Bargaining collectively[1]

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Abstract

Many bargaining situations share the following characteristics: a central party makes an offer to a group of individuals; the proposals are restricted to treat all group members equally; and decisions of the group are reached through a voting process, with the vote binding all the members of the group. Examples include debt restructuring negotiations between a troubled company and its bondholders; shareholder votes on executive compensation; and collective bargaining between a firm and union members. We study how the equilibrium payoffs in such bargaining situations depend on the decision rule adopted by the group.

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1 Introduction

Many bargaining situations share the following characteristics: a central party makes an offer to a group of individuals; the proposals are restricted to treat all group members equally; and decisions of the group are reached through a voting process, with the vote binding all the members of the group. Examples include debt restructuring negotiations between a troubled company and its bondholders; shareholder votes on executive compensation; and collective bargaining between a firm and union members.

In this paper we ask the following question: how does the voting rule used by the group affect bargaining outcomes? Conventional wisdom suggests that voting rules that require the consent of a large fraction of group members improve the group's bargaining position, and hence the welfare of group members. However, this argument has at least two shortcomings. First, voting rules which require a high level of consensus may make agreement hard to achieve, and lead the group to reject even desirable bargaining proposals. Second, as the recent strategic voting literature (Austen-Smith and Banks 1996, Feddersen and Pesendorfer 1997) has shown, the most consensus-driven voting rule of all — unanimity rule — aggregates the information of group members poorly compared to alternatives. In this paper we analyze the extent to which the conventional wisdom is, and is not, correct.

Specifically, we consider the following class of bargaining environments. A large group of *ex ante* identical individuals foresees that at some later date it will have to bargain with an opposing party. While still under the veil of ignorance, the group selects a decision rule. The bargaining game, when it arrives, consists of the opposing party making a take-it-or-leave-it offer, and the group deciding whether to accept or reject the offer. We restrict attention to decision rules in which each member of the group votes on the proposal, which is accepted if and only if the number of votes in favor exceeds a prespecified threshold. Leading examples are the simple majority rule, the two-thirds supermajority rule, and the unanimity rule.

To illustrate our results it is useful to consider the following example. A firm, which is initially wholly owned by a single individual, seeks to restructure its outstanding debt by offering a group of creditors a share of its future cash flow. If creditors decline the offer, they liquidate the firm and obtain \$100, with the original owner receiving nothing. The future cash flow of the firm (if not liquidated) is uncertain: it is either \$100 or \$200, with *ex ante* equal probability. The debtor and each creditor possess private and partially informative information about the relative likelihood of the two valuations.

Suppose first that the creditors are using a majority rule. The recent "strategic voting" literature on how agents vote when in possession of private information has established that as the number of voters grows large the aggregate decision asymptotes to the decision that would have been made under full information. We extend this literature to the case in which the issue being voted over is itself endogenous. For the case of majority voting rules, we show (Section 4) that the firm's choice boils down to the following: either it can offer creditors 1/2 of its cash flow, so that they accept whenever the true cash flow of the firm is \$200; or it can offer creditors all its cash flow, and gain acceptance all the time. Clearly the former is the more attractive option. In equilibrium, then, the firm offers 1/2 of its cash flow. If the true cash flow is \$200 the creditors accept, and receive a payoff of \$100; while if the true cash flow is \$100, creditors reject the offer, and obtain \$100 in liquidation.

This outcome contrasts sharply with that which arises when the unanimous agreement of creditors is required for acceptance. As previous authors (see Feddersen and Pesendorfer 1998) have shown, information aggregation fails when the unanimity rule is used. That is, creditors will reject some offers that (under full information) deliver more than the liquidation value, but accept others that deliver less. Roughly speaking, whether or not creditors are better off employing unanimity rule instead of majority rule depends on whether the firm internalizes these errors or exploits them.

For the parameter values given above, it is the former. As we show (Section 5) the errors made under unanimity are not entirely random. In particular, creditors will always reject an offer of one half of the firm. That is, for low offers there are only mistaken rejections, and no mistaken acceptances. However, as the offer increases, and so becomes more attractive to creditors, the nature of errors shifts from mistaken rejections of good offers to mistaken acceptances of bad offers. This gives the firm the incentive to make an offer strictly better than 1/2 of its cash flow. In this case, the conventional wisdom we alluded to above is correct. In spite — indeed, because — of the unanimity rule's failure to effectively aggregate information, creditors' equilibrium welfare is higher when it is used.

In the above example, the firm makes a relatively low offer (1/2 cash flow) when facing a majority rule. Creditors obtain more using the unanimity rule because it engenders mistakes, and these mistakes in turn dissuade the firm from making a low offer. However, under different circumstances the firm makes a high offer against a majority rule. In such circumstances, the mistakes that arise under unanimity rule hurt the creditors.

For instance, suppose now that the firm's cash flow (absent liquidation) is either $150 or $200. When facing a majority rule the firm must choose between offering 1/2 of its cash flow and gaining acceptance only in the latter case, and offering 2/3 of its cash flow and gaining acceptance always. It is easily seen that it prefers the latter strategy. In this situation, the firm is able to exploit the errors that arise when creditors use the unanimity rule. As we show (Lemma 6) there exists an offer strictly less than 2/3 that the creditors always accept when using the unanimity rule. That is, just as all errors in response to an offer of 1/2 take the form of mistaken rejections, all errors in response to the offer 2/3 take the form of mistaken acceptances. In this instance the conventional wisdom is wrong: far from being the "toughest" voting rule, unanimity is actually the softest. It leads creditors to accept offers that they would otherwise reject. In this case, creditors obtain a better outcome using a majority voting rule.

Our main results generalize the above examples by providing conditions under which the group is better off using unanimity rule. The choice of voting rule has an effect on equilibrium outcomes only if group members are in some way heterogeneous. There are two possibilities. First, they may have different information with respect to the relative desirability of the offer compared to the status quo. Second, their intrinsic preferences over outcomes may differ even conditional on full information. In the above example only the former type of heterogeneity exists. One might be concerned that the intuition we gave for this example, based as it was on information aggregation, is knife-edge. We consider a general model that allows for both types of heterogeneity, and contains pure private values and pure common values frameworks as special cases. The main assumption we make is that the proposer and the group members have diametrically opposing preferences: offers which one side prefers are disliked by the other side. This assumption is satisfied in many common bargaining situations.

For this fairly general set of preferences, we are able to evaluate the asymptotic (as the group size grows large) equilibrium payoffs arising when the group uses a majority rule; and to bound the equilibrium payoffs for the unanimity rule asymptotically. Given economic fundamentals, these results are enough to rank alternate agreement rules from the perspective of the two sides. In cases in which group members' heterogeneity derives primarily from their distinct information (that is, in the close to common values case) we are able to go further, and give a succinct condition for when the group is (and is not) better off using the unanimity rule. As illustrated by our example above, the key determinant is whether, when facing a (hypothetically) fully informed group, the proposer would prefer to make a low offer that is only sometimes accepted, or a high offer that is always accepted.

Inevitably our analysis neglects some important issues. We focus almost exclusively on equilibrium payoffs as the group size grows large. The chief reason for this focus is that it allows us to establish our results with fewer assumptions on preferences and the distributional properties of agents' information. Numerical simulations suggest that the group size needed for our asymptotic results to apply is not large — in many cases the equilibrium with ten agents is very close to the limiting equilibrium.

Related, we ignore the possibility of communication between agents prior to the vote. To some extent this is consistent with our focus on large groups. Moreover, it is worth pointing out that since agents' preferences include a private values component, the extent to which they are able to communicate their private information to each other may be limited. As a result, even if communication generates some public information, there may still be some private information. In general, our results would be qualitatively unchanged if in addition to observing their own private signals, agents also had access to a public signal.[1]

Finally, we take as given the information possessed by group members. As noted above, on a technical level our analysis constitutes an extension of a strategic voting game to allow for the endogeneity of the issue being voted over to the voting rule. Other authors have extended this same basic environment to allow for costly information acquisition,[2] as well as pre-vote communication (see footnote 1). We leave the integration of these distinct and individually important extensions for future work.

Our paper is somewhat related to the extensive recent literature on multilateral bargaining, in which more than two agents must agree on the division of a pie.[3] However, in many negotiations a proposal must treat all members of some group equally, either for technological reasons (e.g., the building of a bridge), or for institutional/legal reasons (e.g., wage determination, debt restructuring). The literature analyzing this important class of bargaining problems is much smaller. In a complete information setting Banks and Duggan (2001) establish equilibrium existence and core equivalence, while Cho and Duggan (2003) and Cardona and Ponsati (2005) establish uniqueness. Closest to us are Manzini and Mariotti (2005), who consider a bargaining game between a group and a

[1] For analysis of communication prior to decision making, see Coughlan (2000), Austen-Smith and Feddersen (2002), Doraszelski *et al* (2003), and Gerardi and Yariv (2005).

[2] See Persico (2004), Martinelli (2005), and Yariv (2004).

[3] The classic paper is Baron and Ferejohn (1989).

central agent, and compare different agreement rules. All the above papers are deterministic complete information models. As such, informational issues do not arise. Moreover, since agreement is always reached, there is no risk of breakdown of agreement from having a "tougher" bargaining stance. In contrast, the possibility of failing to agree to a Pareto improving proposal is central to our analysis and results. Finally, Chae and Moulin (2004) provide a family of solutions to group bargaining from an axiomatic viewpoint. Elbittar *et al* (2004) provide experimental evidence that the choice of voting rule used by a group in bargaining affects outcomes.

In our model, bargaining takes place under two-sided asymmetric information. The literature on bilateral bargaining under asymmetric information is extensive.[4] We add to this literature by considering how the internal organization of one of the parties affects equilibrium outcomes.

On the technical side, our work is closely related to the growing literature on strategic voting — see especially Feddersen and Pesendorfer (1997, 1997, 1998). Our paper contributes to this literature by endogenizing the agenda to be voted upon.

The paper proceeds as follows. Section 2 describes the model. Section 3 analyzes some general properties of the voting stage. Section 4 characterizes the equilibrium outcomes of the bargaining game when the group uses a majority rule. Section 5 conducts the same exercise when the group adopts unanimity rule. Sections 6 and 7 compare outcomes from different rules. Section 8 concludes. All proofs are in Appendix A.

2 Model

There is a single proposer (agent 0), and a group of $n \geq 2$ responders, labelled $i = 1, \ldots, n$. The timing is as follows: (1) Under the veil of ignorance, the coalition of responders fixes a decision making process. As noted, we restrict attention to voting

[4]See Kennan and Wilson (1993) for a review. Of most relevance for our paper are Samuelson (1984), Chatterjee and Samuelson (1987), Evans (1989), Vincent (1989), and Schweizer (1989), all of which study common values environments.

rules: a proposal is accepted and implemented if and only if some fraction α or more of responders vote to accept. That is, the decision rule is completely indexed by the parameter $\alpha \in [0,1]$. Common examples include the simple majority rule, $\alpha = 1/2$; the supermajority rule, $\alpha = 2/3$; and the unanimity rule, $\alpha = 1$. (2) Each agent $i \in \{0, 1, \ldots, n\}$ privately observes a random variable $\sigma_i \in [\underline{\sigma}, \bar{\sigma}]$. As we detail below, the realization of σ_i affects agent i's preferences and/or information. (3) The proposer selects a proposal $x \in [0,1]$. (4) Responders simultaneously cast ballots to accept or reject the proposal. (5) If $n\alpha$ or more responders vote to accept,[5] it is implemented. Otherwise, the status quo prevails.

Preferences

Agent i's relative preferences over the proposal x and the status quo are determined by σ_i and an unobserved state variable $\omega \in \{L, H\}$. We write responder i's utility associated with offer x as $U^\omega(x, \sigma_i, \lambda)$, where $\lambda \in [0,1]$ is a parameter that determines the relative importance of ω and σ_i. We assume that $U^\omega(x, \sigma_i, \lambda)$ is independent of σ_i at $\lambda = 0$ and that $U^L(\cdot, \cdot, \lambda) \equiv U^H(\cdot, \cdot, \lambda)$ at $\lambda = 1$. Likewise, we write $\bar{U}^\omega(\sigma_i, \lambda)$ for responder i's utility under the status quo, and make parallel assumptions for $\lambda = 0, 1$. Note that our framework includes pure *common values* ($\lambda = 0$) and pure *private values* ($\lambda = 1$) as special cases. A key object in our analysis is the utility of a responder from the proposal above and beyond the status quo. Accordingly, we define

$$\Delta^\omega(x, \sigma_i, \lambda) \equiv U^\omega(x, \sigma_i, \lambda) - \bar{U}^\omega(\sigma_i, \lambda).$$

Similarly, we write the proposer's utility from having his offer accepted as $V^\omega(x, \sigma_0)$, and his utility under the status quo as $\bar{V}^\omega(\sigma_0)$. Note that we do not require the relative weights of ω and σ_0 in determining the proposer's preferences to match the relative weights (given by λ) of ω and σ_i in determining responder i's preferences.

[5] Throughout, we ignore the issue of whether or not $n\alpha$ is an integer. This issue could easily be handled formally by replacing $n\alpha$ with $[n\alpha]$ everywhere, where $[n\alpha]$ denotes the smallest integer weakly greater than $n\alpha$. Since this formality has no impact on our results, we prefer to avoid the extra notation and instead proceed as if $n\alpha$ were an integer.

For all preferences $\lambda < 1$, the realization of σ_i provides responder i with useful (albeit noisy) information about the unobserved state variable ω. We assume that the random variables $\{\sigma_i : i = 0, 1, \ldots, n\}$ are independent conditional on ω, and that except for σ_0 (which is observed by the proposer) are identically distributed. Let $F(\cdot|\omega)$ and $F_0(\cdot|\omega)$ denote the distribution functions for the responders and proposer respectively. We assume that both distributions have associated continuous density functions, which we write $f(\cdot|\omega)$ and $f_0(\cdot|\omega)$. The realization of σ_i is informative about ω, in the sense that the monotone likelihood ratio property (MLRP) holds strictly;[6] but no realization is perfectly informative, i.e., $\frac{f(\underline{\sigma}|H)}{f(\underline{\sigma}|L)} > 0$ and $\frac{f(\bar{\sigma}|H)}{f(\bar{\sigma}|L)} < \infty$, with similar inequalities for f_0.

Interpretations

Possible interpretations of the model include the following:

(A) An indebted firm offers n creditors an equity stake x in exchange for the retirement of existing debt claims. If the creditors reject the offer the firm is liquidated. Let $\frac{1}{n}U^{\omega}(x, \sigma_i, \lambda)$ be the value of an x/n share to creditor i, $\frac{1}{n}\bar{U}^{\omega}(\sigma_i, \lambda)$ be the value of receiving $1/n$ of the liquidation value,[7] $V^{\omega}(x, \sigma_0)$ be the debtor's valuation of the remaining $1 - x$ share if his offer is accepted, and $\bar{V}^{\omega}(\sigma_0)$ his payoff in liquidation.

(B) An employer is in wage negotiations with n workers. He offers a wage x, which worker i values at $U^{\omega}(x, \sigma_i, \lambda)$. If the offer is rejected, workers strike: $\bar{U}^{\omega}(\sigma_i, \lambda)$ is worker i's expected payoff from the strike. The firm's total profits if the offer is accepted are $nV^{\omega}(x, \sigma_0)$, and its expected total profits if a strike ensues are $n\bar{V}^{\omega}(\sigma_0)$.

(C) A president proposes a policy x.[8] The proposal is adopted only if passed by the legislature. This requires the support of a sufficient fraction of legislators from the opposing party to the president.

[6]That is, $\frac{f(\sigma|H)}{f(\sigma|L)}$ and $\frac{f_0(\sigma|H)}{f_0(\sigma|L)}$ are strictly increasing in σ.

[7]These preferences are isomorphic under any monotone transformation, and in particular, multiplication by n.

[8]A judicial nominee, for example.

Equilibrium

We examine the pure strategy[9] sequential equilibria of the game just described. Let $\pi_n^*(\cdot;\lambda,\alpha) : [\underline{\sigma},\bar{\sigma}] \to [0,1]$ denote the proposer's offer strategy for the game with n responders using voting rule α and preference parameter λ. As is standard in the strategic voting literature on which we build, we restrict attention to equilibria in which the *ex ante* identical responders behave symmetrically.[10]

Responders are potentially able to infer information about the proposer's observation of σ_0 from his offer, and thus information about the state variable ω. Since only the latter affects responders' preferences, we focus directly on the beliefs about ω after observing an offer x. Let $\beta_n(x;\lambda,\alpha)$ denote the responders' belief that $\omega = H$ after observing offer x in the game with n responders using voting rule α, and preference parameter λ.

