Vote Buying

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Abstract

We examine the consequences of vote buying, as if this practice were allowed and free of stigma. Two parties competing in a binary election may purchase votes via up front binding payments and/or payments (platforms) that are contingent upon the outcome of the election. If voters care only about outcomes and not directly about how they vote, then the party with the largest budget wins at a negligible cost. If up front payments are ruled out and only platforms are allowed, then the winning party depends not only on the relative size of the budgets, but also on the excess support of the party with the a priori majority, where the excess support is measured in terms of the total utility of supporting voters who are in excess of the majority needed to win. If voters care directly about how they vote (as a legislator would), then the determination of the winning party depends on a weighted comparison of the two parties' budgets plus half of the total utility of their supporting voters. We also investigate the endogenous raising of budgets, as well as vote buying in the face of uncertainty.

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1 Introduction

The practice of vote buying appears in many societies and organizations, and in different forms. Obvious examples include direct payments to a voter, donations to a legislator's campaign by a special interest group, the buying of the voting shares of a stock, and the promise of specific programs or payments to voters conditional on the election of a candidate. While we generally think of the trade of goods as being welfare improving; this view is not always held with respect to the buying and selling of votes. In some forms vote buying is considered perfectly legal, while in others it is considered illegal, immoral and undesirable. Our purpose in this paper is to explore the consequences of vote buying. Given this purpose, we consider vote buying in a world in which it is allowed and completely free of stigma in order to see how it would function.

We examine a number of questions about a world with vote buying.

- How does the relative size of the bidders' budgets determine the outcome of an election where vote buying is possible?
- What role do the preferences of the voters play in elections where vote buying is possible?
- How does the outcome of the election depend on whether parties can make up front payments or can only make promises that are contingent on the outcome of the election?
- How does vote buying depend on voters' perception of how likely their vote is to be pivotal?
- What is the impact of having voters care about how their vote is cast (regardless of outcome), as legislators might?
- Is the outcome of a vote buying election efficient?
- What can we say about cases where the bidders' budgets are raised from donations by the voters?

In order to address these questions, we consider the following model. Two parties are each interested in obtaining a majority of votes while spending as little as possible, and subject to not exceeding their respective budgets. Voters have preferences over which party wins, as well as any money payments that they get from the parties. We examine a scenario in which parties compete in up front vote buying as well as one in which the parties may only compete by promises that are contingent upon the outcome of the election. In both scenarios, the parties acquire votes through a sequential, alternating-offers, bidding process, and they are fully informed about each other's budgets and voters' preferences. The selling decision affect voters' payoffs in two ways. First, it determines the payment they get in exchange for their vote. Second, a voter derives utility that is related to her fundamental preferences over outcomes, either because she perceives a positive, even if small, probability of being pivotal, or some "irrational" preference for voting for the preferred outcome even if it has no effect on the outcome. In a large election, we expect the latter effect to be small. So, in a large election with up front vote buying, the winner is the party with the larger budget and, due to the sequential nature of the bidding, the winner ends up paying very little to the voters. In contrast, when the parties compete only through campaign promises (platforms), then the identity of the winner also depends significantly on voters' preferences and substantial promises end up being made to a subset of the voters near the median voter. These voters are the cheapest to sway from one party to the other and they continue to be so through the bidding process.

While the above analysis implies that the outcome of the election could generally be Pareto inefficient, we show that this depends on the source of the parties' budgets. If voters can contribute to the budgets of the parties and equilibrium contributions are monotonic in how much voters like each party, then the party that maximizes the total utility of the voters is the winner. We exhibit one contribution game (among many) where this occurs in equilibrium.

The analysis takes an important turn when we investigate the vote-buying model in situations where voters care non-negligibly about how they vote. This variation is particularly relevant for voting in a legislature in the presence of lobbying. In this interpretation, the parties are two opposing interest groups competing to acquire the votes of legislators. The voters are legislators whose voting preferences are explained by popularity of the two alternative positions among their constituencies, which in turn affect their electablility (see, the literature discussion below for related work on this subject). The problem of identifying the winner in terms of the budgets and preferences turns out to be hard in this case. The interesting insight concerns the tradeoff between a favorable shift in preference towards a party and a change in its budget. Roughly speaking, increasing the median voter's preference for *voting* for Party X over voting for Party Y by the equivalent \$1 has the same effect as increasing party X's budget by \$0.5. In this sense, free money is worth *substantially* more to a party than being liked.

The paper also considers vote buying in situations where there is uncertainty over parties budgets and/or voters' preferences. In the course of analyzing competition in platforms, we characterize the equilibrium for the case in which the parties are uncertain about each other's budgets. The equilibrium outcome here is unique and, in fact, we use it to select among the multiple equilibria in the case of commonly known budgets. We also consider the case in which voters' preferences are unknown to the parties. This has little consequence for vote buying in a large election, where voters' preferences have marginal effect anyway. However, it has important consequences for the outcome of competition in platforms. Perhaps the more interesting qualitative departure from the case of known preferences is that the campaign promises are distributed uniformly across voters as opposed to being concentrated on a subset of "swing" voters near the median.

Three different lines of related literature are the work on Colonel Blotto games, the political science literatures on lobbying (e.g., Groseclose and Snyder (1996)) and vote buying (e.g., Kochin and Kochin (1998)), and the finance literature on corporate control (e.g., Grossman and Hart (1988), Harris and Raviv (1988)). Section...brings a more detailed discussion of some of the relations with those literatures.

2 A Model of Vote Buying

Two "parties," X and Y, compete in an election with an odd number, N, of voters. As mentioned in the introduction, we may think of these parties as candidates in the election, or in other applications as lobbyists or interest groups that support different sides of an issue to be voted on by some group of voters.

The Vote Buying Game

Prior to the election the parties try to influence the voting. Parties may directly buy votes with an up-front payment or they may make campaign promises. The direct purchase of a vote by an up-front payment is a binding agreement that gives the party full control of the vote in exchange for the up-front payment. In contrast, a campaign promise has to be honored by the party only if it is elected and it leaves the voter with full control of the vote. The bidding is an alternating offers process. Party k in its turn announces how much it offers in the form of an up front payment p_i^k to voter i for his or her vote, and how much it promises to pay voter i if it is elected, denoted c_i^k. A fresh offer (or promise) made to a voter cannot be lower than those previously made by the same party to the same voter. There is a smallest money unit $\varepsilon > 0$, so offers can only be made in multiples of ε. The parties finance their up front payments and campaign promises out of budgets denoted B_X and B_Y. The total of the up front payments and campaign promises that a party would have to pay at any stage of the game, assuming that the game were to end at that stage and that party were to win, cannot exceed its budget.[1] At each point in time, given the current promises, each voter will have a unique party that he or she will wish to vote for or sell their vote to (as we discuss below). If party k's up front promise p_i^k has been outbid by the other party, so that voter i currently prefers to sell their vote to the other party, then party k does not have to count this up front promise against its budget. (However, *all* campaign promises (platforms) do need to be honored by the winner.)

When a party makes offers and promises, it observes the past offers and promises received by each voter. The preference of a party is to win at minimal cost. We can think of this as a situation where party k's utility of winning is $W_k - t$ and its utility of losing is $-t$, where $t \leq B_k$ is the total of all payments incurred by party k and $W_k \geq B_k$ is k's value for winning. Without loss of generality, given that payments must be in multiples of ε, we round budgets down to the nearest multiple of ε as any remainder can never be bid. The bidding process ends when two rounds go by without any change in the standing up front payments and campaign promises. Once the bidding process ends, voters simultaneously tender their votes to the parties. The party that collects more than half the votes wins.

Initially, we consider the full information version of the game where the parties' budgets and the voters' preferences are known to the parties when they bid. Later, we relax those assumptions.

Let us discuss the interpretation of campaign promises. We analyze two scenarios in what follows. The first is one where both up front payments and campaign promises are possible. In such situations campaign promises end up being largely dominated by up front payments (see Proposition 1 and Lemma 2). Given our assumption that these both come from the same budget, in this scenario the campaign promises are not so much the platform of a party, but really a form of payment that is made contingent on the party winning rather than on a voter's vote. The second scenario that we analyze is one where campaign promises are the only form of vote buying possible. In that scenario, one can really view these as the platform of a candidate, where these are promises to a particular set of distributive payments conditional on being elected. In that scenario, the differences in the budgets would reflect

[1]The restriction that a party has to treat its campaign promises as if it were to win rules out the case where a party with a negligible budget makes outrageous and impossible promises, and thus bids up the payments made by the other party. Such behavior is also easily ruled out by introducing a small amount of uncertainty in the outcome of the election (for instance due to an error in the counting of votes).

the different resources the candidates might be able to draw upon given their idiosyncratic characteristics.

