A Robust and Optimal Anonymous Procedure for Condorcet's Model*

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Abstract

In Condorcet's model of information aggregation, a group of people decides among two alternatives, with each person getting an independent bit of evidence about which alternative is objectively superior. I define the "supermajority penalty" (SP) procedure and show that it is incentive compatible for all possible preferences and prior beliefs and is in this sense completely robust. I also show that for an unbiased person, the SP procedure is the optimal anonymous incentive compatible procedure when there are significant biases in both directions (when at least one person is biased toward one alternative and at least one person is biased toward the other). The SP procedure is not monotonic, but this is not unusual: I show that when there are significant biases in both directions, all nontrivial anonymous incentive compatible procedures are non-monotonic.

Keywords: Condorcet jury theorem, information aggregation, mechanism design, anonymous mechanisms

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1. *Introduction*

People often make important decisions collectively because they believe that by combining their judgement, they are more likely to make the correct decision. Examples include juries who collectively decide whether to find a person guilty or innocent, journal editors and referees who collectively decide whether to accept a manuscript, and consulting doctors who collectively decide how to treat a patient. In these situations, people's prior beliefs or inclinations might differ, but their interests do not fundamentally conflict; if it were known for certain whether a person was guilty or innocent, or what the best treatment for a patient would be, then all would agree.

A mathematical representation of this kind of collective decision-making was introduced by Condorcet (1785). In his model, there are a group of people who have to decide among two alternatives a and b, and each gets an independent bit of evidence about which alternative is objectively best. Each person's evidence either favors a or b, and each person's evidence is assumed to be correct with probability q, which is the same for everyone. Condorcet showed that the collective procedure which is most likely to yield the best alternative is majority rule: each person reports their evidence, and the alternative which gets the most reports is chosen. If we think of democratic elections as a way of divining a group's collective wisdom, then Condorcet's result can be understood as an argument for majority rule voting.

Starting from Condorcet's original model, a recent line of research has considered whether people want to report their evidence truthfully. This is a practical consideration; for example, in Olympic diving competitions, if the US judge always scores US divers highly regardless of their true performance, the Russian judge might respond by always scoring non-US divers highly regardless of their true performance, thereby degrading the quality of the collective decision for everyone. In Olympic diving, the highest and lowest scores of a panel of judges are discarded and only the remaining scores are averaged to obtain a diver's final score, thereby limiting the ability of a single judge to influence the final score. A procedure is called "incentive compatible" if each person wants to report her information truthfully given the procedure and given that everyone else is reporting truthfully.

If everyone is identical, strategic considerations do not arise (McLennan 1998): no individual wants to deviate from a socially optimal procedure because socially optimal is synonymous with individually optimal. The model becomes nontrivial, and more realistic, when people have different "biases." People might have different prior beliefs over which alternative is best. People might also have different preferences: for example, choosing b when we should have chosen a might be not so bad for you but disastrous for me, and thus I might require a higher "burden of proof" for choosing b. Given people's biases, and given a voting procedure such as majority rule or unanimity rule, we can model the situation as a game in which each person chooses what to report given her evidence.

The usual approach in this line of research on Condorcet's model is to find Nash equilibria of this game, in other words to predict voting behavior given a particular voting procedure. This paper takes the mechanism design approach and asks instead which is the best procedure. This paper restricts itself to anonymous procedures in which the only thing which affects the group's choice of a or b is the total number of people who report a, not who these people are. We also consider only equilibria in which everyone reports their evidence truthfully, and hence everyone's evidence affects the decision equally. Equality is a well-known argument for anonymous procedures (Riker 1982). Another argument is in terms of simplicity: in a group of n people, since each person's evidence favors either a or b, a general procedure must consider 2^n cases, while an anonymous procedure considers only $n+1$ cases, since the total number of people whose evidence favors a ranges from 0 to n.

This paper defines the supermajority penalty (SP) procedure and shows that it is incentive compatible for all possible biases. In this sense, it is completely robust: any individual would report truthfully given this procedure. The SP procedure is anonymous and when there are significant biases in both directions, the SP procedure is in fact the optimal anonymous incentive compatible procedure. Here "optimal" means maximizing the utility of a completely unbiased person, whose priors or preferences do not ex ante favor either alternative, and "both directions" means that at least one person is biased toward a and at least one toward b.

In other words, one can approach Condorcet's situation with no knowledge of the priors and preferences of the people involved and be confident in the SP procedure. Assuming that there is at least one person significantly biased in each direction, from the point of view of an unbiased person, the SP procedure is the optimal anonymous procedure. Even if this assumption is not true, the SP procedure is incentive compatible and still "works."

The SP procedure is called the "supermajority penalty" procedure because if a weak majority of people report a, then the procedure chooses a, but if too many people, a supermajority, report a, then the other alternative b is chosen with some probability, thereby "penalizing" the supermajority. The SP procedure is not monotonic: when the number of reports for an alternative increases, the probability of choosing that alternative sometimes decreases. This non-monotonicity is not intuitive, but the SP procedure is not special in this regard. This paper shows that when there is at least one person significantly biased in each direction, all nontrivial incentive compatible anonymous procedures are non-monotonic. In other words, anonymity and incentive compatibility alone imply non-monotonicity.

2. The model

Condorcet's model has been presented in many papers (here we follow the notation in Chwe 1999). There is a group of people $N = \{1, 2, \ldots, n\}$, where n is odd and at least 3. The group chooses between alternatives a or b. Each person receives private evidence on whether a or b is objectively superior. Each person $i \in N$ has a prior belief that a is superior with probability $\pi_i(a)$ and b is superior with probability $\pi_i(b)$, where $\pi_i(a), \pi_i(b) \in [0, 1]$ and $\pi_i(a) + \pi_i(b) = 1$. Each person's private evidence is correct with probability $q \in (1/2, 1)$. Let $g(d, e)$ be the probability that a person's private evidence supports e when the superior alternative truly is d; thus we have $g(d, e) = q$ if $d = e$ and $g(d, e) = 1 - q$ if $d \neq e$.

Each person reports her private evidence to the procedure, which outputs the group's decision. The decision procedure is thus a function $f : \{a, b\}^n \times \{a, b\} \to [0, 1]$, where $f(r_1, \ldots, r_n, a)$ is the probability of choosing a and $f(r_1, \ldots, r_n, b)$ is the probability of choosing b given the reports $r_1, \ldots, r_n \in \{a, b\}$, and of course $f(r_1, \ldots, r_n, a) + f(r_1, \ldots, r_n, b) = 1$. After the decision is made, each person gets utility $u_i(d, c)$ when the superior alternative truly is d and the alternative chosen is c. We assume that $u_i(a, a) > u_i(a, b)$ and $u_i(b, b) > u_i(b, a)$; in other words, everyone prefers the superior alternative.

Each person's strategy is a choice of what to report given his evidence, a function $s_i : \{a, b\} \to \{a, b\}$. The identity function id, where $id(a) = a$ and $id(b) = b$, is the strategy in which a person always reports his evidence truthfully. Let s_{aa} be the strategy of reporting a all the time ($s_{aa}(a) = s_{aa}(b) = a$) and let s_{bb} be the strategy of reporting b all the time ($s_{bb}(a) = s_{bb}(b) = b$).

Given the procedure f and strategies $s_1, \ldots, s_n$, the probability that the group chooses alternative c given that the superior alternative truly is d is $p_{dc}(f, s_1, \ldots, s_n) = \sum_{(e_1, \ldots, e_n) \in \{a,b\}^n} g(d, e_1) \cdots g(d, e_n) f(s_1(e_1), \ldots, s_n(e_n), c)$. Here $e_1, \ldots, e_n$ is the evidence, $s_1(e_1), \ldots, s_n(e_n)$ are the reports given this evidence, and $f(s_1(e_1), \ldots, s_n(e_n), c)$ is the probability of choosing c given the reports. Hence, given strategies $s_1, \ldots, s_n$, and the procedure f, person i's expected utility is $EU_i(f, s_1, \ldots, s_n) = \pi_i(a)p_{aa}(f, s_1, \ldots, s_n)u_i(a, a) + \pi_i(a)p_{ab}(f, s_1, \ldots, s_n)u_i(a, b) + \pi_i(b)p_{ba}(f, s_1, \ldots, s_n)u_i(b, a) + \pi_i(b)p_{bb}(f, s_1, \ldots, s_n)u_i(b, b)$.

Because we know $p_{ab}(f, s_1, \ldots, s_n) = 1 - p_{aa}(f, s_1, \ldots, s_n)$ and $p_{ba}(f, s_1, \ldots, s_n) = 1 - p_{bb}(f, s_1, \ldots, s_n)$, we have $EU_i(f, s_1, \ldots, s_n) = \pi_i(a)u_i(a, b) + \pi_i(b)u_i(b, a) + \pi_i(a)(u_i(a, a) - u_i(a, b))p_{aa}(f, s_1, \ldots, s_n) + \pi_i(b)(u_i(b, b) - u_i(b, a))p_{bb}(f, s_1, \ldots, s_n)$. The first two terms here are constants and can be dropped. We can then normalize and write

$$EU_i(f, s_1, \ldots, s_n) = \phi_i(a)p_{aa}(f, s_1, \ldots, s_n) + \phi_i(b)p_{bb}(f, s_1, \ldots, s_n) \qquad (*)$$

where $\phi_i(a), \phi_i(b)$ are defined by

$$\phi_i(a) = \frac{\pi_i(a)(u_i(a, a) - u_i(a, b))}{\pi_i(a)(u_i(a, a) - u_i(a, b)) + \pi_i(b)(u_i(b, b) - u_i(b, a))}$$

$$\phi_i(b) = \frac{\pi_i(b)(u_i(b, b) - u_i(b, a))}{\pi_i(a)(u_i(a, a) - u_i(a, b)) + \pi_i(b)(u_i(b, b) - u_i(b, a))}.$$

Note that $\phi_i(a), \phi_i(b) \in [0, 1]$ and $\phi_i(a) + \phi_i(b) = 1$.

The parameters $\phi_i(a), \phi_i(b)$ represent the "bias" of person i. If everyone has the same prior belief $\pi_i(a) = \pi_i(b) = 1/2$, then $\phi_i(a), \phi_i(b)$ correspond to the relative magnitudes of $u_i(a, a) - u_i(a, b)$ and $u_i(b, b) - u_i(b, a)$. For example, if $\pi_i(a) = \pi_i(b) = 1/2$ and $u_i(a, a) = 2$, $u_i(a, b) = 0$, $u_i(b, a) = 0$, $u_i(b, b) = 1$, then $\phi_i(a) = 2/3$ and $\phi_i(b) = 1/3$; person i is biased toward a because her payoff from choosing a correctly is twice that of choosing b correctly. If one has the "standard" utility function $u_i(a, a) = 1, u_i(a, b) = 0, u_i(b, a) = 0, u_i(b, b) = 1$, then $\phi_i(a) = \pi_i(a)$ and $\phi_i(b) = \pi_i(b)$; the bias ϕ_i is simply the prior belief π_i. If $\phi_i(a) > \phi_i(b)$, either because of u_i or π_i, then we say that person i is "biased toward" a. If a person has bias $\phi_i(a) = \phi_i(b) = 1/2$, we call that person unbiased.

The sum of everyone's utility is simply

$$\sum_{i \in N} EU_i(f, s_1, \ldots, s_n) = \Big[\sum_{i \in N} \phi_i(a)\Big]p_{aa}(f, s_1, \ldots, s_n) + \Big[\sum_{i \in N} \phi_i(b)\Big]p_{bb}(f, s_1, \ldots, s_n).$$

Thus utility averaged over the group is equal to the utility of a person who has an average bias. If average bias in a group is $\phi_i(a) = \phi_i(b) = 1/2$, then maximizing the sum of everyone's utility is the same as maximizing the utility of an unbiased person.

We say that the procedure f is incentive compatible for person i if $EU_i(f, id, \ldots, id) \geq EU_i(f, id, \ldots, s_{aa}, \ldots, id)$ and $EU_i(f, id, \ldots, id) \geq EU_i(f, id, \ldots, s_{bb}, \ldots, id)$. In other words, reporting truthfully is at least as good as reporting a always or reporting b always. We do not have to consider the "always lie" strategy s_{ba}, defined by $s_{ba}(a) = b$ and $s_{ba}(b) = a$, because it is easy to show that if one does not gain by deviating from id to either s_{aa} or s_{bb}, then one does not gain by deviating to s_{ba}. If one does not gain by misreporting when one's evidence is for b and one does not gain by misreporting when one's evidence is for a, then one cannot gain by always misreporting. We say the procedure f is incentive compatible if it is incentive compatible for all $i \in N$; in other words, $(id, \ldots, id)$ is a Nash equilibrium.

For $r \in \{a, b\}^n$, define $\alpha(r) = \#\{i \in N : r_i = a\}$; in other words, given the vector of reports r, $\alpha(r)$ is the number of people who report a. We say that the procedure f is anonymous if $\alpha(r) = \alpha(r') \Rightarrow f(r, a) = f(r', a)$. In other words, an anonymous procedure depends only on the number of people who report a or b, not their identities. An anonymous procedure f can be represented by the numbers $\gamma(0), \gamma(1), \ldots \gamma(n)$, where $\gamma(\alpha(r)) = f(r, a)$; in other words, $\gamma(j)$ is the probability that a is chosen given that there are j reports of a.

We say that a procedure f is symmetric if $f(r, a) = f(r', b)$ for all r, r' such that $r_i = a \Leftrightarrow r'_i = b$. In other words, if r, r' are exact "opposites" in that each person's report in r is the opposite of her report in r', then the probability of choosing a given r is the same as the probability of choosing b given r'. If a procedure f is both anonymous and symmetric, then the probability of choosing a given 3 reports of a is the same regardless of who reports a, and is equal to the probability of choosing b given 3 reports of b.

3. Results

Given q, the probability that an individual's evidence is correct, and given n, the number of people in the group, we define the SP procedure f_{SP}. Lemma 1 provides two numbers k and z, which are parameters of f_{SP} (all proofs are in the appendix).

