be sufficiently impatient and the policy space must be bounded. In this setting, we show that, with an odd number of voters and quadratic utilities, if voter and party patience are below a relatively moderate level (discount factors below one half for voters and one third for parties), we again obtain core equivalence. Finally, our last result considers the case at the opposite extreme of the folk theorem, namely, voters with discount factors sufficiently close to zero. In this case, we show that if voters are sufficiently impatient (and parties have discount factors below one third), then policy paths supportable by subgame perfect

[1] To be precise, the second and third results also require that voters place nonzero weight on the future.

equilibrium must be arbitrarily close to the core. As a corollary, if the core is empty and voters are sufficiently impatient, then there exist *no* subgame perfect equilibria of the repeated electoral game.

Repeated elections have been considered in the literature on "electoral accountability," which drops the commitment assumption and modifies other details of the Downsian model, such as adding asymmetric information of one form or another.[2] While many of these models assume a single, or "representative," voter, Duggan (2000) and Bernhardt, Hughson, and Dubey (2002) explicitly allow for a continuum of voters, and the former paper contains simulation results suggesting core equivalence as voters become arbitrarily patient. Aragones and Postlewaite (2000) consider a related model but assume complete information. While these papers assume a one-dimensional policy space, Banks and Duggan (2002) prove existence of equilibria in multiple dimensions and give analytic results on core equivalence. Work on electoral accountability differs from ours not only in removing the commitment assumption, but also in focusing on stationary equilibria.[3]

Kramer (1977) takes a different approach by assuming office-motivated parties that can commit to policy platforms before elections and allowing for multiple dimensions. His model differs from ours, however, in that only the party out of power may choose a platform, while the incumbent party is fixed at its previous position, and in that the parties optimize myopically. Alesina (1988) takes yet another approach by assuming policy-motivated parties that cannot commit to policy platforms, by assuming probabilistic voting, and by considering a specific class of non-stationary equilibria, namely, those using "Nash reversion" punishments. Finally, McKelvey and Ordeshook (1985) and Shotts (2000) are examples of models focusing on the informational aspects of repeated elections with private information.

The organization of the paper is as follows. In Section 2, we lay out the repeated electoral model. The third section describes the equilibrium refinements we impose. In Section 4, we present our results on indeterminacy and in Section 5, we give results on core equivalence. In Section 6, we end

[2]For an expository review of this literature, see Fearon (1999).

[3]As we show, in the repeated Downsian electoral model, stationarity implies core equivalence and, therefore, equilibrium existence problems in multiple dimensions.

with some brief concluding remarks. An appendix contains proofs of all of our propositions, including diagrams of equilibrium constructions used in the proofs of Propositions 2, 3, and 5.

2 The Model

The players in our model are two parties, labeled A and B, and n voters, who participate in an infinite sequence of elections. In each election, the parties simultaneously choose policy platforms from some set X of policy alternatives, generically denoted x, x', etc.[4] We use y to denote a platform choice by party A and z to denote a choice by B. In each period, once the parties have selected platforms, the voters observe these choices and simultaneously cast ballots for A or B. In every period, the election is determined by plurality rule, with the party receiving the most votes implementing its platform for that period. In the event of a tie, parties A and B each win with probability one half, i.e., the winner is decided by the toss of a fair coin in case of a tie. We denote a generic policy in period t by x_t and we let $\mathbf{x} = (x_1, x_2, \ldots)$ denote an infinite path of policies.

A *history* of length t, denoted h_t, in this game is a list of the actions of all players in periods $1, 2, \ldots, t$, i.e., it must list the platforms of the parties, the votes of the voters in each period, and, in case of electoral ties, the outcomes of coin flips to break ties. We define the *initial history,* denoted $h_0 = \emptyset$, as the "empty" list that describes the game at the beginning of period 1. A *finite history* is a history of finite length, whereas an *infinite history* is a list of platforms and votes for every period. We denote the set of all histories of length t by H_t and the set of all finite histories by $H = \bigcup_t H_t$. A *strategy of a party* $P \in \{A, B\}$ is a mapping $\rho_P : H \rightarrow X$, indicating the platform the party will adopt after different histories. A *strategy of a voter* i is a mapping $\sigma_i : H \times X \times X \rightarrow [0, 1]$ which gives, for each finite history and platform pair (y, z) (the platforms of the parties in the current period), the probability of a vote for party A. Letting $\rho = (\rho_A, \rho_B)$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$, a profile of

[4] At this point, we impose no structure on the set X, allowing it to be finite or infinite, perhaps a subset of the real line or a subset of multidimensional Euclidean space.

strategies is (ρ, σ). An *electoral outcome* in period t is (i) the platforms, y_t and z_t, chosen by the parties, and (ii) the winner of the election in t (possibly determined by a tie-breaking coin flip), denoted $w_t \in \{0, 1\}$, where $w_t = 1$ indicates that A won and $w_t = 0$ indicates that B won. Thus, the actual policy implemented in period t can be expressed as $w_t y_t + (1 - w_t) z_t$. Given a finite history h_t, let

$$o(h_t) \quad = \quad ((w_1, y_1, z_1), (w_2, y_2, z_2), \ldots, (w_t, y_t, z_t))$$

denote the sequence of electoral outcomes associated with h_t. A strategy profile (ρ, σ) determines a distribution on infinite sequences of electoral outcomes, where randomness may be introduced by mixed voting strategies and tied elections. This distribution, in turn, determines a distribution on infinite paths of implemented policies.

We assume each voter i has a utility function $u_i \colon X \to \mathbb{R}$ that reflects the voter's preferences over policies in any period. Write $x \,\mathrm{M}\, y$ if x is *plurality-preferred* to y, i.e., if the number of voters with $u_i(x) > u_i(y)$ is greater than the number of voters with $u_i(y) > u_i(x)$. Define the *core,* denoted K, as the set of plurality undominated policies, i.e.,

$$K \quad = \quad \{x \in X \mid y \,\mathrm{M}\, x \text{ for no } y \in X\}.$$

When voter preferences are single-peaked, it is well-known that K consists of the *median* policies, i.e., $x \in K$ if and only if the number of voters with ideal points below x is less than or equal to $n/2$ and the number with ideal points above is also less than or equal to $n/2$. Write $x \,\mathrm{M}^*\, y$ if x is *majority-preferred* to y, i.e., if the number of voters with $u_i(x) > u_i(y)$ is greater than $n/2$. Define the *strong core,* denoted K^*, as the set of majority dominant policies, i.e,

$$K^* \quad = \quad \{x \in X \mid x \,\mathrm{M}^*\, y \text{ for all } y \in X \setminus \{x\}\}.$$

Clearly, $K^* \subseteq K$ and K^* contains at most one element. When n is odd and voter preferences are "linear" ($u_i(x) = u_i(x')$ implies $x = x'$), it is known that $K = K^*$. As well, if n is odd, if X is a convex subset of Euclidean space, and if voter preferences are strictly quasi-concave (as in the standard spatial model), then $K = K^*$.

Voters in our model are fully rational in that they consider the effect of their current vote on future elections in deciding how to vote. We assume that, in order to evaluate these effects, voter preferences over infinite histories are represented by the discounted sum of utilities from policies over time, i.e.,

$$(1-\delta_i)\sum_{t=1}^{\infty}\delta_i^{t-1}[w_t u_i(y_t)+(1-w_t)u_i(z_t)],$$

where $\delta_i \in [0,1)$ is the *discount factor* for voter i. Preferences over lotteries on outcome paths are given by the expected discounted sum of utilities. Party P receives a payoff of one when it wins, zero otherwise. Thus, we assume the parties are probability of winning maximizers. Each has a discount rate $\delta_P \in [0,1)$, and we assume A's preferences over infinite histories are given by

$$(1-\delta_A)\sum_{t=1}^{\infty}\delta_A^{t-1}w_t$$

and B's preferences by

$$(1-\delta_B)\sum_{t=1}^{\infty}\delta_B^{t-1}(1-w_t).$$

Again, preferences over lotteries on outcome paths are given by expected discounted payoffs.

A specification of strategies for parties and voters is a *subgame perfect equilibrium* if it satisfies the following: no party or voter has a different strategy that, following some finite history, yields a distribution over outcome paths with a higher expected discounted sum of utilities. We say a subgame perfect equilibrium *supports* a policy path $\mathbf{x}$ if the distribution on infinite policy paths determined by the equilibrium strategies puts probability one on $\mathbf{x}$. Note that this can happen in one of two ways: either both candidates adopt platform x_t or only party P adopts x_t and wins with probability one. If there is an equilibrium that supports $\mathbf{x}$, then we say the path is *supportable.*

3 Equilibrium Refinements

In this section, we present some restrictions on strategies in order to rule out especially implausible equilibria of the game. It is well-known that, in infinitely repeated games with sufficiently patient players, a large set of outcomes can typically be supported by subgame perfect equilibria (Fudenberg and Maskin (1986)). The standard folk theorem, however, places no limitations on the types of punishments that can be used by the equilibrium supporting a particular outcome. In the context of repeated elections, however, we want to exclude equilibria that are less compelling on the grounds of realism, such as those in which one voter is singled out for voting the wrong way and punished in the future by parties and other voters. Because of this, we focus on equilibria in which such punishments are not used. That is, we focus on equilibria in which the choices of voters and parties in any period are conditioned only on previous electoral outcomes. We refer to this restriction as “outcome stationarity.”