Voter Behavior

The voters are not formally modeled as players in this game, but instead are assumed to sell their votes according to the following simple rule. Each voter i is characterized by a parameter U_i that is interpreted as the difference between the utility she obtains from X's victory in the election and the utility she obtains from Y's victory. U_i can, of course, be either positive or negative. We label voters so that U_i is nonincreasing in i. Under this labeling, we refer to $m = (N+1)/2$ as the median voter. To avoid dealing with ties, which add nothing interesting to the analysis, we assume that, for all i, U_i, αU_i, $U_i/2$ and $\alpha U_i/2$ are not multiples of ε.[2]

If voter i faces final payment promises p_i^k and final campaign promises c_i^k from parties $k = X$ and Y respectively, he will vote for X (and sell the vote to X if $p_i^X > 0$) if

$$p_i^X + \alpha(U_i + c_i^X) > p_i^Y + \alpha c_i^Y, \tag{1}$$

where α is a parameter in $(0, 1]$. Voter i will tender to Y if the strict inequality is reversed.

As we have just said, the voters are not modeled as players. Nevertheless, let us discuss their presumed behavior. Given that this is a two alternative election, voting for their most preferred outcome is a dominant strategy. Here, $U_i + c^X$ and c^Y reflect the relative values of the final election outcomes. These are weighted by α which represents a voter's relative preference weight on final outcome versus up front payments.

One interpretation of α is that it represents the voters' subjective probability of being pivotal. This is not correct as pivot probabilities are endogenous in this game, and in general votes can be purchased in such a way that pivot probabilities are zero (for instance, by buying one extra vote[3]).[4]

Instead, one can interpret α as measuring the preference of a voter voting for the preferred party, even though that vote might not affect the outcome. In large scale elections,

[2]The alternative to ruling out ties by these assumptions is to introduce tie breaking rules. Under various tie-breaking rules that come to mind, the analysis is messier, but does not result in any important change in the conclusions.

[3]Parties could also make payments contingent on the total number of votes purchased, as in Dal-Bo (2003), in a way so that no vote becomes pivotal.

[4]Accounting for pivot probabilities, a voter will compare$p_i^X + \alpha(U_i + c_i^X) + Prob(X \text{ wins} \mid \text{vote X})(U_i + c_i^X) + Prob(Y \text{ wins} \mid \text{vote X})(c_i^Y)$ to $p_i^Y + \alpha c_i^Y + Prob(X \text{ wins} \mid \text{vote Y})(U_i + c_i^X) + Prob(Y \text{ wins} \mid \text{vote Y})(c_i^Y)$. If Pivot probabilities are negligible, then this reduces to the comparison in (1). Note that α is thus not a pivot probability, but some explicit entrance of outcome utilities into the act of voting.

where votes are cast secretly, α might be quite small. We still take it to be positive, so that preferences over final outcomes serve as a tie-breaker. There are some very important applications where α is likely to be large. This happens in elections where votes are reported and recorded openly and publicly, as in some legislative votes or many committee votes. A legislator might care much more about how the vote is cast.[5] For instance, an important case is where $\alpha = 1$ and U_i represents the anticipated cost of voting against the wishes of the legislator's constituents.

Contingent Payments

Another natural form of strategy that the parties might use is one where an up front promise is made and a vote purchased, but where the payment offered is contingent on winning. This is a sort of hybrid of campaign promises and up front promises: the vote is explicitly purchased and controlled as in the case of an up front payment, but where the payment is contingent on winning as is a campaign promise. It is slightly more complicated in terms of how voters value such contingent promises, as the value of the promise is endogenous to the equilibrium outcome; but that turns out to be inconsequential.

The consideration of such contingent payments has little impact on the outcome of the vote buying games in the following sense. If the winner uses such contingent payments instead of up front payments, they end up costing the same. For the loser, they do not cost anything, but still the promises made cannot exceed the budget. The consequence is that the equilibrium winner of the game where contingent payments are allowed turns out to be the same as when they are not considered. The only modification is that the payments in equilibrium may change, as the loser might make some contingent promises that end up being costless, but that the winner ends up having to outbid in equilibrium.

Thus, for ease of exposition, we will not consider such contingent vote purchasing explicitly, but we will return to explicitly mention how our results adjust to the consideration of contingent vote-buying at the end of the paper.

Equilibrium

Strategies are defined in the obvious way. The solution concept is subgame perfect equilibrium.

There are several facts about equilibrium that we can easily deduce. Note that since the sum of payments guaranteed to all voters must go up by at least ε in any two rounds such that the game is not declared to have ended, the bidding process must end after a bounded

[5]This motivation for modeling a preference for voting for one alternative over another is not new here, but emanates from the political science literature. See, for instance, the discussion in Groseclose and Snyder (1996) and some of the references there.

number of rounds. This is thus a finite game with perfect information, and so a pure strategy subgame perfect equilibrium can be found by backward induction. Thus, equilibrium exists in pure strategies. Moreover, this as ties never occur, the equilibrium outcome must be the same for all equilibria in any subgame (again by backward induction). This means that there are well identifiable winners and losers. Finally, an up front payment promise does at least as well as a contingent promise because it is not impacted by α and also can be re-allocated if the other party outbids it.

Proposition **1** *The vote-buying game has an equilibrium in pure strategies. In every equilibrium the same party wins, and the losing party never makes any payment (but may make contingent promises that do not result in payments).*

The proofs of all propositions appear in the appendix.

Interestingly, the fact that in the presence of up front payments, contingent payments are dominated by up front payments, this does not mean that they are irrelevant in determining the outcome of the game. This can be seen in the following example.

Example **1** *Campaign Promises make a Difference in the Payments.*

Consider a three voter society where $\varepsilon = 1$, $U_i = 1/2$ for each i, and $B_X = 90$, while $B_Y = 30$. Let $\alpha = 1$. It is easy to see that X wins in each equilibrium. There is an equilibrium where Y sets $c_i^Y = 10$ for all i, and then X has to offer $p_i^X = 10$ to two voters in order to win.

If we rule out campaign promises and only allow up front payments, then X would still win in all equilibria, but would never pay anything. That follows, since in order to get X to pay something in equilibrium, Y would need to make some promises of up front payments. Once Y has bid, X's final purchase will involve the two cheapest voters and Y will end up buying at least one voter even though she does not win. This cannot be part of an equilibrium as Y could deviate and never make any payments and be better off.

The above example shows that the presence of campaign promises can affect the total payments that the winner needs to make in equilibrium. Nevertheless, as the following proposition shows, the presence of campaign promises does not affect who wins the election.

Proposition **2** *The winner in any equilibrium of the vote-buying game when both up front payments and campaign promises are permitted, is the same as the winner in any equilibrium of a modified version of the game where only up front payments are allowed.*

3 Vote Buying with Negligible Voting Preferences

We first consider the case where voting preferences are negligible. That is, we consider a case where α is small enough so that $|\alpha U_i| < \varepsilon$.

In this case, voters view their vote as having no consequence on its own, and thus are happy to tender to the bidder with the highest offer. Also, campaign promises by parties are essentially dominated by direct purchases. As a result, the party with the highest budget (i.e., highest up to a multiple less than m of ε) wins at a negligible cost.

Let $S_X = |\{i : U_i > 0\}|$ be the number of a priori supporters of X, that is the number of voters who in the absence of any payments would prefer the outcome of X. The analogous number for Y is simply $N - S_X$.

Proposition **3** *In this small α case, party X wins in (every) equilibrium if and only if $B_X \geq B_Y + (m - S_X)\,\varepsilon$. In any equilibrium where X wins, its total payments in any equilibrium are bounded above by $\frac{m\alpha B_Y}{m-1} + m\varepsilon$.*

Note that since budgets appear in multiples of ε, this provides a complete characterization of the winner, as then $B_X < B_Y + (m - S_X)\,\varepsilon$ if and only if $B_Y \geq B_X + (m - S_Y)\,\varepsilon$.

The important aspect of this proposition is that basically the party with the largest budget wins, regardless of the voters' preferences over final outcomes. Also while there is some multiplicity of equilibria in terms of the payments that the winning party needs to make, these payments are a tiny portion of the budget (noting that α is negligible). Thus, the party with the deepest pockets wins and at a negligible cost.

We remark that this is in contrast with the results of Groseclose and Snyder (1996), which reflects the importance of the sequential nature of our game. The small α case here corresponds to a case with small utilities in their case. In their analysis, the first mover would need a budget at least twice that of the second mover in order to win. Essentially, the first mover needs to be able to bid in such a way that the second mover cannot afford to buy any majority. In a game with a more sequential nature, as the one we analyze, the if one party is (temporarily) outbid for some voter, it can remobilize those resources. This back and forth leads parties to be on a more equal footing.