Lemma 1. Given q and n, there uniquely exists $k \in \{0, 1, \ldots, (n-3)/2\}$ such that

$$z = \frac{D_k y^{(n-1)/2} - k(y^{k-1} + y^{n-k})}{(n-k)(y^k + y^{n-1-k}) - k(y^{k-1} + y^{n-k})} \in (0, 1]$$

where $y = q/(1-q)$ and $D_k = (k!(n-k)!)/(((n-1)/2)!)^2$. Also, k is nonincreasing in q, $k < (1-q)n$, and $k/n \to \rho$ as $n \to \infty$, where $\rho \, log \, \rho + (1-\rho) \, log(1-\rho) + log \, 2 = (1/2 - \rho) \, log \, y$.

Definition. Given q and n, define the SP procedure f_{SP} as

$$f_{SP}(r,a) = \begin{cases} 1 & \text{if } \alpha(r) < k \\ z & \text{if } \alpha(r) = k \\ 0 & \text{if } k < \alpha(r) \leq (n-1)/2 \\ 1 & \text{if } (n+1)/2 \leq \alpha(r) < n-k \\ 1-z & \text{if } \alpha(r) = n-k \\ 0 & \text{if } \alpha(r) > n-k \end{cases}$$

where $\alpha(r) = \#\{i \in N : r_i = a\}$, $k \in \{0, 1, \ldots, (n-3)/2\}$ and $z \in (0,1]$ are defined as in Lemma 1, and $f_{SP}(r,b) = 1 - f_{SP}(r,a)$.

In other words, if a weak majority of the reports are for a, then a is chosen with probability 1. However, if the number of reports for a is greater than $n-k$, then this supermajority is "penalized" and a is chosen with probability 0. If the number of reports for a is equal to $n-k$, then a is chosen with probability $1-z$. For example, when $n = 7$ and $q = 2/3$, Lemma 1 gives us $k = 1$ and $z = 95/139$, and the SP procedure f_{SP} is shown in Figure 1.

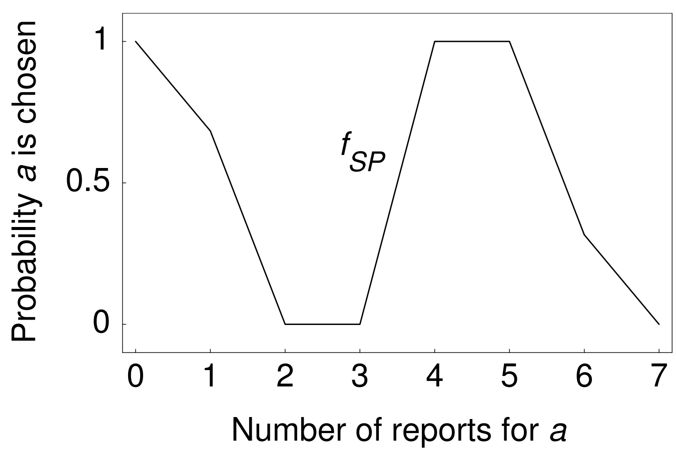

Figure 1. f_{SP} when $n = 7$ and $q = 2/3$ (thus $k = 1$ and $z = 95/139$)

We have two main propositions. Proposition 1 shows the robustness of the SP procedure. Proposition 2 shows its optimality when there is at least one person biased in each direction.

Proposition 1. The procedure f_{SP} is incentive compatible for any $\phi_1, \ldots, \phi_n$. In fact, the incentive compatibility constraints hold with equality.

Proposition 2. Say that there exist $i, j \in N$ such that $\phi_i(a) < 1-q$ and $\phi_j(a) > q$. Then f_{SP} is an anonymous incentive compatible procedure which maximizes EU_0, where

$\phi_0(a) = \phi_0(b) = 1/2$. Also, f_{SP} is the unique maximum for almost all q (all but fewer than $(n-1)/2$ values of q).

What does it mean for a person to have bias outside the interval $[1-q, q]$? If a person's bias is inside this interval, then it is easy to verify that she reports truthfully under the majority rule procedure given that everyone else reports truthfully. So if $\phi_i(a) \in [1-q, q]$ for all $i \in N$, then the optimal anonymous incentive compatible procedure, from the point of view of an unbiased person, is majority rule. But if a person's bias is outside the interval, then majority rule is no longer incentive compatible. If some people have $\phi_i(a)$ slightly greater than q and everyone else has $\phi_j(a) \in [1-q, q]$, then the optimal anonymous incentive compatible procedure allows a to be chosen with some probability when it gets one report short of a majority (Chwe 1999). If one person has $\phi_i(a) < 1-q$ and another has $\phi_j(a) > q$, then Proposition 2 applies and the optimal anonymous incentive compatible procedure is the SP procedure. The SP procedure is the unique optimum for almost all $q \in (1/2, 1)$; there are at most $(n-1)/2$ values of q in the interval $(1/2, 1)$ for which I have not proved uniqueness. I believe that the SP procedure is the unique maximum for all q, but I have not yet proved this slightly stronger statement.

To understand why the SP procedure is incentive compatible for any bias, Lemma 2 is helpful. Say that f is anonymous, and thus can represented by $\gamma(j)$, the probability of choosing a given j reports of a. If f is symmetric as well, and at least one person is significantly biased, it turns out that all of the incentive compatibility constraints boil down to a single equality constraint, which does not depend on ϕ. In other words, as long as someone is significantly biased, people's exact biases do not affect the set of incentive compatible anonymous symmetric procedures. Roughly speaking, f_{SP} is incentive compatible for any belief ϕ because incentive compatibility does not depend on the precise value of ϕ.

Lemma 2. Say that there exists $i \in N$ such that either $\phi_i(a) > q$ or $\phi_i(b) > q$, and say that f is anonymous and symmetric. Then f is incentive compatible if and only if $W(q, \gamma) = 0$, where $\gamma(\alpha(r)) = f(r, a)$ and $W(q, \gamma) = C^{n-1}_{(n-1)/2} q^{(n-1)/2}(1-q)^{(n-1)/2}(2\gamma((n-1)/2) - 1) + \sum_{j=0}^{(n-3)/2} C^{n-1}_j (q^j(1-q)^{n-1-j} + q^{n-1-j}(1-q)^j)(\gamma(j) - \gamma(j+1))$, where C^n_j is the binomial coefficient $C^n_j = n!/((n-j)!j!)$.

Proposition 1 says that the SP procedure satisfies all incentive compatibility constraints with equality: a person does not gain by misreporting, but also does not lose. The SP procedure might seem unusual in this respect. However, Lemma 3 shows that when there is at least one person biased in each direction, incentive compatibility constraints hold with equality for any anonymous incentive compatible procedure, including procedures which are not symmetric. In other words, if this is a concern, it applies to all anonymous incentive compatible procedures, not just the SP procedure.

Lemma 3. Say that there exist $i, j \in N$ such that $\phi_i(a) < 1 - q$ and $\phi_j(a) > q$. If f is anonymous and incentive compatible for persons i and j, then person l's incentive compatibility constraints hold with equality for any ϕ_l.

These lemmas allow us to prove Proposition 3. We say that a procedure f is trivial if there exists a constant $\kappa \in [0, 1]$ such that $f(r, a) = \kappa$ for all $r \in \{a, b\}^n$. In other words, a trivial procedure chooses a with the same probability regardless of the reports. We say that a procedure f is monotonic if $\alpha(r) \leq \alpha(r') \Rightarrow f(r, a) \leq f(r', a)$, where $\alpha(r) = \#\{i \in N : r_i = a\}$. In other words, a procedure is monotonic if the probability it chooses a weakly increases in the number of reports for a. Proposition 3 says that when there are biases in both directions, a nontrivial anonymous incentive compatible procedure cannot be monotonic.

Proposition 3. Say that there exist $i, j \in N$ such that $\phi_i(a) < 1 - q$ and $\phi_j(a) > q$. Say f is an anonymous incentive compatible procedure which is not trivial. Then f is not monotonic.

In other words, when there are biases in both directions, in any nontrivial anonymous incentive compatible procedure, there is at least one scenario in which an additional report for a makes the probability that a is chosen strictly decrease. The non-monotonicity of the SP procedure is nothing special. Non-monotonicity results directly from anonymity and incentive compatibility and does not result from any particular procedure (for a similar result in a different context, see Li, Rosen, and Suen 2001).

4. *Example*

An example is helpful in explaining the results and how they are proved. Say that $n = 3$ and $q = 2/3$. Say we have biases $\phi_1(a) = 1/4$, $\phi_2(a) = 4/7$, and $\phi_3(a) = 10/11$. Note that $\phi_1(a) < 1 - q$ and $\phi_3(a) > q$, and of course $\phi_i(b) = 1 - \phi_i(a)$. Say we have an anonymous procedure γ, which we write as four numbers $\gamma(0), \gamma(1), \gamma(2), \gamma(3)$, where $\gamma(j) \in [0, 1]$ is the probability that the group chooses a given j reports for a.

Remember that a person's expected utility is given by the formula $(*)$ earlier. The first incentive compatibility constraint is that whenever a person receives evidence for b, she should rather report b than report a. Given that she receives evidence for b, her expected utility from reporting b is

$$\phi(a)(\tfrac{1}{3})\Big[(\tfrac{2}{3})^2\gamma(2) + 2(\tfrac{2}{3})(\tfrac{1}{3})\gamma(1) + (\tfrac{1}{3})^2\gamma(0)\Big]$$

$$+\phi(b)(\tfrac{2}{3})\Big[(\tfrac{1}{3})^2(1-\gamma(2)) + 2(\tfrac{1}{3})(\tfrac{2}{3})(1-\gamma(1)) + (\tfrac{2}{3})^2(1-\gamma(0))\Big].$$

The first line is her bias $\phi(a)$ times the probability that a is chosen when a is truly the superior alternative. When a is truly the superior alternative, the probability that she gets evidence b is $\frac{1}{3}$, and we have three cases. In the first case, two other people get evidence a, which happens with probability $(\frac{2}{3})^2$, and the procedure chooses a with probability $\gamma(2)$, and so forth. The second line is her bias $\phi(b)$ times the probability that b is chosen when b truly is the superior alternative. When b is truly the superior alternative, the probability that she gets evidence b is $\frac{2}{3}$, and we similarly have three cases.

If when she receives evidence for b she reports a instead, her expected utility becomes

$$\phi(a)(\tfrac{1}{3})\Big[(\tfrac{2}{3})^2\gamma(3) + 2(\tfrac{2}{3})(\tfrac{1}{3})\gamma(2) + (\tfrac{1}{3})^2\gamma(1)\Big]$$

$$+\phi(b)(\tfrac{2}{3})\Big[(\tfrac{1}{3})^2(1-\gamma(3)) + 2(\tfrac{1}{3})(\tfrac{2}{3})(1-\gamma(2)) + (\tfrac{2}{3})^2(1-\gamma(1))\Big],$$

which is the same as before except that all the arguments of γ increase by 1. By reporting a instead of b, she increases the number of a reports by 1.

So the incentive compatibility constraint is simply that the first expression is greater than or equal to the second; given that $\phi(b) = 1 - \phi(a)$, this inequality is

$$[\tfrac{9}{27}\phi(a) - \tfrac{8}{27}]\gamma(0) + \tfrac{3}{27}\phi(a)\gamma(1) + \tfrac{6}{27}(1-\phi(a))\gamma(2) + [\tfrac{2}{27} - \tfrac{6}{27}\phi(a)]\gamma(3) \geq 0.$$

The second incentive compatibility constraint is that whenever a person receives evidence for a, he should report a rather than b. In a similar manner, we find this constraint to be

$$[\tfrac{6}{27}\phi(b) - \tfrac{2}{27}]\gamma(0) - \tfrac{6}{27}(1-\phi(b))\gamma(1) - \tfrac{3}{27}\phi(b)\gamma(2) + [\tfrac{8}{27} - \tfrac{9}{27}\phi(b)]\gamma(3) \geq 0.$$

We call these two constraints our IC constraints (for "incentive compatibility"). A procedure γ which satisfies the two IC constraints is incentive compatible for a person with bias ϕ.

Assume for a moment (this will be justified later) that γ is symmetric: $\gamma(3) = 1 - \gamma(0)$ and $\gamma(2) = 1 - \gamma(1)$. In other words, the probability that a is chosen given 2 reports of a is the same as the probability that b is chosen given 2 reports of b. We can then simplify to get

$$[\tfrac{5}{9}\phi(a) - \tfrac{10}{27}]\gamma(0) + [\tfrac{1}{3}\phi(a) - \tfrac{10}{27}]\gamma(1) + [-\tfrac{4}{9}\phi(a) + \tfrac{8}{27}] \geq 0$$
$$[\tfrac{5}{9}\phi(b) - \tfrac{10}{27}]\gamma(0) + [\tfrac{1}{3}\phi(b) - \tfrac{10}{27}]\gamma(1) + [-\tfrac{4}{9}\phi(b) + \tfrac{8}{27}] \geq 0.$$

Now we can factor out the term $\phi(a) - \frac{2}{3}$ in the first constraint and the term $\phi(b) - \frac{2}{3}$ in the second constraint and get

$$[\phi(a) - \tfrac{2}{3}](\tfrac{5}{9}\gamma(0) + \tfrac{1}{3}\gamma(1) - \tfrac{4}{9}) \geq 0$$
$$[\phi(b) - \tfrac{2}{3}](\tfrac{5}{9}\gamma(0) + \tfrac{1}{3}\gamma(1) - \tfrac{4}{9}) \geq 0.$$

Note that if $\phi(a), \phi(b) < 2/3$, then the IC constraints boil down to a single inequality: $\frac{5}{9}\gamma(0) + \frac{1}{3}\gamma(1) - \frac{4}{9} \leq 0$. It is easy to verify that majority rule ($\gamma(0) = \gamma(1) = 0$) satisfies this inequality and thus is incentive compatible when a person is not too biased. But when $\phi(a) > 2/3$, the terms $\phi(a) - \frac{2}{3}$ and $\phi(b) - \frac{2}{3}$ have different signs, and similarly when $\phi(b) > 2/3$. Then the IC constraints boil down to a single equality, $\frac{5}{9}\gamma(0) + \frac{1}{3}\gamma(1) - \frac{4}{9} = 0$ (this is Lemma 2). It is easy to see that majority rule does not satisfy this equality. It is also easy to see that if γ satisfies the IC constraints for one particular $\phi(a) > 2/3$ or $\phi(b) > 2/3$, then γ satisfies the IC constraints with equality for any ϕ (this is Lemma 3).

