Multidimensional Voting under Uncertainty*

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Abstract

The nonexistence of equilibria in platform setting games with mutliple issues is one of the more puzzling results in political economics. In this paper we relax the stardard assumption that parties either have perfect information about the electorate or that they behave as expected utility maximizers. We show that equilibria often exist when parties are instead uncertainty averse. What is more, these equilibria can be characterized as a straightforward generalization of the classical median voter result.

1 Introduction

The famous median voter theorem states that in a platform positioning game with a unidimensional issue space played by two vote-share-maximizing parties and a set of voters with single peaked preferences, both parties will announce the policy preferred by the median voter in equilibrium. The assumption that the political spectrum is unidimensional is crucial. Most results for multidimensional games of this sort show that equilibria exist only if the distribution of voters satisfies very strong conditions (Plott (1967), Davis, Hinich and de Groot (1972) and Grandmont (1978)). This is problematic since governments in real existing democracies do decide on many different issues that can in general not be aligned perfectly on a unidimensional spectrum (say left to right).

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In this paper we propose to solve these nonexistence problems by introducing uncertainty aversion into the standard model of multidimensional political competition. We assume that parties are uncertain about the preferences of voters. Consequently any party is uncertain about the vote share it would receive, when announcing some platform x while the other party announces platform y. For the parties it is difficult to predict of the magnitude of the sets of votes for either platform. The Parties are facing a situation of subjective uncertainty. And it is known that in the face of such uncertainty, decision makers do not necessarily act as expected utility maximizers, they exhibit uncertainty aversion. We, therefore, consider it reasonable to model the parties as uncertainty averse. The standard cases of certain parties and expected utility maximizing parties, arise as special cases of the framework proposed here.

Given this alternative assumption on the parties preferences over uncertain outcomes we are able to obtain equilibria for a large range of games played by parties with whose objective is to maximize their vote shares. What is more these equilibria can be characterized by a straightforward extension of the median voter theorem: in equilibrium both parties will propose the policy preferred by the median voter of every dimension. So we give a rigorous justification to the common - but as of yet theoretically unfounded - practice to apply the median voter theorem to isolated issues: If sufficiently much uncertainty prevails, two parties will propose, say, the tax policy preferred by the respective median voter, independently of the voters preferences over the other issues at stake.

2 Political Competition

We model political competition as a two stage game played by two different types of actors, two political parties and a large set of voters. First the two parties simultaneously choose their platforms within some (non-empty) n-dimensional convex issue space $X \subset \mathbb{R}^n$, $n \geq 1$. Then the voters, whose preferences are defined over that same issue space X, cast their votes.

2.1 The Voters

Throughout this paper we assume that each voter's preferences can be represented by a utility function $u_a^g : X \to \mathbb{R}$ with

$$u_a^g(x) := -\sum_{i=1}^{n} g_i(|x_i - a_i|) \text{ for all } x \in X$$

where $a \in X$, $g_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function for all $1 \leq i \leq n$, and $g := (g_i)_{i=1}^n$. We normalize $g_i(0) = 0$ for all i. Since $u_a^g(x)$ is maximized at $x = a$ we call the vector $a \in X$ voter's **ideal point**. Observe that the functions $u_a^g(., x_{-i})$ are single peaked for any

$x_{-i} \in X_{-i}$. In fact any preference relation that can be represented by a function u_a^g is single peaked according the most common notions of single peakedness for multidimensional issue spaces (c.f. Barbera, Gul and Stachetti (1993) and Roemer (2000)). The vector of functions $g = (g_i)_{i=1}^n$ determines the shape of a voter's indifference curves. We say that a voter with a utility u_a^g is of **type** g. We say that the function g_i represents the **attitude** of a voter towards issue i. The indifference curves of two voters of the same type are translations of each other. Also observe that the standard models in which a voters disutility of a platform is the Euclidean distance between his ideal point and that platform is a special case of the present model (the standard model obtains upon setting $g_i(t) := (t)^2$ for all $t \in \mathbb{R}$ and $1 \leq i \leq n$, we denote this type by g°.) The type g for which $g_i(t) := \alpha_i t$ for some $\alpha_i > 0$ for all $1 \leq i \leq n$ will play a major role in this paper. We denote such a type by the vector $\alpha := (\alpha_1, ..., \alpha_n)$. The indifference curves of a voter of this type are diamond shaped.

So any **voter** is characterized by a pair

$$(a, g) \in X \times H.$$

where X is the issue space and H is some finite set of types $H = \{g^1, ..., g^m\}$ an $g^i = (g_1^i, ...g_n^i)$ and $g_j^i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ strictly increasing for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Consequently the entire electorate can be described fully as a set of ideal-point-type-tupels (a, g). The **electorate** is represented by a distribution[1] of voters (a, g)

$$\psi \in \mathbb{P}(X \times H).$$

Example 1: We have 5 voters in a world where only two issues matter, external policy and taxation. We model the issue space X as the set $[0,1]^2$. There are two voters with ideal point $(\frac{1}{2}, \frac{1}{2})$ and type $(1,1)$. There is one voter with ideal point $(0,0)$ and circular indifference curves, another two voters have the same ideal point $(\frac{2}{3}, \frac{1}{3})$ but different types, one is of type $(1,1)$ the other has circular indifference curves. This electorate can be described by the distribution $\psi \in \mathbb{P}([0,1]^2 \times \{g^\circ, (1,1)\})$ with $\psi((\frac{1}{2}, \frac{1}{2}), (1,1)) = \frac{2}{5}$, $\psi((0,0), g^\circ) = \psi((\frac{2}{3}, \frac{1}{3}), (1,1)) = \psi((\frac{2}{3}, \frac{1}{3}), g^\circ) = \frac{1}{5}$.

We are now ready to calculate the vote share of a party given an electorate $\psi \in \mathbb{P}(X \times H)$ and a platform profile $(x, y) \in X \times X$. To do so we define an indicator function $I(\alpha)$ that

[1] We have a metric $d_{\mathbb{R}^n \times H}$ on $\mathbb{R}^n \times H$ with $d_{\mathbb{R}^n \times H}((a, g), (a', g')) := d_{\mathbb{R}^n}(a, a') + d_H(g, g')$ with $d_{\mathbb{R}^n}$ the Euclidean metric on $\mathbb{R}^n$ and $d_H(g, g') = 1$ for $g \neq g'$ and 0 otherwise. The set $X \times H$ is a compact subset of $\mathbb{R}^n \times H$. The Borel $\sigma-$algebra on $\mathbb{R}^n \times H$ induced by the metric $d_{\mathbb{R}^n \times H}$ is equal to the product $\sigma-$algebra on $\mathbb{R}^n \times H$ when $\mathbb{R}^n$ is endowed with the Borel $\sigma-$Algebra and H by the algebra of all subsets. So the set $\mathbb{P}(X \times H)$ is welldefined.

assumes the value 1 when α holds true and 0 otherwise, and we denote the preferences of a voter (a, g) by $\succsim_a^g$. According to our assumption that voters base their decision only on the platforms of the parties all voters in the set $\{(a,g)|I(x \succ_a^g y) = 1\}$, that is all voters that strictly prefer platform x to y, and one half of the indifferent voters, the voters in the set $\{(a,g)|I(x \sim_a^g y) = 1\}$ vote for x. So we define the **vote share (of the party announcing x given that the other party announces y and given the electorate ψ)** as a function $\pi_\psi : X \times X \rightarrow [0, 1]$ where

$$\pi_\psi(x,y) := \int\limits_{X \times H} (I(x \succ_a^g y) + \frac{1}{2} I(x \sim_a^g y))\psi(d(a,g)).$$

The vote share is defined as the mass of all voters strictly preferring x to y plus one half the mass of all indifferent voters.[2]

Example 1 again: Take the electorate defined above, and suppose the two parties propose the platforms $x = (0, \frac{1}{3})$ and $y = (\frac{7}{8}, 0)$ then we have that $\pi_\psi(x,y) = \psi((\frac{1}{2}, \frac{1}{2}), (1,1)) + \psi((0,0), g^\circ) = \frac{3}{5}$.

3 The Parties

The goal of each party is to maximize its vote share. A party's strategy variable is its platform, the issue space X is its strategy space. If the electorate where known, the objective of a party would simply be to maximize $\pi_\psi(x, y)$. The parties of our model do, however, not know the electorate, they are uncertain about the preferences of the voters. The innovation of this paper is to to assume that parties are uncertainty averse.