A symmetric equilibrium is an offer strategy $\pi_n^*(\cdot;\lambda,\alpha)$ for the proposer, a set of responder beliefs $\beta_n(\cdot;\lambda,\alpha)$ and a voting strategy $[\underline{\sigma},\bar{\sigma}] \to \{\text{accept,reject}\}$ for each responder such that the proposer's strategy is a best response to the responders' (identical) strategies; and each responder's strategy maximizes his expected payoff given that all other responders use the same strategy, and his beliefs are $\beta_n(\cdot;\lambda,\alpha)$; and the beliefs themselves are consistent. At a minimum, belief consistency requires that having received an offer x, responders are not more (respectively, less) confident that the state is H than the proposer himself is after he sees the most (respectively, least) pro-H signal $\sigma_0 = \bar{\sigma}$ (respectively, $\sigma_0 = \underline{\sigma}$). That is, for all offers x,

$$\frac{\beta_n(x;\lambda,\alpha)}{1-\beta_n(x;\lambda,\alpha)} \in \left[\frac{f_0(\underline{\sigma}|H)}{f_0(\underline{\sigma}|L)}\frac{\Pr(H)}{\Pr(L)}, \frac{f_0(\bar{\sigma}|H)}{f_0(\bar{\sigma}|L)}\frac{\Pr(H)}{\Pr(L)}\right]. \tag{1}$$

Consequently consistency implies that $\beta_n(x;\lambda,\alpha) \in [\underline{b},\bar{b}]$, for some $0 < \underline{b} < \bar{b} < 1$.

[9] In much of the literature concerning voting by differentially informed individuals, voters are assumed to observe binary signals (see Duggan and Martinelli 2001, and Yılmaz 1999, for exceptions). In such settings non-trivial pure strategy equilibria do not exist. However, in our model voters observe continuous signals. As such, the restriction that voters follow pure strategy equilibria is of no consequence. Moreover, our focus on pure strategy behavior of the proposer is solely for expositional convenience: our main results would hold if the proposer were allowed to follow mixed strategies.

[10] Duggan and Martinelli (2001) give conditions under which the symmetric voting equilibrium is the unique equilibrium for unanimity rule.

Assumptions

We make the following assumptions:

Assumption 1 *Δ^ω, V and $\bar{V}$ are continuously differentiable in their arguments.*

Assumption 2 *$\Delta^H \geq \Delta^L$ and Δ^ω is increasing in σ_i; both relations are strict for $x > 0$.*

Assumption 3 *For all λ, $\Delta^H(0, \bar{\sigma}, \lambda) < 0$ and $\Delta^H(1, \bar{\sigma}, \lambda) > 0$.*

Assumption 4 *For all x, $V^\omega(x, \sigma_0) - \bar{V}^\omega(\sigma_0) \geq 0$ for $\omega = L, H$ and all σ_0.*

Assumption 5 *Δ^ω is strictly increasing and V is strictly decreasing in x.*

Assumption 1 is entirely standard. For future reference, observe that $|\Delta^\omega|$ is bounded above since Δ^ω is continuous in its arguments and has compact domain. Moreover, we will often be interested in the equilibrium properties of economies close to pure common values ($\lambda = 0$). Observe that since σ has compact support, $\Delta^\omega(x, \cdot, \lambda)$ converges uniformly to $\Delta^\omega(x, \cdot, 0)$ as $\lambda \to 0$.

Assumption 2 says responder i is more pro-acceptance when $\omega = H$ than $\omega = L$, and when the realization of σ_i is higher. Since higher values of σ_i are more likely when $\omega = H$ (by MLRP), the content of Assumption 2 (beyond being a normalization) is that the "private" and "common" components of responder utility act in the same direction.

Assumption 3 says that the responders regard the worst offer ($x = 0$) as worthless, i.e., they prefer the status quo. On the other hand, there are some offers which the responders view as worthwhile under some conditions — in particular, responder i prefers the best offer ($x = 1$) to the status quo when $\omega = H$ and $\sigma_i = \bar{\sigma}$.

Assumption 4 says that the proposer strongly dislikes the status quo relative to the range of possible alternatives: regardless of the state, he would prefer to have any proposal $x \in [0, 1]$ implemented.[11]

[11] In general, one can clearly think of a broader range of proposals $[0, \infty)$, but with the proposer preferring the status quo to offers $x \in (1, \infty)$. The content of Assumption 4 is that $x = 1$ is the highest offer the proposer is prepared to make for *any* pair (ω, σ_0). For instance, in our debt renegotiation example, a debtor (the proposer) would prefer being left with any fraction $1-x$ of the firm to liquidation, if (as is typical) in the latter case he is left with nothing.

Finally, Assumption 5 says that the proposer and responders have diametrically opposing preferences: higher x makes the responders more pro-agreement, but reduces the proposer's payoff if his proposal is accepted.

3 The voting stage

Fix a preference parameter λ and a number of responders n. Having observed the proposer's offer x, each responder attaches a subjective probability $b = \beta_n(x;\lambda,\alpha)$ to the state variable ω being H. A central insight of the existing strategic voting literature is that responder i's voting decision depends on the comparison of his expected utilities from accepting and rejecting, conditional on the event of being pivotal. Taking the strategies of other responders as given, let PIV denote the event that his vote is pivotal. Thus responder i votes to accept offer x after observing σ_i if and only if

$$E_b\left[U^{\omega}(x,\sigma_i,\lambda)\,|PIV,\sigma_i\right] \geq E_b\left[\bar{U}^{\omega}(x,\sigma_i,\lambda)\,|PIV,\sigma_i\right], \tag{2}$$

where $\Pr_b$ and E_b denote the subjective probability and expectation given b. Observe that even though responder i does not observe σ_j ($j \neq i$), and does not know whether or not he is actually pivotal, in casting his vote he considers only the payoffs in events in which he is pivotal, and takes into account any information he can thus infer.

Since the random variables σ_i are independent conditional on ω,

$$\Pr_b(\omega|PIV,\sigma_i) = \frac{\Pr_b(\omega,PIV,\sigma_i)}{\Pr_b(PIV,\sigma_i)} = \frac{\Pr(PIV|\omega)\Pr(\sigma_i|\omega)\Pr_b(\omega)}{\Pr_b(PIV,\sigma_i)}. \tag{3}$$

Substituting (3) into inequality (2), and noting that $\Pr_b(H) = b = 1-\Pr_b(L)$, responder i votes to accept proposal x after observing σ_i if and only if

$$\Delta^H(x,\sigma_i,\lambda)\Pr(PIV|H)\,f(\sigma_i|H)\,b + \Delta^L(x,\sigma_i,\lambda)\Pr(PIV|L)\,f(\sigma_i|L)\,(1-b) \geq 0. \tag{4}$$

By MLRP, it is immediate from (4) that in any equilibrium each responder i follows a cutoff strategy, in the sense of voting to accept if and only if σ_i exceeds some critical level. As noted, throughout we focus on symmetric equilibria in which the *ex ante* identical responders follow the same voting strategy. As such, let $\sigma_n^*(x, b, \lambda, \alpha) \in [\underline{\sigma}, \bar{\sigma}]$ denote the common cutoff[12] when the offer is x, responders attach a probability b to $\omega = H$, there are n responders, and the preference parameter and voting rule are λ and α respectively. For clarity of exposition, we will suppress the arguments n, x, b, λ and α unless needed, both for σ^* and other variables introduced below.

Evaluating explicitly, the probability that a responder is pivotal is given by

$$\Pr(PIV|\omega) = \binom{n-1}{n\alpha - 1} (1 - F(\sigma^*(x)|\omega))^{n\alpha-1} F(\sigma^*(x)|\omega)^{n-n\alpha}. \tag{5}$$

The acceptance condition (4) then rewrites to:

$$\begin{aligned} &\Delta^H(x, \sigma_i, \lambda)(1 - F(\sigma^*(x)|H))^{n\alpha-1} F(\sigma^*(x)|H)^{n-n\alpha} f(\sigma_i|H)\, b \\ &+\Delta^L(x, \sigma_i, \lambda)(1 - F(\sigma^*(x)|L))^{n\alpha-1} F(\sigma^*(x)|L)^{n-n\alpha} f(\sigma_i|L)(1-b) \geq 0 \end{aligned} \tag{6}$$

If there exists a $\sigma^* \in [\underline{\sigma}, \bar{\sigma}]$ such that responder i is indifferent between accepting and rejecting the offer x exactly when he observes the signal $\sigma_i = \sigma^*$, then the equilibrium can be said to be a *responsive equilibrium.* That is, a responsive equilibrium exists whenever the equation

$$-\frac{\Delta^H(x, \sigma^*, \lambda)}{\Delta^L(x, \sigma^*, \lambda)} \frac{b}{1-b} \frac{f(\sigma^*|H)}{f(\sigma^*|L)} \frac{1 - F(\sigma^*|L)}{1 - F(\sigma^*|H)} = \left(\frac{(1 - F(\sigma^*|L))^\alpha F(\sigma^*|L)^{1-\alpha}}{(1 - F(\sigma^*|H))^\alpha F(\sigma^*|H)^{1-\alpha}} \right)^n \tag{7}$$

has a solution $\sigma^* \in [\underline{\sigma}, \bar{\sigma}]$. Notationally, we represent a responsive equilibrium by its corresponding cutoff value $\sigma^* \in [\underline{\sigma}, \bar{\sigma}]$.

We turn now to existence and uniqueness of responsive equilibria in the voting stage. For a given subjective probability b that $\omega = H$, it is useful to define the function

$$Z(x, \sigma; n, \alpha, \lambda, b) \equiv \Delta^H(x, \sigma) \frac{b}{1-b} \frac{f(\sigma|H)}{f(\sigma|L)} \left(\frac{F(\sigma|H)}{F(\sigma|L)} \right)^{n-n\alpha} \left(\frac{1 - F(\sigma|H)}{1 - F(\sigma|L)} \right)^{n\alpha-1} + \Delta^L(x, \sigma)$$

If $Z(x, \sigma)$ is positive (negative), and all but one of the responders use a cutoff strategy σ, then the remaining responder i is better off voting to accept (reject) the proposal x

[12] As we show below, there exists a unique cutoff signal.

if he observes $\sigma_i = \sigma$. Similarly, if $Z(x,\sigma) = 0$ then there is a responsive equilibrium in which all responders use the cutoff strategy σ.

By the Theorem of the Maximum, $\max_{\sigma \in [\underline{\sigma},\bar{\sigma}]} Z(x,\sigma)$ and $\min_{\sigma \in [\underline{\sigma},\bar{\sigma}]} Z(x,\sigma)$ are both continuous in x. So we can define[13]

$$\underline{x}_n(\alpha,\lambda,b) = \begin{cases} \min\{x \mid \max_\sigma Z(x,\sigma) \geq 0\} & \text{if } \{x \mid \max_\sigma Z(x,\sigma) \geq 0\} \neq \varnothing \\ 1 & \text{otherwise} \end{cases} \quad (8)$$

$$\bar{x}_n(\alpha,\lambda,b) = \begin{cases} \max\{x \mid \min_\sigma Z(x,\sigma) \leq 0\} & \text{if } \{x \mid \min_\sigma Z(x,\sigma) \leq 0\} \neq \varnothing \\ 0 & \text{otherwise} \end{cases}. \quad (9)$$

Roughly speaking, $\underline{x}_n(\alpha,\lambda,b)$ is the lowest offer that is ever accepted in a responsive equilibrium: if $x < \underline{x}_n(\alpha,\lambda,b)$, then $Z(x,\sigma) < 0$ for all σ. Similarly, $\bar{x}_n(\alpha,\lambda,b)$ is the highest offer that is ever rejected in a responsive equilibrium. Economically, $\underline{x}_n(\alpha,\lambda,b)$ and $\bar{x}_n(\alpha,\lambda,b)$ define the range of offers for which a responsive equilibrium exists:

Lemma 1 ***(Existence and uniqueness)*** *Fix beliefs b, a voting rule α and preferences λ. Then:*

(1) For any n, a responsive equilibrium $\sigma^*(x) \in [\underline{\sigma},\bar{\sigma}]$ exists if and only if $x \in [\underline{x}_n, \bar{x}_n]$. When a responsive equilibrium exists it is the unique symmetric responsive equilibrium.

(2) The equilibrium cutoff $\sigma^*(x)$ is decreasing and continuously differentiable over $(\underline{x}_n, \bar{x}_n)$, with $\sigma^*(\underline{x}_n) = \bar{\sigma}$ and $\sigma^*(\bar{x}_n) = \underline{\sigma}$.

(3) (a) If $\alpha < 1$ and x is such that $\Delta^H(x,\bar{\sigma}) > 0 > \Delta^L(x,\underline{\sigma})$, there exists N such that $x \in (\underline{x}_n, \bar{x}_n)$ for $n \geq N$; (b) if $\alpha = 1$ and x is such that $\Delta^H(x,\bar{\sigma}) > 0 \geq \Delta^H(x,\underline{\sigma})$, there exists N such that $x \in (\underline{x}_n, \bar{x}_n)$ for $n \geq N$.

In addition to responsive equilibria, non-responsive equilibria also exist. Specifically, for any $\alpha > \frac{1}{n}$ there is an equilibrium in which each responder votes to reject regardless of his signal, i.e., $\sigma^* = \bar{\sigma}$. Likewise, for any $\alpha < 1 - \frac{1}{n}$ there is an equilibrium in which each responder votes to accept regardless of his signal, i.e., $\sigma^* = \underline{\sigma}$. We follow the literature

[13] Observe that $\underline{x}_n(\alpha,\lambda,b) > 0$ since, by Assumptions 2 and 3, $Z(0,\sigma) < 0$ for all σ.

and assume that if a responsive equilibrium exists, then it is played. From Lemma 1 it follows that as x increases over the interval $(\underline{x}_n, \bar{x}_n)$ the acceptance probability increases continuously from 0 to 1. We thus assume that when $x \leq \underline{x}_n$ the rejection equilibrium is played, while for $x \geq \bar{x}_n$ the acceptance equilibrium is played. In addition to being intuitive and ensuring continuity, this rule selects the unique trembling-hand perfect equilibrium when $x \leq \underline{x}_n$.[14]

How does the equilibrium respond to changes in responders' beliefs? The following is a straightforward corollary of Lemma 1:

Corollary 1 ***(Change in beliefs)*** *Fix n, α, λ, and suppose that a responsive equilibrium exists given offer x and beliefs b. Then for any beliefs $b' > b$, the acceptance probability is higher.*

The heart of our analysis concerns the effect of the voting rule on the proposer's offer x, and in turn the effect on responder and proposer payoffs. Notationally, we write $\Pi_n^R(x, \lambda, \alpha, b)$ for a responder's expected payoff from offer x under voting rule α, responder preferences λ, and responder beliefs b; and $\Pi_n^P(x, \lambda, \alpha, \sigma_0, b)$ for the proposer's expected payoff after observing σ_0. Before proceeding, we note a second straightforward corollary of Lemma 1:

Corollary 2 ***(Continuity and differentiability of payoffs)*** *Fix a set of responder beliefs b. Then $\Pi_n^R(x, \lambda, \alpha, b)$ and $\Pi_n^P(x, \lambda, \alpha, \sigma_0, b)$ are continuous functions of the offer x, and are differentiable except at the boundaries of the responsive equilibrium range, $\underline{x}_n(\alpha, \lambda, b)$ and $\bar{x}_n(\alpha, \lambda, b)$.*

[14] Formally, for any beliefs b, preference parameter λ and voting rule $\alpha > \frac{1}{2} + \frac{1}{2n}$, if $x \leq \underline{x}_n$ then the only trembling-hand perfect equilibrium is the non-responsive equilibrium in which each responder always rejects. A proof is available on the authors' webpages.

Moreover, although when $x \geq \bar{x}_n$ both the acceptance and rejection equilibria are trembling-hand perfect, the trembles required to support the rejection equilibrium do not satisfy the cutoff rule property we discussed earlier. Indeed, if tremble strategies were required to satisfy the mild monotonicity restriction that voting to accept is weakly more likely after a higher signal, then the acceptance equilibrium would be the *only* trembling-hand perfect equilibrium when $x \geq \bar{x}_n$.