Remember that in our example, we have $\phi_1(a) = 1/4$, $\phi_2(a) = 4/7$, and $\phi_3(a) = 10/11$. An incentive compatible procedure γ must satisfy six inequalities: the two IC inequalities for each of the three values of $\phi(a) = 1/4, 4/7, 10/11$. But since $\phi_1(b) = 3/4 > 2/3$, these six inequalities all reduce down to a single equality, $\frac{5}{9}\gamma(0) + \frac{1}{3}\gamma(1) - \frac{4}{9} = 0$.

Now consider the objective function. Using the formula ($*$), and assuming again that γ is symmetric, an unbiased person with $\phi(a) = \phi(b) = 1/2$ has expected utility

$-\frac{7}{27}\gamma(0) - \frac{2}{9}\gamma(1) + \frac{20}{27}$. Therefore, the incentive compatible procedure which maximizes the expected utility of an unbiased person is the solution to the following constrained optimization problem:

$$\text{Maximize } -\tfrac{7}{27}\gamma(0) - \tfrac{2}{9}\gamma(1) \text{ such that } \tfrac{5}{9}\gamma(0) + \tfrac{1}{3}\gamma(1) - \tfrac{4}{9} = 0$$

where $\gamma(0), \gamma(1) \in [0, 1]$.

To maximize the objective, we would like to set $\gamma(0) = \gamma(1) = 0$, but this would not satisfy the constraint. At least one of the variables $\gamma(0), \gamma(1)$ must be made positive. Which one? It depends on how much a variable hurts the objective relative to how much it helps satisfy the constraint. If we look at the ratios of the coefficients on $\gamma(0)$ and $\gamma(1)$, we find that the ratio for $\gamma(0)$ is $(-\frac{7}{27})/(\frac{5}{9}) = -\frac{7}{15}$ and the ratio for $\gamma(1)$ is $(-\frac{2}{9})/(\frac{1}{3}) = -\frac{2}{3}$. Since the ratio for $\gamma(0)$ is less negative, making $\gamma(0)$ positive is the better "deal," and the maximum is obtained at $\gamma(0) = 4/5$ and $\gamma(1) = 0$.

This is the proof of Proposition 2 in a nutshell. Proposition 1, that γ is incentive compatible for any ϕ, is clear from our observation above that if $\frac{5}{9}\gamma(0) + \frac{1}{3}\gamma(1) - \frac{4}{9} = 0$, then γ satisfies the IC inequalities for any ϕ. To prove Proposition 3, that any incentive compatible procedure is not monotonic, simply write $\frac{5}{9}\gamma(0) + \frac{1}{3}\gamma(1) - \frac{4}{9} = \frac{4}{9}(2\gamma(1) - 1) + \frac{5}{9}(\gamma(0) - \gamma(1)) = \frac{4}{9}(\gamma(1) - \gamma(2)) + \frac{5}{9}(\gamma(0) - \gamma(1))$. Since this expression equals zero, the terms $\gamma(1) - \gamma(2)$ and $\gamma(0) - \gamma(1)$ cannot both be negative.

The only thing remaining is to justify the assumption that γ is symmetric. The argument, roughly speaking, proceeds like this. Say that γ is no longer assumed to be symmetric. In our example, $\phi(a)$ takes on the values $1/4, 4/7, 10/11$. Note that the IC constraints are linear in $\phi(a)$. Hence if γ satisfies the IC constraints for $\phi(a) = 1/4$ and for $\phi(a) = 10/11$, then they satisfy the IC constraints for all convex combinations of $1/4$ and $10/11$, or in other words all $\phi(a) \in [1/4, 10/11]$. Note that since $3/4 \in [1/4, 10/11]$, any incentive compatible γ satisfies the IC constraints for $\phi(a) = 1/4$ and $\phi(a) = 3/4$.

Define a new procedure γ'' as $\gamma''(j) = 1 - \gamma(3 - j)$. In other words, the probability that γ'' chooses a given 2 reports of a is the same as the probability that γ chooses b given 2 reports of b. It is not difficult to show, either through a bit of computation (see Fact 7 in the appendix) or by simply exchanging the names of a and b, that if γ satisfies the IC constraints for $\phi(a) = r$, then γ'' satisfies the IC constraints for $\phi(a) = 1 - r$. Since γ satisfies the IC constraints for $\phi(a) = 1/4$ and $\phi(a) = 3/4$, we know that γ'' satisfies the IC constraints for

$\phi(a) = 3/4$ and $\phi(a) = 1/4$. So both γ and γ'' satisfy the IC constraints for $\phi(a) = 1/4$ and $\phi(a) = 3/4$. Since the IC constraints are linear in γ, if we define another procedure γ' as $\gamma'(j) = (\gamma(j) + \gamma''(j))/2$, we know that γ' satisfies the IC constraints for $\phi(a) = 1/4$ and $\phi(a) = 3/4$.

Note that the procedure γ' is symmetric. We can think of γ' as a "symmetrized" version of γ. Since γ' is symmetric and satisfies the IC constraints for $\phi(a) = 3/4 > 2/3$, by our earlier argument, γ' satisfies the IC constraints for all $\phi(a)$ and hence is incentive compatible. So any incentive compatible γ has a symmetric version γ' which is incentive compatible.

Remember that our objective function is the expected utility of an unbiased person, with $\phi(a) = 1/2$. It is easy to show that an unbiased person gets the same expected utility from γ and γ'', and thus by convexity gets the same expected utility from γ'. We can thus safely restrict ourselves to symmetric γ, because for any incentive compatible γ which is not symmetric, there is a symmetric version γ' which yields the same expected utility. Hence we are done.

To summarize, what makes things simple is that for anonymous and symmetric procedures, the terms $\phi(a) - q$ and $\phi(b) - q$ always factor out of the IC constraints, as illustrated in our example. Hence whether a procedure is incentive compatible depends in a very crude way on $\phi(a)$: for all biases, incentive compatibility boils down to the sign of $W(q, \gamma)$, which is defined in Lemma 2. To understand intuitively "why" the SP procedure has the shape that it does in general, one simply looks at the ratio of the coefficient of $\gamma(j)$ in the objective function EU_0 to the coefficient of $\gamma(j)$ in the constraint $W(q, \gamma)$. This ratio always decreases (becomes more negative) in j, as illustrated in our example: the ratio for $\gamma(0)$ is $-\frac{7}{15}$ and the ratio for $\gamma(1)$ is $-\frac{2}{3}$. Making $\gamma(0)$ positive is always the best "deal," and when this is maxed out at $\gamma(0) = 1$, we resort to the next best deal, $\gamma(1)$, until this is maxed out, and so forth.

5. Comparative statics

The SP procedure depends only on the parameters q and n. How does the SP procedure change as these parameters change? Lemma 1 says that k is nonincreasing in q: as q increases, k stays the same or decreases. Figure 2 shows this for $n = 9$ and q going from 0.501 to 0.8. As q increases, each person's evidence gets stronger and people's biases become relatively less important; thus the "distortion" in the procedure necessary to deal with the biases lessens, and the supermajority penalty regions shrink.

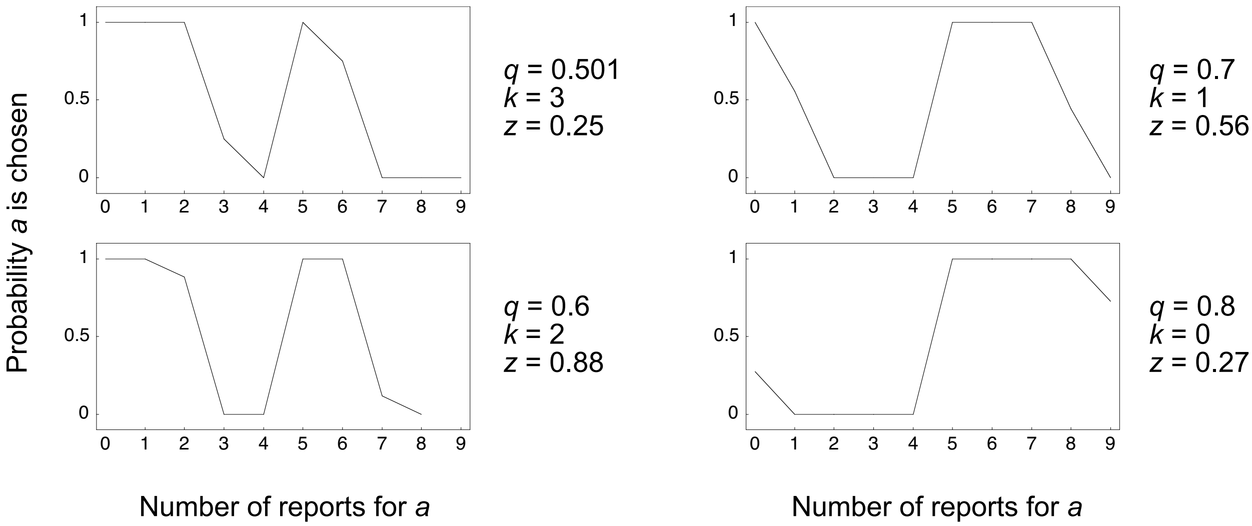

Figure 2. f_{SP} when $n = 9$ and $q = 0.501, 0.6, 0.7, 0.8$

As q approaches 1, we know $k = 0$ because Lemma 1 says that $k < (1-q)n$, and it is also easy to see from the definition of z in Lemma 1 that as q approaches 1, z approaches 0. Hence as q approaches 1, the SP procedure approaches majority rule and its welfare performance approaches that of majority rule. Figure 3 shows EU_0, the expected utility for an unbiased person, as a function of q, for the SP procedure f_{SP} and for majority rule f_{MR}, when $n = 9$. The expected utility from majority rule f_{MR} is the "first best," the best the unbiased person could get if people have no choice but to report truthfully and there were no incentive compatibility problem. For q close to 1, the evidence is very good and the SP procedure performs very close to majority rule. As q decreases below around 0.8, the welfare gap between the two procedures becomes larger as the supermajority penalty regions grow, as shown in Figure 2. For q close to 0.5, the evidence becomes so bad that no procedure is much better than simply choosing a all the time regardless of the evidence, which yields an expected utility of 0.5.

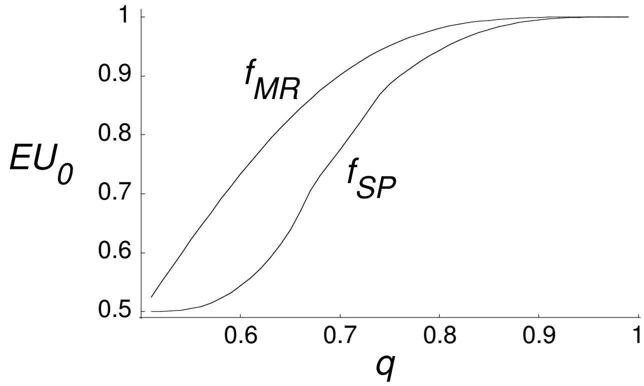

Figure 3. An unbiased person's expected utility EU_0 given f_{SP} and f_{MR} when $n = 9$ and $q = 0.51, 0.52, \ldots, 0.99$

How does the SP procedure change as n changes? Lemma 1 says that as n grows large, $k/n \to \rho$, where ρ is a constant which does not depend on n. Also, note that in expectation we either have qn reports for a (if a is the objectively superior alternative) or $(1-q)n$ reports for a (if b is the objectively superior alternative). By Lemma 1, $k < (1-q)n$, and thus the supermajority penalty regions are not reached in expectation, and are rarely reached as n grows large. Figure 4 shows how the SP procedure changes when $q = 2/3$ and n varies from 5 to 101. When $n = 101$ for example, given that $q = 2/3$, the number of reports for a cluster around 34 (if b is superior) or 67 (if a is superior) and the supermajority penalty rarely occurs.

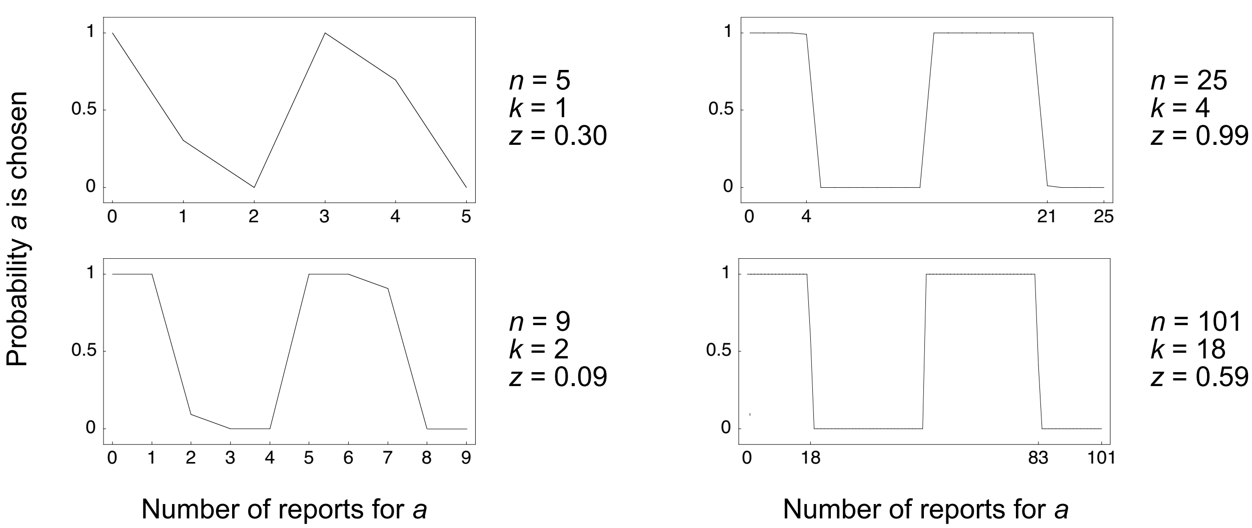

Figure 4. f_{SP} when $q = 2/3$ and $n = 5, 9, 25, 101$

Since the supermajority penalty regions are rarely reached as n grows large, the welfare performance of the SP procedure approaches that of majority rule. Figure 5 shows EU_0 as a function of n, for the SP procedure f_{SP} and for majority rule f_{MR}, when $q = 2/3$.