The proof appears in the appendix, where we show that if $B_X \geq B_Y + (m - S_X)\,\varepsilon$ then a strategy that we call the Least Expensive Majority strategy (LEM), whereby in each stage of the bidding X acquires the least expensive majority (set of m voters), guarantees a victory to X against any bidding strategy that Y might adopt. This implies that, in equilibrium, Y will not enter the bidding except for some bids that will end up surely being outbid or campaign promises that will never be paid.

We remark that in this small α case, if there is uncertainty about exactly how many voters prefer each candidate (S_X is random), then a candidate whose budget is larger than the other candidate's by $N\varepsilon$ wins.[6]

When the voting preferences carry more significant weight (α is non-negligible), the winner is determined by more complicated considerations that involve both the budgets and the preferences. Despite the simplicity of our model, the problem of identifying the winner when voting preferences are significant turns out to be hard. Section 5 reports results on that question.

4 Platforms

Before turning to the large α case where voting preferences are strong, we examine situations where up front purchases of votes are not possible, and only campaign promises can be made. Here parties compete in promises that will be fulfilled if the party wins. Competition in platforms differs from direct vote buying in two ways. First, the payment to the voter is contingent on winning. Second, the payment is independent of the voter's actual vote.

In order to compare the results from platforms and vote buying one needs to relate the budgets. In the previous literature, payments promised by platforms are often assumed to be financed out of the state's resources that are controlled by the winner. Here, this is a special case of our analysis, and more generally we allow the budgets of the parties to differ. This might reflect differences in the candidates' fiscal policy, abilities, or a host of other factors.

The parameter α is now irrelevant and voter i will vote for X if $c_i^X + U_i > c_i^Y$. Without loss of generality, suppose that the median voter is a supporter of party X ($U_m > 0$). Let n be the largest i such that $U_i > 0$. Given a number z, let z^ε be the smallest multiple of ε greater than z. Let

$$T = \sum_{i=m}^{n} U_i^\varepsilon,$$

as pictured in Figure 1.

[[Insert Figure 1 here.]]

T is the minimal sum that Y has to promise to voters in order to secure the support of a minimal majority, in a case where X does not promise anything. T is thus a measure of the preference advantage that X enjoys over Y. We remark that $T > 0$ since $U_m > 0$.

[6] The exact difference required and the exact payments will depend on specifics of the distribution and are not of sufficient interest to explore in detail.

Proposition 4 *If $B_Y \geq B_X + T$ then Y wins; and X wins otherwise.*

This can be deduced from the proof of Proposition 5.

The idea behind Proposition 4 is fairly straightforward. Y must spend at least T in order to buy a majority. After that, X will try to buy some of these votes back (or others, if Y has overspent on these marginal votes), and the competition back and forth will lead to the winner being the party with the largest budget once at least T has been incurred by Y.

Obviously, under complete information about the budgets, there are many equilibria in the game with competition in platforms. Since the loser will not have to fulfill its promises, it is indifferent among all of its feasible platforms and this gives rise to a large set of equilibria. However, in most of these equilibria the loser's behavior is silly: it is optimal only because the loser is certain it will lose. Thus, it is natural to expect that, if there is any, even slight, uncertainty about the relative strength of the parties, the range of equilibrium behaviors will narrow down dramatically. Indeed, Proposition 5 below establishes that the only equilibria that survive uncertainty over the relative size of the budgets involve LEM strategies where the parties purchase the least expensive majority in their turn.

Proposition 5 *If B_X and B_Y are distributed with full support over $\{0, \varepsilon, ..., B\varepsilon\}$, then in any equilibrium:*

(i) Both parties play LEM strategies.

(ii) Y wins if $B_Y \geq B_X + T$ and ends up pledging exactly $B_X + T$, and X wins otherwise and ends up pledging exactly $\max\{B_Y - T + \varepsilon, 0\}$.

If both parties use LEM strategies, then only voters between m and $n+1$ ever receive positive payments, and the total payments received are $\max\{0, B_Y - T + \varepsilon\}$ if X wins and $B_X + T$ if Y wins. That is, the winner commits ε more than the loser who commits all of its budget to a subset of these "near median" voters. If $B_Y < T$ then any strategy by Y is an LEM strategy, and no payments are made (although Y might still make promises).

While payments might be concentrated among the voters between m and n, the particulars of which voters get how much can differ across equilibria. For example, in one equilibrium using LEM strategies in a case where $B_Y > B_X + T$, the final outcome is that Party X ends up offering its entire budget B_X to a single voter, say voter m, and Party Y ends up winning by offering $U_i^\varepsilon + B_X$ to that voter and U_i^ε to all voters $i \in [m, n]$. This happens by having the parties repeatedly outbid each other by a minimal amount for voter m. In another equilibrium of this sort, X's budget is spread equally over voters $i \in [m, n]$, and Y matches all those bids and tops them off by U_i^ε to compensate for these voters' initial preference for X.

The uncertainty over budgets introduced in the analysis of platforms is used primarily as a refinement that rules out "implausible" equilibria, and should be thought as small.

Let us compare the outcomes of up front vote buying and those of competition in platforms. With small α, these differ in two ways. First, the fundamental preferences play a less important (almost no) role in deciding the outcome of up front vote-buying competition, as the winner in the vote-buying contest is determined by the relative size of the budgets, whereas the utility advantage of one candidate over another, T, enters significantly into the calculations of the winner under platforms. Second, the voters get lower payments under vote buying than under competition in platforms. The higher payments accruing to voters under competition in platforms owe to the contingent nature of the promises, which allow the loser to make significant promises as well. In contrast, in vote-buying competition the party that is destined to lose would just lose money if it made significant bids, which allows the winner to collect at no or at very little cost.

5 Significant Voting Preferences.

We now study the case where α is significant. Here, as we have already analyzed the case where only campaign promises are possible, appealing to Proposition 2, we focus on the case where only up front payments are possible.

As mentioned earlier, the case of large αU_i's is relevant for a model of voting in a legislature in the presence of lobbying. In this interpretation the parties are two opposing interest groups competing to acquire the votes of legislators. The voters are legislators whose voting preferences, the αU_i's, are explained by popularity of the two alternative positions among their constituencies, which in turn affect their electability.[7] We do not insist on a legislature as the only application for the large α case, as it might also be that voters simply have nontrivial preferences over how they vote - regardless of being pivotal.

Besides the substantive interest in this case pointed out above, it is also somewhat interesting from an analytical point of view. When the voting preferences carry more significant weight, the identification of the winner entails more complicated considerations that involve

[7] See Groseclose and Snyder () for additional discussion of why legislators might care about how they vote.

both the budgets and the preferences. Despite the simplicity of our model, the problem of identifying the winner in terms of the budgets and preferences turns out to be hard. Nevertheless, we can provide characterizations of the winners of this competition, in the case where the budgets are sufficiently large (as specified below).

The main result we have this case is that when budgets are large enough the winner is determined by comparing the difference in budgets to (approximately) one half the difference in preferences. In order to understand this result, it is useful to understand the structure of the winning strategies. The following example contrasts the optimal strategy here with what might seem to be a good strategy, namely the LEM (least expensive majority) strategy.

Let $\mathcal{V} = \sum \alpha U_i$. $\mathcal{V}$ measures the total preference advantage for X.

Example **2** *Optimal versus Naive Strategies - Why Utility has a Shadow Price of 1/2.*

There are five voters with $\alpha U_1 = \alpha U_2 = \alpha U_3 = 10$ and $\alpha U_4 = \alpha U_5 = 0$. The grid of bids is in tenths. $B_X = 41$ and $B_Y = 55$. According to Corollary 1 below, X should win as

$$B_X + \mathcal{V}/2 + \alpha U_5/2 = \ 41 + 15 + 0 \ = \ 56 >$$

$$B_Y + m\varepsilon = 55 + .3 = 55.3$$

Let us see how X should play to win. Suppose that X follows the naive LEM strategy of always spending the least amount necessary to guarantee a majority at any stage.

Suppose that at the first stage Y makes offers of $p_3^Y = 24.8$, $p_4^Y = 15.1$, and $p_5^Y = 15.1$; to voters 3, 4, and 5, respectively. The cheapest thing to do is for X to buy back voter 3 at a cost of 14.9 (recall that 3 has a utility of 10 for voting for X). Now, Y counters by offering 24.8 to voter 2. The cheapest thing is for X to buy back voter 2 at a cost of 14.9. Now X has committed 29.8 of its budget and has 11.2 left. Now, Y counters by offering 24.8 to voter 1. X is unable to buy any of the voters 1, 4, or 5, and so Y wins.

