

A continuous-time model of multilateral bargaining*

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Abstract

We propose a finite-horizon continuous-time framework for coalitional bargaining that has the following features: (i) Expected payoffs in Markov-perfect equilibrium (MPE) are unique, generating sharp predictions and facilitating comparative statics investigations; (ii) MPE are the only subgame-perfect Nash equilibria (SPNE) of the model that can be approximated by SPNE of nearby discrete-time bargaining models satisfying a genericity condition, providing justification for focusing on MPE in our model; (iii) The model is relatively tractable analytically. We investigate MPE payoffs as the time horizon goes to infinity. In convex games, we connect these limit payoffs to the core of the characteristic function underlying the bargaining game.

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1 Introduction

The idea of explicitly modeling the dynamic aspects of bargaining goes back to Stahl (1972) and Rubinstein (1982). They analyze a bargaining game between two players who take turns at making proposals. The key element of the model is impatience: players care not only about what share of the surplus they acquire, but they prefer to reach an agreement earlier rather than later. Remarkably, any nonzero degree of impatience leads to a unique subgame-perfect Nash equilibrium (SPNE) prediction.

The Stahl-Rubinstein model was extended in many directions.¹ In particular, the literature on multilateral group bargaining examines situations in which there are more than one parties involved, and all of them must agree in order to implement an agreement. A straightforward extension of Rubinstein's model yields a severe multiplicity of SPNE if the number of players is at least three, even though there is a unique stationary SPNE.² Coalitional bargaining investigates more complicated situations, when agreements are possible among subgroups of players, and the surpluses that different coalitions of players can split among each other can differ. In these games, a proposer has to choose both a coalition to approach and a division of the surplus that the coalition generates.³ A simple subclass of coalitional bargaining problems is legislative bargaining (Baron and Ferejohn (1989)), which spawned many applications (see for example Chari *et al.* (1997), and Snyder *et al.* (2005)). In these games, there are only two coalitions: winning ones that generate a fixed positive value, and losing ones that generate zero value.

This paper investigates a model framework for general coalitional bargaining, which is relatively tractable, and has several attractive properties that enhance its applicability. The key features of the model are that time is continuous, and that there is a deadline for negotia-

¹A major extension of the dynamic bargaining model framework that we do not take up in this paper involves incorporating private information held by one or more of the bargaining parties. For early references on bargaining with asymmetric information, see Fudenberg and Tirole (1983), Sobel and Takahashi (1983), Cramton (1984), Rubinstein (1985), Gul *et al.* (1986), and Fudenberg *et al.* (1987). For a relatively recent survey of the topic, see Ausubel *et al.* (2002).

²See for example Osborne and Rubinstein (1990, p63). Krishna and Serrano (1996) modify the game such that players can exit the game with partial agreements, and obtain a unique equilibrium. See also Merlo and Wilson (1995) for extending the bargaining framework with unanimity requirement to stochastic environments, and providing a condition for uniqueness of stationary equilibrium.

³For discrete-time models, see Gul (1989), Chatterjee *et al.* (1993), Moldovanu and Winter (1995), Bloch (1996), Okada (1996), Ray and Vohra (1997) and (1999), Evans (1997), Konishi and Ray (2003), and Gomes (2005). For a continuous-time model, see Perry and Reny (1994).

tions.⁴⁵ In particular, we consider a framework in which players get random opportunities to approach others and make proposals according to independent Poisson arrival procedures.⁶ The model can be considered to be a limit of discrete-time models in which the amount of time between time periods goes to zero, but the probability that some player gets the right to make a proposal at a given time period also goes to zero. The possibility of no one getting the chance to propose at a given period differentiates discrete-time approximations of our continuous model from discrete-time random-arrival coalitional bargaining games typically considered in the literature (see Okada (1996)).

