On the Robustness of Majority Rule and Rule by Consensus†

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Abstract

We show that simple majority rule satisfies the Pareto property, anonymity, neutrality, and (generic) transitivity on a bigger class of preference domains than any other voting rule. If we replace neutrality in the above list of properties with independence of irrelevant alternatives, then the corresponding conclusion holds for unanimity rule (rule by consensus).

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1. Introduction

A *voting rule* is a method for choosing from a set of social alternatives on the basis of voters' preferences. Many different voting rules have been studied in theory and used in practice. But far and away the most popular method has been *simple majority rule*, the rule that chooses alternative x over alternative y if more people prefer x to y than vice versa.

There are, of course, good reasons for majority rule's[1] popularity. It not only is attractively straightforward to use in practice, but satisfies some compelling theoretical properties, among them the *Pareto property* (the principle that if all voters prefer x to y and x is available, then y should not be chosen), *anonymity* (the principle that choices should not depend on voters' labels), and *neutrality* (the principle that the choice between a pair if alternatives should depend only on the pattern of voters' preferences over that pair, not on the alternatives' labels)[2].

But majority rule has a well-known flaw, discovered by the Marquis de Condorcet (1785) and illustrated by the Paradox of Voting (or Condorcet Paradox): it can generate intransitive choices. Specifically, suppose that there are

[1] For convenience, we will omit the modifier "simple" when it is clear that we are referring to *simple* majority rule rather to the many variants, such as the supermajority rules.
[2] In fact, May (1952) established that majority rule is the *unique* voting rule satisfying the Pareto property, anonymity, and neutrality, and a fourth property called *positive responsiveness*—if alternative x is chosen (perhaps not uniquely) for a given configuration of voters' preferences and the only change that is then made to those preferences is to move x up in some voters' preference ordering, x is now uniquely chosen. Without positive responsiveness, there are many voting rules—including all the supermajority rules—that satisfy the properties. We shall come back to May's characterization in section 5.

three voters 1, 2, 3, three alternatives x, y, z, and that voters' preferences are as follows:

1	2	3
x	y	z
y	z	x
z	x	y

(i.e., voter 1 prefers x to y to z, voter 2 prefers y to z to x, and voter 3 prefers z to x to y). Then, as Condorcet noted, a two-thirds majority prefers x to y, y to z, and z to x, so that majority rule fails to select *any* alternative.

Despite the theoretical importance of the Condorcet Paradox, there are important cases in which majority rule avoids intransitivity. Most famously, when alternatives can be arranged linearly and each voter's preferences are *single-peaked* in the sense that his utility declines monotonically in both directions from his favorite alternative, then, following Black (1948), majority rule is transitive for (almost) all[3] configurations of voters' preferences. Alternatively, suppose that, for every three alternatives, there is one that no voter ranks in the middle. This property, called *limited agreement* (see Inada 1969, Sen and Pattanaik 1969), seems to have held in recent French presidential elections, where the Gaullist and Socialist candidates have not inspired much passion, but the National Front candidate, Jean-Marie Le Pen, has attracted either revulsion or admiration, i.e., everybody ranks him either first or last. Whether or not this pattern of preferences has been good for France is open to debate, but it is certainly "good" for majority rule: limited agreement, like single-peakedness, ensures transitivity (almost always).

[3] We clarify what we mean by "almost all" in section 2.

So, majority rule "works well"—in the sense of satisfying the Pareto property, anonymity, neutrality and generic transitivity—for some domains of voters' preferences but not for others. A natural question to ask is how its performance compares with that of other voting rules. Clearly, no voting rule can work well for *all* domains; this conclusion follows immediately from the Arrow impossibility theorem[4] (Arrow, 1951). But we might inquire whether there is a voting rule that works well for a bigger class of domains than does majority rule.[5]

We show that the answer to this question is *no*. Specifically, we establish (Theorem 1) that if a given voting rule F works well on a domain of preferences, then majority rule works well on that domain too. Conversely, if F differs from majority rule[6], there exists some other domain on which majority rule works well and F does not.

Thus essentially majority rule is *uniquely* the voting rule that works well on the most domains; it is, in this sense, the most *robust* voting rule. This property can be viewed as a characterization of majority rule complementing the one given by May (1952) (for more on this, see the discussion and corollary following Theorem 1).

Theorem 1 strengthens a result obtained in Maskin (1995). That earlier proposition requires two rather strong auxiliary assumptions:

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[4] Our formulation of neutrality (see section 3)—which is, in fact, the standard formulation (see Sen, 1971)—incorporates Arrow's *independence of irrelevant alternatives*, the principle that the choice between two alternatives should depend only on voters' preferences for those two alternatives and not on their preferences for other alternatives. We could instead have decomposed neutrality into two separate properties: (i) symmetry with respect to alternatives, and (ii) independence of irrelevant alternatives.

[5] It is easy to find voting rules that satisfy three out of our four properties on *all* domains of preferences. For example, two-thirds majority rule (which deems two alternatives as socially indifferent unless one garners at least a two-thirds majority against the other) satisfies Pareto, anonymity, and neutrality on any domain. Similarly, rank-order voting (see below) satisfy Pareto, anonymity, and generic transitivity on any domain.

[6] More precisely, the hypothesis is that F differs from majority rule in some open neighborhood of preference configurations belonging to a domain on which majority rule works well.

The first is that the number of voters be *odd*. This assumption is needed because Maskin (1995) demands transitivity for *all* preference configurations drawn from a given domain. And as we will see below, even when preferences are single-peaked, intransitivity is possible if the population splits exactly 50-50 between two preference orderings; an odd number of voters prevents this from happening. To capture the idea that such a split is unlikely, we will work with a *continuum* of voters and ask only for *generic* transitivity.

Second, to prove the second half of the proposition, Maskin (1995) makes the strong assumption that the voting rule F being compared with majority rule satisfies Pareto, anonymity, and neutrality on *any* domain. We show that this assumption can be dropped.

Although treating all alternatives alike—as neutrality entails—is a natural constraint in many political and economic settings, it is not always a reasonable assumption. For example, there are cases in which we may wish to treat the status quo differently from other alternatives. For that reason, it is of some interest to investigate which voting rule works best when neutrality is replaced by the weaker assumption of independence of irrelevant alternatives.

Our second major finding (Theorem 2) establishes that, in this modified scenario (where we also impose a mild tie-break consistency requirement), *unanimity rule with an order of precedence* is uniquely the most robust voting rule. To define this rule, fix an ordering of the alternatives, interpreted as the "order of precedence." Then, between two alternatives, the rule will choose the one earlier in the ordering unless voters unanimously prefer the other alternative. Unanimity rule with an order of precedence thus corresponds to the sequential protocol that a committee might follow were it not willing to replace the status quo with another alternative except by consensus.

We proceed as follows. In section 2, we set up the model. In section 3, we define our four properties, Pareto, anonymity, neutrality, and generic transitivity formally. We also characterize when rank-order voting—a major "competitor" of majority rule—satisfies all these properties. In section 4, we establish a lemma, closely related to a result of Sen and Pattanaik (1969) that characterizes when majority rule is generically transitive. We use this lemma in section 5 to establish our main result on majority rule. Finally, we prove the corresponding result for unanimity rule in section 5.

2. The Model

Our model is in most respects a standard social-choice framework. Let X be the set of social alternatives. For technical convenience, we take X to be finite with cardinality $m(\geq 3)$. The possibility of individual indifference often makes technical arguments in the social-choice literature a great deal messier (see for example, Sen and Pattanaik, 1969). We shall simply rule it out by assuming that individual voters' preferences are drawn from a set $\mathfrak{R}$ of *strict orderings*, where $\mathfrak{R}$ is a subset of $\mathfrak{R}_X$, the set of all logically possible strict orderings of X. For any ordering $R \in \mathfrak{R}_X$ and any alternatives $x, y \in X$, the notation xRy denotes "x is preferred to y in ordering R." For example, if we can arrange the social alternatives from "least" to "greatest," i.e., $x_1 < x_2 < \ldots < x_m$,[7] then $\mathfrak{R}$ consists of *single-peaked* preferences (relative to this

[7] We are using the terms "least" and "greatest" figuratively. All we mean is that there is some linear order of the alternatives, e.g., how they line up on the left-right ideological spectrum.

arrangement) if, for all $R \in \mathfrak{R}$, whenever $x_i R x_{i+1}$ for some i, then $x_j R x_{j+1}$ for all $j > i$, and whenever $x_{i+1} R x_i$ for some i, then $x_{j+1} R x_j$ for all $j < i$.