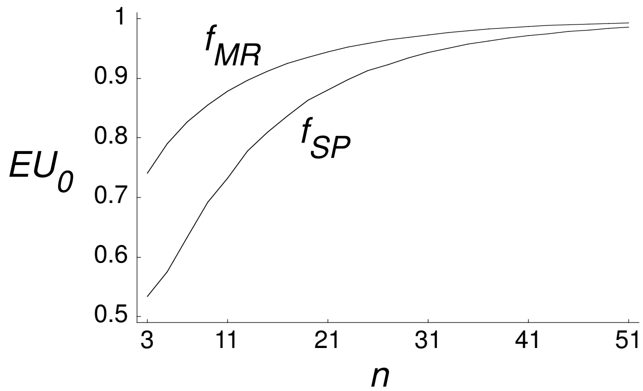

Figure 5. An unbiased person's expected utility EU_0 given f_{SP} and f_{MR} when $q = 2/3$ and $n = 3, 5, \ldots, 51$

6. Discussion

The results in this paper are based on three main assumptions: the welfare criterion is the utility of an unbiased person, the procedure is anonymous in the sense that each person's evidence affects the decision equally, and the procedure is incentive compatible. We discuss these three assumptions in turn.

First, the optimality result of Proposition 2 is from the point of view of an unbiased person. An unbiased person might be considered "objective" in that she cares only about the total probability of "getting it right," the sum of the probability of choosing a when a is superior and the probability of choosing b when b is superior. The utility of an unbiased person is also the objective function assumed by Condorcet in his original model; for example, a person very biased toward a in Condorcet's model would be happiest not with majority rule but with a procedure which chooses a unless a supermajority of reports are for b.

Still, it is natural to ask whether the optimality of the SP procedure depends in a fragile way on the objective function being the utility of a person who is exactly unbiased. Numerical computations show that this is an issue only for q near 0.5. Figure 6 shows the region in which the SP procedure is optimal, in terms of q and $\phi_0(a)$, when $n = 9$. Figure 6 also shows the regions in which the trivial procedures f_A and f_B are optimal, where f_A is the procedure which chooses a with probability 1 regardless of the reports and f_B is the procedure which chooses b with probability 1 regardless of the reports. For q close to 0.5, the optimality of f_{SP} is not very robust. Part of the reason for this is that for low q, the quality of evidence is very low, and hence someone with even the slightest bias toward a finds f_A optimal and someone with even the slightest bias toward b finds f_B optimal. For q greater than around 0.6, f_{SP} is optimal for people whose biases are in a significant band around 0.5. For q greater than around 0.9, f_{SP} is optimal for almost all biases.

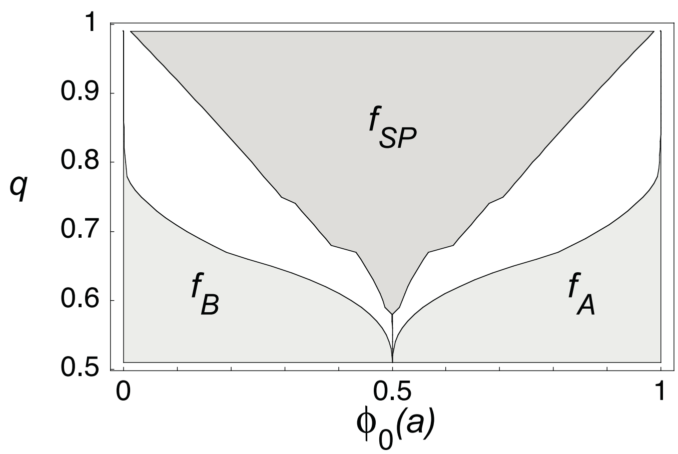

Figure 6. Anonymous incentive compatible procedures which maximize EU_0 given $\phi_0(a) \in [0,1]$ and $q = 0.51, 0.52, \ldots, 0.99$, where $n = 9$

It is also natural to consider other welfare criteria such as total social welfare. Recall that average social welfare is equal to the utility of an individual whose bias is the average of biases in the group. Thus if biases are symmetric around $\phi(a) = \phi(b) = 1/2$ or simply have average $\phi(a) = \phi(b) = 1/2$, then maximizing total social welfare is equivalent to maximizing the utility of an unbiased person. Similarly, maximizing the weighted sum of utilities is equivalent to maximizing the utility of an individual whose bias is the weighted average of biases. Thus in Figure 6, for q greater than around 0.9, f_{SP} is optimal for almost any weighted sum of individual utilities; in other words, most of the Pareto frontier is f_{SP}.

Second, we assume that everyone's evidence matters equally in the decision. Thus we consider only anonymous procedures. But even with an anonymous procedure, if people have different reporting strategies, then each person's evidence does not influence the decision equally. For example, if a person always reports a regardless of her evidence, then her evidence does not affect the decision. Thus in addition, we consider only the equilibrium in which everyone reports truthfully.

But other Nash equilibria are possible. Consider a three person example in which $q = 2/3$. As shown in our example earlier, the SP procedure is given by $\gamma(0) = 4/5, \gamma(1) = 0, \gamma(2) = 1, \gamma(3) = 1/5$, where $\gamma(j)$ is the probability of choosing a given j reports of a. Say $\phi_1(a) = 3/4, \phi_1(b) = 1/4$, $\phi_2(a) = 1/4, \phi_1(b) = 3/4$, and $\phi_3(a) = 1/2, \phi_1(b) = 1/2$. In other words, person 1 is biased toward a, person 2 is biased toward b, and person 3 is unbiased. Note that since the average bias is $\phi(a) = \phi(b) = 1/2$, maximizing total expected utility is equivalent to maximizing person 3's expected utility. By Proposition 1 we know that (id, id, id) is a Nash equilibrium, which gives everyone expected utility $8/15 \approx 0.533$. However, it turns out that (s_{aa}, s_{bb}, id) is also a Nash equilibrium, which gives everyone expected utility $2/3 \approx 0.667$. Here person 1 reports a all the time, and person 2 reports b all the time, and thus their evidence does not affect the decision. This is equivalent to a procedure in which only person 3's vote is counted and thus person 3 acts as a dictator.

The existence of multiple equilibria is of course an issue not just for the SP procedure but for other procedures. For example, if $n = 3$ and everyone is unbiased, given majority rule, the set of Nash equilibria includes (id, id, id), (s_{aa}, s_{bb}, id), (s_{aa}, s_{aa}, s_{aa}), and (s_{bb}, s_{bb}, s_{bb}). To make the seemingly obvious argument that majority rule is the optimal procedure when people are all unbiased, we also have to assume outright the truthful equilibrium.

To fully evaluate a procedure, one should ideally find all equilibria given the procedure. But this is difficult especially when considering all possible procedures. The mechanism design approach allows us to find the best possible procedure together with the best possible equilibrium. We employ the mechanism design approach here, but with the simplifying assumption that everyone's evidence matters equally (the procedure is anonymous and everyone tells the truth). Without this simplifying assumption, finding optimal procedures is not mathematically difficult, but specifying them can require a great deal of detail (since they are defined on 2^n cases) and they also typically depend precisely on people's biases (see for example Chwe 1999). With this simplifying assumption, we find an optimal procedure which is simple, does not depend precisely on people's biases, and is incentive compatible for all biases.

One might say that requiring that everyone's evidence matters equally in the decision is not reasonable; we should expect that a very biased person always reports her bias, and it is simply unrealistic to ask that a procedure encourage truthful reports from even the most biased people. From this point of view, what is interesting about the SP procedure is not so much its optimality but the fact that it even exists. It is a difficult task to get truthful revelation from both a person very biased toward a and a person very biased toward b without being able to tell the two apart. To accomplish this task, naturally the procedure must be substantially "distorted," and the supermajority penalty regions represent this distortion. As shown earlier, the welfare effect of this distortion goes to zero as q and n grow large.

What are optimal incentive compatible procedures if we do not require anonymity? In our three person example, the best procedure for the unbiased person 3 is the same as majority rule except that $f((a, b, b), a) = 4/7$ and $f((a, b, a), b) = 4/7$. In other words, person 1, who is biased toward a, gets to "enforce" her favorite with probability 4/7 even when everyone else votes for b, and similarly person 2, who is biased toward b, gets to "enforce" her favorite with probability 4/7 even when everyone else votes for a. This procedure is in the same spirit as the procedures in Chwe (1999), in that biased individuals are given "special powers." The expected utility of person 3 given this optimal nonanonymous procedure, which is the highest expected utility that an unbiased person can attain in any equilibrium of any procedure, is $44/63 \approx 0.698$.

To summarize our three person example, we can make some welfare comparisons, again from the point of view of an unbiased person. The best incentive compatible procedure assuming that everyone's evidence matters equally (everyone reports truthfully and the procedure is anonymous) is the SP procedure, which yields expected utility $8/15 \approx 0.533$. As mentioned before, given the SP procedure, there exists an equilibrium in which only person 3 reports truthfully, and this yields expected utility $2/3 \approx 0.667$. The procedure in which person 3 is a dictator also yields $2/3 \approx 0.667$. The best incentive compatible procedure among all possible procedures (not assuming anonymity) gives persons 1 and 2 "special powers" and yields expected utility $44/63 \approx 0.698$. For the sake of comparison, the "first best," the best possible procedure if we do not require incentive compatibility, is majority rule, which yields expected utility $20/27 \approx 0.741$. The worst possible procedure, which for example chooses a always regardless of anyone's report, yields expected utility $1/2 = 0.500$. This example shows that our requirement that everyone's evidence matters equally (the procedure is anonymous and we look only at the truth-telling equilibrium) can impose significant welfare losses. Interestingly, to increase total welfare, one should give persons 1 and 2 no power and shut them out entirely, or even better give them "extra" power.

Figure 7 shows nine more examples, where $n = 3, 5, 7$ and $q = 0.6, 0.7, 0.8$. In each example, we assume biases are quite divergent: one person is unbiased, $(n-1)/2$ people have bias $\phi(a) = 1/10$ and $(n-1)/2$ people have bias $\phi(a) = 9/10$. Figure 7 shows EU_0, the expected utility of an unbiased person, or equivalently the average expected utility of the group, for three procedures: the supermajority penalty procedure SP, the optimal nonanonymous incentive compatible procedure NA, and majority rule MR. In other words, majority rule MR is the "first best," the best possible procedure if incentive incompatibility is not an issue. The nonanonymous procedure NA is the "second best," the best incentive compatible procedure. The SP procedure is the "second best" under the additional assumption that each person's evidence matters equally. For all three procedures, we assume the truth-telling equilibrium. In these examples, we calculate the NA procedure with a computer, by numerically solving an optimization problem with 2^n variables. This is why we stop at $n = 7$, which involves 128 variables. Even writing down the NA procedure when $n = 7$ would take a lot of space.

Figure 7 shows that the welfare performance of the SP procedure can be poor; when $q = 0.6$ for example, the expected utility is only slightly higher than 0.5, which one would get from the worst possible procedure. Also, the gap between the SP procedure and the optimal nonanonymous procedure NA is largest when $n = 3$ and also when $q = 0.6$. In other words, the welfare loss from the anonymity assumption is largest for small n and low q. As n increases and q increases, the welfare loss from the anonymity assumption becomes less severe. As mentioned earlier, as n grows large or as q approaches 1, the SP procedure's welfare performance approaches that of majority rule and thus the welfare loss from anonymity goes to zero.

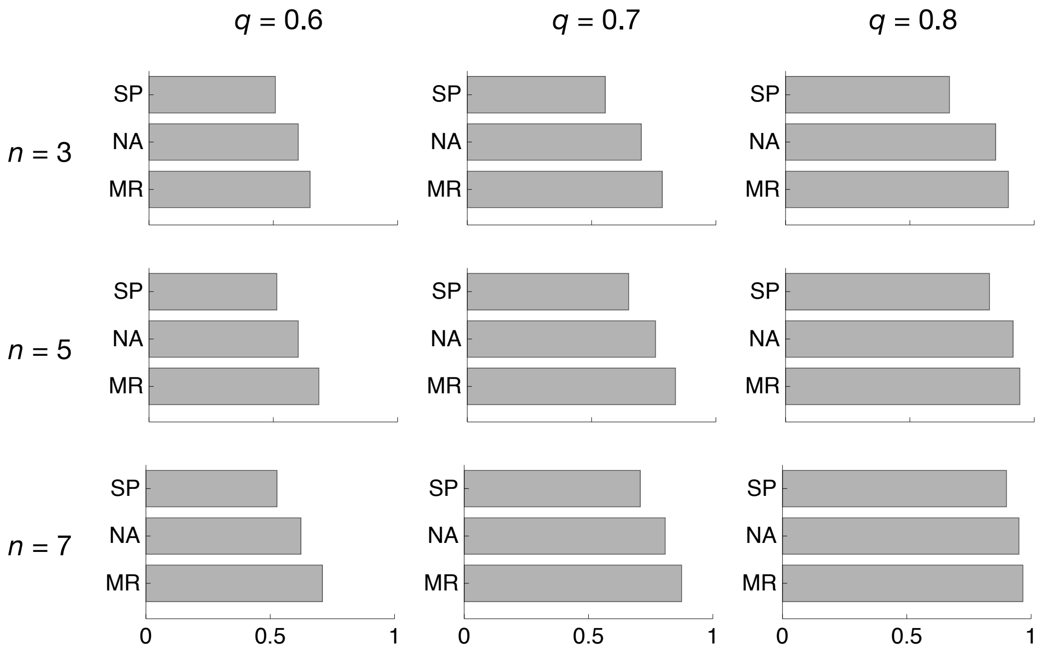

Figure 7. EU_0 under the supermajority penalty procedure (SP), the optimal nonanonymous procedure (NA), and majority rule (MR) when $(n-1)/2$ people have bias $\phi(a) = 1/10$, $(n-1)/2$ people have bias $\phi(a) = 9/10$, and one person has bias $\phi(a) = 1/2$

Thirdly, incentive compatibility is an important assumption in this paper. If we simply assume that people report truthfully and do not worry about their incentives for doing so, then a person's optimal procedure is simply one which chooses a if the total number of reports for a exceeds some cutoff value. This cutoff value is higher if the person's bias toward a is higher; if for example the person is unbiased, the cutoff value is $n/2$ and the optimal procedure is majority rule. Also, as discussed earlier, by assuming incentive compatibility we assume that everyone reports truthfully, which along with our anonymous procedure means that everyone's evidence matters equally. A more comprehensive approach would consider a wide variety of equilibria in which not everyone reports truthfully, but as discussed earlier, doing so would make our inquiry much more complicated. Finding all equilibria for a single specific procedure like majority rule is not trivial; finding all equilibria for a large set of procedures, such as all anonymous procedures, would be a challenge.