4 Majority voting

We first characterize the equilibrium payoffs for any non-unanimity voting rule $\alpha < 1$. Throughout, we refer to any non-unanimity voting rule α as a majority rule. As we will see, asymptotically (in the number of responders n) all such rules generate the same equilibrium outcomes. For $\omega = L, H$, define $\sigma_\omega(\alpha)$ and $x_\omega(\lambda, \alpha)$ implicitly by

$$1 - F(\sigma_\omega(\alpha)|\omega) = \alpha \quad \text{and} \quad \Delta^\omega(x_\omega(\lambda, \alpha), \sigma_\omega(\alpha), \lambda) = 0.$$

That is, conditional on ω there is a probability α that the realization of σ_i exceeds $\sigma_\omega(\alpha)$; and $x_\omega(\lambda, \alpha)$ is the proposal that gives a responder i the same payoff as the status quo, given ω and $\sigma_i = \sigma_\omega(\alpha)$. As such, if the state ω were public information, then an offer just above $x_\omega(\lambda, \alpha)$ would be accepted with probability converging to 1 as the number of responders n grows large.

By Assumption 3, $\Delta^\omega(x, \sigma_\omega, \lambda)$ is strictly negative at $x = 0$, and is strictly increasing in x. Consequently $x_\omega(\lambda, \alpha)$ is well-defined unless $\Delta^\omega(x, \sigma_\omega, \lambda) < 0$ at $x = 1$. For this case, we write $x_\omega(\lambda, \alpha) = \infty$. Note that $\sigma_\omega(\alpha)$ is strictly decreasing. Likewise, $x_\omega(\lambda, \alpha)$ is strictly increasing in α, except at the common values extreme $\lambda = 0$, in which case it is constant in α. Moreover, by Assumption 3, $x_H(\lambda, \alpha) \neq \infty$ for all λ sufficiently small.

Our first result extends Feddersen and Pesendorfer's (1997) finding that under majority rule, the aggregate response of the voting group to an offer x matches that which would be obtained under full information. The key difference relative to their analysis is that the proposal being voted over varies with the number of responders. Because there is no reason to require the proposal to have a well-defined limit, we state our result in terms of the limits infimum and supremum. Formally, let $A_n(x, b, \lambda, \alpha)$ denote the event in which the offer x is accepted:

Lemma 2 ***(Acceptance probabilities under majority)*** *Suppose a majority voting rule $\alpha < 1$ is in effect. Take any $\lambda \in [0, 1]$, and consider a sequence of offers x_n.*

If $\liminf x_n > x_\omega(\lambda,\alpha)$ *then* $\Pr(A_n(x_n)|\,\omega) \to 1$ *and if* $\limsup x_n < x_\omega(\lambda,\alpha)$ *then* $\Pr(A_n(x_n)|\,\omega) \to 0$ *as* $n \to \infty$*.*

Our next result characterizes the proposer's response to the voting behavior described in Lemma 2. To this end, for any σ_0 define

$$W(\sigma_0;\lambda,\alpha) \equiv \Pr(H|\sigma_0)\,V^H(x_H,\sigma_0) + \Pr(L|\sigma_0)\,\bar{V}^L(\sigma_0) - E\left[V^\omega(x_L,\sigma_0)\,|\sigma_0\right]. \quad (10)$$

The function W has the following interpretation: the first two terms are the proposer's expected payoff from offering x_H if this offer is accepted when $\omega = H$ and rejected when $\omega = L$. The final term is the proposer's expected payoff from offering x_L if this offer is always accepted. In Lemma 2 we established that approximately this acceptance behavior is obtained as the number of responders grows large. As such, we should expect that a proposer facing a large coalition will offer x_H whenever $W(\sigma_0;\lambda,\alpha) > 0$; and will offer x_L whenever $W(\sigma_0;\lambda,\alpha) < 0$.

Lemma 3 ***(Equilibrium offer under majority)*** *Suppose a majority voting rule* $\alpha < 1$ *is in effect. Then:*

(1) If $x_L(\lambda,\alpha) \neq \infty \neq x_H(\lambda,\alpha)$*, then for any* $\varepsilon,\delta > 0$ *there exists* $N(\varepsilon,\delta)$ *such that*

(a) If $W(\sigma_0) > \varepsilon$ *and* $n \geq N(\varepsilon,\delta)$ *then* $|\pi_n^*(\sigma_0;\lambda,\alpha) - x_H(\lambda,\alpha)| < \delta$ *and* $\Pr(A_n|\sigma_0,H) > 1-\delta$ *and* $\Pr(A_n|\sigma_0,L) < \delta$*.*

(b) If $W(\sigma_0) < -\varepsilon$ *and* $n \geq N(\varepsilon,\delta)$ *then* $|\pi_n^*(\sigma_0;\lambda,\alpha) - x_L(\lambda,\alpha)| < \delta$ *and* $\Pr(A_n|\sigma_0) > 1-\delta$*.*

(2) If $x_H(\lambda,\alpha) \neq \infty$ *and* $x_L(\lambda,\alpha) = \infty$ *then for any* $\delta > 0$ *there exists* $N(\delta)$ *such that* $|\pi_n^*(\sigma_0;\lambda,\alpha) - x_H(\lambda,\alpha)| < \delta$ *and* $\Pr(A_n|\sigma_0,H) > 1-\delta$ *for all* σ_0 *when* $n \geq N(\delta)$*.*

(3) If $x_L(\lambda,\alpha) = x_H(\lambda,\alpha) = \infty$*, for any* $\delta > 0$ *there exists* $N(\delta)$ *such that* $\Pr(A_n|\sigma_0,\omega) < \delta$ *for all* $\sigma_0, \omega = L, H$ *when* $n \geq N(\delta)$*.*

For use below, we set $W(\sigma_0;\lambda,\alpha) = \infty$ when $x_H(\lambda,\alpha) \neq \infty$ and $x_L(\lambda,\alpha) = \infty$. Lemma 3 says that the proposer will make an offer close to $x_H(\lambda,\alpha)$ (respectively,

$x_L(\lambda,\alpha) > x_H(\lambda,\alpha)$) after observing a σ_0 such that $W(\sigma_0)$ is strictly positive (negative). As stated, it does not cover equilibrium behavior when $W(\sigma_0) = 0$. In general, this knife-edge condition will hold only for finitely many realizations of σ_0. In particular, $W(\sigma_0) = 0$ for at most one value of σ_0 if the proposer's payoffs $V^\omega(x,\sigma_0)$ and $\bar{V}^\omega(\sigma_0)$ are independent of σ_0 — or more generally, if the private values component of proposer payoffs is sufficiently small, i.e., $\left|\frac{\partial}{\partial\sigma_0}V^\omega(x,\sigma_0)\right|$ and $\left|\frac{\partial}{\partial\sigma_0}\bar{V}^\omega(\sigma_0)\right|$ are sufficiently small for all x and σ_0. For the remainder of the paper we make the following mild assumption:

Assumption 6 $W(\sigma_0;\lambda,\alpha) = 0$ *for at most finitely many values of* σ_0 *when* $x_L(\lambda,\alpha) \neq \infty \neq x_H(\lambda,\alpha)$.

From Lemma 3 it is straightforward to establish the limiting expected payoffs of the proposer and the responders under any majority voting rule. Notationally, we write $\Pi_n^{*P}(\lambda,\alpha)$ and $\Pi_n^{*R}(\lambda,\alpha)$ for the proposer's and responders' expected equilibrium payoffs.

Proposition 1 ***(Equilibrium payoffs under majority)*** *Suppose a majority voting rule* $\alpha < 1$ *is in effect and* $x_H(\lambda,\alpha) \neq \infty$. *Then the equilibrium payoffs satisfy:*

$$\begin{aligned}\Pi_n^{*R}(\lambda,\alpha) \;\;\rightarrow\;\; & E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i)\right] + \int_{\sigma_0\ s.t.\ W(\sigma_0)<0} E_{\sigma_i,\omega}\left[\Delta^\omega(x_L,\sigma_i,\lambda)\,|\sigma_0\right] dF_0(\sigma_0) \\ & + \int_{\sigma_0\ s.t.\ W(\sigma_0)>0} \Pr(H|\sigma_0)\, E_{\sigma_i}\left[\Delta^H(x_H,\sigma_i,\lambda)\,|H\right] dF_0(\sigma_0)\end{aligned}$$

$$\begin{aligned}\Pi_n^{*P}(\lambda,\alpha) \;\;\rightarrow\;\; & \int_{\sigma_0\ s.t.\ W(\sigma_0)<0} E_\omega\left[V^\omega(x_L,\sigma_0)\,|\sigma_0\right] dF_0(\sigma_0) \\ & + \int_{\sigma_0\ s.t.\ W(\sigma_0)>0} \left(\Pr(H|\sigma_0)\, V^H(x_H,\sigma_0) + \Pr(L|\sigma_0)\,\bar{V}^L(\sigma_0)\right) dF_0(\sigma_0).\end{aligned}$$

5 Unanimity rule

We now turn to the proposer's offer when he faces responders who employ unanimity rule (i.e., $\alpha = 1$). We first show that if responders' preferences are sufficiently close to the common values extreme (i.e., λ small enough), then compared to their behavior under a majority rule, they accept "low" offers less often but "high" offers more often. Specifically, whereas under majority an offer slightly above $x_H(\lambda, \alpha)$ is accepted with probability approaching one when $\omega = H$, under unanimity such an offer is accepted with vanishingly small probability. Conversely, under majority an offer slightly below $x_L(\lambda, \alpha)$ is rejected with probability approaching one when $\omega = L$; but under unanimity, it is certain to be accepted.

We formally establish these results below. Their main implication is as follows. Consider a realization of σ_0 such that $W(\sigma_0)$ is positive. Against a group using majority the proposer's favorite offer is close to $x_H(\lambda, \alpha)$. However, against a group using unanimity, this — and all lower offers — is rejected almost for sure; but by making a higher offer the proposer can increase the acceptance probability. So in this case, the proposer makes a higher offer when responders use unanimity rule than when they use a majority rule. Conversely, the proposer will actually make a lower offer against the unanimity rule after any σ_0 for which $W(\sigma_0)$ is negative: against a majority rule his favorite offer is close to $x_L(\lambda, \alpha)$, while against unanimity he can assure himself of acceptance with an offer strictly less than $x_L(\lambda, \alpha)$.

These arguments apply only when preferences are sufficiently close to the common values extreme. Consequently, establishing them formally requires us to take what is essentially a double limit: we must allow the preference parameter λ to approach 0 at the same time as the number of responders grows large. A further complication is that, as under majority rules, there is no reason to suppose *a priori* that either $x_n(\lambda)$ or the corresponding acceptance probability converges in either λ or n. Formally, we handle these difficulties by stating our results about acceptance probabilities in terms of

$$\sup_{\Lambda, N} \inf_{\lambda \in (0,\Lambda], n \geq N} \Pr\left(A_n\left(x_n\left(\lambda\right), \lambda\right) | \omega\right), \tag{11}$$

with parallel expressions for offer sequences $x_n(\lambda)$. These expressions serve to give a lower bound on the offer and acceptance probability as n grows large and λ approaches

0 at the same time. To see this, observe that the lower bound

$$\inf_{\lambda\in(0,\Lambda],n\geq N} \Pr\left(A_n\left(x_n\left(\lambda\right),\lambda\right)|\omega\right)$$

increases as N grows and Λ shrinks, and so by taking the supremum of this expression over Λ and N we characterize the limiting behavior of the lower bound of $\Pr\left(A_n|\omega\right)$. Likewise, the following expression captures the limiting behavior of the upper bound of $\Pr\left(A_n|\omega\right)$:

$$\inf_{\Lambda,N} \sup_{\lambda\in(0,\Lambda],n\geq N} \Pr\left(A_n\left(x_n\left(\lambda\right),\lambda\right)|\omega\right).$$

Finally, we want to stress that although we state all our results in terms of the joint limiting behavior of n and λ, analogous versions of our results hold for any fixed preference parameter $\lambda_0 \geq 0$ that is close enough to the common values extreme that $x_H\left(\lambda_0,\alpha=1\right)\neq\infty$.[15]

VOTING UNDER UNANIMITY RULE

Lemma 2 above characterized responders' aggregate response to an arbitrary offer under any majority rule. Lemmas 4, 5 and 6 do the same for the unanimity rule. First, Lemma 4 says that if the offers stay bounded away from $x_H\left(\lambda\right)$ as n grows large, then the probabilities that these offers are accepted likewise stay bounded away from 0.

Lemma 4 ***(Intermediate offers accepted under unanimity)*** *Suppose unanimity rule is in effect ($\alpha = 1$). Take a set of offers $x_n\left(\lambda\right)$. If*

$$\sup_{\Lambda,N} \inf_{\lambda\in(0,\Lambda],n\geq N} x_n(\lambda) - x_H\left(\lambda\right) > 0 \tag{12}$$

then

$$\sup_{\Lambda,N} \inf_{\lambda\in(0,\Lambda],n\geq N} \Pr\left(A_n|H\right) > 0. \tag{13}$$

At first glance Lemma 4 is surprising: one might have conjectured that agreement is impossible when the unanimous consent of a large number of responders is required. The

[15] Proofs are available upon request from the authors. As would be expected, the arguments are slightly easier when the preference parameter is held fixed.

reason why agreement is in fact possible is that each individual votes to accept with a probability that approaches one as the number of responders grows large. Such a strategy is individually rational because, given that many other individuals are voting to accept, each responder concludes that, conditional on being pivotal, there is strong evidence that the offer is worth accepting.

Lemma 4 is established by contradiction. We sketch the argument for a fixed preference parameter λ close to 0. Suppose that, contrary to the claim, the acceptance probability converges to 0, i.e., $(1 - F(\sigma_n^*|H))^n \to 0$. This implies that the equilibrium stays away from the acceptance equilibrium $\sigma^* = \underline{\sigma}$ in the following sense: either (i) $\sigma_n^* \geq \underline{\sigma} + \kappa$, for some $\kappa > 0$, or (ii) $\sigma_n^* > \underline{\sigma}$, and converges to $\underline{\sigma}$ at a "slower" rate than n grows large. In either case it follows (immediately in (i), by l'Hôpital's rule in (ii)) that

$$\frac{(1 - F(\sigma_n^*|L))^n}{(1 - F(\sigma_n^*|H))^n} \to 0 \text{ as } n \to \infty.$$

In other words, conditional on being pivotal, each responder infers that the true realization of ω is almost certainly H. Since by hypothesis the offer is bounded away from $x_H(\lambda)$, the benefit of accepting the offer when $\omega = H$ is likewise bounded away from 0. As such, each responder i votes to accept independent of his own observation σ_i. But then the equilibrium is the acceptance equilibrium $\sigma^* = \underline{\sigma}$, giving a contradiction.

Lemma 5 complements Lemma 4, and says that if instead the proposer's offer converges to $x_H(\lambda)$ as $n \to \infty$, then the probability that it is accepted converges to zero.

Lemma 5 ***(Low offers rejected under unanimity)*** *Suppose unanimity rule is in effect ($\alpha = 1$). Take a set of offers $x_n(\lambda)$. Then (1) if*

$$\sup_{\Lambda, N} \inf_{\lambda \in (0,\Lambda], n \geq N} x_n(\lambda) - x_H(\lambda) \leq 0 \tag{14}$$

then

$$\sup_{\Lambda, N} \inf_{\lambda \in (0,\Lambda], n \geq N} \Pr(A_n) = 0, \tag{15}$$

and (2) if for all $\varepsilon > 0$ there exists N_ε such that[16]

$$\inf_{\Lambda} \sup_{\lambda \in (0,\Lambda], n \geq N_\varepsilon} x_n(\lambda) - x_H(\lambda) \leq \varepsilon \tag{16}$$

then

$$\inf_{\Lambda, N} \sup_{\lambda \in (0,\Lambda], n \geq N} \Pr(A_n) = 0. \tag{17}$$

We sketch the proof of Part (1) for a fixed preference parameter λ. Suppose that contrary to the claim the acceptance probability stays bounded away from 0, even though in the limit the offer x_n is less than $x_H(\lambda)$. That is, $(1 - F(\sigma_n^*|H))^n$ stays bounded away from 0. This means that the equilibrium is "close" to the acceptance equilibrium $\sigma^* = \underline{\sigma}$: either $\sigma_n^* = \underline{\sigma}$, or σ_n^* approaches $\underline{\sigma}$ "faster" than n approaches ∞. It follows that as $n \to \infty$, the acceptance probability in L, $(1 - F(\sigma_n^*|L))^n$, stays bounded away from 0. Consequently, after conditioning on being pivotal each individual i attaches a strictly positive probability to both $\omega = L$ and $\omega = H$. By hypothesis, the offer approaches $x_H(\lambda)$ as $n \to \infty$. So conditional on seeing $\sigma_i = \sigma_H = \underline{\sigma}$, the benefits from accepting the offer approach zero when $\omega = H$, and are strictly negative when $\omega = L$. But then each responder i votes to reject the offer over some neighborhood of observations of σ_i around $\underline{\sigma}$, and so the equilibrium cannot be close to the acceptance equilibrium after all.