For the reason mentioned in the introduction (and elaborated on below), we shall suppose that there is a *continuum* of voters indexed by points in the unit interval $[0,1]$. A *profile* $\boldsymbol{R}$ on $\mathfrak{R}$ is a mapping

$$\boldsymbol{R} : [0,1] \to \mathfrak{R},$$

where $\boldsymbol{R}(i)$ is voter i's preference ordering. Hence, profile $\boldsymbol{R}$ is a specification of the preferences of all voters.

We shall use Lebesgue measure m as our measure of the size of voting blocs. Given alternatives x and y and profile $\boldsymbol{R}$, let

$$q_{\boldsymbol{R}}(x, y) = m\{i \mid x\boldsymbol{R}(i)\, y\}.$$

Then $q_{\boldsymbol{R}}(x, y)$ is the fraction of the population preferring x to y in profile $\boldsymbol{R}$.

Let $\mathscr{C}$ be the set of complete, binary relations (not necessarily transitive or strict) on X. A *voting rule F* is a mapping that, for each profile $\boldsymbol{R}$, assigns a relation $F(\boldsymbol{R}) \in \mathscr{C}$. $F(\boldsymbol{R})$ can be interpreted as the "social preference relation" corresponding to $\boldsymbol{R}$ under F. More specifically, for any profile $\boldsymbol{R}$ and any alternatives $x, y \in X$, the notation "$xF(\boldsymbol{R})\, y$" denotes that x is socially weakly preferred to y under $F(\boldsymbol{R})$. This means that if y is chosen and x is also available, then x is chosen too.

For example, suppose that F^m is *simple majority rule*. Then,

$$xF^m(\boldsymbol{R})\, y \quad \text{if and only if} \quad q_{\boldsymbol{R}}(x, y) \geq q_{\boldsymbol{R}}(y, x).$$

As another example, consider *rank-order voting*. Given (strict) ordering R of X, let $v_R(x)$ be m if x is the top-ranked alternative of R, $m-1$ if x is second-ranked, and so on. That is, a voter with preference ordering R assigns m points to her favorite alternative, $m-1$ points to her next favorite, etc. Thus, given profile $\boldsymbol{R}$, $\int_0^1 v_{\boldsymbol{R}(i)}(x)\,dm(i)$ is alternative x's rank-order score (the total number of points assigned to x) or *Borda count*. If F^{RO} is *rank-order voting*, then

$$xF^{RO}(\boldsymbol{R})\,y \text{ if and only if } \int_0^1 v_{\boldsymbol{R}(i)}(x)\,dm(i) \geq \int_0^1 v_{\boldsymbol{R}(i)}(y)\,dm(i).$$

The notation "$\sim xF(\boldsymbol{R})\,y$" denotes that x is not socially weakly preferred to y, given F and $\boldsymbol{R}$. Hence, if $xF(\boldsymbol{R})\,y$ and $\sim yF(\boldsymbol{R})\,x$, we shall say that x is socially *strictly* preferred to y under $F(\boldsymbol{R})$, which we will usually denote by

$$\frac{F(\boldsymbol{R})}{\begin{matrix} x \\ y \end{matrix}}\ .$$

And if both $xF(\boldsymbol{R})\,y$ and $yF(\boldsymbol{R})\,x$, we shall say that x is socially *indifferent* to y and denote this by

$$\frac{F(\boldsymbol{R})}{x-y}.$$

3. The Properties

We are interested in four standard properties that one may wish a voting rule to satisfy.

Pareto Property on $\mathfrak{R}$: For all $\boldsymbol{R}$ on $\mathfrak{R}$ and all $x, y \in X$, if, for all i, $x\boldsymbol{R}(i)\,y$, then $xF(\boldsymbol{R})\,y$ and $\sim yF(\boldsymbol{R})\,x$, i.e.,

$$\frac{F(\boldsymbol{R})}{\substack{x \\ y}}.$$

In words, the Pareto property requires that if all voters prefer x to y, then society should also (strictly) prefer x to y. Virtually all voting rules used in practice satisfy this property. In particular, majority rule and rank-order voting satisfy it on the unrestricted domain $\Re_X$.

Anonymity on $\Re$: Suppose that $p : [0,1] \to [0,1]$ is a measure-preserving permutation of $[0,1]$ (by "measure-preserving" we mean that, for all $T \subset [0,1]$, $m(T) = m(p(T))$). If, for all $\boldsymbol{R}$, $\boldsymbol{R}^p$ is the profile such that $\boldsymbol{R}^p(i) = \boldsymbol{R}(p(i))$ for all i, then $F(\boldsymbol{R}^p) = F(\boldsymbol{R})$.

In words, anonymity says that social preferences should depend only on the distribution of voters' preferences and not on who has those preferences. Thus if we permute the assignment of voters' preferences by p, social preferences should remain the same. The reason for requiring that p be measure-preserving is to ensure that the fraction of voters preferring x to y be the same for $\boldsymbol{R}^p$ as it is for $\boldsymbol{R}$.

Anonymity embodies the principle that everybody's vote should count equally. It is obviously satisfied on $\Re_X$ by both majority rule and rank-order voting.

Neutrality on $\Re$: For all profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ on $\Re$ and all alternatives *x, y, w, z*, if

$$x\boldsymbol{R}(i)\,y \text{ if and only if } w\boldsymbol{R}'(i)\,z \text{ for all } i$$

then

$$xF(\boldsymbol{R})\,y \text{ if and only if } wF(\boldsymbol{R}')\,y$$

and

$$yF(\boldsymbol{R})x \text{ if and only if } yF(\boldsymbol{R})x.$$

In words, neutrality requires that the social preference between x and y should depend only on the proportions of voters preferring x and preferring y, and not on what the alternatives x and y actually are.

As noted in the introduction, this (standard) version of neutrality embodies independence of irrelevant alternatives, the principle that the social preference between x and y should depend only on voters' preferences between x and y, and not on preferences entailing any other alternative:

Independence of Irrelevant Alternatives (IIA) *on* $\mathfrak{R}$: For all profiles $\boldsymbol{R}$ and $\boldsymbol{R}'$ on $\mathfrak{R}$ and all alternatives x and y, if

$$x\boldsymbol{R}(i)\,y \text{ if and only if } x\boldsymbol{R}'(i)\,y \text{ for all } i,$$

then

$$xF(\boldsymbol{R})\,y \text{ if and only if } xF(\boldsymbol{R}')\,y.$$

Clearly, majority rule satisfies neutrality on the unrestricted domain $\mathfrak{R}_X$. Rank-order voting violates neutrality on $\mathfrak{R}_X$ because, as is well known, it violates IIA on that domain. However, it satisfies neutrality on any domain $\mathfrak{R}$ on which "quasi-agreement" holds.

Quasi-agreement on $\mathfrak{R}$: Within each triple $\{x, y, z\} \subseteq X$, there exists an alternative, say x, such that either (a) for all $R \in \mathfrak{R}$, xRy and xRz; or (b) for all $R \in \mathfrak{R}$, yRx and zRx; or (c) for all $R \in \mathfrak{R}$, either $yRxRz$ or $zRxRy$.

In other words, quasi-agreement holds on domain $\Re$ if, for any triple $\{x, y, z\}$, anybody with preference in $\Re$ either agrees, that, say, *x* is best among the triple, or that *x* is worst, or that *x* is in the middle.

Lemma 1: F^{RO} satisfies neutrality on $\Re$ if and only quasi-agreement holds on $\Re$.

