Racial Gerrymandering and Minority Representation*

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Abstract

We present a model of optimal districting schemes to promote the policy interests of minority groups, incorporating the possibilities of coalition formation at both the electoral and legislative stages. The results show that minorities with relatively little political power prefer to concentrate their voters in a few districts and shift the weight of the bargaining problem to the legislature. Conversely, as minorities gain power, they do best by distributing their voters more evenly across districts. Furthermore, declining majority racism has two effects on minorities: it helps them by making it easier to elect minorities to office, but it may also hurt them by making majority voters more pivotal and therefore increasing their relative power at minorities' expense. The model is then employed to understand the impact of changes in southern politics over the past four decades.

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For political minorities to exercise influence over policy in a majoritarian system, they must, perforce, form coalitions. These coalitions can be constructed at the electoral stage, by working through political parties or peak organizations that amalgamate a variety of interests; or at the legislative stage, through vote trading and logrolling with legislators representing other groups. For a given group at a given time, one of these strategies may be more effective than the other: some groups do best by working through parties, others through legislative bargaining.

Furthermore, electoral institutions can play an important role in favoring one type of coalition building over the other. The classic debate over consociational democracy, for example, is in this vein, asking whether in divided societies it is best to have broad parties that coalesce interests, or parliamentary systems that allow each group to elect representatives from their caste/tribe/subgroup, thus moving the locus of debate off the street and into the legislature.

When designing institutions with the explicit aim of empowering minorities, then, one must consider the possible impact of those institutions on both types of coalition formation. In some cases, institutional change can empower minorities on both dimensions; the simple act of enfranchising minority voters, for instance, means that they can influence both electoral coalitions and legislative bargaining as well. Such moves unambiguously increase minority power over policy making. But in other cases institutional choice means trading off one type of coalition formation against the other at the margin, and in such instances it is an empirical question as to which is relatively more effective.

The particular institution studied here is racial gerrymandering, defined as the allocation of minority voters to districts in the presence of racially polarized voting. The basic question is whether to concentrate minority voters into majority-minority districts, or to spread them out more evenly across districts. According to the framework elaborated above, the former strategy pursues minority influence via legislative bargaining, while the latter relies more on electoral coalitions.

The relation between districting strategies and policy influence was brought into sharp relief by the recent Supreme Court ruling in *Georgia v. Ashcroft*.[1] The question before the Court was whether a proposed redistricting scheme for the Georgia State Senate that "unpacked" minority voters from majority-minority districts violated Section 5 of the 1965 Voting Rights Act. The Court ruled that state legislators were attempting to increase blacks' overall influence on policy, and so the scheme was not "retrogressive," even though it might result in fewer minority representatives elected to office.

Despite the obvious importance of the relationship between institutions

[1] 123 S.Ct. 2498 (2003).

and minority influence over policy, the subject has received scant theoretical attention. This paper therefore presents a formal model of policy making with a political minority, incorporating coalition formation at both the electoral and legislative stages. In the analysis, a districting scheme is implemented which divides various groups of majority and minority voters across districts. Candidates from each group then present themselves for election in each district, offering platforms designed to maximize their probability of being elected. The winners in each district then form a legislature, which in turn passes a redistributive policy.

Voters' payoffs are a function of both their redistributive gains and their ideological attachment to the legislator elected from their district. In other contexts, this ideological attachment might stem from a candidate's expected voting record on issues aligned along a left-right spectrum; here, it can also encompass the degree of racism or in-group racial preferences across voters. We can thus analyze the impact of changing ideological attachments for voters of one group in relation to candidates from another group; i.e., increasing or declining racism.

The results show that minorities with relatively little political power prefer to concentrate their voters in a few districts and shift the weight of the bargaining problem to the legislature. Conversely, as minorities gain power, they do best by distributing their voters more evenly across districts. Furthermore, declining majority racism has two effects on minorities: it helps them by making it easier to elect minorities to office, but it may also hurt them by making majority voters more pivotal and therefore increasing their relative power at minorities' expense.

The next section reviews the relevant literature, after which we define the districting problem in a spatial context and provide some preliminary results. We then develop a formal model that allows for both electoral coalition formation and legislative bargaining. The next section explores the policy implications of our results, analyzing optimal districting schemes under varying circumstances. The final section concludes.

1 Literature Review

The past decade has seen the development of a rich empirical literature on the relationship between racial redistricting and policy outcomes, created largely as a reaction to the Supreme Court's determination in *Shaw v. Reno*[2] that highly concentrated majority-minority districts may constitute unconstitutional racial gerrymanders. Cameron, Epstein and O'Halloran (1994), Lublin (1997), and Epstein and O'Halloran (2000) assert that majority-minority dis-

[2] 509 U.S. 630 (1993).

tricts reduce minorities' influence over policy outcomes, since they tend to elect more republicans in neighboring districts. Canon (1999) challenges this conclusion, arguing that these vote-based studies may miss the behind-the-scenes influence of minority legislators in promoting policy agendas important to their constituents. Such back-room influence is, however, difficult to measure, and recently a number authors have suggested that the promotion of majority-minority districts via Section 5 of the Voting Rights Act may undermine minority influence, even as it increases the number of republican elected to office.[3]

Despite these empirical and legal advances, the formal literature on redistricting lags behind. Most models of redistricting focus on partisan gerrymandering (Musgrove 1977; Owen and Grofman 1988; Sherstuyk 1998; Cox and Katz 1999; Gilligan and Matsusaka 1999); in this context, an optimal gerrymander will trade off mean and variance, trying to obtain as many legislative seats as possible for the majority party, without too much risk that an adverse electoral tide could hand control of the legislature to their opponents.

The exception to this focus on partisan gerrymanders consists of an important series of papers by Shotts (2001; 2002), who examines the impact of majority-minority districting requirements on the actions of strategic gerrymanderers. Shotts examines a unidimensional policy setting in which the minority group in question has extreme preferences at one end of the political spectrum. He finds, surprisingly, that majority-minority requirements have no impact on liberal (pro-minority) gerrymanderers, but the may constrain conservative line-drawers. Thus majority-minority mandates can only work in favor of minority interests.

We expand on this work, moving to a multidimensional setting in order to capture the dynamics of coalition formation at various stages of the policy-making process. We also include ideological preferences for different types of voters, since, as Key (1949) explains, Southern politics after reconstruction was divided not simply because blacks had more liberal policy preferences than whites, but because the underlying bedrock of politics at all levels was a desire to deny blacks any representation at all. Formal models of political institutions in divided societies, then, should incorporate the notion that in such circumstances race can be more than a symbol, more than a summary of policy positions across various issues; often race is *the* issue.

2 Triangles: Districting Made Simple(x)

We first provide a graphical context in which to analyze districting alternatives. For simplicity, we take the electorate to consist of three groups:

[3] See, for instance, Issacharoff (2003), Grofman and Handley (2003), and Epstein and O'Halloran (2004).

$$
\begin{array}{c}
\begin{pmatrix}
WD_1 & BD_1 & R_1 \\
WD_2 & BD_2 & R_2 \\
\vdots & \vdots & \vdots \\
WD_K & BD_K & R_K
\end{pmatrix}
\begin{array}{c} \frac{\mathbf{N}}{K} \\ \frac{\mathbf{N}}{K} \\ \vdots \\ \frac{\mathbf{N}}{K} \end{array} \\
\begin{array}{ccc} \mathbf{N}_{WD} & \mathbf{N}_{BD} & \mathbf{N}_R \end{array}
\end{array}
$$

Table 1: Sample districting matrix

white democrats (WD), black democrats (BD) and republicans (R). These three groups exist in certain proportions in the state overall, and we look at possible districting schemes that divide these voters up into some number of equally-sized districts.

2.1 Districting Alternatives

A valid districting scheme is a matrix, like the one illustrated in Table 1. Each row is a district, giving the number of WD, BD, and R voters (where each of these quantities is, naturally, non-negative). The sum of the entries in each row must equal the total population $\mathbf{N}$ divided by the number of districts K. And the sums of the entries in each column are the group populations $\mathbf{N}_i$.

To visualize the alternatives, we use an equilateral triangle, as in Figure 1, which represents the two-dimensional simplex of possible percentages of each group in a given district. The corners thus indicate districts that contain only one type of voter: WD in the bottom left, BD in the bottom right, and R on top. Point a in the center stands for a district with an equal division of all three types, so that each comprise 1/3 of the district population. The diagonal line is drawn where the BD voters are 1/2 of the district, so all points down and to the right of it are majority-minority; one such point is labeled b.

The state as a whole has a given percent of each type of voter, so it can also be represented as a point on the triangle. Then a valid districting scheme is a set of points that average to the state-wide population proportions. Figure 2 illustrates a state with five districts. The statewide distribution of voters is marked with a large dot labeled "S," while the other five points represent the districts, one of which is majority-minority.

The figure also gives some hints as to districting strategy. Say for example

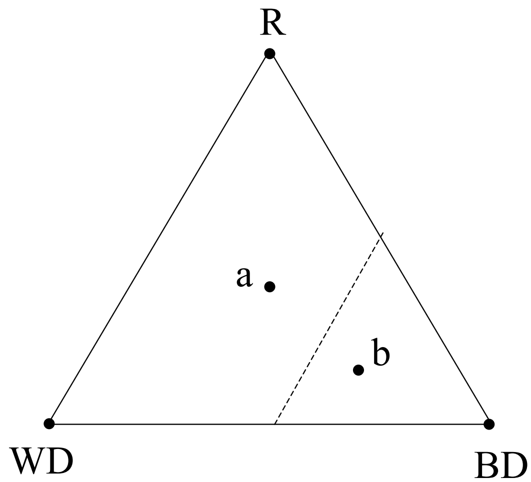


Figure 1: Graphical representation of possible voting districts: a = equal proportions of each type; b = majority-minority district

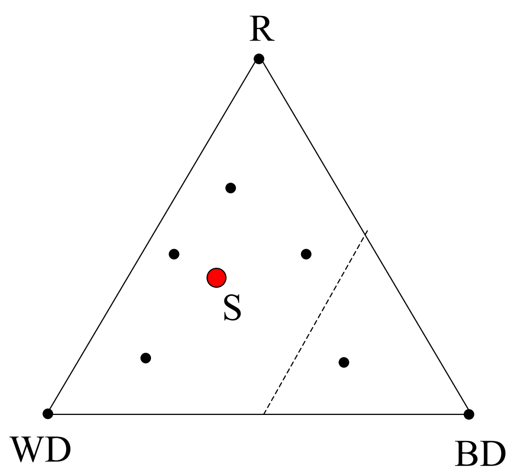


Figure 2: Sample state with five districts, one of which is majority-minority

that at S a white democrat is expected to win. Then if all districts in a state have demographics equal to S — making them microcosms of the state as a whole — white democrats will be the favorites in all races. But if the probability of success for a white democrat in such a district is too close to 50%, risk-averse legislators might prefer to make some districts heavily republican (toward the top of the triangle), allowing them to move the other districts to safer democratic regions (towards the bottom of the triangle).