A range of different approaches have been proposed to model uncertainty averse actors (c.f. Bewley (1986), Gilboa Schmeidler (1989) and Schmeidler (1989)). In this paper we represent the preferences of parties following the modelling approach of Gilboa Schmeidler (1989). In the spirit of this approach we assume that the preferences of parties that are uncertain about the electorate can be represented by $\Pi_\Phi : X \times X \rightarrow \mathbb{R}$ with

$$\Pi_\Phi(x,y) := \min_{\phi \in \Phi} \int\limits_{\mathbb{P}(X \times H)} \pi_\psi(x,y)\phi(d\psi))$$

where $\Phi \subset \mathbb{P}(\mathbb{P}(X \times H))^3$ is a convex and compact set of priors on the electorate ψ.

[2] Observe that the sets $\{(a,g)|I(x \succ_a^g y) = 1\}$ and $\{(a,g)|I(x \sim_a^g y) = 1\}$ are both measurable, as g_j^i strictly increasing for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus the function π_ψ is welldefined for all ψ.

[3] The space $X \times H$ is a compact subset of a metrizable space. So under the under the Prokhorov metric $d_p : \mathbb{P}(X \times H) \times \mathbb{P}(X \times H) \rightarrow \mathbb{R}$ the space $\mathbb{P}(X \times H)$ is itself metrizable and compact and we are able to

The set of all electorates $\mathbb{P}(X \times H)$ represents the set of all states in this context. Given our assumption that parties behave a s vote share maximizers the function $\psi \mapsto \pi_\psi(x,y)$ represents a the utility of a party in each state. A generic element of the set $\mathbb{P}(\mathbb{P}(X \times H))$ is called a belief or prior on the electorate, it is a probability distribution on the set of all electorates (or states) $\mathbb{P}(X \times H)$. The main non-standard axiom in the approach of Gilboa and Schmeidler approach is that when the agent is indifferent between two horse race lotteries he must like and mix of those two at least as good as either one of them. In our context this means that if two parties are indifferent between offering the same platform profile to one unknown electorate or another, they should be at least as well off when offering that same platform profile to a mix between these two unknown electorates.

Observe that the case of expected vote share maximizing parties arises as a special case of this formulation of party preferences. If the set of priors Φ is a singleton $\{\phi'\}$ we have that the objective of any party is to maximize its expected vote share, in this case the utility of a party reduces to

$$\Pi_\Phi(x,y) = \int\limits_{\mathbb{P}(X\times H)} \pi_\psi(x,y)\phi'(d\psi))$$

It turns out that there exists a much simpler representation of these preferences:

Proposition 1: Let the preferences of a party be represented by $\Pi_\Phi(x,y)$. Then we have

$$\Pi_\Phi(x,y) = \min_{\psi\in\Psi} \pi_\psi(x,y).$$

for $\Psi := \{\psi' : \psi' = \int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi) \text{ for some } \phi \in \Phi\}$ a convex subset of $\mathbb{P}(X \times H)$.

The proof of this statement is lengthy and we relegate it to the appendix. However, the main difficulty of the proof lies in the fact that the support of the distributions ψ and ϕ can be large. To gain some intuition on the mechanics of the proof, we demonstrate the gist of Proposition 1 here for the case that the party only considers a finite number of electorates, and that every of one these electorates consists of a finite number of voters. In this case there exists a finite subset $\{\psi_1, ...\psi_{n_\psi}\}$ of $\mathbb{P}(X \times H)$ such that $\text{supp}\phi \subset \{\psi_1, ...\psi_{n_\psi}\}$ for all $\phi \in \Phi$. And there exits a finite subset $\{(a^1, g^1), ..., (a^{n_v}, g^{n_v})\} \in X \times H$ such that $\text{supp}\psi_l \subset \{(a^1, g^1), ..., (a^{n_v}, g^{n_v})\}$ for all $l = 1, ..., n_\psi$. We show that $\int\limits_{\mathbb{P}(X\times H)} \pi_\psi(x,y)\phi(d\psi)) = \pi_{\psi'}(x,y)$ with $\psi' := \int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi)$ for all $\phi \in \Phi$. In this finite case we have $\int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi) = \sum_{l=1}^{n_\psi} \psi_l \phi(\psi_l)$.

define probabilty distributions on the space of all electorates. That is the space $\mathbb{P}(\mathbb{P}(X \times H))$ is welldefined.

$$
\begin{aligned}
\int\limits_{\mathbb{P}(X\times H)} \pi_\psi(x,y)\phi(d\psi)) &= \\
&\sum_{l=1}^{n_\psi}\sum_{k=1}^{n_v}(I(x \succ_{a^k}^{g^k} y) + \frac{1}{2}I(x \sim_{a^k}^{g^k} y))\psi_l((a^k,g^k))\phi(\psi_l) = \\
&\sum_{k=1}^{n_v}(I(x \succ_{a^k}^{g^k} y) + \frac{1}{2}I(x \sim_{a^k}^{g^k} y))\sum_{l=1}^{n_\psi}\psi_l((a^k,g^k))\phi(\psi_l) = \\
&\sum_{k=1}^{n_v}(I(x \succ_{a^k}^{g^k} y) + \frac{1}{2}I(x \sim_{a^k}^{g^k} y))\psi'_k((a^k,g^k)) = \\
&\pi_{\psi'}(x,y).
\end{aligned}
$$

So we can conclude that indeed $\Pi_\Phi(x,y) = \min_{\psi\in\Psi}\pi_\psi(x,y)$ in this case. In a nutshell Proposition 1 says that maximizing the expected vote share is equivalent to maximizing the vote share under the expected distribution of voters. This equivalence is rooted in the fact that the vote share of a party is a linear function of the electorate.

Building on Proposition 1 we represent the preferences of parties over platform profiles in the equivalent but more convenient form of:

$$(x,y) \mapsto min_{\psi\in\Psi}\pi_\psi(x,y)$$

where Ψ is some convex subset of $\mathbb{P}(X \times H)$. At this point it is easy to see that in our context the assumption of expected vote share maximizing parties and parties that know the electorate with certainty yield equivalent results. For we have that for any prior ϕ about the electorate there exists an electorate ψ' such that $\int\limits_{\mathbb{P}(X\times H)} \pi_\psi(x,y)\phi(d\psi)) = \pi_{\psi'}(x,y)$ for all $x, y \in X \times X$. The set Ψ represents a composition of electorates and beliefs, subsequently we will refer to its elements as beliefs about electorates or just electorates interchangeably.

The preferences of any party can be fully specified by the set of beliefs Ψ, and form now on we take this set to be the primitive of our description of the preferences of parties. All further assumption we whish to impose on the preferences of parties we state as assumptions on the set Ψ.

To state these axioms we define $\psi_a \in \mathbb{P}(X)$ and $\psi_g \in \Delta H$ the marginal distribution of a and g respectively by

$$\psi_a(A) := \int_{A\times H}\psi(d(a,g)) \text{ and } \psi_a(\{g^j\}_{j\in K}) := \sum_{j\in K}\int_{X\times g^j}\psi(d(a,g)).$$

We call ψ_a the **distribution of voter ideal** points a. Define

$$\Psi(\Lambda, G) := \{\psi \in \mathbb{P}(X \times H) | \psi_a \in \Lambda \text{ and } \operatorname{supp}\psi_g \subset G\},$$

we call this set $\Psi(\Lambda, G)$ the set of beliefs on the electorate that has been generated by the set of voter ideal point distributions Λ and by the set of types G. The voter ideal point distribution of any element in the set $\Psi(\Lambda, G)$ belongs to Λ and that any voter type that occurs with positive probability according to any $\psi \in \Psi(\Lambda, G)$ belongs to the set G. Our first assumptions state that party beliefs about the electorate can be derived from a set Λ ad a set G.

A1) There exist sets $\Lambda \subset \mathbb{P}(X)$ and $G_i := \{g_{i_k}\}_{k \in K_i}$ with $K_i \in \mathbb{N}$ and $g_{i_k} : \mathbb{R}^+ \to \mathbb{R}$ strictly increasing for all $i_k = 1, ..., K_i$ and $i = 1, ..., n$ such that $\Psi = \Psi(\Lambda, G)$ where $G := \bigtimes_{i=1}^{n} G_i$.

This assumption implies a range of independence assumptions. We assume that parties form their belief about voter ideal point distributions and about voter types independently. We also assume that parties form their beliefs about voter attitudes towards single issues g_i independently form their belief about voter attitudes about other issues. That is we exclude that case that certain attitudes about issues say g_1' only arise in combination with certain other attitudes about other issues say g_2' and g_2''. We also exclude the case that parties believe that voters with ideal points in a particular region would have a higher propensity to be of a certain type.