<u>Proof</u>**:** See appendix.

An ordering *R* is *transitive* if for all $x, y, z \in X$, xRy and yRz imply that xRz. Transitivity demands that if *x* is weakly preferred to *y* and *y* is weakly preferred to *z*, then *x* should be weakly preferred to z.

Transitivity on $\Re$: $F(\boldsymbol{R})$ is transitive for all profiles $\boldsymbol{R}$ on $\Re$.

For our results on majority rule we will, in fact, not require transitivity for *all* profiles in $\Re$ but only for *almost* all. To motivate this weaker requirement, let us first observe that, as mentioned in the introduction, single-peaked preferences do not guarantee that majority rule is transitive for all profiles. Specifically, suppose that $x < y < z$ and consider the profile

$$\begin{array}{cc} \underline{[0,\frac{1}{2})} & \underline{[\frac{1}{2},1]} \\ x & y \\ y & z \\ z & x \end{array}$$

That is, we are supposing that half the voters (those from 0 to $\frac{1}{2}$) prefer *x* to *y* to *z* and that the other half (those from $\frac{1}{2}$ to 1) prefer *y* to *z* to *x*. Note that these preferences are certainly single-peaked relative to the linear arrangement, $x < y < z$. However, the social preference relation under majority rule for this profile is not transitive: *x* is socially indifferent to *y*, *y* is socially strictly preferred to *z*, yet *z* is socially indifferent to *x*. We can denote the relation by:

$$\begin{array}{l} x - y \\ \quad z - x \ . \end{array}$$

Nevertheless, this intransitivity is a knife-edge phenomenon - - it requires that *exactly* as many voters prefer x to y as y to x, and *exactly* as many prefer x to z as prefer z to x. Thus, there is good reason for us to "overlook" it as pathological or *irregular.* And, because we are working with a continuum of voters, there is a formal way in which we can do so, as follows.

Let S be a subset of (0, 1). A profile $\boldsymbol{R}$ on $\Re$ is *regular* with respect to S (which we call an *exceptional set*) if, for all alternatives x and y,

$$q_{\boldsymbol{R}}(x, y) \notin S.$$

That is, a regular profile is one for which the proportions of voters preferring one alternative to another all fall outside the specified exceptional set.

Generic Transitivity on $\Re$: There exists a *finite* exceptional set S such that, for all profiles $\boldsymbol{R}$ on $\Re$ that are regular with respect to S, $F(\boldsymbol{R})$ is transitive.

In other words, generic transitivity requires only that social preferences be transitive for regular profiles, ones where the preference proportions do not fall into some finite exceptional set. For example, majority rule is generically transitive on a domain of single-peaked preferences because if the exceptional set consists of the single point $\frac{1}{2}$— i.e., $S = \{\frac{1}{2}\}$—social preferences are then transitive for all regular profiles.

In view of the Condorcet paradox, majority rule is not generically transitive on domain $\Re_X$. By contrast, rank-order voting is not only generically transitive on $\Re_X$ but fully transitive (i.e., generically transitive with exceptional set $S = f$).

We shall say that a voting rule *works well* on a domain $\mathfrak{R}$ if it satisfies the Pareto property, anonymity, neutrality, and generic transitivity on that domain. Thus, in view of our previous discussion, majority rule works well on a domain of single-peaked preferences, whereas rank-order voting works well on a domain with quasi-agreement.

4. Generic Transitivity and Majority Rule

We will show below (Theorem 1) that majority rule works well on or more domains than (essentially) any other voting rule. To establish this result, it will be useful to have a characterization of precisely when majority rule works well, which amounts to asking when majority rule is generically transitive. We have already seen in the previous section that a single-peaked domain ensures generic transitivity. And we noted in the introduction that the same is true when the domain satisfies limited agreement. But single-peakedness and limited agreement are only sufficient conditions for generic transitivity; what we want is a condition that is both sufficient and necessary.

To obtain that condition, note that, for any three alternatives *x*, *y*, *z*, there are six logically possible strict orderings, which can be sorted into two Condorcet "cycles"[8]:

$$\begin{array}{ccc|ccc} x & y & z & x & z & y \\ y & z & x & z & y & x \\ z & x & y & y & x & z \end{array}$$

cycle 1 cycle 2

We shall say that a domain $\mathfrak{R}$ satisfies the *no-Condorcet-cycle* property [9] if it contains no Condorcet cycles. That is, for each triple of alternatives at least one ordering is missing

[8] We call these *Condorcet* cycles because they constitute preferences that give rise to the Condorcet paradox

[9] Sen and Pattanaik (1969) refer to this condition as *extremal restriction.*

from each of cycles 1 and 2 (more precisely for each triple $\{x, y, z\}$, there do not exist orderings $\{R, R', R''\}$ in $\Re$ that, when restricted to $\{x, y, z\}$, generate cycle 1 or cycle 2).

Lemma 2: Majority rule is generically transitive on domain $\Re$ if and only if $\Re$ satisfies the no-Condorcet-cycle property.[10]

Proof**:** If there existed a Condorcet cycle in $\Re$, then we could reproduce the Condorcet paradox. Hence, the no-Condorcet-cycle property is clearly necessary.

To show that it is sufficient, we must demonstrate, in effect, that the Condorcet paradox is the *only* thing that can interfere with majority rule's generic transitivity. To do this, let us suppose that F^m is not generically transitive on domain $\Re$. Then, in particular, if we let $S = \{\frac{1}{2}\}$ there must exist a profile $\boldsymbol{R}$ on $\Re$ that is regular with respect to $\{\frac{1}{2}\}$ but for which $F^m(\boldsymbol{R})$ is *intransitive*. That is, there exist $x, y, z \in X$ such that $xF^m(\boldsymbol{R})yF^m(\boldsymbol{R})zF^m(\boldsymbol{R})x$, with at least one strict preference. But because $\boldsymbol{R}$ is regular with respect to $\{\frac{1}{2}\}$, $xF^m(\boldsymbol{R})y$ implies that

(1) $$q_{\boldsymbol{R}}(x, y) > \tfrac{1}{2},$$

that is, over half the voters prefer *x* to *y*. Similarly, $yF^m(\boldsymbol{R})z$ implies that

(2) $$q_{\boldsymbol{R}}(y, z) > \tfrac{1}{2},$$

meaning that over half the voters prefer *y* to *z*. Combining (1) and (2), we conclude that there must be some voters in $\boldsymbol{R}$ who prefer *x* to *y* to *z*, i.e.,

(3) $$\begin{matrix} x \\ y \\ z \end{matrix} \in \Re.$$ [11]

[10] For the case of an odd and finite number of voters, Sen and Pattanaik (1969) establish that the no-Condorcet-cycle property is necessary and sufficient for majority rule to be transitive.

By similar argument, it follows that

$$\begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \in \Re .$$

Hence, $\Re$ contains a Condorcet cycle, as was to be shown.

Q.E.D.

5. The Robustness of Majority Rule

To establish our main finding about majority rule, we need one final concept. Given profile $\boldsymbol{R}$ and ordering R let $w(R;\boldsymbol{R}) = U\{i \mid \boldsymbol{R}(i) = R\}$. That is, $w(R;\boldsymbol{R})$ is the proportion of voters in profile $\boldsymbol{R}$ with ordering R. For $e > 0$, let the e-neighborhood of $\boldsymbol{R}$—which we will denote by $N_e(\boldsymbol{R})$—be the set of profiles $\boldsymbol{R}'$ whose proportions differ from those of $\boldsymbol{R}$ by less than e:

$$N_e(\boldsymbol{R}) = \{\boldsymbol{R}' \mid |w(R;\boldsymbol{R}) - w(R;\boldsymbol{R}')| < e \text{ for all } R\}.$$

Theorem 1: Suppose that voting rule F works well on domain $\Re$. Then, majority rule F^m works well on $\Re$ too. Conversely, suppose that F differs from F^m on some open neighborhood. More precisely, assume that there exist a profile $\boldsymbol{R}^*$ (on a domain $\Re^m$ on which F^m works well) and $e > 0$ such that

(4) $$F(\boldsymbol{R}) \neq F^m(\boldsymbol{R}) \text{ for all } \boldsymbol{R} \in N_e(\boldsymbol{R}^*).$$

Then, there exists a domain $\Re'$ on which F^m works well, but F does not.