Definition 1 (OS) *A strategy profile* (ρ, σ) *satisfies* outcome stationarity *if for all* t *and any two histories* $h_t \in H_t$ *and* $h'_t \in H_t$ *such that* $o(h_t) = o(h'_t)$*,*

1. *for each party* P*,* $\rho_P(h_t) = \rho_P(h'_t)$*, and*
2. *for all* (y, z) *and all* i*,* $\sigma_i(h_t, y, z) = \sigma_i(h'_t, y, z)$*.*

In other words, outcome stationarity requires that after any two histories with identical sequences of outcomes, the specified platform choice of each party is the same and the choices of the voters can only be conditioned on the party's choices of platforms in the current period. Thus, following a finite history h_t and platform choices of the parties, y and z, each voter i can calculate the expected discounted sum of utilities if A is elected and if B is elected in the current period, given the strategies of the other players. Denote these continuation values by $v_i(h_t, y, z, 1)$ and $v_i(h_t, y, z, 0)$, respectively. This formulation will be very useful in what follows.

Even restricting the available strategies to those that satisfy outcome stationarity, it is possible to establish a folk theorem-like result. In fact, the result is much stronger than the standard folk theorem because it holds for any $\delta_i \geq 0$, not just a particular range of values.

Proposition 1 *If $n \geq 3$, every policy path is supportable by a subgame perfect equilibrium satisfying (OS).*

The proof of this proposition is straightforward. Let $\mathbf{x}$ be any given policy path. Both parties choose a strategy to play x_t in period t, regardless of h_t. For the voters, as long as both parties choose the prescribed platforms in each period (or both deviate), the voters randomize their vote. If a party deviates from x_t, then in that and all later periods, all voters vote for the non-deviating party. Clearly, no player can gain by deviating. Subgame perfection holds because (as $n \geq 3$) it is a Nash equilibrium in the voting subgame for all voters to vote for A, regardless of their preferences over candidate platforms; similarly, it is Nash for all to vote for B. It is clear that this a subgame perfect equilibrium satisfying outcome stationarity, but it requires some voters to vote against their preferred party. The standard response to this in a one-shot model is to impose "sincere" voting, which is equivalent to elimination of weakly dominated strategies.

In our repeated setting, imposing sincere voting in each stage game is not satisfactory, as the voters are fully rational and anticipate the effect of their votes on the choices of the parties in later periods. Rather, we consider only subgame perfect equilibria in which each voter, while taking the strategies of all players in the future as fixed, essentially eliminates weakly dominated strategies in the voting subgame. In other words, when the continuation value to player i of having A elected in the current period is strictly higher than the continuation value of electing B, voter i votes for A, and similarly for B. This is equivalent to requiring that all voters act as if they were pivotal in the current period. Following Baron and Kalai (1993), we refer to this restriction as "stage game weak dominance."

Definition 2 (WD) *A strategy profile (ρ, σ) satisfies* stage game weak dominance *if for every t, every history $h_t \in H_t$, and every platform choices y and z,*

1. $(1-\delta_i)u_i(y) + \delta_i v_i(h_t, y, z, 1) > (1-\delta_i)u_i(z) + \delta_i v_i(h_t, y, z, 0)$ *implies* $\sigma_i(h_t, y, z) = 1$, *and*

2. *the reverse inequality implies* $\sigma_i(h_t, y, z) = 0$.

Stage game weak dominance requires voters with strict preference to act accordingly, but it does not restrict the actions of indifferent voters. While it true that the choice of such a voter is irrelevant to that voter, it can dramatically affect the choices of the parties. In fact, in one dimension, we can modify the argument for Proposition 1 to again support every policy path. If either party, say B, deviates from the given policy in period t to any other policy, then, in all future periods, both parties locate at the median, where the voters always vote for A. Such "Nash reversion" equilibria satisfy stage game weak dominance, but they depend critically on the possibility that the parties are treated asymmetrically, even when they adopt identical platforms (and are expected to do so after every history). We therefore impose a last restriction, augmenting stage game weak dominance, which we call "party symmetry."

Definition 3 (PS) *A strategy profile* (ρ, σ) *satisfies* party symmetry *if for every* t*, every history of length* t*, and every platform choices* y *and* z*,* $(1-\delta_i)u_i(y)+\delta_i v_i(h_t, y, z, 1) = (1-\delta_i)u_i(z)+\delta_i v_i(h_t, y, z, 0)$ *implies* $\sigma_i(h_t, y, z) = 1/2$.

That is, when indifferent between the two parties, each voter flips a fair coin to decide. Note that, as a consequence, if the inequality in Definition 2 holds for a plurality of voters after some history and platform pairs, then party A wins with probability greater than one half.

4 Indeterminacy

We find the result that "anything can happen" under three sets of conditions, each using a different logic to support policy paths. Our first result assumes only that voters are somewhat patient, placing more weight on the future than on the present.

Proposition 2 *Let $\delta_i > 1/2$ for all voters. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

The equilibrium constructed in the appendix can be described roughly as follows. We specify that both parties choose platform x_t in period t, unless some party has deviated. In any period where one party, say A, has deviated to $y \neq x_t$, future policy platforms depend on several factors. If $x_t \,\mathrm{M}\, y$, then the deviation is ignored and the parties "return to the equilibrium path." If $y \,\mathrm{M}\, x_t$, then future policy platforms depend on which party wins: if the deviating party, A, wins, then the parties adopt x_t in all future periods; if party B wins, then the parties move to the deviant platform, y, in all future periods. Along the equilibrium path, voters are indifferent between the parties and so flip coins to decide their ballots, giving the parties expected discounted equilibrium payoffs of one half. If A deviates to y, a voter votes for A if $u_i(x_t) > u_i(y)$, votes for B if the opposite inequality holds, and flips a coin if equality holds. Given the strategies of the parties, and given that $\delta_i > 1/2$, these voting strategies are best responses satisfying (WD) and (PS). Given the strategies of the voters, no party has an incentive to deviate: if A deviates to $y \,\mathrm{M}\, x_t$, for example, then a plurality of voters will vote for B, so A wins with probability less than one half in period t (and will win with probability one half in the future), giving it an expected discounted payoff less than one half.

Whereas the preceding result restricts the patience of voters and allows for arbitrary party discount factors, the next relies on somewhat patient parties and assumes only that voters put some positive weight on the future. Interestingly, the construction used to support arbitrary policy paths uses the existence of a strong core point: equilibrium uniqueness in one-shot Downsian elections can lead to indeterminacy in infinitely repeated elections.

Proposition 3 *Let $\delta_i > 0$ for all voters, let $\delta_P \geq 1/2$ for both parties, and let $K^* \neq \emptyset$. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

The equilibrium constructed for Proposition 3 uses some of the same ideas as does Proposition 2, but it is somewhat more complex, so we leave the reader to the formal description in the appendix. One difference between the constructions is of note. To prove Proposition 2, we specified party strategies so that the parties always choose the same platform and, unless one party deviated, voters treat the parties symmetrically. In proving Proposition 3, we also specify that parties always choose the same platform, but in some subgames one party wins with probability one and one wins with probability zero. This is consistent with voter incentives because the parties' current platforms are identical and, by design, the expectations about the future are worse for a majority party if the latter party wins. Note that without the difference in future policies, and without positive discount factors for the voters, our party symmetry condition would make this impossible. A difficulty that arises is the potential for the "losing" party to deviate profitably. Because voters may be arbitrarily close to myopic, we cannot use future policies to induce voters to punish deviations, so we are forced to assume a strong core point.

Next, we show that, if typical regularity conditions are imposed, then Proposition 3 is actually robust to the assumption that the strong core is nonempty. Indeed, if voter discount factors are positive and voter preferences are sufficiently close to admitting a strong core, then every policy path is supportable. In the following, we say the sequence $\{u_i^m\}$ of utility functions converges *uniformly* to u_i if

$$\sup_{x \in X} |u_i^m(x) - u_i(x)| \quad \to \quad 0.$$

We say the sequence $\{(u_1^m, \ldots, u_n^m)\}$ of vectors of utility functions converges uniformly to $(u_1, \ldots, u_n)$ if $u_i^m \to u_i$ uniformly for every voter. We write $K^*(u_1, \ldots, u_n)$ for the strong core at the vector $(u_1, \ldots, u_n)$.

Proposition 4 *Assume $X \subseteq \mathbb{R}^d$ is compact and convex. Let $\delta_i = \delta > 0$ for all voters, let $\delta_P \geq 1/2$ for both parties, let $K^*(u_1, \ldots, u_n) \neq \emptyset$, and let $(u_1^m, \ldots, u_n^m) \to (u_1, \ldots, u_n)$ uniformly. Assume u_i^m is concave and u_i is continuous for all voters. Then, for high enough m, every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

Our last result in this section allows for arbitrary discount factors of parties and any positive discount factors for voters, and instead uses a condition on voter utilities. We say utility functions are *majority unbounded below* if there exists a majority $C \subseteq N$ such that, for all $c \in \mathbb{R}$, there exists $x \in X$ satisfying $u_i(x) \leq c$ for all $i \in C$, i.e.,

$$\inf_{x \in X} \max_{i \in C} u_i(x) \quad = \quad -\infty.$$

This condition is satisfied if, for example, $X = \mathbb{R}^d$, each u_i is concave, and some policy is Pareto dominated. To see this, take any $x^0 \in X$ outside the Pareto set, so there exists $x' \in X$ with $u_i(x') > u_i(x^0)$ for every voter. Letting $x^1 = 2x^0 - x'$, we have $x^0 = (1/2)x' + (1/2)x^1$, and, by concavity,

$$u_i(x^0) \quad \geq \quad (1/2)u_i(x') + (1/2)u_i(x^1)$$

for every voter. Equivalently,

$$u_i(x^1) \quad \leq \quad u_i(x') - 2(u_i(x') - u_i(x^0)).$$

Defining $x^k = (k+1)x^0 - kx'$ for each integer $k \geq 2$, we can deduce

$$u_i(x^k) \quad \leq \quad u_i(x') - k(u_i(x') - u_i(x^0)).$$

Since $u_i(x') - u_i(x^0) > 0$ for all i, we can make each $u_i(x^k)$ arbitrarily low by picking k sufficiently high, fulfilling the definition of majority unboundedness below. Since these assumptions on voter preferences are rather weak, we see that unboundedness of the policy space is essentially sufficient for the condition. In fact, as long as X is a closed subset of Euclidean space and each u_i is continuous, it is necessary as well.