The standard case of an electorate with any voter ideal point distribution μ where all voters are of the same type with circular indifference curves fulfils A1), just set $G_i = \{g_i | g_i(t) = t^2\}$ for all i. Given A1 we can continuously decrease the uncertainty in a model, by decreasing the size of the sets Λ and G continuously until both contain only singletons. Finally the assumption is much stronger than we need it. But at this point it would be unnecessarily complicated to state a weaker form of this assumption, as A1 allows us to state the subsequent theorems in a straightforward manner. We denote the only electorate ψ in $\Psi(\{\mu\}, \{\alpha\})$ by $\psi := \mu * \alpha$. In this electorate every voter is of type α, electorates of this type will play a major role in our proofs.

A2) The distribution of voter ideal points ψ_a is nonatomic and $\operatorname{supp}(\psi_a) = X$ for any $\psi \in \Psi$.

We take A2 purely for convenience. The first part of it makes sense in the context of large electorates, or when the knowledge of parties about the voter ideal points lacks precision. Parties might still act as if they where facing a continuum of voters. The second part says that according to any belief about the electorate there is a positive measure of voters in any nonatomic subset $A \subset X$.

3.1 Equilibrium

We summarize a **voting game with uncertainty averse parties** by (n, X, Λ, G), where n denotes the dimensionality of the issue space X, Λ denotes the set of all of the party's priors on the distribution of voter ideal points and $G = \overset{n}{\underset{i=1}{\times}} G_i$ denotes the finite set of voter types that parties consider. We denote the set of political equilibria in such a game by $PE(n, X, \Lambda, G)$, where a (**political) equilibrium** is a platform profile (x, y) such that neither party can increase its payoff by deviating from its platform. So (x, y) is an equilibrium if

$$x \in \arg\max_{z \in X} \left(\min_{\psi \in \Psi(\Lambda, G)} \pi_\psi(z, y) \right)$$

and

$$y \in \arg\max_{z \in X} \left(1 - \min_{\psi \in \Psi(\Lambda, G)} \pi_\psi(x, z) \right).$$

Observe that we assumed here that both parties subscribe to the same set of priors on the electorate. This seems the most natural extension of the common prior assumption (Harsanyi 1967-8) to games with uncertainty averse actors. However, we take this assumption purely for convenience here, our results can easily be extended to a range of games in which parties hold different priors about the electorates.[4]

4 Uncertainty about voter types

In this section we study games $(n, X, \{\mu\}, G)$, that is those games in which parties are uncertain about the voter types, but do have full information about the distribution of ideal points (or, equivalently, act as expected utility maximizers with respect to the distribution of voter ideal points). We first restrict our attention to this *one* type of uncertainty to show what degree of type-uncertainty alone suffices to obtain political equilibria.

[4] The set of priors not only reflects what parties think about the distribution of the electorate but it also reflects their degree of uncertainty aversion. So even two actors that have access to the same amount of information might base their decisions on different sets of priors, simply because one is more uncertainty averse than the other. An investigation of the common prior assumption in this context could be based on Ghiradato (2003) approach to seperating ambiguity form ambituigy attitude. The common prior assumtion should be applied to the ambiguity in the problem while the ambiguity attitude is subjective to the players.

We start by showing that any 2-dimensional game with parties that are sufficiently uncertain about the shape of voter preferences has an equilibrium. The condition we consider is that in any set of attitudes about the separate issues G_i there are some differentiable functions h_i and k_i with $h_i'(x) \geq \alpha_i \geq k_i'(x)$ for some fixed $\alpha_i > 0$. If this condition is met we say that **type-uncertainty around** α prevails. To illustrate this concept, take a voter with ideal point a and two different platforms x and x' that differ only with respect to the i'th issue: $a_i < x_i = x_i' - d_i$ and $x_j = x_j'$ for all $j \neq i$. Given type uncertainty around α parties do not know if $u_a^g(x) \leq u_a^g(x') + \alpha_i d_i$ or $u_a^g(x) \geq u_a^g(x') + \alpha_i d_i$. In other words, parties do not know if the utility of a voter decreases by more or less than $d \bullet \alpha$ when the platform is further distanced from the ideal point of the voter by some vector $d = (d_1, ...d_n)$. Parties need not be unreasonably uncertain for this condition to be fulfilled; they might, for example, know with certainty that the utility decrease remains within certain bounds, such as $u_a^g(x') + \gamma_i d \leq u_a^g(x) \leq u_a^g(x') + \beta_i d$ for some $\beta_i > \alpha_i > \gamma_i < \alpha_i$.

In fact, parties need not be uncertain at all to fulfill our definition of type-uncertainty around α. To see this, consider the extreme case that parties know that all voters are of the same type α, with $u_a^\alpha(x) := -\sum_{i=1}^{n} \alpha_i |x_i - a_i|$ In this case the set G and therefore also the set $\Psi(\{\mu\}, G)$ are singletons.

Theorem 1. Let $(2, X, \{\mu\}, G)$ be a game where parties are type-uncertain around α. Then $(2, X, \{\mu\}, G)$ has a political equilibrium.

There is no hope of demonstrating this theorem using Nash's Existence Theorem, or any of its relatives, for best response correspondences in a game $(2, X, \{\mu\}, G)$ can be quite erratic, they are, in particular, generally not convex-valued. We proceed by means of a different strategy: first we show that there is only one candidate for an equilibrium, and then we show that in our game there does not exist any preferred deviation for either party from that platform profile.

4.1 Characterization of Equilibria

We proceed by characterizing the set of equilibria of a game of political competition $(n, X, \{\mu\}, G)$. To do so we need to introduce the notion of the median vector. For any probability-distribution μ on some $X \subset \mathbb{R}^n$ we call the vector of the medians of all marginal distributions μ_i, the vector $(m(\mu_i))_{1 \leq i \leq n}$, the **median vector of** μ. We denote this by $m(\mu)$. Throughout section 3 we normalize the median vector of μ to $0 : m(\mu) = 0$.

Proposition 2. Let (x, y) be an equilibrium of a game $(n, X, \{\mu\}, G)$, then $x = y = 0$.

Proof: Observe first that, for every $(x, y) \in PE(n, X, \{\mu\}, G)$, we need to have that have $\pi_\psi(x,y) = \frac{1}{2}$ for all $\psi \in \Psi(\{\mu\}, G)$, for otherwise any party with a lower vote share for some $\psi \in \Psi(\{\mu\}, G)$ would be better off by deviating to the platform of the other party. Suppose that $(x,y) \neq (0,0)$ say $y_1 \neq 0$. Let $x' := (0, y_2, ..., y_n)$. So all voters will base their decision only on their attitude towards the first issue $g_1(|t_1 - a_1|)$. Since 0 is median of μ_1, and since a voter with ideal point a prefers (a_1, y_{-1}) to y for all $y \in X$, at least half of the voters will vote for x'. And since supp(μ) a convex set, we have $\pi_\psi(x', y) > \frac{1}{2} = \pi_\psi(x,y)$ for all $\psi \in \Psi(\{\mu\}, G)$, contradicting $(x,y) \in PE(n, X, \{\mu\}, G)$. ■

The intuition behind Proposition 2 is that any party can choose to compete with the other in only one dimension. Thus, the only protection against such "one-dimensional attacks" is to propose with respect to every issue i the pertaining median $m(\mu_i) = 0$. Also observe that certainty case is covered by Proposition 2: G might be a singleton.[5] Observe, furthermore, that the proof did not make any use of the particular construction of the set of party beliefs as we assumed it in A1. In fact Proposition 2 holds true for any set of party beliefs Ψ with $\psi_a = \mu$ for some fixed $\mu \in \mathbb{P}(X)$ and all $\psi \in \gneqq$. Proposition 2 settles the question of the characterization of political equilibrium. We are now ready to prove the existence of an equilibrium.

4.2 The Existence of Equilibrium

A sufficient condition for Theorem 1 to hold is that for any deviation x form the median vector 0 there exists an electorate $\psi \in \Psi(\Lambda, G)$ such that $\pi_\psi(x, 0) < \frac{1}{2}$. If there is one such type profile we have that $\min\limits_{\psi \in \Psi(\Lambda, G)} \pi_\psi(x, 0) < \frac{1}{2}$ while $\min\limits_{\psi \in \Psi(\Lambda, G)} \pi_\psi(0, 0) = \frac{1}{2}$. So, uncertainty aversion proves a strong force towards both parties announcing the same platform. After all, the entire uncertainty is eliminated when both parties announce the same platform since in this case they both receive half the vote under *any* assumption on the electorate.