Similarly, the requirement that some majority-minority districts be created means that these districts must be located near the bottom right, pushing the other districts up and to the left, possibly increasing the likelihood that republicans will win elsewhere.

2.2 Voting and Elections

Let us now turn to the question of which type of candidate will win, given district characteristics. We analyze a two-stage electoral cycle, consisting of a primary pitting a white democratic candidate against a black democrat, with the winner facing a republican in the general election. For the time being, we make the following simplifying assumptions.

1. In the primary election, all BD voters cast their ballots for the BD candidate;
2. In the general election, all BD voters cast their ballots against the R candidate, so they vote for whichever type of Democrat won the primary;
3. In the primary, a fraction a of WD voters cross over to cast their ballots for the BD candidate, with $0 \leq a \leq \frac{1}{2}$;
4. In a BD vs. R general election, a fraction b of WD voters cast their ballots for the BD candidate, with $a \leq b \leq 1$;
5. In a BD vs. WD general election, a fraction c of WD voters cast their ballots for the WD candidate, with $b \leq c \leq 1$; and
6. In the general election, all R voters cast their ballots for the R candidate.

These conditions are fairly natural: the BD and R voters are extreme and so will vote only for candidates of their own party. Furthermore, the BD voters are homogeneous enough that they cross over and vote for WD candidates at a lower rate than WD voters will vote for a black candidate.[4]

[4]The results below do not change qualitatively if we allow for black crossover as well; see the analysis in Section 5.3 below.

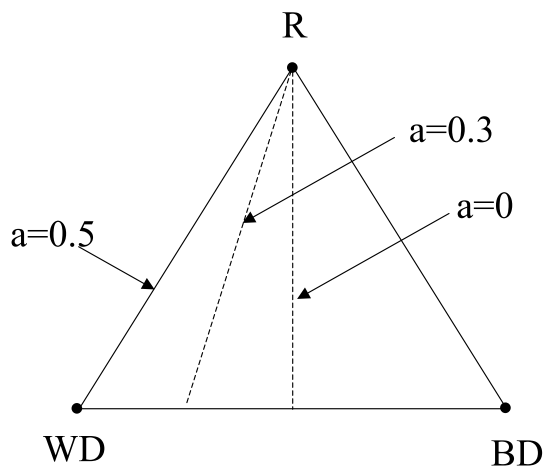


Figure 3: Regions in which minority candidate wins primary, by degree of white crossover (a)

WD voters in a BD vs. R general election will vote for a BD candidate at a higher rate than they crossed over in the primary, since they in general prefer a WD candidate to a republican. And WD voters will vote for a WD candidate versus a republican opponent at a higher rate than they voted for a BD candidate.

Given these assumptions, the BD candidate will win the primary in a district with N_{BD} black democrats and N_{WD} white democrats if:

$$\begin{aligned} N_{BD} + aN_{WD} &\geq (1-a)N_{WD} \\ \frac{N_{BD}}{N_{WD}} &\geq 1-2a. \end{aligned} \tag{1}$$

The greater the value of a, the fewer the number of BD voters relative to WD voters are necessary for a BD to win the election. And the critical ratio reaches 0 when $a = 1/2$; if a were any greater, then a BD candidate could win even with no black voters in the district (which would be nice, of course, but this does not yet describe reality).

The set of points in the triangle satisfying Equation 1 is illustrated in Figure 3. In general, this region will be to the right of a line starting at the top apex and going down to the bottom side. At $a = 0$ it will bisect the triangle, and then grow smoothly until it contains the entire triangle at $a = 1/2$.

For the minority candidate to then win the general election against a republican opponent, normalizing the district size to 1 and using the fact that $N_R = 1 - N_{WD} - N_{BD}$, we need:

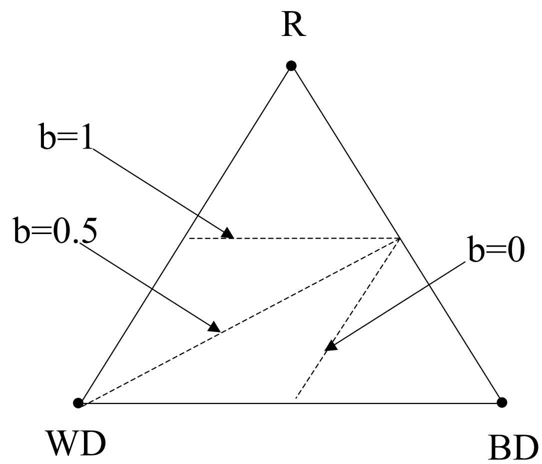


Figure 4: Regions in which minority candidate wins general election, by degree of white crossover (b)

$$\begin{aligned}
N_{BD} + bN_{WD} &\geq (1-b)N_{WD} + N_R \\
N_{BD} &\geq (1-2b)N_{WD} + (1 - N_{WD} - N_{BD}) \\
2N_{BD} &\geq 1 - 2bN_{WD} \\
N_{BD} + bN_{WD} &\geq 1/2. \qquad (2)
\end{aligned}$$

This denotes a region demarcated by a line starting at the midpoint of the right side of the triangle, where $N_{BD} = N_R = 1/2$. At $b = 1$, the line is horizontal, meaning that black and white democratic voters are perfect substitutes, so republicans can only win if they comprise more than half the district. At $b = 0$, the line goes down to the midpoint of the bottom edge of the triangle, meaning that blacks can only win if they are over half the population. These possibilities, along with $b = 1/2$, are illustrated in Figure 4.

Calculations similar to equation 2 show that a white democrat will win the general election if $N_{BD} + cN_{WD} \geq 1/2$. Combining the primary and general election effects, Figure 5 shows a typical scenario for who wins the overall election, drawn for $a = 0.3$, $b = 0.8$, and $c = 1$. Note the asymmetry on the left-hand side of the figure, due to the fact that it is easier for a white democrat to defeat a republican opponent than it is for a black democrat to win, given greater white support for WD candidates as opposed to BD candidates.

One last issue concerns strategic voting, or lack thereof, in the primary. The assumptions above imply that no one votes strategically; the net crossover

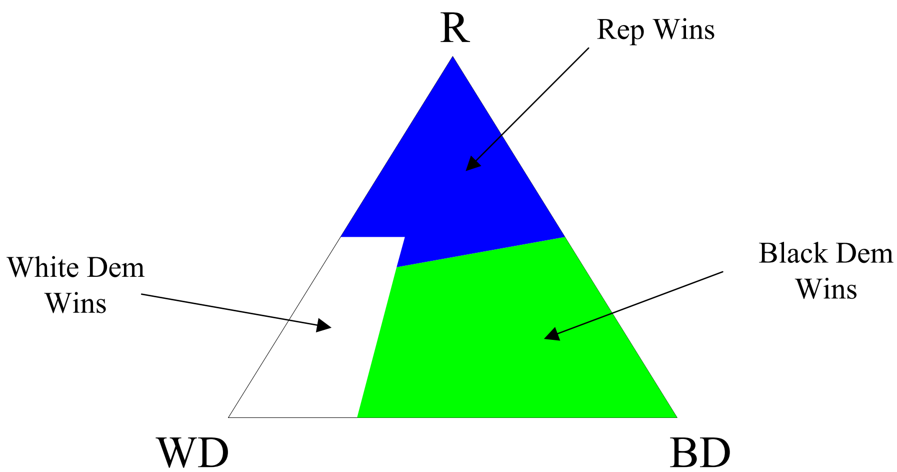


Figure 5: Overall election winners

is a regardless of what happens in the general election. For most points in the triangle this makes no difference, but there is a region in which the lack of strategic voting does alter the results, as indicated in Figure 6. For points in the shaded region in the middle of the triangle, naive voting would have a BD winning the primary and then losing to a republican in the general election. Were the primary voters sophisticated, they would elect a WD instead, who would go on to beat the republican in the general. This type of sophisticated voting is not much in evidence, so we will assume sincere primary voting hereafter and indicate which of our results depend on this lack of sophistication.

2.3 Some Preliminary Analysis

The triangle diagrams also offer a few interesting and cautionary tales, as further illustrated in Figure 7. One diagonal line, running along a contour of constant N_{BD}, goes from an area where a WD wins, to BD, on to R. So there is more to districting than just specifying the percent of black voters; the composition of the rest of the district matters as well. Even more to the point, adding republicans to a district can be good for black voters; as the figure shows, this can allow a black democrat to win the primary, and then the general, whereas a white democrat would have won before. Thus, supposedly conservative shifts in districting can aid minority constituents.

The other diagonal line moves straight towards the BD corner, so it represents adding more black voters to a district while keeping the ratio of WD to R voters constant. Again, the impact is non-monotonic. We first move

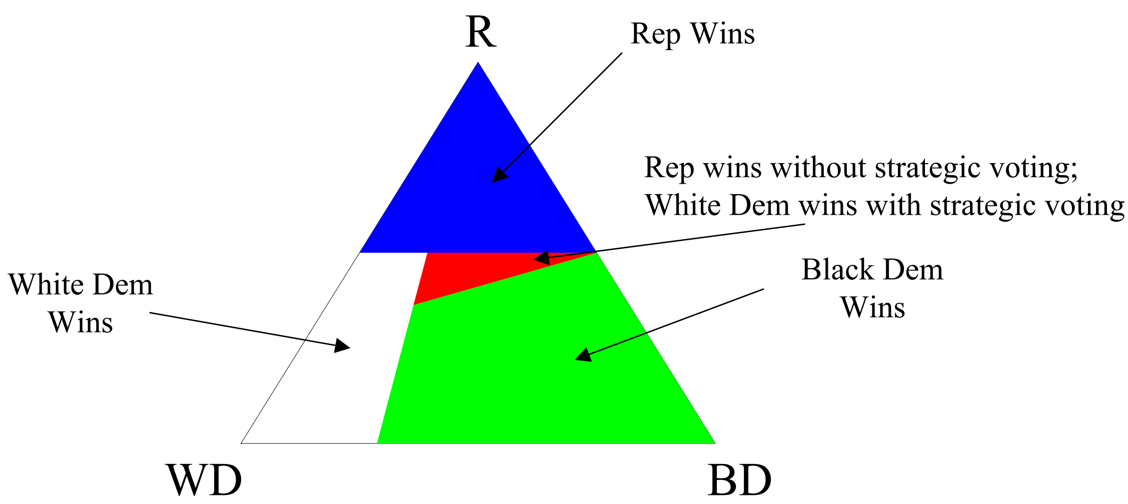


Figure 6: Region in which strategic voting in the primary can affect the election winner

from a situation where a republican wins, to a white democrat, back to a republican, and finally to a black democrat. So a pro-minority shift can have adverse consequences if the entire district composition is not taken into account.[5]

[5]This non-monotonicity requires some degree of sincere voting; if all voters were sophisticated, then the line could not re-enter the republican region.