Once an offer is made, we assume that the approached parties react immediately, and all of them have to accept the proposal in order for an agreement to be reached. Once an agreement is reached (by some coalition), the game ends. If a proposal is rejected by any of the approached players, the game continues, and players wait for the next arrival. Two highlighted special cases that fit into this framework are n -player group bargaining, where only the grand coalition can generate positive surplus, and legislative bargaining, where any large enough coalition of players (in the case of simple majority, voting coalitions involving more than half of the players) can end the game by reaching an agreement. Another example is a patent race in which several different coalitions of players have the opportunity to develop the same technology, but the race ends after some coalitions successfully obtained a patent.

The main results we show for this framework are the following. A Markov perfect equilibrium (MPE), that is an SPNE in which strategies only depend on the payoff-relevant part of the game history, always exists, and expected payoffs in MPE are uniquely determined. This greatly facilitates the applicability of the model, and in particular comparative statics exercises with respect to the parameters of the model (the time horizon for negotiations, arrival rates for proposals, and the characteristic function indicating the values of different coalitions).⁷ Furthermore, we show that the MPE are the only SPNE of the model that can

⁴Our motivation for imposing a deadline is twofold. First, in many real-world bargaining situations, there are natural deadlines that end negotiations. If the NHL (National Hockey League) and the NHL Players' Association do not reach an agreement by a certain date, then the season needs to be canceled, as happened in 2004. For reaching an out-of-court settlement, the announcement of the verdict poses a final deadline. Second, we use the resulting model for selecting among SPNE of the infinite-horizon model: we pay highlighted attention to characterizing limit equilibrium payoffs as the deadline gets infinitely far away, and investigate which SPNE payoffs of the infinite-horizon game can be attained this way. We note that several papers investigate deadline effects in discrete-time bargaining models: Fershtman and Seidmann (1993) examine bilateral bargaining with a particular commitment; Ma and Manove (1993) study bilateral bargaining with imperfect control over the timing of offers; Norman (2002) investigates legislative bargaining with deadline; finally, Yildiz (2003) and Ali (2006) consider long finite horizon games in which players disagree over their bargaining powers.

⁵For continuous-time bargaining models in the existing literature, see Perry and Reny (1993,1994) and Sákovics (1993). These models differ in substantial ways from the one proposed in the current paper. See a related discussion in Subsection 3.1.

⁶In our model, players with higher arrival rates can propose more frequently in expectation. This might be either a consequence of institutional features, like certain members of a legislature (party leaders or other elected officials within the legislature) enjoying preferential treatment in initiating proposals, or of how much attention and resources a player can devote to the bargaining procedure at hand. For models in which the right to make an offer is endogenous, see Board and Zwiebel (2005) and Yildirim (2007).

⁷In a companion paper (Ambrus and Lu (2010)), we apply our model to legislative bargaining with long

be approximated by SPNE of nearby discrete-time bargaining models satisfying a regularity condition, that holds generically. This provides a justification for focusing on MPE in the continuous-time model if one regards it as a limit of discrete-time environments.

By providing microfoundations for stationary SPNE in convex and legislative bargaining games, the paper fills a gap in the literature. Both the literature on general coalitional bargaining and the literature on legislative bargaining in stationary environments primarily focus on analyzing stationary SPNE, because of the severe multiplicity and relative complexity of SPNE. However, despite the large number of papers using the solution concept in multilateral bargaining games, there is little work on formally justifying this practice.⁸ In fact, Norman (2002) provides negative results in this direction: he shows that in legislative bargaining, there can be many non-Markovian SPNE, even if the game is finite. Moreover, even when one restricts attention to specifications of the game that have unique SPNE in finite horizons, expected equilibrium payoffs in general do not converge, as the horizon goes to infinity, to stationary SPNE payoffs of the infinite-horizon version of the game.