The full proof that this sufficient condition holds can be found in the appendix, here we give a sketch of that proof. First we show that given our assumption that parties are type uncertain around α, for any platform $x \in X$ there exists some $\psi^x \in \Psi(\Lambda, G)$ such that $\pi_{\psi^x}(x, 0) \leq \pi_{\mu * \alpha}(x, 0)$. So while the type α might not be contained in G for any x there exists some electorate $\psi \in \Psi(\Lambda, G)$ that gives the deviating party no more vote share than it would get in the case that all voters where of type α.This actually holds independently of the dimension of the issue space of the game and we conclude that whenever $(0, 0)$ is an equilibrium of the a game $(n, X, \{\mu\}, \{\alpha\})$ it also is an equilibrium of the game $(n, X, \{\mu\}, G)$ for any $n \in \mathbb{N}$.

To complete our proof we need to show that $PE(2, X, \{\mu\}, \{\alpha\})$ is always nonempty.

[5] If a distribution μ has a generalized median it coincides with the median vector.

Without loss of generality we only investigate deviations $q \gg 0$. First we establish that for any deviation with $q_1\alpha_1 \neq q_2\alpha_2$ less than half the electorate votes for the deviator: in this case the partisans of the deviation can either be all found above the median line $x_2 = 0$ or to the right of the median line $x_1 = 0$.

insert figures 1 and 2

Finally for the case of $q_1\alpha_1 = q_2\alpha_2$ the set of voters preferring q to 0 is a subset of the positive quadrant while the set of voters preferring 0 to q is a superset of the negative quadrant. The proof is concluded by observation that the positive and the negative quadrant of any two dimensional distribution (with median vector 0) contain an equal amount of probability mass.

The fact that $\pi_{\mu*\alpha}(x, 0) < \frac{1}{2}$ in any 2-dimensional game is of some interest in its own right. It implies that $PE(2, X, \{\mu\}, \{\alpha\}) \neq \emptyset$ for *all* μ. This stands in sharp contrast with many of the prior results on multidimensional games. Using two different - but equally contrived - assumptions on all voters types we obtain two results that differ dramatically: in the case that all voters indifference curves a circles political equilibria nearly always fail to exist, in the alternative case that all voters indifference curves are diamonds there always is a political equilibrium.

Let us quickly discuss the question if this existence result can be extended to some larger class of voter types. Plott (1967) already showed that for any kind of differentiable utilities there is little hope to obtain equilibria. Apparently though non-differentiabilities do not provide the key to the existence problem either:

Remark 2: Take a game $(2, X, \{\mu\}, \{g\})$, with $g := (g_i)_{i=1,\dots,n}$. Assume that all attitudes g_i are either strictly concave or strictly convex. Then there exists a distribution of voter ideal points μ such that $PE(2, X, \{\mu\}, \{g\}) = \varnothing$.

It follows from this observation that our existence result for two dimensional games with the constant type α presents a knife edge result in the class of two-dimensional games. It does not extend to higher dimensions either. If it would we could apply the proof of Theorem 1 to games of any dimension, since the two-dimensionality of the game was only used to establish that any game $(2, X, \{\mu\}, \{\alpha\})$ has an equilibrium. All other arguments of the proof apply to games of any dimension. To obtain existence results for higher dimensional issue spaces it appears that we need to introduce some uncertainty about the distribution of ideal points. But before doing so let us conclude this section with the example of a 3-dimensional game with type-uncertainty around α that does not have an equilibrium. Observe, that this example at the same time shows that the result $PE(2, X, \{\mu\}, \{\alpha\}) \neq \emptyset$ does not extend to higher dimensions.

Example 2. Take the game $(3, X, \{\mu\}, \{\alpha\})$, with $X = [-1, 1]^3$ and μ given by the following chart, where the upper row denotes subspaces of $[-1, 1]^3$ and the lower row the probability mass in those subspaces. We assume that the conditional distribution in any of the subspaces S is uniform.

S	$[0,1]^3$	$[0,1]^2 \times [-1,0)$	$[0,1] \times [-1,0) \times [0,1]$	$[0,1] \times [-1,0)^2$
$\mu(S)$	.3	05	.05	.1

f	$[-1,0)^3$	$[-1,0)^2 \times [0,1]$	$[-1,0) \times [0,1] \times [-1,0)$	$[-1,0) \times [0,1]^2$
$\mu(A_f)$	.2	.15	.15	0

Assume that $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$. Observe that $m(\mu) = 0$ and therefore by Proposition 2 $(0, 0)$ is the only candidate for any equilibrium. But $(-\frac{1}{1000}, -\frac{1}{1000}, -\frac{1}{1000})$ is a preferred deviation from this platform profile as nearly all voters in $[-1, 0)^3$, $[-1, 0)^2 \times [0, 1]$, $[-1, 0) \times [0, 1] \times [-1, 0)$ and $[0, 1] \times [-1, 0)^2$ will vote for the deviator, and these sets contain 60% of the electorate.

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5 Uncertainty about the distribution of ideal points

In this section we relax the assumption that parties know the exact distribution of ideal points μ. We ask what would happen, if the parties had only some vague idea about the real μ. Given our prior existence result for 2-dimensional games we shall focus here on the 3-dimensional case. The main result in this context will be that if the two uncertainty averse parties know sufficiently little about the distribution *and* the types of voters then political equilibria exist in 3-dimensional issue spaces.

5.1 Characterization of equilibria

As in the prior case median vectors will play an important role in the characterization of equilibria. In this case, however, many distributions of voter-ideal points matter, and therefore, many median vectors have to be taken into account. We, thus, concentrate on the **median set** $M(\Lambda)$, which is defined as

$$M(\Lambda) := \{x \in \mathbb{R}^n : \min_{\mu \in \Lambda} m_k(\mu) \leq x_k \leq \max_{\mu \in \Lambda} m_k(\mu) \text{ for all } 1 \leq k \leq n\}.$$

In a sense $M(\Lambda)$ is the set of all platforms *in-between* the median platforms of the different distributions μ in Λ.

Proposition 3: Let (x, y) be an equilibrium of a game (n, X, Λ, G), then $x, y \in M(\Lambda)$.

Proof: Observe first that, for every $(x, y) \in PE(n, X, \Lambda, G)$, we have $\pi_\psi(x, y) = \frac{1}{2}$ for every $\psi \in \Psi(\Lambda, G)$, for otherwise any party with a lower vote share according to any prior on the electorate would be better off by deviating to the platform of the other party. Suppose that $y \notin M(\Lambda)$, say $y_1 < \min_{\mu \in \Lambda} m_1(\mu)$. Then by the same argument as in the proof of Proposition 2 a deviation from x to $x' := (\min_{\mu \in \Lambda} m_1(\mu), y_2, ..., y_n)$ yields for every $\psi \in \Psi(\Lambda, G)$ a vote share $\pi_\psi(x', y) > \frac{1}{2}$, contradicting $(x, y) \in PE(n, X, \Lambda, G)$. ■

We did not succeed to show that both parties have to announce the same platform in any equilibrium. It seems however quite implausible that there would be any equilibria in which parties propose two different platforms, since in any such equilibrium (x, y) we would have that $\pi_\mu^f(x, y) = \frac{1}{2}$ for *all* f, μ. It is quite hard to imagine that this condition would be fulfilled for two platforms $x \neq y$.[6]

It would be natural to assume that in any equilibrium both parties must announce the median vector of some distribution in Λ. It is easy to show that it is not true. The following game $(2, X, \Lambda, \{\alpha\})$ has an equilibrium (x, x) for which there does not exist any $\mu \in \Lambda$ such that $x = m(\mu)$. Let $\Lambda = \{\mu | \mu = \lambda\mu_1 + (1 - \lambda)\mu_2 \text{ for } \lambda \in [0, 1]\}$ with supp(μ_1) =supp(μ_2) = $[-1, 2]^2$ and $\mu_1([-1, 1]^2) = \mu_2\ ([0, 2]^2) = .999$. Finally the distribution of μ_1 conditional on $[-1, 1]^2$ as well as the distribution of μ_2 conditional on $[0, 2]^2$ are uniform. Then (x, x) with $x = (\frac{1}{3}, \frac{2}{3})$ is an equilibrium, but for all $\mu \in \Lambda$ we have $m_1(\mu) = m_2(\mu)$.

