Pivots Versus Signals in Elections*

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September 29, 2007

Abstract

Incentives in voting models typically hinge on the event that a voter is pivotal. But voting can also influence the behavior of elites in subsequent periods. In this paper, we consider a model in which voters have private information about their policy preferences and an election is held in each of two periods. In this setting, a vote in the first period can have two types of consequences; it may be pivotal in deciding who wins the first election and it provides a signal that informs the beliefs that candidates running in the second election use when selecting equilibrium platforms. Pivot events are exceedingly unlikely, but when they occur the effect of a single vote is enormous, since it determines the electoral outcome. In contrast, vote totals always have some signaling effect on future policies, but the effect of a single vote is always very small. We investigate whether the former, pivot, effect or the latter, signaling, effect drives equilibrium voting behavior in large electorates.

*Prepared for the Annual Conference on Political Economy at the University of Rochester's Wallis Institutue, October 2007. For helpful comments, we thank Patrick Hummel and seminar participants at Berkeley, the Hoover Institution, the 2006 MPSA annual meeting, and the 2007 APSA annual meeting. Marc Meredith provided excellent research assistance on numerical simulations early in the project. We are indebted to Keith Krehbiel and Nolan McCarty for profound observations on how this work fits into the larger literature in political science.

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1 Introduction

In nearly all models of voting the payoff from casting a particular ballot hinges exclusively on pivot events. These are events in which the election is tied or nearly tied, so that a single vote can determine the outcome. In decision-theoretic models a voter decides whether and how to vote based on exogenous probabilities of ties between candidates. Game theoretic models endogenize the equilibrium probability that a vote is pivotal. And several recent influential papers focus on information that a voter can infer from the fact that he is pivotal, and analyze electoral equilibria when voters condition on being pivotal.[1]

The pivot-based literature on elections is vast, but the models have two features in common: (i) when a voter is pivotal, the action she takes has a large impact on her payoff, but (ii) pivot events are very unlikely. The large impact is due to the fact that in a pivot event, a single vote can determine the outcome of the election. The low probability arises from the fact that in a large election it is exceedingly unlikely that two candidates will receive the same number of votes or differ by exactly one vote.

Although pivot based models dominate the game-theoretic literature on elections, the infrequency of pivot events in all but the smallest elections raises a natural question: is electoral behavior driven by more than just concerns about being pivotal? If pivot events do not actually drive the calculus of voters then a large and growing literature on voting theory may be focused on second-order concerns.

Why would voters care about anything other than a pivot event? Consider, for example, the buildup to the 2006 midterm election in the United State. Pundits speculated that voters' dissatisfaction with President Bush's handing of the war in Iraq would cost the Republican party its majority in Congress. While Republicans' electoral losses may in fact have been a direct result of voters' desire to change the composition of the legislature, another explanation is that voters cast ballots for Democrats in order to send Bush a message, and encourage him to change policy.

An emerging literature, based on the intuition that vote totals matter in elections that don't end in a tie, offers an alternative perspective to the dominant pivot-based theories of elections. Theorists

[1] Decision theoretic models include the work of Downs (1957), Tullock (1967), Riker and Ordeshook (1968), and Myerson and Weber (1993). Examples of game theoretic models include Palfrey and Rosenthal (1983), Myerson (1998, 2000), Campbell (1999), and Borgers (2004). Models involving information aggregation include Feddersen and Pesendorfer (1996, 1997, 1999), Austen-Smith and Banks (1996), Dekel and Piccione (2000), and Battaglini (2005).

of elections have explored several different ways that vote totals could affect downstream electoral or policy outcomes.

One modelling approach is a common values setup, in which voters use their votes to convey information about the state of the world. Piketty (2000) develops a two period model of referendum voting in which voters communicate policy information to each other as they vote. Razin (2003) analyzes a model of mandates in which vote totals convey information to the winner of an election and thus affect the policies she enacts. In this model, the functional form of the election winner's response to signals is a primitive of the game. Razin investigates limiting behavior for large electorates and characterizes two types of equilibria. In one type of equilibrium voters' behavior is "conventional" in the sense that a voter whose private signal indicates that liberal policies are good tends to vote for a liberal candidate. In any limit of these conventional equilibria, the behavior of voters converges to coin flipping.[2] The other equilibria are "unconventional" since voters respond perversely to their private signals: upon observing information that favors liberal policies, a voter becomes more likely to vote for the *conservative* candidate.

The equilibria of the Piketty and Razin models suggest that a desire to influence the decisiveness of victory, and not just the identity of the winner, can remain in large elections. However, in both models all voters have identical preferences and the effects that their votes can have on future policy outcomes are determined directly by modelling assumptions, rather than being determined by strategic choices made by competing political elites who observe election outcomes.

Another modelling approach is a private values setup, in which voters can use their votes to affect future candidates' positions, and thus policy outcomes. Each election thus serves two purposes: to select a winner and to act as a poll about voters' preferences. In this vein, Meirowitz and Tucker (2007) analyze a model of alternating primary and presidential elections, in which voters use their votes to signal dissatisfaction with an incumbent and thereby induce him to exert costly effort to make himself more appealing in a subsequent election.

Castanheira (2003) adapts Piketty's model to a private values setting, and uses it to analyze voting for losers in an election with four candidates. In his model, there are four possible distributions of

[2]This result is Razin's Proposition 4, part (i).

voters in the electorate, and voters may choose to vote for a candidate who is almost certain to lose in the first election since their vote may determine which of four positions candidates will adopt in a second period election. In Castanheira's model, as in Piketty's model, the signaling effect of first period electoral behavior is based on the very low probability event that a vote is informationally pivotal, i.e., although there is signaling in these models, the signaling is fundamentally based on pivot events, which are extremely unlikely to occur.

In contrast, Shotts (2006) develops a private values model of repeated elections in which each voter's actions, by conveying information about his preferences, always have some small effect on future policy outcomes. Small magnitude, high probability signaling effects work quite differently from large magnitude low probability pivot effects. However, in large elections, a single vote has a vanishingly small effect on politicians' beliefs about voter preferences, and Shotts does not address the question of whether signaling motivations are actually relevant in large elections.

Moreover, a voter may find herself cross-pressured when a vote for the candidate she prefers sends the wrong message to politicians, i.e., she may wish to vote for one candidate for signaling reasons and a different one for pivot reasons. Because both the likelihood that she is pivotal and the effect of a single vote on candidates' vote shares are small it is not clear how tradeoffs between pivot and signaling effects balance out in equilibria for large elections. Thus, although recent papers have moved beyond pivot-based theories of voting, they do not shed light on the question of whether equilibrium behavior in large repeated elections with private values is driven by pivot or signaling considerations.

In this paper, we analyze a private values model of repeated elections with both pivot and signaling motivations and show that the latter dominates with a large electorate. While our paper is related to that of Meirowitz (2005), which focuses on signaling motivations in public opinion, it is, however, closest to that of Shotts (2006). Shotts analyzes elections with a fixed population and focuses on equilibria in which moderates abstain, to signal that they are moderate. In contrast, we analyze a model without the possibility of abstention and focus on limiting behavior for large electorates.

It turns out that the assumption that voters cannot abstain is not crucial for our results. Hummel (2007) builds on the present paper and Shotts (2006) to study limiting behavior when voters can choose to abstain. Hummel shows that although the type of equilibrium with abstention characterized by Shotts exists for any finite population, in the limit abstention vanishes and voter behavior converges to the behavior we characterize here. Hummel also derives a useful lower bound for the magnitude of signaling effects in large elections.

Having discussed the relevant literature, we now summarize our contribution. The key elements of our model are as follows. Each voter in our model has private information about his own policy preferences, and in each of two elections he casts a ballot for one of two available alternatives. Following the first election, two office-motivated candidates compete for a second office, by staking policy positions, and the second election is held. Candidates in the second period base their policy positions on beliefs about the distribution of preferences in the electorate. In equilibrium these beliefs are informed by the vote totals in the first election, and each voter's vote in the first period thus has a small signaling effect on second period policy.

Our main result is that as the electorate gets large the equilibrium converges to the equilibrium of a slightly different game in which the first period outcome is payoff irrelevant, as in the case of a first period poll. In other words, in large elections behavior is driven by the signaling motivation and not the pivot motivation.

This result has potentially important implications for the literature on pivot-based models of elections, since most of the interesting equilibria in such models rely heavily on the fact that a voter only cares about events in which his vote is pivotal. In our model, in contrast, the effect of pivot events on equilibrium voter behavior is relatively unimportant compared to the effect of signaling concerns. At the very least, future research needs to take seriously the possibility that pivot events are not of first-order importance when rational voters take into account the future effects of their votes.

The paper proceeds as follows. Section 2 introduces the model and in Section 3 we present two concrete examples of how signaling and pivot effects work. Section 4 proves equilibrium existence. Section 5 describes the intuition behind our main result, which is proved in Section 6. Section 7 discusses the result.

2 The Model

Consider an electorate with an odd number of voters $n \geq 3$. It will be convenient to use the fact that $n = 2m+1$ for some integer m. Let the set of voters be N, and let each voter $i \in N$ have an ideal point, $v_i \in [0,1]$. We assume that the ideal points are iid draws from a distribution function, $F(\cdot)$, which is strictly increasing and continuously differentiable, and has a continuous density $f(\cdot)$ on the support $[0,1]$. Each voter's utility over policy, x, in a given period is $u_i(x) = -\gamma(|x - v_i|)$ where $\gamma : [0,1] \to \mathbb{R}_+$ is strictly increasing, convex and differentiable. Since expected utility is defined only up to positive affine transformations we make the innocuous but convenient assumption that $\gamma'(1) = 1$. The voter's total utility is simply the sum of his policy utility in the two periods.

In the first period election, two fixed alternatives are available. We denote the locations of the alternatives by $L, R \in [0,1]$ (with $L \leq R$). If voters care only about the first period, or are myopic, elimination of weakly dominated strategies yields a unique equilibrium, in which all voters to the left of $x_p = \frac{L+R}{2}$ vote L and all voters to the right of x_p vote R. We call this the *pivot cutpoint.* We, however, are interested in the dependencies across elections, and thus consider a model with two periods, building on Shotts (2006).

In the second period, two office motivated candidates select policy platforms and then the electorate votes. The candidates are assumed to know only the distribution $F(\cdot)$ from which the n ideal points are drawn, the size of the electorate, n, and the voters' first period actions. From Calvert (1985), we know that for a game in which two office motivated candidates believe that $F_{median}(\cdot)$ is the distribution of the median voter's ideal point, in any Nash equilibrium with weakly undominated voting the candidates will both locate at $F^{-1}_{median}(\frac{1}{2})$.[3] In the two-period signaling game that we study, in any Perfect Bayesian equilibrium, the distribution of the median depends on the first-period votes via Bayes' Rule. At any history in which $F_{median}(\cdot)$ is a distribution consistent with Bayes' Rule following the observed first period voting, the second period candidates both locate at the point $F^{-1}_{median}(\frac{1}{2})$. While Shotts (2006) focuses on equilibria with abstention in the first period, we restrict the set of actions available to voters so that they must vote either L or R; this enables us to focus on a particularly simple class of equilibria,

[3] One such equilibrium has each voter flipping a fair coin when indifferent. Given strategies for the first period, the second period behavior is standard, and well understood (Calvert 1985, Shotts 2006).

involving only a single cutpoint. In particular, we focus on a class of equilibria in which all voters use the same type specific monotone voting strategy.