Proposition 5 *Assume that voter utilities are majority unbounded below and that $\delta_i > 0$ for all voters. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

The equilibrium used in the proof is similar to that used to prove Proposition 2. Now, however, if A deviates to $y \, \mathrm{M} \, x_t$ and wins, then we specify that both parties move to a platform that is significantly worse than x_t for a majority of voters, a construction that is made possible by majority unboundedness below. This gives a majority of voters an incentive to vote against A and eliminates any incentive for the parties to deviate. Note that majority unboundedness below, and therefore the logic of Proposition 5, can hold only when X is infinite.

5 Core Equivalence

In this section, we first consider policy paths supportable by subgame perfect equilibria satisfying a stationarity condition stronger than outcome stationarity. We say a specification of strategies satisfies "strong stationarity" if parties use history-independent strategies and voters condition only on the platforms of the parties in the current period.

Definition 4 (SS) *A strategy profile (ρ, σ) satisfies* strong stationarity *if for all t and any two histories $h_t \in H_t$ and $h'_t \in H_t$,*

1. *for each party P, $\rho_P(h_t) = \rho_P(h'_t)$, and*
2. *for all (y, z) and all i, $\sigma_i(h_t, y, z) = \sigma_i(h'_t, y, z)$.*

In a subgame perfect equilibrium satisfying strong stationarity, it is clear that $v_i(h_t, y, z, P)$ is independent of h_t, y, z, and P: regardless of their values, the parties will each choose some y and z in period $t+1$, and in every period thereafter, and any voter who conditions only on those platforms will vote the same way in every period. Similarly, the parties' expected discounted payoffs are constant across all histories.

The next proposition shows that strengthening outcome stationarity to strong stationarity brings us back to the Downsian core equivalence result. Note that the "only if" direction in the following proposition does not rely on party symmetry because strong stationarity and weak dominance are enough to imply that all voters must vote sincerely in every period.

Proposition 6 *A policy path* $\mathbf{x}$ *is supportable by a subgame perfect equilibrium satisfying (SS), (WD), and (PS) if and only if* $x_t \in K$ *for all* t*.*

An implication is that every policy path through the core can be supported by a subgame perfect equilibrium, in fact, by a strongly stationary one. In the remainder of this section, we drop the restriction of strong stationarity and give sufficient conditions for the opposite inclusion, namely, that all supportable policy paths must lie in the core. From Propositions 2-5, we know that we must assume the policy space is bounded and that we must impose restrictions on voter and party patience.

Our next result imposes rather weak restrictions on discount factors and adds the following strong, but standard, conditions: the number of voters is odd, the policy space is Euclidean, voter utilities are quadratic, i.e., $u_i(x) = -||x - \tilde{x}_i||^2$ for each i, where $\tilde{x}_i$ is i's "ideal point," and the ideal points of the voters are distinct, i.e., $\tilde{x}_i = \tilde{x}_j$ implies $i = j$. Furthermore, the proposition requires the non-emptiness of the core, an assumption that is automatically satisfied in one dimension but quite restrictive in multiple dimensions.

Proposition 7 *Assume that* n *is odd, that* $X \subseteq \mathbb{R}^d$ *is bounded, that the utility functions* u_i *are quadratic with distinct ideal points, and that* $K \neq \emptyset$*. Let* $\delta_i = \delta < 1/2$ *for all voters, and let* $\delta_P < 1/3$ *for the parties. Then* $\mathbf{x}$ *is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS) if and only if* $x_t \in K^* = K$ *for all* t*.*

In Proposition 7, we assume voters are minimally impatient, in the sense that they place more weight on the present than the future, and we impose a somewhat stronger restriction on party discount factors. In fact, we can prove core equivalence in the environment of the proposition by interchanging those assumptions: party discount factors are less than one half and the voter discount factor is less than one third. Admittedly, however, the assumptions of quadratic utility and a non-empty core limit the interest of such a result. The next proposition reaches essentially the same conclusion, assuming minimally impatient parties and imposing a stronger restriction on the voters' discount rates, without those background assumptions.

In stating the result, we use a measure of how easily a policy path $\mathbf{x}$ can be profitably deviated from. To define this measure, let $\mathcal{M}$ denote the collection of all majority coalitions, and let $P_C(x)$ denote the set of policies, x', such that $u_i(x') > u_i(x)$ for all $i \in C$. Define

$$\psi(\mathbf{x}) \quad = \quad \sup_{t \in \mathbb{N}} \max_{C \in \mathcal{M}} \sup_{x' \in P_C(x_t) \cup \{x_t\}} \min_{i \in C} u_i(x') - u_i(x_t),$$

where $\mathbb{N}$ is the natural numbers. In words, for every x' preferred by a majority C to x_t, we find the minimum utility difference between x' and x_t for members of C; we then find the policy x' and majority C for which this minimum difference is maximized to measure how far x_t is from being a core point; and we then find the policy in $\mathbf{x}$ furthest from being a core point. That is our measure of how easily $\mathbf{x}$ may be "broken." Note that $\psi(\mathbf{x}) \geq 0$. In the following proposition, let $\overline{u}_i = \sup_{x \in X} u_i(x)$ and $\underline{u}_i = \inf_{x \in X} u_i(x)$. Note that the proposition does not rely on party symmetry.

Proposition 8 *Assume each u_i is bounded, and let $\delta_P < 1/2$ for the parties. If $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (OS) and (WD), then*

$$\frac{\psi(\mathbf{x})}{\overline{u}_i - \underline{u}_i + \psi(\mathbf{x})} \quad \leq \quad \delta_i$$

for some voter i.

It is interesting to compare this result to Proposition 2, which does not impose any restrictions on party discount factors and holds, in particular, when $\delta_P < 1/2$ for both parties. Because all policy paths are supportable under the conditions of the latter proposition, it follows that, under those conditions, the inequality in Proposition 8 must be satisfied by all paths. Indeed, it is: given any path $\mathbf{x}$, it can be checked that $\psi(\mathbf{x}) \leq \overline{u}_i - \underline{u}_i$ for some voter i; for that voter,

$$\frac{\psi(\mathbf{x})}{\overline{u}_i - \underline{u}_i + \psi(\mathbf{x})} \quad \leq \quad \frac{1}{2},$$

which is less than δ_i under the assumptions of Proposition 2.

We can characterize the paths for which $\psi(\mathbf{x}) = 0$ as follows. Define the *weak core,* denoted K°, as the set of majority undominated policies, i.e.,

$$K^\circ \;\; = \;\; \{x \in X \mid y \, \mathrm{M}^* \, x \text{ for no } y \in X\}.$$

Clearly, $K^* \subseteq K \subseteq K^\circ$. Under standard assumptions, such as n odd and strict quasi-concavity, we have equivalence of these three sets. It is straightforward to verify that $\psi(\mathbf{x}) = 0$ if and only if $x_t \in K^\circ$ for all t. Thus, when $K^\circ \neq \emptyset$ and $x_t \in K^\circ$ for all t, the condition in Proposition 8 is unrestrictive: it is satisfied regardless of the voters' discount factors. When $x_t \notin K^\circ$ for some t, the proposition implies that, when voter discount rates are low enough, the path $\mathbf{x}$ cannot be supported by a subgame perfect equilibrium satisfying our conditions. Put differently, the only policy paths that are supportable for all discount rates lie in the weak core. Note that the fraction on the lefthand side of the inequality in Proposition 8 is increasing in $\psi(\mathbf{x})$, implying that, the further a policy path is from being in the core, the higher voter discount rates must be to support it. The fraction is decreasing in $\overline{u}_i - \underline{u}_i$, reflecting potentially stronger incentives that future outcomes may have on voters, dissuading them from voting for a deviating party.

We can say more if we impose some very weak regularity conditions on the policy space and voter utilities.

Corollary 1 *Assume X is compact and each u_i is continuous, and let $\delta_P < 1/2$ for the parties. If $K^\circ = \emptyset$, then there exists $\underline{\delta} > 0$ such that, when $\delta_i < \underline{\delta}$ for all voters, no policy path is supported by a subgame perfect equilibrium satisfying (OS) and (WD).*

The proof of the corollary uses the observation that, by a version of the Theorem of the Maximum (Aliprantis and Border, 1994, Lemma 14.28), the function

$$f(x) \;\; = \;\; \max_{C \in \mathcal{M}} \sup_{y \in P_C(x) \cup \{x\}} \min_{i \in C} u_i(y) - u_i(x)$$

is lower semicontinuous in x. Since X is compact, f has a minimum on X, say at policy x'. Moreover, since $K^\circ = \emptyset$, the value of f at x' is positive.

Now, given any policy path $\mathbf{x}$, note that $\psi(\mathbf{x}) \geq f(x') > 0$. Then, setting

$$\underline{\delta} \;=\; \frac{f(x')}{(\max_{i \in N} \overline{u}_i - \underline{u}_i) + f(x')},$$

the corollary is proved. In fact, though the corollary is not stated in quite such terms, we can use these arguments to show that there does not exist any subgame perfect equilibrium satisfying our conditions — not even ones that induce non-degenerate distributions on policy paths.