Thus far we have focused on the aggregate response of responders to offers close to $x_H(\lambda)$. We now turn to the opposite extreme of offers close to $x_L(\lambda)$. Recall that under majority voting, the proposer makes an offer close to $x_L(\lambda)$ whenever $W(\sigma_0) < 0$, but never makes an offer that is significantly greater. Our next result shows that when the proposer faces unanimity rule, the maximum offer he will ever make is lower than $x_L(\lambda)$. Specifically, we show that there is an offer below $x_L(\lambda)$ that responders always accept if n is sufficiently large.

Lemma 6 ***(Upper bound on offers under unanimity)*** *Suppose unanimity rule is in effect ($\alpha = 1$), and $x_L(\lambda) \neq \infty$ for all λ small enough. Then there exists $\kappa > 0$ and*

[16] The hypothesis (16) of Part 2 of Lemma 5 says, loosely, that $\lim_{\lambda \to 0} x_n(\lambda) - x_H(\lambda)$ converges to a constant $C \leq 0$ as $n \to \infty$.

$\Lambda > 0$ *such that the offer* $x_L(\lambda) - \kappa$ *is accepted with certainty for all* $\lambda \leq \Lambda$ *and all* n. *As such,* $\pi_n^*(\sigma_0; \lambda) \leq x_L(\lambda)$.

Lemmas 5 and 6 both relate to the failure of information aggregation under unanimity rule. In Lemma 5, failure of information aggregation leads offers above x_H to be rejected even when $\omega = H$. Conversely, in Lemma 6 failure of information aggregation leads offers below x_L to be accepted even when $\omega = L$.

Equilibrium offers under unanimity

From Lemmas 4 and 5, offers that converge to $x_H(\lambda)$ are rejected, while offers that remain bounded away are accepted with a probability that stays bounded away from 0. Since the proposer prefers acceptance to rejection, his equilibrium offers stay bounded away from $x_H(\lambda)$:

Lemma 7 ***(Equilibrium offers under unanimity)*** *Suppose unanimity rule is in effect (*$\alpha = 1$*). Then there exists* $\kappa > 0$ *and* N *such that for all* σ_0*,*

$$\sup_{\Lambda} \inf_{\lambda \in (0,\Lambda], n \geq N} \pi_n^*(\sigma_0; \lambda) - x_H(\lambda) \geq \kappa.$$

That is, the proposer's offer is bounded uniformly away from $x_H(\lambda)$ *when* n *is sufficiently large and* λ *is sufficiently small.*

Responder payoffs

When σ_0 is such that $W(\sigma_0) > 0$, for any majority rule α responders would receive a higher offer if instead they employed unanimity rule (for large n). Does this mean that responders are better off under unanimity rule? The main complication is that under unanimity responders' aggregate decision is frequently suboptimal *ex post*, in the sense that offers that should be accepted are rejected, and *vice versa*. As such, it is conceivable that a higher offer would increase the error rate to such an extent that it actually lowers responder welfare. Our next result explicitly characterizes the effect of a change in the offer on responders' welfare under unanimity. It shows that in spite of changes in error rate, the net effect is positive.

Lemma 8 *Suppose unanimity rule is in effect ($\alpha = 1$). Then holding responder beliefs fixed, for $x \neq \underline{x}_n(\alpha, \lambda, b), \bar{x}_n(\alpha, \lambda, b)$,*

$$\begin{aligned}\frac{\partial \Pi_n^R}{\partial x} &\geq b \Pr(A_n(x)|H) E\left[\frac{\partial}{\partial x}\Delta^H(x, \sigma, \lambda)|H, \sigma \geq \sigma^*\right] \\ &\quad + (1-b)\Pr(A_n(x)|L) E\left[\frac{\partial}{\partial x}\Delta^L(x, \sigma, \lambda)|L, \sigma \geq \sigma^*\right].\end{aligned}$$

We are now ready to state our main conclusion in this section. As we have seen, the proposer's equilibrium offers stay bounded away from $x_H(\lambda)$ when he faces unanimity rule (Lemma 7). Moreover, when responders use unanimity rule, higher offers improve their welfare (Lemma 8). From these observations, the responders' equilibrium payoff under unanimity rule is bounded away from their status quo payoff, $E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right]$.

Proposition 2 ***(Lower bound for the responders' payoff under unanimity)*** *There exists $\gamma > 0$ such that*

$$\sup_{N,\Lambda} \inf_{\lambda \in (0,\bar{\Lambda}], n \geq N} \Pi_n^{*R}(\lambda, \alpha = 1) - E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] \geq \gamma.$$

Finally, Lemma 6 gives an upper bound for the proposer's offer against unanimity rule. An immediate consequence is the following upper bound for the responders' payoff:

Proposition 3 ***(Upper bound for the responders' payoff under unanimity)*** *Suppose $x_L(\lambda, \alpha) \neq \infty$ for all λ small enough. Then there exists $\gamma > 0$ and $\Lambda > 0$ such that for all $\lambda \leq \Lambda$ and all n,*

$$\Pi_n^{*R}(\lambda, \alpha = 1) \leq E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] + E_{\sigma_i,\omega}\left[\Delta^\omega(x_L(\lambda), \sigma_i, \lambda)\right] - \gamma.$$

6 Comparing majority and unanimity voting rules

Propositions 1, 2 and 3 characterize payoffs under majority and unanimity rules. We use these results to analyze the effect of the voting rule α on the welfare of the two negotiating parties. We consider the following three representative cases:

Case I: If it were public information that $\omega = L$, the proposer would be *unable* to make an offer that the responders would accept. That is, $x_L = \infty$.

Case II: If it were public information that $\omega = L$, the proposer would be *able* to make an offer that the responders would accept. Moreover, he prefers to have the offer x_L accepted always than to have the less generous offer x_H accepted only when $\omega = H$; and this is true independent of his information σ_0.

Case III: If it were public information that $\omega = L$, the proposer would be *able* to make an offer that the responders accept. However, the proposer prefers to have the offer x_H accepted only when $\omega = H$ than to have the more generous offer x_L accepted always; and this is true independent of his information σ_0.

In all three cases, we focus on circumstances in which the responder coalition is large, and responder heterogeneity derives primarily from distinct private information, i.e., $\lambda \approx 0$. Such an assumption is natural in many applications. In particular, whenever responders receive a financial claim as an outcome of bargaining, and these claims are *ex post* tradeable, responder preferences will be close to the common values extreme.

Since all majority voting rules $\alpha < 1$ asymptotically deliver the same payoffs at the common values extreme, our results concern the comparison between an arbitrary majority rule $\alpha < 1$, and unanimity rule $\alpha = 1$. (Moreover, recall that although in general x_ω depends on the voting rule α, it is independent of α at the common values extreme $\lambda = 0$.)

Case I: Proposer unable to make a satisfactory offer when $\omega = L$

Directly from Proposition 1, for any majority voting rule $\alpha < 1$,

$$\Pi_n^{*R}(\lambda, \alpha) \to E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] + \Pr(H)\, E_{\sigma_i}\left[\Delta^H(x_H(\lambda, \alpha), \sigma_i, \lambda)\,|H\right]$$

as the number of responders grows large. The first term is the responders' payoff under the status quo. In general, the second term can be positive or negative. However, by definition, $\Delta^H(x_H(\lambda, \alpha), \sigma_H, \lambda) = 0$, and Δ^H is independent of σ_i when $\lambda = 0$. Consequently $E_{\sigma_i}\left[\Delta^H(x_H(\lambda), \sigma_i, \lambda)\,|H\right]$ approaches 0 as $\lambda \to 0$, and so the responders' payoff approaches their status quo payoff. Put differently, against a majority rule the proposer is able to reduce the responders' payoff all the way to their outside option.

In contrast, when responders use unanimity rule, from Proposition 2 we know their payoff is bounded away from their status quo payoff as n grows large and preferences approach the common values extreme. Consequently:

Proposition 4 *Fix a majority rule $\alpha < 1$. If $x_L(\lambda, \alpha) = \infty$ for λ small enough, then there exists $\gamma > 0$ and $\bar{\lambda} > 0$ such that whenever $\lambda \leq \bar{\lambda}$, for all n large enough the responders are better off under unanimity: $\Pi_n^{*R}(\lambda, 1) \geq \Pi_n^{*R}(\lambda, \alpha) + \gamma$.*

Proposition 4 says that, in this case, responders prefer unanimity rule to any majority rule. However, and in spite of the opposing preferences of the proposer and the responders, this does not by itself imply that the proposer prefers to face majority rule. In particular, suppose that the proposer's relative valuation of x were much higher when $\omega = L$ than $\omega = H$. Offers slightly above x_H are accepted with higher probability when $\omega = L$ under unanimity rule than under a majority rule, but with lower probability when $\omega = H$. As such, it is quite possible that a change from a majority rule to the unanimity rule would increase the welfare of both negotiating parties: the responders gain because they receive a higher offer, while the proposer gains because agreement is reached when $\omega = L$, which he values highly.

Nonetheless, under many circumstances effects of this type do not arise. In particular, consider the following additional assumption, which is satisfied in standard "split-the-dollar" type bargaining games:

Assumption 7 *(i) $V^\omega(x, \sigma_0)$ and $\bar{V}(\sigma_0)$ are independent of σ_0; (ii) $V^\omega(x = 1, \sigma_0) = \bar{V}(\sigma_0)$; (iii) $U^\omega(x, \sigma_i, \lambda = 0)$ and $V^\omega(x, \sigma_0)$ are linear in x; (iv) $U_x^\omega(x, \sigma_i, \lambda = 0) / V_x^\omega(x, \sigma_0)$ is independent of ω.*

Assumption 7 says that (i) the proposer's payoff has no private value component, (ii) the proposer is indifferent between the status quo and having his most generous offer

$x = 1$ accepted, (iii) both the proposer and responders are risk-neutral in x, and (iv) the relative value of changes in the offer x for the responders and the proposer is always the same. Assumption 7 is sufficient (but not necessary) to imply the following result:

Proposition 5 ***(Proposer's payoff in a split-the-dollar environment)*** *Fix a majority voting rule $\alpha < 1$. If Assumption 7 holds and $x_L(\lambda, \alpha) = \infty$ for λ small enough, then there exists $\bar{\lambda} > 0$ such that whenever $\lambda \leq \bar{\lambda}$, for all n large enough the proposer is worse off against unanimity rule: $\Pi_n^{*P}(\lambda, 1) < \Pi_n^{*P}(\lambda, \alpha)$.*

A useful way to view Proposition 5 is as follows. Given that $x_L = \infty$, agreement on any $x \in [0, 1]$ when $\omega = H$, and disagreement when $\omega = L$, are *ex post* Pareto efficient outcomes. As such, a majority rule is asymptotically *ex post* Pareto efficient. Assumption 7 is enough to ensure that *ex post* Pareto efficiency implies *ex ante* Pareto efficiency. Since responders do better under majority rule than unanimity rule, it follows that the proposer must do worse.

Case II: Proposer able and willing to make a satisfactory offer when $\omega = L$

Formally, this case arises when $W(\sigma_0; \lambda, \alpha) < 0$ for all σ_0. Against any majority rule $\alpha < 1$, from Lemma 3 the proposer's offer converges to x_L as the responder coalition grows large. That is, the proposer is both *able* and *willing* to make an offer that is accepted when $\omega = L$. From Proposition 1, as the number of responders grows large

$$\Pi_n^{*R}(\lambda, \alpha) \rightarrow E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right] + E_{\sigma_i,\omega}\left[\Delta^{\omega}(x_L(\lambda, \alpha), \sigma_i, \lambda)\right].$$

In contrast, when responders use unanimity rule there exists an offer strictly less than x_L that the responders will always accept (see Lemma 6). As such, the proposer clearly prefers to face unanimity rule (even without additional conditions such as Assumption 7). Moreover, the responders' payoff under unanimity is bounded away from

$$E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right] + E_{\sigma_i,\omega}\left[\Delta^{\omega}(x_L(\lambda, \alpha = 1), \sigma_i, \lambda)\right]$$

from above (see Proposition 3). Thus the responders receive both a higher offer and a higher payoff under a majority rule than under unanimity rule.

Loosely speaking, in this case the proposer is able to exploit the failure of unanimity rule to aggregate responders' information. Formally:

Proposition 6 *Fix a majority voting rule $\alpha < 1$. If $W(\sigma_0; \lambda = 0, \alpha) < 0$ for all σ_0, then there exists $\bar{\lambda} > 0$ such that whenever $\lambda \leq \bar{\lambda}$, for all n large enough the responders are worse off and the proposer is better off under unanimity rule: $\Pi_n^{*R}(\lambda, 1) < \Pi_n^{*R}(\lambda, \alpha)$ and $\Pi_n^{*P}(\lambda, 1) > \Pi_n^{*P}(\lambda, \alpha)$.*

Case III: Proposer able but unwilling to make a satisfactory offer when $\omega = L$

Formally, this case arises when $x_L(\lambda, \alpha) \neq \infty$ and $W(\sigma_0; \lambda, \alpha) > 0$ for all σ_0. Against any majority rule $\alpha < 1$, from Lemma 3 the proposer's offer converges to x_H as the responder coalition grows large. That is, although the proposer is able to make an offer that would be accepted by responders who knew that $\omega = L$, when facing a majority rule he is *unwilling* to do so.

For the responders, the welfare comparison exactly matches that of Case I, for the same reasons. As such, the responder coalition prefers to use unanimity rule:

Proposition 7 *Fix a majority voting rule $\alpha < 1$. If $W(\sigma_0; \lambda = 0, \alpha) > 0$ for all σ_0, then there exists $\gamma > 0$ and $\bar{\lambda} > 0$ such that whenever $\lambda \leq \bar{\lambda}$, for all n large enough the responders are better off under unanimity: $\Pi_n^{*R}(\lambda, 1) \geq \Pi_n^{*R}(\lambda, \alpha) + \gamma$.*

Finally, there are conditions under which the proposer prefers to face majority rule, and conditions under which he prefers to face unanimity rule. The key distinction relative to Case I is that since $x_L \neq \infty$, the equilibrium outcome under majority rule is *ex post* Pareto inefficient when $\omega = L$ — both the proposer and responders would be better off if they agreed to x_L. As such, moving from a majority rule to unanimity rule subjects the proposer's equilibrium payoff to two offsetting effects. On the one hand, the proposer raises his offer beyond x_H, his offer against a majority coalition. But on the other hand, unanimity improves the prospects for agreement when $\omega = L$, thereby moving the outcome when $\omega = L$ closer to Pareto efficiency. It is easy to produce examples in which either one of these effects dominates. In particular, even when Assumption 7 holds there are instances in which the equilibrium outcome under a majority voting rule is Pareto dominated by the equilibrium outcome under unanimity rule.

Summary

Based on the results above, the following table lists the preferred voting rule of each of the two negotiating parties.

	Responder coalition	Proposer
Case I ("unable")	Unanimity	Majority (under Assumption 7)
Case II ("able and willing")	Majority	Unanimity
Case III ("able but unwilling")	Unanimity	Ambiguous

For low offers agreement is harder to reach under unanimity rule. In the common values setting, it is not impossible, however. Unanimity engenders a better offer from the proposer, making the responders better off. For higher offers, however, the so-called swing voter's curse has the implication that agreement is *more* likely under unanimity. In this case, unanimity is not the tougher rule — it is the softer one. Using unanimity rule then lowers the responders' payoff. Which of the two cases applies depends on the offer the proposer would make against fully informed responders. In particular, if agreement is *not* Pareto efficient when $\omega = L$ then unanimity is better for responders.