In such an equilibrium, first-period voting strategies are characterized by a cutpoint x_c with voters to the left ($v_i < x_c$) voting L and voters to the right ($v_i > x_c$) voting R. If voting satisfies this cutpoint, i.e. it is monotone, then the number of votes for R, denoted $\#R$, captures all of the publicly available information about voter ideal points, and is a sufficient statistic for the second-period candidates' problem of inferring the distribution of the median voter's ideal point from first-period behavior. We denote such a posterior distribution as $F_{median}(\cdot \mid \#R; x_c)$.

Before proceeding we provide a few comments about this function. Given that $\#R$ of n voters have ideal points to the right of (greater than) x_c, the median is less than x_c if and only if $\#R \leq m = \frac{n-1}{2}$. Similarly the median is greater than x_c if and only if $\#R \geq m+1$. In the former case, the median is the $(m+1)$'th lowest ideal point of the $n - \#R$ voters with ideal points less than x_c, i.e., the median ideal point is the $(m+1)$'th order statistic from $n - \#R$ draws from the distribution $H^-(x; x_c) = \frac{F(x)}{F(x_c)}$ with support $[0, x_c]$. Similarly in the latter case, the median is the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from the distribution $H^+(x; x_c) = \frac{F(x)-F(x_c)}{1-F(x_c)}$ with support $[x_c, 1]$.

As previously mentioned, Calvert's result shows that given x_c, $\#R$, and a belief mapping $F_{median}(\cdot|\#R; x_c)$ sequential rationality of the candidates and weakly undominated voting by the voters implies that the second period policy is $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$. In characterizing a cutpoint perfect Bayesian equilibrium with weakly undominated second period voting strategies it is sufficient to characterize a first period cutpoint $x_c \in [0,1]$ such that if every voter other than i is using the strategy with cutpoint x_c it is optimal for voter i to do so as well. Checking this condition hinges on the fact that in an equilibrium of this form second period candidates both locate at the point $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$.

The equilibrium cutpoint balances two effects that influence first period voting. The pivot effect captures the incentive to vote for L if $|L - v_i| < |R - v_i|$ and R if the opposite is true. The signaling motivation captures the incentive to vote for R if, given i's expectations about the actions of the other voters, increasing $\#R$ is likely to move the second period policy $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$ towards v_i, and to vote for L if increasing $\#R$ is likely to move the second period policy away from v_i. The pivot effect is the product of the probability that i is pivotal and the payoff difference between the policy L and R.

In contrast to the pivot effect, which captures a low probability event with a non trivial payoff in that event, the signaling motivation takes into account the fact that i's vote will *always* have an effect on the second period policy. However, the signaling effect is small for each of the possible realizations of the votes cast by $N \backslash \{i\}$. For some realizations of the votes by $N \backslash \{i\}$ increasing $\#R$ will be attractive to i, while for other realizations of the these votes increasing $\#R$ will be unattractive to i.

The goal of this paper is to compare high impact, low probability pivot events versus low impact, high probability signaling effects and determine which type of effect dominates in large elections. In particular we investigate the limiting behavior of the cutpoint x_c as n tends to infinity. We find that the limiting cutpoint corresponds to the equilibrium cutpoint in a different game in which the first period is irrelevant (or, equivalently, $L = R$) so that, in the limit, the cutpoint for voter behavior is identical to what it would be if voters were motivated purely by signaling concerns. Thus, we find that while equilibrium voting involves a balancing of these two motivations, in a very strong sense, equilibrium voting in large elections is driven by voters' desire to influence the inferences of observers and not by their desire to influence the election at hand.

3 Two Examples

Before analyzing the model, we illustrate it with two simple examples.

Example 1. We start with what is essentially a decision-theoretic version of the model, in which there is just one voter, with ideal point v_i and a linear loss function $\gamma(|x - v_i|) = |x - v_i|$. Suppose the exogenously-fixed first period candidate locations are $L = \frac{1}{2}$ and $R = 1$. The second period candidates believe that the single voter's ideal point is drawn from a uniform distribution on $[0, 1]$. Thus, if the voter's strategy is monotone, with cutpoint x_c, the second period policy will be $\frac{x_c}{2}$ if i votes for L and $\frac{1+x_c}{2}$ if i votes for R. For i to be indifferent between voting L and R when her ideal point is $v_i = x_c$, the following equality must hold:

$$-|x_c - L| - \left|x_c - \frac{x_c}{2}\right| = -|R - x_c| - \left|x_c - \frac{1 + x_c}{2}\right|.$$

For $L = \frac{1}{2}$ and $R = 1$ this equality is solved at $x_c = \frac{2}{3}$.

Example 2. To illustrate how pivot and signaling effects work in the model, we now consider the simplest variant where a vote has a probabilistic effect on both first and second period outcomes. While this example cannot resolve the horse race between the signaling and pivot effects as the number of voters gets large, all of the relevant incentives and quantities of interest are present. Consider $n = 3$ and assume that voters $i \in \{1, 2, 3\}$ have ideal points that are *i.i.d.* draws from a uniform distribution on $[0, 1]$. Assume that each voter has a linear loss function $\gamma(|x - v_i|) = |x - v_i|$. The first-period election is between two candidates, with exogenously-fixed policy positions $L = \frac{1}{2}$ and $R = 1$.

We first consider two benchmark cases: a pure pivot model and a pure signaling model. In a pure pivot model there is a unique voting equilibrium in weakly undominated strategies: a voter votes for the closer candidate, i.e., she votes for L if her ideal point is to the left of $\frac{L+R}{2} = 0.75$ and votes R if her ideal point is to the right of 0.75. So $x_p = 0.75$ is the *pivot cutpoint.*

For a pure signaling model, all that matters is how a vote affects $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$ through $\#R$. If voters only care about the outcome of the second period election then in the three voter example there is a unique equilibrium, specified by the *signaling cutpoint,* $x_s = 1/2$.

To check that this cutpoint is an equilibrium in the game in which only the second period outcome affects voter payoffs, we confirm that a voter with $v_i = 1/2$ is indifferent between voting L and R, given the other actors' strategies. Focusing on voter $i = 1$, assume that the other voters are using this cutpoint strategy. The signaling effect of $i's$ first-period vote thus depends on the other voters' first-period actions:

- With probability $F(x_s) \cdot F(x_s) = \frac{1}{2} \cdot \frac{1}{2} = 0.25$ the other two voters vote L. In this case the second-period policy outcome will be 0.25, if i votes L. This is true because the second-period candidates' posterior belief given $\#R = 0$ and $x_s = 1/2$ is that all three voters' ideal points are uniform draws from $[0, 0.5]$. If i votes R the second-period policy outcome will be $F^{-1}_{median}\left(\frac{1}{2}|1; 1/2\right) = 0.35$, since there is a 50% chance that both of the L voters, and hence the median, will be to the left of 0.35. Thus, the signaling effect of voting R if both other voters vote L is to move the second-period policy outcome from 0.25 to 0.35.

- With probability $F(x_s) \cdot (1 - F(x_s)) + (1 - F(x_s)) \cdot F(x_s) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = 0.5$ the other two voters split their votes. In this case if i votes L the second-period policy outcome will be 0.35 and if she votes R the policy outcome will be 0.65.

- With probability $(1-F(x_s))\cdot(1-F(x_s))=\frac{1}{2}\cdot\frac{1}{2}=0.25$ the other two voters vote R. In this case if i votes L the second-period policy outcome will be 0.65 and if i votes R the policy outcome will be 0.75.

Thus, if a voter with ideal point v_i votes L her expected second-period utility is $-0.25\cdot|v_i-0.25|-0.5\cdot|v_i-0.35|-0.25\cdot|v_i-0.65|$. If she votes R her expected utility is $-0.25\cdot|v_i-0.35|-0.5\cdot|v_i-0.65|-0.25\cdot|v_i-0.75|$. A voter at $v_i=x_s=1/2$ is indifferent between voting L and voting R. It is straightforward to confirm that any voter left of $1/2$ strictly prefers to vote L and a voter to the right prefers to vote R.

It is worth noting three features of signaling effects that will show up in our later analysis of large elections. First, which action, L or R, better promotes the voter's policy interests in the second period depends on the other voters' actions, as well as the cutpoint x_s. For a voter with $v_i=1/2$, if the other two voters vote L then voting R is optimal, whereas if the others vote R then voting L is optimal, and if the others split their votes then the voter is indifferent. Second, the different signaling effects are not equally likely to occur, but rather occur with different probabilities. Third, since the other voters' actions are simply draws from a binomial, in a large election, the most likely realized vote totals are those where L receives a share close to $F(x_s)$ of the votes and R receives a share close to $1-F(x_s)$ of the votes. All three of these properties of signaling effects hold regardless of the cutpoint for voter behavior in the first period.

In a model with both pivot and signaling effects, equilibrium behavior hinges on the *combined cutpoint* x_c. As shown in Figure 1, in the three-voter example, $x_c\approx 0.65$.[4] For this x_c the *pivot probability* is $2\cdot 0.65\cdot(1-0.65)=0.455$. For a voter with $v_i=x_c$ the utility difference between the two possible first-period policy outcomes, L and R, is $-|v_i-L|+|v_i-R|=-\left|0.65-\frac{1}{2}\right|+|0.65-1|=0.2$. So i receives, in expectation, $0.2\cdot 0.455\approx 0.09$ more first-period utility by voting L than by voting R.

[Insert Figure 1 about here]

The second-period signaling effect is a bit more complicated to compute:

[4] It is just coincidence that this value for x_c is the same, to two decimal places, as the second period policy outcome under one of the action profiles in the pure signaling model. For the general model, they need not be the same.

- With probability $F(x_c) \cdot F(x_c) = 0.65 \cdot 0.65 = 0.4225$ the other two voters vote L. In this case the second-period policy outcome will be 0.325 if i votes L. If i votes R the second-period policy outcome will be 0.463.

- With probability $F(x_c) \cdot (1 - F(x_c)) + (1 - F(x_c)) \cdot F(x_c) = 2 \cdot 0.65 \cdot (1 - 0.65) = 0.455$ the other two voters split their votes. In this case if i votes L the second-period policy outcome will be 0.463 and if i votes R the policy outcome will be 0.756.

- With probability $(1 - F(x_c)) \cdot (1 - F(x_c)) = (1 - 0.65) \cdot (1 - 0.65) = 0.1225$ the other two voters vote R. In this case if i votes L the second-period policy outcome will be 0.756 and if i votes R the policy outcome will be 0.825.

Thus, if a voter with ideal point v_i votes L his expected second-period utility is $-0.4225 \cdot |v_i - 0.325| - 0.455 \cdot |v_i - 0.463| - 0.1225 \cdot |v_i - 0.756|$, which equals -0.24 for $v_i = 0.65$. And if i votes R his expected utility is $-0.4225 \cdot |v_i - 0.463| - 0.455 \cdot |v_i - 0.756| - 0.1225 \cdot |v_i - 0.825|$, which equals -0.15 for $v_i = 0.65$. The difference is equal to 0.09, and for $v_i = x_c$ it exactly counteracts the first-period utility gain that the voter receives by voting L rather than R. Thus at x_c the pivot and signaling effects cancel each other out and the voter is indifferent.