6 Conclusion

We have shown that, if voters are somewhat patient, or if parties are somewhat patient and the strong core is close to non-empty, or if the policy space is unbounded, then there is a subgame perfect equilibrium of the infinitely repeated electoral game. This is true regardless of the dimensionality of the policy space or voter preferences, providing a solution to the equilibrium existence problem. This sword is double-edged, however, for, in fact, *every* path of policies can be supported by a class of subgame perfect equilibria. As a consequence, the sharp predictions of the median voter theorem — and more generally core equivalence in multiple dimensions with a non-empty core — are endangered. We show that the median voter theorem holds if stationarity is imposed or if parties and voters are sufficiently impatient, but then we lose existence of equilibria in multiple dimensions when the core is empty. To achieve a general equilibrium existence result that preserves the median voter theorem in a model of infinitely repeated elections, we conclude that the background assumptions of the Downsian model must be re-examined. As in the electoral accountability approach, alternatives may involve policy motivations for candidates, dropping the commitment assumption (as in the literature on citizen-candidates), allowing for imperfect information about voter preferences (as in the literature on probabilistic voting), or some combination of these directions.

A Proofs of Propositions

Proposition 2 *Let $\delta_i > 1/2$ for all voters. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

Proof: Let $\mathbf{x} = (x_1, x_2, \ldots)$ be any policy path. We construct a subgame perfect equilibrium to support $\mathbf{x}$ by labeling each finite history, h_t, with a "state," $s(h_t)$ and by labeling each history and a platform pair (y, z) with a state $s(h_t, y, z)$. This labeling rule, defined recursively below, will simplify our specification of strategies. The set S of states will be

$$S \;=\; (X \times \{\infty\}) \cup (X \times \{A, B\} \times X) \cup (\mathbb{Z}_+ \times \{*\}),$$

where $\mathbb{Z}_+$ is the non-negative integers. We label the initial history with $(0, *)$, i.e., $s(h_0) = (0, *)$. If h_t is labeled $(t, *)$, we interpret this to mean "the parties have followed the desired path of play through period t and will continue to do so." In this case, given platforms (y, z), define the next state as follows.

- h_t labeled $(t, *)$:

$$s(h_t, y, z) \;=\; \begin{cases} (x_t, A, y) & \text{if } y \,\mathrm{M}\, x_{t+1} = z \\ (x_t, B, z) & \text{if } z \,\mathrm{M}\, x_{t+1} = y \\ (t+1, *) & \text{else.} \end{cases}$$

Thus, if the parties both choose x_{t+1}, or if one party deviates to something not plurality-preferred to x_{t+1}, or if both parties deviate to other platforms, the state continues to reflect that we follow the desired path of play. (Such deviations are harmless below.) If h_t is labeled (x, ∞), we interpret this to mean "the parties are supposed to choose x in period $t + 1$ and will do so ever after." Define the state transition rule as follows.

- h_t labeled (x, ∞):

$$s(h_t, y, z) \;=\; \begin{cases} (x, A, y) & \text{if } y \,\mathrm{M}\, x = z \\ (x, B, z) & \text{if } z \,\mathrm{M}\, x = y \\ (x, \infty) & \text{else.} \end{cases}$$

If (h_t, y, z) is labeled $(t+1, *)$, then following voting and the selection of the winner in period $t+1$, we simply label the new history h_{t+1} with $(t+1, *)$, as below.

- (h_t, y, z) labeled $(t+1, *)$:

$$s(h_{t+1}) \quad = \quad (t+1, *).$$

If (h_t, y, z) is labeled (x, ∞), then following voting and the selection of the winner in period $t+1$, we label the new history h_{t+1} with (x, ∞).

- (h_{t+1}, y, z) labeled (x, ∞):

$$s(h_t, y, z) \quad = \quad (x, \infty).$$

Thus, if we begin period $t+1$ in states $(t+1, *)$ or (x, ∞), and the parties choose as required (x_{t+1} in the former case, x in the latter), then the state at the beginning of period $t+2$ is independent of the outcome of voting. If (h_t, y, z) is labeled (x, A, x'), we interpret this to mean "the parties were supposed to choose x in $t+1$ but A deviated to x'." Then, after voting and the selection of the winner P in period $t+1$, we label the new history h_{t+1} as follows.

- (h_t, y, z) labeled (x, A, x'):

$$s(h_{t+1}) \quad = \quad \begin{cases} (x, \infty) & \text{if } P = A \\ (x', \infty) & \text{if } P = B. \end{cases}$$

Thus, if A deviates from x to x' and wins, then the state moves to (x, ∞), i.e., the original policy outcome ever after. If A deviates and B wins, then the state moves to (x', ∞), i.e., A's deviation forever. Since $x' \,\mathrm{M}\, x$, this will give a plurality of voters an incentive to vote against A. If (h_t, y, z) is labeled (x, B, x'), which we interpret to mean "the parties were supposed to choose x in $t+1$ but B deviated to x'," we label h_{t+1} as follows.

- (h_t, y, z) labeled (x, B, x'):

$$s(h_{t+1}) \quad = \quad \begin{cases} (x, \infty) & \text{if } P = B \\ (x', \infty) & \text{if } P = A. \end{cases}$$

We next specify strategies for parties and voters.

1. Parties:

   (a) If h_t is labeled $(t, *)$, then the parties adopt platforms $y_{t+1} = z_{t+1} = x_{t+1}$.

   (b) If h_t is labeled (x, ∞), then both adopt $y_{t+1} = z_{t+1} = x$.

2. Voters:

   (a) If (h_t, y, z) is labeled $(t + 1, *)$, then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.

   (b) If (h_t, y, z) is labeled (x, ∞), then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.

   (c) If (h_t, y, z) is labeled (x, A, x'), then voter i votes for A if $u_i(x) > u_i(x')$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(x) = u_i(x')$.

   (d) If (h_t, y, z) is labeled (x, B, x'), then voter i votes for B if $u_i(x) > u_i(x')$; i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(x) = u_i(x')$.

This construction is diagrammed in Figure 1, where arrows denote transitions between states as a function of party platforms and election returns. Hollow arrows indicate the path of play.

[Figure 1 about here.]

We now verify that the above specification of strategies is, indeed, subgame perfect and satisfies (OS), (WD), and (PS). By the one-shot deviation principle (Fudenberg and Tirole, 1991), we need show only that no party or voter can achieve a higher expected discounted payoff by a "one-shot deviation" following any history. That is, we need to show, given an arbitrary history, that no party or voter can profit by deviating in the following period and returning to the above strategy thereafter. Consider a voter i's decision after (h_t, y, z) labeled $(t+1, *)$. Regardless of the winner in period $t+1$, according to the above strategies, the parties will choose the same platforms in subsequent periods, namely, $x_{t+2},\ x_{t+3},\ \ldots,$ so $v_i(h_t, y, z, 1) = v_i(h_t, y, z, 0)$. It is a best response to vote for A if

$$(1-\delta_i)u_i(y) + \delta_i v_i(h_t, y, z, 1) \quad \geq \quad (1-\delta_i)u_i(z) + \delta_i v_i(h_t, y, z, 0),$$

which is equivalent to $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$, so 2(a) gives the voters best responses. Similarly, after (h_t, y, z) labeled (x, ∞), the above strategies specify that the parties both choose x forever, and 2(b) is a best response. After (h_t, y, z) labeled (x, A, x'), the policy path depends on which party wins in period $t+1$. If A wins, then, according to the above strategies, both parties will choose x thereafter, so $v_i(h_t, y, z, 1) = u_i(x)$; if B wins, then both parties will choose x' thereafter, so $v_i(h_t, y, z, 1) = u_i(x')$. Thus, it is a best response for i to vote for A if

$$(1-\delta_i)u_i(x') + \delta_i u_i(x) \quad \geq \quad (1-\delta_i)u_i(x) + \delta_i u_i(x').$$

Since $\delta_i > 1/2$, this is equivalent to $u_i(x) \geq u_i(x')$, and it is a best response to vote for A if $u_i(x) \geq u_i(x)$, likewise for B if $u_i(x') \geq u_i(x)$, as in 2(c). Note that, since $x' \,\mathrm{M}\, x$ by construction, the latter holds for a plurality of voters, so B will win with probability greater than one half in period $t+1$. The analysis is similar after (h_t, y, z) labeled (x, B, x'), but, in that case, A wins with probability greater than one half in period $t+1$. We conclude that the strategies specified above for voters are best responses after all histories.

Consider the decision of a party, say A, after a history h_t labeled $(t, *)$. According to the strategies specified above, the parties both choose x_{t+1} in period $t+1$ and follow $\mathbf{x}$ thereafter, the voters flip coins to decide between parties in all periods, and A's expected discounted payoff is one half. If A deviates by choosing platform $y \neq x_{t+1}$ and following the above strategy thereafter, there are two possibilities. First, if $y \,\mathrm{M}\, x_t$, then (h_t, y, x_t) is labeled with (x_t, A, y). By 2(c), with some probability $\pi_A < 1/2$, party A wins, the new history h_{t+1} is labeled (y, ∞), and the voters randomize thereafter, giving the party an expected discounted payoff of one half. With probability $\pi_B = 1 - \pi_A > 1/2$, party B wins, and the new history is labeled (y, ∞).