Our uniqueness result for MPE differs sharply from existing uniqueness results in the bargaining literature. In finite-horizon coalitional bargaining games, there generically is a unique SPNE, as shown in Norman (2002) in the context of legislative bargaining. This is because for generic vectors of recognition probabilities, no player is ever indifferent between approaching any two coalitions of players, so strategies and continuation payoffs can be simply computed by backward induction.⁹ This is not the case in our continuous-time framework: in MPE, indifferences are generated endogenously, for open sets of arrival vectors, for nondegenerate intervals of time during the game. For this reason, SPNE payoffs in our games are not unique, even generically. In light of this, we find it surprising that MPE payoffs are still unique in the continuous-time game, and the arguments needed to show this are unrelated to those establishing generic uniqueness of SPNE in finite-horizon discrete games. Moreover, our uniqueness result is also very different from uniqueness results for stationary SPNE in special classes of infinite-horizon coalitional bargaining games, for example the main result of Eraslan (2002) in the Baron and Ferejohn (1989) legislative bargaining context. It is well known that those uniqueness results do not extend to general random-proposer coalitional bargaining games. On the other hand, the uniqueness of MPE payoffs in our finite horizon games holds for coalitional bargaining games with general characteristic functions.

We obtain further results in games with convex characteristic functions. In this context we show that, for low enough discount rate, as the time horizon goes to infinity, MPE payoffs converge to stationary equilibrium payoffs of the infinite-horizon game. This, together with our previous results, establishes a two-step justification for focusing on stationary SPNE in

finite-time horizon.

⁸Baron and Kalai (1993) show that the stationary equilibrium is the unique simplest equilibrium in the Baron and Ferejohn legislative bargaining game. Chatterjee and Sabourian (2000) show that noisy Nash equilibrium with complexity costs leads to the unique stationary equilibrium in n-person group bargaining games (that is, when unanimity is required for an agreement). See also Baron and Ferejohn (1989) for informal arguments for selecting the stationary equilibrium in their game.

⁹The flipside of this simplicity is that strategies (including which coalitions to approach) and continuation values typically do not converge as the time horizon goes to infinity, instead “jump around”.

these games: given any sequence of SPNE expected payoffs of regular finite-horizon discrete-time games, first taking the limit to continuous time and then taking the limit as the time horizon goes to infinity achieves expected payoffs in a stationary SPNE of the infinite-horizon continuous model.

In convex games, we also show that as the discount rate goes to 0, the infinite-horizon limit MPE payoff converges to a point of the core of the underlying characteristic function, for any vector of arrival rates. Conversely, it also holds that for any point of the core, there is a vector of arrival rates such that as the discount rate goes to 0, the infinite-horizon limit MPE payoff converges to the given point. Hence, by varying the arrival rates, we can establish an exact equivalence between points of the core of the underlying characteristic function and infinite-horizon limit MPE payoffs of the continuous-time bargaining game. We show by example that this core equivalence result does not generally hold for nonconvex games with nonempty core. Our results complement existing ones on noncooperative foundations of the core in coalitional bargaining games, such as in Chatterjee *et al.* (1993), Perry and Reny (1994), or Yan (2002). However, our results are novel in that for any vector of arrival rates, the limit MPE payoffs are unique, but varying the arrival rates establishes an exact equivalence between limit MPE payoffs and the core. In some papers, like in Perry and Reny (1994), for a given specification of the model, there can be a severe multiplicity of equilibrium payoffs (including all points of the core), making comparative statics analysis more difficult than in our model. In other models, only one direction of the equivalence relationship holds: either that all equilibrium payoffs in a class of games correspond to points of the core (as in Chatterjee *et al.* (1993)), or that all points of the core can be supported as equilibrium outcomes (as in Yan (2002)).

2 The model

The underlying cooperative game

Consider a bargaining situation with set of players $N = \{1, 2, \dots, n\}$ and characteristic function $V : 2^N \rightarrow \mathbb{R}_+$, where $V(C)$ for $C \subset N$ denotes the surplus that players in C can generate by themselves (without players in $N \setminus C$). We refer to elements of 2^N as *coalitions*. We assume that if $C_1 \subset C_2$, then $V(C_1) \leq V(C_2)$. Occasionally, we will refer to the collection (N, V) as the underlying cooperative game behind the dynamic bargaining model investigated. The core of the underlying cooperative game is defined as: $\mathcal{C}(V) = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i \in C} x_i \geq V(C) \forall C \subset N \text{ and } \sum_{i \in N} x_i = V(N)\}$.