In our definition of a procedure, people report either a or b simultaneously and thus it might be thought that we are ruling out some procedures, for example procedures with multiple rounds, in which people can condition their own messages on the previous messages of others, or procedures in which people can report a greater variety of messages other than just plain a or b. However, our approach does indeed include such procedures and in fact all possible procedures, again assuming that each person's evidence affects the decision equally. The reason for this is often referred to as the "revelation principle" (see for example Myerson 1991, p. 260). Say we have a possibly quite complicated procedure and say that people's strategies, which are also possibly quite complicated, are an equilibrium. Then their strategies and the procedure together create a function which assigns to every evidence profile $r \in \{a,b\}^n$ a probability that a is finally chosen. Call this function $f(r,a)$ (and of course define $f(r,b) = 1 - f(r,a)$). Then f must be incentive compatible. If f is not incentive compatible, then a person with evidence a (for example) could gain by acting as if she had evidence b. This is not possible because we are in an equilibrium and no one can gain by deviating. In other words, any possible equilibrium of any possible procedure must result in an incentive compatible f. Thus in this paper, we are considering all possible equilibria of all possible procedures (again, under the assumption that each person's evidence affects the decision equally).

By the way, in our model we allow people to have different prior beliefs, and the revelation principle is more typically used when prior beliefs are identical. The revelation principle simply says that in any equilibrium of any procedure, a person with evidence a cannot gain by acting as if she had evidence b. This simple fact is true regardless of whether priors are identical or not. However, I should clarify the claim that the revelation principle allows us to consider all possible procedures. One interpretation of heterogeneous priors is that they arise because of differential information: I might be more biased toward a than you because I have some idiosyncratic information favoring a. Under this interpretation, people with different priors might usefully share this idiosyncratic information. This idiosyncratic information is not in our model and we do not consider procedures in which this unspecified idiosyncratic information is shared. The only information which people can share is the information explicitly specified in the model: each person's evidence, which is correct with probability q. Of course, to avoid any interpretive issues about heterogeneous priors, as mentioned earlier, one can assume identical priors and say that biases come from heterogeneous preferences.

7. Conclusion

This paper starts with Condorcet's original model, adds strategic voting and heterogeneous prior beliefs and preferences, and finds an optimal and surprisingly robust voting procedure. This robustness is not due to our particular procedure, but is an "artifact" of Condorcet's model itself. Other extensions of Condorcet's model include giving each person a continuous, not binary, signal about which alternative is superior (for example Duggan and Martinelli 2001 and Li, Rosen, and Suen 2001) and giving some people more informative signals than others (for example Ben-Yashar and Milchtaich 2006). The assumption of binary signals in our paper greatly simplifies the consideration of anonymous procedures. When signals are binary, an anonymous procedure is simply a function of the total number of reports for a. With continuous signals, an anonymous procedure is a function of everyone's continuous signals which is symmetric in its arguments, a much more complicated mathematical object. If we allow continuous signals, we might get similar results: for example, in a two person model with continuous signals, one can show that a monotonic procedure cannot be incentive compatible (Li, Rosen, and Suen 2001). If people have different quality evidence, which we would represent in our model by letting each person have a different value of q, then of course the results here would not hold; the SP procedure does depend on the specific value of q (in fact q and n are the only parameters the SP procedure does depend on). Finding optimal incentive compatible procedures when signals are continuous or when people have different quality evidence is a question for future work. Another line of research explicitly models uncertainty in people's biases; it is possible that uncertainty about biases can help in getting people to truthfully reveal their evidence (Austen-Smith and Feddersen 2006). Since the SP procedure is incentive compatible for all biases, it provides a minimum level of performance when biases are uncertain.

Despite its power, surprisingly few papers have used the mechanism design approach to analyze Condorcet's model (examples are Chwe 1999, Li, Rosen, and Suen 2001, and Wolinsky 2002). The mechanism design approach has several substantive advantages over starting with a single particular procedure like majority rule or unanimity rule. For example, since in real life people discuss and argue before voting, "pre-play communication" should be considered. But allowing this changes the strategic situation dramatically (Coughlan 2000). In fact, if pre-play communication is unrestricted, then it really doesn't matter what the voting rule is: all voting rules except for unanimity rules generate the same set of equilibrium outcomes (Gerardi and Yariv 2006). Thus pre-play communication should be considered an integral part of the procedure itself. Since there are many possible kinds of pre-play communication, including bilateral conversations, group announcements, and straw polls, it might seem almost impossible to find which procedure is socially optimal. But the mechanism design approach does exactly this. Also, the mechanism design approach asks the more profound question of what is the best possible procedure, not what behavior is given a particular procedure. But it is mathematically simpler, just a linear programming problem.

Most of the existing work on Condorcet's model is "conservative" in that it considers already well-known procedures such as majority rule and unanimity rule. But the entire point of Condorcet's original argument is to derive the optimal procedure, not assume it. When we add strategic voting, mechanism design allows us to keep the spirit of Condorcet's original question. This paper shows that if we assume strategic voting and anonymity in the sense of everyone's evidence affecting the decision equally, Condorcet's model cannot be understood as supporting well-known procedures such as majority rule as long as there is at least one person biased in each direction. The SP procedure is the optimal anonymous procedure for a very large set of biases (when there is at least one person biased in each direction) and is incentive compatible for all biases. The SP procedure is not monotonic, but no incentive compatible anonymous procedure is monotonic when there is at least one person biased in each direction.

Would anyone actually use a non-monotonic procedure in practice? Non-monotonic procedures are not that unusual. In a similar model in which experts who all have similar biases report to a decision maker with a different bias, the optimal procedure for the decision maker is non-monotonic (Wolinsky 2002). When deciding among more than two alternatives, some popular procedures are non-monotonic, such as plurality voting with a runoff among the top two candidates (Riker 1982). If one is wedded to monotonicity, the results here might be understood as showing the limitations of procedures which treat everyone's evidence equally: given biases in both directions, one cannot retain monotonicity without treating some people's evidence differently from others. As discussed earlier, procedures which treat everyone's evidence equally can also be quite limited in terms of social welfare. Perhaps our results illustrate the limitations of Condorcet's model itself; although it has been used often as a foundation for modeling voting and decision making, it can yield surprisingly nonintuitive results.

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Appendix

Lemma 1. Given q and n, there uniquely exists $k \in \{0, 1, \ldots, (n-3)/2\}$ such that

$$z = \frac{D_k y^{(n-1)/2} - k(y^{k-1} + y^{n-k})}{(n-k)(y^k + y^{n-1-k}) - k(y^{k-1} + y^{n-k})} \in (0, 1]$$

where $y = q/(1-q)$ and $D_k = (k!(n-k)!)/(((n-1)/2)!)^2$. Also, k is nonincreasing in q, $k < (1-q)n$, and $k/n \to \rho$ as $n \to \infty$, where $\rho\, log\, \rho + (1-\rho)\, log(1-\rho) + log\, 2 = (1/2-\rho)\, log\, y$.

Proof. We write z as a function of k and y and multiply numerator and denominator by $y^{-(n-1)/2}$ to get $z_k(y) = num_k(y)/den_k(y)$, where $num_k(y)$ and $den_k(y)$ are defined as

$$num_k(y) = D_k - k(y^{k-n/2-1/2} + y^{n/2-k+1/2})$$

$$den_k(y) = (n-k)(y^{k-n/2+1/2} + y^{n/2-k-1/2}) - k(y^{k-n/2-1/2} + y^{n/2-k+1/2}).$$

Note that $num_k(y)$ is strictly decreasing in y for $y > 1$ and $k \in \{1, \ldots, (n-3)/2\}$, because $(d/dy)(num_k(y)) = -(k/y)(n/2-k+1/2)(-y^{k-n/2-1/2} + y^{n/2-k+1/2}) < 0$.

We prove five useful facts. Fact A is $num_{k+1}(y) \leq 0 \Leftrightarrow num_k(y) \leq den_k(y)$. This is because $num_{k+1}(y) \leq 0$ is equivalent to $D_{k+1}/(k+1) \leq y^{k-n/2+1/2} + y^{n/2-k-1/2}$ and $num_k(y) \leq den_k(y)$ is equivalent to $D_k/(n-k) \leq y^{k-n/2+1/2} + y^{n/2-k-1/2}$, and it is easy to verify from the definition of D_k that $D_{k+1}/(k+1) = D_k/(n-k)$. Similarly, we have Fact B: $num_{k+1}(y) > 0$ is equivalent to $num_k(y) > den_k(y)$.

Fact C is that $num_k(y) \geq 0$ is equivalent to

$$\frac{n-k}{k} \cdot \frac{n-k-1}{k+1} \cdots \frac{(n+1)/2}{(n-1)/2} \geq y^{k-n/2-1/2} + y^{n/2-k+1/2}.$$

To see this, note that $num_k(y) \geq 0$ is equivalent to $D_k/k \geq y^{k-n/2-1/2} + y^{n/2-k+1/2}$ and

$$D_k/k = \frac{(n-k)!}{((n-1)/2)!} \frac{(k-1)!}{((n-1)/2)!} = \frac{n-k}{k} \cdot \frac{n-k-1}{k+1} \cdots \frac{(n+1)/2}{(n-1)/2}.$$

Similarly, we have Fact D: $num_k(y) > 0$ is equivalent to

$$\frac{n-k}{k} \cdot \frac{n-k-1}{k+1} \cdots \frac{(n+1)/2}{(n-1)/2} > y^{k-n/2-1/2} + y^{n/2-k+1/2}.$$

Fact E is $num_{k+1}(y) \geq 0 \Rightarrow num_k(y) > 0$ for $k \in \{0, \ldots, (n-3)/2\}$. If $num_{k+1}(y) \geq 0$, from Fact C we have

$$\frac{n-k-1}{k+1} \cdot \frac{n-k-2}{k+2} \cdots \frac{(n+1)/2}{(n-1)/2} > y^{n/2-k-1/2}$$

since $y^{k-n/2+1/2} > 0$. Note that there are $n/2 - k - 1/2$ terms on the left hand side of this inequality; it is also easy to see that these terms $\frac{n-k-1}{k+1}, \ldots, \frac{(n+1)/2}{(n-1)/2}$ are all less than $(n-k)/k$. Hence $[(n-k)/k]^{n/2-k-1/2} > y^{n/2-k-1/2}$ and thus we get the inequality $(n-k)/k > y$. Fact C gives us the inequality

$$\frac{n-k-1}{k+1} \cdot \frac{n-k-2}{k+2} \cdots \frac{(n+1)/2}{(n-1)/2} \geq y^{k-n/2+1/2} + y^{n/2-k-1/2}.$$

We multiply these inequalities together and get

$$\frac{n-k}{k} \cdot \frac{n-k-1}{k+1} \cdot \frac{n-k-2}{k+2} \cdots \frac{(n+1)/2}{(n-1)/2} > y^{k-n/2+3/2} + y^{n/2-k+1/2}.$$

But $y^{k-n/2+3/2} > y^{k-n/2-1/2}$ since $y > 1$ and thus by Fact D we have $num_k(y) > 0$.

Since $q \in (1/2, 1)$, we have $y = q/(1-q) \in (1, \infty)$. Let $y \in (1, \infty)$ and show that there uniquely exists $k \in \{0, 1, \ldots, (n-3)/2\}$ such that $z_k(y) \in (0, 1]$.

Note that $num_0(y) = D_0 > 0$. Since $num_{(n-1)/2}(1) = D_{(n-1)/2} - (n-1) = (n+1)/2 - (n-1) = (3-n)/2 \leq 0$ and $num_{(n-1)/2}(y)$ is strictly decreasing for $y > 1$, we know $num_{(n-1)/2}(y) < 0$. Hence there exists $k \in \{0, \ldots, (n-3)/2\}$ such that $num_k(y) > 0$ and $num_{k+1}(y) \leq 0$. Note that this k is unique: say $num_k(y) > 0$, $num_{k+1}(y) \leq 0$, $num_{k'}(y) > 0$ and $num_{k'+1}(y) \leq 0$ for $k < k'$. Then we have $num_{k+1}(y) \leq 0$ and $num_{k'}(y) > 0$, which contradicts Fact E.

Show $z_k(y) \in (0, 1]$. Since $num_{k+1}(y) \leq 0$, we know $num_k(y) \leq den_k(y)$ by Fact A. Since $num_k(y) > 0$, we know $z_k(y) = num_k(y)/den_k(y) \in (0, 1]$.

