Investing in Failure: Equilibrium Constraints on Prior Beliefs in Bargaining Problems*

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Abstract

We consider the bargaining problem between two players who may invest *ex ante* in the value of the disagreement payoff of the bargaining protocol. As is the case in war and other interesting political environments, we assume that the disagreement payoffs have interdependent components that are a function of both investments. We characterize necessary conditions on equilibrium investment strategies in this environment, describe how investments and the probability of disagreement must vary across mechanisms, and specify what equilibrium conditions imply for constraints on various aspects of the design environment. We also demonstrate how to construct consistent sets of investment strategies, disagreement payoff functions, and incentive compatible mechanisms.

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1 Introduction

A large class of problems can be understood using models of bargaining with a disagreement point. Countries that cannot reach a settlement over disputed territory can go to war. Firms that fail to reach an agreement can use the courts, bureaucracy, or a legislature to try and resolve the dispute. In each of these contexts the parties can influence–usually at a cost–their odds of success in a war, court trial, agency hearing, or lobbying battle. It is well known that the institution governing the interaction can influence the probability of disagreement as well as the likely settlements. When players posses asymmetric information about the disagreement point the particular form of this uncertainty can also influence the odds of bargaining failure and lotteries over settlements. In most contexts, however, the utility of disagreement depends, at least partially, on actions that the parties take prior to disagreement. Countries can invest in military capacity, firms can experience and document losses, collect evidence, or establish relationships with corrupt government officials. Accordingly, it makes sense to think about disagreement payoffs as endogenous– the result of investment decisions that are made prior to negotiating.

To illustrate the point we begin with an example where our problem has traction. Consider two countries involved in a dispute over a prize of value 1. Each country can have high or low capacity to fight. Typically, one would model the situation by assuming that each nation had a "type" that was either high or low. In our example, the countries interact in a simple institution. The institution has proposal power and is able to offer a single settlement of 1/2 to each country. If both countries accept this settlement then they receive their share and the game ends. If either (or both) countries refuses the settlement, then both countries fight. We assume that the expected payoffs from conflict are given by a contest function that is represented in the following matrix

capacity	l	h
l	s, s	l, w
h	w, l	b, b

with the first and second entries corresponding to the payoff of coutries 1 and 2 respectively. Assume that $0 < l < b < s < \frac{1}{2} < w < w + l < 1$. This structure captures the ideas that wars are always inefficient (as they destroy resources), wars between strong states are more destructive than wars between weak states, and that it is better to be strong than weak if you are going to fight. A natural way to proceed would be to assume that the types are generated by an exogenous distribution; for simplicity we might assume that each nation has high capacity with probability 1/2 and that the types are independent. We might wonder if their is an equilibrium in which high types chose to reject the settlement and low types decide to accept the settlement. In fact this behavior is sequentially rational as long as $s \leq \frac{1}{2} \leq w$. This last conclusion is obtained by noting that a country's decision to reject a settlement is only payoff relevant when the other country is a low type (given the conjectured strategy profile). So when the above inequality holds neither type has an incentive to deviate from the proposed strategy profile. Given that the types occur with equal probability and this strategy, the expected payoff of each country is just $\frac{1}{4}\left[\frac{1}{2} + l + w + b\right]$. Moreover, the probability of not reaching a settlement (i.e. the probability of war) is $\frac{3}{4}$. Two comparative statics are immediate: equilibrium expected payoffs are increasing in the terms, l, w, b and the probability of war is constant (as long as changes in the parameters don't alter the validity of the inequality above). Now, suppose the probability that each state has high capacity is the result of strategic (equilibrium) play predicated on the expectation that countries make arming decisions which are hidden actions and then play the game described above. In this case, it seems natural to assume that selection of h imposes a cost of c on a nation. In this case, for a country to be willing to randomize over the types it must be the case that the expected payoff from l is exactly c less than the expected payoff from h. Here, we say expected payoff because a country does not know which level the other country will select. A little algebra leads to the conclusion that the probability that h is chosen must be $p = \frac{1-2(w-c)}{1-2(w-b+l)}$. Since our verification that neither state had an incentive to deviate did not hinge on the probability of h the conjectured strategies for playing the negotiation game are still sequentially rational. One observation is that the equilibrium value of p is now not typically 1/2. In other-words the assumption that the types are generated by any particular lottery is typically not consistent with the assumption that the investments are generated by equilibrium behavior. Of course if the parameters c, w, b, l all happen to solve the equation $\frac{1}{2} = \frac{1-2(w-c)}{1-2(w-b+l)}$ the assumption would be fine; but this is a rather knife edged assumption. Here (using the fact that a nation must be indifferent across the strategies of l and h) we can write the equilibrium expected utility of the game for a country as

$$\left(\frac{1-2(w-c)}{1-2(w-b+l)}\right)l+\left(1-\frac{1-2(w-c)}{1-2(w-b+l)}\right)s.$$

This expression simplifies to

$$s+(l-s)\left(\frac{1-2(w-c)}{1-2(w-b+l)}\right).$$

A few observations surface if we think about the consequences of varying some of the parameters. Observe that this payoff depends on s while the equilibrium payoff in the game with exogenous types does not. Moreover this expression is not linear, or even increasing, in the parameters l, w, b in the model that treated the lotteries over types as exogenous the equilibrium payoffs are linear in these parameters. Finally, the probability of war is

$$\left[1-\left(\frac{1-2(w-c)}{1-2(w-b+l)}\right)\right]^2$$

which depends on the features of the contest function (l, w, b).This is in contrast to the game with exogenous types, where the probability of war does not vary (locally) with the parameters of the game. This toy example, serves only to illustrate the motivation for and potential bite of considering choices that influence a country/player's outside option as being endogenous–determined by equilibrium play– and thus potentially dependent on expectations about downstream play.

It is not entirely surprising that expectations about the negotiating process can influence investment decisions. The literature on the hold-up problem considers investment by buyers and or sellers prior to trade (Gul, 2001; Segal and Whinston, 2002). In the hold-up problem the investments typically influence the value of agreement, not the value of the outside option and investment by the seller typically only influences the seller's value of agreement and not the buyer's. Here, we consider problems where the investments influence the value of the disagreement payoffs, or outside option, and influence both players' payoffs for disagreement. The closest connection we are aware of in the literature is Plott (1987). In that article Plott considers a model in which two parties invest in legal fees and then a contest function determines the trial winner. He shows that the choice of legal rules, in particular who pays the legal fees after a trial, can influence the quantity of legal expenses that the parties absorb and points out that the English rule may be inefficient. The Plott model differs from ours in that it does not model interactions in the trial or allow for settlements to be reached. In this way the Plott model is a special mechanism in our set-up where the probability of disagreement is always one. The papers by Plott and Gul, as well as the example described above, illustrate that institutions governing how conflicts are resolved can have systematic upstream effects on investment decisions. In other words, when the outside options depend on decisions which might depend on the bargaining game, additional restrictions on beliefs about outside options and play in these games may surface. We investigate this link between game forms and investments using a mechanism design approach. Specifically, we consider situations where players can make "hidden action" investments in anticipation of their play in a bargaining game that can either divide a pie or end in the outside options. In this paper we focus on characterizing restrictions that necessarily follow from equilibrium play, but do not focus on characterizing equilibrium play or even expressing sufficient conditions for equilibria of a particular form. It is interesting, however, that necessity implies significant amounts of structure for this problem.