What was wrong with this strategy? The problem is that while X bought the cheapest voters at each stage, X also kept freeing up a large amount of Y's budget for Y to spend elsewhere, while X's budget was committed. X needs to worry not only about what X is spending at any given stage, but also about how much of Y's budget is freed up. Effectively, freeing up a unit of Y's budget is "just" as bad for X as spending an extra unit of X's budget.

So, instead of following the strategy of buying the cheapest voters, let X always follow a strategy of measuring the "shadow price" of a voter as the amount that X must spend *plus the amount of Y's budget that is freed up.*

If X had followed that strategy, then in response to Y's first stage offers above, X would have purchased voter 4 (or voter 5) at a price of 15.2. Then Y would have only 15.1 free - while X would still have 25.8 of uncommitted budget. Y would have to buy one of voters 1 or 2. Regardless of which one Y buys, X could outbid Y on either one of these. By simply offering 10 to each of voters 1 and 2, X could make sure that Y could not buy a majority from that point on. So, now X wins.

So, indeed, keeping track of the shadow price is a good strategy. In fact, for large budgets it is an optimal strategy in that it guarantees a win for whichever candidate should win according to Proposition 6. Let us see how we get from this understanding of "shadow prices" to the expressions underlying Proposition 6.

X is now keeping track of the offer that X has to make to buy a voter given the current offer of Y, plus the amount of Y's budget that is freed up. The amount that X has to offer to buy a given voter i when Y has an offer of p_i^Y in place is $p_i^Y - \alpha U_i$. The amount of Y's budget that is freed up is p_i^Y. So the " shadow price" is $2p_i^Y - \alpha U_i$. Dividing through by 2 gives us $p_i^Y - \frac{\alpha U_i}{2}$. This translates into " strength" of Y being Y's budget less the $\frac{\alpha U_i}{2}$'s of the majority of voters that are most favorable to Y. Similarly X's " strength" is X's budget plus the $\frac{\alpha U_i}{2}$'s of the majority of voters that are most favorable to X.

There are some slight adjustments to account for the grid size and some other details that are covered in the formal proof of the following results that we provide in the appendix.

Proposition **6** *X wins if*

$$B_X - B_Y \geq -\mathcal{V}/2 - \alpha U_N/2 + m\varepsilon \text{ and} \tag{2}$$

$$B_X \geq \left|\frac{m\alpha U_1}{2}\right| - \frac{\sum_{i=m+1}^{N} \alpha U_i}{2} - \frac{\alpha U_N}{2} + m\varepsilon \tag{3}$$

and Y wins if

$$B_X - B_Y \leq -\mathcal{V}/2 - \alpha U_1/2 - m\varepsilon \text{ and} \tag{4}$$

$$B_Y \geq \left|\frac{m\alpha U_N}{2}\right| + \frac{\sum_{i=1}^{m-1} \alpha U_i}{2} + \frac{\alpha U_1}{2} + m\varepsilon. \tag{5}$$

If the budgets are large enough so that (3) and (5) are satisfied, then we have the following corollary.

Corollary **1** *If the budgets are large enough so that (3) and (5) are satisfied, then X wins if*

$$B_X - B_Y \geq -\mathcal{V}/2 - \alpha U_N/2 + m\varepsilon$$

and Y wins if

$$B_X - B_Y \quad \leq \quad -\mathcal{V}/2 - \alpha U_1/2 - m\varepsilon.$$

The interesting feature is the relationship between the relative value to a party resulting from increasing the preferences of voters towards a party by some amount and that resulting from increasing the party's budget by the same amount. Very roughly, when preferences are known, increasing the median voter's preference for a given party by \$1 is equivalent to increasing the budget of that party by \$0.5. Thus money is worth *much* more to a party than being liked, even if voters are likely to be pivotal or care intensely about how they vote.

Let us make one remark about the results here compared to those in Proposition 3. The small α case is a special case of the above results. With small α, $\mathcal{V}$ is negligible relative to the budgets, and the comparison boils down to a comparison of the budgets. Note also, that then the optimal strategy simplifies to the LEM strategy, but the LEM is only optimal in that special case.

The next example shows that Proposition 6 is not valid without the assumption of large enough budgets.

EXAMPLE **3** *Large versus Small Budgets*

Consider a society where $B_Y = 0$. Let there be 3 voters. Let $\alpha U_1 = -10$, $\alpha U_i = -20$, and $\alpha U_3 = -30$. Let $B_X = 30.2$ and have the grid be in $\varepsilon = 0.1$. Here X can win by buying voters 1 and 2 at prices of 10.1 and 20.1.

In this example

$$B_X + \frac{\mathcal{V}}{2} + \frac{\alpha U_1}{2} = -5 < B_Y - m\varepsilon = -.2,$$

and so if we applied the expressions from Proposition 6, we would mistakenly conclude that Y should win. Those expressions cannot be applied when the budgets are small. The problem is that with a small budget, a candidate cannot take advantage of the utility of voters and the game changes. In the appendix, we offer a conjecture for the correct expressions for the case of small budgets.

We close this section with an example showing that while voters preferences only count half as much as monetary budgets, having minority support that is very strong can be enough to help a candidate overcome having a smaller budget than the opposition.

EXAMPLE **4** *The party with a smaller budget and minority support can win*

There are three voters and let $\varepsilon = .1$.

$U_1 = U_2 = 10$ while $U_3 = -60$, and $\alpha = 1$.

The budgets are $B_X = 200$ and $B_Y = 190$. So X has a larger budget and starts with the support of the majority of voters. However, applying Proposition 6, we see that

$$B_X + \frac{\mathcal{V}}{2} + \frac{\alpha U_1}{2} = 185 < B_Y - m\varepsilon = 190 - .2.$$

Here, the strong support of the third voter for Y is a big asset. Very roughly, the game boils down to one where X has to win the support both voters 1 and 2, while Y needs only to get one of them.

6 Efficiency and Endogenous budgets

We now discuss some issues about the efficiency of vote buying.

One important aspect about our characterizations of the competition between parties, is that the relative budgets of the parties is an important determinant of the winner, and that voters' preferences are not fully reflected in the determination of the outcome (and sometimes not reflected at all). For instance, in a small α case where voters strongly support X, but Y has a slightly larger budget, Y still wins. Thus, even if we take the budget of the parties to represent the utility of some unmodeled agents, the outcome of a vote-buying equilibrium can result in a Pareto inefficient decision. Generally, there is no tight relationship between the vote-buying equilibrium outcome and overall welfare, if we take the parties' budgets to be exogenous.

The conclusion that vote-buying equilibria can be Pareto inefficient, and are not always appropriately related to voters' preferences over outcomes, derives from the fact that we have taken budgets to be exogenous. If we endogenize the budgets, then we can fully account for all preferences, and we also reach very different conclusions about the Pareto efficiency of the vote-buying equilibria. In particular, if voter contributions are monotonic in total utility then the outcome is efficient, at least in the small α case. Here we outline a very simple game that provides such an equilibrium.

We now stick to the case where only up front vote buying is permitted; appealing to Proposition 2, as our concern is whether the efficient candidate wins.

Consider the following variation on the vote-buying game, which we call the *Campaign Donation Vote Buying Game*.

(1) There is some ordering over voters, according to which voters sequentially choose an amount to donate to each party, where voter i's donations are denoted $(d_i^X, d_i^Y) \in$

$[0, |U_i|]$. Donations are made in a series of rounds, and voters can increase their promised donations in any round. Any increase must be at least in multiples of ε, or the remaining budget that a voter has if that is smaller than ε. The donation part of the game ends when there is a round with no increases in donations.[8]

(2) The parties' budgets are $B_X = \sum_i d_i^X$ and $B_Y = \sum_i d_i^Y$.

(3) The parties play the vote-buying game.

Let U_X be X's support in terms of total utility of voters ($U_X = \sum_i [U_i]^+$), and U_Y be Y's support in terms of total utility of voters ($U_Y = \sum_i [-U_i]^+$).

Let us first consider the small α case.

Proposition 7 *Party X wins in the campaign donations vote-buying game if and only if $U_X \geq U_Y + (m - S_X)\varepsilon$. Only the winning party receives campaign donations, and those donations are at most $m\varepsilon$.*

Thus, up to a small factor, the outcome of the vote-buying game is now the one that maximizes the overall total utility of the society. Interestingly, this now offers a potential Pareto improvement, not only over the case where budgets are exogenous, but also over the case where there is no vote buying. For instance, in the absence of vote buying it is possible for a candidate to be elected who has a majority of supporters but who lacks the majority of support in terms of total utility. When voters can donate to candidates campaigns, this is no longer possible.

Let us now turn to the case where α might be significant. We treat the case where U_X satisfies (3) in the place of B_X, and U_Y satisfies (5) in the place of B_Y.