Basic description of the noncooperative game

The dynamic bargaining game we investigate is defined as follows. The game is set in continuous time, starting at $-T < 0$.¹⁰ There is a Poisson arrival process associated with each player i , with arrival rate $\lambda_i > 0$. The processes are independent from each other. For

¹⁰We use the nonstandard notation of negative time because fixing the deadline at zero facilitates a convenient way of keeping track of reservation values at time t , independently of the length of the game. This is because in MPE the latter only depends on the time remaining before the deadline, not on when the game started. This notation allows us to have increasing t as time progresses.

future reference, we define $\lambda \equiv \sum_{i=1}^n \lambda_i$. (In an abuse of notation, we will also refer to λ as the vector of arrival rates.) Whenever the process realizes for a player i , she can make an offer $x = (x_1, x_2, \dots, x_n)$ to a coalition $C \subseteq N$ satisfying $i \in C$. The offer x must have the following characteristics:

1. $x_j \geq 0$ for all $1 \leq j \leq n$;
2. $\sum_{j=1}^n x_j \leq V(C)$.

Players in $C \setminus \{i\}$ immediately and sequentially accept or reject the offer (the order in which they do so turns out to be unimportant). If everyone accepts, the game ends, and all players in N are paid their shares according to x . If an offer is rejected by at least one of the respondents, it is taken off the table, and the game continues with the same Poisson arrival rates. If no offer has been accepted at time 0, the game ends, and all players receive payoff 0.

We assume that players discount future payoffs using a constant discount rate $r \in (0, \infty)$.¹¹

For a formal definition of strategies in the above game, see the Appendix.

3 Examples

3.1 Group bargaining

In order to facilitate understanding of the model framework, we start with the simplest possible specification. Here we assume that $V(N) = 1$, and that $V(C) = 0 \forall C \neq N$. Since only the grand coalition can generate value, the acceptance of every player is required for any outcome with nonzero payoffs.

As is well-known in the literature, if the number of players is at least 3, in an alternating-offer bargaining game with infinite horizon, any division of the surplus can be supported in subgame-perfect Nash equilibrium (SPNE), if players are patient enough. The same conclusion holds in our continuous-time framework with random arrivals.¹² In stark contrast to this, in the game with deadline, there is a unique SPNE for any vector of arrival rates. Players in the unique SPNE play Markovian strategies. The proof (in the Appendix, as are all other proofs in the paper) uses a similar argument as in Shaked and Sutton (1984).

Theorem 1: In any SPNE, the n -player group bargaining game ends at the first realization of the Poisson arrival process for any player. After any arrival, an offer is made to N and all players accept. SPNE payoff functions are unique, with player i receiving $\frac{\lambda_i + r}{\lambda + r} + \frac{\lambda - \lambda_i}{\lambda + r} e^{(\lambda + r)t}$ when she makes the offer at time t , and $\frac{\lambda_i}{\lambda + r} (1 - e^{(\lambda + r)t})$ when she is not the proposer.

¹¹The presence of a deadline, together with the possibility of no arrival occurring over any given time horizon, implies that most of our conclusions also apply to a model with no discounting ($r = 0$). See an earlier circulated version of this paper in which we focused on the case of no discounting.

¹²In particular, the type of construction in p63 of Osborne and Rubinstein (1990), originally by Shaked, supports even the most extreme allocation in which one player gets all the surplus when the discount rate is $r = 0$.