After that history, according to 1(b) above, both parties choose y and, by 2(b), voters randomize between the parties thereafter. Thus, A's expected discounted payoff from deviating is

$$\begin{aligned}&\pi_A((1-\delta_A)(1)+\delta_A(1/2))+\pi_B((1-\delta_A)(0)+\delta_A(1/2)\\ &= \ (1-\delta_A)(\pi_A)+\delta_A(1/2),\end{aligned}$$

which is less than one half, since $\pi_A < 1/2$. Second, if not $y \, \mathrm{M} \, x_t$, then (h_t, y, x_t) is labeled with $(t+1, *)$. By 2(a), A wins with probability $\pi_A \leq 1/2$ and B wins with probability $\pi_B \geq 1/2$. By 1(a) and 2(a), both parties follow the path $\mathbf{x}$ and voters randomize between them thereafter. Thus, A's expected discounted payoff from deviating is less than or equal to one half. The logic following a history h_t labeled (x, ∞) is similar: if a party deviates to a platform plurality-preferred to x, then it will win in period $t+1$ with probability less than one half and win half the time thereafter; if it deviates to a platform not plurality-preferred to x, then it can do no better than win half the time in $t+1$, and it wins half the time thereafter. We conclude that party A, likewise party B, has no profitable one-shot deviations.

Thus, the above specification of strategies is a subgame perfect equilibrium, and it clearly supports $\mathbf{x}$. That it satisfies (OS) follows from two observations: the transition rule for states only depends on past electoral outcomes; and strategies, in turn, only depend on states. Finally, (WD) and (PS) are clear from the preceding arguments. ∎

Proposition 3 *Let $\delta_i > 0$ for all voters, let $\delta_P \geq 1/2$ for both parties, and let $K^* \neq \emptyset$. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

Proof: Let $K^* = \{x^*\}$ and fix an arbitrary alternative $x^0 \in X \setminus \{x^*\}$. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be any policy path. As in the proof of Proposition 2, we label each history, h_t, with a state $s(h_t)$ and each history and platform pair (y, z) with a state $s(h_t, y, z)$. The set S of states is

$$\begin{aligned}S \ &= \ \{(x^*, \infty), (E, A), (E, B)\} \cup (\mathbb{Z}_+ \times \{*\}) \cup (\{O_1\} \times \{A, B\})\\ &\quad \cup (\{O_2\} \times X) \cup (\{O\} \times \{A, B\} \times X).\end{aligned}$$

Following the proof of Proposition 2, states evolve according to a transition rule, defined recursively as follows. The initial history is assigned the state $(0, *)$. Suppose that, following history h_t, the parties choose platforms (y, z). As before, $(t, *)$ is interpreted to mean "the parties have followed the desired path of play through period t and will continue to do so."

- If h_t is labeled $(t, *)$, then

$$s(h_t, y, z) \quad = \quad \begin{cases} (E, A) & \text{if } y \,\mathrm{M}\, x_{t+1} = z \\ (E, B) & \text{if } z \,\mathrm{M}\, x_{t+1} = y \\ (t+1, *) & \text{else.} \end{cases}$$

As before, (x^*, ∞) means "the parties are supposed to choose x^* in period $t + 1$ and will do so every after."

- If h_t is labeled (x^*, ∞), then

$$s(h_t, y, z) \quad = \quad (x^*, \infty).$$

We interpret (O_1, A) to mean "party A deviated from the desired path in the previous period and won the election," and similarly for (O_1, B).

- If h_t is labeled (O_1, A), then

$$s(h_t, y, z) \quad = \quad \begin{cases} (O, A, y) & \text{if } y \neq z = x^* \\ (O, B, z) & \text{if } z \neq y = x^* \\ (O_1, A) & \text{else.} \end{cases}$$

- If h_t is labeled (O_1, B), then

$$s(h_t, y, z) \quad = \quad \begin{cases} (O, A, y) & \text{if } y \neq z = x^* \\ (O, B, z) & \text{if } z \neq y = x^* \\ (O_1, B) & \text{else.} \end{cases}$$

We interpret (O_2, x) to mean "some party deviated from the desired path of play; this was followed by another deviation; and from now on the parties will both choose x."

- If h_t is labeled (O_2, x), then

$$s(h_t, y, z) \quad = \quad \begin{cases} (E, A) & \text{if } y \,\mathrm{M}\, z = x \\ (E, B) & \text{if } z \,\mathrm{M}\, y = x \\ (O_2, x) & \text{else.} \end{cases}$$

Now suppose that, following h_t and platform choices (y, z), the winning party is P. The resulting history, h_{t+1}, is labeled as follows. As before, if the parties have followed the desired path, then the transition is independent of the winner in period $t+1$.

- If (h_t, y, z) is labeled $(t+1, *)$, then

$$s(h_{t+1}) \quad = \quad (t+1, *).$$

The transition is similar if we are in a state that called for the strong core point ever after.

- If (h_t, y, z) is labeled (x^*, ∞), then

$$s(h_{t+1}) \quad = \quad (x^*, \infty).$$

We interpret (E, A) to mean "party A deviated from the desired path of play in the previous period." In this case, the label of h_{t+1} will depend on the winner in period $t+1$. Similar remarks hold for (E, B).

- If (h_t, y, z) is labeled (E, A), then

$$s(h_{t+1}) \quad = \quad \begin{cases} (x^*, \infty) & \text{if } P = B \\ (O_1, A) & \text{if } P = A. \end{cases}$$

- If (h_t, y, z) is labeled (E, B), then

$$s(h_{t+1}) \quad = \quad \begin{cases} (x^*, \infty) & \text{if } P = A \\ (O_1, B) & \text{if } P = B. \end{cases}$$

If party A deviated from the desired path in the previous period and won, the label of h_{t+1} depends on the winner in period $t+1$, and similarly if party B deviated and won.

- If (h_t, y, z) is labeled (O_1, A), then

$$s(h_{t+1}) = \begin{cases} (O_2, x^0) & \text{if } P = A \\ (O_1, A) & \text{if } P = B. \end{cases}$$

- If (h_t, y, z) is labeled (O_1, B), then

$$s(h_{t+1}) = \begin{cases} (O_2, x^0) & \text{if } P = B \\ (O_1, B) & \text{if } P = A. \end{cases}$$

We interpret (O, A, y) to mean "party A deviated from the desired path of play in the previous period, won, and has deviated again, this time to platform y." In this case, the label of h_{t+1} again depends on the winner in period $t + 1$. Similar remarks hold for (O, B, z).

- If (h_t, y, z) is labeled (O, A, y), then

$$s(h_{t+1}) = \begin{cases} (O_2, y) & \text{if } P = A \\ (O_1, A) & \text{if } P = B. \end{cases}$$

- If (h_t, y, z) is labeled (O, B, z), then

$$s(h_{t+1}) = \begin{cases} (O_2, z) & \text{if } P = B \\ (O_1, B) & \text{if } P = A. \end{cases}$$

If some party deviated from the desired path of play, followed by another deviation, and the parties are to choose x forever, then the transition is independent of the winner.

- If (h_t, y, z) is labeled (O_2, x), then

$$s(h_t, y, z) = (O_2, x).$$

We next specify strategies for parties and voters.

1. Parties:

(a) If h_t is labeled $(t, *)$, then the parties adopt platforms $y_{t+1} = z_{t+1} = x_{t+1}$.

(b) If h_t is labeled (x^*, ∞) or (O_1, A) or (O_1, B), then both adopt $y_{t+1} = z_{t+1} = x^*$.

(c) If h_t is labeled (O_2, x), then both parties choose $y_{t+1} = z_{t+1} = x$.

2. Voters:

(a) If (h_t, y, z) is labeled $(t+1, *)$, (x^*, ∞), (E, A), (E, B), or (O_2, x), then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.

(b) If (h_t, y, z) is labeled (O_1, A), then voter i votes for B if

$$(1 - \delta_i)u_i(z) + \delta_i u_i(x^*) > (1 - \delta_i)u_i(y) + \delta_i u_i(x^0);$$

voter i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

(c) If (h_t, y, z) is labeled (O_1, B), then voter i votes for A if

$$(1 - \delta_i)u_i(y) + \delta_i u_i(x^*) > (1 - \delta_i)u_i(z) + \delta_i u_i(x^0);$$

voter i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

(d) If (h_t, y, z) is labeled (O, A, y), then voter i votes for B if

$$(1 - \delta_i)u_i(z) + \delta_i u_i(x^*) > u_i(y);$$

voter i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

(e) If (h_t, y, z) is labeled (O, B, z), then voter i votes for A if

$$(1 - \delta_i)u_i(y) + \delta_i u_i(x^*) > u_i(z);$$

voter i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

This construction is diagrammed in Figure 2.

[Figure 2 about here.]

To show that these strategies form a subgame perfect equilibrium, it is sufficient, by the one-shot deviation principle, to show there is no history at which a party or voter can profitably deviate that period and return to the above strategy thereafter. Consider a voter i's decision after a history (h_t, y, z) labeled $(t+1, *)$. Regardless of the winner in period $t+1$, according to the above strategies, the parties will choose the same platforms thereafter, namely, x_{t+2}, x_{t+3}, ..., so $v_i(h_t, y, z, 1) = v_i(h_t, y, z, 0)$. It is a best response to vote for A if

$$(1-\delta_i)u_i(y) + \delta_i v_i(h_t, y, z, 1) \quad \geq \quad (1-\delta_i)u_i(z) + \delta_i v_i(h_t, y, z, 0),$$

which is equivalent to $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$, so 2(a) gives the voters best responses. Similarly, after (h_t, y, z) labeled (x^*, ∞), the above strategies specify that the parties both choose x^* forever, and 2(a) is a best response. After (h_t, y, z) labeled (O_2, x), the parties both choose x forever, so 2(a) is a best response. Next, consider a history h_t and platform pair (y, z) labeled (E, A). If B wins, then the state moves to (x^*, ∞), and both parties choose x^* thereafter. Thus, $v_i(h_t, y, z, 0) = u_i(x^*)$. If A wins, then the state becomes (O_1, A), and, by 1(b), the parties both choose x^* in period $t+2$. Then, by 2(b), voter i votes for B if $u_i(x^*) > u_i(x^0)$, so a majority of voters vote for B, and B wins with probability one. According to our transition rule, the state remains (O_1, A), and B continues to win thereafter with platform x^*. Thus, $v_i(h_t, y, z, 1) = u_i(x^*)$. Again, voting for A is a best response if $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$, so 2(a) is a best response. Applying the same argument, 2(a) is also a best response in state (E, B).