The current paper draws on progress made in a few related papers. Meirowitz and Sartori (2008) consider models in which investment decisions are hidden actions and players bargain after making their investment decisions. They find a strong condition that is necessary and sufficient for the existence of equilibria in which disagreement/war is avoided. Moreover, the possibility of disagreement in equilibrium and the presence of investment strategies that involve randomizing are shown to be equivalent. This finding justifies our focus on restrictions that must hold in an equilibrium in which the players randomize in their effort choices. In our context, many standard bargaining games will only posses equilibria in which the investments are in pure strategies if probability of bargaining failure is 0. Given this result, analysis of interesting problems (in which the threat of war is real) involves randomization in effort choices. The Meirowitz and Sartori paper, however, is limited as it does not provide tight characterizations of equilibrium play, they instead focus on whether or not war can be avoided. As a consequence the paper does not tell us much about the relationship between institutional choice and equilibrium behavior. Another literature that is relevant to this study involves the use of mechanism design to establish "game free" results. One of the most influential applications of this approach is Myerson and Satterthwaite (1983) in which it is shown that agreement cannot be guaranteed in problems of bilateral trade. In the study of negotiations and war fighting Banks (1990) shows that in problems with one sided asymmetric information, the equilibrium settlements and probability of fighting must be monotone in the unobserved capacity of the privately informed nation in any equilibrium to any bargaining game. More recently, Fey and Ramsay (2008) consider problems in which both players posses private information and investigate when it is possible to construct institutions possessing equilibria in which the probability of war is 0. A key finding of Fey and Ramsay is that with interdependent values it is not possible to construct mechanisms with equilibria in which the probability of war is zero. In these papers a nation or both players has private information, but the beliefs about these types are exogenous. Together, the conclusions from Fey and Ramsay and Meirowitz and Sartori, justify a focus on problems with interdependent values and investment decisions that are not predictable (involve mixed strategies).

The paper begins by defining the general bargaining problem and characterizing a coherent view of institutions in this setting. Our analysis proceeds by first taking as given a fixed lottery over types and analyzing the induced problem of mechanism design with interdependent values. In an approach analogous to backward induction, we then use the results from this analysis to characterize equilibrium investment strategies and ultimately state and prove our main result.

2 The model

The first step is to define the set of situations for which our analysis applies. Consider the interaction between two players in anticipation of a negotiation. Each player, $i \in \{1, 2\}$ must first select a level of investment $a_i \in \mathbb{R}_+$ that will contribute to their disagreement payoff. In the case of international conflict, for example, this investment may be spent on arms. To allow investments to be in mixed strategies we write $F_i(\cdot)$ to capture the distribution function of a_i. We assume that the cost of investment a_i is given by $c_i(a_i)$ where $c_i(\cdot)$ is a strictly increasing and differentiable function. By $c_i'(a_i)$ we denote the first derivative of the cost at a_i and by $c_i^{-1}(\cdot)$ we denote the inverse of the cost function. The investment choices are assumed to be hidden actions–player i knows its choice of a_i but it does not observe the choice by player $-i$. The players then negotiate over a resource or prize that is under dispute. Without loss of generality we assume the prize is of size 1. Below we will discuss how to think about the details of a negotiation process and for now we just observe that the result of a successful negotiation is a pair of settlement payoffs (t_1, t_2). Given a settlement t_1, t_2 and a pair of investments a_1, a_2 the payoffs are $t_1 - c_1(a_1)$ and $t_2 - c_2(a_2)$ respectively. If the players fail to reach an agreement they can invoke the costly outside option. If they do so, investment levels will influence the payoffs from disagreement. Rather than specifying a particular functional form for these payoffs, we just assume that the expected payoff to player i from a disagreement when its investment level is a_i, and the investment level of the other player is a_{-i}, is given by $p_i(a_i, a_{-i}) - c_i(a_i)$. We impose the natural assumption that $p_i(\cdot, \cdot)$ is strictly increasing in the first argument and strictly decreasing in the second argument. Throughout, we assume that these functions are twice continuously differentiable. In order to connect with the literature on inefficient bargaining failure, it is natural to assume that $p_1(\mathbf{a}) + p_2(\mathbf{a}) < 1$ for all $\mathbf{a} \in \mathbb{R}^2_+$. None of the results in this paper, however, actually require this assumption. The arguments do, however, hinge on this sum having an upper bound.

Our approach to the analysis proceeds without imposing or assuming any particular model of negotiation. We think of a negotiation procedure, protocol, game or "institution" as a sequence of interactions that must eventually either distribute settlements to players or result in the disagreement outcome. We impose one important restriction on the class of mechanisms. We assume that the mechanism cannot monitor the players private information. We draw upon *the revelation principle* to establish results about a large class of games or strategic interactions in which the investment decisions are hidden actions by focusing on *direct revelation mechanisms* in which the reports are unverifiable.

Formally we say a direct revelation mechanism is a pair of message spaces M_1, M_2 and a triple of mappings

$$\begin{aligned} t_1 &: M_1 \times M_2 \to \mathbb{R}_+ \\ t_2 &: M_1 \times M_2 \to \mathbb{R}_+ \\ q &: M_1 \times M_2 \to [0,1]. \end{aligned}$$

So a direct revelation mechanism in our setting is then a cheap talk game where M_i is player i's possible hidden actions, q is the probability of disagreement, and t_i is i's report contingent "transfer" or "payoff." We could assume that transfers are only positive when q is zero, but as t_i can depend on reports, our simpler description is without loss of generality. We also note that in our setting the outcome of a disagreement, i.e., its payoff to the players, depends on the level of investment that each player has made in the investment stage but not on the reports of the players. If we think of the problem without the revelation principle, then the assumption is that outside options depend on investments but not on how the players bargain.

Before turning to the analysis an observation is worth making. We do not require that the transfers or payoffs satisfy budget balance. We also do not require that transfers are non-negative. Finally, we do not require that the players are willing to accept settlements that are distributed by the mechanism. To be clear we remain agnostic about these issues. We take this approach because the current paper highlights consequences of incentive compatibility and optimality of investments. Additional structure and concern about participation constraints, or ex-post individual rationality constraints would impose additional restrictions and might involve conditions that are even stronger than the incentive compatibility conditions that we work with. Importantly, however, the results we prove cannot be relaxed by adding more structure of this form. Since we do not focus on sufficient conditions for particular types of equilibria (or even existence of equilibria) this additional structure is not needed. Subsequent work that proceeds in the other direcetion will surely need to deal with these added complications.