This example illustrates the basic tension between pivot and signaling effects in our model. In this three voter example, the equilibrium cutpoint is $x_c \approx 0.65$, which lies between the signaling cutpoint, $x_p = 0.5$, and the pivot cutpoint, $x_s = 0.75$. The question is how a sequence of equilibrium cutpoints $\{x_m\}$ will behave in the limit as the population size $n = 2m + 1$ gets large.

The difficulty in answering this question is that in large elections both the pivot effect and the signaling effect become small; the probability of a pivot event goes to zero and the distance that second-period candidates move in response to a single vote also goes to zero. The question is which converges faster.

4 Preliminary Results

In this section we establish two lemmas that are useful in establishing existence of a particular type of equilibrium for any n as well as in proving the main result about the limiting behavior of this type of equilibrium. We then present the existence result. Our analysis focuses on a particular class of equilibria.

Definition 1 (Symmetric Cutpoint Strategy) *Voters use a symmetric cutpoint strategy if there exists a point $x_c \in [0,1]$ such that for all $i \in N$*

(1) if $x_c = 0$ then i votes L if $v_i = 0$, and i votes R if $v_i > 0$

(2) if $x_c \in (0,1]$, then i votes L if $v_i < x_C$, and votes R if $v_i \geq x_c$.

Given that all other voters use a symmetric cutpoint strategy with cutpoint x_c, optimal behavior for a voter with ideal point v_i depends on the difference in her expected utility between voting R and voting L in the first period. Using $a_i^1 \in \{L, R\}$ to denote voter $i's$ first period action, we can express this difference as

$$u_{dif}\left(v_i\right) \equiv u(a_i^1 = R|v_i) - u(a_i^1 = L|v_i) = u_{dif1}\left(v_i\right) + u_{dif2}\left(v_i\right) \tag{1}$$

where

$$u_{dif1}\left(v_i\right) \equiv \binom{2m}{m} F(x_c)^m \left(1 - F(x_c)\right)^m \left(\gamma(|L - v_i|) - \gamma(|R - v_i|)\right) \tag{2}$$

and

$$u_{dif2}\left(v_i\right) \equiv \sum_{k=0}^{2m} \binom{2m}{k} \left(F(x_c)\right)^{2m-k} \left(1 - F(x_c)\right)^k \left(\begin{array}{c} \gamma(\left|F_{median}^{-1}(\frac{1}{2}|k, x_c) - v_i\right|) \\ -\gamma(\left|F_{median}^{-1}(\frac{1}{2}|k+1, x_c) - v_i\right|) \end{array} \right). \tag{3}$$

Thus $u_{dif1}\left(v_i\right)$ captures the first period effect of voting: the pivot probability is $\binom{2m}{m}\left(F(x_c)\right)^m \left(1 - F(x_c)\right)^m$ and the utility difference between the two candidates is $\gamma(|L - v_i|) - \gamma(|R - v_i|)$ for a voter with ideal point v_i. Likewise, $u_{dif2}\left(v_i\right)$ captures the second period effect: the probability that k other voters vote R is $\binom{2m}{k}\left(F(x_c)\right)^{2m-k}\left(1 - F(x_c)\right)^k$ and the utility difference between voting R versus L in this event is $\gamma(\left|F_{median}^{-1}(\frac{1}{2} \mid k; x_c) - v_i\right|) - \gamma(\left|F_{median}^{-1}(\frac{1}{2} \mid k+1; x_c) - v_i\right|)$.

Remark 1: Deriving $F_{median}(y|\#R; x_c)$ for $x_c \in (0,1)$

To understand $u_{dif2}\left(v_i\right)$ it is important to see how $F_{median}(y|\#R; x_c)$ depends on $\#R$ and x_c. This function can be characterized in terms of order statistics. We note that for fixed $x_c \in (0,1)$, the distribution of the median is constructed as follows. Given that there are $n - \#R$ draws with values strictly less than x_c and $\#R$ draws with values greater than or equal to x_c we know that the median is less than x_c if $\#R$ is strictly less than $m+1$ and it is greater than x_c otherwise. If $\#R < m+1$, the median is the $m+1$ largest of $n-\#R$ draws from the conditional (on $y < x_c$) distribution $H^-(y; x_c) = \max\{0, \min\{1, \frac{F(y)}{F(x_c)}\}\}$. Accordingly if $\#R < m+1$, $F_{median}(y|\#R; x_c) = H^-_{m+1,n-\#R}(y; x_c)$, which is the distribution of the $(m+1)$'th order statistic from $n - \#R$ draws from the distribution function $H^-(\cdot; x_c)$. Similarly if $\#R \geq m+1$, the median is the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from the conditional (on $y > x_c$) distribution $H^+(y; x_c) = \max\{0, \min\{1, \frac{F(y)-F(x_c)}{1-F(x_c)}\}\}$. So if $\#R \geq m+1$ then $F_{median}(y|\#R; x_c) = H^+_{m-(n-\#R),\#R}(y; x_c)$, which is the distribution of the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from the distribution function $H^+(\cdot; x_c)$.■

Remark 2: Deriving $F_{median}(y|\#R; x_c)$ for $x_c \in \{0, 1\}$

Now consider extremal cutpoints, $x_c \in \{0, 1\}$. If $x_c = 0$ then according to Definition 1, all voters with ideal points in $(0, 1]$ vote R and voters with ideal point $v_i = 0$ vote L. Accordingly, if $\#R < m+1$ then the median voter's ideal point is 0 with probability 1 and $F_{median}(y|\#R; 0)$ is constant at 1 for all $y \in [0, 1]$. In this case, we define $F^{-1}_{median}(\frac{1}{2} \mid \#R; 0) = 0$ and it is clear that equilibrium second period candidate locations are at 0. If $\#R \geq m+1$ then the median is the $(m+1-(n-\#R))$'th order statistic from $\#R$ draws from $F(\cdot)$. This distribution corresponds to $H^+(y; x_c) = \frac{F(y)-F(0)}{1-F(0)}$ with support $[0, 1]$. If $x_c = 1$ then according to Definition 1, all voters with ideal point 1 vote R and all voters with ideal points $v_i \in [0, 1)$ vote L. Accordingly, if $\#R \geq m+1$ then the median voter's ideal point is at 1 with probability 1 and $F_{median}(y|\#R; 1)$ is constant at 0 for all $y \in [0, 1)$ and equal to 1 at $y = 1$. In this case we define $F^{-1}_{median}(\frac{1}{2} \mid \#R; 1) = 1$. If $\#R < m+1$ then the median is the $(m+1)$'th order statistic from $n - \#R$ draws from $F(\cdot)$. This distribution corresponds to $H^-(y; x_c) = \frac{F(y)}{F(1)}$ on $[0, x_c]$.■

One way to see how the distribution function $F_{median}(\cdot|\cdot; \cdot)$ behaves as the arguments $\#R$ and x_c change is to consider the case of the uniform, $F(y) = y$ on $[0, 1]$. Figure 2 plots the function $H^-(y; x_c)$ for $x_c \in \{0, 0.65, 1\}$. Figure 3 plots $H^-_{m+1,n-\#R}(y; x_c)$ for $n = 11, m = 5$ and $\#R \in \{2, 3\}$. This figure shows how increasing $\#R$ shifts the second period candidates' beliefs about the location of the median to the right, thereby causing $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c)$ to increase.

[Insert Figures 2 and 3 about here]

Combining Remarks 1 and 2, we can express the second period policy location as a function of x_c and $\#R$ when voters use a symmetric cutpoint strategy.

$$\chi(x_c, \#R) = \begin{cases} 0 \text{ if } x_c = 0 \text{ and } \#R < m+1 \\ 1 \text{ if } x_c = 1 \text{ and } \#R \geq m+1 \\ \{y : H^{-}_{m+1,n-\#R}\ (y; x_c) = \frac{1}{2}\} \text{ if } x_c \in (0,1] \text{ and } \#R < m+1 \\ \{y : H^{+}_{m+1-(n-\#R),\#R}\ (y; x_c) = \frac{1}{2}\} \text{ if } x_c \in [0,1) \text{ and } \#R \geq m+1. \end{cases}$$

The first lemma builds on this derivation to establish properties of the distribution of the median and the above mapping

Lemma 1 (Properties of Second Period Policy Outcomes) *If voters use a symmetric cutpoint strategy with cutpoint x_c then*

(1) For each $\#R < m+1$ and $y \in [0,1]$, $F_{median}(y|\#R; x_c)$ is weakly decreasing in x_c for $x_c \in [0, y)$ and strictly decreasing for $x_c \in [y, 1]$ and for each $\#R \geq m+1$ and $y \in (0,1)$, $F_{median}(y|\#R; x_c)$ is weakly decreasing in x_c for $x_c \in (y, 1]$ and strictly decreasing in $x_c \in [0, y]$.

(2) If $\#R_1 < \#R_2$ (both in $0, 1, 2, ..., n$) then for each $x_c \in (0,1)$, for some set $A_{x_c} \subset [0,1]$ with positive lebesgue measure $F_{median}(y|\#R_1; x_c) > F_{median}(y|\#R_2; x_c)$ if $y \in A_{x_c}$ and $F_{median}(y|\#R_1; x_c)$ $\geq F_{median}(y|\#R_2; x_c)$ for all $y \in [0,1]$. For $x_c \in \{0, 1\}$, $F_{median}(y|\#R_1; x_c) \geq F_{median}(y|\#R_2; x_c)$ for all $y \in [0,1]$.

(3) For any $\#R \in \{0, 1, 2, ..., n-1\}$ and $x_c \in [0,1]$, $F^{-1}_{median}(\frac{1}{2} \mid \#R; x_c) \leq F^{-1}_{median}(\frac{1}{2} \mid \#R+1; x_c)$.

(4) $F_{median}(y|\#R; x_c)$ is continuous in x_c on $(0,1)$ for each $\#R \in \{0, ..., n\}$ and $y \in [0,1]$ as well as continuous in y on $[0,1]$ for each $\#R \in \{0, ..., n\}$ and $x_c \in (0,1)$.

(5) The mapping $\chi(x_c, \#R)$ is a function from $[0,1] \times \{1, 2...., n\}$ into $[0,1]$ and it is continuous in x_c.

Proof:

(1) Assume $\#R < m+1$. From our derivation of $F_{median}(y|\#R; x_c)$ in Remark 1 this distribution takes on the value 1 if $y \geq x_c$ and $H^{-}_{m+1,n-\#R}\ (y; x_c)$ otherwise. Thus the conclusion that it is weakly

decreasing for $x_c \in [y,1]$ is immediate. Consider $x_c < x_c'$. Since

$$\frac{F(y)}{F(x_c')} < \frac{F(y)}{F(x_c)}$$

$H^-(y;x_c') < H^-(y;x_c)$ for all $x < x_c$ and thus the former first order stochastically dominates the latter on $[0,x_c]$. This ordering of $H^-(y;x_c')$ and $H^-(y;x_c)$ implies that the distributions of order statistics, $H^-_{m+1,n-\#R}(y;x_c')$ and $H^-_{m+1,n-\#R}(y;x_c)$ are also ordered by first order stochastic dominance (see for example Theorem 4.4.1 of David and Nagaraja, p. 75). An analogous argument holds in the case of $\#R \geq m+1$, establishing that $H^+_{m+1-(n-\#R),\#R}(y;x_c')$ and $H^+_{m+1-(n-\#R),\#R}(y;x_c)$ are ordered by first order dominance.