Proposition 8 *In the large budget case, party X wins in the campaign donations vote-buying game if*

$$U_X - U_Y \quad \leq \quad -\alpha U_N/3 + \frac{2}{3}m\varepsilon$$

[8]We cap voters' donations at their total utility and require minimal increases simply for convenience, as it keeps the game finite. One could alternatively consider the infinite game where voters could make arbitrary increases in donations in any given period (and would have to assign a largely negative utility to the infinite path where the game never ends). In equilibrium, voters would never make payments exceeding their total utility in any case, and although they might make higher payments off the equilibrium path, the equilibrium outcome would remain unchanged.

and Y wins if

$$U_X - U_Y \ \leq \ -\alpha U_1/3 - \frac{2}{3}m\varepsilon.$$

Only the winning party receives campaign donations, and those donations are the minimum necessary for the winning party to buy a majority (without an opposition).

The proof of Proposition 8 is an easy extension of the proof of Proposition 7, noting (2) and (4), and that $\mathcal{V} = U_X - U_Y$.

Again, except in the case where the total utility is almost evenly balanced, the efficient party wins in equilibrium when budgets are endogenized. Thus, approximate efficiency is guaranteed when budgets are endogenous.

Let us emphasize that the above results also show that the donation game exhibits minimal donations. While this depends on the complete information environment and the fact that there is no access related motivation for donations, this still provides important intuition for why both campaign donations and spending might be dwarfed relative to the utility value of the outcome of an election.

7 Unknown preferences

Our analysis for the large voting preference case, has focused on situations where the voting preferences are known. We close with an analysis that examines the case where voters' preferences are private information. Again, we examine the case of direct vote buying.

Suppose that, for all i, αU_i is an independent draw from a continuous distribution F with connected support. Suppose also that both $x - [1 - F(x)]/f(x)$ and $x + F(x)/f(x)$ are increasing for all x.

Here the parties are symmetric. The resulting equilibrium has an intuitive relationship to that of Myerson (1993), except that voters' preferences (which were absent in Myerson's analysis) enter here in an important way. Essentially voters preferences give a boost to the party who is expected to have median support.

Proposition **9** *For any $\delta > 0$, there is $N(\delta)$ such that for all $N > N(\delta)$ the following hold (1) If $B_Y > B_X + F^{-1}(0.5)N/2 + \delta$, then Y wins with probability of at least $1 - \delta$. (2) If $B_X > B_Y - F^{-1}(0.5)N/2 + \delta$, then X wins with probability of at least $1 - \delta$.*

The result is almost a complete characterization for large N, as the budgets cover most possible budget differences except than those that fall in an interval of size 2δ.

We note that when δ is sufficiently small, the party who is likely to lose will not enter the bidding and the winning party will bid the minimum necessary to secure majority with sufficiently high probability. Thus, we again can see a result that echoes the earlier ones, where we see minimal spending in equilibrium.

We can also endogenize the budgets here in the obvious way. As well, we can consider the case of platforms, where the analysis will be similar, but the relevant preferences are those over outcomes, and α drops out.

8 Concluding Discussion

Now that we have presented our results, we return to discuss some things that we deferred earlier.

Contingent Payments

We mentioned that another natural form of strategy that the parties might use is one where an up front promise is made and a vote purchased, but where the payment offered is contingent on winning. As claimed earlier, the consideration of such contingent payments has little impact on the outcome of the vote buying games in the following sense. All of the Propositions extend to the additional consideration of contingent payments, modulo the fact that the payments by the winner might be larger in Proposition 3. (Note that Propositions 4 and 5 only consider campaign promises, and so no up front promises would be considered, contingent or otherwise.) The idea of the proof is the following: suppose that the winner changed from X to Y due to the introduction of such contingent promises. Then in equilibrium, any of Y's promises turn out not to be contingent. By using non-contingent promises according to the original equilibrium strategy X can defeat Y's strategy. While this stops short of being a proof, it provides the essential ideas.

Other Analyses of Vote-Buying Games:

Colonel Blotto Games: A "Colonel Blotto Game" is one where two opposing armies simultaneously allocate forces among n fronts. Any given front is won by the army that committed a larger force to it and the overall winner is the army that wins a majority of the fronts. This model can be readily interpreted as a model of electoral competition, where each party wins the voters to whom it made the larger promise and the overall winner of the election is the party that managed to win a majority of the votes.

Indeed formal models of electoral competition with promises using this framework date back at least to Gross and Wagner's (1950) continuous version of a Colonel Blotto game.

The difficulty in using the Colonel Blotto Game to deduce anything about vote buying is that, even in the simplest setting with identical voters and candidates, such games are notoriously difficult to solve.[9] The existing analyses are of symmetric mixed strategy equilibria in which voters are treated identically (from an ex ante point of view) and the parties are equally likely to win, which does not provide much insight into vote-buying behavior.

Our decision to model the competition as a sequential bidding process is partly motivated by the need to find a more workable model that also allows to consider asymmetries among candidates and/or voters. The sequential model introduces of course some new technical complications[10], but it allows us to deal with asymmetries without dealing with the unwieldy mixed strategy equilibria of the simultaneous version.

The idea of considering sequential vote buying games is not new to us, and appears in Grossman and Hart (1988) and Harris and Raviv (1988) [discussed below], as well as Groseclose and Snyder (1996). Groseclose and Snyder present a model of vote buying in a legislature. This model is like the direct purchase part of our model except for the bidding procedure, which in their model ends after two rounds. The main insight is that the second mover has a substantial advantage. The first mover has to purchase a supermajority of voters in order to successfully block the response of the second mover. Thus, for example, in the $\alpha = 0$ version of the model, the first mover would need twice the budget of the second mover in order to win, since the second mover should not be able to purchase the least expensive 50%. As is evident from the above analysis, our more symmetric bidding process neutralizes the effect of the order of moves and consequently gets different results both with respect to the identity of the winner, and how much they pay and which voters they buy.

There are other articles that are related in that they address the broad efficiency and distributional considerations that also motivate us. But those discussions that we found are so distant in terms of their focus and framework that they should be considered largely complementary to our discussion it might not be useful to try to relate them to our analysis. For example, Kochin and Kochin (1998) offer a logic for the prohibition of vote buying, which is based on the costs of buying votes and forming blocking coalitions. This, they argue, can lead to inefficient decisions depending on the source of costs and how they are

[9]see Laslier and Picard (2002) and Szentes and Rosenthal (2001) for some characterizations of equilibria). Myerson (1993) circumvents some of the technical difficulties of Colonel Blotto games by allowing candidates to meet the budget constraint on average, rather than exactly.

[10]Our setting can be seen as an ascending all-pay auctions over multiple goods, where there is an extreme form of complementarity among the goods: they are valuable only if a majority is purchased.

distributed. They suggest that in the absence of any costs, vote buying will always lead to efficient decisions, although the specific vote buying process is not modeled.[11]and[12]

In terms of the structure of the vote-buying game, the strand of the literature on corporate control (Harris and Raviv(1988), Grossman and Hart (1988)) is also closely related to our analysis. They examine settings in which two alternative management teams–the incumbent team and a rival–are competing to gain control of the corporation through acquisition of a majority of the shareholders' votes. The alternative teams are the counterparts of our parties and the private benefits that these teams would extract from controlling the corporation are the counterparts of the parties' valuations for being elected. Since we are not going to discuss the substance of the corporate control issues, let us continue to call the players parties X and Y as we have done thoughout. To understand the model of Harris and Raviv[13] (henceforth HR) consider our basic model with constant U_i, say $U_i = U$, for all i. The bidding procedure is different. It consists of two rounds (like Groseclose and Snyder's model–henceforth GS), but the offers are public price commitments to buy up to 50% of the votes at the announced price (in contrast with the personalized offers in GS's and our models). Unlike in our model, the voters are players. After observing the price offers made by the two parties, the voters decide simultaneously how to tender their votes, which they may split in any way they wish between the parties. The winner is a party that buys $N/2$ or more votes and still has some free cash. Their main result is the characterization of an equilibrium in which Party X wins if $B_X + NU/2 > B_Y$ and Party Y wins if the strict inequality is reversed[14]. If $B_X + NU/2 > B_Y$ and X is the designated first mover, it offers to buy $N/2$ votes at a price of $2B_Y/N - U$ per vote. In equilibrium, Y gives up and does not make any offer in response. If Y wanted to contest X's offer, its most aggressive response would be offering to buy $N/2$ votes for the

[11]This idea is also implicit in arguments by Tobin (1970), who suggests that a market for votes would allow power to be concentrated among the rich - suggesting some frictions in borrowing.