3 Results

The analytic convenience of the revelation principle is that it allows us to learn about equilibria to any mechanism by focusing on just "truthful" equilibria to direct mechanisms. The latter are studied by way of incentive compatibility conditions, which ensure that players are willing to truthfully report their private information to the mediator in a direct mechanism. We begin with a fairly standard description of incentive compatible behavior in a direct mechanism, treating the distribution functions $F_i(\cdot)$ as fixed. Let F_i be player i's mixed strategy equilibrium distribution over the hidden action. Recall our direct mechanism is a pair of functions $t_i(m_i, m_j) : R_+^2 \to [0,1]$ that describes the report contingent transfer to i and a function $q(m_i, m_j) : R_+^2 \to [0,1]$ that determines the probability of disagreement. Expected utility to i of making a report m_i in this direct mechanism, given investment a_i , can then be written as

$$U_i(m_i|a_i) = \int [t_i(m_i, m_j) + q(m_i, m_j)p(a_i, m_j)]dF_j(m_j).$$

It is convenient to define

$$T_i(m_i) = \int T_i(m_i, m_j)dF_j(m_j),$$
$$P_i(m_i|a_i) = \int q(m_i, m_j)p(a_i, m_j)dF_j(m_j),$$

where $T_i(m_i)$ is the expected transfers when player i reports m_i and $P_i(m_i|a_i)$ is the expected value of disagreement, including the chance it happens, conditional on a_i. Thus we can describe a player's interim expected utility of a report m_i as

$$U_i(m_i|a_i) = T_i(m_i) + P_i(m_i|a_i)$$

In a slight abuse of notation, we let $U_i(a_i) = U_i(a_i|a_i)$. While we will impose no particular structure on $q(\cdot,\cdot)$ and F_1, F_2 as our goal is to determine things that must be true in any equilibrium, we will require that equilibrium is well defined in the sense that a mechanism is sufficiently well behaved so that expected utilities are well-defined. In particular, we will assume that $q(m_i, m_j)p(a_i, m_j)$ is integrable (in particular that $\int q(m_i, m_j)p(a_i, m_j)dF_j(m_j)$ is a real number). Note that this also implies that $\int q(m_i, m_j)dF_j(m_j) isarealnumber.$[1]

The direct revelation mechanism framework is useful as it allows us to characterize incentives across game forms and various equilibria. Throughout the paper we invoke the following revelation principle.

Revelation Principle *(Myerson, 1979) If there exists a game with equilibrium investing decisions given by the mixtures F_1 and F_2 and the lottery $G(t_1, t_2, p_1, p_2)$ over transfers and disagreement payoffs, then there is a direct mechanism possessing an equilibrium in which investing strategies are given by F_1 and F_2 and the states report truthfully $m_i(a_i) = a_i$, which induces the same lottery over the outcomes.*

For fixed investment strategies the argument involves the standard composition strategy as found in Myerson (1979). Since investment decisions are privately observed and reports are unverifiable this first stage introduces no additional complications. Without loss of generality we will proceed by looking at equilibrium incentives in direct mechanisms and focus on Bayesian Nash equilibria to the induced games.

It is important to notice what a direct mechanism represents. The theorem states that in our study of lotteries over settlements and bargaining failure it is sufficient describe an equilibrium to a game as a triple $< t_1, t_2, q >$. This means two triples $< t_1, t_2, q >$ and $< t_1', t_2', q' >$ can represent equilibria from completely different game forms or two different equilibria selected from a given game form. Our results apply either way. The fact that these results are relevant for both constructions is important. This is because when we talk about triples as "institutions" we speak to two distinct visions of institutions in the literature. In the applied literature comparison of institutions is usually a statement about the equilibrium correspondence of two distinct extensive form games. For example, we may consider the difference of equilibrium payoffs and the probability of war across game forms where countries negotiate bilaterally and when they negotiate within a framework of a third party organization like the African Union. Similarly, in the law and economics literature

[1]This connection follows from the dominated convergence theorem and the fact that since $p(\cdot, \cdot)$ is bounded by 1 the latter is bounded by 1 plus the former integral.

one might compare litigation games under American and English rules for fees. In each case, we can describe an equilibrium of each game form with its own mappings $< t_1, t_2, q >$.

There is also another view, more common in political economy, which takes institutions to be an equilibrium to some game or super-game, where there may exist opportunities for renegotiation or focal equilibrium selection (Calvert, 1995). Here, when comparing two different equilibria to such a game, we also have two different sets of mappings. That is, the method works equally well for comparing across equilibria of a single game and comparing equilibria across different games. It is prudent to be careful in deciding which interpretation one wishes to invoke as the interpretations of the results depends on this choice.

A second observation that we make regards the supports of investment strategies. Since investments matter only in that they influence the payoff $p_i(\cdot, \cdot)$ from bargaining failure, and they impose a cost $c_i(\cdot)$, any investment a_i with $c_i(a_i) > 1 \geq \max_{a_i} p_i(a_i, 0)$ is strictly dominated by $a_i' = 0$. Accordingly, we know that in any equilibrium the investments will be bounded between 0 and $c_i^{-1}(1) = b$. Accordingly, when proving results about equilibria we know that the investments have a support that is compact. In particular, we can assume that the investments have a support contained in some interval $[\underline{a}_i, \overline{a}_i] \subseteq [0, b]$.

We can now proceed with our results. Our first result characterizes the value of playing a bargaining game as a function of the pre-play investment choices. Before proving the formal result it is useful to describe the intuition behind the result and howit follows from incentive compatibility conditions. For a fixed pair of distributions F_1, F_2 suppose $< t_1', t_2', q' >$ is an equilibrium to a game. By the Revelation Principle there is a direct mechanism $< t_1, t_2, q >$ such that players truthfully report and the mechanism induces the same lottery over outcomes. Consider player i's strategy. In this direct mechanism incentive compatibility implies that for any $a_i > a_i'$

$$U_i(a_i) = T_i(a_i) + P_i(a_i|a_i) \geq T_i(a_i') + P_i(a_i'|a_i)$$

and

$$U_i(a_i') = T_i(a_i') + P_i(a_i'|a_i') \geq T_i(a_i) + P_i(a_i|a_i').$$

By standard arguments, if at a_i' the function $q(\cdot, a_j)$ is continuous for almost every a_j in the support of F_j then

$$\frac{\partial U_i(a_i')}{\partial a_i} = \int \left[\frac{\partial p_i(a_i', a_j)}{\partial a_i}\right] q(a_i', a_j) dF_j(a_j).$$

It is not unreasonable to worry that, given the complications that may arise from mixtures that do not have densities, this envelope theorem result will not hold. The following theorem builds on this intuition and illustrates that this relationship holds for almost every a_i even for our slightly more complicated case. The proof shows that the well know envelope theorem of Milgrom and Segal (2002) applies to our environment with probability distributions that arise from mixed strategies and q functions that need not be continuous.