Remark: Without the requirement that $\boldsymbol{R}^*$ belong to a domain on which majority rule works well, the converse assertion above would be false. In particular, consider a voting

[11] To be precise, formula (3) says that there exists an ordering in $\Re$ in which x is preferred to y and y is preferred to z. However, because F^m satisfies IIA we can ignore the alternatives other than x, y, z.

rule that coincides with majority except for profiles that contain a Condorcet cycle. It is easy to see that such a rule works well on any domain for which majority rule does.

<u>Proof</u>**:** Suppose first that F works well on $\Re$. If, contrary to the theorem, F^m does not work well on $\Re$, then, from Lemma 2, there exists a Condorcet cycle in $\Re$:

$$\begin{matrix} x \\ y \\ z \end{matrix}, \begin{matrix} y \\ z \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix} \in \Re . \tag{5}$$

Let S be the exceptional set for F on $\Re$. Because S is finite (by assumption), we can find an integer n such that, if we divide the population into n equal groups, any profile for which all the voters in each particular group have the same ordering in $\Re$ must be regular with respect to S.

Let $\left[0,\tfrac{1}{n}\right]$ be group 1, $\left(\tfrac{1}{n},\tfrac{2}{n}\right]$ be group 2, …, and $\left(\tfrac{n-1}{n},1\right]$ be group n. Consider a profile $\boldsymbol{R}_1$ on $\Re$ such that all voters in group 1 prefer y to x and all voters in the other groups prefer x to y. That is, the profile is

$$\begin{array}{cccc} \underline{1} & \underline{2} & \cdots & \underline{n} \\ y & x & & x \\ x & y & & y \end{array} . \tag{7}$$

From (5), such a profile exists on $\Re$. From neutrality (implying IIA), the social preferences $F(\boldsymbol{R}_1)$ do not depend on voter's preferences over other alternatives.

There are three cases either (i) x is socially strictly preferred to y under $F(\boldsymbol{R}_1)$; (ii) x is socially indifferent to y under $F(\boldsymbol{R}_1)$; or (iii) y is socially strictly preferred to x under $F(\boldsymbol{R}_1)$.

Case (i): $\begin{array}{c} \underline{F(\boldsymbol{R}_1)} \\ x \\ y \end{array}$

Consider a profile $\boldsymbol{R}_1^*$ on $\mathfrak{R}$ in which all voters in group 1 prefer *x* to *y* to *z*; all voters in group *z* prefer *y* to *z* to *x*; and all voters in the remaining groups prefer *z* to *x* to *y*. That is,

[12]

(8)
$$\boldsymbol{R}_1^* = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{array}$$

Notice that, in profile $\boldsymbol{R}_1^*$, voters in group 1 prefer *x* to *z* and that all other voters prefer *z* to *x*. Hence, neutrality and the case (*i*) hypothesis imply that *z* must be socially strictly preferred to *x* under $F\left(\boldsymbol{R}_1^*\right)$, i.e.,

(9)
$$\begin{array}{c} \underline{F\left(\boldsymbol{R}_1^*\right)} \\ z \\ x \end{array}$$

Observe also that, in $\boldsymbol{R}_1^*$, voters in group 2 prefer *y* to *x* and all other voters prefer *x* to *y*. Hence from anonymity and neutrality and the case (*i*) hypothesis, we conclude that *x* must be socially strictly preferred to *y* under $F\left(\boldsymbol{R}_1^*\right)$, i.e.,

(10)
$$\begin{array}{c} \underline{F\left(\boldsymbol{R}_1^*\right)} \\ x \\ y \end{array}.$$

Now (9), (10), and generic transitivity imply that *z* is socially strictly preferred to *y* under $F\left(\boldsymbol{R}_1^*\right)$, i.e.,

(11)
$$\begin{array}{c} \underline{F\left(\boldsymbol{R}_1^*\right)} \\ y \\ z \end{array}$$

[12] This is not quite right because we are not specifying how voters rank alternatives other than *x*, *y*, and *z*. But from IIA, these other alternatives do not matter.

But (8), (11), and neutrality imply for any profile such that

$$\begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ y & y & z & & z \\ z & z & y & & y \end{array},$$

z must be socially strictly preferred to y. Hence, from neutrality, for any profile $\boldsymbol{R}_2$ on $\mathfrak{R}$ such that

(12)
$$\begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ y & y & x & & x \\ x & x & y & & y \end{array},$$

x must be socially strictly preferred to y, i.e.,

(13)
$$\begin{array}{c} \underline{F(\boldsymbol{R}_2)} \\ x \\ y \end{array}$$

That is, we have shown that if x is socially strictly preferred to y when just *one* out of n groups prefers y to x (as in (7)), then x is again socially strictly preferred to y when *two* groups out of n prefer y to x (as in (12)).

Now choose $\boldsymbol{R}_2^*$ on $\mathfrak{R}$ so that

(14)
$$\boldsymbol{R}_2^* = \begin{array}{cccccc} \underline{1} & \underline{2} & \underline{3} & \underline{4} & \cdots & \underline{n} \\ x & y & y & z & & z \\ y & z & z & x & & x \\ z & x & x & y & & y \end{array}.$$

Arguing as above, we can use (12) – (14) to show that x is socially strictly preferred to y if *three* groups out of n prefer y to x. Continuing iteratively, we conclude that x is socially preferred to y even if $n-1$ groups out of n prefer y to x, which, in view of neutrality, violates the case (i) hypothesis. Hence case (i) is impossible.

Case (ii):
$$\begin{array}{c} \underline{F(\boldsymbol{R}_1)} \\ y \\ x \end{array}$$

But from the case (*i*) argument, case (*ii*) leads to the same contradiction as before. Hence we are left with

Case (*iii*): $\begin{array}{c} F(\boldsymbol{R}_1) \\ \hline x - y \end{array}$

Consider a profile $\hat{\boldsymbol{R}}$ on $\Re$ such that

$$\hat{\boldsymbol{R}} = \begin{array}{cccc} 1 & \cdots & n-1 & n \\ \hline x & & x & y \\ y & & y & z \\ z & & z & x \end{array} .$$

From anonymity, neutrality and the case (*iii*) hypothesis, we conclude that *x* is socially indifferent to *y* and *y* is socially indifferent to *z* under $F(\hat{\boldsymbol{R}})$, i.e.,

(15) $$\begin{array}{c} F(\hat{\boldsymbol{R}}) \\ \hline x - y \end{array} .$$

and

(16) $$\begin{array}{c} F(\hat{\boldsymbol{R}}) \\ \hline y - z \end{array} .$$

But the Pareto property implies that *y* is socially *strictly* preferred to *z* under $F(\hat{\boldsymbol{R}})$, which together with (15) and (16) contradicts generic transitivity. We conclude that case (*iii*) is impossible too, and so F^m must work well on $\Re$ after all, as claimed.

Turning to the converse, suppose that there exist domain $\Re^m$, profile $\boldsymbol{R}^*$ on $\Re^m$, and $e > 0$ such that F^m works well on $\Re^m$ and (4) holds. From (4), we can partition voters into *n* equal group and assign everyone within a group the same ordering in such a way that the resulting profile $\boldsymbol{R}^\circ$ on $\Re^m$ is regular and $F(\boldsymbol{R}^\circ) \neq F^m(\boldsymbol{R}^\circ)$. Furthermore, because F^m is assumed to work well on $\Re^m$, we can assume that *F* does too (otherwise, we can take $\mathfrak{R}' = \mathfrak{R}^m$). Hence, from neutrality of F, we can assume that $\boldsymbol{R}^\circ$ consists of just two orderings R' and R''. More formally, there exist integers n and k with

(17) $$n - k > k ,$$

alternatives $x, y \in X$ and orderings $R', R'' \in \mathfrak{R}$ with

(18) $$y\, R' x \text{ and } x\, R'' y$$

such that if we take

(19) $$\boldsymbol{R}^\circ = \frac{1}{R'} \cdots \frac{k}{R'} \quad \frac{k+1}{R''} \cdots \frac{n}{R''},$$

then y is weakly socially preferred to x under $F(\boldsymbol{R}^\circ)$, i.e.,

(20) $$y\, F(\boldsymbol{R}^\circ)\, x \quad .$$

Notice that $F(\boldsymbol{R}^\circ) \neq F^m(\boldsymbol{R}^\circ)$.