[12]Philipson and Snyder (1996) also find Pareto improvements from vote buying. They model a specialist system for vote buying, and a one dimensional policy space, and find that, if the distribution of ideal points is skewed enough, then the equilibrium with vote buying differs from the equilibrium without vote buying (the median ideal point). This difference reflects the ability of an intense minority to obtain a policy it prefers in exchange for side payments.

[13]The related model of Grossman and Hart does not seem to have an explicit equilibrium model for the particular case that would be close to our model (what they call competition in restricted offers between parties with significant private benefits).

[14]In HR's model the parties do not have budgets but rather private benefits up to which they are willing to spend. However, with the two round bidding procedure in HR, the distinction between budgets and private values is not important and hence we will refer to them as budgets. This distinction might be important in multi-round bidding procedures such as ours, where out of equilibrium behavior might involve bidding above one's value.

price $2B_Y/N$, which would exhaust its entire budget. This offer would not defeat X's offer, since in the ensuing subgame each voter would tender half his votes to X and half to Y. In this situation each voter is pivotal and hence indifferent between tendering to Y and getting the price $2B_Y/N$ and tendering to X and getting the price $2B_Y/N - U$ plus the utility U. While the identity of the winner does not depend on which party is designated to move first, the actual payments made in equilibrium depend importantly on this feature. If X is the second mover when $B_X + NU/2 > B_Y$, then Y does not offer anything in the first round and X wins at zero cost.

Notice that this equilibrium relies importantly on every voter becoming pivotal in the continuation following Y's out of equilibrium response to X's offer. However, the model has other equilibria. Consider the subgame following Y's response. Observe that, if the prices offered by X and Y are different from one another, then the tendering game has an equilibrium in which voters allocate their votes so as to equate their expected revenue (price times probably of not being rationed) across the parties. In this equilibrium no voter is pivotal and the party offering the higher price wins. Going backwards, this implies that in the overall equilibrium the party with the higher budget wins (with a payment that depends again on whether it moves first or second). In a sense this equilibrium seems more robust than the equilibrium that HR focus on, since the everybody-pivot-equilibrium in the subgame would not survive noise (e.g., some fraction of the voters who tender randomly), while the high-price-wins-equilibrium just described will survive it.

To compare HR with our and with GS models, observe that the relevant version of our (and hence also GS) model for this comparison is the $\alpha = 0$ case, since HR's model has no voting preferences (the U in HR's model represents underlying outcome preferences and not voting preferences). Also, since we view pivot considerations as marginal in either of these models, we will focus on equilibria of the HR model that do not rely on pivot considerations. Given this, the important differences are that HR considers quantity restricted uniform price offers, while GS and we consider personalized offers, and that HR and GS consider the two round procedure while we consider multi-round-open-ended procedure. The result is that in GS the first mover needs at least twice the budget of the second mover to win in this environment. In both HR and our model, the highest budget wins, but the payments may differ. In our model the winner pays nothing, while in HR the winner's payment might be substantial if it moves first.

The above discussion exposes the modeling relations between the direct purchase part of our model and the HR paper.

Finally, let us mention that our model also sheds light on some empirical observations.

One of our findings, is that when the two parties are aware of the size of each other's budget, and have a fairly accurate feeling for the voters' preferences, then the price of a vote will be negligible. This reflects the all-pay flavor of the bidding - that is, the loser pays for any votes that they buy. Whenever one party is fairly sure that they will be outbid in the end, they will not enter the bidding. This is consistent with some broad stylized facts that we see both in political elections and stock shares. For instance, Ansolabehere, de Figueiredo and Snyder (2002) document the paucity of money being contributed to political campaigns and find that the largest part of the relatively small donations to campaigns comes from individuals and has little impact on legislator's votes (a puzzle first pointed out by Tullock (1972)). A related puzzle arises in the price of stock shares, where the price of voting shares is generally similar to that of non-voting shares (Lamont and Thaler (2001)).[15] While our stylized analysis is certainly not the only explanation for these stylized facts, it does provide some strong intuition for them.

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10 Appendix

Proof of Proposition 1: The facts that the vote-buying game has an equilibrium in pure strategies follows from the fact that this is a finite game of perfect information, and hence we can find such an equilibrium via backwards induction.

The fact that in every equilibrium the same party wins, also follows from a backward induction argument. Each terminal node has a unique winner (as the αU_i's are not a multiple of ε and so voters are never indifferent), and parties prefer to win regardless of the payments necessary. Thus, in any subgame, working by induction back from nodes whose successors are only terminal nodes, there is a unique winner. It then follows directly that the losing party never makes any payments, as they could otherwise deviate to offer nothing and guarantee no payment. ▮

Proof of Proposition 2: By Proposition 1, we know that there is a unique winner in every equilibrium of the unmodified game. Without loss of generality, say that X is the winner and Y is the loser, of the game where both forms of promises are permitted. Consider a game where X is permitted to make both forms of promises and Y is only permitted to make up front payments. As this only imposes a restriction on Y's strategies, X remains the winner of all equilibria of this game.[16] Next we note that there exists an equilibrium in this game where at any node X only makes up front payment promises (or no promises), as at any point an up front payment is at least as attractive to a voter as an equivalent campaign

[16]More formally, start with an equilibrium in the larger game. Trim the tree so that we eliminate any actions of Y that result in campaign promises. By backward induction, in any subgame of the resulting tree if X won previously, X still wins, while if Y won, then either Y still wins or else X wins. As X won previously in the overall game, X still wins.

promise and is at least as flexible for X as it is no more binding.[17] This (properly trimmed) remains an equilibrium of the game (with the trimmed tree) where no campaign promises are permitted. ∎

Proof of Proposition 3: By Proposition 2, we can determine the winner by examining the game with only up front payments. We then come back to bound the winner's payments in the game where campaign promises are also possible.

Suppose that $B_X \geq B_Y + (m - S_X)\,\varepsilon$. We show that then X has a strategy that guarantees a win. As a symmetric argument applies to show that Y wins if $B_X < B_Y + (m - S_X)\,\varepsilon$, this implies the if and only if statement. We show that the LEM strategy whereby in each stage of the bidding X acquires the least expensive available smallest majority (i.e., m voters), and purchases voters who prefer Y whenever the cost is the same, guarantees a victory to X against any bidding strategy that Y might adopt. This implies immediately that, in equilibrium, Y will only make offers if she expects X to overbid all her offers. As X bids for only the least expensive voters this can occur only if $S_X \leq m$. In this case X will have spend at least $\varepsilon\,(m - S_X)$ to purchase the majority. There are equilibrium in which X spends up to εm. In these equilibria Y bids ε for up to S_X voters i for which $\alpha U_i > 0$, and X buys them back.

We now argue that X wins with the LEM strategy above. A "current winner" at a point in the bidding process will refer to the party that would win if the process terminated at that point, and an "active offer" will refer to an offer that would be taken by a voter in the equilibrium of the selling game that would be played if the process were stopped at that point. Observe that if Y is the current winner and has a sum B committed in active offers, then X has to commit at most $B + (m - S_X)\,\varepsilon$ to become a current winner. To see this suppose that Y is the current winner, let p_Y be the m^{th} highest active offer that Y has outstanding, where we rank voters with identical offers from Y higher if they prefer Y to X, i.e., if $U_i < 0$. Let voter j be the target of that l^{th} highest offer. Let p_X be the highest active offer that X must have in order to become the current winner in the least expensive way, and let voter i be the target of that offer.

If $U_j > 0$ then $p_X \leq p_Y$ for otherwise, it would be cheaper for X to acquire j's vote instead of i's vote. (Recall that when faced with the same offers the voter sells to her preferred party.) Since to become current winner X needs only m active offers, it follows

[17] To be careful, we need to keep track of Y's responses to X's actions. However, given that Y can only make up front payments, using a backward induction argument we can establish that in any subgame X's chance of winning (which is either 0 or 1 in any subgame) can only go up by a switch from a campaign promise to an equivalent up front payment.

that its cost would be at most $p_X m \leq p_Y m \leq B$, where $p_Y m \leq B$ since to be a current winner Y must have at least m active offers with p_Y being the m^{th} highest offer.

If $U_j < 0$ the argument is similar, but requires a little care in counting. In this case assume that $k \leq S_X$ of the voters who prefer X have active offers from Y. By the ranking described above, these voters have an offer of at least $p_Y + \varepsilon$. Now consider those voters not receiving any of the m highest active offers from Y. These include $S_X - k$ voters who prefer X and whose offers from Y must be at most $p_Y - \varepsilon$. Therefore to purchase enough votes X needs at most $(p_Y + \varepsilon)\, m - (S_X - k)\, \varepsilon$, where $p_Y m + k\varepsilon \leq B$, since to be a current winner Y must have at least m active offers with p_Y being the m^{th} highest offer and at least k voters have active offers of $p_Y + \varepsilon$. Therefore $(p_Y + \varepsilon)\, m - (S_X - k)\, \varepsilon = p_Y m + \varepsilon\, (m - (S_X - k))$ $\leq B - k\varepsilon + \varepsilon\, (m - (S_X - k)) = B + \varepsilon\, (m - S_X)$.