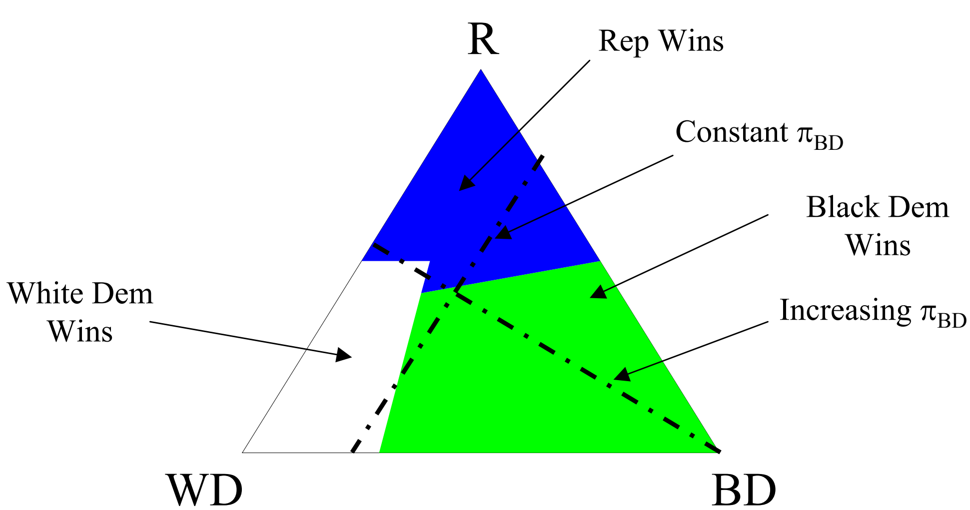


Figure 7: Analysis of election winners

3 A Model of Optimal Districting

The triangle plots help us understand the districting problem and some of its subtleties. We now move to a more systematic analysis of redistricting, specifically from the perspective of the policy gains that can accrue to minority voters. The question is, given a state with a certain percentage of BD, WD, and R voters, which districting plans maximize minority voters' overall utility? Is it better for minorities to have a lot of influence in a few districts, or more modest influence over a wider area? Will optimal schemes elect many or few members of the minority group to office? And how do utility-maximizing schemes change with changes in social conditions, such as increased minority participation and increasing (or decreasing) crossover voting?

To address these questions, we generalize the triangle analysis above by placing it in the context of electoral competition and legislative policy bargaining. Voters in our model have certain ideological attachments to different candidates, and these candidates then compete for office by promising group-specific policy benefits. This approach seems well-suited to our purposes: it captures both the fact that voters of one race may prefer representatives of the same race, as well as competition over policy outcomes. As we will see, it allows us to address the impact of increasing or decreasing racism, greater registration and turnout by minority constituents, and changing partisan attachments in the electorate.

3.1 Districts

Assume a population of voters, V, divided into a given number of identifiable groups Θ; these may be defined according to voters' ethnicity, language, economic status, political party, etc. Thus there is a partition from the set of voters V to groups, $\nu : V \to \Theta$.

For simplicity, we assume here a state population divided along racial and partisan lines with voter types $\Theta = \{BD, WD, R\}$, for black democrat, white democrat, and republican, respectively. Their statewide populations are $\mathbf{N}_{BD}$, $\mathbf{N}_{WD}$, and $\mathbf{N}_R$, with $\sum_i \mathbf{N}_i = \mathbf{N}$, the total state population. Since population proportions must sum to 1, we can represent the mix of voter types statewide—or in any given district—as a point in the two-dimensional simplex, S^2.

A district is a vector $\mathbf{d} = (N_{BD}, N_{WD}, N_R)$ of voters, $N_i \geq 0$. Let $\mathcal{D}$ be the set of all possible districts, and assume that the state will be divided into K districts, K odd, with N_{ik} representing the number of voters of type i in district k. Then a *districting scheme* is a function $\mathbf{D} : S^2 \to \mathcal{D}^K$, yielding a vector $(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_K)$ of districts. Furthermore, a *valid districting scheme* is a districting scheme such that in any given district, $\sum_i N_{ik} = \mathbf{N}/K$, and across districts $\sum_k N_{ik} = \mathbf{N}_i$. Equivalently, as in the triangle analysis above, the average of the percentages of each group in the K districts must equal their statewide population proportion $\mathbf{N}_i/\mathbf{N}$.

3.2 Candidates and Elections

In each of the K districts there are three candidates competing for a seat in the legislature, and these candidates are also of types BD, WD, and R. Candidates try to maximize their vote share, with platforms that offer a proportion T_i of the district's redistributive benefits to voters of type i. Denote the redistributive platform of candidate j's party towards group i in district k as T_{ijk}; then campaign platforms must satisfy $\sum_i T_{ijk} = 1$ for each j and k.

Candidates reach office according to a two-stage electoral cycle. Each district first holds a primary election, in which the BD candidate faces a WD opponent, and then a general election where the primary winner squares off against the republican. Only WD and BD voters cast ballots in the primary, whereas all voters take part in the general election. We assume that candidates are committed to a single platform for the entire electoral cycle; they cannot, for instance, change platforms between the primary and general elections.

Represent a candidate by a vector $c = (\theta, T_{BD}, T_{WD}, T_R)$, where $\theta \in \{BD, WD, R\}$ is the candidate's type, and let $\mathcal{C}$ be the set of all possible candidates. Let $\mathbf{c}_k$ be the vector of three candidates from district k, and $\mathbf{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_K\}$ be the entire set of $(3K)$ candidates in all districts. Then an election is a mapping $\mathbf{L} : \mathcal{D}^K \times \mathcal{C}^{3K} \to \mathcal{C}^K$, producing a representative for each district with a given type and committed to a given platform.

To smooth out the response functions, we assume probabilistic voting, so that the probability a candidate wins a given election rises with the expected proportion of votes she receives. Given expected vote proportion v, let the probability of winning the election be $\Psi(v)$, with $\Psi' > 0$, $\Psi(0) = 0$, $\Psi(1) = 1$, and $\Psi(1 - v) = 1 - \Psi(v)$. We assume here the simplest linear function $\Psi(v) = v$, so that, for instance, a candidate expecting to receive 60% of the vote wins with a 60% probability.[6]

The winners of the K district elections then go to a legislature $\mathbf{L} \in \mathcal{C}^K$. If we take candidates' equilibrium strategies as given, then elections transform a districting scheme into a legislature; that is, $\mathbf{L} = \mathbf{L}(\mathbf{D}(S^2))$.

[6]The qualitative results derived below do not depend on our assumption of probabilistic voting. See, for instance, the results in Table 4.

3.3 Legislative Policy Making

The legislature then passes a redistributive policy, dividing K dollars across all districts. They do so via a Baron-Ferejohn (1989) open rule bargaining process: a legislator is selected at random to offer a proposed division of the legislative pie. The entire legislature then votes on the proposal, and if it is adopted then the game ends. If it is rejected by majority vote, then discounting occurs (all payoffs are lowered by a factor of δ, $0 < \delta \leq 1$), and the game starts again with another member chosen at random to make an offer. In this game, members try to maximize the benefits going to their district.

The outcome of this legislative process will be a vector $(B_1, B_2, \ldots, B_K)$ of district-specific benefits, with $B_k \geq 0$ and $\sum_k B_k \leq K$. So the legislative policy function is $\mathbf{P} : \mathcal{C}^K \to \Re_+^K$. This follows from the results of the elections, which in turn depend on the districting scheme, so $\mathbf{P} = \mathbf{P}(\mathbf{L}(\mathbf{D}(S^2)))$.

Any funds allocated to district k in the legislative process are divided according to the platform adopted by that district's representative. So if the type j representative from district k ran on a platform promising T_{ijk} to members of group i, then voters in this group will receive $T_{ijk} * B_k$ in total benefits, with individual benefits $b_{ij} = (T_{ijk} * B_k)/N_{ik}$.

3.4 Voters

Voters from group i receive utility $U_i(\cdot)$ from this redistributive policy outcome, identical for each member of the group. In particular, assume that the utility from consumption is given by:

$$U_i(b) = \kappa_i \frac{b^{1-\epsilon}}{1-\epsilon}$$

where $\epsilon > 0$. Then the marginal utility of an extra dollar of consumption is

$$U_i'(b) = \kappa_i(b^{-\epsilon}).$$

As b increases from 0 to ∞ the marginal utility falls from ∞ to 0. A one percent increase in b causes an ϵ percent decrease in marginal utility, so ϵ captures the degree of diminishing returns in private consumption. Furthermore, the parameter κ captures the tradeoff between ideological and consumption benefits; higher values of κ imply that voters are more responsive to distributive as opposed to ideological benefits.

Voters also benefit from their ideological attachment to the winning candidate in their district. Each voter is assumed to receive an ideological benefit X^j for a candidate of type j. Thus, for instance, a voter with ideological preference of X^{BD} for black candidates and X^R for republicans gets extra utility

$X^{BD} - X^{R}$ from seeing a black democrat win office instead of a republican.[7] The voter will therefore prefer the black democrat unless the republican offers her sufficiently greater consumption value:

$$E[U_i(b_{iR})] - E[U_i(b_{iBD})] > X^{BD} - X^{R}.$$

Define the critical value, or "cutpoint" X_i for group i in an election between candidates of types 1 and 2 by:

$$X_i^e = U_i(b_{i1}) - U_i(b_{i2}),$$

where e is the type of election being contested. Then group i voters with values of X less than X_i^e will vote for candidate 1, while the others will vote for candidate 2. Let $\Phi_i^e(\cdot)$ be the cumulative distribution of voters of group i in an election of type e, so that, given the campaign platforms, a proportion $\Phi_i^e(X_i)$ will vote for candidate 1. Given N_i voters of type i, this candidate will receive $N_i \Phi_i^e(X_i)$ votes from group i, with total votes of:

$$V_1^e = \sum_{i \in \Theta} N_i \Phi_i^e(X_i).$$

Similarly, the opposing candidate will get votes:

$$V_2^e = \sum_{i \in \Theta} N_i [1 - \Phi_i^e(X_i)] = \sum_{i \in \Theta} N_i - V_1^e.$$

The distribution functions $\Phi_i^e(X_i)$ play an important role in the analysis to follow. They indicate the ideological preference of a given voter i for one candidate over another. These preferences could arise partly from a spatial policy model, measuring the degree to which voters agree with the policy choices of their representative. But they could also arise to some degree from racial voting preferences: voters might want to support candidates of one race over those of another race. In the legal literature, this is what is meant by polarized voting; the willingness, or lack thereof, of voters to cross over and vote for candidates of another race. We assume for simplicity that if the distribution of type i voters in the entire population is $\Phi_i(\cdot)$, then this is also the distribution of the type i voters in any given district.[8]

Notice that the rates at which different types of voters cast their ballots for various candidates are given by the $\Phi_i^e(0)$ functions for group i in an election of type e, where for convenience we label the primary as election $e = 1$, a BD vs. R general election as type $e = 2$, and a WD vs. R general

[7] This difference may, of course, be negative.