After (h_t, y, z) labeled (O_1, A), the policy path depends on which party wins in period $t+1$. If B wins, then the state remains (O_1, A), and the above strategies specify that both parties will choose x^* thereafter, so $v_i(h_t, y, z, 1) = u_i(x^*)$. If A wins, then the state moves to (O_2, x^0), where, by 1(c), the parties both choose x^0 thereafter, implying $v_i(h_t, y, z, 1) = u_i(x^0)$. Thus, it is a best response for i to vote for B if

$$(1-\delta_i)u_i(z)+\delta_i u_i(x^*) \;\; \geq \;\; (1-\delta_i)u_i(y)+\delta_i u_i(x^0),$$

and 2(b) is a best response. Note that, if $y \neq x^* = z$, then a majority of voters vote for B, and B wins with probability one. For (O_1, B), the same argument shows that 2(c) is a best response. Note that, in that case, if $z \neq x^* = y$, then a majority of voters vote for A, and A wins with probability one.

Now consider (h_t, y, z) labeled (O, A, y). If B wins in period $t+1$, the state moves to (O_1, A) and both parties adopt x^* in all following periods. This implies that $v_i(h_t, y, z, 1) = u_i(x^*)$. If A wins, then the state moves to (O_2, y), where both parties choose y thereafter, so $v_i(h_t, y, z, 1) = u_i(y)$. Thus, it is a best response for i to vote for B if

$$(1-\delta_i)u_i(z)+\delta_i u_i(x^*) \;\; \geq \;\; (1-\delta_i)u_i(y)+\delta_i u_i(y),$$

and voting for A is a best response if the opposite weak inequality holds, as in 2(d). Note that, if $y \neq x^* = z$, then a majority of voters vote for B, and B wins with probability one. A similar calculation shows that 2(e) is a best response when the state is labeled (O, B, z). In that case, if $z \neq x^* = z$, then a majority of voters vote for A, and A wins with probability one. We conclude that the above voting strategies specify best responses after all histories.

Turning to the parties, consider the decision of a party, say A, after a history h_t labeled $(t, *)$. According to the strategies specified above, the parties both choose x_{t+1} in period $t+1$ and follow $\mathbf{x}$ thereafter, the voters flip coins to decide between parties in all periods, and A's expected discounted payoff is one half. If A deviates to y such that y such that $y \, \mathrm{M} \, x_t$, then the state moves to (E, A) and, by 2(a), party A wins with probability $\pi_A > 1/2$. The state then moves to (O_1, A), and, according to the strategies specified above, A wins with probability zero thereafter. Party B wins with probability $\pi_B = 1 - \pi_A < 1/2$, after which the state moves to (x^*, ∞), and A's expected discounted payoff is then one half. Thus, A's expected discounted payoff from deviating is

$$\pi_A((1-\delta_A)(1)+\delta_A(0))+\pi_B(1/2),$$

which is no greater than one half, since $\delta_A \geq 1/2$. If A deviates to y such that not $y \, \mathrm{M} \, x_{t+1}$, then the state remains at $(t, *)$. By 2(a), A wins in period $t+1$ with probability less than or equal to one half. Regardless of the winner, the state stays at $(t, *)$, and A's expected discounted payoff is then one half. Thus, the deviation is not profitable.

After a history labeled (x^*, ∞), according to the above strategies, both parties choose x^* and A wins with probability one half in every period, yielding an expected discounted payoff of one half. If A deviates to $y \neq x^*$, then the state remains at (x^*, ∞), and, by 2(a), a majority of voters vote for B, and A wins with probability zero. After that, A's expected discounted payoff is again one half, so the deviation is not profitable.

Next, take a history h_t labeled (O_1, A). According to the strategies specified above, party A wins with probability zero in period $t+1$ and in all future periods. If A deviates to $y \neq x^*$, then the state moves to (O, A, y). By 2(d), voter i votes for B if $u_i(x^*) > u_i(y)$, where we use the fact that, by 1(b), $z = x^*$. This inequality holds for a majority of voters, so B wins with probability one in period $t+1$ after A deviates. The state then moves back to (O_1, A), where A continues to lose forever. Thus, the deviation is not profitable. After a history labeled (O_1, B), A's expected discounted payoff is one, so A clearly has no profitable deviation.

Lastly, after a history h_t labeled (O_2, x), according to the strategies specified above, both parties choose x forever, and A's expected discounted payoff is one half. If A deviates to y such that $y \, \mathrm{M} \, x$, then the state moves to (E, A). (We treat this deviation as we would treat a deviation from the desired path.) By 2(a), A wins with probability $\pi_A > 1/2$ in period $t+1$, in which case the state moves to (O_1, A), where B wins in all subsequent periods. Party B wins with probability $\pi_B < 1/2$, in which case the state moves to (x^*, ∞) and A's expected discounted payoff is one half. Thus, A's expected discounted payoff from deviating is

$$\pi_A((1-\delta_A)(1) + \delta_A(0)) + \pi_B(1/2),$$

which is not profitable. If A deviates to y such that not $y \, \mathrm{M} \, x$, then, by 2(a), A wins in period $t+1$ with probability no greater than one half, the state remains at (O_2, x), and A's expected discounted payoff is again one half. Thus, the deviation is not profitable. We conclude that party A, and likewise party B, has no profitable one-shot deviations.

Thus, the above specification of strategies is a subgame perfect equilibrium, and it clearly supports $\mathbf{x}$. That it satisfies (OS) follows from two observations: the transition rule for states only depends on past electoral outcomes; and strategies, in turn, only depend on states. Finally, (WD) and (PS) are clear from the preceding arguments. ∎

Proposition 4 *Assume $X \subseteq \mathbb{R}^d$ is compact and convex. Let $\delta_i = \delta > 0$ for all voters, let $\delta_P \geq 1/2$ for both parties, let $K^*(u_1, \ldots, u_n) \neq \emptyset$, and let $(u_1^m, \ldots, u_n^m) \to (u_1, \ldots, u_n)$ uniformly. Assume u_i^m is concave and u_i is continuous for all voters. Then, for high enough m, every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

Proof: The proof is very close to the proof of Proposition 3, so we only sketch the changes needed here. Let $K^*(u_1, \ldots, u_n) = \{x^*\}$. As before, we fix a policy $x^0 \neq x^*$. For high enough m, note that a majority of voters will have $u_i^m(x^*) > u_i^m(x^0)$. Thus, as before, if both parties choose x^* in state (O_1, A), then B will win with probability one; similarly, A will win with probability one in state (O_1, B). But to ensure that, e.g., B winning with probability one in state (O_1, A) is an equilibrium in that subgame, we must ensure that A cannot win with positive probability by deviating from x^*. Originally, when x^* was the strong core point, this was relatively easy: we defined the parties' strategies so that, if A deviated to $y \neq x^*$, then the voters would expect y forever if A won and x^* forever if B won. Naturally, a majority would then vote against A. Now, however, there may be a policy y that is majority-preferred to x^* at utility profile $(u_1^m, \ldots, u_n^m)$. After a deviation by A, therefore, we now specify strategies for the parties such that, were A to win, they would choose x^0 thereafter. Specifically, we have the state move from (O, A, y) to (O_2, x^0). We claim that, for high enough m, this specification will ensure that A cannot profitably deviate to such a y, because, as before, a majority would vote against A after such a deviation.

To establish the claim, suppose that, for each m, there exists a policy y^m with

$$(1-\delta)u_i^m(y^m) + \delta u_i^m(x^0) \quad \geq \quad u_i(x^*)$$

for at least half of the voters, allowing party A to deviate and win with positive probability. Letting $x^m = (1-\delta)y^m + \delta x^0$, concavity of the u_i^m implies

$$u_i^m(x^m) \quad \geq \quad u_i^m(x^*)$$

for all such i. By compactness of X, $\{y^m\}$ has a subsequence with some limit, say y, and then $\{x^m\}$ has a corresponding subsequence with limit $x = (1-\delta)y + \delta x^0 \neq x^*$. By uniform convergence to, and continuity of, u_i, the above inequality implies

$$u_i(x) \quad \geq \quad u_i(x^*)$$

for at least half of the voters, contradicting our assumption that x^* is the strong core point at $(u_1, \ldots, u_n)$. Therefore, for high enough m, there is no platform to which A can deviate in state (O_1, A) and win with positive probability. Of course, the same approach can be used in state (O_1, B). Lastly, we must support (x^*, x^*) as an equilibrium in state (x^*, ∞), trivial when x^* is the strong core point. To do this, we simply use the same method used to support (x_{t+1}, x_{t+1}) in state $(t, *)$, i.e., we move to state (E, A) if A deviates to y such that $y \,\mathrm{M}\, x^*$, allowing that party to win in that period with positive probability but with probability zero thereafter. ∎

Proposition 5 *Assume that voter utilities are majority unbounded below and that $\delta_i > 0$ for all voters. Then every policy path is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS).*

Proof: Fix $\delta > 0$ and an arbitrary alternative $x^0 \in X$. Given these choices, let $\underline{x}(x, x')$ be an alternative in X satisfying

$$u_i(\underline{x}(x, x')) < \frac{(1-\delta_i)u_i(x) + \delta_i u_i(x^0) - (1-\delta_i)u_i(x')}{\delta_i}$$

for some majority C and all $i \in C$. That such an alternative exists is a consequence of majority unboundedness from below. Let $\mathbf{x} = (x_1, x_2, \ldots)$ be any policy path. As in the proofs of our earlier propositions, we label each history, h_t, with a state $s(h_t)$, and we label each history h_t and platform pair (y, z) with a state $s(h_t, y, z)$. The set S of states is

$$S \quad = \quad (X \times \{\infty\}) \cup (X \times \{A, B\} \times X) \cup (\mathbb{Z}_+ \times \{*\}).$$

We label the states recursively as follows. The initial history is assigned the state $(0, *)$. States are assigned as follows, with interpretations as in the proof of Proposition 2.