(2) To establish strict monotonicity in $\#R$, consider two integers, $\#R_1$ and $\#R_2$, with $0 \leq \#R_1 < \#R_2 \leq n$. If $\#R_1 < m+1 \leq \#R_2$ then the support of $F_{median}(\cdot|\#R_1;x_c)$ is $[0,x_c]$ and the support of $F_{median}(\cdot|\#R_2;x_c)$ is $[x_c,1]$. Since the distribution $F(\cdot)$ is strictly increasing on $[0,1]$, $F_{median}(y|\#R_1;x_c) > 0$ for all $y \in (0,x_c)$ while $F_{median}(y|\#R_1;x_c) = 1$ for all $y \geq x_c$. Similarly, $F_{median}(y|\#R_2;x_c) = 0$ for all $y \leq x_c$ and $F_{median}(y|\#R_2;x_c) < 1$ for $y \in (x_c,1)$. Thus, $F_{median}(y|\#R_2;x_c) \leq F_{median}(y|\#R_1;x_c)$, with a strict inequality for any $y \notin \{0,1\}$.

Suppose instead that $\#R_1 < \#R_2 < m+1$. The relevant comparison is now between $H^-_{m+1,n-\#R_1}(y;x_c)$ and $H^-_{m+1,n-\#R_2}(y;x_c)$. To see that these two distributions are ordered by first order stochastic dominance, we can partition $n-\#R_1$ draws from $F(\cdot)$ into two sets: first $n-\#R_2$ draws are taken and then another $\#R_2-\#R_1$ are taken. Because $F(\cdot)$ is strictly increasing on $[0,1]$, the probability that one of the $\#R_2-\#R_1$ draws is less than the $m+1$ highest draw of the first $\#R_2-\#R_1$ draws is strictly positive, and thus $H^-_{m+1,n-\#R_2}(y;x_c) < H^-_{m+1,n-\#R_1}(y;x_c)$ for y on $[0,x_c]$. This implies that $F_{median}(y|\#R_2;x_c) \leq F_{median}(y|\#R_1;x_c)$ with a strict inequality if $y \in A_{x_c} = [0,x_c)$ if $\#R_1 < \#R_2 < m+1$. A similar argument holds for $A_{x_c} = (x_c,1]$ and $m+1 \leq \#R_1 < \#R_2$.

The result for $x_c \in \{0,1\}$ follows from Remark 2.

(3) Follows immediately from (2).

(4) Continuity of $F_{median}(y|\#R;x_c)$ in x_c on $(0,1)$ for each $\#R \in \{0,...,n\}$ and $y \in [0,1]$ as well as continuity in y on $[0,1]$ for each $\#R \in \{0,...,n\}$ and $x_c \in (0,1)$ follows from the assumption that $F(\cdot)$ is strictly increasing and continuously differentiable and the fact that the distribution of an order statistic from a differentiable distribution function has a density. In particular, for $\#R < m+1$ the distribu-

tion $F_{median}(y|\#R;x_c)$ has density $h^-_{m+1,n-\#R}(y;x_c) = k\left[\frac{\partial}{\partial y}\left(\frac{F(y)}{F(x_c)}\right)\right]\frac{F(y)}{F(x_c)}^a(1-\frac{F(y)}{F(x_c)})^b$ for integers k,a,b. For $\#R \geq m+1$, the distribution $F_{median}(y|\#R;x_c)$ has density $h^+_{m+1-(n-\#R),\#R}(y;x_c) = k'\left[\frac{\partial}{\partial y}\left(\frac{F(y)-F(x_c)}{1-F(x_c)}\right)\right]\left(\frac{F(y)-F(x_c)}{1-F(x_c)}\right)^{a'}(1-\left(\frac{F(y)-F(x_c)}{1-F(x_c)}\right))^{b'}$ for some k',a',b'. Since we have assumed that $F(\cdot)$ has a continuous density, for fixed $\#R$, as long as $x_c \in (0,1)$ the above densities are well defined and thus the distribution functions are continuous.

(5) To show that $\chi(x_c,\#R)$ is defined on its domain we must show that $\{y : H^-_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in (0,1]$ and $\#R < m+1$ and that $\{y : H^+_{m+1-(n-\#R),\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in [0,1)$ and $\#R \geq m+1$. In the first case, consider $x_c \in (0,1]$ and $\#R < m+1$. From the proof of part 4 of this lemma we see that $H^-_{m+1,n-\#R}(y;x_c)$ has a continuous density function that is strictly positive as long as $y < x_c$. So the function $H^-_{m+1,n-\#R}(y;x_c)$ is continuous and strictly increasing in y on $[0,x_c]$ with $0 = H^-_{m+1,n-\#R}(0;x_c) < \frac{1}{2} < H^-_{m+1,n-\#R}(x_c;x_c) = 1$. This means that the set $S^-(x_c,\#R) = \{y \in [0,1] : H^-_{m+1,n-\#R}(y;x_c) \in (0,1)\}$ is non-empty for $x_c \in (0,1]$ and $\#R < m+1$. Moreover, by the intermediate value theorem this means that the set $\{y : H^-_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in (0,1]$ and $\#R < m+1$. An analogous argument establishes that $\{y : H^+_{m+1-(n-\#R),\#R}(y;x_c) = \frac{1}{2}\}$ is non-empty if $x_c \in [0,1)$ and $\#R \geq m+1$. To show that $\chi(x_c,\#R)$ is a function it is sufficient to note that $F_{median}(\cdot|\#R;x_c)$ has the property that for each value of x_c and $\#R$ there are 2 numbers, $a_1,a_2 \in [0,1]$ such that $F_{median}(\cdot|\#R;x_c)$ is constant at 0 on $[0,a_1]$, $F_{median}(\cdot|\#R;x_c)$ is strictly increasing on $[a_1,a_2]$ and $F_{median}(\cdot|\#R;x_c)$ is constant at 1 on $[a_2,1]$. This means that the equation $F_{median}(\cdot|\#R;x_c) = \frac{1}{2}$ has at most one solution (and it is in $[a_1,a_2]$).

To establish continuity we consider two cases. First assume that $\#R < m+1$. By part 4 of this lemma, for a fixed y, $H^-_{m+1,n-\#R}(y;x_c)$ is continuous in x_c on $(0,1)$ and thus this and the fact that it is strictly increasing (and has a density) in y on a neighborhood of the point $\{y : H^-_{m+1,n-\#R}(y;x_c) = \frac{1}{2}\}$ implies by way of the implicit function theorem that the solution $\chi(x_c,\#R)$ is continuous in x_c if $x_c \in (0,1)$ and $\#R < m+1$. Continuity at $x_c = 0$ follows from the fact $\chi(x_c,\#R) \leq x_c$ if $\#R < m+1$ and thus $\lim_{x_c\to 0}\chi(x_c,\#R) = 0$ and $\chi(0,\#R) = 0$. Continuity at $x_c = 1$ follows from the fact that $H^-_{m+1,n-\#R}(y;1)$ is defined and for each y, $H^-_{m+1,n-\#R}(y;x_c)$ is continuous in x_c at 1. An analogous argument about $H^+_{m+1-(n-\#R),\#R}(y;x_c)$ establishes continuity in the case of $\#R \geq m+1$.■

The next result establishes some properties of the utility difference function in Equation 1. Since this result simply uses conclusions from Lemma 1 in standard ways the proof is in the appendix.

Lemma 2 (Properties of Utility Difference Function) *If voters use a symmetric cutpoint strategy with cutpoint x_c then*

(1) $u_{dif}(v_i)$ is continuous and weakly increasing in v_i.

(2) (Lipschitz property) $\forall \tilde{v}_i, \hat{v}_i \in [0,1]$, $|u_{dif}(\tilde{v}_i) - u_{dif}(\hat{v}_i)| \leq 4\,|\tilde{v}_i - \hat{v}_i|$.

(3) $u_{dif}(0) \leq 0$ and $u_{dif}(1) \geq 0$.

We can now state our first main result. The proof, which applies a standard fixed point argument to the function $u_{dif}(\cdot)$, is in the appendix.

Proposition 1 *There exists an equilibrium in which voters use a symmetric cutpoint strategy in the first period.*

5 Intuition for the Convergence Result

Having established existence, we now turn to the question of equilibrium behavior in large electorates, i.e., as $m \to \infty$. We suppress the c subscript and let x_m denote the cutpoint in a symmetric cutpoint strategy equilibrium with $n = 2m + 1$ voters. Our interest is then in $\lim_{m\to\infty} x_m$ (if it exists). We show that this limit is equal to the point $F^{-1}(\frac{1}{2})$. Since this limit does not depend on the first period candidate locations L and R, it is also the limit of cutpoints for equilibria in the pure signaling game.

The proof proceeds by contradiction. We show that if a sequence of cutpoints does not converge to $F^{-1}(\frac{1}{2})$ then these cutpoints cannot be equilibrium cutpoints for infinitely many values of m because voters at the cutpoint x_m, who must be indifferent in equilibrium, will strictly prefer to vote for one candidate over the other. To be more precise, we show that if any subsequence of equilibrium cutpoints converges to a point other than $F^{-1}(\frac{1}{2})$ then for m sufficiently large a voter with ideal point x_m will strictly prefer to vote for one candidate over the other. Once it is established that no subsequence converges to a point other than $F^{-1}(\frac{1}{2})$ it follows that every subsequence, and thus the actual sequence, converges to $F^{-1}(\frac{1}{2})$. This brief section serves as a roadmap for the proof, presenting an informal version of the argument. The next section contains a proof of the main result.

Suppose that in a large electorate voters behave according to a cutpoint x_m which is converging to a number $Z > F^{-1}(\frac{1}{2})$. We show that for large values of m a voter at x_m will strictly prefer to vote R. There are three types of effects that the voter must consider.

The first consideration is a pivot effect, which we label PV. Since the election is not expected to be a tie, i.e., $x_m \neq F^{-1}(\frac{1}{2})$, and the population size is large, the probability of this pivot event is exceedingly small in a large electorate.

The second consideration involves bad signaling effects from voting R; whenever more than half of the other voters vote R, the second period policy will be to the right of x_m, so if a voter with $v_i = x_c$ votes R this will move second period policy to the right, i.e., away from his ideal point, as established in part 3 of Lemma 1. However, because $x_m > F^{-1}(\frac{1}{2})$, more than half of the votes are expected to go to L, and thus in a large electorate bad signaling effects are extremely unlikely to occur. We find an upper bound on the probability-weighted sum of these bad signaling effects, and label it $UBBS$ (upper bound for bad signaling).

The third consideration involves good signaling effects from voting R; whenever more than half of the other voters vote L, the second period policy will be to the left of x_m, so if a voter with $v_i = x_m$ votes R this will move second period policy to the right, i.e., towards his ideal point. Since $x_m > F^{-1}(\frac{1}{2})$, more than half of the votes are expected to go to L, and thus in a large electorate it is extremely likely that the signaling effect of voting R will be good. We find a lower bound on the probability-weighted sum of these good signaling effects, and label it $LBGS$ (lower bound for good signaling).