5.2 The Existence of Equilibrium

To state sufficient conditions for the existence of equilibria in games with distribution uncertainty we have to introduce some more concepts. We call a voter a leftist if her ideal point lies with respect to *all* issues below the median vector, that is, a voter with ideal point $a \in X$ is a **leftist (rightist)** iff $a_i < m(\mu_i)$ $(a_i > m(\mu_i))$ for all issues i. Given some distribution μ we denote the subset of all leftists by A_l^μ. The subset of all rightists is defined analogously and is denoted by A_r^μ. A distribution is called **left leaning (right leaning)** if

[6] In Bade (2003) we show that if we model uncertainty aversion in a different way following Bewley (1986) equilibria with both parties announcing different platforms should always be expected to arise.

$\mu(A_l^\mu) > \mu(A_r^\mu)$, $(\mu(A_r^\mu) > \mu(A_l^\mu))$. A distribution with equally many leftists and rightists $(\mu(A_l^\mu) = \mu(A_r^\mu))$ is called **balanced**.

Theorem 2. Let $(3, X, \Lambda, G)$ be a game where parties are type-uncertain around α, and assume that there exist some left leaning μ_l and some right leaning μ_r in Λ. Then $(3, X, \Lambda, G)$ has a political equilibrium.

The proof proceeds similarly to that of Theorem 1. Using type-uncertainty around α we show that $PE(3, X, \{\mu\}, \{\alpha\}) \subseteq PE(3, X, \{\mu\}, G)$ for any fixed μ. The structure of the preferences of parties implies that $PE(3, X, \{\mu\}, G) \subseteq PE(3, X, \Lambda, G)$ for all $\mu \in \Lambda$. So if we can show that $PE(3, X, \{\mu\}, \{\alpha\})$ is non-empty for some $\mu \in \Lambda$ we are done. The main difficulty is that $PE(3, X, \{\mu\}, \{\alpha\})$ might be empty (see Example 1). In fact we can show that $(3, X, \{\mu\}, \{\alpha\})$ has an equilibrium if and only if μ is balanced. Now, given our assumption that the parties are uncertain as to whether the electorate is left or right leaning we can show that there exists some balanced μ^* in Λ, and we have $PE(3, X, \{\mu^*\}, \{\alpha\}) = (m(\mu^*), m(\mu^*))$. And by the prior arguments we have $(m(\mu^*), m(\mu^*)) \in PE(3, X, \Lambda, G)$.

The following example shows that the conditions given in this theorem are sufficient but not necessary for the existence of an equilibrium.

Example 3: Take the following game of 3-dimensional political competition $(3, [-1, 1]^3, \Lambda, G)$. Let all μ in Λ be right leaning. Assume that for all $v = (a, b, c)$ with $a, b, c \in \{-.2, .2\}$ there exists a distribution μ^v in Λ such that $m(\mu^v) = v$ and let for any of these distributions be at least .6 of the probability mass be concentrated in $B_{.05}(m(\mu^v))$ the 0.05−ball around $m(\mu^v)$.

We claim that the platform profile $(0, 0)$ is a political equilibrium of this game. Observe that for any deviation y from 0 there exists quadrant Q^* such that all voters in that quadrant are voting for 0. Note that there exists some distribution μ^{v^*} in Λ such that $v^* \in B_{.05}(m(\mu^{v^*})) \subset Q^*$. Consequently we have that $\min\limits_{\mu\in\Lambda} \pi_\mu(y, 0) \leq 1 - \mu^{v^*}(Q^*) < 1 - \mu^{v^*}(B_{.05}(m(\mu^{v^*}))) = .4 < \frac{1}{2}$ and $(0, 0)$ is an equilibrium even though by construction all distributions $\mu \in \Lambda$ are right leaning. Observe also, that we did not have to impose any condition on G to establish that $(0, 0)$ is an equilibrium, in particular type-uncertainty around was not needed and G might even be any singleton.

Roughly speaking, Theorem 2 establishes that any 3-dimensional game played amongst parties that are neither certain if the electorate leans to the right or left nor whether marginal utilities of voters in the i'th dimension diminish by more or less than some fixed α_i has an equilibrium. Example 2 shows that this amount of uncertainty is sufficient but not necessary for the existence of equilibria. But from Proposition 3 we know that in any equilibrium, whether the sufficient conditions are fulfilled or not, both parties have to announce a policy from the median set. This means that with respect to every issue i the parties announce the ideal of a voter that is the median voter of the marginal distribution μ_i of some $\mu \in \Lambda$.

6 Higher Dimensional Games

The question of under which conditions equilibria exist in games of higher-dimensional political competition naturally arises. Based on our analysis of the three dimensional case it is easy to state a condition for the existence political equilibria in n-dimensional games with uncertain parties. To do so we introduce a notation of all „quadrants" of the distribution μ. To this end define the function $sgn : \mathbb{R}^n \to \{1, 0, -1\}^n$ by $sgn(x) = (sgn(x_i))_{1 \leq i \leq n}$, and let $A_f^\mu := \{x | sgn(x - m(\mu)) = f\}$ for all $f \in \{1, -1\}^n$. The expressions A_f^μ describe the quadrants around the median vector of a distribution. Observe that $A_{(1,1,1)}^\mu = A_r^\mu$ and $A_{(-1,-1,-1)}^\mu = A_l^\mu$. We call a distribution $\mu \in \mathbb{P}(X)$ **equilibrated** if there is the same amount of probability mass in each pair of opposing quadrants, $\mu(A_f) = \mu(A_{-f})$ for all $f \in \{1, -1\}^n$.

Theorem 4: Let (n, X, Λ, G) be a game where parties are type-uncertain around α and let there exist some equilibrated μ in Λ. Then (n, X, Λ, G) has an equilibrium.

Admittedly the condition that there exist some equilibrated μ in Λ is not be very appealing. However, it seems realistic to assume that the uncertainty about electorates increases with the dimensionality of the issue space. And in our model equilibria exist more often as uncertainty increases. Finally, as in our prior results on games with uncertainty the conditions in Theorem 4 are sufficient but not necessary for the existence of an equilibrium So this should at least give some indication that it is likely that also in higher dimensional issue spaces equilibria would exist.

7 Conclusion

In this paper we showed that uncertainty aversion is a useful modelling tool to mitigate the non-existence problems of voting theory with multidimensional issue spaces. Given that parties are sufficiently uncertain equilibria exist in games of multi-issue political competition with two parties. What is more, in these equilibria both parties announce issue by issue the policy preferred by some relevant median voter. So this theory can be used to justify the common practice to look at certain issues in isolation when modelling democratic processes.

The power of the uncertainty aversion assumption lies in the fact that the two parties face no uncertainty when they both announce the same platform. Therefore, the parties in this model face a similar choice as the subjects of the famous mind experiment by Ellsberg[7]: they can chose between a lottery of given odds (adopting the same platform as the other party and both half the vote for sure) or they can go for the uncertain option of proposing a different platform, in this case their chances of success are no longer clear. So if they are sufficiently uncertainty averse, any party will always stick to the median vector as its policy, given that also the other party proposes the median vector.

The conditions for the existence of equilibria given in the theorems of this paper are sufficient but not necessary for the existence of equilibria. It is hoped that in the future more stringent conditions for the existence of equilibria will be established. A promising venue could be to restrict the set of permissible distributions and then ask what amount of uncertainty is sufficient to establish that equilibria exist. In particular we hope to show that under the assumption that the society is not polarized (following Caplin and Nalebuff (1988)) a small amount of uncertainty on voter ideal point distribution is sufficient to establish the existence of equilibria in n-dimensional games.

Another extension of this research could be to apply the same model of uncertainty aversion to solve different but related non-existence problems in political economy. Some models of ideologically motivated parties for example is plagued by similar non-existence problems as the Downs model. The challenge in extending the present framework to such models lies in defining the utilities of parties.

Furthermore, in a companion paper (Bade 2003) we show that with a different approach towards modelling uncertainty averse actors (following Bewley 1986), there might be equilibria in which office motivated parties announce different platforms. Our explanation for platform divergence does not need any *ad hoc* assumptions on the ideological motivation or parties, the driving force of this result are non-convexities in the preferences of voters.

8 Literature

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9 Appendix

Proof of Proposition 1: The integral $\int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi)$ is well-defined as the identity function on $\mathbb{P}(X \times H)$ is an integrable function.