Theorem 1 *If $< t_1, t_2, q, F_1, F_2 >$ is an equilibrium then (1) for almost every a_i in $[0, b] \subseteq [\underline{a}_i, \overline{a}_i]$ the derivitive of the value function exists and is given by*

$$U_i'(a_i) = \int_{\underline{a_j}}^{a_j} \left[\frac{\partial p_i(a_i, a_j)}{\partial a_i}\right] q(a_i, a_j) dF_j(a_j)$$

and (2) net of costs, the value function is given by

$$U_i(\widehat{a}_i) = U_i(\underline{a}_i) + \int_{\underline{a}_i}^{\widehat{a}_i} \int_{\underline{a}_j}^{a_j} \left[\frac{\partial p_i(a_i, a_j)}{\partial a_i}\right] q(a_i, a_j) dF_j(a_j) da_i$$

Proof. Fix F_1, F_2 and suppose $< t_1', t_2', q' >$ is an equilibrium to such a game. By the Revelation Principle there is a direct mechanism $< t_1, t_2, q >$ such that players truthfully report and the mechanism induces the same lottery over outcomes. We use Milgrom and Segal (2002, Thrm 2) to establish claims (1) and (2).

Milgrom and Segal state three sufficient conditions (which we state in our notation):

(i) $U_i(a_i) = U_i(m_i|a_i)$ is absolutely continuous and differentiable in a_i for all m_i;

(ii) there exists an integrable function $g : [a_i, a_i] \to \mathbb{R}$ such that $\left|\frac{\partial U_i(m_i|a_i)}{\partial a_i}\right| \leq g(a_i)$ for all m_i and almost every a_i; and

(iii) that an optimal response, m_i, exists for each type a_i.

Condition (iii) is satisfied in any incentive compatible direct mechanism. The rest of the proof focuses on verifying that the first two conditions are satisfied and proceeds in in three steps. First we show that

$$\frac{\partial U_i}{\partial a_i}(m_i|a_i) = \frac{\partial}{\partial a_i}\int_{\underline{a}_j}^{\bar{a}_j} p_i(a_i,t)q(a_i,t)dF_j(t) = \int_{\underline{a}_j}^{\bar{a}_j} \frac{\partial p_i}{\partial a_i}(a_i,t)q(a_i,t)dF_j(t).$$

Second we show that $U_i(m_i|a_i)$ is Lipschitz continuous in a_i, which reduces to showing that the derivative of U_i with respect to a_i is bounded. We then use these conclusions to show that $U_i(m_i|a_i)$ is absolutely continuous in a_i and bounded by a linear function and thus equal to the integral of its derivative almost everywhere.

To begin, recall that we have assumed that q and F_1, F_2 are sufficiently well behaved that all of the relevant integrals exist. Second observe that the investment level only enters $U_i(m_i|a_i)$ through the term $P_i(m_i|a_i)$, so we can ignore the term $T_i(m_i)$. We now characterize the derivative of $U_i(m_i|a_i)$ with respect to a_i.

Lemma 1 $\frac{\partial U_i}{\partial a_i}(m_i|a_i) = \int \frac{\partial p(a_i,m_j)}{\partial a_i} q(m_i,m_j)dF_j(m_j)$ *at every* a_i, m_i.

Proof. We verify that standard conditions for interchanging the order of integration and differentiation are satisfied. We use the versions presented in Durrett (1995, Thrm 9.1):

(a) $\int |q(m_i,m_j)p(a_i,m_j)|\, dF_j(m_j) < \infty$;

(b) $\frac{\partial p(a_i,m_j)}{\partial a_i} q(m_i,m_j)$ exists and is continuous in a_i for each m_i, m_j;

(c) $\int \frac{\partial p(a_i,m_j)}{\partial a_i} q(m_i,m_j)dF_j(m_j)$ is continuous in a_i for every m_i;

(d) and for each a_i, m_i there is some $\delta > 0$ s.t.

$$\int\int_{-\delta}^{\delta} \left| \frac{\partial p(a_i+\theta,m_j)}{\partial a_i} q(m_i,m_j)\right| d\theta dF_j(m_j) < \infty.$$

Condition (a) holds as both p and q are non-negative so

$$\int |q(m_i, m_j)p(a_i, m_j)|\, dF_j(m_j) = \int q(m_i, m_j)p(a_i, m_j)dF_j(m_j)$$

where we have assumed that the latter is finite.

Conditions (b) and (c) follow from the assumption that p is continuously differentiable.

Condition (d) is verified as follows. Since p is twice continuously differentiable in all its arguments, we know that the first derivative of p is continuous. Since we are focusing on a compact set this derivative attains a finite maximum. Thus we may concluded that the derivative of p is bounded and p is Lipshitz continuous. As q is also bounded above by 1 and the mixed strategy dF_j integrates to 1, there exists a constant M s.t.

$$\begin{aligned}
\int \int_{-\delta}^{\delta} \left| \frac{\partial p(a_i + \theta, m_j)}{\partial a_i} q(m_i, m_j) \right| d\theta dF_j(m_j) &< \\
\int \int_{-\delta}^{\delta} M\theta d\theta dF_j(m_j) &= \\
\int 2\delta M dF_j(m_j) &= 2\delta M < \infty,
\end{aligned}$$

satisfying the last condition of Durrett, proving our identity. ■

Next we show that $U_i(m_i, a_i)$ is also Lipschitz continuous.

Lemma 2 *$U_i(m_i|a_i)$ is Lipshitz continuous.*

Proof. Because p is Lipschitz continuous, there exists a number k such that

$$\frac{\partial p}{\partial a_i}(a_i, m_j) < k.$$

Moreover since $0 \leq q(m_i, m_j) \leq 1$, F_j integrates to 1, and we have the derivative of

$U_i(m_i|a_i)$ as described in Lemma 1, we get

$$\frac{\partial U_i}{\partial a_i}(m_i|a_i) = \int_{\underline{a}_j}^{\bar{a}_j} \frac{\partial p}{\partial a_i}(a_i,t)q(m_i,t)dF_j(t) < k\int_{\underline{a}_j}^{\bar{a}_j} q(m_i,t)dF_j(t) < k[\bar{a}_j - \underline{a}_j]$$

and $U_i(m_i|a_i)$ is thus Lipshitz continuous. ■

So Lemma 2 implies that (ii) is satisfied with $g(a_i) = Ka_i$ for some constant K. Moreover, since Lipshitz continuity implies absolute continuity on the interval and Lemma 1 establishes the differentiability of $U_i(m_i|\cdot)$ for each m_i condition (i) is satisfied, completing the proof.

■

With these results we see that treating the distribution over investment levels as fixed we can now focus on the study of what types of investment strategies are actually possible in an equilibrium. What we find is that the equilibrium conditions from strategic investment pin down a number of characteristics of the equilibrium.