[8] We also assume that the number of voters in each district is large enough that we can calculate expected voter utility as the integral of $\Phi_i(\cdot)$ with respect to voter types.

Group	Election	BD	WD	R
BD	Primary	BD^1	$1 - BD^1$	
WD	Primary	WD^1	$1 - WD^1$	
BD	BD/R	BD^2		$1 - BD^2$
WD	BD/R	WD^2		$1 - WD^2$
R	BD/R	R^2		$1 - R^2$
BD	WD/R		BD^3	$1 - BD^3$
WD	WD/R		WD^3	$1 - WD^3$
R	WD/R		R^3	$1 - R^3$

Table 2: Crossover rates

as $e = 3$. For instance, in a BD vs. WD primary, a proportion $\Phi^1_{BD}(0)$ of black voters will vote for the BD candidate, and the remaining $1 - \Phi^1_{BD}(0)$ will vote for the WD candidate.

We redefine these quantities as *crossover* rates, in accordance with the usual standard for voting studies, letting θ^e represent the rate at which voters of group θ vote for the more liberal candidate in election e.[9] Thus, a proportion WD^1 of white democrats cross over to vote for the BD candidate in the primary, while $1 - WD^1$ vote for the white candidate. Similarly, a proportion R^2 of republican voters prefer the black democrat in a general election. For reference, a table of these crossover rates is given in Table 2.

Then, for instance, the BD candidate will be expected to win the primary if:

$$\begin{aligned} N_{BD}BD^1 + N_{WD}WD^1 &\geq N_{BD}(1 - BD^1) + N_{WD}(1 - WD^1) \\ N_{BD}(2BD^1 - 1) &\geq N_{WD}(1 - 2WD^1) \\ \frac{N_{BD}}{N_{WD}} &\geq \frac{1 - 2WD^1}{2BD^1 - 1}, \end{aligned}$$

similar to Equation 1 above. We make only a few basic assumptions about the relative magnitudes of these crossover variables, based on the idea that white

[9] Assuming for the purposes of definition that black democrats are more liberal than white democrats, who are more liberal than republicans.

democrats are closer in ideology to black democrats than are republicans. This leads to: $BD^1 > WD^1$; $BD^2 > WD^2 > R^2$; and $BD^3 > WD^3 > R^3$.

Let Ψ_θ^e represent the probability a type θ candidate wins election e, and Ψ_θ be the probability that the candidate wins overall. Given that the proportion of votes a candidate receives equals her probability of winning, we have:

$$\Psi_{BD}^1 = \frac{N_{BD}BD^1 + N_{WD}WD^1}{N_{BD} + N_{WD}} \tag{3}$$

$$\Psi_{WD}^1 = 1 - \Psi_{BD}^1$$

$$\Psi_{BD}^2 = \frac{N_{BD}BD^2 + N_{WD}WD^2 + N_R R^2}{N_{BD} + N_{WD} + N_R} \tag{4}$$

$$\Psi_R^2 = 1 - \Psi_{BD}^2$$

$$\Psi_{WD}^3 = \frac{N_{BD}BD^3 + N_{WD}WD^3 + N_R R^3}{N_{BD} + N_{WD} + N_R}$$

$$\Psi_R^3 = 1 - \Psi_{WD}^3$$

$$\Psi_{BD} = \Psi_{BD}^1 \Psi_{BD}^2 \tag{5}$$

$$\Psi_{WD} = \Psi_{WD}^1 \Psi_{WD}^3$$

$$\Psi_R = 1 - \Psi_{BD} - \Psi_{WD}.$$

These equations define a surface on S^2 similar to that illustrated in Figure 5, but with smoothly increasing probabilities of election for each type, rather than sharply demarcated regions.

The overall utility for a voter of type i with distributive benefits b_i and a representative of type j is the sum of their ideological and distributive benefits: $\mathcal{U}_i = X_i^j + E[U_i(b_i)]$. Voters are assumed to cast their ballots sincerely for the candidate offering them higher utility.

3.5 Evaluating Plans

To summarize, the order of play is as follows:

1. Given statewide population parameters $\mathbf{N}_{BD}$, $\mathbf{N}_{WD}$, and $\mathbf{N}_R$ and number of districts K, a valid districting scheme $\mathbf{D}$ is enacted.

2. Candidates of type j in district k adopt platforms offering consumption shares T_{ijk} for $i, j \in \{WD, BD, R\}$.

3. Voters elect candidates in primary and general elections, yielding a legislature $\mathbf{L}$.

4. The legislature passes a redistributive policy $\mathbf{P}$.

5. All players receive their utilities, and the game ends.

We will evaluate districting plans according to their impact on minority voters, assuming that a social planner wishes to maximize minority voters' overall welfare. Let L_k be the legislator elected from district k, and let $\theta(L_k)$ be her type. Then given the utility functions above, the social planner selects:

$$\mathbf{D}^* \in \underset{\mathbf{D}\in\mathcal{D}^K}{\operatorname{argmax}} \sum_{i=1}^{\mathbf{N}_{BD}} X_i^{\theta[L_k(\mathbf{D})]} + E\left[U_i(b_i) \mid \mathbf{P}\left(\mathbf{L}\left(\mathbf{D}\right)\right)\right].$$

The social planner, then, must allocate the different types of voters across districts, taking into account the impact of the chosen districting scheme on minority voters' distributive and ideological utilities. For instance, concentrating minority voters into a few districts will increase the probability of electing minority representatives to office, at the potential cost of electing more republicans elsewhere. This strategy also promises large distributive benefits in the concentrated-minority districts, but makes it less likely that these representatives will be included in winning legislative coalitions. Spreading voters out means that minorities have the opportunity to influence outcomes in more districts, but it also raises the possibility that they will be marginalized everywhere, electing no minorities to office and gaining only paltry distributive benefits. The question is how these considerations trade off under changing ideological distributions, population proportions of the different groups, and variations in group power.

4 Platforms and Policy Benefits

We solve the game from the last stage forward. The legislative game is elementary; in equilibrium, the legislator chosen to make the first offer constructs a random coalition of $\frac{K-1}{2}$ other legislators and keeps the remainder for herself.[10] Let l be the legislator who makes the offer, C be the legislators selected to be in the coalition, and D be the remaining legislators. Then equilibrium offers to share the K being distributed are:

$$B_k = \begin{cases} \frac{(2-\delta)K-\delta}{2} & \text{if } k = l; \\ \delta & \text{if } k \in C; \\ 0 & \text{if } k \in D. \end{cases}$$

Since the game is symmetric, each legislator has an expected return of 1 from the legislative bargaining session. This in turn means that if a group is promised T_{ijk} in transfers from a given candidate's platform, then this is also their expected total legislative payout if that candidate is elected to office.

[10] See Baron and Ferejohn (1989).

Candidates then adopt platforms to maximize their votes, balancing their offers to the various groups. In equilibrium the candidates adopt identical redistributive platforms: $T_{i1k} = T_{i2k}$ for each group i in a given district k. Consequently, voters cast their ballots for the candidate for whom they have the higher ideological affinity to start with.

Furthermore, the share of the benefits offered to group i in equilibrium is

$$T_{ij} = \frac{\pi_i N_i}{\sum_j \pi_j N_j}, \tag{6}$$

where

$$\pi_i = [\kappa_i \phi_i(0)]^{1/\epsilon} \tag{7}$$

and $\phi(\cdot) = \Phi'(\cdot)$. The π_i parameters can be thought of as each group's political power. The value of π_i increases for groups with larger values of κ_i, so groups get a bigger share of the legislative pie the more they care about distributive as opposed to ideological issues.

A group's power also grows with $\phi_i(0)$, which is the density of their distribution function at the point where voters are indifferent between the two candidates running for office. This term captures the "swinginess" or "pivotality" of a group: the greater the percentage of a group's members who are indifferent between the candidates, or close to it, the more benefits the group receives. The intuition behind this result is straightforward. First, in equilibrium, the candidates offer the same platform to voters, so this will make no difference in voters' decisions.[11] Since the offers T_i cancel out, those voters who are indifferent between the parties in equilibrium are those for whom $X_i^j = 0$ in the first place. When deciding whether to transfer funds from one group to another, then, it is these marginal voters who will gain or lose; hence the candidates pay off the groups in ratios proportional to their $\phi(0)$ values.

5 Equilibrium Analysis

To understand the properties of this equilibrium, we break the analysis into three stages. We first examine the implications of Equation 6 for per-voter distributive benefits T_{ij}. Ignoring for the moment the ideological benefits of electing different types of representatives, we ask how one would allocate minority voters across districts so as to maximize their total (or average) distributive returns. We next turn to the analysis of the ideological utility that arises from the different types of representatives elected to office. Finally,

[11]Indeed, the most remarkable aspect of the equilibrium is that there exists a pure strategy in platforms that is adopted by both candidates. See Dixit and Londregan (1996, pp. 1149-50) for the details.

we put these two pieces together to describe districting plans that maximize minorities' overall utility.