Theorem 1 implies that player i 's expected payoff converges to $\frac{\lambda_i}{\lambda+r}$ as $T \rightarrow \infty$. Moreover, a player's expected payoff, both unconditionally and conditionally on getting an arrival, is monotonically increasing in her arrival rate, at all times. The fact that in our model, a player's ability to make offers more frequently increases her expected payoff is in contrast with the predictions of the models in Perry and Reny (1993) and Sákovics (1993). In the latter models, a player that can only speak infrequently (both to propose and to respond) obtains a higher share of the surplus because she can credibly threaten to impose a higher time cost on other players, should her offer be rejected. The intuition is that in these models, the longer a player's waiting time is, the more costly it is for her opponent to reject an offer, as the opponent would have to wait a long time before her counteroffer could be accepted. By contrast, in our model, approached players respond to offers instantaneously (time only lapses between two offers, and not between an offer and a response); hence, the above effect is not present.

Figure 1 below depicts the limit of expected MPE continuation payoff functions of the game as $r \rightarrow 0$, if $n = 3$ and arrival rates are $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{3}$, and $\lambda_3 = \frac{1}{6}$. Continuation values at the deadline are 0 for all players. Going back in time, continuation values start increasing at the rate corresponding to arrival rates, and converge to the relative arrival rates.

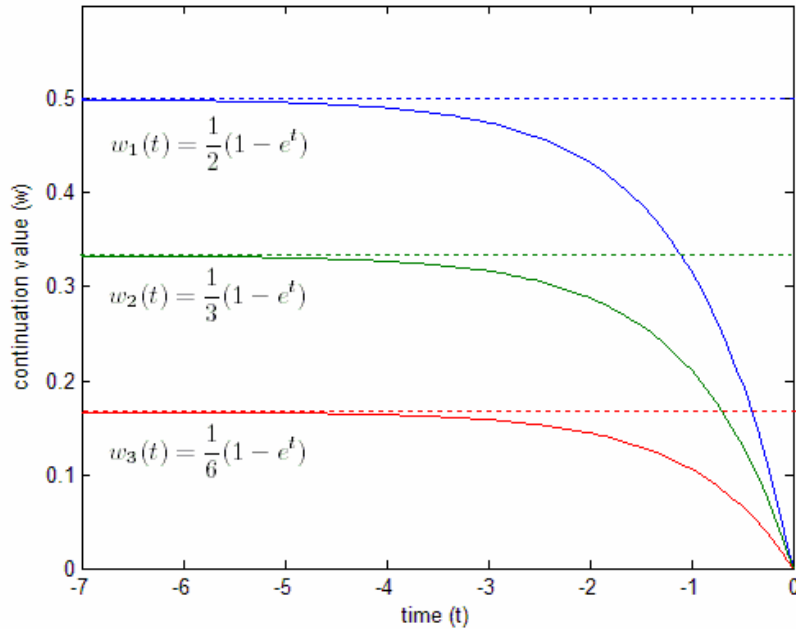


Figure 1.

Expected payoffs conditional on arrival follow a reverse pattern. Any player close to the deadline can keep most of the pie to herself, and payoffs conditional on proposing are monotonically decreasing in the time left before the deadline.

3.2 Examples of coalitional bargaining

In this subsection, we present two examples that demonstrate the role of subcoalitions in determining play and expected payoffs in MPE. In both examples, expected MPE payoffs are unique. This is a general feature of our model, which we establish in the next section. In the examples, we focus on the limit case when $r \rightarrow 0$ (the qualitative conclusions drawn from the examples remain the same for low enough r). We also normalize $V(N) = 1$.

1. Subcoalitions imposing bounds on limit equilibrium payoffs

Consider a game in which $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $V(\{1\}) = \frac{1}{2}$, $V(\{1, 2\}) = \frac{5}{6}$, and let the value of all coalitions other than the grand coalition be 0. MPE continuation payoffs are depicted in Figure 2. Going back in time from the deadline, all players' payoffs start increasing at the same rate, corresponding to the common Poisson arrival rate. However, when player 3's continuation payoff reaches $\frac{1}{6}$, her marginal contribution to the grand coalition, the other two players stop approaching her with probability 1, in a way that keeps player 3's continuation payoff constant at $\frac{1}{6}$.¹³ The other two players' continuation payoffs keep increasing until player 2's payoff reaches $\frac{1}{3}$, which is her marginal contribution to the value of coalition $\{1, 2\}$. At this point, player 1 starts proposing with positive probability to the singleton coalition involving only herself (that is, excluding player 2), and player 2's continuation payoff is kept constant at $\frac{1}{3}$. Finally, player 1's payoff converges to $\frac{1}{2}$, the value she can generate by herself. As $T \rightarrow \infty$, the probability that the grand coalition is approached goes to 1, since as the proposer surplus shrinks, players 2 and 3 need to be excluded with smaller and smaller probability for their expected continuation payoffs to be held constant. This reveals an interesting nonmonotonicity with respect to efficiency: near the deadline, all players approach the efficient grand coalition, and far away from the deadline, all players approach the grand coalition with probability close to 1. However, for intermediate time horizons, inefficient subcoalitions can form with high probability.