This implies that, when X follows that LEM strategy, it can always outbid Y to become the current winner. Since the bidding process must end after a bounded number of rounds, X must win. Since X must buy $m - S_X$ votes, he must spend at least $\max\{(m - S_X)\, \varepsilon, 0\}$. If Y makes an offer to any of the votes that X purchased, then it would cost X more to repurchase that vote than to purchase a different one, and after X's purchase of a different vote Y will eventually lose and have to pay something, which is worse for Y than not purchasing in the first place (by hypothesis) so in equilibrium Y will not purchase back a vote that X purchased. If Y purchases a vote from i such that $U_i > 0$ then X is indifferent between purchasing this vote back at cost ε and purchasing a different vote from j with $U_j < 0$, so, as noted, there is an equilibrium where Y offers ε to some of the $S_X \leq m$ voters and X purchases them back, leading to total cost of up to $m\varepsilon$.

Now, let us come back to bound the payments that X makes when X wins in the game where both up front payments and campaign promises are possible. X can still follow an LEM strategy, and that will still win. As Y surely loses, Y will not be making any binding up front payments in equilibrium. Thus, consider the ending promises that are made by Y. It must that X has bought a least expensive majority, meaning that the maximum price paid for any voter in this majority is at most the minimum price of the voters not purchased. Any promises made by Y to the voters that X did not purchase must have been made in the form of campaign promises. The highest the minimum cost could be is then $\frac{\alpha B_Y}{m-1} + \varepsilon$. The claimed expression then follows directly. ∎

Proof of Proposition 5: The proof is based on three lemmas. First, we characterize the outcomes resulting when at least one player follows LEM strategies. Second, we conclude that there is an equilibrium in which both play LEM strategies. Third, we prove that in any equilibrium LEM strategies are played by both.

Lemma 1 *1. If $B_Y \geq B_X + t$, then*

(a) If X uses an LEM strategy then with an LEM strategy Y wins and spends $B_X + t$.

(b) If X adopts an LEM strategy, then to win Y must spend at least $B_X + t$.

(c) If Y uses the LEM strategy then X cannot win.

2. If $B_Y < B_X + t$, then

(a) If Y uses an LEM strategy then with an LEM strategy X wins and spends $B_Y - t + \varepsilon$.

(b) If Y adopts the LEM strategy then to win X must spend at least $B_Y - t + \varepsilon$.

(c) If X uses the LEM strategy then Y cannot win.

Proof of Lemma 1: 1a and 2a follow immediately from the nature of the LEM strategies: Y initially must buy (we use the term buy to indicate voters who are convinced by the platform to vote for the buying party) $n - m + 1$ of the voters from m to n at cost t; X then must buy one voter with an additional cost of ε (either one of those bought by Y or possibly $n + 1$ if $|U_{n+1}| < \varepsilon$); Y then must buy a voter back at additional cost ε; and so on. Iff $B_Y \geq B_X + t$ will this process end with Y winning.

1b is proved by induction on B_X as follows. Clearly, 1b is true for $B_X = 0$ and any t. Suppose it is true for $B_X \leq K$ and for all t, and consider $B_X = K + \varepsilon$. Let T be the sum spent by Y in its first step. Clearly, $T \geq t$. Following its LEM strategy X pays some S such that $\varepsilon \leq S \leq T - t + \varepsilon$. If X's budget is such that it cannot purchase a majority then any payment more than t by Y in the first step is redudnant. Otherwise, after X's purchase, the situation is equivalent to an initial configuration with $t' = \varepsilon$, $B'_Y = B_Y - T$ and $B'_X = B_X - S \geq B_X - (T - t + \varepsilon)$. Since $B'_X \leq K$, by the inductive assumption Y must spend from this point on at least $B'_X + \varepsilon$ and hence Y's overall expenditure will be $B'_X + \varepsilon + T$. Now, this and $B'_X \geq B_X - (T - t + \varepsilon)$ imply that Y's overall expenditure is at least $B_X - (T - t + \varepsilon) + \varepsilon + T = B_X + t$. So Y cannot benefit from spending more than t, and as noted above can lose. (Note that Y spending t initally is an LEM strategy for Y.)

For all x, Part $2x$ is the counterpart of $1x$. In particular, $2b$ is analogous to $1b$. Finally, $1c$ follows from $2b$. This completes the proof of the lemma. ▮

Lemma 2 *LEM strategies for both parties constitute an equilibrium.*

Proof of Lemma 2: For $B_Y \geq B_X + t$, $1a$ and $1b$ of Lemma 1 imply that Y's LEM strategy is best response against X's LEM strategy. $1c$ implies that X's LEM strategy is best response against Y's LEM strategy. Analogously, $2a$–$2c$ of Lemma 1 imply that X's and Y's LEM strategies are mutual best responses when $B_Y < B_X + t$. ▮

Lemma **3** *All equilibria use LEM strategies.*

Proof of Lemma 3: The proof is by induction on B (the number of multiples of ε that bounds B_X and B_Y). For $B = 1$ the proposition is obviously true. Suppose that it is true for $B = K$; we now prove that it holds for $B = K + 1$.

If $B_Y < t$, then the claim follows immediately. Otherwise, in the first step Y promises some $T \geq t$. The new situation then is $t' < 0$, $B'_X = B_X \leq (K+1)\varepsilon$ and $B'_Y = B_Y - T \leq K\varepsilon$. If $B_X < |t'|$, then by definition the parties follow LEM strategies from that point on. Otherwise, to become the current winner X spends $S > |t'|$. This results in the configuration $t'' \in (0, S + t']$, $B''_X = B'_X - S = B_X - S \leq K\varepsilon$ and $B''_Y = B'_Y = B_Y - T \leq K\varepsilon$. Notice that if X is playing a best response, then $t'' \leq K\varepsilon$, since if X makes $t'' = (K+1)\,\varepsilon$ then X wins at a cost that with positive probability is higher than necessary (recall that Y's budget was bounded by $(K+1)\,\varepsilon$). Therefore, X's best response would result in $t'' \leq K\varepsilon$.

Thus, following X's move, the inductive assumption applies and Y wins iff $B''_Y \geq B''_X + t''$ at incremental cost (from here on) of $B''_X + t''$; X wins otherwise at incremental cost of $B''_Y - t'' + \varepsilon$. Translating this to the original data, Y wins if $B_Y - T \geq B_X - S + t''$, in which case its *overall* expenditure (from the start) will be $B_X - S + t'' + T$, and X wins if and only if $B_Y - T < B_X - S + t''$, in which case its overall expenditure will be $\max\{B_Y - t'', 0\} + S + \varepsilon$. Observe that, subject to the constraint $S \geq |t'|$, X's winning probability is maximized and its expected expenditure is uniquely minimized at $S = |t'|$, which is exactly what is required by an LEM strategy for X. Now, going back to Y's first move, this implies that Y will win iff $B_Y - T > B_X + t'$, at overall expense of $T + B_X - |t'| - \varepsilon$. Now, subject to the constraint $T \geq t$, Y's winning probability is maximized and its expected expenditure is uniquely minimized at $T = t$, which again corresponds only to LEM strategies for Y. ▮

This completes the proof of Proposition 5. ▮

Proof of Proposition 6: Let us show that X has a strategy that guarantees that X wins if (2) and (3) are satisfied. The other case is analogous.

Let us describe a strategy that X can follow to guarantee a win. Have X allocate offers in the following way. Let t be the period. X will identify a set of voters S_t to " buy" that has cardinality exactly m. X will make the minimal necessary offers to buy these votes.

To complete the proof we need only describe how X should select S_t, and then show that if X has followed this strategy in past periods, then X will have enough budget to cover the required payments regardless of the strategy of Y.

Let p_i^Y be the current offer that Y has to voter i. Set this to 0 in the case where Y has never made a viable offer to the voter, or in a case where X already has the best standing offer to the voter. Similarly define p_i^X.

X selects to whom to make offers by looking for those with that minimize the sum of what X has to offer, plus what offers of Y's that X frees up. In particular, let S_t be the set of voters than minimizes $\sum_{i \in S_t} 2p_i^Y - \alpha U_i$. This is equivalent to choosing the m voters that have the smallest values of

$$p_i^Y - \frac{\alpha U_i}{2}.$$

In the case where there are some i's that are tied under the above criterion, let X lexicographically favor voters with lower indices. To complete the proof, we simply need to show that this strategy is within X's budget in every possible situation, presuming that X has followed this strategy up to time t.[18]

Notice that the cost of a voter $i \in S_t$ to X is at most

$$\left[p_i^Y - \alpha U_i\right]^+ + \varepsilon. \tag{6}$$

The expression $\left[p_i^Y - \alpha U_i\right]^+$ captures the fact that it could be that $p_i^Y < \alpha U_i$ in which case no offer is necessary.