We show next that for any probability $\phi \in \mathbb{P}(\mathbb{P}(X \times H))$ we have $\int\limits_{\mathbb{P}(X\times H)} \pi_{\psi}(x,y)\phi(d\psi)) = \pi_{\psi'}(x,y)$ with $\psi' := \int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi)$.

$$\begin{aligned}
\int\limits_{\mathbb{P}(X\times H)} \pi_\psi(x,y)\phi(d(\psi)) &= \\
&\int\limits_{\mathbb{P}(X\times H)}\int\limits_{X\times H} (I(x \succ_a^g y) + \frac{1}{2} I(x \sim_a^g y))\psi(d(a,g))\phi(d\psi) =
\end{aligned}$$

$$\begin{aligned}
\int\limits_{\mathbb{P}(X\times H)} \left(\psi\left(\{(a,g)|I(x \succ_a^g y)=1\}\right) + \frac{1}{2}\psi\left(\{(a,g)|I(x \sim_a^g y)=1\}\right)\right)\phi(d\psi) &= \\
\int\limits_{\mathbb{P}(X\times H)} \psi\left(\{(a,g)|I(x \succ_a^g y)=1\}\right)\phi(d\psi) + \frac{1}{2}\int\limits_{\mathbb{P}(X\times H)} \psi\left(\{(a,g)|I(x \sim_a^g y)=1\}\right)\phi(d\psi) &= (*)
\end{aligned}$$

Now we can apply our definition: $\psi' := \int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi)$ and obtain

$$\begin{aligned}
(*) &= \psi'\left(\{(a,g)|I(x \succ_a^g y)=1\}\right) + \psi'\left(\{(a,g)|I(x \sim_a^g y)=1\}\right) = \\
\int\limits_{X\times H} (I(x &\succ_a^g y) + \frac{1}{2} I(x \sim_a^g y))\psi'(d(a,g)) = \\
&\pi_{\psi'}(x,y).
\end{aligned}$$

It follows that

$$\Pi_\Phi(x,y) = \min_{\psi\in\Psi} \pi_\psi(x,y).$$

for $\Psi := \{\psi : \psi = \int\limits_{\mathbb{P}(X\times H)} \psi\phi(d\psi)$ for some $\phi \in \Phi\}$. All that remains to show is that Ψ is indeed a convex subset of $\mathbb{P}(X \times H)$ and. We show first that $\Psi \subset \mathbb{P}(X \times H)$

1.
$$\psi'(\varnothing) = \int\limits_{\mathbb{P}(X\times H)} \psi(\varnothing)\phi(d\psi) = \int\limits_{\mathbb{P}(X\times H)} 0\phi(d\psi) = 0$$

2.
$$\psi'(X \times H) = \int\limits_{\mathbb{P}(X\times H)} \psi(X \times H)\phi(d\psi) = \int\limits_{\mathbb{P}(X\times H)} 1\phi(d\psi) = 1$$

3. Now take $\{A_i\}_{i=1}^{\infty}$ a countable set of mutually disjoint Borel sets in $X \times H$. We have

$$\begin{aligned}
\psi'(\cup_{i=1}^{\infty} A_i) &= \\
\int\limits_{\mathbb{P}(X \times H)} \psi(\cup_{i=1}^{\infty} A_i)\phi(d\psi) &= \\
\int\limits_{\mathbb{P}(X \times H)} \Sigma_{i=1}^{\infty} \psi(A_i)\phi(d\psi) &= \\
\int\limits_{\mathbb{P}(X \times H)} \lim_{n\to\infty} \Sigma_{i=1}^{n} \psi(A_i)\phi(d\psi) &= (*)
\end{aligned}$$

Now define $f_n(\sigma) := \Sigma_{i=1}^{n} \psi(A_i)$, Observe that $f_1 \le f_2 \le f_3 \le ...$, so by the monotone convergence theorem we have that:

$$\begin{aligned}
(*) &= \\
\lim_{n\to\infty} \int\limits_{\mathbb{P}(X \times H)} \Sigma_{i=1}^{n} \psi(A_i)\phi(d\psi) &= \\
\lim_{n\to\infty} \sum_{i=1}^{n} \int\limits_{\mathbb{P}(X \times H)} \psi(A_i)\phi(d\psi) &= \\
\sum_{i=1}^{\infty} \int\limits_{\mathbb{P}(X \times H)} \psi(A_i)\phi(d\psi) &= \\
&= \sum_{i=1}^{\infty} \psi'(A_i).
\end{aligned}$$

4. Show that Ψ convex. Take $\alpha, \beta \in \Psi$ and any $\lambda \in [0,1]$. There exist ϕ_α, ϕ_β in Φ such that $\alpha = \int \psi\phi_\alpha(d\psi)$ and $\beta = \int \psi\phi_\beta(d\psi)$. We have:

$$\begin{aligned}
\lambda\alpha + (1-\lambda)\beta &= \\
\lambda \int\limits_{\mathbb{P}(X \times H)} \psi\phi_\alpha(d\psi) + (1-\lambda) \int\limits_{\mathbb{P}(X \times H)} \psi\phi_\beta(d\psi) &= \\
& \int\limits_{\mathbb{P}(X \times H)} \psi(\lambda\phi_\alpha + (1-\lambda)\phi_\beta)(d\psi).
\end{aligned}$$

But as Φ is a convex set we have that $\lambda\phi_\alpha + (1-\lambda)\phi_\beta \in \Phi$ and therefore we conclude that $\lambda\alpha + (1-\lambda)\beta \in \Psi$. ■

Proof of Theorem 1: Before starting with the proof of Theorem 1 we separately show two main steps of that proof as Lemma 1 and Lemma 2:

Lemma 1: Any 2-dimensional game $(2, X, \{\mu\}, \{\alpha\})$ has an equilibrium.

Proof: Suppose some profitable deviation $q \gg 0$ existed. Let A be the set of voters that are indifferent between 0 and q. Given the type profile α we have that all voters $a' \notin A$ for which there exists an $a \in A$ such that $a' \ll a$ strictly prefer 0 to q. If $A \cap \{a|a_1 = 0\} = \varnothing$ or $A \cap \{a|a_2 = 0\} = \varnothing$ then since the A is a connected set either all voters in $\{a|a_1 \leq 0\}$ or all voters in $\{a|a_2 \leq 0\}$ will prefer 0 to q. Figures 1 and 2 give two examples for these two cases; the dotted graphs represent the set A, the shaded areas represent the sets $\{a|a_1 \leq 0\}$ and $\{a|a_2 \leq 0\}$ respectively. Since $(0,0)$ is the median vector we have $\mu(\{a|a_1 \leq 0\}) \geq \frac{1}{2}$ and $\mu(\{a|a_2 \leq 0\}) \geq \frac{1}{2}$ so in either case at least half the electorate votes for 0 and therefore such a deviation to q cannot raise the deviating party's vote share.

Let us now consider the remaining case in which $A \cap \{a|a_1 = 0\} \neq \varnothing$ and $A \cap \{a|a_2 = 0\} \neq \varnothing$. This only holds for deviations q such that $q_1\alpha_1 = q_2\alpha_2$. In this case all voters in $\{a|a_1 \leq 0 \text{ and } a_2 \geq q_2\}$ and in $\{a|a_1 \geq q_1 \text{ and } a_2 \leq 0\}$ are indifferent between 0 and q.

Since we assume that all indifferent voters vote for either platform with equal probability we only need to look at the voters that strictly prefer one platform to the other. The set of voters strictly preferring q to 0 is a subset of $\{a|a_1 > 0, a_2 > 0\}$ whereas the set strictly preferring 0 to q is a superset of $\{a|a_1 < 0, a_2 < 0\}$. But since $(0,0)$ is the median vector of the nonatomic μ we have $\mu(\{a|a_1 > 0, a_2 > 0\}) = \mu(\{a|a_1 < 0, a_2 < 0\})$. Consequently it cannot be that such a deviation increases the vote share. But by the same arguments no other deviation q raises the vote share to the deviating party and $(0,0)$ is a political equilibrium. Q.E.D.

Lemma 2: Take a game $(n, X, \{\mu\}, G)$ with type uncertainty around α, then for all x in X there exists a type profile ψ^x in $\Psi(\{\mu\}, G)$ such that $\pi_{\psi^x}(x, 0) \leq \pi_{\mu * \alpha}(x, 0)$.