To give the idea of the proof, let us assume for the time being that F satisfies the Pareto property, anonymity, and neutrality on the *unrestricted* domain $\mathfrak{R}_X$. Consider $z \notin \{x, y\}$ and profile $\boldsymbol{R}^{\circ\circ}$ such that

(21) $$\boldsymbol{R}^{\circ\circ} = \begin{array}{c} 1 \\ \hline z \\ y \\ x \end{array} \cdots \begin{array}{c} k \\ \hline z \\ y \\ x \end{array} \quad \begin{array}{c} k+1 \\ \hline z \\ x \\ y \end{array} \cdots \begin{array}{c} n-k \\ \hline z \\ x \\ y \end{array} \quad \begin{array}{c} n-k+1 \\ \hline x \\ z \\ y \end{array} \cdots \begin{array}{c} n \\ \hline x \\ z \\ y \end{array}.$$ [13]

Then from (18)-(21), anonymity, and neutrality, we have

(22) $$yF(\boldsymbol{R}^{\circ\circ})\, x \text{ and } xF(\boldsymbol{R}^{\circ\circ})\, z.$$

From the Pareto property, we have

[13] We have again left out the alternatives other than x, y, z, which we are entitled to do by IIA. To make matters simple, assume that the orderings of $\boldsymbol{R}^{\circ\circ}$ are all the *same* for these other alternatives. Suppose furthermore that, in these orderings, x, y, z, are each preferred to any alternative not in $\{x, y, z\}$.

(23) $$\frac{F\left(\boldsymbol{R}^{\circ\circ}\right)}{\begin{matrix} z \\ y \end{matrix}}.$$

But, by construction, $\boldsymbol{R}^{\circ\circ}$ is regular with respect to F's exceptional set. Thus, (22) and (23) together imply that F violates generic transitivity on $\Re' = \left\{ \begin{matrix} z \\ y \\ x \end{matrix}, \begin{matrix} z \\ x \\ y \end{matrix}, \begin{matrix} x \\ z \\ y \end{matrix} \right\}$. Yet, from Lemma 2 (see also footnote 11), F^m is generically transitive on $\Re'$, which implies that $\Re'$ is a domain on which F^m works well but F does not. Thus, we are done in the case in which F always satisfies the Pareto property, anonymity and neutrality.

However, if F does not always satisfy these properties, then we can no longer infer (22) from (18)-(21), and so must argue less directly (although we shall still make use of the same basic idea).

Consider R' and R'' of (18). Suppose first that there exists alternative $z \in X$ such that

(24) $$zR'y \qquad \text{and} \qquad zR''x.$$

Let w be the alternative immediately below z in ordering R'. If $w \neq x$, let R'_* be the strict ordering that is identical to R' except that w and z are now *interchanged* (so that $w R'_* z$). By construction of R'_*, the domain $\{R', R'', R''_*\}$ does not contain a Condorcet cycle, and so, from Lemma 2, F^m works well on this domain. Hence, we can assume that F works well on this domain too (otherwise, we are done). Notice that neutrality of F and (20) then imply that if we replace R' by R''_* in profile $\boldsymbol{R}^\circ$ (to obtain profile $\boldsymbol{R}^\circ_*$) we must have

(25) $$yF\left(\boldsymbol{R}^\circ_*\right)x.$$

Now, if w_* is the alternative immediately below z in R'_* and $w_* \neq x$, we can perform the same sort of interchange as above to obtain R'_{**} and $\boldsymbol{R}^{\circ}_{**}$ and so conclude that F^m and F work well on $\{R', R''_*, R''_{**}\}$

and that

(26) $$yF\left(\boldsymbol{R}^{\circ}_{**}\right)x.$$

By such a succession of interchanges, we can, in effect, move z "downward" while still ensuring that F and F^m work well on the corresponding domains and that the counterparts to (20), (25) and (26) holds. The process comes to end, however, once the alternative immediately below z in R' (or R''_*, R''_{**}, etc.) is x. Furthermore, this must happen after finitely many interchanges (since X is finite). Hence, we can assume without loss of generality that $w = x$ (i.e., that x is immediately below z in R').

Let R''' be the strict ordering that is identical to R' except that x and z (which we are assuming are adjacent in R') are now interchanged. From the above argument, F^m works well on $\mathfrak{R}' = \{R', R'', R'''\}$ and we can suppose that F does too. Hence, from the same argument we used for $\boldsymbol{R}^{\circ\circ}$ above, we can conclude that

(27) $$yF\left(\boldsymbol{R}^{\circ\circ}_*\right)x \text{ and } xF\left(\boldsymbol{R}^{\circ\circ}_*\right)$$

and

(28) $$\frac{F\left(\boldsymbol{R}^{\circ\circ}_*\right)}{\begin{matrix} z \\ y \end{matrix}},$$

where $\boldsymbol{R}^{\circ\circ}_*$ is the profile

$$\frac{1}{R'}\cdots\frac{k}{R'} \quad \frac{k+1}{R''}\cdots\frac{n-k}{R''} \quad \frac{n-k+1}{R'''}\cdots\frac{n}{R'''}.$$

But (27) and (28) generate the same contradiction we obtained following (23). Thus, we are done in the case where (24) holds.

Next, suppose that there exists $z \in X$ such that

(29) $$x R' z \qquad \text{and} \qquad y R'' x.$$

But this case is the mirror image of the case where (24) holds. That is, just as in the previous case we generated R''' with

(30) $$x R''' z R''' y$$

through a finite succession of interchanges in which z moves *downwards* in $R^{\boldsymbol{\prime}}$, so we can now generate R''' satisfying (30) through a finite succession of interchanges in which z moves *upwards* in $R^{\boldsymbol{\prime}}$. If we then take $\mathfrak{R}' = \{R', R^{\boldsymbol{\prime}}, R'''\}$, we can furthermore conclude, as when (24) holds, that F^{m} and F work well on $\mathfrak{R}'$. But, paralleling the argument for $\boldsymbol{R}_{*}^{\circ\circ}$, we can show that

$$y F\left(\boldsymbol{R}_{**}^{\circ\circ}\right) z \qquad \text{and} \qquad z F\left(\boldsymbol{R}_{**}^{\circ\circ}\right) y$$

and

$$\begin{array}{c} F\left(\boldsymbol{R}_{**}^{\circ\circ}\right) \\ \hline x \\ z \end{array},$$

where $\boldsymbol{R}_{**}^{\circ\circ}$ is the profile

$$\frac{1}{R'} \cdots \frac{k}{R'} \quad \frac{k+1}{R''} \cdots \frac{n-k}{R''} \quad \frac{n-k+1}{R'''} \cdots \frac{n}{R'''},$$

implying that $F\left(\boldsymbol{R}_{**}^{\circ\circ}\right)$ is intransitive. This contradicts the conclusion that F works well on $\mathfrak{R}'$, and so again we are done.

Finally, suppose that there exists $z \in X$

such that

(31) $$zR'y \quad \text{and} \quad xR''z\,R''y.$$

As in the preceding case, we move *z* upwards in R' through a succession of interchanges. Only *this* time, the process when *z* and *x* are interchanged to generate $\hat{R}'$ such that

(32) $$z\hat{R}''x\hat{R}''y.$$

As in the previous cases, we can conclude that *F* and F^m work well on $\{R', R'', \hat{R}''\}$.