The amount that must be offered to a voter can only rise or stay constant over time, and so if some voters were " purchased" by X in the past and have not been subsequently purchased by Y, then these voters are still among the cheapest m available in the current period time and would still be selected under X's strategy (including the lexicographic tie-breaking).

Let i^* denote the most "expensive" $i \in S_t$ in terms of the "adjusted price" $p_i^Y - \frac{\alpha U_i}{2}$. If there are several voters tied for this distinction, pick the one with the lowest index. So, $i^* \in \arg\max_{i \in S_t} \left\{p_i^Y - \frac{\alpha U_i}{2}\right\}$, and let $\overline{S}_t$ be the complement of S_t union $\{i^*\}$.

Given the algorithm followed by X, we know that

$$p_i^Y - \frac{\alpha U_i}{2} \leq p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$$

[18]This implies the proposition, as it means that either Y will not respond and the game will end with X the winner, or else X will get to move again and can again follow the same strategy. As the game must end in a finite number of periods, this implies that X must win.

for every $i \in S_t$. This can be rewritten as

$$p_i^Y \leq p_{i^*}^Y - \frac{\alpha U_{i^*}}{2} + \frac{\alpha U_i}{2} \tag{7}$$

for each $i \in S_t$.

Equations (6) and (7) imply that the amount required by X to follow this strategy at this stage is at most

$$\sum_{i \in S_t} \left[p_{i^*}^Y - \frac{\alpha U_{i^*}}{2} - \frac{\alpha U_i}{2} \right]^+ + m\varepsilon \tag{8}$$

If we can get an upper bound on the expression $p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$, then we have an upper bound on how much X has to pay. So we want to maximize $p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$ subject to the following constraints:

(1) $p_i^Y - \frac{\alpha U_i}{2} \geq p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$ for every $i \notin S_t$,

(2) $p_i^Y \geq \alpha U_i + p_i^X$, and

(3) $\sum_{i \in \overline{S}_t} p_i^Y \leq B^Y$.

To get an upper bound, we ignore (2), and relax (3) by replacing B_Y with $\bar{B}_Y = \max\left\{B_Y, \left|\frac{m\alpha U_1}{2}\right| + \frac{\sum_{i=1}^m \alpha U_i}{2}\right\}$. The solution then involves spending all of $\bar{B}^Y$ in a manner that equalizes $p_i^Y - \frac{\alpha U_i}{2}$ with $p_{i^*}^Y - \frac{\alpha U_{i^*}}{2}$ for each $i \notin S_t$. (This is feasible due to the lower bound imposed on $\bar{B}_Y$; it is not necessarily feasible for B_Y, but still gives a bound). Thus, we end up with

$$p_i^Y = x^Y\left(\overline{S}_t\right) + \alpha U_i/2,$$

for each $i \in \overline{S}_t$, where

$$x^Y(\overline{S}_t) = \frac{\bar{B}^Y - \sum_{i \in \overline{S}_t} \frac{\alpha U_i}{2}}{m} \tag{9}$$

From (8), for X's strategy to be feasible it is sufficient that

$$B^X \geq \sum_{i \in S_t} \left[x^Y\left(\overline{S}_t\right) - \alpha U_i/2\right]^+ + m\varepsilon.$$

Substituting for x^Y from (9), this becomes

$$B^X \geq \bar{B}^Y - \sum_{i \in S_t \cup \overline{S}_t} \alpha U_i/2 + m\varepsilon.$$

This simplifies to

$$B^X \geq \bar{B}^Y - \sum_i \alpha U_i/2 - \alpha U_{i^*}/2 + m\varepsilon,$$

which has an upper bound when $i^* = N$, and which then yields the claimed expressions by substituting the definition of $\bar{B}_Y$. ▮

The conjecture for smaller budgets: Recall that we have ordered the voters so that $V_i = \alpha U_i$'s are non-increasing. Consider any $S \subset N$, denote $[z]^+ = \max\{z, 0\}$, and let

$$\begin{aligned} x^X(S) &= \max\left\{x \mid \sum_{i\in S} \max\left(\left[x - \frac{V_i}{2}\right]^+; -V_i\right) \leq B_X\right\} \\ x^Y(S) &= \max\left\{x \mid \sum_{i\in S} \max\left(\left[x + \frac{V_i}{2}\right]^+; V_i\right) \leq B_Y\right\} \end{aligned}$$

Let $N^X = \{1, \ldots m\}$ and $N^Y = \{m, \ldots, N\}$. Note that these are the cheapest majorities for X and Y to " buy," respectively. Let $N^{X\prime} = N^X \cup \{n\} \setminus \{m\}$ and $N^{Y\prime} = N^Y \cup \{1\} \setminus \{m\}$. These are slight modifications of the cheapest majorities where we have substituted one voter in each case.

Conjecture 1 *X wins if $x^X(N^{X\prime}) \geq x^Y(N^Y) + \varepsilon$ and Y wins if $x^Y(N^{Y\prime}) \geq x^X(N^X) + \varepsilon$.*

Proof of Proposition 9: The outline of a proof is as follows.

CLAIM 1: Suppose that Party Y offers a constant price x to all voters. The least expensive way for Party X to assure itself expected share σ of the vote would be offering a constant price to all voters.

PROOF OF THE CLAIM: The problem of finding bids p_i^X that Party X will make to voter i to assure expected share σ at minimum cost is as follows.

$$\min_{\{p_i^X\}} \sum_i p_i^X[1 - F(x - p_i^X)] \text{ s.t. } \sum_i \frac{1 - F(x - p_i^X)}{N} \geq \sigma$$

The first order conditions can be written as

$$p_i^X + \frac{1 - F(x - p_i^X)}{f(x - p_i^X)} = \frac{\lambda}{N}$$

This can be rewritten as

$$x - p_i^X - \frac{1 - F(x - p_i^X)}{f(x - p_i^X)} = x - \frac{\lambda}{N}$$

Since by assumption the LHS is increasing in $x - p_i^X$ and the RHS is constant, the p_i^X's that satisfy this condition are the same for all i.

CLAIM 2: If $(0.5+\eta)N[\frac{B_X}{(0.5-\eta)N} + F^{-1}(0.5-\eta)] < B_Y$, then Y can obtain expected share $(0.5+\eta)$ of the vote at each stage.

PROOF OF CLAIM: Suppose that it is Y's turn. If Y can offer all voters the same price $p = B_X/(0.5-\eta)N + F^{-1}(0.5-\eta)$, then Y can win in one step. This is so since, by the previous claim, X's least expensive way of getting at least $(0.5-\eta)N$ is by offering the same price to all voters. The minimal constant price required here is $B_X/(0.5-\eta)N$ which exactly exhausts X's budget. Now, since $B_X \frac{0.5+\eta}{0.5-\eta} + (0.5+\eta)NF^{-1}(0.5-\eta) < B_Y$, the price p is feasible for Y when only $(0.5+\eta)N$ voters (or slightly more) accept it. Thus, if p is infeasible at that stage, then there are more than $(0.5+\eta)N$ voters who would prefer to sell to Y at that price. But this means that there is a lower price $p' < p$ that gives Y an expected majority of $(0.5+\eta)N$. Since $(0.5+\eta)Np' < (0.5+\eta)Np < B_Y$, the price p' is feasible.

Now, for $\delta > 0$, there exists sufficiently small $\eta > 0$ such that $(0.5+\eta)N[\frac{B_X}{(0.5-\eta)N} + F^{-1}(0.5-\eta)] < B_X + dN/2 + \delta$. Therefore, if η is sufficently small, $B_Y > B_X + dN/2 + \delta$ together with Claim 2 imply that Y can obtain expected share of $(0.5+\eta)$. When N is made sufficiently large (here we mean that B_X and B_Y increase proportionately with N), expected share of $(0.5+\eta)$ means arbitrarily large probability of winning. Therefore, there exists $N(\delta)$ such that, for $N > N(\delta)$, Y's winning probability is above $1-\delta$.

Similar analysis can be done for Part (2) of the proposition. The condition that $x + F(x)/f(x)$ is increasing implies that, when X makes a constant offer to all voters, Y's least expensive way to secure a given expected share is by making a constant offer as well. Then an argument like that of Claim 2 would establish that if, $(0.5+\eta)N[\frac{B_Y}{(0.5-\eta)N} - F^{-1}(0.5+\eta)] < B_X$, then X can obtain a share of $(0.5+\eta)$ at each stage. ∎