5.1 Minority Power and Distributive Benefits

The characterization of districting schemes that provide the most benefits to minorities depends on the behavior of the function in Equation 6 on the two-dimensional simplex S^2. That is, we are interested in its behavior on the surface

$$N_{BD} + N_{WD} + N_R = P,$$

where $P = \mathbf{N}/K$ is the total district population. We thus rewrite Equation 6, minorities' distributive benefits in a given district, as

$$\begin{aligned} f(N_{BD}, N_{WD}) &= \frac{\pi_{BD}N_{BD}}{\pi_{BD}N_{BD} + \pi_{WD}N_{WD} + \pi_R N_R} & (8) \\ &= \frac{\pi_{BD}N_{BD}}{\pi_{BD}N_{BD} + \pi_{WD}N_{WD} + \pi_R(P - N_{BD} - N_{WD})} \\ &= \frac{\pi_{BD}N_{BD}}{(\pi_{BD} - \pi_R)N_{BD} + (\pi_{WD} - \pi_R)N_{WD} + P\pi_R}. & (9) \end{aligned}$$

Note that the denominator in Equation 9, which we will denote as Σ, is always positive (this is clear once one realizes that it is the same as the denominator in 8). The derivatives of this equation with respect to the groups' relative powers are straightforward:

$$\begin{aligned} \frac{\partial f}{\partial \pi_{BD}} &= \frac{N_{BD}\pi_R(P - N_{BD} - N_{WD}) + N_{BD}N_{WD}\pi_{WD}}{\Sigma^2} > 0 \\ \frac{\partial f}{\partial \pi_{WD}} &= -\frac{N_{BD}N_{WD}\pi_{BD}}{\Sigma^2} < 0 \\ \frac{\partial f}{\partial \pi_R} &= -\frac{N_{BD}\pi_{BD}(P - N_{BD} - N_{WD})}{\Sigma^2} < 0. \end{aligned}$$

Thus increases in the minority group's power are beneficial, while increasing the power of either other group decreases the minority's utility.

We now turn to the districting question: how to maximize minority voters' utility by changing the numbers of different types of voters across districts. That is, we seek a valid districting scheme $\tilde{\mathbf{D}}^*$ such that

$$\tilde{\mathbf{D}}^* \in \operatorname*{argmax}_{\mathbf{D} \in \mathcal{D}^K} \sum_{i=1}^{N_{BD}} E\left[U_i\left(b_i\right) \mid \mathbf{P}\left(\mathbf{L}\left(\mathbf{D}\right)\right)\right].$$

The utility-maximizing scheme may not be unique; let the set of all such schemes be $\tilde{\mathcal{D}}^*$, with $\tilde{\mathbf{D}}^*$ a representative element. To determine the characteristics of such a $\tilde{\mathbf{D}}^*$, we first take the derivatives of Equation 9 with respect to group populations:

$$\frac{\partial f}{\partial N_{BD}} = \frac{\pi_{BD}[\pi_R(P - N_{WD}) + \pi_{WD}N_{WD}]}{\Sigma^2} > 0 \quad (10)$$

$$\frac{\partial f}{\partial N_{WD}} = \frac{\pi_{BD}N_{BD}(\pi_R - \pi_{WD})}{\Sigma^2}. \quad (11)$$

Equation 10 is always positive, as one would expect; adding more minority voters to a district increases their share of distributive benefits. Importantly, Equation 11 is positive if and only if $\pi_R > \pi_{WD}$, so blacks benefit if voters from the more powerful non-minority group are replaced with voters from the less powerful group. In fact, if we let $\text{int}(T)$ represent the interior of any set T, then we have:

Proposition 1. *For any* $\tilde{\mathbf{D}}^* \in \tilde{\mathcal{D}}^*$, $\pi_{WD} \neq \pi_R \Rightarrow \exists\, \mathbf{d} \in \tilde{\mathbf{D}}^*$ *such that* $\mathbf{d} \notin \text{int}(S^2)$. *Further, if* $N_{BDi} \neq N_{BDj}, \forall\, \mathbf{d}_i, \mathbf{d}_j \in \tilde{\mathbf{D}}^*, i \neq j$, *then there is at most one* $\mathbf{d} \in \tilde{\mathbf{D}}^*$ *such that* $\mathbf{d} \in \text{int}(S^2)$.

Proof. Begin with a valid districting scheme $\mathbf{D}$, and assume to the contrary that $N_{\theta k} > 0$ for all districts k and all θ. Without loss of generality, further assume that $\pi_R > \pi_{WD}$, so that republicans are more powerful than white democrats. Take two districts k_1 and k_2, with N_{BD1} black democrat voters in k_1 and N_{BD2} in k_2, $N_{BD1} > N_{BD2}$.

Consider the effect of moving one republican voter from k_1 to k_2, while at the same time moving a white democrat out of k_2 and into k_1. Such a move clearly preserves the validity of the districting scheme, and by Equation 11 it will increase the utility of black district k_1 voters by

$$N_{BD1}\frac{\pi_{BD}N_{BD1}(\pi_R - \pi_{WD})}{\Sigma^2} = N_{BD1}^2\frac{\pi_{BD}(\pi_R - \pi_{WD})}{\Sigma^2},$$

while decreasing the utility of black district k_2 voters by

$$N_{BD2}\frac{\pi_{BD}N_{BD2}(\pi_R - \pi_{WD})}{\Sigma^2} = N_{BD2}^2\frac{\pi_{BD}(\pi_R - \pi_{WD})}{\Sigma^2}.$$

Given the assumption that $N_{BD1} > N_{BD2}$, the former quantity is larger than the latter, leading to an increase in the average distributive payoffs to black voters. Thus the proposed $\mathbf{D}$ cannot be optimal. ∎

The essence of the proof is to note that, given any two districts in the interior of S^2, one can increase average payoffs to black voters by shifting voters from the more powerful non-black group out of the district with the higher N_{BD}, and voters from the less powerful non-black group in. This will continue until one of the districts collides with a side of the triangle. If there are still interior points, and if the black populations of any two districts are unequal, then continue this process until at most one district remains in the interior of the triangle. The proposition further implies that in optimal districting schemes, the highest concentrations of black voters will share districts with the less powerful non-black group.

All that remains to completely characterize $\tilde{\mathcal{D}}^*$ is to determine the optimal distribution of minority voters across districts. The surfeit of boundary conditions makes the usual maximization solution via Lagrange multipliers opaque, but we can gain insight into the solution by examining the concavity/convexity of the payoff function with respect to the number of black voters in the district. We thus calculate the determinants of the principal minors of the Hessian matrix:

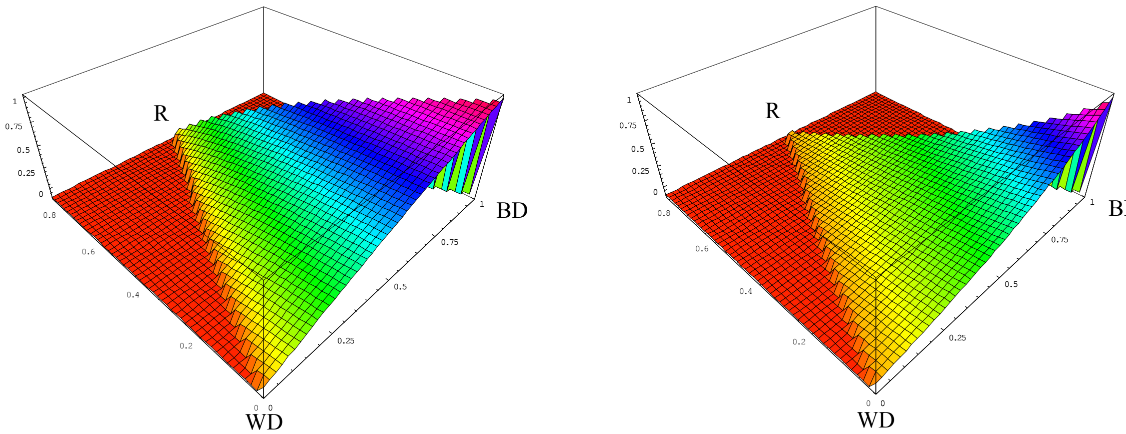
$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial N_{BD}^2} & \frac{\partial^2 f}{\partial N_{BD} \partial N_{WD}} \\ \frac{\partial^2 f}{\partial N_{WD} \partial N_{BD}} & \frac{\partial^2 f}{\partial N_{WD}^2} \end{bmatrix}.$$

Substituting from Equations 10 and 11, we have:

$$\begin{aligned}
\frac{\partial^2 f}{\partial N_{BD}^2} &= \frac{2\pi_{BD}(\pi_R - \pi_{BD})[\pi_R(P - N_{WD}) + \pi_{WD}N_{WD}]}{\Sigma^3} \\
\frac{\partial^2 f}{\partial N_{BD}\partial N_{WD}} &= \frac{\pi_{BD}(\pi_R - \pi_{WD})[P\pi_R + N_{BD}(\pi_R - \pi_{BD}) + N_{WD}(\pi_R - \pi_{WD})]}{\Sigma^3} \\
\frac{\partial^2 f}{\partial N_{WD}^2} &= \frac{2N_{BD}\pi_{BD}(\pi_{WD} - \pi_R)^2}{\Sigma^3} \\
\det(H) &= \frac{\partial^2 f}{\partial N_{BD}^2} * \frac{\partial^2 f}{\partial N_{WD}^2} - \left(\frac{\partial^2 f}{\partial N_{BD}\partial N_{WD}}\right)^2 = \frac{\pi_{BD}^2(\pi_R - \pi_{WD})^2}{\Sigma^4}.
\end{aligned}$$

The determinant of the entire H matrix is clearly positive, but the value of $\partial^2 f/\partial N_{BD}^2$ is indeterminate, indicating that the H matrix can be positive definite, negative definite, or neither, depending on the parameter values. For the purposes of optimization, this means that the surface could be either concave or convex. Examples of each are given in Figure 8. The left-hand figure, drawn with $\pi_{BD} = 7$, $\pi_{WD} = 6$, and $\pi_R = 3$, shows a concave version of the payoffs, while the right-hand figure, drawn with $\pi_{BD} = 3$, $\pi_{WD} = 6$, and $\pi_R = 7$, is convex.