We consider the ratio of bad signaling plus pivot effects to good signaling effects, and show that this ratio

$$\frac{PV + UBBS}{LBGS}$$

can be expressed as a limit of the form

$$\lim_{m\to\infty} \frac{P_{tie} + (m+1)\, P_{tie}}{cP}. \tag{4}$$

In this expression, $P_{tie} = \binom{2m}{m} F(x_m)^m \,(1 - F(x_m))^m$ is the probability of an exact tie among the other $2m$ voters given the cutpoint x_m. In the denominator, P is the probability of a certain type of good signaling effect, and P goes to zero much more slowly than P_{tie}. The c in the denominator is a constant that does not depend on m. At the end of the proof we show that Equation 4 is bounded by an expression of the form $(m+2)\, q^{\left(\frac{1}{2}-c_1\right)2m}$ with constants $q \in (0,1)$, and $c_1 \in (0, 1/2)$. Thus the limit of Equation 4 is 0, which means that for a voter with ideal point x_m (and voters with ideal points close enough to x_m) it will be optimal to deviate and vote R.

6 The Convergence Result

Our main result is

Proposition 2 $\lim_{m\to\infty} x_m = F^{-1}(\frac{1}{2})$.

Proof: Assume by way of a contradiction that the cutpoints do not converge to the point $M \equiv F^{-1}(\frac{1}{2})$. Because $x_m \in [0,1]$, $\forall m$, the Bolzano-Weierstrass Theorem implies that there exists some number $Z \in [0,1]$ with $Z \neq M$ such that a subsequence $\{x_{m'}\} \to Z$. We focus on such a subsequence, ignoring the residual portion of the original sequence. Thus the assumption that Proposition 2 is false equates to the claim that $\{x_m\} \to Z$. Either $Z < M$ or $Z > M$ and in the remainder of the proof we focus on the latter case; the argument for the former case is virtually identical and is thus omitted.

Our goal is to show that there exists a $\bar{m}$ such that if $m > \bar{m}$ then a voter with ideal point x_m has a strict preference to vote for R. Once this claim is established, the continuity of the utility functions established in Lemma 2 implies that for $m > \bar{m}$ there exists a $\delta_m > 0$ such that if $v_i \in (x_m - \delta_m, x_m + \delta_m)$ a voter with ideal point v_i prefers to vote for R when everyone else uses the cutpoint x_m. Thus for some voters to the left of x_m voting L is not a best response, contradicting the hypothesis that x_m is an equilibrium cutpoint when the population size is $2m+1$. This contradiction means that we cannot have a subsequence of cutpoints converging to any $Z \neq M$ and thus the sequence of cutpoints converges to M.

For each m, consider a voter, i, with ideal point x_m. Given that voters to the left of x_m vote L and voters to the right of x_m vote R, the probability of any individual voting R is

$$p_m \equiv 1 - F(x_m).$$

Since $x_m > F^{-1}(1/2)$ we know that $p_m < \frac{1}{2}$.

We start by analyzing the utility function of a voter with ideal point x_m. If exactly m voters other than i vote R then the election is tied, and i's vote is pivotal in determining the first period policy.

However, in terms of first period motivations, which depend on the candidate locations L and R, it is not clear whether i prefers to vote L or vote R in the event that he is pivotal. In terms of second period motivations, which depend on candidate locations given $\#R$, it is also unclear whether i prefers to vote L or vote R in the event that he is pivotal.

In contrast, the voter's preferences are clear for events in which he is not pivotal. If $m-1$ or fewer of the other voters vote for R the second period policy will be to the left of x_m regardless of i's vote and if at least $m+1$ of the other voters vote for R then the second period policy will be to the right of x_m regardless of i's vote (as established in Remark 1). These facts and the monotonicity of the second period policy in $\#R$ (part 3 of Lemma 1) imply that if $m-1$ or fewer of the other voters vote for R then a vote for R moves the second period policy closer to i's ideal point. On the other hand, if at least $m+1$ of the other voters vote for R, then a vote for R moves the second period away from i's ideal point. Note that in either of these cases, i's vote cannot move policy far enough to leapfrog her ideal point, x_m (see Remark 1).

Following Equations 1, 2, and 3, given the conjectured equilibrium for population size $n = 2m+1$, the utility difference between voting R versus voting L for a voter with ideal point $v_i = x_m$ in the equilibrium with population size $2m+1$ is

$$u^m_{dif}(x_m) \equiv u^m_{dif1}(x_m) + u^m_{dif2}(x_m)$$

which we re-write as

$$\begin{aligned} u^m_{dif}(x_m) \quad = \quad & \binom{2m}{m}(1-p_m)^m {p_m}^m \left(\gamma(|L-x_m|) - \gamma(|R-x_m|)\right) \\ & + \sum_{k=0}^{m-1}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right) - x_m\right|)\right) \\ & + \binom{2m}{m}(1-p_m)^m {p_m}^m \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m;x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m+1;x_m\right) - x_m\right|)\right) \\ & + \sum_{k=m+1}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right) - x_m\right|)\right). \end{aligned} \tag{5}$$

The first line of Equation 5 is the pivot effect. The second line represents good signaling effects of voting R when $m-1$ or fewer other voters vote R. The third line represents the indeterminate signaling effect when the $2m$ other voters split their votes equally between L and R. The fourth line represents bad signaling effects, when $m+1$ or more other voters vote R.

Our ultimate goal is to show that there exists an $\bar{m}$ such that for $m > \bar{m}$, $u^m_{dif}(x_m) > 0$. To simplify the expression in Equation 5, we first simplify each component of the utility function, by finding bounds that limit how severe the bad effects of voting R can be, along with a lower bound on good effects of voting R.

Pivot effect. The pivot effect can be either positive or negative, depending on the positions of the two first-period candidates. A bound based on the fact that L, R, and x_m are all in the interval $[0,1]$, will be sufficient:

$$\binom{2m}{m}(1-p_m)^m {p_m}^m (\gamma(|L-x_m|)-\gamma(|R-x_m|)) > -\binom{2m}{m}(1-p_m)^m {p_m}^m\gamma'(1) = -\binom{2m}{m}(1-p_m)^m {p_m}^m. \tag{6}$$

Recall that we have normalized utility so that $\gamma'(1) = 1$ and we have assumed that this function is convex, allowing us to treat the maximal possible utility difference between L and R as 1.

Bad and indeterminate signaling effects. For any $k \in \{1,..,n-1\}$, $F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right) \in (0,1)$ and $F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right) \in (0,1)$, so

$$\sum_{k=m+1}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k;x_m\right)-x_m\right|)-\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1;x_m\right)-x_m\right|)\right)$$
$$+\binom{2m}{m}(1-p_m)^m{p_m}^m\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m;x_m\right)-x_m\right|)-\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|m+1;x_m\right)-x_m\right|)\right)$$
$$> -\sum_{k=m}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\gamma'(1) = -\sum_{k=m}^{2m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k$$

We will use repeatedly the fact that the *binomial expansion is monotonic.* In particular, since $p_m < 1/2$, for any $k \in \{m+1,...,2m\}$, $\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k < \binom{2m}{m}(1-p_m)^m{p_m}^m$. As a consequence of this fact, the total of the bad and indeterminate signaling effects must be strictly greater than

$$-(m+1)\binom{2m}{m}(1-p_m)^m{p_m}^m. \tag{7}$$

Good signaling effects. We now develop a lower bound on good signaling effects. Fix any points A and B in the unit interval such that $M < A < B < Z$. For any m, let A_m represent the largest number less than A such that for some integer $a_m < 2m+1$ it is the case that $A_m = F^{-1}_{median}\left(\frac{1}{2}|a_m;x_m\right)$. Likewise, let B_m represent the largest number less than B such that for some integer $b_m < 2m+1$ it is the case that $B_m = F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)$. For the b_m identified in the definition of B_m let $C_m = F^{-1}_{median}\left(\frac{1}{2}|b_m+1;x_m\right)$.

For m sufficiently large it turns out that $M < A_m < B_m < B \leq C_m < Z$, and since we are interested in the limit as $m \to \infty$ we henceforth focus on values of m that are large enough for this inequality to hold. Since $A_m \leq B_m < B < Z$ are true by construction, establishing that these inequalities eventually hold requires only that we show that (i) $M < A_m$, (ii) $B \leq C_m < Z$ and (iii) $A_m < B_m$ for m large enough. Each of these results follows from the fact that for any $q \in (0,1)$ the sequence $\left|F^{-1}_{median}\left(\frac{1}{2}|(2m+1)q+1; x_m\right) - F^{-1}_{median}\left(\frac{1}{2}|(2m+1)q; x_m\right)\right|$ converges to 0. To see this note that in the first instance this fact allows us to conclude that if $A_m < M$ infinitely often then it is not the case that for every m, $A < F^{-1}_{median}\left(\frac{1}{2}|a_m+1; x_m\right)$, which contradicts the definition of A_m (and a_m). In the second instance, if $Z < C_m$ infinitely often then this fact means that we must have $B < B_m$ for infinitely many values of m (contradicting the definition of B_m) and if $C_m < B$ infinitely often then this fact means that for infinitely many values of m, $B_m < C_m < B$, contradicting the definition of B_m. In the third instance the fact that $A < B$ implies that if $A_m = B_m$ then $A_m > A$ or $F^{-1}_{median}\left(\frac{1}{2}|b_m+1; x_m\right) < B$ for infinitely many m (contradicting the definition of A_m or B_m).

For fixed m the set of profiles for other voters that, given $i's$ vote, can result in a policy between A_m and C_m consists of profiles for which the number of other voters that vote R is in the set $\{a_m, a_m+1, ..., b_m-1, b_m\}$. Although we cannot analytically solve for the policy distance between $F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right)$ and $F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right)$ for particular values of k, we do know that

$$\sum_{k=a_m}^{b_m}\left(F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right)\right) = C_m - A_m > B - A.$$

Since we have $A_m < B_m < B \leq C_m$ the last inequality above is due to the fact that $A_m < A < B$. We re-write the good signaling effects term from Equation 5 as

$$\sum_{k=0}^{m-1}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m\right|)\right)$$

$$\begin{aligned}
= \quad & \sum_{k=0}^{a_m-1}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m\right|)\right) \\
& + \sum_{k=a_m}^{b_m}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m\right|)\right) \\
& + \sum_{k=b_m+1}^{m-1}\binom{2m}{k}(1-p_m)^{2m-k}{p_m}^k\left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m\right|)\right).
\end{aligned}$$

Because $B < Z$ and $\gamma(\cdot)$ is convex the slope of the utility difference to i with $v_i > \frac{Z+B}{2}$ for points in $[0, B]$ is at least $\underline{\gamma} := \gamma'(\frac{Z-B}{2})$. Since $\gamma(\cdot)$ is strictly increasing $\gamma'(\frac{Z-B}{2}) > 0$. Because x_m is assumed to converge to Z, for m sufficiently large $v_i > \frac{Z+B}{2}$. Thus, the good signaling effects term is eventually greater than

$$\sum_{k=a_m}^{b_m} \binom{2m}{k} (1-p_m)^{2m-k} {p_m}^k \left(\left| F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m \right| - \left| F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m \right| \right) \underline{\gamma}.$$

Since $M < A_m < C_m$ and $p_m < 1/2$, we know from monotonicity of the binomial expansion that the event in which b_m others vote for R is the least likely of the set of profiles of the $2m$ other voters in which i's vote can result in a policy in the interval $[A_m, C_m]$. Thus the above expression is greater than

$$\binom{2m}{b_m} (1-p_m)^{2m-b_m} p_m^{b_m} \sum_{k=a_m}^{b_m} \left(\left| F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m \right| - \left| F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m \right| \right) \underline{\gamma}. \quad (8)$$