- If h_t is labeled $(t, *)$, then

$$s(h_t, y, z) \quad = \quad \begin{cases} (x_t, A, y) & \text{if } y \neq x_{t+1} = z \\ (x_t, B, z) & \text{if } z \neq x_{t+1} = y \\ (t+1, *) & \text{else.} \end{cases}$$

- If h_t is labeled (x, ∞), then

$$s(h_t, y, z) \quad = \quad \begin{cases} (x, A, y) & \text{if } y \neq x = z \\ (x, B, z) & \text{if } z \neq x = y \\ (x, \infty) & \text{else.} \end{cases}$$

After votes are cast and a winner, P, is determined, states move as follows.

- If (h_t, y, z) is labeled $(t+1, *)$, then

$$s(h_{t+1}) \quad = \quad (t+1, *).$$

- If (h_t, y, z) is labeled (x, ∞), then

$$s(h_{t+1}) \quad = \quad (x, \infty).$$

If A deviates from x to x' and wins, we specify the new state as $(\underline{x}(x, x'), \infty)$, while, if A deviates and B wins, we define the new state as (x^0, ∞). This will give a majority of voters an incentive to vote against A after such a deviation. Similarly comments hold for deviations by B.

- If (h_t, y, z) is labeled (x, A, x'), then

$$s(h_{t+1}) = \begin{cases} (\underline{x}(x, x'), \infty) & \text{if } P = A \\ (x^0, \infty) & \text{if } P = B. \end{cases}$$

- If (h_t, y, z) is labeled (x, B, x'), then

$$s(h_{t+1}) = \begin{cases} (\underline{x}(x, x'), \infty) & \text{if } P = B \\ (x^0, \infty) & \text{if } P = A. \end{cases}$$

We next specify strategies for parties and voters.

1. Parties:

   (a) If h_t is labeled $(t, *)$, then the parties adopt platforms $y_{t+1} = z_{t+1} = x_{t+1}$.

   (b) If h_t is labeled (x, ∞), then both adopt $y_{t+1} = z_{t+1} = x$.

2. Voters:

   (a) If (h_t, y, z) is labeled $(t+1, *)$, then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.

   (b) If (h_t, y, z) is labeled (x, ∞), then voter i votes for A if $u_i(y) > u_i(z)$; i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if $u_i(y) = u_i(z)$.

   (c) If (h_t, y, z) is labeled (x, A, x'), then voter i votes for A if

   $$(1-\delta_i)u_i(x') + \delta_i u_i(\underline{x}(x, x')) > (1-\delta_i)u_i(x) + \delta_i u_i(x^0);$$

   voter i votes for B if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

   (d) If (h_t, y, z) is labeled (x, B, x'), then voter i votes for B if

   $$(1-\delta_i)u_i(x') + \delta_i u_i(\underline{x}(x, x')) > (1-\delta_i)u_i(x) + \delta_i u_i(x^0);$$

   voter i votes for A if this inequality is reversed; and i votes for the parties with equal probabilities if equality holds.

This construction is diagrammed in Figure 3.

[Figure 3 about here.]

To show that these strategies form a subgame perfect equilibrium, it is sufficient, by the one-shot deviation principle, to show there is no history after which a party or voter can profit by deviating that period and returning to the above strategy thereafter. Consider a voter i's decision after a history (h_t, y, z) labeled $(t+1, *)$. Regardless of the winner in period $t+1$, according to the above strategies, the parties will choose the same platforms thereafter, namely, x_{t+2}, x_{t+3}, $\ldots$, so $v_i(h_t, y, z, 1) = v_i(h_t, y, z, 0)$. It is a best response to vote for A if

$$(1-\delta_i)u_i(y) + \delta_i v_i(h_t, y, z, 1) \quad \geq \quad (1-\delta_i)u_i(z) + \delta_i v_i(h_t, y, z, 0),$$

which is equivalent to $u_i(y) \geq u_i(z)$, and voting for B is a best response if $u_i(z) \geq u_i(y)$, so 2(a) gives voters best responses. Similarly, after (h_t, y, z) labeled (x, ∞), the above strategies specify that the parties both choose x forever, and 2(b) is a best response. After (h_t, y, z) labeled (x, A, x'), the policy path depends on which party wins in period $t+1$. If A wins, then, according to the above strategies, both parties will choose $\underline{x}(x, x')$ thereafter, so $v_i(h_t, y, z, 1) = u_i(\underline{x}(x, x'))$; if B wins, then both parties will choose x^0 thereafter, so $v_i(h_t, y, z, 1) = u_i(x^0)$. Thus, it is a best response to vote for A if

$$(1-\delta_i)u_i(x') + \delta_i u_i(\underline{x}(x, x')) \quad \geq \quad (1-\delta_i)u_i(x) + \delta_i u_i(x^0),$$

and voting for B is a best response if the opposite weak inequality holds, so 2(c) gives the voters best responses. Note that, given our definition of $\underline{x}(x, x')$, a majority of voters vote for B after A deviates, so B wins with probability one. The analysis is similar after (h_t, y, z) labeled (x, B, x'), but, in that case, A wins with probability one after B deviates. We conclude that the strategies specified above for voters are best responses after all histories.

Turning to the parties, consider a history h_t labeled $(t, *)$. If a party chooses x_{t+1} and follows the above strategy thereafter, then, since voters flip coins to decide between parties in all periods, the party's expected discounted payoff is one half. If one, say A, deviates from x_{t+1} to $y \neq x_{t+1}$, then (h_t, y, x_t) is labeled with (x_t, A, y). By 2(c), a majority of voters vote for B, which wins with probability one, and the new history h_{t+1} is labeled (x^0, ∞). By 1(b), both parties choose then choose x^0 thereafter, and, by 2(b), voters randomize between the parties forever. Thus, A's expected discounted payoff from deviating is $(1 - \delta_i)(0) + \delta_i(1/2) < 1/2$. The logic following a history h_t labeled (x, ∞) is similar. We conclude that party A, likewise party B, has no profitable one-shot deviations.

Thus, the above specification of strategies is a subgame perfect equilibrium, and it clearly supports $\mathbf{x}$. That it satisfies (OS) follows from two observations: the transition rule for states only depends on past electoral outcomes; and strategies, in turn, only depend on states. Finally, (WD) and (PS) are clear from the preceding arguments. ∎

Proposition 6 *A policy path* $\mathbf{x}$ *is supportable by a subgame perfect equilibrium satisfying (SS), (WD), and (PS) if and only if* $x_t \in K$ *for all* t.

Proof: Suppose $\mathbf{x}$ is supported by a subgame perfect equilibrium satisfying (WD) and (SS), but that $x_t \notin K$ for some t. In period t, therefore, either both parties choose x_t or only one does and wins with probability 1. To deal with both cases, it is enough to assume that at least one party chooses x_t and wins with probability greater than or equal to one half. Without loss of generality, suppose A is the other party, which wins with probability $p_A < 1/2$ in period t. We claim that A can increase its expected discounted payoff by deviating to y such that $y \, \mathrm{M} \, x_t$ in period t and returning to its original strategy thereafter. Assumptions (WD) and (SS) together imply that voters vote sincerely in each period. That is, voter i votes for A if $u_i(y) > u_i(x_t)$. Since this holds for a plurality of voters, A wins with probability strictly greater than 1/2 in period t. So A achieves a higher payoff in period t. By (SS), party A's expected payoff for all later periods is independent of A's choice in period t. Therefore A's deviating strategy is profitable, a contradiction. Therefore, $x_t \in K$ for all t.

Now suppose $x_t \in K$ for all t and consider the following specification of strategies. Let $y_t = z_t = x_t$ after all histories, and, after any finite history and platform choices (y, z), let voter i vote for A if $u_i(y) > u_i(z)$, for B if the inequality is reversed, and for both parties with equal probability if $u_i(y) = u_i(z)$. Again, we use the one-shot deviation principle to establish that this specification is a subgame perfect equilibrium. After any finite history h_t and any (y, z), voter i's continuation value, given these strategies, is just the discounted sum $v_i = \sum_{t'=t+1}^{\infty} \delta_i^{t'-t-1} u_i(x_{t'})$. It is then a best response for i to vote for A if

$$(1 - \delta_i)u_i(y) + \delta_i v_i \quad \geq \quad (1 - \delta_i)u_i(z) + \delta_i v_i,$$

or equivalently, $u_i(y) \geq u_i(z)$. Similarly, it is a best response for i to vote B if $u_i(z) \geq u_i(y)$. Thus, we have specified best responses for the voters. Note that, if the above strategies are followed, then each party wins with probability one half after every history. Suppose a party, say A, deviates to platform $y \neq x_{t+1}$ after any history of length t and then returns to the above strategy. Since $x_{t+1} \in K$, it is not the case that $y \,\mathrm{M}\, x_{t+1}$. According to the above strategies, therefore, A would win in period $t + 1$ with probability no greater than one half and would win with probability one half thereafter. We conclude that the specified strategies are best responses for the parties, and that we have a subgame perfect equilibrium. The properties (SS), (WD), and (PS) are evident from the preceding arguments. ∎

Proposition 7 *Assume that n is odd, that $X \subseteq \mathbb{R}^d$ is bounded, that the utility functions u_i are quadratic with distinct ideal points, and that $K \neq \emptyset$. Let $\delta_i = \delta < 1/2$ for all voters, and let $\delta_P < 1/3$ for the parties. Then $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (OS), (WD), and (PS) if and only if $x_t \in K^* = K$ for all t.*

Proof: That $\mathbf{x}$ can be supported if $x_t \in K^*$ for all t follows from Proposition 6. Consider any subgame perfect equilibrium, and let X^* be the set of possible equilibrium policy outcomes: $x \in X^*$ if and only if there is some finite history after which one party adopts x and wins with positive probability.