Theorem 2 *In any equilibrium to any game, if a_i is in the support of i's mixed strategy then*

$$c_i'(a_i) = \int \frac{\partial p_i(a_i,a_j)}{\partial a_i} q(a_i,a_j)dF_j(a_j) \tag{1}$$

Proof. Suppose $< t_1', t_2', q', F_1, F_2 >$ is an equilibrium. By the Revelation Principle there is a direct mechanism (t,q) such that players truthfully report and the mechanism induces the same lottery over outcomes. We focus on such a direct mechanism. For F_1, F_2 to constitute equilibrium mixed strategies it must be the case that for every pair of arming levels , a_i and a_i' in a set that occurs with probability one under F_i $U_i(a_i) - U_i(a_i') = c_i(a_i) - c_i(a_i')$. Taking limits and applying Theorem 2 we obtain

$$\int \frac{\partial p_i(a_i,a_j)}{\partial a_i} q(a_i,a_j)dF_j(a_j) = c_i'(a_i)$$

■

This means there is a clear relationship between q and F whenever a_i is in the support of player i's mixed strategy. To make the exposition slightly more natural we write every a_i when we should say something like F_i-almost every a_i. As p and

c are exogenous, we conclude that q andF_j have to be "offsetting" in a very specific sense for any two equilibria that use the same investment levels for a fixed p and c.

3.1 Comparative Statics and Separability

Suppose we have two equilibria $< t_1, t_2, q, F_1, F_2 >$ and $< t_1, t_2, q, F_1, F_2 >$ in which investment level a_i is in the supports of both F_i and $\widehat{F}_i$. Then Theorem 2 implies that

$$\int \frac{\partial p_i(a_i, a_j)}{\partial a_i} q(a_i, a_j) dF_j(a_j) = \int \frac{\partial p_i(a_i, a_j)}{\partial a_i} \widehat{q}(a_i, a_j) dF'_j(a_j).$$

Multiplying both sides by 1 yields

$$\int q(a_i, a_j) dF_j(a_j) \frac{\int \frac{\partial p_i(a_i,a_j)}{\partial a_i} q(a_i, a_j) dF_j(a_j)}{\int q(a_i, a_j) dF_j(a_j)} = \int \widehat{q}(a_i, a_j) dF_j(a_j) \frac{\int \frac{\partial p_i(a_i,a_j)}{\partial a_i} \widehat{q}(a_i, a_j) d\widehat{F}_j(a_j)}{\int \widehat{q}(a_i, a_j) d\widehat{F}_j(a_j)}.$$

Letting *disagreemnt* and $\widehat{disagreement}$ denote the events in which a settlement is not reached in the two equilibria, and $\Pr(disagreement \mid a_i)$ and $\Pr(\widehat{disagreement} \mid a_i)$ denote the equilibrium probabilities of disagreement conditional on investing at level a_i in the two equilibria we have

$$\frac{\Pr(disagreement \mid a_i)}{\Pr(\widehat{disagreement} \mid a_i)} = \frac{\mathbb{E}(\frac{\partial p_i(a_i,a_j)}{\partial a_i} \mid \widehat{disagreement})}{\mathbb{E}(\frac{\partial p_i(a_i,a_j)}{\partial a_i} \mid disagreement)}.$$

In other words, the likelihood of disagreement and the marginal value of investing given no settlement are in some sense compliments. That is, if the probability of no settlement given a_i is higher in one equilibrium than the other, than the expected marginal effect from a_i must be ordered in the opposite way across these equilibria.

The strength and implications of this relationship can be most easily seen if we impose some assumptions on the exogenous functions, p and c. Specifically if the cross partial derivative of p is 0, i.e $p_i(a_i, a_j)$ is of the form $g(a_i) + h(a_j)$ then the probability of no settlement given a_i is constant. In particular, in this seperable case, Theorem 2 implies the following result:

Corollary 1 *If $p_i(a_i, a_j) = g(a_i) + h(a_j)$ then the probability of disagreement given investment level a_i is the same in every equilibrium in which investment level a_i is in the support of i's investment strategy. In particular, the following formula holds*

$$\Pr(disagreement \mid a_i) = \frac{c_i'(a_i)}{g'(a_i)}.$$

In this separable case an increase in the marginal cost of investing at level a_i or a decrease in in the marginal value of a_i from disagreement must result in an increase in the probability of no settlement given a_i. For the seperable case, this means that holding fixed technology c, p variation in the game form q, t only effects the likelihood of bargaining failure through its effect on the investment strategies. The law of iterative expectations tells us that

Corollary 2 *In the separable case, if both $< t_1, t_2, q, F_1, F_2 >$ and $< t_1', t_2', q', F_1, F_2 >$ are equilibria then they both induce the same probability of disagreement.*

In particular in this case we have $\Pr(disagreement) = \int \frac{c_i(a_i)}{g(a_i)} dF_i(a_i)$.

While the separable case is knife-edged, this result illustrates that the pathway by which the choice of game form, q and t, can influence the conditional probability of bargaining failure depends on the investments being compliments or substitutes (at least locally). We find the effect of institutions on conditional probabilities of no settlement is continuous in the magnitude of the cross partial of p. In particular, let $\mathbb{P}(a_i) = \max_{a_j} \frac{\partial p_i(a_i,a_j)}{\partial a_i} - \min_{a_j} \frac{\partial p_i(a_i,a_j)}{\partial a_i}$. This value is finite since p is continuously differentiable and the support is compact. This value is 0 in the seperable case. Moreover, let $\mathbb{E}p(a_i) = \int \frac{\partial p_i(a_i,a_j)}{\partial a_i} dF_j(a_j)$ and $\mathbb{E}\widehat{p}(a_i) = \int \frac{\partial p_i(a_i,a_j)}{\partial a_i} d\widehat{F}_j(a_j)$. Define, the difference in conditional probability of war between two equilibria as

$$\Delta(a_i) := \left| \int \frac{\partial p_i(a_i, a_j)}{\partial a_i} q(a_i, a_j) dF_j(a_j) - \int \frac{\partial p_i(a_i, a_j)}{\partial a_i} \widehat{q}(a_i, a_j) d\widehat{F}_j(a_j) \right|.$$

Recall that q and F are bounded between 0 and 1, implying that the following three inequalities hold,

$$\begin{aligned}
\left|\int \frac{\partial p_i(a_i,a_j)}{\partial a_i} q(a_i,a_j)dF_j(a_j) - \mathbb{E}p(a_i)\int q(a_i,a_j)dF_j(a_j)\right| &< \mathbb{P}(a_i), \\
\left|\int \frac{\partial p_i(a_i,a_j)}{\partial a_i} \widehat{q}(a_i,a_j)d\widehat{F}_j(a_j) - \mathbb{E}\widehat{p}(a_i)\int \widehat{q}(a_i,a_j)d\widehat{F}_j(a_j)\right| &< \mathbb{P}(a_i), \\
|\mathbb{E}p(a_i) - \mathbb{E}\widehat{p}(a_i)| &< \mathbb{P}(a_i).
\end{aligned}$$

The first two of these inequalities imply that

$$\Delta(a_i) < \left|\mathbb{E}p(a_i)\int q(a_i,a_j)dF_j(a_j) - \mathbb{E}\widehat{p}(a_i)\int \widehat{q}(a_i,a_j)d\widehat{F}_j(a_j)\right| + 2\mathbb{P}(a_i),$$

and the third implies that

$$\left|\mathbb{E}p(a_i)\int q(a_i,a_j)dF_j(a_j) - \mathbb{E}\widehat{p}(a_i)\int \widehat{q}(a_i,a_j)d\widehat{F}_j(a_j)\right| < \\
\mathbb{P}(a_i)\left|\left(\int q(a_i,a_j)dF_j(a_j) - \int \widehat{q}(a_i,a_j)d\widehat{F}_j(a_j)\right)\right|.$$

Combining these last two inequalities we obtain

$$\begin{aligned}
\Delta(a_i) &< \mathbb{P}(a_i)\Delta(a_i) + 2\mathbb{P}(a_i), \\
\Delta(a_i)(1-\mathbb{P}(a_i)) &< 2\mathbb{P}(a_i), \\
\Delta(a_i) &< \frac{2\mathbb{P}(a_i)}{(1-\mathbb{P}(a_i))}.
\end{aligned}$$

Thus we see that as $\mathbb{P}(a_i)$ vanishes so too does $\Delta(a_i)$. This bounding condition is summarized in the following proposition.