Note that because $\forall k \in \{a_m, a_m+1, ..., b_m\}$, $F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) < F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) < x_m$,

$$\begin{aligned} &\sum_{k=a_m}^{b_m} \left(\left| F^{-1}_{median}\left(\frac{1}{2}|k; x_m\right) - x_m \right| - \left| F^{-1}_{median}\left(\frac{1}{2}|k+1; x_m\right) - x_m \right| \right) \\ &= F^{-1}_{median}\left(\frac{1}{2}|b_m+1; x_m\right) - F^{-1}_{median}\left(\frac{1}{2}|a_m; x_m\right) \\ &= C_m - A_m \\ &> B - A. \end{aligned}$$

Thus, we can rewrite Equation 8 to get the following lower bound for good signaling effects:

$$\underline{\gamma}(B-A) \binom{2m}{b_m} (1-p_m)^{2m-b_m} p_m^{b_m}. \quad (9)$$

Having derived bounds on pivot effects, bad signaling effects, and good signaling effects (Equations 6, 7, and 9, respectively) we now substitute these bounds into the utility difference expression in Equation 5 to get

$$u^m_{dif}(x_m) > -\binom{2m}{m} (1-p_m)^m {p_m}^m - (m+1) \binom{2m}{m} (1-p_m)^m {p_m}^m + \underline{\gamma}(B-A) \binom{2m}{b_m} (1-p_m)^{2m-b_m} p_m^{b_m}.$$

To show that there exists an $\bar{m}$, such that for $m > \bar{m}$, $u^m_{dif}(x_m) > 0$, it is sufficient to show that

$$\lim_{m\to\infty} \frac{\binom{2m}{m} (1-p_m)^m {p_m}^m + (m+1) \binom{2m}{m} (1-p_m)^m {p_m}^m}{\underline{\gamma}(B-A) \binom{2m}{b_m} (1-p_m)^{2m-b_m} p_m^{b_m}} = 0.$$

Combining terms in the numerator, and noting that $\underline{\gamma}(B-A)$ is strictly greater than zero and unaffected by m, it is sufficient to show that $\lim_{m\to\infty}(m+2)\frac{\binom{2m}{m}(1-p_m)^m {p_m}^m}{\binom{2m}{b_m}(1-p_m)^{2m-b_m}p_m^{b_m}}=0$. For convenience, define $\Psi_m=(m+2)\frac{\binom{2m}{m}(1-p_m)^m {p_m}^m}{\binom{2m}{b_m}(1-p_m)^{2m-b_m}p_m^{b_m}}$. Rearranging yields

$$
\begin{aligned}
\Psi_m &= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\binom{2m}{m}}{\binom{2m}{b_m}} \\
&= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\frac{2m!}{m!m!}}{\frac{2m!}{b_m!(2m-b_m)!}} \\
&= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{b_m!(2m-b_m)!}{m!m!} \\
&= (m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\prod_{j=1}^{m-b_m}(2m-b_m-j+1)}{\prod_{j=1}^{m-b_m}(m-j+1)}.
\end{aligned}
$$

Taking the largest of the $m-b_m$ terms on the top of the product and the smallest of the $m-b_m$ terms on the bottom we see that

$$\Psi_m<(m+2)\left(\frac{p_m}{1-p_m}\right)^{m-b_m}\frac{\prod_{j=1}^{m-b_m}(2m-b_m)}{\prod_{j=1}^{m-b_m}(b_m+1)}=(m+2)\left[\left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right)\right]^{m-b_m}. \quad (10)$$

To find $\lim_{m\to\infty}\Psi_m$ we establish results about the terms $\left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right)$ and the exponent $m-b_m$. Two intermediate limiting results will be established. The first is the fact that $B_m\to B\in(M,Z)$. The second, which builds on the first, is that $\lim_{m\to\infty}\frac{b_m}{2m}\in(0,\frac{1}{2}]$.

Recall that B_m is the largest number less than B such that for some integer $b_m<2m+1$ it is the case that $B_m=F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)$. We now claim that $B_m\to B$. To see this note that if for some $\varepsilon>0$, $\left|F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)-B\right|>\varepsilon$ infinitely often we obtain the following contradiction. Since $\max_{x\in[0,1]}|F_{median}(x|b_m;x_m)-F_{median}(x|b_m+1;x_m)|\to 0$ there is some m' s.t. $\left|F^{-1}_{median}\left(\frac{1}{2}|b_{m'};x_{m'}\right)-F^{-1}_{median}\left(\frac{1}{2}|b_{m'}+1;x_{m'}\right)\right|<\varepsilon$ and thus $F^{-1}_{median}\left(\frac{1}{2}|b_{m'}+1;x_{m'}\right)<B$, contradicting the definition of $B_{m'}$. We have, thus, shown that $B_m=F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)$ is converging to B, which is in (M,Z). This implies that b_m must also have a limit.

We use this fact to establish eventual bounds on the ratio $\frac{b_m}{2m}$. Suppose now that $\frac{b_m}{2m}$ converges to a number greater than $\frac{1}{2}$. By Remark 1, $b_m>m+1$ implies that $F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)>x_m$ and since we have assumed that $x_m\to Z$, and we have $F^{-1}_{median}\left(\frac{1}{2}|b_m;x_m\right)\to B$ we must have $B\geq Z$ contradicting the definition of B (that $B<Z$). Also it is clear that $b_m\to 0$ is not possible since $B>0$ and $B_m\to B$. Thus $c_1:=\lim\frac{b_m}{2m}\in(0,\frac{1}{2}]$.

We now work on the limit of the terms in brackets from equation 10, using the fact that $\lim_{m\to\infty} p_m = 1-F(Z)$ and $\lim_{m\to\infty} \frac{b_m}{2m} = c_1$,

$$\begin{aligned} \lim_{m\to\infty}\left(\frac{p_m}{1-p_m}\right)\left(\frac{2m-b_m}{b_m+1}\right) &= \lim_{m\to\infty}\left(\frac{p_m}{1-p_m}\right)\left(\frac{1-\frac{b_m}{2m}}{\frac{b_m+1}{2m}}\right) \\ &= \frac{1-F(Z)}{F(Z)}\cdot\frac{1-c_1}{c_1}. \end{aligned} \tag{11}$$

We wish to show that $\frac{1-F(Z)}{F(Z)}\cdot\frac{1-c_1}{c_1}\in(0,1)$, and because $\max\{1-F(Z),c_1\}\le\frac{1}{2}$ it suffices to show that $c_1 > 1-F(Z)$. We show that if $c_1 \le 1-F(Z)$ then $B \le M$.

First, from Theorem 2.5 in David and Nagaraja (2003, p. 17) the distribution of the $(m+1)$'th ideal point (i.e. the median) from $2m+1$ (i.e. n) draws from $F(\cdot)$ conditional on the fact that $b_m < m+1$ realized ideal points are greater than x_m is just the distribution of the $(m+1)$'th draw from $2m+1-b_m$ draws from the distribution $\frac{F(\cdot)}{F(x_m)}$ on $[0,x_m]$, which we denote as $X_{m+1,2m+1-b_m}$.

To disprove the possibility that $c_1 \le 1-F(Z)$ we assume that there is a $\delta\ge 0$ such that $c_1+\delta = 1-F(Z)$. We use a very weak consequence of Sen's result on the asymptotic normality of sample quantiles for non i.i.d. draws (Sen 1968, p. 1725, Theorem 2.1).[5] to conclude that $X_{m+1,2m+1-b_m}$ is asymptotically normal with mean ξ where ξ solves $\frac{F(\xi)}{F(Z)} = \frac{1}{2F(Z)+2\delta} < \frac{1}{2}$. This is true since we have just established that $X_{m+1,2m+1-b_m}$ is an order statistic from $\frac{F(\cdot)}{F(x_m)}$, which is converging to $\frac{F(\cdot)}{F(Z)}$ by assumption, and $\lim\frac{m+1}{2m+1-b_m} = \frac{1}{2(1-c_1)} = \frac{1}{2(F(Z)+\delta)}$. Thus $X_{m+1,2m+1-b_m}$ has the same limiting distribution as $X_{\frac{n}{2F(Z)+2\delta},n}$. This means that $X_{m+1,2m+1-b_m}$ is asymptotically normal with mean ξ where ξ solves $\frac{F(\xi)}{F(Z)} = \frac{1}{2F(Z)+2\delta}\le\frac{1}{2}$ and thus since $F(\cdot)$ is strictly increasing on its support, $F(\xi)\le\frac{1}{2}$ implies that $\xi\le M$. But this means that $B\le M$, which is not possible. Given that we have established $\frac{1}{2}\ge c_1 > 1-F(Z)$, we conclude that $\frac{1-F(Z)}{F(Z)}\cdot\frac{1-c_1}{c_1}\in(0,1)$.

[5] Sen extends Bahadur's (1966) result on the asymptotic normality of a qauntile of a sample to the case of non i.i.d random variables. Bahadur's result states that if $F(x)$ is a probability distribution that is twice differentiable and $F(\xi)=p$ (with density $f(\xi)>0$) and $X_{r,n}$ is the r th order statistic from n draws and Z_n is the number of observations greater than ξ then for $r/n\to p\in(0,1)$ the random variable $X_{r,n}$ is equal to $\xi+\frac{Z_n-n(1-p)}{nf(\xi)}+R_n$, where R_n vanishes, and this random variable is asymptotically normal. Sen's generalization goes well beyond but includes the case of independent draws from distributions that vary with n, $F_n(\cdot)$. This is the only part of the extension we need. The challenge of these results is pinning down the rate of convergence of R_n. We care only about the fact that $X_{r,n}$ is asymptotically normal with mean ξ.