By our assumptions that n is odd and that voters have quadratic utility functions, the strong core and core coincide with the ideal point of some voter, indexed k. By $K \neq \emptyset$, by the assumption of common discount factors, and by (WD), Lemma 1 of Banks and Duggan (2001) then implies that this core voter is "decisive" after all histories. That is, A wins with probability one after history h_t and platform choices (y, z) if

$$(1-\delta)u_k(y) + \delta v_k(h_t, y, z, 1) \quad > \quad (1-\delta)u_k(z) + \delta v_k(h_t, y, z, 0),$$

and B wins with probability one if the inequality is reversed. Furthermore, if equality holds for the core voter, then, by the assumption of distinct ideal points, the above inequality and the reverse inequality hold for equal numbers of voters. Thus, in this case, the parties win with equal probabilities. It follows that, if $x \in X^*$, then there is a history after which some party wins with probability at least one half by choosing x. Let

$$\underline{u} \quad = \quad \inf\{u_k(x) \mid x \in X^*\},$$

which is finite, since X is bounded. Letting $\tilde{x}_k$ denote the core point (and the ideal point of the core voter), suppose that $\underline{u} < u_k(\tilde{x}_k)$. By construction, there exist a sequence $\{h_{t_m}\}$ of histories and a sequence $\{x_m\}$ of policies such that (i) for all m, some party, say P_m, wins with probability at least one half by adopting x_m after h_{t_m}, and (ii) $u_k(x_m) \to \underline{u}$. Assume without loss of generality that $P_m = A$ along a subsequence. Indexing that subsequence again by m, we have $P_m = A$ for all m, i.e., B wins with probability less than or equal to one half in the period following each history h_{t_m}. We claim that, for high enough m, B can win with probability one after h_{t_m} by deviating to $\tilde{x}_k$. If not, then, because the core voter is decisive, we must have

$$(1-\delta)u_k(\tilde{x}_k) + \delta v_k(h_{t_m}, x_m, \tilde{x}_k, 0) \quad \leq \quad (1-\delta)u_k(x_m) + \delta v_k(h_{t_m}, x_m, \tilde{x}_k, 1)$$

for some subsequence (also indexed by m). Since

$$\underline{u} \;\leq\; v_k(h_{t_m}, x_m, \tilde{x}_k, 0) \quad \text{and} \quad v_k(h_{t_m}, x_m, \tilde{x}_k, 1) \;\leq\; u_k(\tilde{x}_k),$$

this implies

$$(1-\delta)u_k(\tilde{x}_k) + \delta\underline{u} \quad \leq \quad (1-\delta)u_k(x_m) + \delta u_k(\tilde{x}_k),$$

or equivalently,

$$\frac{u_k(\tilde{x}_k) - u_k(x_m)}{u_k(\tilde{x}_k) - \underline{u}} \quad \leq \quad \frac{\delta}{1-\delta}.$$

Taking limits, we have

$$\lim_{m\to\infty} \frac{u_k(\tilde{x}_k) - u_k(x_m)}{u_k(\tilde{x}_k) - \underline{u}} \quad = \quad 1,$$

but $\delta < 1/2$ implies $\delta/(1-\delta) < 1$. This contradiction establishes the claim. Let h_{t_m} be any history such that B can win with probability one in the period following. The party's expected discounted payoff is less than or equal to $(1-\delta_P)(1/2) + \delta_P(1)$ if it does not deviate, while it's expected discounted payoff from deviating is at least $(1-\delta_P)(1) + \delta_P(0)$. Since $\delta_P < 1/3$, the deviation is profitable, a contradiction. Therefore, $\underline{u} = u_k(\tilde{x}_k)$, and we conclude that $X^* = \{\tilde{x}_k\}$. ∎

Proposition 8 *Assume each u_i is bounded, and let $\delta_P < 1/2$ for the parties. If $\mathbf{x}$ is supportable by a subgame perfect equilibrium satisfying (OS) and(WD), then*

$$\frac{\psi(\mathbf{x})}{\overline{u}_i - \underline{u}_i + \psi(\mathbf{x})} \quad \leq \quad \delta_i$$

for some voter i.

Proof: Consider a policy path $\mathbf{x}$ supported by a subgame perfect equilibrium satisfying (OS) and (WD), and suppose the inequality in the statement of the proposition is violated for all voters. Since the lefthand side of the inequality is strictly increasing and continuous in $\psi(\mathbf{x})$, we may take a period t, a majority C, and a policy x' such that

$$\delta_i \quad < \quad \frac{\min_{j\in C} u_j(x') - u_j(x_t)}{\overline{u}_i - \underline{u}_i + \min_{j\in C} u_j(x') - u_j(x_t)}$$

for all $i \in N$. In particular, for all $i \in C$,

$$\delta_i \;<\; \frac{u_i(x') - u_i(x_t)}{\overline{u}_i - \underline{u}_i + u_i(x') - u_i(x_t)}. \tag{1}$$

Letting h_{t-1} denote the equilibrium path of play in the first $t-1$ periods, we have, by assumption, that x_t is the policy outcome in period t with probability one. One of the parties, say A, must have an expected discounted payoff starting from the beginning of period t of less than or equal to one half. Thus, because $\delta_A < 1/2$, party B wins in period t with some positive probability, and we conclude that B's platform is x_t. We claim that A can deviate to x' in period t and win with probability one. Indeed, for all $i \in C$, the inequality

$$(1-\delta_i)u_i(x') + \delta_i v_i(x', x_t, 1) \;>\; (1-\delta_i)u_i(x_t) + \delta_i v_i(x', x_t, 0)$$

is implied by

$$(1-\delta_i)u_i(x') + \delta_i \underline{u}_i \;>\; (1-\delta_i)u_i(x_t) + \delta_i \overline{u}_i,$$

which is equivalent to (1). By (WD), therefore, A wins in period t after deviating to x'. The expected discounted payoff from deviating is at least $(1-\delta_A)(1) + \delta_A(0) = 1-\delta_A > 1/2$, so the deviation is profitable, a contradiction. ∎

References

[1] A. Alesina (1988) "Credibility and Policy Convergence in a Two-Party System with Rational Voters," *American Economic Review,* 78: 796-805.

[2] C. Aliprantis and K. Border (1994) *Infinite Dimensional Analysis,* New York: Springer-Verlag.

[3] E. Aragones and A. Postlewaite (2000) "Campaign Rhetoric: A Model of Reputation," mimeo.

[4] J. Banks (1995) "Singularity Theory and Core Existence in the Spatial Model," *Journal of Mathematical Economics,* 24: 523-536.

[5] J. Banks and J. Duggan (2001) "A Lemma on Social Choice over Lotteries," mimeo.

[6] J. Banks and J. Duggan (2002) "A Multidimensional Model of Repeated Elections," mimeo.

[7] D. Baron and E. Kalai (1993) "The Simplest Equilibrium of a Majority Rule Division Game," *Journal of Economic Theory,* 61: 290-301.

[8] D. Bernhardt, E. Hughson, and S. Dubey (2002) "Term Limits and Pork Barrel Politics," *Journal of Public Economics,* forthcoming.

[9] D. Black (1958) *The Theory of Committees and Elections,* Cambridge: Cambridge University Press.

[10] G. Cox (1984) "Non-collegial Simple Games and the Nowhere Denseness of the Set of Preference Profiles having a Core," *Social Choice and Welfare,* 1: 159-164.

[11] A. Downs (1957) *An Economic Theory of Democracy,* New York: Harper and Row.

[12] J. Duggan (2000) "Repeated Elections with Asymmetric Information," *Economics and Politics,* 12: 109-136.

[13] J. Fearon (1999) "Electoral Accountability and the Control of Politicians: Selecting Good Types Versus Sanctioning Poor Performance," in Przeworski et al., eds., *Democracy, Accountability, and Representation,* Cambridge: Cambridge University Press.

[14] D. Fudenberg and E. Maskin (1986) "The Folk Theorem in Repeated Games with Discounting or with Incomplete Information," *Econometrica,* 54: 533-556.

[15] D. Fudenberg and J. Tirole (1991) *Game Theory,* Cambridge: MIT Press.

[16] G. Kramer (1977) "A Dynamical Model of Political Equilibrium," *Journal of Economic Theory,* 16: 310-334.

[17] M. Le Breton (1987) "On the Core of Voting Games," *Social Choice and Welfare,* 4: 295-305.

[18] R. McKelvey and P. Ordeshook (1967) "Sequential Elections with Limited Information," *American Journal of Political Science,* 29: 480–512.

[19] C. Plott (1967) "A Notion of Equilibrium and its Possibility under Majority Rule," *American Economic Review,* 57: 787-806.

[20] A. Rubinstein (1979) "A Note on the Nowhere Denseness of Societies having an Equilibrium under Majority Rule," *Econometrica,* 47: 511-514.

[21] N. Schofield (1983) "Generic Instability of Majority Rule," *Review of Economic Studies,* 50: 695-705.

[22] K. Shotts (2000) "A Signaling Model of Repeated Elections," mimeo.