Theorem 3 *As the cross partial of* p *as parameterized by* $\mathbb{P}(a_i)$ *vanishes, so does the difference in conditional probability of bargaining failure across equilibria,* $\Delta(a_i)$.

3.2 Admissability

To summarize, thus far we have focused on necessary conditions that must hold in situations where mixing over investments is optimal given rational expectations about bargaining behavior and bargaining behavior constitutes an equilibrium given "correct" beliefs about the investment strategies. In particular, Theorem 2 shows that there is a very specific relationship between the probability of settlements and the investment strategies used in equilibrium. Moreover, this relationship depends on the exogenous technology p and c. In this section we elaborate on the structure of this relationship and consider a slightly different aspect of the problem. What must be true about mixed strategies over investments that will fulfill the necessary conditions of equilibrium strategies? What must be true about the technology and function determining the probability of settlement, q, in order for it to be possible that some equilibrium involves this function q. The approach here, then builds on the insight that both q and F_1, F_2 are endogenous and so the existence of equilibria in which a particular function q obtains hinge on the possibility of finding some functions F_1, F_2 that satisfy the condition in Theorem 2 with this particular q.

Recall from above that Theorem 2 states that if $< t_1, t_2, q, F_1, F_2 >$ is an equilibrium then F_1 and F_2 solve the following integral equations

$$\begin{aligned}
\int \frac{\partial p_1(a_1, a_2)}{\partial a_1} q(a_1, a_2) dF_2(a_2) &= c_1'(a_1) \text{ for each } a_1 \text{ in the support of } F_1 \\
\int \frac{\partial p_2(a_2, a_1)}{\partial a_2} q(a_1, a_2) dF_1(a_1) &= c_2'(a_2) \text{ for each } a_2 \text{ in the support of } F_2
\end{aligned}$$

It is instructive to rewrite these conditions as an orthogonality condition. Then most of the intuition behind our characterization can be gleamed from the following heuristic argument. If we pretended that the supports were finite and could still use the condition in Theorem 2, then the mixtures over effort would be a non-trivial solution to a homogeneous system of linear equation. In particular, we could think about the linear algebra problem where the existence of solutions to the system would hinge on finding a lottery (vector) that is orthogonal to the vectors capturing $\frac{\partial p_i(a_i,a_j)}{\partial a_i} q(a_i, a_j) - c_i'(a_i)$ for each a_i. Since we are working with functions instead of vectors, results about codimension do not directly apply, but much of our intuition from finite dimensional vector spaces does carry through.

To proceed we let $b_i = c_i^{-1}(1)$. So in any undominated investment strategy by i the support, S_i is a subset of $[0, b_i]$. When it is clear that we are focusing on just the condition for one player, we drop the subscript on b_i. Let $\mathcal{L}^2$ denote the set of square-integrable functions from $[0, b]$ into $\mathbb{R}^1$. Then for any two functions $x, y \in \mathcal{L}^2$ the mapping $\langle x \mid y \rangle = \int_0^b x(t)y(t)dt$ is an inner product. Accordingly, with the norm $\|x\| = \sqrt{\langle x \mid x \rangle}$ we have a Hilbert Space. For convenience we focus on distributions over investments that have densities, f_1, f_2 on supports contained in $[0, b_1], [0, b_2]$ respectively. The case of densities is particularly interesting given that most applied models assume that types are drawn from a density. This analysis then seeks to provide foundations for this common modeling assumption. For any set $A \subset \mathcal{L}^2$, the orthogonal complement of A, denoted $A^{\perp}$ is the set $\{y \in \mathcal{L}^2 : \forall x \in A, \langle x \mid y \rangle = 0\}$. This is the set of elements of $\mathcal{L}^2$ that are orthogonal to each element of A.

We treat payoff functions (p_1, p_2), cost functions (c_1, c_2), and incentive compatible mechanisms (t_1, t_2, q) as parameters and define a new function of (a_i, a_{-i}), $h_i(a_i, a_{-i}|q, p, c) = \frac{\partial p_i(a_i, a_j)}{\partial a_i} q(a_i, a_j) - c_i'(a_i)$. So $h_i(a_i, \cdot|q, p, c)$ is a function in $\mathcal{L}^2$. For fixed q, p, c we can then think about the set of such functions, $H_i(q, p, c) = \bigcup\limits_{a_i \in S_i} h_i(a_i \cdot |q, p, c)$. To recap, we will restate the condition in Theorem 2 as a system of inner product conditions, just as a system of linear equations of the form $Ax = 0$ can be conceived of a finite number of orthogonality conditions. The analogue to the matrix A is the set $H_i(q, p, c)$. Our analysis is then analogues to the study of how elements of the matrix A influence solutions to the system. Next let $\mathcal{H}_i(q, p, c)$ denote the subspace of $\mathcal{L}^2$ spanned by $H_i(q, p, c)$. We then have the following necessity result.

Theorem 4 *For a fixed set of primitives,* $p_1(\cdot, \cdot), p_2(\cdot, \cdot), c_1(\cdot), c_2(\cdot)$ *the following are true:*

(a)If $< t_1, t_2, q, f_1, f_2 >$ *is an equilibrium then*

(i) $\mathcal{H}_1(q, p, c)$ *and* $\mathcal{H}_2(q, p, c)$ *cannot be equal to* $\mathcal{L}^2$

(ii) for $i = 1, 2,$ $f_i \in \mathcal{H}_{-i}^{\perp}(q, p, c)$ *and* $\mathcal{H}_{-i}(q, p, c)$ *is in the orthogonal complement of the span of* f_i

(iii) if in addition the function q *is never 0 or 1, then for any* $\varepsilon > 0$ *there*

exists a functions q' such that $\|q - q'\| < \varepsilon$ and there are no equilibria with these primitives and q'

(b) if for fixed q, there are equilibria with two different densities, f_i and f_i' then there is also an equilibrium with $\alpha f_i + (1-\alpha) f_i'$ for any $\alpha \in (0,1)$.