We now find a lower bound, as a function of m, on the exponent $m - b_m$ in Equation 10. We first tighten the bound on c_1 showing that in fact $c_1 < \frac{1}{2}$. By way of a contradiction suppose now that $\frac{b_m}{2m} \to \frac{1}{2}$. We can divide this sequence into two subsequences. The first subsequence contains $\{\frac{b_t}{2t}\}_{t=1}^{\infty}$ with t satisfying $b_t \leq t$, and the second contains $\{\frac{b_j}{2j+1}\}_{j=1}^{\infty}$ with t satisfying $b_j > j$. So the first subsequence consists of the cases where the median ideal point is an intermediate order statistic of the form $X_{b_t,t}$ from the parent distribution $\frac{F(\cdot)}{F(x_m)}$ on $[0, x_m]$ with $\frac{k_t}{t} \to 1$ and the second subsequence consists of the cases where the median ideal point is an intermediate order statistic of the form $X_{b_j-j,j}$ from the parent distribution $\frac{F(y)-F(x_m)}{1-F(x_m)}$ on $[x_m, 1]$ with $\frac{b_j-j}{j} \to 0$. For intermediate order statistics, the relevant analogue to the result by Sen used above is due to Watts (1980). To apply Watts's Theorem 1 we first consider a slight modification of the subsequences with the parent distributions $\frac{F(\cdot)}{F(Z-\varepsilon)}$ on $[0, Z-\varepsilon]$ and $\frac{F(y)-F(Z+\varepsilon)}{1-F(Z+\varepsilon)}$ on $[Z+\varepsilon, 1]$ respectively. Here ε is chosen to satisfy $Z-B > 2\varepsilon$. Watts's result, restated in our notation, states that as long as the convergence of $\frac{k_t}{t} \to 1$ and $\frac{b_j-j}{j} \to 0$ is slow enough, the relevant order statistics converge to $Z - \varepsilon + \delta_t$ and $Z + \varepsilon + \delta_j$ in probability (where both δ_t and δ_j vanish at a known rate).[6] These two convergence results imply that even when $\frac{k_t}{t} \to 1$ and $\frac{b_j-j}{j} \to 0$ converge slowly $F_{median}(Z - 2\varepsilon | b_t; Z - \varepsilon) \to 0$ and $F_{median}(Z + 2\varepsilon | b_j; Z + \varepsilon) \to 1$. Accordingly it is not possible for B_m to converge to B with $B < Z - 2\varepsilon$. Now we relax the assumption that the sequence of draws is from identical parent distributions and allow for the fact that the draws are from $\frac{F(\cdot)}{F(x_m)}$ on $[0, x_m]$ and $\frac{F(y)-F(x_m)}{1-F(x_m)}$ on $[x_m, 1]$. To accommodate this fact it is sufficient to observe that the actual order statistics in the subsequence $\{\frac{b_t}{2t}\}_{t=1}^{\infty}$ (as well as $\{\frac{b_j}{2j+1}\}_{j=1}^{\infty}$) first order stochastically dominate the subsequence $\{\frac{b_t}{2t}\}_{t=1}^{\infty}$ with parent distribution $\frac{F(\cdot)}{F(Z-\varepsilon)}$ on $[0, Z-\varepsilon]$.[7] Thus, we cannot have $B_m \to B$ if $\frac{b_m}{2m} \to \frac{1}{2}$, so it cannot be that $\frac{b_m}{2m} \to \frac{1}{2}$, so $c_1 \in (0, \frac{1}{2})$. Thus $\lim_{m\to\infty} \frac{m-b_m}{2m} = \frac{1}{2} - c_1$, and if we fix a $\delta = \frac{\frac{1}{2}-c_1}{2}$ there exists a m_1 such that for all $m > m_1$, $\frac{m-b_m}{2m} > \frac{1}{2} - c_1 - \delta$, i.e.,

$$m - b_m > \left(\frac{1}{2} - c_1\right) 2m. \tag{12}$$

[6] Bounds on the rate of $\frac{k_t}{t} \to 1$ and $\frac{b_j-j}{j} \to 0$ are critical to showing that a limiting distribution exists. For example when $\frac{b_j-j}{\log^3 j}$ has a finite limit the intermediate order statistic behaves like an extreme value and may not possess a limiting distribution. For our purposes the worst case scenarios are the ones when the convergence is slow. If $\frac{b_m}{2m} \to \frac{1}{2}$ more quickly, it is even harder to support $B_m \to B$ with $B \neq Z$ while $x_m \to Z$.

[7] This additional, rather trivial, step is neccesitated by our inability to find an extension of Watts's result to the case of non i.i.d draws from a parent distribution.

To conclude the argument we can combine results from Equations 10, 11, and 12 to get our result:

$$\begin{aligned}\lim_{m\to\infty}\Psi_m &\leq \lim_{m\to\infty}(m+2)\left[\frac{1-F(Z)}{F(Z)}\cdot\frac{1-c_1}{c_1}\right]^{\left(\frac{1}{2}-c_1\right)2m} \\ &= 0.\end{aligned}$$

This last step follows from the observation that the relevant limit is of the form $\lim_{m\to\infty}(m+2)(\zeta(m))^{\eta(m)}$ where $\lim_{m\to\infty}\zeta(m)\in(0,1)$ and $\lim_{m\to\infty}\eta(m)=\infty$.■

7 Discussion

One can imagine other signaling motivations in elections.[8] In fact Razin (2003) discusses a model in which the signaling motivation is exogenous. The goal of our paper, in contrast, is to consider a game where the signaling motivation is endogenous and see which effect – pivot or signaling – dominates. A natural extension of our analysis would be a model in which the second period candidates do not converge in equilibrium, e.g., because they face uncertainty about voter preferences and have policy motivations. We conjecture that the proof technique employed in establishing Proposition 2 could be extended to address this case, but such an analysis is beyond the scope of this paper.

At a broader level, our result has important implications for theories of elections. Put bluntly, it may be the case that existing electoral models' focus on pivot events is misplaced. Of course, the purely pivot-based variant of our model is substantially more simplified than the sophisticated one-shot pivot-based models that other authors have used to analyze issues such as turnout, multiple candidates, sequential versus simultaneous voting, and voters' correlated private information. However, all of these analyses are fundamentally based on low-probability pivot events, so our main result suggests that it might be fruitful to rethink some of the more sophisticated pivot-based models, as well as the insights about representation and efficiency that they yield, when there is a large electorate and a signaling motivation is present.

[8] We should note that at least two authors have conjectured that signaling may remain important in large elections. Fowler and Smirnov (2007) analyze a decision theoretic model of voting in situations where vote totals affect future policies via an exogenously set reaction function. Although Fowler and Smirnov are interested in the importance of the signaling motivation they assume the problem away by hard wiring the pivot probability at zero so their model cannot be used to analyze the relative importance of pivot and signaling effects. Our finding breathes additional life into the relevance of their conclusions.

8 Appendix

The following lemma is used in the proof of Proposition 1.

Lemma 2:

(1) $u_{dif}(v_i)$ is continuous and weakly increasing in v_i.

(2) [Lipschitz property] $\forall \tilde{v}_i, \hat{v}_i \in [0,1]$, $|u_{dif}(\tilde{v}_i) - u_{dif}(\hat{v}_i)| \leq 4\ |\tilde{v}_i - \hat{v}_i|$.

(3) $u_{dif}(0) \leq 0$ and $u_{dif}(1) \geq 0$.

Proof: We first prove separate versions of this result for $u_{dif1}(v_i)$ and $u_{dif2}(v_i)$, then combine them to get the desired result for $u_{dif}(v_i) = u_{dif1}(v_i) + u_{dif2}(v_i)$.

For $u_{dif1}(v_i)$, note that $\gamma(|L - v_i|) - \gamma(|R - v_i|)$ is continuous and weakly increasing in v_i since γ is continuous and strictly increasing and $L \leq R$. Thus, because $\binom{2m}{m}(F(x_c))^m (1 - F(x_c))^m \in [0,1]$, $u_{dif1}(v_i)$ is continuous and weakly increasing in v_i. For the Lipschitz property, note that

$$\begin{aligned} |u_{dif1}(\tilde{v}_i) - u_{dif1}(\hat{v}_i)| &= |\gamma(|L - \tilde{v}_i|) - \gamma(|R - \tilde{v}_i|) - \gamma(|L - \hat{v}_i|) + \gamma(|R - \hat{v}_i|)| \\ &= |\gamma(|L - \tilde{v}_i|) - \gamma(|L - \hat{v}_i|) - \gamma(|R - \tilde{v}_i|) + \gamma(|R - \hat{v}_i|)| \end{aligned}$$

which is less than or equal to $2\ |\tilde{v}_i - \hat{v}_i|$ because $\gamma'(1) = 1$ and $\gamma(\cdot)$ is convex. Finally, since $L \leq R$, $u_{dif1}(0) \leq 0$ and $u_{dif1}(1) \geq 0$.

For $u_{dif2}(v_i)$, note that by part 3 of Lemma 1, $\forall k \in \{1, ..., 2m\}$, $F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) \leq F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right)$, so $\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - v_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - v_i\right|)$ is continuous and weakly increasing in v_i. Thus the probability-weighted sum,

$$\sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - v_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - v_i\right|)\right)$$

is continuous and weakly increasing in v_i. For the Lipschitz property, note that $|u_{dif2}(\tilde{v}_i) - u_{dif2}(\hat{v}_i)|$ equals

$$\left| \begin{array}{c} \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \tilde{v}_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \tilde{v}_i\right|)\right) - \\ \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \hat{v}_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \hat{v}_i\right|)\right) \end{array} \right|$$

which simplifies to

$$\left| \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(\begin{array}{c} \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \tilde{v}_i\right|) - \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - \hat{v}_i\right|) \\ -\gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \tilde{v}_i\right|) + \gamma(\left|F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right) - \hat{v}_i\right|) \end{array} \right) \right|.$$

Recall that $\gamma'(1) = 1$ and $\gamma(\cdot)$ is convex, so the above less than or equal to

$$\sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \cdot 2 |\tilde{v}_i - \hat{v}_i| \leq 2 \cdot |\tilde{v}_i - \hat{v}_i|$$

For $u_{dif2}(0) \leq 0$, recall from part 3 of Lemma 1 that $\forall k$, $F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) \leq F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right)$ so

$$\begin{aligned} u_{dif2}(0) &= \sum_{k=0}^{2m} \binom{2m}{k} (F(x_c))^{2m-k} (1 - F(x_c))^k \left(F^{-1}_{median}\left(\frac{1}{2}|k; x_c\right) - F^{-1}_{median}\left(\frac{1}{2}|k+1; x_c\right)\right) \\ &\leq 0 \end{aligned}$$

By a similar argument $u_{dif2}(1) \geq 0$.

Because $u_{dif1}(v_i)$ and $u_{dif2}(v_i)$ are continuous and weakly increasing in v_i, so is $u_{dif}(v_i) = u_{dif1}(v_i) + u_{dif2}(v_i)$. And because $|u_{dif1}(\tilde{v}_i) - u_{dif1}(\hat{v}_i)| \leq 2|\tilde{v}_i - \hat{v}_i|$ and $|u_{dif2}(\tilde{v}_i) - u_{dif2}(\hat{v}_i)| \leq 2|\tilde{v}_i - \hat{v}_i|$, we have $|u_{dif}(\tilde{v}_i) - u_{dif}(\hat{v}_i)| \leq 4|\tilde{v}_i - \hat{v}_i|$. Finally, $u_{dif1}(0) \leq 0$ and $u_{dif2}(0) \leq 0$ imply that $u_{dif}(0) \leq 0$, and likewise $u_{dif1}(1) \geq 0$ and $u_{dif2}(1) \geq 0$ imply $u_{dif}(1) \geq 0$.■

Proposition 1: There exists an equilibrium in which voters use a symmetric cutpoint strategy in the first period.

Proof: Consider the correspondence

$$\phi(x_c) = \{v_i \in [0,1] : u_{dif}(v_i) = 0 \text{ when } N\backslash i \text{ use the symmetric cutpoint strategy specified by } x_c\}$$

Note that $\phi(x_c) : [0,1] \to [0,1]$ is nonempty for all $x_c \in [0,1]$, by parts 1 and 3 of Lemma 2 and the Intermediate Value Theorem. Also, since $u_{dif}(x)$ is continuous and weakly increasing, $\phi(x_c)$ is convex-valued. So, to apply Kakutani's fixed point theorem, and conclude that there exists an equilibrium, i.e., an $x_c^* \in \phi(x_c^*)$, all we need to do is to establish that $\phi(x_c)$ is upper hemi-continuous.

Consider a sequence of points $\{x_c^t\} \to \tilde{x}_c$ and a sequence $\{y^t\} \to \tilde{y}$ where $y^t \in \phi(x_c^t), \forall t$. We need to show that $\tilde{y} \in \phi(\tilde{x}_c)$.

For each t, following the definition of $u_{dif}(v_i)$ in Equation 1, let $u^t_{dif}(v_i)$ be the utility difference function given cutpoint x_c^t and let $\tilde{u}_{dif}(v_i)$ be the utility difference function given cutpoint $\tilde{x}_c$.