7 Private values

In general, adoption of unanimity rule affects the group's payoffs in two distinct ways. On the one hand, unanimity makes agreement harder to obtain. On the other hand, this "toughness" may be useful in negotiation. The previous section identifies a fairly general set of circumstances under which the latter effect dominates: whenever preferences are close to common values, and the proposer is either unable or unwilling to make a satisfactory offer when $\omega = L$, the increase in the proposer's offer (relative to majority) more than compensates for the increased probability of mistakenly rejecting the offer.

One way to think about this result is that when responders vote strategically, the requirement of unanimity is not as inimical to agreement as it might at first seem. Recall that each responder conditions his or her vote only on the circumstances under which it is actually pivotal. Given unanimity rule, this means that a responder considers the impact of voting to accept an offer conditional on all other responders accepting — in other words, conditional on all other responders viewing the offer as attractive. Such a responder will vote to accept unless his own signal is very pro status quo.

In contrast, as we move to a situation in which responders are further away from the common values benchmark, we reach a situation in which agreement is indeed extremely difficult to obtain under unanimity. This is most easily seen at the extreme of fully private values preferences ($\lambda = 1$): each responder will vote to accept only if his own valuation of the offer on the table exceeds his payoff under the status quo.

Formally, suppose that $\Delta^H (x = 1, \underline{\sigma}, \lambda = 1) < 0$, so that a responder who received the most pro status quo realization of σ_i prefers the status quo to even the most generous offer. Under this assumption, it can be shown that if responders have preferences sufficiently close to pure private values, then the agreement probability converges to zero as the number of responders grows large. Moreover, under additional mild assumptions there exists *some* majority rule $\hat{\alpha} < 1$ that Pareto dominates unanimity whenever preferences are close to private values and the number of responders is sufficiently large.[17]

8 Concluding remarks

There are many instances in which a group of individuals is engaged in collective bargaining. In such instances, it is often tempting to model the group as a single individual. One way to view our paper is as an exploration of the extent to which this approach

[17]A proof is available on the authors' webpages.

is justified. When the group uses a majority rule, and the main source of intra-group heterogeneity is different information, it is indeed the case that the response of a large group to an offer made by the opposing party matches that of a single individual endowed with the same information. However, by adopting unanimity rule the group will cause its joint behavior to diverge from that of an individual. Our results suggest that under some circumstances such a purposeful deviation is beneficial to the group.

A somewhat related issue is the extent to which our results would change if instead of analyzing a take-it-or-leave game we allowed for alternating offers. However, to do so would introduce the non-trivial complication that the group would have to decide on a procedure for selecting a counter-offer. In particular, what is the optimal way for a group to decide which member(s) should make the offer?[18] We leave this undoubtedly important issue for future research.

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[18]See Haller and Holden (1997) for an analysis of a related issue: should an agreement reached by a group representative by subject to ratification by the rest of the group?

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A Appendix

We repeatedly use the following result. The proof is straightforward and available from the authors' webpages.

Lemma 9 *$F(\sigma|H)/F(\sigma|L)$ is increasing in σ, and is bounded above by 1. Consequently, $F(\sigma|H) \leq F(\sigma|L)$, and is strict if $\sigma \in (\underline{\sigma}, \bar{\sigma})$. Moreover, $(1 - F(\sigma|H))/(1 - F(\sigma|L))$ is increasing in σ, and is bounded above by $f(\bar{\sigma}|H)/f(\bar{\sigma}|L) > 1$.*

Proof of Lemma 1: First note that if $Z(x,\sigma) = 0$, then it must be the case that $\Delta^H(x,\sigma) > 0$ by Assumption 2. This implies that $Z(x,\sigma)$ is strictly increasing in σ whenever $Z(x,\sigma) \geq 0$. In turn, $Z(x,\sigma') < 0$ for all $\sigma' < \sigma$ if $Z(x,\sigma) = 0$.

Part 1: By definition, if $x < \underline{x}_n$ then $Z(x,\cdot) < 0$, while if $x > \bar{x}_n$ then $Z(x,\cdot) > 0$. For $x \in [\underline{x}_n, \bar{x}_n]$ we claim that $Z(x,\sigma) = 0$ for some unique σ, which we write as $\sigma^*(x)$. Existence is immediate, since $\max_\sigma Z(x,\sigma) \geq 0 \geq \min_\sigma Z(x,\sigma)$, and $Z(x,\sigma)$ is continuous in σ. Uniqueness follows from the result we have just shown that $Z(x,\sigma)$ is strictly increasing in σ whenever $Z(x,\sigma) \geq 0$.

Part 2: To see that $\sigma^*(x)$ is decreasing, consider x and $x' > x$ in $(\underline{x}_n, \bar{x}_n)$. Since $Z(x, \sigma^*(x)) = 0$, it follows that $Z(x', \sigma^*(x)) > 0$. Since $Z(x',\sigma)$ is increasing in σ it must be the case that $\sigma^*(x') < \sigma^*(x)$. By the Implicit Function Theorem, $\sigma(x)$ is continuously differentiable over $(\underline{x}_n, \bar{x}_n)$. To see $\sigma^*(\underline{x}_n) = \bar{\sigma}$, suppose to the contrary that $\sigma^*(\underline{x}_n) < \bar{\sigma}$. By definition $Z(\underline{x}_n, \sigma^*(\underline{x}_n)) = 0$, and so $Z(\underline{x}_n, \bar{\sigma}) > 0$. By continuity there exists an $x < \underline{x}_n$ such that $Z(x, \bar{\sigma}) > 0$ as well. This contradicts the definition of $\underline{x}_n$. Likewise, to see $\sigma^*(\bar{x}_n) = \underline{\sigma}$ suppose to the contrary that $\sigma^*(\bar{x}_n) > \underline{\sigma}$. By definition $Z(\bar{x}_n, \sigma^*(\bar{x}_n)) = 0$ which implies that $Z(\bar{x}_n, \underline{\sigma}) < 0$. By continuity there exists an x such that $x > \bar{x}_n$ and $Z(x,\sigma) < 0$, contradicting the definition of $\bar{x}_n$.

Part 3: Part (a) is immediate from the observation that as $n \to \infty$,

$$\frac{f(\sigma|H)}{f(\sigma|L)} \left(\frac{F(\sigma|H)}{F(\sigma|L)} \right)^{n-n\alpha} \left(\frac{1-F(\sigma|H)}{1-F(\sigma|L)} \right)^{n\alpha-1}$$

converges to 0 and ∞ respectively for $\sigma = \underline{\sigma}, \bar{\sigma}$. Part (b) is proved similarly. ∎

Proof of Lemma 2: We prove the lemma in four steps.

Claim 1 *If* $\limsup x_n < x_H(\lambda)$ *then* $\liminf \sigma_n^* > \sigma_H$.

Proof: By hypothesis, there exists ε such that $x_n \leq x_H(\lambda) - \varepsilon$ for all n large enough. Suppose now to the contrary that $\liminf \sigma_n^* \leq \sigma_H$. So for any $\delta > 0$, there exists a subsequence of σ_n^* such that $\sigma_n^* \leq \sigma_H + \delta$. By definition $\Delta^H(x_H(\lambda), \sigma_H, \lambda) = 0$; so for δ

small enough, there exists $\hat{\varepsilon}$ such that $\Delta^H(x_n, \sigma_n^*, \lambda) < -\hat{\varepsilon}$. Moreover, $\Delta^L(x_n, \sigma_n^*, \lambda) \leq \Delta^H(x_n, \sigma_n^*, \lambda)$. Consequently $Z(x_n, \sigma_n^*) < 0$. As such, σ_n^* is not a responsive equilibrium; and since $x_n \leq \bar{x}_n$ then σ_n^* is not an acceptance equilibrium either. The only remaining possibility is that σ_n^* is a rejection equilibrium — but then $\sigma_n^* = \bar{\sigma}$, which gives a contradiction when δ is chosen small enough. ∎

Claim 2 *If* $\limsup x_n < x_L(\lambda)$ *then* $\liminf \sigma_n^* > \sigma_L$.

Proof: By hypothesis, there exists ε such that $x_n \leq x_L(\lambda) - \varepsilon$ for all n large enough. Suppose now to the contrary that $\liminf \sigma_n^* \leq \sigma_L$. So for any $\delta > 0$, there exists a subsequence of σ_n^* such that $\sigma_n^* \leq \sigma_L + \delta$. By definition $\Delta^L(x_L(\lambda), \sigma_L, \lambda) = 0$; so for δ small enough, there exists $\hat{\varepsilon}$ such that $\Delta^L(x_n, \sigma_n^*, \lambda) < -\hat{\varepsilon}$. Next, define

$$\phi = \max_{\sigma \in [\underline{\sigma}, \sigma_L + \delta]} \frac{(1 - F(\sigma|H))^\alpha F(\sigma|H)^{1-\alpha}}{(1 - F(\sigma|L))^\alpha F(\sigma|L)^{1-\alpha}}$$

Note that the function $(1-q)^\alpha q^{1-\alpha}$ is increasing for $q \in (0, 1-\alpha)$ and decreasing for $q \in (1-\alpha, 1)$. Recall that by definition $F(\sigma_L|L) = 1 - \alpha$, and by Lemma 9 $F(\sigma|H) < F(\sigma|L)$ for all $\sigma \in (\underline{\sigma}, \bar{\sigma})$. It follows that $\phi < 1$ for δ chosen small enough, and so

$$\left(\frac{(1 - F(\sigma^*|H))^\alpha F(\sigma^*|H)^{1-\alpha}}{(1 - F(\sigma^*|L))^\alpha F(\sigma^*|L)^{1-\alpha}} \right)^n \leq \phi^n \to 0.$$

Since σ_n^* is bounded away from $\bar{\sigma}$, then $1 - F(\sigma_n^*|H)$ is bounded away from 0. By belief consistency, $\frac{\beta_n(x_n)}{1-\beta_n(x_n)}$ is bounded away from infinity. Consequently $Z(x_n, \sigma_n^*) < 0$ for n sufficiently large. A contradiction then follows as in Claim 1. ∎

Claim 3 *If* $\liminf x_n > x_L(\lambda)$ *then* $\limsup \sigma_n^* < \sigma_L$.

Proof: By hypothesis, there exists ε such that $x_n \geq x_L(\lambda) + \varepsilon$ for all n large enough. Suppose that contrary to the claim $\limsup \sigma_n^* \geq \sigma_L$. So for any δ, there exists a subsequence such that $\sigma_n^* \geq \sigma_L - \delta$. By definition, $\Delta^L(x_L(\lambda), \sigma_L, \lambda) = 0$; so for δ small enough, there exists $\hat{\varepsilon}$ such that $\Delta^L(x_n, \sigma_n^*, \lambda) > \hat{\varepsilon}$. Moreover, $\Delta^H(x_n, \sigma_n^*, \lambda) \geq$

$\Delta^L(x_n, \sigma_n^*, \lambda)$. Consequently $Z(x_n, \sigma_n^*) > 0$ for n sufficiently large. So σ_n^* cannot be a responsive equilibrium; and since $x_n \geq \underline{x}_n$ it is not a rejection equilibrium either. The only remaining possibility is that σ_n^* is an acceptance equilibrium — but then $\sigma_n^* = \underline{\sigma}$, which gives a contradiction when δ is chosen small enough. ∎

Claim 4 *If* $\liminf x_n > x_H(\lambda)$ *then* $\limsup \sigma_n^* < \sigma_H$.

Proof: By hypothesis, there exists ε such that $x_n \geq x_H(\lambda) + \varepsilon$ for all n large enough. Suppose now to the contrary that $\limsup \sigma_n^* \geq \sigma_H$. So for any $\delta > 0$, there exists a subsequence of σ_n^* such that $\sigma_n^* \geq \sigma_H - \delta$. By definition $\Delta^H(x_H(\lambda), \sigma_H, \lambda) = 0$; so for δ small enough, there exists $\hat{\varepsilon}$ such that $\Delta^H(x_n, \sigma_n^*, \lambda) > \hat{\varepsilon}$. Next, define

$$\phi = \min_{\sigma \in [\sigma_H - \delta, \bar{\sigma}]} \frac{(1 - F(\sigma|H))^\alpha F(\sigma|H)^{1-\alpha}}{(1 - F(\sigma|L))^\alpha F(\sigma|L)^{1-\alpha}}$$

Recall that by definition $F(\sigma_H|H) = 1 - \alpha$. By arguments similar to those in Claim 2, it follows that $\phi > 1$ for δ chosen small enough, and so

$$\left(\frac{(1 - F(\sigma^*|H))^\alpha F(\sigma^*|H)^{1-\alpha}}{(1 - F(\sigma^*|L))^\alpha F(\sigma^*|L)^{1-\alpha}} \right)^n \geq \phi^n \to \infty.$$

From Lemma 9, the term $\frac{1-F(\sigma|L)}{1-F(\sigma|H)}$ lies above $f(\bar{\sigma}|L)/f(\bar{\sigma}|H)$. By belief consistency, $\frac{\beta_n(x_n)}{1-\beta_n(x_n)}$ is bounded away from zero. Consequently $Z(x_n, \sigma_n^*) > 0$ for n sufficiently large. A contradiction then follows as in Claim 3. ∎

Proof of Lemma 3: We focus on Part 1a. (Part 1b and 2 are proved by similar arguments, which we omit for conciseness. Part 3 is immediate from Lemma 2.) The main idea is straightforward: for any σ_0 such that $W(\sigma_0) > 0$, the proposer prefers offering $x_H(\lambda, \alpha)$ and gaining acceptance if and only if $\omega = H$ to offering $x_L(\lambda, \alpha)$ and gaining acceptance all the time. Given the limiting behavior of responders established in Lemma 2, intuitively it follows that the proposer's offer converges to $x_H(\lambda, \alpha)$ as the number of responders grows large. The main difficulty encountered in the formal proof is establishing uniform convergence: for any $\varepsilon, \delta > 0$, there is some $N(\varepsilon, \delta)$ such that when $n \geq N(\varepsilon, \delta)$, the proposer's offer lies within δ of $x_H(\lambda, \alpha)$ for all σ_0 such that $W(\sigma_0) > \varepsilon$.

Take any $\varepsilon, \delta > 0$. Throughout the proof, we omit all λ and α arguments for readability. We define $\Delta_0^\omega(x, \sigma_0) \equiv V^\omega(x, \sigma_0) - \bar{V}^\omega(\sigma_0)$, the proposer's gain to offer x being accepted conditional on ω.

Preliminaries: The first part of the proof consists of defining bounds which we will use to establish uniform convergence below. Choose μ, δ_1, δ_2, $\delta_3 \in (0, \delta]$ such that $x_H + \mu < x_L - \mu$, and for all σ_0 for which $W(\sigma_0) > \varepsilon$,

$$\Pr(H|\sigma_0) V^H\left(x_H + \frac{\mu}{2}, \sigma_0\right) + \Pr(L|\sigma_0) \bar{V}^L(\sigma_0) \geq E\left[V^\omega(x_L - \mu, \sigma_0) |\sigma_0\right] + \frac{\varepsilon}{2}, \tag{18}$$

$$\delta_1 \Delta_0^H\left(x^H + \frac{\mu}{2}, \sigma_0\right) \leq \frac{\varepsilon}{4}, \tag{19}$$

$$\delta_2\left(\Pr(H|\sigma_0) \Delta_0^H(0, \sigma_0) + \Pr(L|\sigma_0) \Delta_0^L(0, \sigma_0)\right) < (1 - \delta_1) \Pr(H|\sigma_0) \Delta_0^H\left(x_H + \frac{\mu}{2}, \sigma_0\right), \tag{20}$$

$$\Pr(H|\sigma_0)\left((1 - \delta_1) \Delta_0^H\left(x^H + \frac{\mu}{2}, \sigma_0\right) - \Delta_0^H(x_H + \mu, \sigma_0)\right) > \Pr(L|\sigma_0) \delta_3 \Delta_0^L(x_H + \mu, \sigma_0), \tag{21}$$

$$\begin{aligned} & \Pr(H|\sigma_0)\left((1 - \delta_1) V^H\left(x^H + \frac{\mu}{2}, \sigma_0\right) + \delta_1 \bar{V}^H(\sigma_0)\right) + \Pr(L|\sigma_0) \bar{V}^L(\sigma_0) \\ > \; & \Pr(H|\sigma_0)\left((1 - \delta) V^H(x_H - \mu, \sigma_0) + \delta \bar{V}^H(\sigma_0)\right) \\ + \; & \Pr(L|\sigma_0)\left(\delta_3 V^L(x_H - \mu, \sigma_0) + (1 - \delta_3) \bar{V}^L(\sigma_0)\right). \end{aligned} \tag{22}$$

A choice of μ, δ_1, δ_2, δ_3 exists such that (18), (19), (20), (21), and (22) hold as follows. First, choose μ such that (18) holds, along with

$$V^H\left(x_H + \frac{\mu}{2}, \sigma_0\right) > (1 - \delta) V^H(x_H - \mu, \sigma_0) + \delta \bar{V}^H(\sigma_0). \tag{23}$$

It is possible to choose $\mu > 0$ that satisfies these two inequalities for all σ_0 since $|V_x^\omega|$ is bounded. The same argument applies in choosing $\delta_1, \delta_2, \delta_3$ below. Second, choose δ_1 such that (19) holds, along with

$$(1 - \delta_1) \Delta_0^H\left(x^H + \frac{\mu}{2}, \sigma_0\right) - \Delta_0^H(x_H + \mu, \sigma_0) > 0, \tag{24}$$

$$\left((1-\delta_1) V^H\left(x^H+\frac{\mu}{2}, \sigma_0\right)+\delta_1 \bar{V}^H(\sigma_0)\right)-\left((1-\delta) V^H(x_H-\mu, \sigma_0)+\delta \bar{V}^H(\sigma_0)\right)>0, \tag{25}$$

where (25) is possible by (23). Third, choose δ_2 such that (20) holds. Finally, choose δ_3 such that (21) and (22) hold, which is possible by (24) and (25) respectively.