Show $z_j(y) \notin (0, 1]$ or is undefined for $j \in \{0, \ldots, (n-3)/2\}$, $j \neq k$. Since $j \neq k$, we have three possible cases: (i) $num_j(y) > 0$ and $num_{j+1}(y) > 0$, (ii) $num_j(y) \leq 0$ and $num_{j+1}(y) \leq 0$, and (iii) $num_j(y) \leq 0$ and $num_{j+1}(y) > 0$. By Fact E, (iii) cannot happen. Also by Fact E, if $num_{j+1}(y) = 0$, then $num_j(y) \leq 0$ is impossible and hence case (ii) reduces to $num_j(y) \leq 0$ and $num_{j+1}(y) < 0$. In case (i), we have $num_j(y) > den_j(y)$ by Fact B; since $num_j(y) > 0$, if $den_j(y) > 0$, then $z_j(y) > 1$, if $den_j(y) < 0$, then $z_j(y) < 0$, and if $den_j(y) = 0$, then $z_j(y)$ is undefined. In case (ii), we have $num_j(y) < den_j(y)$ by Fact A; since $num_j(y) \leq 0$, if $den_j(y) > 0$, then $z_j(y) \leq 0$, if $den_j(y) < 0$, then $z_j(y) > 1$, and if $den_j(y) = 0$, then $z_j(y)$ is undefined.

To show that k is nonincreasing in q, it suffices to show that k is nonincreasing in y, since $y = q/(1-q)$. Let $y < y'$. Remember that k is chosen so that $num_k(y) > 0$ and $num_{k+1}(y) \leq 0$. Choose k' so that $num_{k'}(y') > 0$ and $num_{k'+1}(y') \leq 0$. We show $k \geq k'$

by contradiction. Say that $k < k'$. Since $num_{k'}(y') > 0$ and $k+1 \leq k'$, by Fact E we have $num_{k+1}(y') > 0$. Hence by Fact D we have

$$\frac{n-k-1}{k+1} \cdots \frac{(n+1)/2}{(n-1)/2} > (y')^{k-n/2-1/2} + (y')^{n/2-k+1/2}.$$

But $num_{k+1}(y) \leq 0$, and hence by Fact D we have

$$\frac{n-k-1}{k+1} \cdots \frac{(n+1)/2}{(n-1)/2} \leq y^{k-n/2-1/2} + y^{n/2-k+1/2}.$$

Thus we have $(y')^{k-n/2-1/2} + (y')^{n/2-k+1/2} < y^{k-n/2-1/2} + y^{n/2-k+1/2}$, which contradicts $y < y'$ since $y, y' > 1$.

Since $num_k(y) > 0$, by the reasoning in the proof of Fact E, we have $(n-k)/k > y$ and thus $k < n/(y+1) = (1-q)n$ since $y = q/(1-q)$. To show $k/n \to \rho$ as $n \to \infty$, note that as n grows large, k is given by $num_k(y) = 0$, which is $(k-1)!(n-k)! = (((n-1)/2)!)^2 y^{n/2-k+1/2}$ as n grows large. Using Stirling's approximation $log\, m! \approx m\, log\, m - m$, we have

$$(k-1)\, log(k-1) + (n-k)\, log(n-k) = (n-1)\, log((n-1)/2) + (n/2-k+1/2)\, log\, y.$$

If we let $k = \rho n$, we get

$$(\rho n-1)\, log(\rho n-1) + (1-\rho)n\, log((1-\rho)n) = (n-1)\, log((n-1)/2) + (n/2-\rho n+1/2)\, log\, y.$$

As n grows large, we have $\rho n\, log(\rho n) + (1-\rho)n\, log((1-\rho)n) = n\, log(n/2) + n(1/2-\rho)\, log\, y$, and thus $\rho\, log(\rho n) + (1-\rho)\, log((1-\rho)n) = log(n/2) + (1/2-\rho)\, log\, y$. We simplify to get $\rho\, log\, \rho + (1-\rho)\, log(1-\rho) + log\, 2 = (1/2-\rho)\, log\, y$. ■

To prove the remaining lemmas and the propositions, we need the following notation. Since we consider anonymous procedures f, we write $f(r,a) = \gamma(\alpha(r))$, where $\gamma(j)$ is the probability that the procedure choses a given that there are j reports of a. Given biases ϕ, define

$$h_\phi(j) = \phi(a)q^j(1-q)^{n-j} - \phi(b)q^{n-j}(1-q)^j.$$

We write $h(j)$ when there is no ambiguity about ϕ. Let C^n_j be the binomial coefficient $C^n_j = n!/((n-j)!j!)$.

Let $A(\phi,\gamma) = EU_i(f, id, id_{-i}) - EU_i(f, s_{aa}, id_{-i})$ be person i's utility difference between playing id and s_{aa} given that everyone else plays id. By our formula $(*)$, we have $A(\phi,\gamma) = \phi(a)[p_{aa}(f, id, id_{-i}) - p_{aa}(f, s_{aa}, id_{-i})] + \phi(b)[p_{bb}(f, id, id_{-i}) - p_{bb}(f, s_{aa}, id_{-i})]$.

Show that $p_{aa}(f, id, id_{-i}) - p_{aa}(f, s_{aa}, id_{-i}) = \sum_{j=0}^{n-1} C_j^{n-1} q^j (1-q)^{n-j} (\gamma(j) - \gamma(j+1))$. To see this fact, note that $p_{aa}(f, id, id_{-i})$ and $p_{aa}(f, s_{aa}, id_{-i})$ are different only when $e_i = b$ (when $e_i = a$, in both id and s_{aa}, person i makes the same report, a). Since the superior alternative is truly a, the probability that $e_i = b$ is $1 - q$. The probability that j other people have evidence a is $C_j^{n-1} q^j (1-q)^{n-1-j}$. Hence the probability that $e_i = b$ and that j people in total have evidence a is $C_j^{n-1} q^j (1-q)^{n-j}$. Given this, if person i plays id, then the group chooses a with probability $\gamma(j)$; if person i plays s_{aa}, then the group chooses a with probability $\gamma(j+1)$; thus the fact is demonstrated. Similarly, we find that $p_{bb}(f, id, id_{-i}) - p_{bb}(f, s_{aa}, id_{-i}) = \sum_{j=0}^{n-1} C_j^{n-1} (1-q)^j q^{n-j} (\gamma(j+1) - \gamma(j))$.

Thus $A(\phi, \gamma) = \sum_{j=0}^{n-1} C_j^{n-1} h_\phi(j)(\gamma(j) - \gamma(j+1))$. If we let $B(\phi, \gamma) = EU_i(f, id, id_{-i}) - EU_i(f, s_{bb}, id_{-i})$, in a similar manner we find $B(\phi, \gamma) = \sum_{j=1}^{n} C_{j-1}^{n-1} h_\phi(j)(\gamma(j) - \gamma(j-1))$. Thus our two incentive constraints are simply $A(\phi, \gamma) \geq 0$ and $B(\phi, \gamma) \geq 0$.

We need eight facts. We have Fact 1: $h((n-1)/2) = (\phi(a) - q) q^{(n-1)/2} (1-q)^{(n-1)/2}$. This is true because

$$h((n-1)/2) = \phi(a) q^{(n-1)/2} (1-q)^{(n+1)/2} - (1 - \phi(a)) q^{(n+1)/2} (1-q)^{(n-1)/2}$$

$$= (\phi(a)(1-q) - (1 - \phi(a))q) q^{(n-1)/2} (1-q)^{(n-1)/2}$$

$$= (\phi(a) - q) q^{(n-1)/2} (1-q)^{(n-1)/2}.$$

Next we have Fact 2: $h(j) + h(n-1-j) = (\phi(a) - q)[q^j (1-q)^{n-1-j} + q^{n-1-j} (1-q)^j]$. To see this, note that $h(j) + h(n-1-j) =$

$$\phi(a) q^j (1-q)^{n-j} - (1 - \phi(a)) q^{n-j} (1-q)^j + \phi(a) q^{n-1-j} (1-q)^{j+1} - (1 - \phi(a)) q^{j+1} (1-q)^{n-1-j}$$

$$= \phi(a)[q^j (1-q)^{n-j} + q^{n-j} (1-q)^j + q^{n-1-j} (1-q)^{j+1} + q^{j+1} (1-q)^{n-1-j}] - q^{n-j} (1-q)^j - q^{j+1} (1-q)^{n-1-j}$$

$$= \phi(a)[q^j (1-q)^{n-1-j} ((1-q) + q) + q^{n-1-j} (1-q)^j (q + (1-q))] - q(q^{n-1-j} (1-q)^j + q^j (1-q)^{n-1-j})$$

$$= (\phi(a) - q)[q^j (1-q)^{n-1-j} + q^{n-1-j} (1-q)^j].$$

Next we have Fact 3: $h((n+1)/2) = (q - \phi(b)) q^{(n-1)/2} (1-q)^{(n-1)/2}$. This is true because

$$h((n+1)/2) = (1 - \phi(b)) q^{(n+1)/2} (1-q)^{(n-1)/2} - \phi(b) q^{(n-1)/2} (1-q)^{(n+1)/2}$$

$$= ((1 - \phi(b))q - \phi(b)(1-q)) q^{(n-1)/2} (1-q)^{(n-1)/2}$$

$$= (q - \phi(b)) q^{(n-1)/2} (1-q)^{(n-1)/2}.$$

We have Fact 4: $h(j) + h(n+1-j) = (q - \phi(b))[q^{j-1}(1-q)^{n-j} + q^{n-j}(1-q)^{j-1}]$. We get this because $h(j) + h(n+1-j) =$

$$(1-\phi(b))q^j(1-q)^{n-j} - \phi(b)q^{n-j}(1-q)^j + (1-\phi(b))q^{n+1-j}(1-q)^{j-1} - \phi(b)q^{j-1}(1-q)^{n+1-j}$$

$$= -\phi(b)[q^j(1-q)^{n-j} + q^{n-j}(1-q)^j + q^{n+1-j}(1-q)^{j-1} + q^{j-1}(1-q)^{n+1-j}] + q^j(1-q)^{n-j} + q^{n+1-j}(1-q)^{j-1}$$

$$= -\phi(b)[q^{j-1}(1-q)^{n-j}((q+(1-q)) + q^{n-j}(1-q)^{j-1}((1-q)+q)] + q(q^{j-1}(1-q)^{n-j} + q^{n-j}(1-q)^{j-1})$$

$$= (q - \phi(b))[q^{j-1}(1-q)^{n-j} + q^{n-j}(1-q)^{j-1}].$$

We have Fact 5: if $\gamma(j) = 1 - \gamma(n-j)$, then $A(\phi, \gamma) = (\phi(a) - q)W(q, \gamma)$, where

$$\begin{aligned} W(q,\gamma) = {} & C^{n-1}_{(n-1)/2} q^{(n-1)/2}(1-q)^{(n-1)/2}(2\gamma((n-1)/2) - 1) \\ & + \sum_{j=0}^{(n-3)/2} C^{n-1}_j (q^j(1-q)^{n-1-j} + q^{n-1-j}(1-q)^j)(\gamma(j) - \gamma(j+1)). \end{aligned}$$

We can write

$$\begin{aligned} A(\phi,\gamma) &= C^{n-1}_{(n-1)/2} h((n-1)/2)(\gamma((n-1)/2) - \gamma((n+1)/2)) \\ &\quad + \Big(\sum_{j=0}^{(n-3)/2} + \sum_{j=(n+1)/2}^{n-1}\Big) C^{n-1}_j h(j)(\gamma(j) - \gamma(j+1)) \\ &= C^{n-1}_{(n-1)/2} h((n-1)/2)(\gamma((n-1)/2) - \gamma((n+1)/2)) \\ &\quad + \sum_{j=0}^{(n-3)/2} C^{n-1}_j h(j)(\gamma(j) - \gamma(j+1)) + C^{n-1}_{n-1-j} h(n-1-j)(\gamma(n-1-j) - \gamma(n-j)). \end{aligned}$$

We know that $C^{n-1}_j = C^{n-1}_{n-1-j}$. We also know that $\gamma(n-1-j) - \gamma(n-j) = 1 - \gamma(j+1) - (1 - \gamma(j)) = \gamma(j) - \gamma(j+1)$ and $\gamma((n-1)/2) - \gamma((n+1)/2) = \gamma((n-1)/2) - (1 - \gamma((n-1)/2)) = 2\gamma((n-1)/2) - 1$. Thus

$$\begin{aligned} A(\phi,\gamma) = {} & C^{n-1}_{(n-1)/2} h((n-1)/2)(2\gamma((n-1)/2) - 1) \\ & + \sum_{j=0}^{(n-3)/2} C^{n-1}_j (h(j) + h(n-1-j))(\gamma(j) - \gamma(j+1)). \end{aligned}$$

But by Facts 1 and 2, this is equal to $(\phi(a) - q)W(q, \gamma)$.

We have Fact 6: if $\gamma(j) = 1 - \gamma(n-j)$, then $B(\phi,\gamma) = (\phi(b) - q)W(q,\gamma)$, where $W(q,\gamma)$ is defined as in Fact 5. We can write

$$\begin{aligned}
B(\phi,\gamma) &= C^{n-1}_{(n-1)/2}h((n+1)/2)(\gamma((n+1)/2) - \gamma((n-1)/2)) \\
&\quad + \Big(\sum_{j=1}^{(n-1)/2} + \sum_{j=(n+3)/2}^{n}\Big)C^{n-1}_{j-1}h(j)(\gamma(j) - \gamma(j-1)) \\
&= C^{n-1}_{(n-1)/2}h((n+1)/2)(\gamma((n+1)/2) - \gamma((n-1)/2)) \\
&\quad + \sum_{j=1}^{(n-1)/2} C^{n-1}_{j-1}h(j)(\gamma(j) - \gamma(j-1)) + C^{n-1}_{n-j}h(n+1-j)(\gamma(n+1-j) - \gamma(n-j)).
\end{aligned}$$

We know that $C^{n-1}_{j-1} = C^{n-1}_{n-j}$. We also know that $\gamma(n+1-j) - \gamma(n-j) = 1 - \gamma(j-1) - (1 - \gamma(j)) = \gamma(j) - \gamma(j-1)$ and $\gamma((n+1)/2) - \gamma((n-1)/2) = 1 - \gamma((n-1)/2) - \gamma((n-1)/2) = 1 - 2\gamma((n-1)/2)$. Thus

$$\begin{aligned}
B(\phi,\gamma) = {} & C^{n-1}_{(n-1)/2}h((n+1)/2)(1 - 2\gamma((n-1)/2)) \\
& + \sum_{j=1}^{(n-1)/2} C^{n-1}_{j-1}(h(j) + h(n+1-j))(\gamma(j) - \gamma(j-1)).
\end{aligned}$$

By Facts 3 and 4, this is equal to $(q - \phi(b))(-W(q,\gamma))$.