We first note that $\left\{u^t_{dif}(v_i)\right\}$ converges pointwise to $\tilde{u}_{dif}(v_i)$. The first part of the utility difference function is

$$u^t_{dif1}(v_i) = \binom{2m}{m} \left(F(x_c^t)\right)^m \left(1 - F(x_c^t)\right)^m (\gamma(|L - v_i|) - \gamma(|R - v_i|))$$

which converges pointwise to

$$\tilde{u}_{dif1}(v_i) = \binom{2m}{m} (F(\tilde{x}_c))^m (1 - F(\tilde{x}_c))^m (\gamma(|L - v_i|) - \gamma(|R - v_i|))$$

since $\{x_c^t\} \to \tilde{x}_c$. The second part is

$$\begin{aligned} u_{dif2}^t(v_i) &\equiv \sum_{k=0}^{2m} \binom{2m}{k} \left(F(x_c^t)\right)^{2m-k} \left(1 - F(x_c^t)\right)^k \\ &\quad \cdot \left(\gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k; x_c^t\right) - v_i\right|) - \gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k+1; x_c^t\right) - v_i\right|)\right) \end{aligned}$$

which (by part 5 of Lemma 1, continuity of $\gamma(\cdot)$, and the fact that $\{x_c^t\} \to \tilde{x}_c$) converges pointwise to

$$\begin{aligned} \tilde{u}_{dif2}(v_i) &\equiv \sum_{k=0}^{2m} \binom{2m}{k} (F(\tilde{x}_c))^{2m-k} (1 - F(\tilde{x}_c))^k \\ &\quad \cdot \left(\gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k; \tilde{x}_c\right) - v_i\right|) - \gamma(\left|F_{median}^{-1}\left(\frac{1}{2}|k+1; \tilde{x}_c\right) - v_i\right|)\right). \end{aligned}$$

Now we suppose that $\tilde{y} \notin \phi(\tilde{x}_c)$, and derive a contradiction. If $\tilde{y} \notin \phi(\tilde{x}_c)$ then either $\tilde{u}_{dif}(\tilde{y}) > 0$ or $\tilde{u}_{dif}(\tilde{y}) < 0$. Without loss of generality suppose the former. Then since $u_{dif}^t(v_i)$ converges pointwise to $\tilde{u}_{dif}(v_i)$ there exists T such that for all $t > T$, $u_{dif}^t(\tilde{y}) > \frac{\tilde{u}_{dif}(\tilde{y})}{2}$. By the Lipschitz property in part 2 of Lemma 2, for all $t > T$, $u_{dif}^t(\tilde{y}) - u_{dif}^t(\tilde{y} - \delta) \leq 4\delta$ for any $\delta > 0$. Setting $\delta = \frac{\tilde{u}_{dif}(\tilde{y})}{8}$ we have that for $t > T$, $u_{dif}^t(\tilde{y}) - u_{dif}^t(\tilde{y} - \delta) < \frac{\tilde{u}_{dif}(\tilde{y})}{2}$, so $u_{dif}^t(\tilde{y} - \delta) > u_{dif}^t(\tilde{y}) - \frac{\tilde{u}_{dif}(\tilde{y})}{2} > 0$. Thus, since $y^t \in \phi(x_c^t)$ or, equivalently, $u_{dif}^t(y^t) = 0$, and $u_{dif}^t(v_i)$ is weakly increasing in v_i, we conclude that $y^t < \tilde{y} - \delta$ for all $t > T$, which means that $\{y^t\}$ cannot converge to $\tilde{y}$, a contradiction.■

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<u>Figure 1</u>: Equilibrium in Example 2, Combined Model

First period candidates: L=1/2, R=1

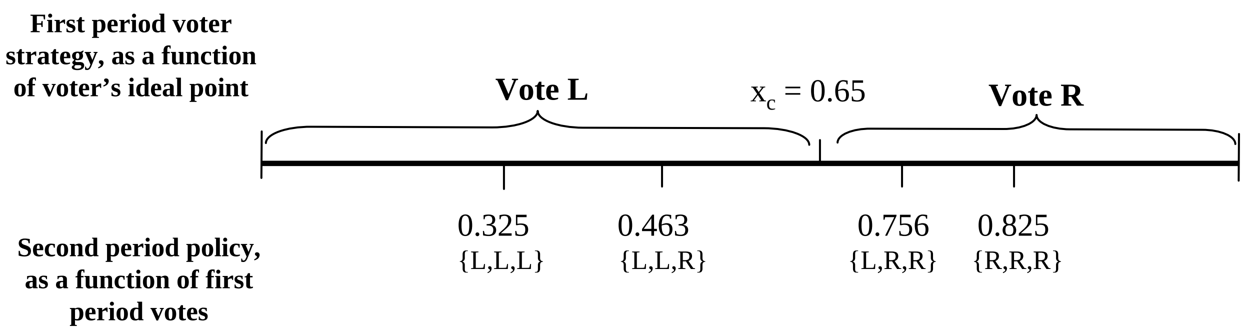
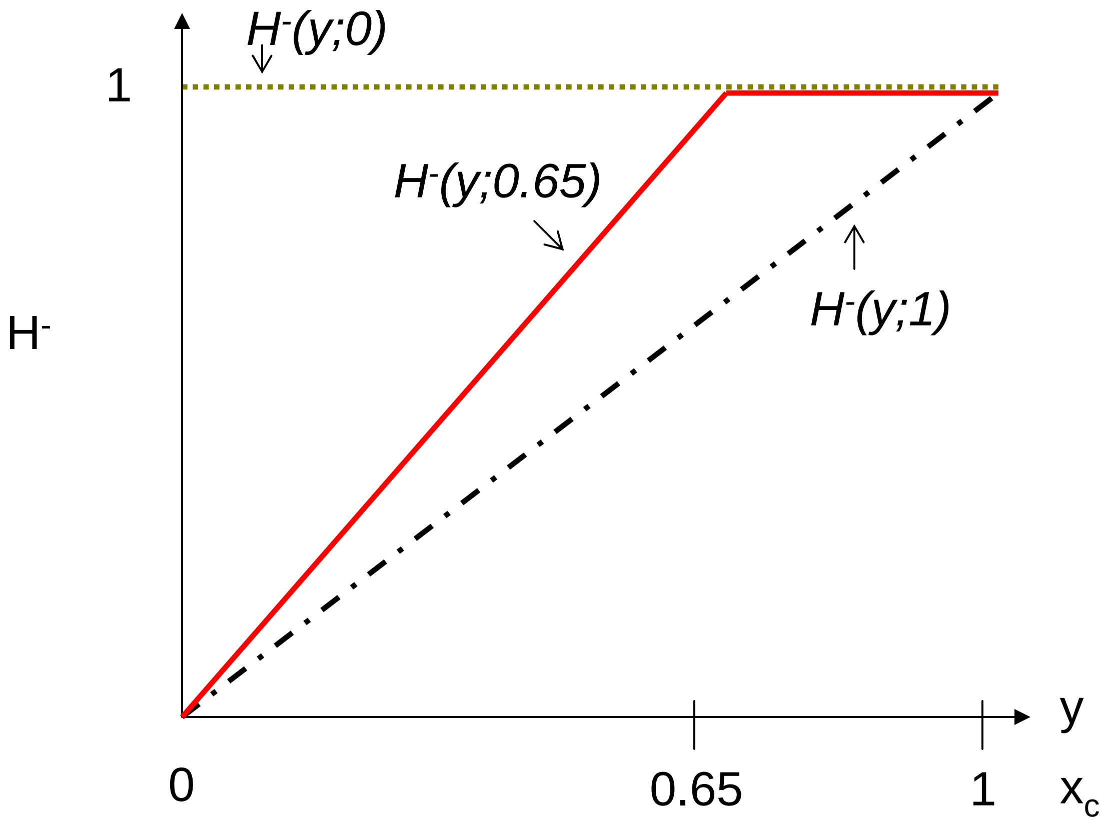
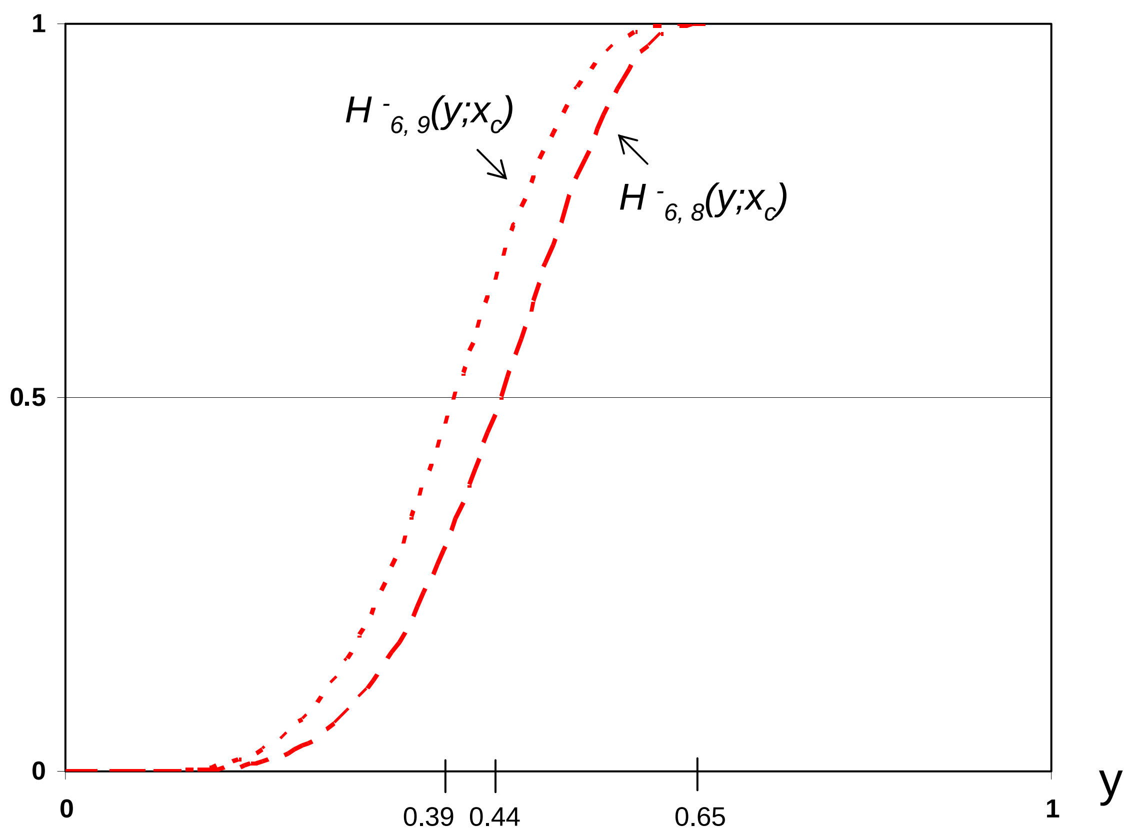


Figure 2



Examples of $H^{-}(y;x_c)$ for values of x_c in *{0,0.65,1}*

Figure 3



Examples of $H^{-}_{m+1,n-\#R}(y;x_c)$ for $x_c = 0.65$

N=11, m+1=6, and #R is either 2 (for the left dashed line) or 3 (for the right dashed line)

For $\#R = 3$, $F^{-1}_{median}(1/2|\#R;x_c) = 0.39$

For $\#R = 4$, $F^{-1}_{median}(1/2|\#R;x_c) = 0.44$