Proof. Observe that the conditions in Theorem 2 is equivalent to

$$\langle h_i(a_i, \cdot | q, p, c) \mid f_{-i}(\cdot) \rangle = 0 \text{ for each } a_i \in S_i.$$

(a):

(i) If $\mathcal{H}_i(q,p,c) = \mathcal{L}^2$ then $\mathcal{H}_i^{\perp}(q,p,c)$ is empty and there are no functions f_{-i} satisfying the necessary condition in the theorem.

(ii) This characterization of the solutions is definitional.

(iii) A well known result of Hilbert spaces (stated in our notation) is that if $A \subset \mathcal{L}^2$ and A contains an open ball then $A^{\perp} = \{0\}$ where the $\mathbf{0}$ element does not integrate to 1 and thus cannot be a density. This implies that if $< t_1, t_2, q, f_1, f_2 >$ is an equilibrium there is a function q' arbitrarily close to q for which f_{-i} does not solve $\langle h_i(a_i, \cdot; q', p, c) \mid f_{-i} \rangle = 0$. But this only implies that any particular solution will no longer be a solution after some small perturbation of q. To see that there are some small perturbations that result in no solutions the argument differs. The finite dimensional analogue of the following argument is simple: arbitrarily close to every n by n matrix of rank less than n is a matrix of full rank. We start with the fact that since $\mathcal{L}^2$ is separable any total set (orthonormal basis) of a subspace of $\mathcal{L}^2$ is countable.[2] So let $\mathcal{F}_i \subset \mathcal{H}_i^{\perp}(q,p,c)$ be a total set of $\mathcal{H}_i^{\perp}(q,p,c)$. The key here is that $\mathcal{F}$ forms a a set that is dense in $\mathcal{H}_i^{\perp}(q,p,c)$ and it is countable. Using diagonalization we can define a function $W(\cdot)(a_{-i}) : [0,b] \to \mathcal{F}_i$ with the property that on each interval in $[0,b]$ the function is surjective. So for each value of a_{-i}, $W(\cdot)(a_{-i})$ is a function in the $\mathcal{H}_i^{\perp}(q,p,c)$, and over any interval O in $[0,b]$, $W(O)(a_{-i})$ is a total set of $\mathcal{H}_i^{\perp}(q,p,c)$. For $\delta > 0$ let

$$q^{\delta}(a_i, a_{-i}) = q(a_i, a_{-i}) + \delta \left(\frac{\partial p_i(a_i, a_{-i})}{\partial a_i} \right)^{-1} W(a_i)(a_{-i}).$$

[2]See theorem 3.6-4 of Kreyszig (1978). A convenient example of a total set is the polynomials with rational coefficients.

By continuity of the inner product (and norm), for each $\varepsilon > 0$ there is some δ s.t. $\left\| h_i(a_i, \cdot; q^\delta, p, c) - h_i(a_i, \cdot; q, p, c) \right\| < \varepsilon$. Moreover, since q is in the interior of $(0,1)$ for small enough δ, q^δ is still in $[0,1]$. But we assert that the orthogonal complement of $\mathcal{H}_i(q^\delta, p, c)$ is empty. This conclusion implies (by way of part (ii) of this theorem) that there are no densities with supports contained in $[0, b_{-i}]$ that solve the condition in Theorem 2 with q^δ and these primitives. To prove this assertion we now show that there are no functions in $\mathcal{L}^2$ satisfying

$$\left\langle h_{-i}(a_i, \cdot; q^\delta, p, c) \mid f \right\rangle = 0$$

for each a_i in any interval. We focus on the conditions that have to hold for an arbitrary interval O of values of a_i.

By linearity we have

$$\left\langle h_i(a_i, \cdot; q^\delta, p, c) \mid f \right\rangle = \\ \int_0^{b_{-i}} \left[\frac{\partial p_i(a_i, a_j)}{\partial a_i} q(a_i, a_j) - c_i'(a_i) \right] f(t)dt + \delta \int_0^1 W(a_i)(a_{-i}) f(t)dt.$$

There are two cases: either an arbitrary function f is in $\mathcal{H}_i^\perp(q, p, c)$ or it is not. In the former case we know that the first term of the right hand side is 0 but the second term is not always 0–as $f \in \mathcal{H}_i^\perp(q, p, c)$ implies that $f \notin \mathcal{H}_i^\perp(q, p, c)^\perp$ and thus since $\mathcal{F}_i$ is a total set any density (thus not the 0 function), f cannot be orthogonal to every function in $\mathcal{F}_i$. And thus since $W(\cdot)(a_{-i})$ is surjective on each interval O, the second term of the right hand side is strictly positive for some value of a_i in each interval (and the first term is non negative). This means that f is not in $\mathcal{H}_i^\perp(q^\delta, p, c)$. In the latter case, the first term of the right hand side is not always 0 (otherwise f would have been a solution with q). Moreover, $f \notin \mathcal{H}_i^\perp(q, p, c)$ implies that $f \in \mathcal{H}_i(q, p, c)$ and is orthogonal to every function in $\mathcal{F}_i$ so the second term is 0, so again f is not in $\mathcal{H}_i^\perp(q^\delta, p, c)$.

(b) The result follows from the linearity of orthogonal compliments and the fact that if f_i and f_i' are non negative and integrate to 1, then any convex combination of these functions also satisfies these conditions. ■

3.2.1 Example

From our result we observe that there are a substantial number of constrains regarding the relationship between an equilibrium probability of bargaining failure, the cost function, the technology of disagreement payoffs, and the mixing strategies. Starting with the equation from Theorem 2, we have

$$\int_{S_2} \frac{\partial p}{\partial a_1}(a_1, t) q(a_1, t) f_2(t) dt = c, \tag{2}$$

whenever the mixed strategy can be represented by a smooth density and costs are linear. We may use this equation answer several types of questions. For example, what must q and p_i be like in an equilibrium where player j mixes uniformly over an interval of investments? Or, what p and f would support a particular equilibrium probability of bargaining failure for particular levels of investment?

To illustrate how one might conduct this type of analysis, we see what types of technology and densities could support a mechanism with an "upper-triangle" probability of disagreement, where bargaining failure occurs with probability 1 if the sum of investments exceeds 1 and zero otherwise. Consider the case where types are drawn from densities over an interval of investments $[0, b]$ with $b > 2$. In this example, b is the supremum of the un-dominated arming strategies. Recall an investment level that costs more than the value of the outside option obtained when the other player invests 0 is dominated. With this q and a smooth mixture with support on all un-dominated strategies we can then rewrite (2) such that

$$\int_{1-a}^{b} \frac{\partial p}{\partial a_1}(a_1, t) f_2(t) dt = c \tag{3}$$

for all a in the support of the mixed strategy. Next observe that if $\frac{\partial p}{\partial a_1}(a_1, t) f_2(t) dt = \frac{c}{(b-1+a)}$, then (3) will hold.[3] Thus one possible density is given by

$$f_2(t) = (\frac{\partial p}{\partial a_1}(a_1, t))^{-1} \frac{c}{(b-1+a)}.$$

[3] Other representations of f_2 are feasible.