Let $\underline{b}$ and $\bar{b}$ respectively denote the most pro-L and pro-H beliefs possible. Fix a realization of σ_0 such that $W(\sigma_0) \geq \varepsilon$. Define the following offer sequences, which we use throughout the proof:

$$x_n \equiv \pi_n^*(\sigma_0), \quad x_n^{H+} \equiv x_H+\frac{\mu}{2}, \quad x_n^{H-} \equiv x_H-\mu, \quad x_n^{L-} \equiv x_L-\mu.$$

By Lemma 2, $\Pr\left(A_n\left(x_n^{H+}, \underline{b}\right)|H\right) \rightarrow 1$ and $\Pr\left(A_n\left(x_n^{H+}, \bar{b}\right)|L\right) \rightarrow 0$; $\Pr\left(A_n\left(x_n^{H-}, \bar{b}\right)|\omega\right) \rightarrow 0$ for $\omega=L, H$; and $\Pr\left(A_n\left(x_n^{L-}, \bar{b}\right)|L\right) \rightarrow 0$. Thus there exist N_1, N_2, N_3 such that $\Pr\left(A_n\left(x_n^{H+}, \underline{b}\right)|H\right) \geq 1-\delta_1$ and $\Pr\left(A_n\left(x_n^{H+}, \bar{b}\right)|L\right) \leq \delta_1$ for $n \geq N_1$; $\Pr\left(A_n\left(x_n^{H-}, \bar{b}\right)|\omega\right) \leq \delta_2$ for $\omega=L, H$ and $n \geq N_2$; and $\Pr\left(A_n\left(x_n^{L-}, \bar{b}\right)|L\right) \leq \delta_3$ for $n \geq N_3$. Let $N(\varepsilon, \delta)=\max\{N_1, N_2, N_3\}$. Note that $N(\varepsilon, \delta)$ depends only on ε and δ, and not σ_0.

Part A: If $W(\sigma_0) \geq \varepsilon$ then $\Pr(A_n(\pi_n^*(\sigma_0))|L) \leq \delta_3 \leq \delta$ when $n \geq N(\varepsilon, \delta)$.

Proof: If $x_n \leq x_n^{L-}$ then $\Pr(A_n(x_n)|L) \leq \Pr\left(A_n\left(x^{L-}, \bar{b}\right)|L\right) \leq \delta_3$ for $n \geq N(\varepsilon, \delta)$. Consequently it suffices to show that $x_n \leq x_n^{L-}$ for all $n \geq N(\varepsilon, \delta)$. If this were not the case, there must exist some $m \geq N(\varepsilon, \delta)$ such that $x_m > x_m^{L-}$. By Assumption 4 the proposer is always better off when his offer is accepted; and so if $x_m > x_m^{L-}$ the proposer's expected payoff is bounded above by $E\left[V^\omega\left(x_m^{L-}, \sigma_0\right)|\sigma_0\right]$. In contrast, since $m \geq N(\varepsilon, \delta)$, the proposer's payoff from the offer x_m^{H+} is bounded below by

$$\begin{aligned}
& \Pr(H|\sigma_0)\left((1-\delta_1) V^H\left(x_m^{H+}, \sigma_0\right)+\delta_1 \bar{V}^H(\sigma_0)\right)+\Pr(L|\sigma_0) \bar{V}^L(\sigma_0) \\
= & \Pr(H|\sigma_0)\left(V^H\left(x_m^{H+}, \sigma_0\right)-\delta_1 \Delta_0^H\left(x_m^{H+}, \sigma_0\right)\right)+\Pr(L|\sigma_0) \bar{V}^L(\sigma_0) \\
\geq & \Pr(H|\sigma_0) V^H\left(x_m^{H+}, \sigma_0\right)+\Pr(L|\sigma_0) \bar{V}^L(\sigma_0)-\frac{\varepsilon}{4}
\end{aligned}$$

where the inequality follows by (19) (and the fact that $\Pr(H|\sigma_0) \leq 1$). By (18) this lower bound exceeds $E\left[V^\omega\left(x_m^{L-}, \sigma_0\right)|\sigma_0\right]$, contradicting the optimality of x_n.

Part B: If $W(\sigma_0) \geq \varepsilon$ then $|\pi_n^*(\sigma_0) - x_H| \leq \mu \leq \delta$ for all $n \geq N(\varepsilon, \delta)$.

Proof: First, we claim that $x_n > x_n^{H-}$ whenever $n \geq N(\varepsilon, \delta)$. If this were not the case, there must exist some $m \geq N(\varepsilon, \delta)$ such that $x_m \leq x_m^{H-}$. The acceptance probability of x_m given ω is consequently less than that of x_m^{H-} under the most pro-acceptance beliefs $\bar{b}$, which is in turn less than δ_2. The acceptance probability of x_m^{H+} given H is at least $1 - \delta_1$. It follows from (20) that the proposer's payoff is higher under x_m^{H+} than under x_m. But this contradicts the optimality of the proposer's offer x_m. Second, we claim that $x_n \leq x_H + \mu$ whenever $n \geq N(\varepsilon, \delta)$. If not, there exists $m \geq N(\varepsilon, \delta)$ such that $x_m > x_H + \mu$. By Part A, proposer's payoff under x_m is bounded above by

$$\Pr(H|\sigma_0)\left(\Delta_0^H\left(x^H + \mu, \sigma_0\right) + \bar{V}^H(\sigma_0)\right) + \Pr(L|\sigma_0)\left(\delta_3\Delta_0^L\left(x^H + \mu, \sigma_0\right) + \bar{V}^L(\sigma_0)\right).$$

In contrast, since $m \geq N(\varepsilon, \delta)$, the proposer's payoff from the offer x_m^{H+} is bounded below by

$$\Pr(H|\sigma_0)\left((1-\delta_1)\Delta_0^H\left(x_m^{H+}, \sigma_0\right) + \bar{V}^H(\sigma_0)\right) + \Pr(L|\sigma_0)\bar{V}^L(\sigma_0),$$

which exceeds the payoff from the offer x_m by (21), contradicting optimality of x_m.

Part C: If $W(\sigma_0) \geq \varepsilon$, then $\Pr(A_n(\pi_n^*(\sigma_0))|H) \geq 1 - \delta$ for all $n \geq N(\varepsilon, \delta)$.

Proof: Suppose that contrary to the claim, there exists $m \geq N(\varepsilon, \delta)$ such that $\Pr(A_m(x_m)|H) < 1 - \delta$. By Part A, $\Pr\left(A_n\left(x_m^{L-}, \bar{b}\right)|L\right) \leq \delta_3$, and by Part B, $x_m \geq x_m^{H-}$ and hence proposer's payoff is bounded above by

$$\Pr(H|\sigma_0)\left((1-\delta)V^H\left(x_m^{H-}, \sigma_0\right) + \delta\bar{V}^H(\sigma_0)\right) + \Pr(L|\sigma_0)\left(\delta_3 V^L\left(x_m^{H-}, \sigma_0\right) + (1-\delta_3)\bar{V}^L(\sigma_0)\right).$$

In contrast, under the offers x_m^{H+}, the proposer's payoff is bounded below by

$$\Pr(H|\sigma_0)\left((1-\delta_1)V^H\left(x_m^{H+}, \sigma_0\right) + \delta_1\bar{V}^H(\sigma_0)\right) + \Pr(L|\sigma_0)\bar{V}^L(\sigma_0).$$

By (22) the latter is strictly greater, contradicting the optimality of the offers x_n. ∎

Proof of Proposition 1: Immediate from Lemma 3 ∎

Proof of Lemma 4: The basic idea of the proof is as follows. In Claim 1, we choose Λ and N such that when $\lambda \in (0, \Lambda]$ and $n \geq N$ the voting equilibrium is "well-behaved." In Claim 2, we show that if $\inf_{\lambda \in (0,\Lambda], n \geq N} \Pr(A_n|H) = 0$, then this would allow us to form a sequence in which the number of responders grows large "faster" than the probability that each responder votes to accept approaches 1. In Claim 3, we then show just the opposite: given the "well-behaved" properties we established in Claim 1, then the number of responders must grow large "slower" than the probability that each responder votes to accept approaches 1.

Claim 1: There exists (Λ, N) and $\delta > 0$ such that for all $\lambda \in (0, \Lambda]$ and $n \geq N$,

$$\left(\frac{1 - F(\sigma_n^*(x_n(\lambda), \lambda)|L)}{1 - F(\sigma_n^*(x_n(\lambda), \lambda)|H)}\right)^n \geq \delta \text{ if } \sigma_n^*(x_n(\lambda), \lambda) > \underline{\sigma} \tag{26}$$

$$\inf_{\lambda \in (0,\Lambda]} \sigma_n^*(x_n(\lambda), \lambda) \leq (\underline{\sigma} + \bar{\sigma})/2. \tag{27}$$

Proof of Claim 1: By hypothesis (condition (12)), for all $\varepsilon > 0$ sufficiently small there exists $\Lambda > 0$ and N_0 such that $x_n(\lambda) - x_H(\lambda) \geq \varepsilon$ whenever $\lambda \in (0, \Lambda]$ and $n \geq N_0$.

Note that $\sigma_H = \underline{\sigma}$, so $\Delta^H(x_H(\lambda), \underline{\sigma}) = 0 < \Delta^H(x_H(\lambda), \bar{\sigma})$. Let $\underline{b}$ be the most pro-L beliefs. From Lemma 1, there exists $N_1 \geq N_0$ such that if $n \geq N_1$ and $\lambda \in (0, \Lambda]$, then there is a responsive voting equilibrium when the offer is $x_n(\lambda) \geq x_H(\lambda) + \varepsilon$ and the beliefs are $\underline{b}$. Consequently, by Corollary 1 it follows that if $\lambda \in (0, \Lambda]$ and $n \geq N_1$, then given the actual beliefs b associated with offer $x_n(\lambda)$, either a responsive equilibrium exists, or the equilibrium is the acceptance equilibrium.

If the equilibrium for some $x_n(\lambda)$ and λ is the acceptance equilibrium, then $\sigma_n^*(x_n(\lambda), \lambda) = \underline{\sigma}$. On the other hand, if the equilibrium is a responsive equilibrium, then since $\sigma_n^*(x_n(\lambda), \lambda) \geq \underline{\sigma} = \sigma_H$ and $x_n(\lambda) - x_H(\lambda) \geq \varepsilon$, and by definition $\Delta^H(x_H(\lambda), \sigma_H, \lambda) = 0$, it follows that there exists $\hat{\varepsilon} > 0$ such that $\Delta^H(x_n(\lambda), \sigma_n^*, \lambda) > \hat{\varepsilon}$. By assumption Δ^L is bounded, and the remaining terms on the lefthand side of the equilibrium condition (7) are bounded away from 0. Consequently, there exists $\delta > 0$ and $N_2 \geq N_1$ such that inequality (26) holds for all $\lambda \in (0, \Lambda]$ and $n \geq N_2$. To complete the proof of Claim 1, it suffices to show that

$$\inf_{\lambda\in(0,\Lambda]} \sigma_n^*\left(x_n\left(\lambda\right),\lambda\right) \to \underline{\sigma} \text{ as } n \to \infty. \tag{28}$$

Suppose to the contrary that for all $\lambda \in (0,\Lambda]$ there exists a subsequence of $\sigma_n^*\left(x_n\left(\lambda\right),\lambda\right)$ which stays bounded away from $\underline{\sigma}$. Clearly along this subsequence the equilibria are responsive. From Lemma 9, $\frac{1-F(\sigma|L)}{1-F(\sigma|H)} < 1$ for any $\sigma \in (\underline{\sigma},\bar{\sigma})$, and converges to $f\left(\bar{\sigma}|L\right)/f\left(\bar{\sigma}|H\right) < 1$ as $\sigma \to \bar{\sigma}$. This gives a contradiction to (26).

Claim 2: Let Λ and N be as defined in Claim 1. If

$$\inf_{\lambda\in(0,\Lambda]} \Pr\left(A_n|H\right) = 0, \tag{29}$$

then there exists a sequence (λ_m, n_m) with $\lambda_m \in (0,\Lambda]$ and $n_m \geq N$ such that as $m \to \infty$,

$$\begin{aligned} \lim \left(1 - F\left(\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right)|H\right)\right)^{n_m} &= 0, &(30)\\ \lim n_m &= \infty, &(31)\\ \lim \sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right) &= \underline{\sigma}, &(32)\\ \lim \left(\frac{1 - F\left(\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right)|L\right)}{1 - F\left(\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right)|H\right)}\right)^{n_m} &= \hat{\delta} > 0. &(33)\end{aligned}$$

Proof of Claim 2: Equality (29) can only hold if there exists some sequence (λ_m, n_m) for which $\lambda_m \in (0,\Lambda]$ and $n_m \geq N$ and (30) holds. If n_m were bounded above, then (30) could hold only if $\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right) \to \bar{\sigma}$. Since this would contradict (27), it follows that $n_m \to \infty$. If $\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right) = \underline{\sigma}$ then $F\left(\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right)|H\right) = 0$; so to avoid violating (30), we need $\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right) > \underline{\sigma}$ for all m large enough. Without loss, choose the sequence (λ_m, n_m) so that it has this property directly. Since $\sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right) > \underline{\sigma}$ and $n_m \to \infty$, (26) implies that $\lim \sigma_{n_m}^*\left(x_{n_m}\left(\lambda_m\right),\lambda_m\right) = \underline{\sigma}$.

The sequence defined by (33) is bounded, and so by Bolzano-Weierstrass has a convergent subsequence. Again, without loss, assume that the sequence (λ_m, n_m) was chosen so that it has this property directly. By (26), the limit of expression (33) is above δ completing the proof of the claim.