Take $\hat{\boldsymbol{R}}^{\circ\circ}$ such that

$$\hat{\boldsymbol{R}}^{\circ\circ} = \frac{1}{R'} \cdots \frac{k}{R'} \quad \frac{k+1}{\hat{R}''} \cdots \frac{n-k}{\hat{R}''} \quad \frac{n-k+1}{R''} \cdots \frac{n}{R''}.$$

Then, as in the arguments about $\boldsymbol{R}_*^{\circ\circ}$ and $\boldsymbol{R}_{**}^{\circ\circ}$, we infer that $F\left(\hat{\boldsymbol{R}}^{\circ\circ}\right)$ is intransitive, a contradiction of the conclusion that *F* works well on $\{R', R'', \hat{R}''\}$. This completes the proof when (31) holds. The remaining possible cases involving *z* are all repetitions of one or another of the cases already treated.

Q.E.D.

We have already mentioned May's (1952) characterization of majority rule (see footnote 2). In view of Theorem 1, we can provide an alternative characterization. Specifically, call two voting rules *F* and F' *generically the same* on domain $\mathfrak{R}$ if the set of profiles $\boldsymbol{R}$ on $\mathfrak{R}$ for which $F(\boldsymbol{R}) = F'(\boldsymbol{R})$ is open and dense (put another way, we are requiring that if $F(\boldsymbol{R}^*) \neq F'(\boldsymbol{R}^*)$ then there should *not* exist $e > 0$, such that $F(\boldsymbol{R}) \neq F'(\boldsymbol{R})$ for all $R \in N_e(R^*)$). Call *F* *maximally robust* if there exists no voting rule that (*i*) works well on every domain on which F works well and (*ii*) works well on some domain on which F does not work well. Theorem 1 implies:

Corollary**:** Any maximally robust voting rule is generically the same as majority rule.

6. Unanimity Rule

The symmetry inherent in neutrality is often a reasonable and desirable property--we would presumably want to treat all candidates in a presidential election the same. However, there are also many circumstances in which it is natural to favor certain alternatives. The rules for changing the U.S. Constitution are a case in point. They have been deliberately devised so that, at any time, the current version of the Constitution—the status quo—is difficult to revise.

Accordingly, let us relax neutrality and just impose IIA. We will require the following additional weak condition on voting rules:

Tie-break Consistency: Given voting rule F, there exists an ordering R_F such that, for all $x, y \in X$ and all $\boldsymbol{R}$ on $\mathfrak{R}$ for which $q_{\boldsymbol{R}}(x, y) = q_{\boldsymbol{R}}(y, x)$, we have

$xR_F^* y$ if and only if $xF(\boldsymbol{R}^*)\, y$.

Tie-break consistency requires that in situations where the population splits 50-50 between two alternatives, the "tie" be broken (or not broken as the case may be) consistently in the sense that it be done *transitively* (note that, given IIA, the only aspect of the condition that is restrictive is the stipulation that R_F be an *ordering*—which entails transitivity). That is, if x is chosen over y when the population splits between x and y, and y is chosen over z when the population splits between y and z, then x should be chosen over z when the population splits between x and z. Observe that because the likelihood that the population will split *exactly* is very low, tie-break consistency is not a terribly demanding condition.

Let R_* be a strict ordering of *X*. We shall denote *unanimity rule with order of precedence* R_* by $F_{R^*}^U$ and define it so that, for all profiles $\boldsymbol{R}$ and all alternatives *x* and *y*, $xF_{R^*}^U(\boldsymbol{R})\,y$ if and only if either $x\boldsymbol{R}(i)\,y$ for all $i \in [0,1]$ or else there exist *j* such that $x\boldsymbol{R}(j)\,y$ and $xF_{R^*}^U(\boldsymbol{R})\,y$. That is, between *x* and *y*, the alternative earlier in the order of precedence R_* will be chosen unless voters unanimously prefer the other alternative.

$F_{R^*}^U$ can be implemented by the following procedure. Begin with alternative x_1 as the status quo (where $x_1 R_* x_2 \cdots R_* x_m$). At each stage (there are $m-1$ in all), compare the current status quo with the next alternative in the order R_*. If everyone prefers this next alternative, then it becomes the new status quo; otherwise, the old status quo remains in place.

We shall say that a voting rule *works satisfactorily* on a domain $\mathfrak{R}$ if it satisfies the Pareto property, anonymity, IIA, and transitivity on $\mathfrak{R}$.[14]

Just as Lemmas 1 and 2 characterize when rank-order voting and majority rule work well, Lemma 3 tells us when unanimity rule with an order of precedence works satisfactorily:

Lemma 3: Unanimity rule with order of precedence R_* works satisfactorily on domain $\mathfrak{R}$ if and only if, for all triples $\{x, y, z\}$ with

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[14] There is an obvious sense in which to work *satisfactorily* is a less demanding requirement than to work *well*, since the former imposes only IIA rather than the stronger condition, neutrality. Note, however, that working satisfactorily requires *exact* transitivity, whereas working well only *generic* transitivity.

(33) $$\begin{array}{c} R_* \\ \hline x \\ y \\ z \end{array} \quad ,$$

there do not exist both R' and R'' in $\Re$ such that

(34) $$\begin{array}{c} R' \\ \hline y \\ z \\ x \end{array} \qquad \begin{array}{c} R'' \\ \hline z \\ x \\ y \end{array} \quad .$$

Proof: Suppose that, for some triple $\{x, y, z\}$ satisfying (33), there exist R' and R'' in $\Re$ satisfying (34). Consider profile $\hat{\boldsymbol{R}}$ such that

$$\hat{\boldsymbol{R}} = \begin{array}{c} \left[0, \frac{1}{2}\right) \\ \hline y \\ z \\ x \end{array} \qquad \begin{array}{c} \left[\frac{1}{2}, 1\right] \\ \hline z \\ x \\ y \end{array} \quad .$$

Because $xR_* y$ and voters from $\frac{1}{2}$ to 1 prefer x to y,

we have

(35) $$\begin{array}{c} F_{R_*}^U\left(\hat{\boldsymbol{R}}\right) \\ \hline x \\ y \end{array} \quad .$$

Similarly, we have

(36) $$\begin{array}{c} F_{R_*}^U\left(\hat{\boldsymbol{R}}\right) \\ \hline y \\ z \end{array} \quad .$$

But because everyone prefers z to x, we have

$$\begin{array}{c} F_{R_*}^U\left(\hat{\boldsymbol{R}}\right) \\ \hline z \\ x \end{array} \ ,$$

which together with (35) and (36) contradicts transitivity. We conclude that if (34) holds, then a *necessary* condition for $F_{R^*}^U$ to work satisfactorily on $\mathfrak{R}$ is that either R' or R'' be missing from $\mathfrak{R}$.

Conversely, suppose that $F_{R_*}^U$ does *not* work satisfactorily on $\mathfrak{R}$. Because this voting rule always satisfies Pareto, anonymity, and IIA, there must exist $\{x, y, z\}$ satisfying (33) and a profile $\boldsymbol{R}^*$ such that either

(37)
$$\begin{array}{c} F_{R_*}^U\left(\boldsymbol{R}^*\right) \\ \hline x \\ y \\ z \\ x \end{array}$$

or

(38)
$$\begin{array}{c} F_{R_*}^U\left(\boldsymbol{R}^*\right) \\ \hline x \\ z \\ y \\ x \end{array} \quad .$$

Suppose first that (38) holds. Then, from (33), we must have

$$z\boldsymbol{R}^*(i)\,y\boldsymbol{R}^*(i)\,x \text{ for all } i \in [0,1],$$

which contradicts the hypothesis that $xF_{R_*}^U\left(\boldsymbol{R}^*\right)z$. Hence, (37) must hold. Then because, by assumption, xR_*z, we infer that

(39) $$z\boldsymbol{R}^*(i)\,x \text{ for all } i \in [0,1].$$

Because $xF_{R_*}^U\left(\boldsymbol{R}^*\right)y$ and $yF_{R_*}^U\left(\boldsymbol{R}^*\right)z$, there must exist i' and i'' such that

(40) $$x\boldsymbol{R}^*(i')\,y \text{ and } y\boldsymbol{R}^*(i'')\,z.$$

But (39) and (40) imply:

$$\begin{array}{c}\boldsymbol{R}^*(i')\\ \hline z\\ x\\ y\end{array} \quad \text{and} \quad \begin{array}{c}\boldsymbol{R}^*(i'')\\ \hline y\\ z\\ x\end{array}$$

Hence, when (37) holds, that not both R' and R'' belong to $\mathfrak{R}$ is also a *sufficient* condition for $F_{R_*}^U$ to work satisfactorily on $\mathfrak{R}$.