We have Fact 7: Say $\phi''(a) = \phi(b)$ and $\phi''(b) = \phi(a)$ and $\gamma''(j) = 1 - \gamma(n-j)$. Then $A(\phi,\gamma) = B(\phi'',\gamma'')$ and $B(\phi,\gamma) = A(\phi'',\gamma'')$. From the definition of $B(\phi,\gamma)$, we have

$$\begin{aligned}
B(\phi'',\gamma'') &= \sum_{j=1}^{n} C^{n-1}_{j-1}h_{\phi''}(j)(\gamma''(j) - \gamma''(j-1)) \\
&= \sum_{j=1}^{n} C^{n-1}_{j-1}h_{\phi''}(j)(1 - \gamma(n-j) - (1 - \gamma(n-j+1))) \\
&= \sum_{j=1}^{n} C^{n-1}_{j-1}h_{\phi''}(j)(\gamma(n-j+1) - \gamma(n-j)).
\end{aligned}$$

It is easy to show that $h_{\phi''}(j) = -h_{\phi}(n-j)$ and that $C^{n-1}_{j-1} = C^{n-1}_{n-j}$, and thus $B(\phi'',\gamma'') = \sum_{j=1}^{n} C^{n-1}_{n-j}h_{\phi}(n-j)(\gamma(n-j) - \gamma(n-j+1))$. We change variables to get $B(\phi'',\gamma'') = \sum_{j=0}^{n-1} C^{n-1}_{j}h_{\phi}(j)(\gamma(j) - \gamma(j+1)) = A(\phi,\gamma)$. We show $B(\phi,\gamma) = A(\phi'',\gamma'')$ similarly.

Finally, we have Fact 8: if $y > 1$ and $x > 0$, $y^{2x} - xy^x \, log\, y - 1 > 0$. Since this holds with equality when $x = 0$, it suffices to show that $(d/dx)(y^{2x} - xy^x \, log\, y - 1) = y^x(2y^x \, log\, y - logy - x(log\, y)^2) > 0$, or in other words $2y^x \, log\, y - log\, y - x(log\, y)^2) > 0$. This is true when $x = 0$; since $(d/dx)(2y^x \log y - logy - x(\log y)^2) = (\log y)^2(2y^x - 1) > 0$, we are done.

Lemma 2. Say that there exists $i \in N$ such that either $\phi_i(a) > q$ or $\phi_i(b) > q$, and say that f is anonymous and symmetric. Then f is incentive compatible if and only if $W(q,\gamma) = 0$, where $\gamma(\alpha(r)) = f(r,a)$ and $W(q,\gamma) = C^{n-1}_{(n-1)/2} q^{(n-1)/2}(1-q)^{(n-1)/2}(2\gamma((n-1)/2) - 1) + \sum_{j=0}^{(n-3)/2} C^{n-1}_j (q^j(1-q)^{n-1-j} + q^{n-1-j}(1-q)^j)(\gamma(j) - \gamma(j+1))$, where C^n_j is the binomial coefficient $C^n_j = n!/((n-j)!j!)$.

Proof. Since f is symmetric, we have $\gamma(j) = 1 - \gamma(n-j)$ and hence by Facts 5 and 6 we have $A(\phi,\gamma) = (\phi(a) - q)W(q,\gamma)$ and $B(\phi,\gamma) = (\phi(b) - q)W(q,\gamma)$. Say $\phi_i(a) > q$. It is clear that $\phi_i(b) < q$. If f is incentive compatible, we have $A(\phi_i,\gamma) = (\phi_i(a) - q)W(q,\gamma) \geq 0$ and $B(\phi_i,\gamma) = (\phi_i(b) - q)W(q,\gamma) \geq 0$. Since $\phi_i(a) - q > 0$ and $\phi_i(b) - q < 0$, we have $W(q,\gamma) = 0$. To show the other direction, say $W(q,\gamma) = 0$. By Facts 5 and 6, we have $A(\phi,\gamma) = 0$ and $B(\phi,\gamma) = 0$ for all ϕ. Hence f is incentive compatible. If $\phi_i(b) > q$, the proof is similar. ■

Lemma 3. Say that there exist $i, j \in N$ such that $\phi_i(a) < 1 - q$ and $\phi_j(a) > q$. If f is anonymous and incentive compatible for persons i and j, then person l's incentive compatibility constraints hold with equality for any ϕ_l.

Proof. Let $r = \min\{\phi_i(b), \phi_j(a)\}$. We know $r > q$. Define biases ϕ_r and ϕ_{1-r} as $\phi_r(a) = r$, $\phi_r(b) = 1 - r$ and $\phi_{1-r}(a) = 1 - r$, $\phi_{1-r}(b) = r$. It is easy to see that $\phi_r(a), \phi_{1-r}(a) \in [\phi_i(a), \phi_j(a)]$. Since f satisfies $A(\phi_i,\gamma) \geq 0, B(\phi_i,\gamma) \geq 0, A(\phi_j,\gamma) \geq 0, B(\phi_j,\gamma) \geq 0$, since $A(\phi,\gamma)$ and $B(\phi,\gamma)$ are linear in $\phi(a)$, and $\phi_r(a), \phi_{1-r}(a) \in [\phi_i(a), \phi_j(a)]$, we know f satisfies $A(\phi_r,\gamma) \geq 0$, $B(\phi_r,\gamma) \geq 0$ and $A(\phi_{1-r},\gamma) \geq 0$, $B(\phi_{1-r},\gamma) \geq 0$.

Define $\gamma''(j) = 1 - \gamma(n-j)$. By Fact 7 we have $A(\phi_r,\gamma'') = B(\phi_{1-r},\gamma) \geq 0$ and $A(\phi_{1-r},\gamma'') = B(\phi_r,\gamma) \geq 0$. Define $\gamma'(j) = (\gamma(j) + \gamma''(j))/2$. It is easy to see that $\gamma'(j)$ is symmetric and hence by Facts 5 and 6, we have $A(\phi_r,\gamma') = (r-q)W(q,\gamma')$ and $A(\phi_{1-r},\gamma') = (1-r-q)W(q,\gamma')$.

Now $A(\phi_r,\gamma') = (A(\phi_r,\gamma) + A(\phi_r,\gamma''))/2 \geq 0$ and $A(\phi_{1-r},\gamma') = (A(\phi_{1-r},\gamma) + A(\phi_{1-r},\gamma''))/2 \geq 0$. Hence $(r-q)W(q,\gamma') \geq 0$ and $(1-r-q)W(q,\gamma') \geq 0$. Since $r - q > 0$ and $1 - r - q < 0$, we have $W(q,\gamma') = 0$. Hence $A(\phi_r,\gamma') = 0$. But since $A(\phi_r,\gamma) \geq 0$ and

$A(\phi_r, \gamma'') \geq 0$, we have $A(\phi_r, \gamma) = A(\phi_r, \gamma'') = 0$. Similarly, we conclude that $A(\phi_{1-r}, \gamma) = 0$ and also that $B(\phi_r, \gamma) = 0$ and $B(\phi_{1-r}, \gamma) = 0$.

Now for any ϕ_l, we have $\phi_l(a) = \lambda\phi_r(a) + (1-\lambda)\phi_{1-r}(a)$, where $\lambda \in \Re$ (that is, $\phi_l(a)$ is a linear, not necessarily convex, combination of $\phi_r(a)$ and $\phi_{1-r}(a)$), and thus $A(\phi_l, \gamma) = \lambda A(\phi_r, \gamma) + (1-\lambda)A(\phi_{1-r}, \gamma) = 0$ and similarly $B(\phi_l, \gamma) = 0$. ■

Proposition 1. The procedure f_{SP} is incentive compatible for any $\phi_1, \ldots, \phi_n$. In fact, the incentive compatibility constraints hold with equality.

Proof. Since f_{SP} is anonymous and symmetric, by Lemma 2 it suffices to show that $W(q, \gamma_{SP}) = 0$, where $\gamma_{SP}(\alpha(r)) = f_{SP}(r)$. We compute

$$\begin{aligned} W(q, \gamma_{SP}) = {} & C^{n-1}_{(n-1)/2} q^{(n-1)/2}(1-q)^{(n-1)/2}(-1) \\ & + C^{n-1}_{k-1}(q^{k-1}(1-q)^{n-k} + q^{n-k}(1-q)^{k-1})(1-z) \\ & + C^{n-1}_{k}(q^{k}(1-q)^{n-1-k} + q^{n-1-k}(1-q)^{k})(z). \end{aligned}$$

It is easy to show that this is zero given the definition of z and k in Lemma 1. ■

Proposition 2. Say that there exist $i, j \in N$ such that $\phi_i(a) < 1-q$ and $\phi_j(a) > q$. Then f_{SP} is an anonymous incentive compatible procedure which maximizes EU_0, where $\phi_0(a) = \phi_0(b) = 1/2$. Also, f_{SP} is the unique maximum for almost all q (all but fewer than $(n-1)/2$ values of q).

Proof. Let F be the set of anonymous incentive compatible procedures. Let $F' \subset F$ be the set of symmetric anonymous incentive compatible procedures. First show that f_{SP} maximizes EU_0 over F'. By Lemma 2, if f is symmetric and anonymous, then f is incentive compatible if and only if $W(q, \gamma) = 0$. Hence we have the single constraint $W(q, \gamma) = 0$, along with the constraints $\gamma(j) \in [0, 1]$. Since f is symmetric, our choice variables are $\gamma(0), \gamma(1), \ldots, \gamma((n-1)/2)$.

From our formula $(*)$, we have

$$
\begin{aligned}
EU_0 &= \phi_0(a)p_{aa}(f, id, \ldots, id) + \phi_0(b)p_{bb}(f, id, \ldots, id) \\
&= \phi_0(a)\sum_{j=0}^{n} C_j^n q^j (1-q)^{n-j}\gamma(j) + \phi_0(b)\sum_{j=0}^{n} C_j^n (1-q)^j q^{n-j}(1-\gamma(j)) \\
&= \sum_{j=0}^{n} C_j^n (1-q)^j q^{n-j} + \sum_{j=0}^{n} C_j^n[\phi_0(a)q^j(1-q)^{n-j} - \phi_0(b)(1-q)^j q^{n-j}]\gamma(j) \\
&= \sum_{j=0}^{n} C_j^n (1-q)^j q^{n-j} + \sum_{j=0}^{n} C_j^n h_0(j)\gamma(j).
\end{aligned}
$$

Since this first term is constant in $\gamma(j)$, maximizing EU_0 is equivalent to maximizing $\sum_{j=0}^{n} C_j^n h_0(j)\gamma(j)$. But

$$
\begin{aligned}
\sum_{j=0}^{n} C_j^n h_0(j)\gamma(j) &= \sum_{j=0}^{(n-1)/2} C_j^n h_0(j)\gamma(j) + C_{n-j}^n h_0(n-j)\gamma(n-j) \\
&= \sum_{j=0}^{(n-1)/2} C_j^n h_0(j)(2\gamma(j)-1)
\end{aligned}
$$

because $C_j^n = C_{n-j}^n$, $h_0(n-j) = -h_0(j)$ from the definition of $h_\phi(j)$, and $\gamma(n-j) = 1-\gamma(j)$ since f is symmetric.

We write the Lagrangian

$$
L = \sum_{j=0}^{(n-1)/2} C_j^n h_0(j)(2\gamma(j)-1) + \lambda W(q,\gamma) + \sum_{j=0}^{(n-1)/2} (\mu(j)-\nu(j))\gamma(j).
$$

By the Kuhn-Tucker theorem, if we can find $\lambda \in \Re$ and $\mu(j), \nu(j) \geq 0$ such that $\partial L/\partial\gamma(j) = 0$, where $\mu(j) = 0$ if $\gamma(j) > 0$ and $\nu(j) = 0$ if $\gamma(j) < 1$, then f_{SP} maximizes EU_0 over F'. In other words, if we let

$$
M = \sum_{j=0}^{(n-1)/2} C_j^n h_0(j)(2\gamma(j)-1) + \lambda W(q,\gamma)
$$

it suffices to find $\lambda \in \Re$ such that $\partial M/\partial\gamma(j) \geq 0$ if $\gamma(j) > 0$, $\partial M/\partial\gamma(j) \leq 0$ if $\gamma(j) < 1$, and $\partial M/\partial\gamma(j) = 0$ if $\gamma(j) \in (0,1)$.