Proof: We begin by showing that for all ideal points a such that:

$$u_a^\alpha(0) > u_a^\alpha(x)$$

there exists some $g \in G$ such that $u_a^g(0) > u_a^g(x)$. Split the set $\{1, ..., n\}$ into two disjoint sets H, K with $|x_i - a_i| \geq |a_i|$ for all $i \in H$ and $|a_i| > |x_i - a_i|$ for all $i \in K$. Observe first of all that $u_a^\alpha(x) > u_a^\alpha(y)$ implies

$$\sum_{i \in H} \alpha_i(|x_i - a_i| - |a_i|) > \sum_{i \in K} \alpha_i(|a_i| - |x_i - a_i|).$$

Then since the slope of h_i is always at least α_i we have that

$$h_i(|x_i - a_i|) - h_i(|a_i|) \geq \alpha_i|x_i - a_i| - \alpha_i|a_i| \text{ for all } i \in H.$$

By the same logic we have

$$\alpha_i|a_i| - \alpha_i|x_i - a_i| \geq k_i(|y_i - a_i|) - k_i(|a_i|) \text{ for all } i \in K.$$

Jointly these three inequalities imply:

$$\sum_{i \in H} h_i(|x_i - a_i|) - h_i(|a_i|) > \sum_{i \in K} k_i(|a_i|) - k_i(|x_i - a_i|)$$

Now choose $g_i = h_i$ for $i \in H$ and $g_i = k_i$ for $i \in K$, the above chain of inequalities yields: $\sum_{i=1}^{n} (g_i(|x_i - a_i|) - g_i(|a_i|)) > 0$ or $u_a^g(0) > u_a^g(x)$.

Now observe that for all ideal points a such that

$$u_a^\alpha(0) = u_a^\alpha(x)$$

we have for g with $g_i = h_i$ for all $i \in H$ and $g_i = k_i$ for all $i \in K$ where the sets H and K have been defined as above, that $u_a^g(0) \geq u_a^g(x)$.

Define a conditional probability $r : X \times G \to [0, 1]$ such that $r(a, g(a)) = 1$ where $g_i(a) = h_i$ for all $i \in H$ and $g_i(a) = k_i$ for all $i \in K$ where the sets H and K have been defined as above and $r(a, g) = 0$ if $g \neq g(a)$. Define $\psi^x := \mu \times r$.

Then we have by construction that

$$\begin{aligned}
\pi_{\psi^x}(x, 0) &= \\
1 - \mu(a|u_a^{g(a)}(0) &> u_a^{g(a)}(x)) - \frac{1}{2}\mu(a|u_a^{g(a)}(0) = u_a^{g(a)}(x)) \leq \\
1 - \mu(a|u_a^\alpha(0) &> u_a^\alpha(x)) - \frac{1}{2}\mu(a|u_a^\alpha(0) = u_a^\alpha(x)) = \\
& \pi_{\mu * \alpha}(x, 0). \blacksquare
\end{aligned}$$

Proof of Theorem 1: By Proposition 1 we know that the only candidate for an equilibrium is $(0, 0)$. The payoff to a party that deviates to x is $\min_{\psi \in \Psi(\{\mu\}, G)} \pi_\psi(x, 0)$. By Lemma

2 we know that $\pi_{\mu*\alpha}(x,0)$ is an upper bound on this payoff. Finally by Lemma 1 we know that according to the type profile α no profitable deviation form $(0,0)$ exists. To summarize: $\min_{\psi\in\Psi(\{\mu\},G)} \pi_\psi(x,0) \leq \pi_{\psi^x}(x,0) \leq \pi_{\mu*\alpha}(x,0) \leq \frac{1}{2} = \min_{\psi\in\Psi(\{\mu\},G)} \pi_\psi(0,0)$ where ψ^x is the type profile constructed in Lemma 2. So there is not profitable deviation, and $(0,0)$ is an equilibrium of $(2, X, \{\mu\}, G)$. ■

Proof of Theorem 2: The following Lemma 3 is a major building block of the proof of Theorem 2.

Lemma 3: A 3-dimensional game $(3, X, \{\mu\}, \{\alpha\})$ has an equilibrium if and only if μ is balanced.

Proof: As in section 5 we define the „quadrants" of the distribution μ. We define the function $sgn : \mathbb{R}^n \to \{1,0,-1\}^n$ by $sgn(x) = (sgn(x_i))_{1\leq i\leq n}$, and let $A_f^\mu := \{x|sgn(x - m(\mu)) = f\}$ for all $f \in \{1,-1\}^n$. As Lemma 3 covers games with certainty we can, for ease of exposition, revert to the normalization $m(\mu) = 0$. We therefore also drop μ from the notation of the sets of all leftists and rightist and now write A_l and A_r and from all quadrants and write A_f.

We first show that $\mu(A_l) = \mu(A_r)$ if and only if $\mu(A_f) = \mu(A_{-f})$ for all f, then we show that this condition is necessary and sufficient for $(0,0)$ being an equilibrium.

Let $\mu(A_l) = \mu(A_r)$. Define 4 variables $D_f = \mu(A_f) - \mu(A_{-f})$ for all f with $f_1 = 1$. Since 0 is the median vector of μ we have that $\sum_{f_i=1} D_f = 0$ for all $i = 1,2,3$. Given $D_{(1,1,1)} = \mu(A_r) - \mu(A_l) = 0$ this reduces to a system of 3 linearly independent equations in 3 unknowns, the only solution is $D_f = \mu(A_f) - \mu(A_{-f}) = 0$ for all f.

Given $\mu(A_f) = \mu(A_{-f})$ for all f we show that for any deviation from $(0,0)$ the party remaining at 0 gets at least half the vote share. First we derive a condition under which all voters in some A_f vote for 0. Then we use this condition to show that given the choice between the platforms $q \neq 0$ and 0 for any f either all voters in A_f or all voters in A_{-f} vote for 0 or no voter in either A_f or A_{-f} strictly prefers q to 0.

All voters in A_f vote for 0 iff for all $a \in A_f$ the utility from platform $0 : -\sum_{i=1}^{3} \alpha_i |a_i|$ is larger than the utility from the other platform: $-\sum_{i=1}^{3} \alpha_i |q_i - a_i|$. So let us calculate

$$\sup_{a\in A_f} \sum_{i=1}^{3} \alpha_i |a_i| - \sum_{i=1}^{3} \alpha_i |q_i - a_i|$$

Since for all $a \in A_f$ we have $a_i < 0$ if $f_i = -1$ and $a_i > 0$ if $f_i = 1$ the above expression reduces to:

$$\sup_{a\in A_f}\left(\sum_{f_i=-1}-\alpha_i a_i-\alpha_i(q_i-a_i)+\right.$$

$$\left.+\sum_{0\le a_i\le q_i}\alpha_i a_i-\alpha_i(q_i-a_i)+\sum_{q_i<a_i}\alpha_i a_i+\alpha_i(q_i-a_i)\right)=$$

$$=\sup_{a\in A_f}\left(\sum_{f_i=-1}-\alpha_i q_i+\sum_{0\le a_i\le q_i}2\alpha_i a_i-\alpha_i q_i+\sum_{q_i<a_i}\alpha_i q_i\right)=$$

$$=\sum_{i=1}^{3}f_i\alpha_i q_i$$

So if this expression is negative we are done all voters in A_f vote for 0. If this expression is positive we have

$$-\sum_{i=1}^{3}f_i\alpha_i q_i=\sum_{i=1}^{3}-f_i\alpha_i q_i<0$$

and therefore all voters in A_{-f} vote for 0. Finally if $\sum_{i=1}^{n}f_i\alpha_i q_i=0$ then

$$\begin{aligned}\sup_{a\in A_f}\sum_{i=1}^{3}\alpha_i|a_i|-\sum_{i=1}^{3}\alpha_i|q_i-a_i| &= \\ \sup_{a\in A_{-f}}\sum_{i=1}^{3}\alpha_i|a_i|-\sum_{i=1}^{3}\alpha_i|q_i-a_i| &= 0.\end{aligned}$$

and consequently no voter in either A_f or A_{-f} strictly prefers platform q.

Given the assumption that $\mu(A_{-f})=\mu(A_f)$ for all f we can now show that $\sum_{f_1=1}\mu(A_f)$ represents a lower bound on $\pi(0,q)$. This is so since for any f with $f_1=1$ either a mass of voters $\mu(A_f)$ or a mass of voters $\mu(A_{-f})=\mu(A_f)$ or a mass of voters $\frac{1}{2}(\mu(A_f)+\mu(A_{-f}))=\mu(A_f)$ votes for 0. On the other hand since 0 is the median vector of μ we know that $\sum_{f_i=1}\mu\left(A_f\right)=\frac{1}{2}$ and we have $\pi(0,q)\ge\frac{1}{2}$ and therefore no deviation from $(0,0)$ that raises the vote share of the deviating party exists.