Q.E.D.

We can now establish our second major result:

Theorem 2: Suppose that *F* satisfies tie-break consistency. There exists a strict ordering R_* such that for all domains $\mathfrak{R}$ on which *F* works satisfactorily, $F_{R_*}^U$ works satisfactorily on $\mathfrak{R}$ too. Furthermore, if there exist a domain $\mathfrak{R}^U$ on which $F_{R_*}^U$ works satisfactorily and profile $\boldsymbol{R}$ on $\mathfrak{R}^U$ such that $F(\boldsymbol{R}) \neq F_{R_*}^U(\boldsymbol{R})$, then there exists a domain $\mathfrak{R}'$ on which $F_{R_*}^U$ works satisfactorily but *F* does not.

<u>Proof</u>: Given voting rule *F*, let R_F be the corresponding "tie-break" ordering prescribed by tie-break consistency. Choose a strict ordering R_* consistent with R_F, i.e., let R_* be a strict ordering such that, for all $x, y \in X$

(41) if xR_*y then xR_Fy .

Consider $\{x, y, z\}$ with

(42) $$\begin{array}{c}R_*\\ \hline x\\ y\\ z\end{array}$$

and suppose that *F* works satisfactorily on domain $\mathfrak{R}$. From Lemma 3, $F_{R_*}^U$ works satisfactorily on $\mathfrak{R}$ provided that whenever R' and R'' are two strict orderings such that

(43) $\dfrac{R'}{\begin{matrix} y \\ z \\ x \end{matrix}}$ and $\dfrac{R''}{\begin{matrix} z \\ x \\ y \end{matrix}}$,

then not both R' and R'' can belong to $\mathfrak{R}$. Thus, to establish the first assertion of the Theorem, it suffices to show that if (43) holds, either R' and R'' must be missing from $\mathfrak{R}$.

Suppose to the contrary that $R', R'' \in \mathfrak{R}$. Consider the profile $\hat{\boldsymbol{R}}$ on $\mathfrak{R}$ such that

(44) $$\hat{\boldsymbol{R}} = \frac{\left[0,\tfrac{1}{2}\right)}{R'} \quad \frac{\left[\tfrac{1}{2},1\right]}{R''}$$

From (41) and (42) we have

(45) $$xR_F\, yR_F\, z$$

(although the rankings in (45) may not be strict).

Hence, from (44) and (45), tie-break consistency implies that

(46) $$xF\left(\hat{\boldsymbol{R}}\right) yF\left(\hat{\boldsymbol{R}}\right) z \quad .$$

But because everyone in $\hat{\boldsymbol{R}}$ prefers *z* to *x*, the Pareto property gives us

$$\frac{F\left(\hat{\boldsymbol{R}}\right)}{\begin{matrix} z \\ x \end{matrix}} \quad ,$$

which, together with (46), means that $F\left(\hat{\boldsymbol{R}}\right)$ is not transitive, a contradiction. Thus the first assertion of the theorem is indeed established.

To prove the converse, consider profile $\boldsymbol{R}$ and domain $\mathfrak{R}^U$ such that

(47) $\boldsymbol{R}$ is on $\mathfrak{R}^U$

(48) $F^U_{R_*}$ works satisfactorily on $\mathfrak{R}^U$

and

(49) $$F(\boldsymbol{R}) \neq F^{U}_{R_*}(\boldsymbol{R}) \quad .$$

If there are multiple such $\boldsymbol{R}$ and $\mathfrak{R}^U$ choose a pair $\left(\boldsymbol{R}_F, \mathfrak{R}^U_F\right)$ and alternatives x_F, y_F to solve

(50) $$\max q_{\boldsymbol{R}}(x, y)$$

subject to (47) - (49) and

(51) $$x F^{U}_{R_*}(\boldsymbol{R})\, y \text{ and } y F(\boldsymbol{R})\, x \quad .$$

From (51), we have

(52) $$x_F R_* y_F \quad .$$

Let R_{**} be the *opposite* of R_*, i.e., for all x, y

$$x R_* y \text{ if and only if } y R_{**} x \quad .$$

Let z_F be the alternative just below y_F in ordering R_* (if y_F is the lowest alternative in R_*, the argument is very similar). Let $\overline{R}_*$ be the ordering that coincides with R_* except that y_F and z_F are interchanged. Finally, let $\hat{R}_*$ be the ordering that coincides with R_* except that x_F and y_F are interchanged.

It is a matter of straightforward verification to check that, for all $R \in \left\{R_{**}, \boldsymbol{R}_*, \hat{R}_*\right\}$ and all x, y, z if

$$\begin{array}{c} R_* \\ \hline x \\ y \\ z \end{array} \quad ,$$

then we have neither

(53) $$\begin{array}{c} R \\ \hline y \\ z \\ x \end{array}$$

nor

(54) $$\begin{array}{c} R \\ \hline z \\ x \\ y \end{array} \quad ,$$

which, from Lemma 3, implies that $F^U_{R_*}$ is transitive on $\bar{\Re}^U_F = \Re^U_F \cup \{R_*, R_{**}, \bar{R}_*\}$.

We know, from (41) and (52), that $x_F R_F y_F$. There are two cases.

Case I: $$\begin{array}{c} R_F \\ \hline x_F - y_F \end{array}$$

Because $x_F R_* z_F$, (41) implies that

(55) $$x_F R_F z_F \quad .$$

Consider the profile

$$\boldsymbol{R}^1 = \begin{array}{cc} [0, \frac{1}{2}) & [\frac{1}{2}, 1] \\ \hline R_* & R_{**} \end{array}$$

From (55), we have

(56) $$x_F F\left(\boldsymbol{R}^1\right) z_F \quad .$$

From the Pareto property, we have

$$\begin{array}{c} F\left(\boldsymbol{R}^1\right) \\ \hline z_F \\ y_F \end{array} \quad .$$

Finally, from the Case I hypothesis, we have

(57) $$\begin{array}{c} F\left(\boldsymbol{R}^1\right) \\ \hline x_F - y_F \end{array} \quad .$$

But combining (55) – (57) we conclude that $F\left(\boldsymbol{R}^1\right)$ is intransitive, and so, if Case I holds, we can take $\mathfrak{R}' = \overline{\mathfrak{R}}_F^U$ to complete the proof.

Case II:

$$\begin{array}{c} R_F \\ \hline x_F \\ y_F \end{array}$$

Recall from our choice of $\boldsymbol{R}_F$ that $F\left(\boldsymbol{R}_F\right)$ and $F_{R_*}^U\left(\boldsymbol{R}\right)$ rank x_F and y_F differently. Thus, in view of (52) and the Case II hypothesis, we cannot have $q_{\boldsymbol{R}_F}\left(x_F, y_F\right) = q_{\boldsymbol{R}_F}\left(y_F, x_F\right) = \frac{1}{2}$. We must therefore have either

(58) $$q_{\boldsymbol{R}_F}\left(x_F, y_F\right) > q_{\boldsymbol{R}_F}\left(y_F, x_F\right)$$

or

(59) $$q_{\boldsymbol{R}_F}\left(x_F, y_F\right) < q_{\boldsymbol{R}_F}\left(y_F, x_F\right) \quad .$$

Suppose first that (58) holds. Because $F_{R_*}^U$ works satisfactorily on $\overline{\mathfrak{R}}_{R_*}^U$, we can assume that F does too (otherwise, we can take $\mathfrak{R}' = \overline{\mathfrak{R}}_F^U$ and we are done).