We have $\partial M/\partial\gamma(j) =$

$$
2C_j^n h_0(j) + \lambda[C_j^{n-1}(q^j(1-q)^{n-j-1} + q^{n-j-1}(1-q)^j) - C_{j-1}^{n-1}(q^{j-1}(1-q)^{n-j} + q^{n-j}(1-q)^{j-1})].
$$

Since $h_0(j) = (1/2)(q^j(1-q)^{n-j} - q^{n-j}(1-q)^j)$, and by simplifying further, we have the formula

$$\frac{\partial M}{\partial \gamma(j)} = \frac{C_j^n}{n}\Big(n(q^j(1-q)^{n-j} - q^{n-j}(1-q)^j)$$
$$+\lambda[(n-j)(q^j(1-q)^{n-j-1} + q^{n-j-1}(1-q)^j) - j(q^{j-1}(1-q)^{n-j} + q^{n-j}(1-q)^{j-1})]\Big).$$

If we let $y = q/(1-q) > 1$, we can write this as

$$\frac{\partial M}{\partial \gamma(j)} = \frac{C_j^n}{n}(1-q)^{n-1}\Big(n(1-q)(y^j - y^{n-j}) + \lambda[(n-j)(y^j + y^{n-j-1}) - j(y^{j-1} + y^{n-j})]\Big).$$

Define

$$\theta(j) = \frac{y^{n-j} - y^j}{(n-j)(y^j + y^{n-j-1}) - j(y^{j-1} + y^{n-j})}.$$

We show that $\theta(j)$ is positive and strictly increasing for $j \in [0, n/2)$. Note that $\theta(0) = (y^n - 1)/(n(1 + y^{n+1})) > 0$ since $y > 1$. It suffices to show that $\partial\theta(j)/\partial j > 0$ for $j \in [0, n/2)$. We can write

$$\theta(j) = \frac{y^n - y^{2j}}{(n-j)(y^{2j} + y^{n-1}) - j(y^{2j-1} + y^n)}.$$

After some computation, we find

$$\frac{\partial\theta(j)}{\partial j} = \frac{(1+y)y^{4j-1}(y^{2n-4j} - (2n-4j)y^{n-2j} - 1)}{((n-j)(y^{2j} + y^{n-1}) - j(y^{2j-1} + y^n))^2}$$

which has the same sign as $y^{2n-4j} - (2n-4j)y^{n-2j} - 1$. This is positive by Fact 8 (let $x = n - 2j > 0$). Also note that the numerator of $\theta(j)$ is positive for $j \in [0, n/2)$ and thus the denominator of $\theta(j)$ is positive for $j \in [0, n/2)$.

We can write

$$\frac{\partial M}{\partial \gamma(j)} = \frac{C_j^n}{n}(1-q)^{n-1}\Big(-n(1-q)\theta(j) + \lambda\Big)[(n-j)(y^j + y^{n-j-1}) - j(y^{j-1} + y^{n-j})].$$

Note that the first two factors are positive and the last factor is the denominator of $\theta(j)$ and thus positive for $j \in [0, n/2)$. Hence the sign of $\partial M/\partial\gamma(j)$ is the sign of $-n(1-q)\theta(j) + \lambda$.

Let $\lambda = n(1-q)\theta(k)$. From the definition of f_{SP}, we have $\gamma_{SP}(j) = 1$ for $0 \le j \le k-1$, $\gamma(k) = z$, and $\gamma_{SP}(j) = 0$ for $k+1 \le j \le (n-1)/2$. It thus suffices to show that $-n(1-q)\theta(j) + \lambda \ge 0$ for $0 \le j \le k-1$ and $-n(1-q)\theta(j) + \lambda \le 0$ for $k+1 \le j \le (n-1)/2$. But this is true because $\theta(j)$ is strictly increasing for $j \in [0, n/2)$.

Now show that f_{SP} uniquely maximizes EU_0 over F'. Say that another procedure f maximizes EU_0 over F'. By the Kuhn-Tucker theorem, there exists $\lambda \in \Re$ such that

$\partial M/\partial\gamma(j) \geq 0$ if $\gamma(j) > 0$, $\partial M/\partial\gamma(j) \leq 0$ if $\gamma(j) < 1$, and $\partial M/\partial\gamma(j) = 0$ if $\gamma(j) \in (0,1)$, where M is defined above. Recall that the sign of $\partial M/\partial\gamma(j) \geq 0$ is the sign of $-n(1-q)\theta(j) + \lambda$ and strictly decreases for $j \in [0, n/2)$ because $\theta(j)$ strictly increases for $j \in [0, n/2)$. Hence there are three possible cases: $-n(1-q)\theta(j) + \lambda < 0$ for all j, $-n(1-q)\theta(j) + \lambda > 0$ for all j, or

$$\begin{aligned} \exists k \in \{0,1,\ldots,(n-1)/2\} \text{ such that } & -n(1-q)\theta(j) + \lambda > 0 \text{ for } j < k, \\ & -n(1-q)\theta(k) + \lambda \geq 0, \\ & -n(1-q)\theta(j) + \lambda < 0 \text{ for } j > k. \end{aligned}$$

In the first case, we have $\gamma(j) = 0$ for all $j \in \{0,1,\ldots,(n-1)/2\}$, which violates $W(q,\gamma) = 0$ and is thus not feasible. In the second case, we have $\gamma(j) = 1$ for all $j \in \{0,1,\ldots,(n-1)/2\}$, which violates $W(q,\gamma) = 0$ and is thus not feasible. In the third case, we have $\gamma(j) = 1$ for $j \in \{0,1,\ldots,k-1\}$, $\gamma(k) = z \in (0,1]$, and $\gamma(j) = 0$ for $j \in \{k+1,\ldots,(n-1)/2\}$. By Lemma 1, there is a unique k and z which makes $W(q,\gamma) = 0$, so we are done.

Now show that f_{SP} uniquely maximizes EU_0 over F for all but fewer than $(n-1)/2$ values of q. Say that γ maximizes EU_0 over F. Define γ'' as $\gamma''(j) = 1 - \gamma(n-j)$ and show γ'' is incentive compatible. Given ϕ, it suffices to show that $A(\gamma'',\phi) = 0$ and $B(\gamma'',\phi) = 0$. Define ϕ'' as $\phi''(a) = \phi(b)$ and $\phi''(b) = \phi(a)$. Since γ is incentive compatible, by Lemma 3 we have $A(\gamma,\phi'') = 0$ and $B(\gamma,\phi'') = 0$. By Fact 7 we have $A(\gamma'',\phi) = B(\gamma,\phi'') = 0$ and $B(\gamma'',\phi) = A(\gamma,\phi'') = 0$.

Define γ' as $\gamma'(j) = (\gamma(j) + \gamma''(j))/2$. Since γ and γ'' are incentive compatible, by convexity, γ' is incentive compatible. It is easy to see that EU_0 given γ is equal to EU_0 given γ'', and thus by convexity, EU_0 given γ' is equal to EU_0 given γ. Since γ' is symmetric and γ maximizes EU_0 over F, γ' maximizes EU_0 over F'. Since γ_{SP} uniquely maximizes EU_0 over F', we have $\gamma' = \gamma_{SP}$, in other words $(\gamma(j) + 1 - \gamma(n-j))/2 = \gamma_{SP}(j)$. Show that $\gamma = \gamma_{SP}$. Note that for j such that $\gamma_{SP}(j) = 0$, we must have $\gamma(j) = 0$ and $\gamma(n-j) = 1$, and hence $\gamma_{SP}(j) = \gamma(j)$. For j such that $\gamma_{SP}(j) = 1$, we must have $\gamma(j) = 1$ and $\gamma(n-j) = 0$, and hence $\gamma_{SP}(j) = \gamma(j)$. Hence it suffices to show that $\gamma(k) = z$ and $\gamma(n-k) = 1 - z$.

We know $(\gamma(k)+1-\gamma(n-k))/2 = \gamma_{SP}(k) = z$, and hence $1-\gamma(n-k)-z = z-\gamma(k)$. Since γ is incentive compatible, by Lemma 3 we have $0 = A(\phi, \gamma)$. In other words,

$$\begin{aligned}
0 = &C_{k-1}^{n-1}(h(k-1)(1-\gamma(k)) + h(n-k)\gamma(n-k)) \\
&+ C_k^{n-1}(h(k)\gamma(k) + h(n-k-1)(1-\gamma(n-k))) \\
&- C_{(n-1)/2}^{n-1} h((n-1)/2).
\end{aligned}$$

Since γ_{SP} is incentive compatible, we have $0 = A(\phi, \gamma_{SP})$, or in other words

$$\begin{aligned}
0 = &C_{k-1}^{n-1}(h(k-1)(1-z) + h(n-k)(1-z)) \\
&+ C_k^{n-1}(h(k)z + h(n-k-1)z) \\
&- C_{(n-1)/2}^{n-1} h((n-1)/2).
\end{aligned}$$

We can subtract these two equations to get

$$\begin{aligned}
0 = &C_{k-1}^{n-1}(h(k-1)(z-\gamma(k)) + h(n-k)(\gamma(n-k)-1+z)) \\
&+ C_k^{n-1}(h(k)(\gamma(k)-z) + h(n-k-1)(1-\gamma(n-k)-z)).
\end{aligned}$$

Since $1-\gamma(n-k)-z = z-\gamma(k)$, we have

$$0 = (z-\gamma(k))[C_{k-1}^{n-1}(h(k-1)-h(n-k)) + C_k^{n-1}(-h(k)+h(n-k-1))].$$

It suffices to show that $C_{k-1}^{n-1}(h(k-1)-h(n-k)) + C_k^{n-1}(-h(k)+h(n-k-1)) \neq 0$ for all but fewer than $(n-1)/2$ values of q, because then we can conclude that $z-\gamma(k) = 0$ and thus $\gamma(k) = z$ and $\gamma(n-k) = 1-z$.

Since $C_{k-1}^{n-1}(h(k-1)-h(n-k)) + C_k^{n-1}(-h(k)+h(n-k-1))$

$$= \frac{(n-1)!}{k!(n-k)!}[k(h(k-1)-h(n-k)) + (n-k)(-h(k)+h(n-k-1)]$$

it suffices to show that $k(h(k-1)-h(n-k)) + (n-k)(-h(k)+h(n-k-1)) = 0$ for at most $(n-1)/2$ values of q. With some calculation, we find that this expression is equal to $V_k(y)y^k(1-q)^n(\phi(a)(1/y-1)+1)$, where $V_k(y) = y^{n-2k}(n-k-ky)+k-(n-k)y$ and $y = q/(1-q) > 1$. We have $\phi(a)(1/y-1)+1 \neq 0$, and since there are at most $(n-1)/2$ possible values of k, it suffices to show that $V_k(y) = 0$ for at most one value of y such that $y > 1$. We calculate $V_k'(y) = y^{n-2k-1}((n-k)(n-2k)-k(n-2k+1)y)-(n-k)$ and $V_k''(y) = y^{n-2k-2}(n-2k)((n-k)(n-2k-1)-k(n-2k+1)y)$. Show that for $y > 1$, if $V_k'(y) \leq 0$, then $V_k''(y) < 0$. If $V_k'(y) \leq 0$, we have $y^{n-2k-1}((n-k)(n-2k)-k(n-2k+1)y) \leq n-k$.

If the left hand side of this inequality is negative or zero, then $(n-k)(n-2k-1)-k(n-2k+1)y<0$, and thus $V_k''(y)<0$. If the left hand side of this inequality is positive, then $(n-k)(n-2k)-k(n-2k+1)y<y^{n-2k-1}((n-k)(n-2k)-k(n-2k+1)y)$ and so $(n-k)(n-2k)-k(n-2k+1)y<n-k$, and thus $(n-k)(n-2k-1)-k(n-2k+1)y<0$ and so $V_k''(y)<0$.

Hence for $y>1$, if $V_k'(y)\leq 0$, then $V_k'(y')<0$ for all $y'>y$. Note that $V_k(1)=0$. If $V_k'(1)\leq 0$, then $V_k'(y)<0$ for all $y>1$, and so $V_k(y)<0$ for all $y>1$. If $V_k'(1)>0$, then either $V_k(y)>0$ for all $y>1$ or there exist roots of $V_k(y)$ which are greater than 1. Let y^* be the smallest such root (there are a finite number of roots since $V_k(y)$ is a polynomial in y). Since $V_k'(1)>0$, we cannot have $V_k'(y^*)>0$ because then (since $V_k(y)$ is continuous) y^* would not be the smallest root. So we must have $V_k'(y^*)\leq 0$. Hence $V_k'(y')<0$, and therefore $V_k(y)<0$, for $y'>y^*$. Hence $V_k(y)$ has at most one root greater than 1. ■

Proposition 3. Say that there exist $i,j\in N$ such that $\phi_i(a)<1-q$ and $\phi_j(a)>q$. Say f is an anonymous incentive compatible procedure which is not trivial. Then f is not monotonic.

Proof. Say f is anonymous, incentive compatible, and not trivial. Let $\gamma(\alpha(r))=f(r,a)$. Say f is monotonic, in other words $\gamma(j)\leq\gamma(j+1)$ for all j. As in the proof of Lemma 3, define $\gamma''(j)=1-\gamma(n-j)$ and define $\gamma'(j)=(\gamma(j)+\gamma''(j))/2$. Note that γ' is symmetric. It is easy to see that γ' is monotonic: since $\gamma(n-j)\geq\gamma(n-j-1)$, we have $1-\gamma(n-j)\leq 1-\gamma(n-j-1)$ and thus $\gamma''(j)\leq\gamma''(j+1)$, and since $\gamma(j)\leq\gamma(j+1)$, we have $\gamma'(j)\leq\gamma'(j+1)$. Since f is not trivial, there exists k such that $\gamma(k)<\gamma(k+1)$. Since f is monotonic, we have $\gamma(n-k-1)\leq\gamma(n-k)$ and thus $\gamma''(k)\leq\gamma'(k+1)$. Therefore $\gamma'(k)<\gamma'(k+1)$.

The proof of Lemma 3 shows that $W(q,\gamma')=0$. We have $W(q,\gamma')=C^{n-1}_{(n-1)/2}q^{(n-1)/2}(1-q)^{(n-1)/2}(\gamma'((n-1)/2)-\gamma'((n+1)/2))+\sum_{j=0}^{(n-3)/2}C^{n-1}_j(q^j(1-q)^{n-1-j}+q^{n-1-j}(1-q)^j)(\gamma'(j)-\gamma'(j+1))$. This expression is the weighted sum of $\gamma'(j)-\gamma'(j+1)$ terms, where all of the weights are greater than zero. Since $\gamma'(j)-\gamma'(j+1)\leq 0$ for all j, it must be that $\gamma'(j)-\gamma'(j+1)=0$ for all j, which contradicts $\gamma'(k)<\gamma'(k+1)$. ■