Claim 3: If (λ_m, n_m) is a sequence that satisfies (31) - (33), then

$$\lim \left(1 - F\left(\sigma^*_{n_m}\left(x_{n_m}(\lambda_m), \lambda_m\right)|H\right)\right)^{n_m} > 0$$

Proof of Claim 3: For readability, for the remainder of the proof we suppress the arguments $(x_{n_m}(\lambda_m), \lambda_m)$ in writing $\sigma^*_{n_m}$. It suffices to show that $\lim \ln \left(1 - F\left(\sigma^*_{n_m}|H\right)\right)^{n_m}$ is bounded away from $-\infty$. To evaluate this limit, we use the discrete version of l'Hôpital's rule, which gives[19]

$$\lim \ln \left(1 - F\left(\sigma^*_{n_m}|H\right)\right)^{n_m} = \lim \frac{n_{m+1} - n_m}{\frac{1}{\ln 1 - F\left(\sigma^*_{n_{m+1}}|H\right)} - \frac{1}{\ln 1 - F\left(\sigma^*_{n_m}|H\right)}} \tag{34}$$

Define $J(\sigma) = \ln(1 - F(\sigma|L)) - \ln(1 - F(\sigma|H))$ and $K(\sigma) = \ln(1 - F(\sigma|H))$. Expanding, the right hand side of (34) is equal to

$$\lim (n_{m+1} - n_m) \frac{J\left(\sigma^*_{n_m}\right) J\left(\sigma^*_{n_{m+1}}\right)}{J\left(\sigma^*_{n_m}\right) - J\left(\sigma^*_{n_{m+1}}\right)} \frac{J\left(\sigma^*_{n_m}\right) - J\left(\sigma^*_{n_{m+1}}\right)}{K\left(\sigma^*_{n_m}\right) - K\left(\sigma^*_{n_{m+1}}\right)} \frac{K\left(\sigma^*_{n_m}\right)}{J\left(\sigma^*_{n_m}\right)} \frac{K\left(\sigma^*_{n_{m+1}}\right)}{J\left(\sigma^*_{n_{m+1}}\right)}.$$

By (32), $\sigma^*_{n_m} \to \underline{\sigma}$, and so[20]

$$\frac{J\left(\sigma^*_{n_m}\right) - J\left(\sigma^*_{n_{m+1}}\right)}{K\left(\sigma^*_{n_m}\right) - K\left(\sigma^*_{n_{m+1}}\right)} \to \frac{f(\underline{\sigma}|L)}{f(\underline{\sigma}|H)} - 1 \tag{35}$$

$$\frac{K\left(\sigma^*_{n_m}\right)}{J\left(\sigma^*_{n_m}\right)} \to \left(\frac{f(\underline{\sigma}|L)}{f(\underline{\sigma}|H)} - 1\right)^{-1} \tag{36}$$

By strict MLRP, $f(\underline{\sigma}|L)/f(\underline{\sigma}|H) > 1$. Applying the discrete version of l'Hôpital's rule a second time,

$$\lim \ln \left(\frac{1 - F\left(\sigma^*_{n_m}|L\right)}{1 - F\left(\sigma^*_{n_m}|H\right)}\right)^{n_m} = \lim (n_{m+1} - n_m) \frac{J\left(\sigma^*_{n_m}\right) J\left(\sigma^*_{n_{m+1}}\right)}{J\left(\sigma^*_{n_m}\right) - J\left(\sigma^*_{n_{m+1}}\right)}. \tag{37}$$

Since this limit is bounded away from $-\infty$ by property (33), it follows that the limit of expression (34) is also. This completes the proof of Claim 3.

[19] This is necessary since $\sigma^*_{n_m}$ is not differentiable with respect to n_m. The discrete version of l'Hôpital's rule holds provided that $\frac{1}{\ln 1 - F\left(\sigma^*_{n_m}(\lambda_m)|H\right)} \to \infty$, which is satisfied since $\sigma^*_{n_m}(\lambda_m) \to \underline{\sigma}$.

[20] The next two limits follow since $\ln(1 - F(\sigma|R)) = -F(\sigma|R) + o\left(F(\sigma|R)^2\right)$ and $F(\sigma|R) = (\sigma - \underline{\sigma}) f(\underline{\sigma}|R) + o\left((\sigma - \underline{\sigma})^2\right)$; consequently $\ln(1 - F(\sigma|R)) = -(\sigma - \underline{\sigma}) f(\underline{\sigma}|R) + o\left((\sigma - \underline{\sigma})^2\right)$.

Proof of Main Result: Take Λ and N as defined in Claim 1. If condition (29) is satisfied, then by Claim 2 there exists a sequence (λ_m, n_m) that satisfies (30) - (33) hold. But then by Claim 3, (30) cannot be satisfied, a contradiction. Consequently, condition (29) cannot hold, i.e.,

$$\inf_{\lambda\in(0,\Lambda],n\geq N} \Pr(A_n|H) > 0.$$

This gives the result. ∎

Proof of Lemma 5:

Part 1: Suppose to the contrary that (15) does not hold. So there exists Λ and N such that $\inf_{\lambda\in(0,\Lambda],n\geq N} \Pr(A_n|H) > 0$. From (14), we can construct a sequence (λ_m, n_m) for which $\lambda_m \in (0,\Lambda]$ and $n_m \geq N$, with $n_m \to \infty$ and $\inf_m x_{n_m}(\lambda_m) - x_H(\lambda_m) \leq 0$. By Bolzano-Weierstrass the sequences $x_{n_m}(\lambda_m) - x_H(\lambda_m)$ and $\Pr(A_{n_m}(x_{n_m}(\lambda_m), \lambda_m) H)$ have a convergent subsequences. Thus without loss, assume that (λ_m, n_m) is chosen directly so that limits exist, and

$$\lim x_{n_m}(\lambda_m) - x_H(\lambda_m) \;\leq\; 0, \tag{38}$$

$$\lim \Pr(A_{n_m}(x_{n_m}(\lambda_m), \lambda_m)|H) \;>\; 0. \tag{39}$$

As in the proof of Lemma 4 (39) implies $\sigma^*_{n_m}(x_{n_m}(\lambda_m), \lambda_m) \to \underline{\sigma}$ since $n_m \to \infty$. For the remainder of the proof we suppress the arguments $(x_{n_m}(\lambda_m), \lambda_m)$ for clarity. Moreover, by definition $\Delta^H(x_H(\lambda), \sigma_H, \lambda) = 0$ and so $\Delta^L(x_H(\lambda), \sigma_H, \lambda) < 0$. Since $\sigma_H = \underline{\sigma}$, and beliefs are bounded, it follows that for n_m large enough a responder would strictly prefer to reject an offer of $x_{n_m}(\lambda_m)$ after observing σ_H. Consequently the equilibrium does not feature unconditional acceptance, and is instead responsive: $\sigma^*_{n_m} > \underline{\sigma}$ for n_m large, and hence, the equilibrium condition (7) must hold. Since $\sigma^*_{n_m} > \underline{\sigma} = \sigma_H$ and $\Delta^H(x_H(\lambda), \sigma_H, \lambda) = 0$ by definition, it follows that $\lim \Delta^H\left(x_{n_m}(\lambda_m), \sigma^*_{n_m}, \lambda_{n_m}\right) \leq 0$ by (38). If the inequality is strict we have an immediate contradiction to the equilibrium condition. Otherwise, since $\frac{\beta(x_n)}{1-\beta(x_n)}$ is bounded away from infinity, the equilibrium condition (7) implies that

$$\lim \left(\frac{1-F\left(\sigma^*_{n_m}|L\right)}{1-F\left(\sigma^*_{n_m}|H\right)} \right)^{n_m} = 0 \tag{40}$$

Define $J(\sigma)$ and $K(\sigma)$ as in the proof of Lemma 4. By equation (37) of the same proof,

$$\lim \ln \left(\frac{1-F\left(\sigma^*_{n_m}|L\right)}{1-F\left(\sigma^*_{n_m}|H\right)} \right)^{n_m} = \lim \left(n_{m+1}-n_m\right) \frac{J\left(\sigma^*_{n_m}\right) J\left(\sigma^*_{n_{m+1}}\right)}{J\left(\sigma^*_{n_m}\right) - J\left(\sigma^*_{n_{m+1}}\right)}$$

By (39), $\lim \left(1-F\left(\sigma^*_{n_m}|H\right)\right)^{n_m}$ exists and is strictly positive. By the discrete version of l'Hôpital's rule,

$$\lim \ln \left(1-F\left(\sigma^*_{n_m}|H\right)\right)^{n_m} = \lim \left(n_{m+1}-n_m\right) \frac{K\left(\sigma^*_{n_m}\right) K\left(\sigma^*_{n_{m+1}}\right)}{K\left(\sigma^*_{n_m}\right) - K\left(\sigma^*_{n_{m+1}}\right)} > -\infty.$$

By (35) and (36) (again, see the proof of Lemma 4), it follows that $\lim \ln \left(\frac{1-F\left(\sigma^*_{n_m}|L\right)}{1-F\left(\sigma^*_{n_m}|H\right)} \right)^{n_m}$ exists and is bounded away from $-\infty$. But this contradicts (40), completing the proof.

Part 2: Suppose to the contrary that (17) does not hold: there exists $\delta > 0$ such that for any Λ and N, $\sup_{\lambda \in (0,\Lambda], n \geq N} \Pr\left(A_n\right) \geq \delta$. So we can construct a sequence $\left(\lambda_m, n_m\right)$ with $\lambda_m \to 0$ and $n_m \to \infty$ such that $\Pr\left(A_{n_m}\left(x_{n_m}\left(\lambda_m\right), \lambda_m\right)|H\right) \geq \delta$. By (16), $\limsup_m x_{n_m}(\lambda_m) - x_H\left(\lambda_m\right) \leq 0$. As in Part 1, by Bolzano-Weierstrass we can assume without loss that the subsequence $\left(\lambda_m, n_m\right)$ is chosen so that equations (38) and (39) hold. The remainder of the proof exactly parallels that of Part 1. ∎

Proof of Lemma 6: Observe that $x_L\left(\lambda = 0, \alpha = 1\right) > 0$ since $\Delta^L\left(x = 0, \sigma_L, \lambda = 0\right) < 0$ by Assumption 3, Δ^L is strictly increasing in x by Assumption 5 and $\Delta^L\left(x_L\left(0,1\right), \sigma_L, \lambda = 0\right) = 0$ by definition. Since $\sigma_L\left(\alpha = 1\right) = \underline{\sigma}$ by choosing x sufficiently close to $x_L\left(0,1\right)$, the term $\Delta^L\left(x, \underline{\sigma}, \lambda = 0\right)$ can be made negative. Consequently, there exists $\kappa > 0$ such that

$$-\frac{\Delta^H\left(x_L\left(0,1\right)-\kappa, \underline{\sigma}, \lambda = 0\right)}{\Delta^L\left(x_L\left(0,1\right)-\kappa, \underline{\sigma}, \lambda = 0\right)} \frac{f\left(\underline{\sigma}|H\right) \Pr\left(H\right)}{f\left(\underline{\sigma}|L\right) \Pr\left(L\right)} \frac{f\left(\underline{\sigma}|H\right)}{f\left(\underline{\sigma}|L\right)} > 1.$$

By continuity, it follows that there exists $\Lambda > 0$ such that for all $\lambda \leq \Lambda$,

$$-\frac{\Delta^H\left(x_L\left(\lambda,1\right), \underline{\sigma}, \lambda\right)}{\Delta^L\left(x_L\left(\lambda,1\right), \underline{\sigma}, \lambda\right)} \frac{f\left(\underline{\sigma}|H\right) \Pr\left(H\right)}{f\left(\underline{\sigma}|L\right) \Pr\left(L\right)} \frac{f\left(\underline{\sigma}|H\right)}{f\left(\underline{\sigma}|L\right)} > 1. \tag{41}$$

By MLRP $\frac{1-F(\sigma|H)}{1-F(\sigma|L)} \geq 1$ for any σ (see Lemma 9). Since for any consistent beliefs $b \geq \underline{b}$, and both $\frac{f(\sigma|H)}{f(\sigma|L)}$ and $\frac{1-F(\sigma|H)}{1-F(\sigma|L)}$ are increasing in σ, it follows that by (41) that $Z(x_L(\lambda,1)-\kappa,\sigma;n,\alpha=1,\lambda,b)>0$ for all σ. Thus $x_L(\lambda,1)-\kappa > \bar{x}_n(\alpha=1,\lambda,b)$, i.e., the offer $x_L(\lambda,1)-\kappa$ is always accepted. ∎

Proof of Lemma 7: The proof is by contradiction. Suppose that contrary to the claim, for all N and all $\kappa>0$ there exists σ_0 such that

$$\sup_{\Lambda} \inf_{\lambda\in(0,\Lambda],n\geq N} \pi_n^*(\sigma_0;\lambda)-x_H(\lambda)<\kappa. \tag{42}$$

As such, for all N and all $\kappa>0$ there exists σ_0 such that for any $\Lambda>0$, there exists $n\geq N$ and $\lambda\in(0,\Lambda]$ such that $\pi_n^*(\sigma_0;\lambda)-x_H(\lambda)<\kappa$. This in turn implies that for any N, there exists n_N, λ_N and σ_0^N such that $n_N\geq N$, $\lambda_N\leq 1/N$, and $\pi_{n_N}^*\left(\sigma_0^N;\lambda_N\right)-x_H(\lambda_N)\leq\frac{1}{N}$. That is, along the sequence $\left(n_N,\lambda_N,\sigma_0^N\right)$ the number of responders becomes infinite, preferences approach the common values extreme, and $\pi_{n_N}^*\left(\sigma_0^N;\lambda_N\right)-x_H(\lambda_N)\to 0$. By Lemma 5, it follows that[21] $\Pr\left(A_{n_N}\left(\pi_{n_N}^*\left(\sigma_0^N;\lambda_N\right),\lambda_N\right)\right)\to 0$ as $N\to\infty$. Choose $\varepsilon>0$ such that for λ small enough, $x_H(\lambda)+\varepsilon<1$. Consider the alternate sequence of offers $\tilde{x}_{n_N}(\lambda_N)=x_H(\lambda_N)+\varepsilon$. From Lemma 4,[22] $\liminf\Pr\left(A_{n_N}\left(\tilde{x}_{n_N}(\lambda_N),\lambda_N\right)|H\right)>0$. Since (by Assumption 4) the proposer strictly prefers acceptance of any offer $x<1$ to rejection, it follows that for N large enough the offer $\tilde{x}_{n_N}(\lambda_N)$ is preferred to the offer $\pi_{n_N}^*\left(\sigma_0^N;\lambda_N\right)$ when there are N responders and preferences are λ_N. But this contradicts the optimality of the equilibrium offers, completing the proof. ∎

Proof of Lemma 8: Fix a set of beliefs b and an offer x. When x is such that the equilibrium is either a non-responsive rejection equilibrium, or a non-responsive acceptance equilibrium, the result follows trivially. Below, we focus on the case in which there is a

[21] Observe that in Lemma 5 both the hypothesis and the result are stated in terms of the full set of offers $\{x_n(\lambda):n,\lambda\}$. Here, we are using a variant where both the hypothesis and result hold for a particular sequence $\{x_{n_N}(\lambda_N):N=1,2,\ldots\}$. It is easily verified that the proof of Lemma 5 can be adopted, with only minor modifications, to cover this case.