The importance of this difference is clear. If we wish to maximize the overall return to minorities, then in the concave case a social planner would divide minority voters more evenly across districts than she would with a convex payoff function. Note that the difference between the curvatures of the two surfaces lies in the relative power of minorities compared with the other groups: concave for more powerful minorities, and convex for less powerful. This forms the basis of:



(a) Concave: $\pi_{BD} = 7$, $\pi_{WD} = 6$, $\pi_R = 3$ (b) Convex: $\pi_{BD} = 3$, $\pi_{WD} = 6$, $\pi_R = 7$

Figure 8: Concave and convex payoff functions

Proposition 2. *If* $\pi_{BD} = \max_{\theta \in \Theta}\{\pi_\theta\}$*, then* T_{ij} *is concave on* S^2*; if* $\pi_{BD} = \min_{\theta\in\Theta}\{\pi_\theta\}$*, then* T_{ij} *is convex.*

Proof. Consider a districting scheme $\mathbf{D}$ and a district $\tilde{\mathbf{d}} = (N_{BD}, N_{WD}, N_R)$. Define $\alpha = N_{WD}/(N_{WD}+N_R)$, and let $\ell = \{\mathbf{d} \in \mathcal{D} \mid N_{WD}/(N_{WD}+N_R) = \alpha\}$. Thus ℓ is a line running through $\tilde{\mathbf{d}}$, connecting it to $(1, 0, 0)$ (the corner of S^2 where the district is entirely composed of BD voters), keeping the ratio of WD to R voters constant throughout. This defines a parameterized path:

$$\begin{aligned} g(t) &= \frac{\pi_{BD}t}{\pi_{BD}t + \pi_{WD}\alpha(1-t) + \pi_R(1-\alpha)(1-t)} \\ &= \frac{\pi_{BD}t}{[\pi_{BD} - \alpha\pi_{WD} - (1-\alpha)\pi_R]t + \alpha\pi_{WD} + (1-\alpha)\pi_R}. \end{aligned}$$

We then calculate:

$$\begin{aligned} g'(t) &= \frac{\pi_{BD}[\alpha\pi_{WD} + (1-\alpha)\pi_R]}{\{[\pi_{BD} - \alpha\pi_{WD} - (1-\alpha)\pi_R]t + \alpha\pi_{WD} + (1-\alpha)\pi_R\}^2} \\ g''(t) &= -\frac{2\pi_{BD}[\pi_{BD} - \alpha\pi_{WD} - (1-\alpha)\pi_R][\alpha\pi_{WD} + (1-\alpha)\pi_R]}{\{[\pi_{BD} - \alpha\pi_{WD} - (1-\alpha)\pi_R]t + \alpha\pi_{WD} + (1-\alpha)\pi_R\}^3}. \quad (12) \end{aligned}$$

The expression in Equation 12 is negative (positive) if $\pi_{BD} > (<)\alpha\pi_{WD} - (1-\alpha)\pi_R$; that is, when blacks' power is greater (less) than the weighted average of the other groups' powers, based on district population. Clearly, $\pi_{BD} = \max_{\theta\in\Theta}\{\pi_\theta\}$ implies that $g''(t) < 0$ for all t, indicating that the entire surface is concave; conversely, $\pi_{BD} = \min_{\theta\in\Theta}\{\pi_\theta\}$ indicates that the surface is convex. ■

Since optimal values of N_{BD} on a concave surface will be less dispersed than on a convex surface, we have the result that as minority voters gain power, all else being equal, optimal gerrymanders for distributive benefits divide these voters more equally across districts.[12] Formally, let

$$V(\mathbf{D}) = \max_{\mathbf{d}_i,\mathbf{d}_j \in \mathbf{D}^*} N_{BD_i} - N_{BD_j}$$

be the maximum difference between the BD population of any two districts in an optimal districting scheme. Then $\frac{\partial V}{\partial \pi_{BD}} \leq 0$, so that minority voters are (weakly) spread out less as their power increases. Combining these results with Proposition 1, we can say that optimal districting schemes will concentrate black voters in a few districts when their power is low, spread them out when their power is high, and combine them as much as possible with the less powerful of the other two groups.

These results are illustrated in Table 3, which details optimal districts for varying levels of groups' power, done for a state with three districts in which the population proportions of BD, WD, and R voters are 25%, 40%, and 35%, respectively. The power of WD voters in the simulations is fixed at $\pi_{WD} = 3$, while the other two groups' power varies between 1 and 5. Note that, as predicted, the variance $V(\mathbf{D})$ declines and blacks' utility rises within each set of observations as π_{BD} increases, and that, where possible, black voters are put into districts with more voters from the less powerful of the other groups.

5.2 Ideological Benefits

We now turn to the ideological benefit minority voters gain from their representative. We first examine the impact of different districting schemes on the election of minority representatives to office:

Proposition 3.

1. *$\partial\Psi_{BD}/\partial N_{BD} > 0$, so increasing the number of black democrats in a district always increases the probability of electing a black democrat.*

2. *The sign of $\partial\Psi_{BD}/\partial N_{WD}$ is indeterminate, so substituting white democrats for republicans can increase or decrease the chances of electing a black democrat.*

3. *There exists $\widehat{R}^2 > 0$ such that $R^2 < \widehat{R}^2 \Rightarrow \Psi_{BD}$ is convex on S^2.*

[12] In fact, optimal districts when T_{ij} is convex put all minority voters into as few districts as possible. Conversely, when T_{ij} is concave, $N_{BD} > 0$ for all districts.

π_{BD}	π_{WD}	π_R	n^1_{BD}	n^1_{WD}	n^1_R	n^2_{BD}	n^2_{WD}	n^2_R	n^3_{BD}	n^3_{WD}	n^3_R	$V(D)$	Util.
1	3	1	75%	0%	25%	0%	76%	24%	0%	44%	56%	75%	0.25
2	3	1	44%	0%	56%	31%	20%	49%	0%	100%	0%	44%	0.32
3	3	1	0%	100%	0%	39%	0%	61%	36%	20%	44%	39%	0.39
4	3	1	0%	100%	0%	38%	0%	62%	37%	20%	43%	38%	0.43
5	3	1	30%	0%	70%	15%	85%	0%	30%	35%	35%	15%	0.48
1	3	2	0%	20%	80%	75%	0%	25%	0%	100%	0%	75%	0.20
2	3	2	75%	0%	25%	0%	37%	63%	0%	83%	17%	75%	0.25
3	3	2	42%	0%	58%	33%	20%	47%	0%	100%	0%	42%	0.30
4	3	2	35%	0%	65%	11%	89%	0%	29%	31%	40%	24%	0.35
5	3	2	30%	0%	70%	19%	81%	0%	26%	39%	35%	11%	0.40
1	3	3	0%	20%	80%	0%	100%	0%	75%	0%	25%	75%	0.17
2	3	3	0%	40%	60%	75%	0%	25%	0%	80%	20%	75%	0.22
3	3	3	15%	1%	85%	30%	61%	9%	30%	58%	12%	15%	0.25
4	3	3	25%	69%	6%	25%	48%	27%	25%	3%	72%	0%	0.31
5	3	3	25%	55%	20%	25%	33%	42%	25%	32%	43%	0%	0.36
1	3	4	0%	45%	55%	75%	25%	0%	0%	50%	50%	75%	0.17
2	3	4	0%	76%	24%	75%	25%	0%	0%	19%	81%	75%	0.22
3	3	4	0%	0%	100%	0%	95%	5%	75%	25%	0%	75%	0.25
4	3	4	39%	61%	0%	36%	59%	5%	0%	0%	100%	39%	0.29
5	3	4	7%	0%	93%	33%	55%	12%	35%	65%	0%	28%	0.33
1	3	5	0%	11%	89%	0%	84%	16%	75%	25%	0%	75%	0.17
2	3	5	0%	46%	54%	0%	49%	51%	75%	25%	0%	75%	0.22
3	3	5	0%	70%	30%	0%	25%	75%	75%	25%	0%	75%	0.25
4	3	5	0%	0%	100%	40%	60%	0%	35%	60%	5%	40%	0.29
5	3	5	38%	62%	0%	0%	0%	100%	37%	58%	5%	38%	0.33

Table 3: Districting plans that maximize minority voters' distributive benefits, under the assumptions: $N_{BD} = 25\%$, $N_{WD} = 40\%$, $N_R = 35\%$.

Proof. Substituting Equations 3 and 4 into Equation 5, we get:

$$\begin{aligned}
\frac{\partial \Psi_{BD}}{\partial N_{BD}} &= \frac{\partial(\Psi^1_{BD}\Psi^2_{BD})}{\partial N_{BD}} \\
&= \frac{\partial \Psi^1_{BD}}{\partial N_{BD}}\Psi^2_{BD} \cdot \Psi^1_{BD}\frac{\partial \Psi^2_{BD}}{\partial N_{BD}} \\
&= \frac{N_{WD}(BD^1 - WD^1)}{(N_{BD}+N_{WD})^2} \cdot \frac{\sum_\theta \theta^2 N_\theta}{\sum_\theta N_\theta} + \\
&\quad \frac{N_{BD}BD^1 + N_{WD}WD^1}{N_{BD}+N_{WD}} \cdot \frac{BD^2 - R^2}{\sum_\theta N_\theta},
\end{aligned}$$

which is clearly positive, as each of its terms are positive. Similar calculations with respect to WD voters yield:

$$\begin{aligned}
\frac{\partial \Psi_{BD}}{\partial N_{WD}} &= \frac{\partial \Psi^1_{BD}}{\partial N_{WD}}\Psi^2_{BD} \cdot \Psi^1_{BD}\frac{\partial \Psi^2_{BD}}{\partial N_{WD}} \\
&= -\frac{N_{BD}(BD^1 - WD^1)}{(N_{BD}+N_{WD})^2} \cdot \frac{\sum_\theta \theta^2 N_\theta}{\sum_\theta N_\theta} + \\
&\quad \frac{N_{BD}BD^1 + N_{WD}WD^1}{N_{BD}+N_{WD}} \cdot \frac{WD^2 - R^2}{\sum_\theta N_\theta}. \qquad (13)
\end{aligned}$$

Let $N_{BD}BD^1 + N_{WD}WD^1 \equiv V_P$ be the expected number of votes a BD candidate receives in the primary, and $N_{BD}BD^2 + N_{WD}WD^2 + N_R R^2 \equiv V_G$ be the votes she expects in the general election. Then Equation 13 is positive if and only if:

$$\frac{V_P}{V_G} > \frac{N_{BD}}{N_{BD}+N_{WD}} \cdot \frac{BD^1 - WD^1}{WD^2 - R^2}, \qquad (14)$$

which depends on the particular parameter values in question. Finally,

$$\frac{\partial^2 \Psi_{BD}}{\partial N^2_{BD}} = \frac{2\,N_{WD}\,(BD^1 - WD^1)\,(BD^2\,N_{WD} - P\,R^2 - N_{WD}\,WD^2)}{(N_{BD}+N_{WD})^3\,P}, \qquad (15)$$

which is positive when

$$\frac{N_{WD}}{P} > \frac{R^2}{BD^2 - WD^2}. \qquad (16)$$

Hence $R^2 < (BD^2 - WD^2)N_{WD}/P$ implies that Ψ_{BD} is convex on S^2. ■

The results on adding black voters to a district are not surprising; they can only increase the probability that a BD wins both the primary and general elections. Neither are the results on adding WD voters mysterious;

these voters provide support for BD candidates in the general election, but favor WD candidates in the primary, and it is only when the former effect dominates the latter that the overall chances of electing a BD to office rise. Straightforward as this assertion may be, its logical counterpart (really just a restatement under different terms) may still surprise some observers: one may be able to increase the probability of electing a black democrat from a given district by increasing the number of republican voters.