Hence, if $\boldsymbol{R}$ is a profile on $\overline{\mathfrak{R}}_F^U$ such that

(60) $$q_{\boldsymbol{R}}\left(x_F, y_F\right) = q_{\boldsymbol{R}_F}\left(x_F, y_F\right),$$

anonymity and neutrality of F imply that

(61) $$y_F\, F\left(\boldsymbol{R}\right) x_F \quad .$$

Let $\overline{R}_{**}$ be the ordering that coincides with $\mathfrak{R}_{**}$ except that x_F and y_F are interchanged. One can verify mechanically that for all $R \in \left\{R_{**}, \overline{R}_*, \hat{R}_*, \overline{R}_{**}\right\}$ and all x, y, z, if

$$\begin{array}{c} R_* \\ \hline x \\ y \\ z \end{array}$$

then we do not have

(62) $$\begin{array}{c} R \\ \hline y \\ z \\ x \end{array} \quad .$$

Hence, from Lemma 3, $F_{R_*}^U$ works satisfactorily on $\hat{\Re}_{R_*}^U = \left\{R_*, R_{**}, \bar{R}_*, \hat{R}_*, \bar{R}_{**}\right\}$, and so we can assume that the same is true of *F*. Hence, if $\boldsymbol{R}$ is a profile on $\hat{\Re}_{R_*}^U$ satisfying (60), we can infer (61). Consider $\boldsymbol{R}^2$ such that

$$\boldsymbol{R}^2 = \begin{array}{c} \left[0,\frac{1}{2}\right) \\ \hline \bar{R}_* \end{array} \quad \begin{array}{c} \left[\frac{1}{2}, q_{\boldsymbol{R}_F}\left(x_F, y_F\right)\right) \\ \hline \bar{R}_{**} \end{array} \quad \begin{array}{c} \left[q_{\boldsymbol{R}_F}\left(x_F, y_F\right), 1\right] \\ \hline R_{**} \end{array}$$

Because $q_{\boldsymbol{R}^2}\left(x_F, y_F\right) = q_{\boldsymbol{R}_F}\left(x_F, y_F\right)$, the above argument implies that

(63) $$y_F F\left(\boldsymbol{R}^2\right) x_F \quad .$$

From the Pareto property

(64) $$\begin{array}{c} F\left(\boldsymbol{R}^2\right) \\ \hline z_F \\ y_F \end{array} \quad .$$

Furthermore, because $q_{\boldsymbol{R}^2}\left(x_F, z_F\right) = \frac{1}{2}$, (41) and the fact that $x_F R_* z_F$ imply that

(65) $$x_F F\left(\boldsymbol{R}^2\right) z_F \quad .$$

But (63) – (65) contradict the transitivity of $F\left(\boldsymbol{R}^2\right)$, and so we can take $\Re' = \hat{\Re}_{R_*}^U$ when (58) holds.

Finally, assume that (59) holds. If there exists $b < \frac{1}{2}$ and a profile $\boldsymbol{R}$ on $\hat{\mathfrak{R}}^{U}_{R_*}$ such that

(66) $$q_{\boldsymbol{R}}(y_F, z_F) = b$$

and

(67) $$z_F F(\boldsymbol{R})\, y_F \quad ,$$

consider profile $\boldsymbol{R}^3$ such that

$$\boldsymbol{R}^3 = \frac{\left[0, q_{\boldsymbol{R}_F}(x_F, y_F)\right)}{\bar{R}_*} \quad \frac{\left[q_{\boldsymbol{R}_F}(x_F, y_F), q_{\boldsymbol{R}_F}(x_F, y_F) + b\right)}{\hat{R}_*} \quad \frac{\left[q_{\boldsymbol{R}_F}(x_F, y_F) + b, 1\right]}{R_{**}}$$

Because $q_{\boldsymbol{R}^3}(y_F, z_F) = b$, (66) and (67) imply that

(68) $$z_F F(\boldsymbol{R}^3)\, y_F \quad .$$

Because $q_{\boldsymbol{R}^3}(x_F, y_F) = q_{\boldsymbol{R}_F}(x_F, y_F)$, we have

(69) $$y_F F(\boldsymbol{R}^3)\, x_F \quad .$$

Now, $q_{\boldsymbol{R}^3}(x_F, z_F) = q_{\boldsymbol{R}_F}(x_F, y_F) + b$ and so from the choices of x_F, y_F, and $\boldsymbol{R}_F$ and the fact that

$$\frac{F^{U}_{R_*}(\boldsymbol{R}^3)}{\substack{x_F \\ z_F}} \quad ,$$

we must have

(70) $$\frac{F(\boldsymbol{R}^3)}{\substack{x_F \\ z_F}} \quad .$$

But (68) – (70) contradict the transitivity of $F(\boldsymbol{R}^3)$.

Thus assume that, for all $b < \frac{1}{2}$ and profiles $\boldsymbol{R}$ on $\hat{\mathfrak{R}}^U_{R_*}$ with

(71) $$q_{\boldsymbol{R}}(y_F, z_F) = b \quad ,$$

we have

(72) $$y_F F(\boldsymbol{R}) z_F \quad .$$

If there exists $d \in \left(0, \frac{1}{2}\right)$ and profile $\boldsymbol{R}'$ on $\hat{\mathfrak{R}}^U_{R_*}$ such that

(73) $$q_{\boldsymbol{R}'}(x_F, z_F) = d$$

and

(74) $$z_F F(\boldsymbol{R}') x_F ,$$

then consider profile $\boldsymbol{R}^4$ such that

$$\boldsymbol{R}^4 = \frac{[0,d)}{R_*} \quad \frac{[d,1]}{\bar{R}_{**}} \quad .$$

From the Pareto property,

(75) $$\frac{F\left(\boldsymbol{R}^4\right)}{\begin{matrix} x_F \\ y_F \end{matrix}}$$

From (71) and (72), we have

(76) $$y_F F\left(\boldsymbol{R}^4\right) z_F \quad .$$

From (73) and (74), we have

(77) $$z_F F(\boldsymbol{R})^4 x_F \quad .$$

But (75) – (77) contradict the transitivity of $F\left(\boldsymbol{R}^4\right)$. So we conclude that, for all

$d \in \left(0, \frac{1}{2}\right)$, if $\boldsymbol{R}'$ on $\hat{\mathfrak{R}}^U_{R_*}$ satisfies (73),

then

(78)
$$\begin{array}{c} F(\boldsymbol{R}') \\ \hline x_F \\ y_F \end{array} \quad .$$

Finally, consider profile $\boldsymbol{R}^5$ such that

$$\boldsymbol{R}^5 = \begin{array}{c} \left[0, q_{\boldsymbol{R}_F}\left(x_F, y_F\right)\right) \\ \hline \overline{R}_* \end{array} \quad \begin{array}{c} \left[q_{\boldsymbol{R}_F}\left(x_F, y_F\right), 1\right] \\ \hline R_{**} \end{array} \quad .$$

From the Pareto property, we have

(79)
$$\begin{array}{c} F\left(\boldsymbol{R}^5\right) \\ \hline z_F \\ y_F \end{array} \quad .$$

Because $q_{\boldsymbol{R}^5}\left(x_F, y_F\right) = q_{\boldsymbol{R}_F}\left(x_F, y_F\right)$, we have

(80)
$$y_F F\left(\boldsymbol{R}^5\right) x_F \quad .$$

Finally, because $q_{\boldsymbol{R}^5}\left(x_F, z_F\right) < \frac{1}{2}$, (73) and (75) imply

(81)
$$\begin{array}{c} F\left(\boldsymbol{R}^5\right) \\ \hline x_F \\ z_F \end{array} \quad .$$

Now, (79) – (81) contradict the transitivity of $F\left(\boldsymbol{R}^5\right)$, and so we can take $\mathfrak{R}' = \hat{\mathfrak{R}}^U_{R_*}$.

Q.E.D.

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