[22] Parallel remarks apply here as in footnote 21.

responsive voting equilibrium. For the remainder of the proof, we write $b(H)$ for b and $b(L)$ for $1-b$. The average responder's expected payoff can be decomposed as

$$\begin{aligned}\Pi_n^R(x,\lambda,\alpha,b) &= \sum_\omega b(\omega)\left(E\left[\bar{U}^\omega(\sigma_i,\lambda)|\omega\right]\right)\\ &\quad+\sum_\omega b(\omega)\Pr(A_n(x,b)|\omega)\,E\left[\frac{1}{n}\sum_{i=1}^n \Delta^\omega(x,\sigma_i,\lambda)|A_n(x,b),\omega\right].\end{aligned}$$

Now,

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\Pr(A_n(x,b)|\omega)\,E\left[\frac{1}{n}\sum_{i=1}^n\Delta^\omega(x,\sigma_i,\lambda)|A_n(x,b),\omega\right]\right)\\ =\ &\frac{\partial}{\partial x}\left(\int_{\sigma^*}\cdots\int_{\sigma^*}\frac{1}{n}\sum_{i=1}^n\Delta^\omega(x,\sigma_i,\lambda)\,dF(\sigma_1|\omega)\ldots dF(\sigma_n|\omega)\right)\\ =\ &\frac{\partial}{\partial x}\left((1-F(\sigma^*|\omega))^{n-1}\int_{\sigma^*}\Delta^\omega(x,\sigma,\lambda)\,dF(\sigma|\omega)\right)\\ =\ &-\frac{\partial\sigma^*}{\partial x}f(\sigma^*|\omega)(n-1)(1-F(\sigma^*|\omega))^{n-2}\int_{\sigma^*}\Delta^\omega(x,\sigma,\lambda)\,dF(\sigma|\omega)\\ &-\frac{\partial\sigma^*}{\partial x}(1-F(\sigma^*|\omega))^{n-1}\Delta^\omega(x,\sigma^*,\lambda)f(\sigma^*|\omega)\\ &+(1-F(\sigma^*|\omega))^{n-1}\int_{\sigma^*}\Delta_x^\omega(x,\sigma,\lambda)\,dF(\sigma|\omega).\end{aligned}\tag{43}$$

Note that $\frac{\partial}{\partial x}\Pr(A_n(x,b)|\omega)=\frac{\partial}{\partial x}(1-F(\sigma^*|\omega))^n=-n\frac{\partial\sigma^*}{\partial x}f(\sigma^*|\omega)(1-F(\sigma^*|\omega))^{n-1}$, and

$$(1-F(\sigma^*|\omega))^{n-1}\int_{\sigma^*}\Delta_x^\omega(x,\sigma,\lambda)\,dF(\sigma|\omega)=\Pr(A_n(x,b)|\omega)\,E\left[\Delta_x^\omega(x,\sigma,\lambda)|\omega,\sigma\geq\sigma^*\right].$$

After substitution, expression (43) rewrites as

$$\begin{aligned}&\frac{\partial}{\partial x}\Pr(A_n(x,b)|\omega)\left(\frac{n-1}{n}E\left[\Delta^\omega(x,\sigma,\lambda)(x,\sigma)|\omega,\sigma\geq\sigma^*\right]+\frac{1}{n}\Delta^\omega(x,\sigma^*,\lambda)\right)\\ &+\Pr(A_n(x,b)|\omega)\,E\left[\Delta_x^\omega(x,\sigma,\lambda)|\omega,\sigma\geq\sigma^*\right].\end{aligned}$$

Consequently

$$\begin{aligned}\frac{\partial\Pi_n^R}{\partial x} &= +\frac{n-1}{n}\sum_\omega b(\omega)\frac{\partial}{\partial x}\Pr(A_n(x,b)|\omega)\,E\left[\Delta^\omega(x,\sigma,\lambda)|\omega,\sigma\geq\sigma^*\right]\\ &\quad+\frac{1}{n}\sum_\omega b(\omega)\frac{\partial}{\partial x}\Pr(A_n(x,b)|\omega)\,\Delta^\omega(x,\sigma^*,\lambda)\\ &\quad+\sum_\omega b(\omega)\Pr(A_n(x,b)|\omega)\,E\left[\Delta_x^\omega(x,\sigma,\lambda)|\omega,\sigma\geq\sigma^*\right].\end{aligned}$$

Now, from the equilibrium condition (7):

$$\frac{\frac{\partial \Pr(A_n(x,b)|H)}{\partial x}}{\frac{\partial \Pr(A_n(x,b)|L)}{\partial x}} = \frac{f(\sigma^*|H)(1-F(\sigma^*|H))^{n-1}}{f(\sigma^*|L)(1-F(\sigma^*|L))^{n-1}} = -\frac{\Delta^L(x,\sigma^*,\lambda)}{\Delta^H(x,\sigma^*,\lambda)}\frac{b(L)}{b(H)}$$

and so

$$\sum_\omega b(\omega)\frac{\partial}{\partial x}\Pr(A_n(x,b)|\omega)\Delta^\omega(x,\sigma^*,\lambda) = 0.$$

Since $\Delta^\omega(x,\sigma,\lambda)$ is increasing in σ (Assumption 2), it follows that

$$\sum_\omega b(\omega)\frac{\partial}{\partial x}\Pr(A_n(x,b)|\omega)E[\Delta^\omega(x,\sigma,\lambda)|\omega,\sigma\geq\sigma^*]\geq 0$$

also. The result then follows. ∎

Proof of Proposition 2: For any λ and n, consider an equilibrium $(\pi_n^*, \sigma_n^*, \beta_n)$. We start with some preliminary bounds: From Lemma 7, we know that there exists $\kappa_x > 0$ and N_x such that for all σ_0,

$$\sup_\Lambda \inf_{\lambda\in(0,\Lambda],n\geq N_x} \pi_n^*(\sigma_0;\lambda,\alpha=1) - x_H(\lambda) \geq \kappa_x.$$

For any λ, define a sequence of offers by $\hat{x}_n(\lambda) = x_H(\lambda) + \kappa_x/2$. Let $\underline{b}$ and $\bar{b}$ denote the most pro-L and pro-H beliefs possible respectively. From Lemma 4, there exists $\kappa_A > 0$ and N_A such that

$$\sup_\Lambda \inf_{\lambda\in(0,\Lambda],n\geq N_A} \Pr(A_n(\hat{x}_n(\lambda),\underline{b})|H) \geq \kappa_A.$$

Since $\frac{\partial}{\partial x}\Delta^H$ is bounded uniformly away from 0, there exists $\delta > 0$ such that $E\left[\frac{\partial}{\partial x}\Delta^H(x,\sigma_i,\lambda)|H,\sigma_i\geq\sigma^*\right] > \delta$ for all x and λ. By Lemma 5, choose N_L such that

$$\inf_\Lambda \sup_{\lambda\in(0,\Lambda],n\geq N_L} \Pr\left(A_n\left(x_H(\lambda),\bar{b}\right)|L\right)\max\left|\Delta^L\right| \leq \underline{b}\kappa_A\delta\frac{\kappa_x}{4}. \tag{44}$$

Finally, define $\hat{N} = \max\{N_x, N_A, N_L\}$. We show below that for any σ_0,

$$\sup_\Lambda \inf_{\lambda\in(0,\Lambda],n\geq\hat{N}} \Pi_n^R(\pi_n^*(\sigma_0),\lambda,\alpha,\beta_n(\pi_n^*(\sigma_0))) - E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i,\lambda)\right] \geq \underline{b}\kappa_A\delta\frac{\kappa_x}{4}. \tag{45}$$

Inequality (45) is enough to establish the Proposition, as follows. First, inequality (45) implies that there exists $\bar{\Lambda}$ such that

$$\inf_{\lambda\in(0,\bar{\Lambda}],n\geq\hat{N}} \Pi_n^R\left(\pi_n^*\left(\sigma_0\right),\lambda,\alpha,\beta_n\left(\pi_n^*\left(\sigma_0\right)\right)\right)-E_{\sigma_i,\omega}\left[\bar{U}^\omega\left(\sigma_i,\lambda\right)\right]\geq \underline{b}\kappa_A\delta\frac{\kappa_x}{8}. \tag{46}$$

From (46),

$$\begin{aligned}
&\sup_{\Lambda>0}\inf_{\lambda\in(0,\Lambda],n\geq\hat{N}} \Pi_n^{*R}\left(\lambda\right)-E_{\sigma_i,\omega}\left[\bar{U}^\omega\left(\sigma_i,\lambda\right)\right]\\
=\ &\sup_{\Lambda>0}\inf_{\lambda\in(0,\Lambda],n\geq\hat{N}}\int \Pi_n^R\left(\pi_n^*\left(\sigma_0\right),\lambda,\beta_n\left(\pi_n^*\left(\sigma_0;\lambda\right)\right)\right)dF\left(\sigma_0\right)-E_{\sigma_i,\omega}\left[\bar{U}^\omega\left(\sigma_i,\lambda\right)\right]\\
\geq\ &\sup_{\Lambda\in(0,\bar{\Lambda}]}\inf_{\lambda\in(0,\Lambda],n\geq\hat{N}}\int \Pi_n^R\left(\pi_n^*\left(\sigma_0\right),\lambda,\beta_n\left(\pi_n^*\left(\sigma_0;\lambda\right)\right)\right)dF\left(\sigma_0\right)-E_{\sigma_i,\omega}\left[\bar{U}^\omega\left(\sigma_i,\lambda\right)\right]\\
\geq\ &\sup_{\Lambda\in(0,\bar{\Lambda}]}\int\inf_{\lambda\in(0,\Lambda],n\geq\hat{N}} \Pi_n^R\left(\pi_n^*\left(\sigma_0\right),\lambda,\beta_n\left(\pi_n^*\left(\sigma_0;\lambda\right)\right)\right)dF\left(\sigma_0\right)-E_{\sigma_i,\omega}\left[\bar{U}^\omega\left(\sigma_i,\lambda\right)\right]\\
\geq\ &\sup_{\Lambda\in(0,\bar{\Lambda}]}\int \underline{b}\kappa_A\delta\frac{\kappa_x}{8}dF\left(\sigma_0\right)=\underline{b}\kappa_A\delta\frac{\kappa_x}{8}.
\end{aligned}$$

Setting $\gamma=\underline{b}\kappa_A\delta\frac{\kappa_x}{8}$ gives the result.

Proof of inequality (45): Consider a particular realization of σ_0. For the remainder of the proof, we write $x_n^*\left(\lambda\right)$ for $\pi_n^*\left(\sigma_0;\lambda\right)$, and drop λ where doing so will create no confusion. Likewise, let $b_n\left(\lambda\right)$ denote the responders' equilibrium beliefs at the offer $x_n^*\left(\lambda\right)$, i.e., $b_n\left(\lambda\right)=\beta_n\left(x_n^*\left(\lambda\right),\lambda\right)$.

Arithmetically, the responders' expected payoff from the equilibrium offer x_n^* given the equilibrium beliefs b_n can be decomposed as

$$\begin{aligned}
\Pi_n^R\left(x_n^*,\lambda,b_n\right)=\Pi_n^R\left(x_H\left(\lambda\right),\lambda,b_n\right) &+ \Pi_n^R\left(\hat{x}_n\left(\lambda\right),\lambda,b_n\right)-\Pi_n^R\left(x_H\left(\lambda\right),\lambda,b_n\right)\\
&+ \Pi_n^R\left(x_n^*,\lambda,b_n\right)-\Pi_n^R\left(\hat{x}_n\left(\lambda\right),\lambda,b_n\right).
\end{aligned} \tag{47}$$

From Lemma 8, the derivative of Π_n^R between $x_H\left(\lambda\right)$ and $\hat{x}_n\left(\lambda\right)$ is positive, and so $\Pi_n^R\left(\hat{x}_n\left(\lambda\right),\lambda,b_n\right)-\Pi_n^R\left(x_H\left(\lambda\right),\lambda,b_n\right)\geq 0$. By the construction of $\hat{x}_n\left(\lambda\right)$, the acceptance probability given offer $\hat{x}_n\left(\lambda\right)$ and beliefs b_n satisfies

$$\sup_{\Lambda}\inf_{\lambda\in(0,\Lambda],n\geq N_A}\Pr\left(A_n\left(\hat{x}_n\left(\lambda\right),b_n\right)|H\right)\geq\kappa_A,$$

since under beliefs b_n acceptance is (weakly) more likely than under $\underline{b}$ (see Corollary 1). Consequently, since $x_n^* - \hat{x}_n(\lambda) \geq \frac{\kappa_x}{2}$ for λ chosen small enough, by Lemma 8,

$$\sup_{\Lambda} \inf_{\lambda \in (0,\Lambda], n \geq \hat{N}} \Pi_n^R(x_n^*, \lambda, b_n) - \Pi_n^R(\hat{x}_n(\lambda), \lambda, b_n) \geq \underline{b}\kappa_A \delta \frac{\kappa_x}{2}.$$

Finally,

$$\begin{aligned}
& \Pi_n^R(x_H(\lambda), \lambda, b_n) - E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] \\
= \; & b_n \Pr(A_n(x_H(\lambda), b_n) | H) E\left[\Delta^H(x_H(\lambda), \sigma_i, \lambda) | A_n(x_H(\lambda), b_n), H\right] \\
& + (1 - b_n) \Pr(A_n(x_H(\lambda), b_n) | L) E\left[\Delta^L(x_H(\lambda), \sigma_i, \lambda) | A_n(x_H(\lambda), b_n), L\right].
\end{aligned}$$

Recall that $\Delta^H(x_H(0), \sigma_i, 0) = 0$ for all σ_i, and so for any N,

$$\sup_{\Lambda} \inf_{\lambda \in (0,\Lambda], n \geq N} E\left[\Delta^H(x_H(\lambda), \sigma_i, \lambda) | A_n(x_H(\lambda), b_n), H\right] = 0.$$

From (44), regardless of whether the expression $E\left[\Delta^L(x_H(\lambda), \sigma_i, \lambda) | A_n(x_H(\lambda), b_n), L\right]$ is positive or negative,

$$\sup_{\Lambda} \inf_{\lambda \in (0,\Lambda], n \geq N_L} \Pi_n^R(x_H(\lambda), \lambda, b_n) - E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] \geq -\underline{b}\kappa_A \delta \frac{\kappa_x}{4}.$$

From (47), it follows that

$$\sup_{\Lambda} \inf_{\lambda \in (0,\Lambda], n \geq \hat{N}} \Pi_n^R(x_n^*, \lambda, \alpha, b_n) - E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] \geq \underline{b}\kappa_A \delta \frac{\kappa_x}{4}.$$

The result then follows. ∎

Proof of Proposition 3: Immediate from Lemma 6 and Lemma 8. ∎

Proof of Proposition 4: Proposition 4 is a special case of Proposition 7, which is proved below.

Proof of Proposition 5: The proof is omitted for reasons of space. It is available from the authors' webpages.

Proof of Proposition 6: From Proposition 3, there exists $\gamma > 0$ and $\Lambda > 0$ such that whenever $\lambda \leq \Lambda$,

$$\Pi_n^{*R}(\lambda, 1) - E_{\sigma_i,\omega}\left[\bar{U}^\omega(\sigma_i, \lambda)\right] \leq E_{\sigma_i,\omega}\left[\Delta^\omega(x_L(\lambda, 1), \sigma_i, \lambda)\right] - \gamma.$$

As such, there exists $\Lambda_1 \leq \Lambda$ such that whenever $\lambda \leq \Lambda_1$,

$$\Pi_n^{*R}(\lambda, 1) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right] \leq E_{\sigma_i,\omega}\left[\Delta^{\omega}(x_L(0,1), \sigma_i, 0)\right] - \frac{\gamma}{2}.$$

Moreover, there exists Λ_α such that when $\lambda \leq \Lambda_\alpha$

$$\left|E_{\sigma_i,\omega}\left[\Delta^{\omega}(x_L(0,\alpha), \sigma_i, 0)\right] - E_{\sigma_i,\omega}\left[\Delta^{\omega}(x_L(\lambda,\alpha), \sigma_i, \lambda)\right]\right| < \frac{\gamma}{8}.$$

As such, when $\lambda \leq \min\{\Lambda_\alpha, \Lambda_1\}$ from Proposition 1 there exists N_α such that when $n \geq N_\alpha$,

$$\begin{aligned} \Pi_n^{*R}(\lambda, \alpha) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right] &\geq E_{\sigma_i,\omega}\left[\Delta^{\omega}(x_L(\lambda,\alpha), \sigma_i, \lambda)\right] - \frac{\gamma}{4} \\ &> \Pi_n^{*R}(\lambda, 1) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right]. \end{aligned}$$

The comparison of the proposer's payoff is straightforward. ∎

Proof of Proposition 7: From Proposition 2, there exists $\gamma_0 > 0$, N_1 and $\Lambda_1 > 0$ such that

$$\inf_{\lambda \in (0,\Lambda_1], n \geq N_1} \left(\Pi_n^{*R}(\lambda, 1) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right]\right) \geq \frac{\gamma_0}{2}.$$

Choose Λ_α such that $\Pr(H) E_{\sigma_i}\left[\Delta^H(x_H(\lambda,\alpha), \sigma_i, \lambda) | H\right] \leq \frac{\gamma_0}{8}$, for all $\lambda \in (0, \Lambda_\alpha]$. Choose $\Lambda_W > 0$ such that $\min_{\sigma_0 \in [\underline{\sigma}, \bar{\sigma}]} W(\sigma_0; \lambda, \alpha) > 0$ whenever $\lambda \leq \Lambda_W$. Then from Proposition 1, for any $\lambda \leq \min\{\Lambda_\alpha, \Lambda_W\}$ there exists $N_\alpha(\lambda)$ such that $\Pi_n^{*R}(\lambda, \alpha) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right] \leq \frac{\gamma_0}{4}$ whenever $n \geq N_\alpha(\lambda)$. Consequently, whenever $\lambda \leq \min\{\Lambda_\alpha, \Lambda_W, \Lambda_1\}$ and $n \geq \max\{N_\alpha(\lambda), N_1\}$,

$$\Pi_n^{*R}(\lambda, 1) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right] \geq \frac{\gamma_0}{2} > \frac{\gamma_0}{4} \geq \Pi_n^{*R}(\lambda, \alpha) - E_{\sigma_i,\omega}\left[\bar{U}^{\omega}(\sigma_i, \lambda)\right].$$

Setting $\gamma = \gamma_0/4$ and $\bar{\lambda} = \min\{\Lambda_\alpha, \Lambda_W, \Lambda_1\}$ completes the proof. ∎