The fact that $\Psi(\cdot)$ is convex at low levels of republican crossover (R^2) implies that under these conditions, districting schemes that maximize the number of black democrats elected will concentrate minority voters in as few districts as possible. This accords with empirical findings on the subject (see for instance Cameron, Epstein and O'Halloran, 1994), although it has never been shown in a general theoretical context before. Two interesting points emerge from the analysis here: first, the relation between electing black democrats and concentrating minority voters depends on low crossover rates; when R^2 is higher, optimal schemes for descriptive representation spread minority voters more evenly across districts. Second, the convexity of $\Psi(\cdot)$ derives from the two-step primary-general election process. Adding black voters to a district, that is, increases the chances a BD candidate wins both the primary *and* general elections, and since $\Psi(\cdot)$ is the product of these two probabilities, adding black voters at the margin has a quadratic impact on the overall chances of electing BD's to office.[13]

To examine the expected ideological utility that members of a group gain from their representatives, define the average utility per voter of a given type i for a j type representative:

$$\overline{X}_i^j = \int_{-\infty}^{\infty} X_i^j d[\Phi(X_i)].$$

Then the total utility to voters electing a type j representative is $N_{ij}\overline{X}_i^j$. For convenience, recalibrate utilities so that $\overline{X}_{BD}^{BD} = 1$ and $\overline{X}_{BD}^{R} = 0$, and define $\beta \equiv \overline{X}_{BD}^{WD}$. Overall expected utility for minority voters includes both the type elected and their average attachment to representatives of that type:

$$\begin{aligned} E(X) &= \Psi_{BD} * \overline{X}_{BD}^{BD} + \Psi_{WD} * \overline{X}_{BD}^{WD} + \Psi_R * \overline{X}_{BD}^{R} \\ &= \Psi_{BD} + \beta\Psi_{WD}. \end{aligned}$$

It is natural to ask whether the districting schemes that maximize minority voters' overall expected ideological utility are the same as those that elect minority representatives.

[13] In fact, looking at the primary and general elections independently, we see that the election function is actually concave in N_{BD} for the primary and linear in N_{BD} for the general, making the overall convexity all the more interesting.

Proposition 4. *There exists* $\widehat{\beta} > 0$ *such that* $\beta < \widehat{\beta} \Rightarrow E(X)$ *is convex on* S^2.

Proof. We examine the second derivative of ideological utility with respect to N_{BD}:

$$\frac{\partial^2 E(X)}{\partial N_{BD}^2} = \frac{2\, N_{WD}\ (BD^1 - WD^1)}{\left(N_{BD} + N_{WD}\right)^3 P} \cdot \\ \{N_{WD}(BD^2 - WD^2) - P\left[R^2 + \alpha\left(N_{WD}\left(BD^3 - WD^3\right) - R^3\right)\right]\}. \quad (17)$$

Let γ represent the latter expression in Equation 17. Then the entire expression $\partial^2 E(X)/\partial N_{BD}^2$ will be positive when γ is positive, implying that:

$$N_{WD}(BD^2 - WD^2) > P\left[R^2 + \alpha\left(N_{WD}\left(BD^3 - WD^3\right) - R^3\right)\right].$$

As we let R^2 and R^3 go to 0, this becomes:

$$\frac{BD^2 - WD^2}{BD^3 - WD^3} > \alpha P.$$

■

When the extra utility of electing a black democrat is high enough (β is close to 0), the $E(X)$ function is convex, and districting schemes that maximize overall utility coincide with those that elect as many black democrats to office. Conversely, when it is more important to avoid electing republicans (β is close to 1), then the function becomes concave, and optimal schemes spread minority voters more across districts. As partisan concerns rise, then, minority voters prefer to work more through electoral coalitions, joining with WD voters to minimize the number of republicans elected to office.

5.3 Optimal Districts

We now combine the previous results in this section to examine districting schemes that maximize minority voters' overall utility, from both distributive and ideological benefits. On the one hand, these benefits are additive, so it might seem that the task is simply to add up the effects examined above. But this rosy scenario is complicated by the fact that the two effects are inextricably linked: groups receive greater distributive benefits if their "swinginess," their density at $\phi(0)$ rises, but this quantity also indicates the amount of crossover voting by members of that group.

This observation cuts two ways. First, as majority voters are more and more willing to cross over and vote for minority candidates, the chances of electing minorities to office rise, which increases the average ideological utility of BD voters. However, this greater willingness to cross over means that majority voters are now more pivotal, so they will receive larger shares of distributive benefits B_k in equilibrium. From minorities' point of view, then, the price to be paid for greater electoral support from other groups is a loss of distributive benefits.

Second, the more politically cohesive are black democrats—the more they vote only for black democrats running for office—the less pivotal they are compared to other groups, and thus the less they get paid off. In this sense, the model captures the notion that the most loyal democratic supporters are also the most easily "taken for granted" by their elected representatives. Thus decreased racial polarization in voting patterns is a mixed blessing for minorities, involving as it does a tradeoff between ideological and distributive benefits.

How do these considerations affect the nature of optimal districting schemes as minorities gain power? We know that the distributive payoff function T_{ijk} becomes concave as π_{BD} rises; how does this interact with ideological utility, given that $E(X)$ is convex under certain circumstances?

Proposition 5. *Districting schemes that maximize minorities' overall utility concentrate minority voters less as their power increases.*

Proof. We know from Proposition 2 that T_{ijk} becomes concave as π_{BD} rises; we wish to determine the conditions under which overall utility $\mathcal{U}_{BD} = U(T_{ijk}) + E(X)$ is concave on S^2 with respect to π_{BD}. Recall from Equation 7 that

$$\pi_i = [\kappa_i \phi_i(0)]^{1/\epsilon},$$

so that π_{BD} can increase either through a rise in κ_i or a rise in $\phi_i(0)$.

Taking the former, a rise in κ_i indicates that black voters prefer more distributive to ideological benefits at the margin. Since voters' overall utility is given by

$$X + \kappa_i \frac{b^{1-\epsilon}}{1-\epsilon},$$

an increase in κ_i indicates that the weight placed on distributive returns increases relative to ideology. This means that the concavity of T_{ijk} will eventually dominate the sum, even if $E(X)$ is convex, making $\mathcal{U}_{BD}$ concave in π_{BD}.

Taking the latter, an increase in $\phi_i(0)$ indicates that minority voters are becoming more pivotal; meaning that their voting rates BD^e decline for each election type e. Taking the total derivative of Equation 17 with respect to BD^e yields

$$\frac{\partial \left(\frac{\partial^2 E(X)}{\partial N_{BD}^2} \right)}{\partial BD^e} = \frac{2N_{WD}\gamma + 2N_{WD}^2(1-\alpha)(BD^1 - WD^1)}{(N_{BD} + N_{WD}^3)} > 0. \tag{18}$$

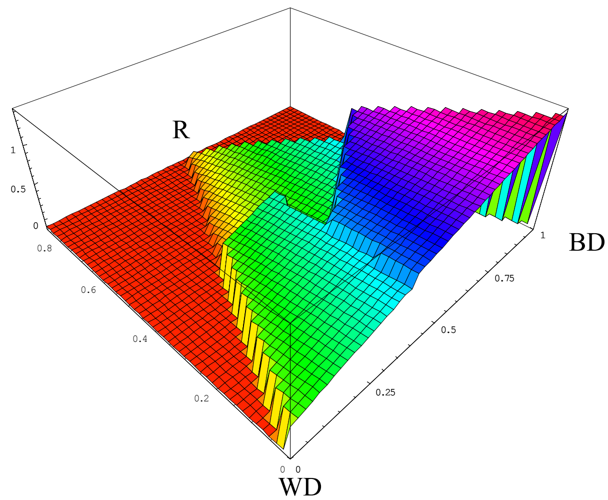


Figure 9: Minority voters' total expected payoffs, including ideological benefits

The expression in Equation 18 is positive, so lower values of BD^e will make the surface of $E(X)$ more concave, again implying that $\mathcal{U}_{BD}$ becomes concave on S^2. ■

Figure 9 is the left-hand (concave) graph from Figure 8, combined with the ideological utility to black voters from different types of representatives, similar to Figure 5. Thus, we have Figure 8 with "cliffs," showing black voters' extra utility from electing a white democrat or black democrat, using republicans as the baseline. This extra dimension (literally and figuratively) to the analysis adds an incentive to create concentrated minority districts, to the point where BD candidates can get elected, or at least having a WD elected rather than a Republican. These districts are of the type in which BD candidates can just eke out a win; that is, they sit near the border of the region in which BD candidates win, so as to not waste extra votes that could be more fruitfully used elsewhere.

To illustrate Proposition 5, then, we calculate optimal districting schemes for the same values of BD, WD, and R power as in Table 3, and using the same overall population proportions. The extra utility of electing a WD is assumed to be 0.1 and 0.3 for electing a BD, relative to a baseline of 0 for a republican. And the WD primary crossover rate is 20%, while the general election crossover rates are 50%.

Note that the variances are generally higher in this table as compared with Table 3, due to the increased desire to concentrate minorities up to the point where a BD candidate can be elected in some districts. Note also that the rule stating that $V(D)$ weakly decreases within each subgroup of

π_{BD}	π_{WD}	π_R	n^1_{BD}	n^1_{WD}	n^1_R	n^2_{BD}	n^2_{WD}	n^2_R	n^3_{BD}	n^3_{WD}	n^3_R	$V(D)$	Util.
1	3	1	0%	61%	39%	75%	0%	25%	0%	59%	41%	75%	0.42
2	3	1	0%	61%	39%	46%	11%	43%	29%	48%	23%	46%	0.50
3	3	1	46%	8%	46%	29%	48%	23%	0%	65%	35%	46%	0.57
4	3	1	28%	47%	26%	31%	38%	31%	16%	35%	49%	15%	0.62
5	3	1	17%	36%	47%	28%	43%	28%	29%	41%	29%	12%	0.67
1	3	2	47%	13%	41%	0%	64%	36%	28%	44%	28%	47%	0.37
2	3	2	29%	41%	29%	46%	13%	41%	0%	65%	35%	46%	0.46
3	3	2	28%	43%	28%	44%	14%	42%	3%	63%	34%	41%	0.52
4	3	2	30%	41%	29%	32%	37%	32%	13%	42%	45%	18%	0.57
5	3	2	29%	42%	29%	17%	35%	48%	29%	43%	28%	12%	0.63
1	3	3	28%	47%	25%	0%	63%	37%	47%	11%	42%	47%	0.35
2	3	3	48%	21%	31%	0%	53%	47%	27%	46%	27%	48%	0.43
3	3	3	5%	54%	41%	30%	45%	25%	40%	21%	39%	35%	0.48
4	3	3	10%	40%	50%	30%	49%	21%	35%	31%	35%	24%	0.54
5	3	3	12%	45%	42%	28%	44%	28%	35%	31%	35%	22%	0.58
1	3	4	0%	50%	50%	27%	46%	27%	48%	24%	28%	48%	0.34
2	3	4	0%	50%	50%	27%	46%	27%	48%	24%	28%	48%	0.41
3	3	4	0%	50%	50%	43%	33%	23%	32%	37%	32%	43%	0.46
4	3	4	14%	36%	50%	30%	45%	24%	31%	39%	31%	17%	0.51
5	3	4	28%	46%	25%	30%	40%	30%	17%	33%	50%	13%	0.56
1	3	5	35%	34%	31%	0%	50%	50%	40%	36%	24%	40%	0.32
2	3	5	0%	50%	50%	36%	29%	34%	39%	41%	21%	39%	0.39
3	3	5	34%	31%	34%	0%	50%	50%	41%	39%	21%	41%	0.45
4	3	5	0%	0%	100%	39%	61%	0%	36%	59%	5%	39%	0.49
5	3	5	38%	62%	0%	0%	0%	100%	37%	58%	5%	38%	0.53