Now suppose μ where not balanced, that is assume $\mu(A_l)=x$ and $\mu(A_r)=y$ with $x>y$. Let the values of all $\mu(A_f)$ be given by the following chart:

f	$(1,1,1)$	$(1,1,-1)$	$(1,-1,1)$	$(1,-1,-1)$
$\mu(A_f)$	y	z	w	$\frac{1}{2}-z-w-y$

f	$(-1,-1,-1)$	$(-1,-1,1)$	$(-1,1,-1)$	$(-1,1,1)$
$\mu(A_f)$	x	$z+(y-x)$	$w+(y-x)$	$\frac{1}{2}-z-w-y-(y-x)$

When a deviator plays $\lambda(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3})$ with $\lambda > 0$ against 0, then all voters in $A_l, A_{(1,-1,-1)}, A_{(-1,-1,1)}$ and $A_{(-1,1,-1)}$ are voting for 0. In the limit for $\lambda \to 0$ only these voters will vote for 0. So in the limit the vote share of the remaining party is $\frac{1}{2}+y-x$. Since the remaining parties vote share decreases continuously in λ there exists some $\lambda^* > 0$ such that $\pi(\lambda^*(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}), 0) > \frac{1}{2}$ and $(0,0)$ cannot be an equilibrium. ■

Before proceeding with the proof of Theorem 2 let us remark that Lemma 3 generalizes Lemma 1 as any 2-dimensional distribution of voter ideal points is balanced. Secondly, observe that balancedness of μ does not imply $\mu(A_f) = \mu(A_{-f})$ for all f for higher dimensional issue spaces.

Proof of Theorem 2: We start by showing that there exists a balanced μ in Λ. Define $\mu_\lambda = \lambda\ \mu_l + (1-\lambda)\mu_r$. Define

$$\begin{aligned} f &: [0,1] \to [-1,1] \\ f(\lambda) &= \mu_\lambda(A_l^{\mu_\lambda}) - \mu_\lambda(A_r^{\mu_\lambda}) \end{aligned}$$

a continuous function. Clearly: $f(1) > 0$ and $f(0) < 0$ so there exists some $\lambda^b \in (0,1)$ such that $f(\lambda^b) = 0$. Observe that μ_{λ^b} is balanced. Since Λ convex we also have that $\mu_{\lambda^b} \in \Lambda$.

By the same argument as forwarded in the proof of Theorem 1 we know that $(m(\mu_{\lambda^b}), m(\mu_{\lambda^b}))$ is an element of $PE(3, X, \{\mu_{\lambda^b}\}, G)$. Finally

$$\min_{\psi \in \Psi(\Lambda, G)} \pi_\psi(0,0) \geq \min_{\psi \in \Psi(\{\mu_{\lambda^b}\}, G)} \pi_\psi(x, 0)$$

and therefore $(m(\mu_{\lambda^b}), m(\mu_{\lambda^b}))$ is also an element of $PE(3, X, \Lambda, G)$. Q.E.D.

Proof of Remark 2:

We have to consider three cases: g_1 and g_2 both strictly concave, both strictly convex or g_1 strictly convex and g_2 strictly concave.

Take first case $u_a^g(x) = -g_1(|x_1 - a_1|) - g_2(|x_2 - a_2|)$ with g_1 and g_2 strictly concave. Pick three points a, b, x in $\mathbb{R}^2$ such that

$$\begin{gathered} x \gg 0, \\ a_2 = x_2,\ a_1 < 0 \text{ and } g_1(x_1 - a_1) - g_1(-a_1) < g_2(x_2), \\ b_1 = x_1,\ b_2 < 0 \text{ and } g_2(x_2 - b_2) - g_2(-b_2) < g_1(x_1). \end{gathered}$$

Observe that such a triple of vectors exists since $g_1(x_1 - a_1) - g_1(a_1)$ as well as $g_2(x_2 - b_2) - g_2(b_2)$ go to zero as a_1 and b_2 become very small. Pick some small ε such that all voters with ideal points in either $B_\varepsilon(a)$ or $B_\varepsilon(b)$ strictly prefer x to 0 and such that $x_i - \varepsilon > 0$ for $i = 1, 2$, $a_1 + \varepsilon < 0$ and $b_2 + \varepsilon < 0$.[8] Now define an atomless Borel probability measure μ with support $[-c, c]^2$ for $c = \max\{|a_1|, |a_2|, |b_1|, |b_2|, |x_1|, |x_2|\} + 2\varepsilon$ by

$$\mu(B_\varepsilon(a)) = \mu(B_\varepsilon(b)) = \mu(B_\varepsilon(x)) = \mu(B_\varepsilon(-x)) = \frac{1}{5}$$

and

$$\begin{aligned} \mu([0,c]^2 \backslash B_\varepsilon(x)) &= \mu([0,c] \times [-c,0] \backslash B_\varepsilon(b)) = \\ \mu([-c,0] \times [0,c] \backslash B_\varepsilon(a)) &= \mu([-c,0]^2 \backslash B_\varepsilon(-x)) = \frac{1}{20}. \end{aligned}$$

Then $(0,0)$ is the median vector of μ, but it is not a political equilibrium. To see that observe that by construction $\pi(x, 0) \geq \mu(B_\varepsilon(a)) + \mu(B_\varepsilon(b)) + \mu(B_\varepsilon(x)) = \frac{3}{5} > \frac{1}{2}$.

For the second case $u_a(x) = -g_1(|x_1 - a_1|) - g_2(|x_2 - a_2|)$ with g_1 and g_2 strictly convex follow the same argument through with the modification that now the points x, a and b have to fulfill

$$\begin{gathered} x \gg 0, \\ a_1 = -1,\ a_2 > 0 \text{ and } g_1(x_1 + 1) - g_1(1) < g_2(a_2) - g_2(|x_2 - a_2|), \\ b_2 = -1,\ b_1 > 0 \text{ and } g_2(x_2 + 1) - g_2(1) < g_1(b_1) - g_1(|x_1 - b_1|). \end{gathered}$$

Observe that such vectors x, a and b exist since $g_i(a_i) - g_i(|x_i - a_i|)$ becomes large as a_i becomes large.

In the third case $u_a(x) = -g_1(|x_1 - a_1|) - g_2(|x_2 - a_2|)$ with g_1 strictly concave and g_2 strictly convex, the condition on the points x, a and b is:

$$\begin{gathered} x \gg 0, \\ a_2 = x_2,\ a_1 < 0 \text{ and } g_1(x_1 - a_1) - g_1(-a_1) < g_2(x_2), \\ b_2 = -1,\ b_1 > 0 \text{ and } g_2(x_2 + 1) - g_2(1) < g_1(b_1) - g_1(|x_1 - b_1|). \end{gathered}$$

In both cases the construction of μ follows in complete analogy and we are done. Q.E.D.

Given the proof of Remark 2 it is easy to construct an n-dimensional (n>2) distribution μ that $(n, X, \{\mu\}, \{g\})$ does not have an equilibrium where all g_i either strictly convex or strictly concave. Pick $X = [-1, 1]^n$. By the above proof we can pick some two dimensional distribution μ' such that $(2, [-1, 1]^2, \{\mu'\}, \{g'\})$ does not have an equilibrium, where $(g'_1, g'_2) = (g_1, g_2)$. Now define $\mu := \mu' * \prod_{i=3}^{n} \mu_i$with μ_i the uniform distribution on $[-1, 1]$. Then again by

[8] Such an ε exists since voters a or b *strictly* prefer x to 0, and since g_1 and g_2 are continious functions.

the proof of Remark 2 $(n, [-1,1]^n, \{\mu\}, \{g'\})$ does not have an equilibrium.

Proof of Theorem 4: Following the proof of Theorem 3, observe that once we had established that $\mu(A_f) = \mu(A_{-f})$ for all f in Lemma 3 we made no more use of either balancedness or 3-dimensionality in the proofs of Lemma 3. So we note the following Lemma 4 in passing:

Lemma 4: Take an n-dimensional game $(n, X, \{\mu\}, \{\alpha\})$ with $m(\mu) = 0$. This game has an equilibrium $\mu(A_f) = \mu(A_{-f})$ for all f .

The proof of Theorem 4 proceeds like that of Theorem 2 replacing Lemma 3 by Lemma 4.