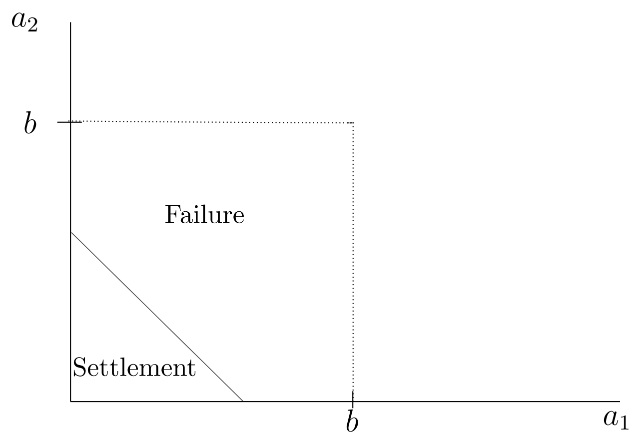

Figure 1: Example of an upper triangle equilibrium $q(a_1, a_2)$.

But obviously f_2 cannot depend on a_1 implying that our p for this q must satisfy

$$\frac{\partial}{\partial a_1}\Big[\frac{\partial p}{\partial a_1}(a_1, t)(b - 1 + a_1)\Big] = 0,$$

The product rule yields

$$\frac{\partial^2 p}{\partial a_1^2}(a_1, t)(b - 1 + a_1) + \frac{\partial p}{\partial a_1} = 0, \text{ for all } t, a_1. \tag{4}$$

This last equation is a partial differential equation in a_1 with a family of solutions

$$p(a, t) = g(t) + h(t)\ln(b - 1 + a_1). \tag{5}$$

To continue with our example, but account for this constraint, let

$$g(t) = \alpha, h(t) = \beta(b + 1 - a_1) \tag{6}$$

giving

$$p(a_1, t) = \alpha + \beta(b + 1 - a_1)\ln(b - 1 + a_1) \tag{7}$$

$$\frac{\partial p}{\partial a_1}(a_1, t) = \frac{\beta(1 + b - t)}{b - 1 + a_1}. \tag{8}$$

Given this function, f_2 integrates to one if

$$\beta = k \ln(1+b)$$

and thus we have a candidate solution,

$$f_2(t) = \frac{1}{\ln(b+1)(1+b-t)}. \tag{9}$$

Assuming a symmetric equilibrium, we can write down the probability of bargaining failure as

$$1 - \int_0^1 \int_{1-j}^0 \frac{1}{\ln(b+1)(b-i+1)} \frac{1}{\ln(b+1)(b-j+1)} didj, \tag{10}$$

with the closed form solution

$$1 - \frac{1}{\ln(b+1)^2}[Li_2(\frac{b+1}{2b+1}) - Li_2(\frac{b}{2b+1}) \\ + \ln(b)\ln(\frac{b+1}{2b+1}) - \ln(\frac{b}{2b+1})\ln(b+1)],$$

where $Li_2(z) = \int_z^0 \frac{\ln(1-t)}{t}dt$ is the dilogarithm. Figure (2) presents graphically the probability of bargaining failure as a function of the upper bound of the support (b).[4]

Similar exercises can be done by assuming a particular distribution of actions or a particular technology leading to disagreement payoffs as a function of investment.

4 Conclusion

In many applied contexts the choice of bargaining institutions may also influence actions that the agents take in order to influence their outside options. For example when bargaining failure is likely, an agent has a stronger incentive to invest in making her outside option better, and this investment in turn can make the outside option more attractive –and, thus, influence the incentives while bargaining. As this sim-

[4]We should point out the while the marginal cost c does not appear in the formula for the probability of failure, it implicetely appears as the dominance bound b depends on the cost c.

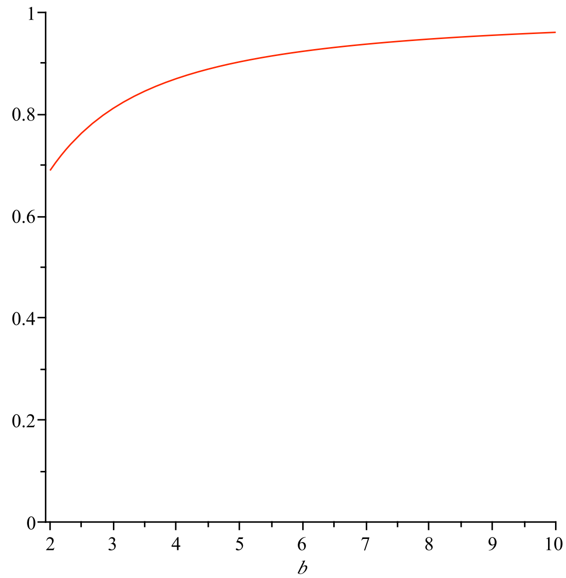

Figure 2: Probability of bargaining failure as a function of the upper bound of support (b).

ple statement suggests, equilibrium investment decisions and equilibrium bargaining behavior are linked. In contrast, much of the literature applying bargaining theory to topics like negotiating in the shadow of war posits an exogenous distribution over types. Here, we seek to uncover constraints on the types of lotteries over types and expectations over equilibrium play that are consistent with equilibrium play to a game in which the investments are endogenous. The paper focuses on problems in which two players can each make investment decisions which are hidden actions and the outside option of each player is assumed to depend on the investments of both players.

Using the envelope theorem to reach a convenient representation of any equilibrium payoffs as a function of investments, and the fact that if players randomize over investment levels in equilibrium they must be indifferent between these decisions, we obtain a system of equations that are necessary for equilibrium play. The system relates features that we treat as exogenous, like the technology relating investments to outside values and the cost function for investment, to features that we treat as endogenous, like lotteries over investments and behavior in the bargaining game. For the case of a separable outside value technology–where players have interdependent values, but investments are both weak compliments and weak substitutes–the probability of war conditional on a level of investment is the same in all equilibria to all games (holding fixed the exogenous features). This result is robust in the sense that small departures from separability result in small departures from the constant conditional probability of war result.

The structure of the relationship between the exogenous and endogenous features can be interpreted in a convenient manner. Equilibrium lotteries over investment must solve a continuum of inner product problems in a functional space. The structure of this type of problem allows us to conceive of ways to determine what types of lotteries are needed to support a particular type of equilibrium mapping from investments levels into bargaining failure for any particular set of exogenous features. Alternatively, we can evaluate what types of exogenous features must hold to support a particular description of endogenous features.

We hope the results here will cause applied theorists to give more thought to the relationships between beliefs about types and equilibrium play. Going forward, the structure of the relationships between equilibrium investments and bargaining behavior might be incorporated into design problems by considering a Bayesian mechanism design problem in which the lotteries over type are no longer exogenous but instead constrained by the equalities characterized here. A second avenue of closely related work pertains to sufficiency. We have focused only on necessary conditions that stem from incentive compatibility and optimality of investment decisions and in some results we have focused on necessary conditions for equilibria in which investments have densities. Equilibria of this form, of course need not exist (as is proven in the last corollary). Sufficiency results that provide much leverage may, however, require a bit more structure than we have currently imposed, but in general this direction has yet to be pursued.

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