Table 4: Districting plans that maximize minority voters' overall utility, under the assumptions: $N_{BD} = 25\%$, $N_{WD} = 40\%$, $N_R = 35\%$, $\overline{X}^{BD}_{BD} = 0.3$, $\overline{X}^{WD}_{BD} = 0.1$, $\overline{X}^{R}_{BD} = 0$, $\chi^1_{WD} = 20\%$, $\chi^2_{WD} = \chi^3_{WD} = 50\%$.

five simulations still holds in general, although it (barely) fails in the last subgroup. However, it is still the case that when minorities are the least powerful group, at least one district has no minority voters in it; and when minorities are the most powerful, $N_{BDk} > 0$ for all districts k.

6 Application: Changes in the Southern Landscape

We now use the framework developed in the previous sections to examine the impact of various changes that have taken place in the South over the past three decades: increasing black voter registration, the defection of many white democrats to the republican party, and decreasing white racism. For each development, we analyze its impact on minority electoral success, policy benefits, and optimal redistricting plans.

6.1 Increasing Black Registration

Prior to the passage of the 1965 Voting Rights Act (VRA), many southern states enacted laws to, *de facto*, disenfranchise blacks. Such devices as the grandfather clause, poll taxes, and white-only primaries, not to mention direct intimidation, minimized blacks' participation in politics. When one form of discrimination was outlawed, the states would switch to another.

This continued until the VRA swept away all such "tests and devices," and its Section 5 preclearance provisions required covered states to obtain the permission of the federal government before adopting any new law that might impact minorities' ability to vote. The most direct result of passing the VRA was thus to greatly increase blacks' participation, to the point where now, in most areas of the South, minorities register and vote at rates at or above those of white voters.

In the model above, the impact of such an increase in statewide $\mathbf{N}_{BD}$ is (usually) unambiguous: it acts just like an increase in power π_{BD}, and so both increases the flow of benefits to minority constituents and makes it easier to elect minorities to office, thus increasing their ideological benefits. The electoral benefits were illustrated in Figure 7: increasing the percent of black voters while keeping the ratio of WD and R voters constant nearly always makes it easier to elect black candidates. We say "nearly always" because, as pointed out in the earlier discussion of the figure, there are some exceptional regions where, under sincere voting, republicans get elected rather than Democrats. Other than this, though, the overall effect should be to increase descriptive black representation.

The shift in legislative pork barrel benefits is illustrated in the top half of Figure 10, where the horizontal axis shows the ideological distributions of the WD and BD voters, with the 0, or indifference, point in the middle. The

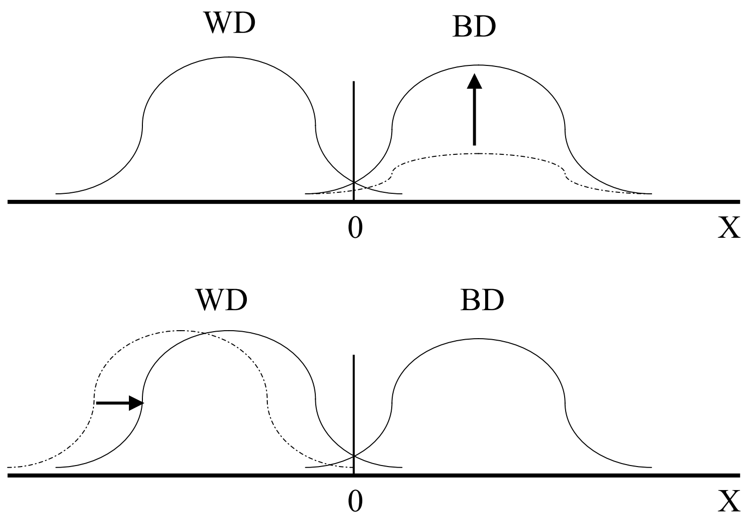


Figure 10: Impact on relative power of WD and BD voters of increase in black registration (top) and decreasing white racism (bottom).

increase in the size of the black electorate increases their power by raising $\phi_{BD}(0)$ while also increasing the number of districts that could elect a BD candidate.

Further, according to the model, the first response of district-drawers as the number of blacks registered and voting increases should be to create concentrated minority districts, and indeed this is what happened in the 1970's and 1980's, with one rule of thumb stating that districts had to be at least 65% black to be "effectively" majority-minority.

As black participation continues to increase, the response should be to concentrate minority voters less, spreading them out more evenly across districts. Again, this is happening, but not without considerable resistance, with divided opinions both within the black community and without. Some worry that reducing the black majorities in these districts will dilute their influence over policy and reduce the number of blacks in office, thereby giving back some of the hard-won gains of the civil rights movement. Others see it as a natural progression of blacks into mainstream politics, and a way to spread their influence over greater areas. This debate continues to be fought, and the recent Supreme Court case *Georgia v. Ashcroft* mentioned in the introduction will hardly put it to rest.

6.2 The Rebirth of Southern Republicanism

The second notable development concerns the breakup of the formerly "Solid South" democratic party. Since Reconstruction, southerners had identified the republicans as the party of Lincoln and the North, and thus voted nearly unanimously for Democratic candidates. But Democratic support of the

VRA and other civil rights measures in the 1960's led inexorably to the defection of many southerners to the republicans, who after all had a solid conservative message that appealed to many voters.

The electoral impact of this shift has been investigated in Figure 7: as we move from a situation where WD voters dominate the political landscape to one where R voters are in evidence as well, it becomes easier to elect BD candidates to office. Thus the electoral and hence ideological impact on black voters is positive.

In addition, Equation 11 above shows that blacks will benefit in terms of policy benefits if the politically stronger group of non-minority voters is replaced by the weaker group, where again power is measured in terms of $\phi_i(0)$. Given that WD voters are more centrist than republicans, the change is indeed in the desired direction, allowing blacks to compete more equally for their share of the legislative pie. Thus the switch from white democrats to republicans is also in blacks' favor.

6.3 Decreasing White Racism

Finally, we come to the increased willingness of white voters of all stripes to vote for minority candidates, due to steadily decreasing racism in the South. Figures 3 and 4 above show the impact of such changes in the values of a and b, both of which expand the region in which a BD candidate can win the general election. Thus decreasing racism does help blacks win office, and indeed, the number of elected blacks in the South has skyrocketed since the adoption of the VRA.[14]

But, as mentioned above and illustrated in the bottom half of Figure 10, the impact on distributive benefits is not straightforward. For white voters to become less racist means that they have less ideological aversion to blacks' holding office, which means that their distribution of X values will shift to the right, as in the figure. This is turn will, at first, increase white voters' density at $X = 0$, meaning that they will enjoy more legislative benefits as they become more pivotal. This will continue until the central hump of the distribution passes the 0 point, past which decreased racism also leads to less of a share of the legislative pie.

Since less than 50% of white voters reliably support black candidates, though, we may assume that we are still on the upward slope of the distribution function. Hence white voters may be gaining increasing benefits from candidates' platforms at black voters' expense. Of course the tradeoff in terms of increased descriptive representation may well be worthwhile, but it is still interesting to note that decreased racism is not an unalloyed good

[14] See the essays Davidson and Grofman (1994) for detailed state-by-state analyses attributing the rise in black office holding directly to the VRA.

for minority voters. In fact, there have long been rumblings that Democrats in office, white and black alike, take their black constituents for granted and give them less than their fair share of insiders' benefits. This may be true, and if so, the model given here provides a plausible rationale for why it would happen.

7 Conclusion

To conclude we return to our motivating example—the recent *Georgia v. Ashcroft* decision—to see how it relates to the analysis above. The courts have interpreted Section 5 of the VRA to mean that the federal government should preclear a proposed change in state laws, including redistricting, if and only if that change does not retrogress from the existing status quo. If we interpret retrogression to mean that the change will decrease the minority's expected utility, then our model predicts that, in general, a rise in minority power should dictate in favor of spreading minority voters out across districts, just as the challenged Georgia redistricting plan did in *Ashcroft*. Indeed, keeping the previous districting scheme, which concentrated black voters into relatively few districts, may well decrease black voters' utility, wasting their votes and helping to elect more republicans to the legislature.

Of course, it may be that the plan passed went too far in spreading out black voters; after all, such a change might trade off descriptive representation — electing blacks to office, which we term ideological benefits — for substantive representation — passing policies preferred by the minority community, which we term distributive benefits. If voters value highly the ideological returns to having many minority office-holders, then they may be unwilling to make such a tradeoff. However, the biggest outcry against the plan came not from blacks voters or legislators (43 out of the 46 black legislators in the State Assembly voted in favor of the plan), but from republicans, against whom the plan was mainly directed.

We end by pointing to a number of possible extensions to our model. First, our current legislative model is simple, so as to focus on the logic of changing preferences and group powers without building in an incentive to form party-based coalitions in the legislature. Hence there are no real legislative parties or permanent coalitions. But if one wanted to investigate the impact of changing legislative rules, committee powers, or party leadership on districting, then these elements could be incorporated into the legislative model.

One could also examine the impact of other electoral rules. Here, minorities must win first a primary and then a general election via plurality votes to gain office. One could just as well use single transferable votes, multi-member districts, approval voting, or any other popular election method and see how that changes the results, both in getting minorities elected and in the policy favors paid by candidates of one type to their supporters of another type. As the saying goes, we leave these for future investigation.

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