¹³In cases where player 3's relative arrival rate is very high, her continuation payoff may momentarily increase above $\frac{1}{6}$.

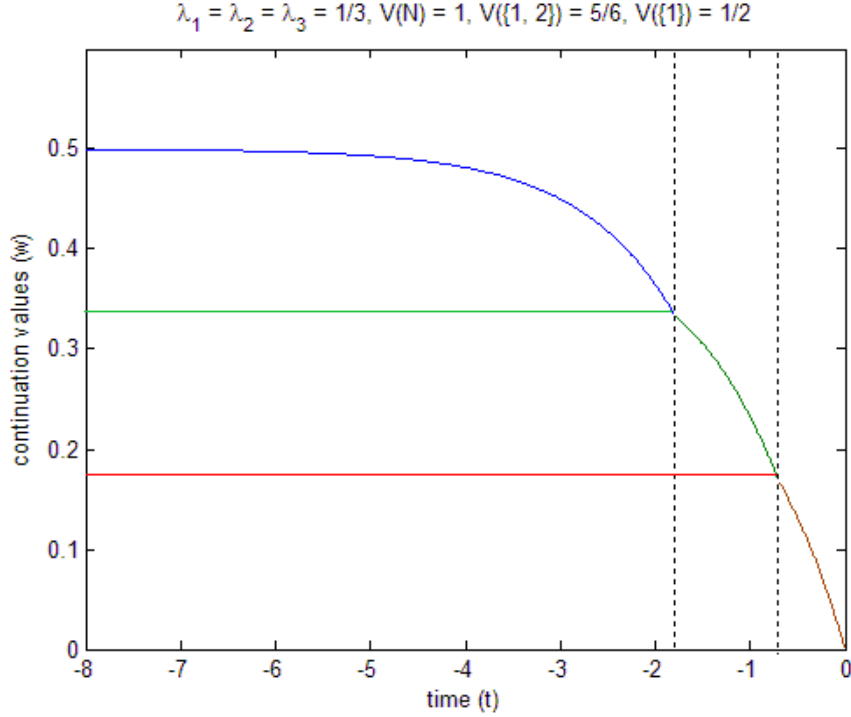


Figure 2.

The example demonstrates that the values subcoalitions can generate by themselves can act as lower bounds on how much players of the coalition can expect in MPE, if players are patient enough and the time horizon is long enough. In the previous example of group bargaining, relative expected payoffs are purely determined by the arrival rates (relative likelihoods of being the proposer). In games with more complicated characteristic functions, both the arrival rates and the values of subcoalitions play a role in shaping expected payoffs in MPE.

2. *Limit inefficiency*

The next example features a situation in which the underlying cooperative game has an empty core. Let $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$, $V(\{1, 2\}) = V(\{1, 3\}) = V(\{2, 3\}) = \frac{3}{4}$ and $V(\{1\}) = V(\{2\}) = V(\{3\}) = 0$. Here, each player's marginal contribution to the grand coalition is $\frac{1}{4}$. Hence, once MPE continuation values reach $\frac{1}{4}$, all players switch to proposing to 2-player coalitions with probabilities that keep everyone's continuation payoff constant at this level. This implies that players make inefficient agreements even in the limit as the deadline gets far away (and players are infinitely patient).

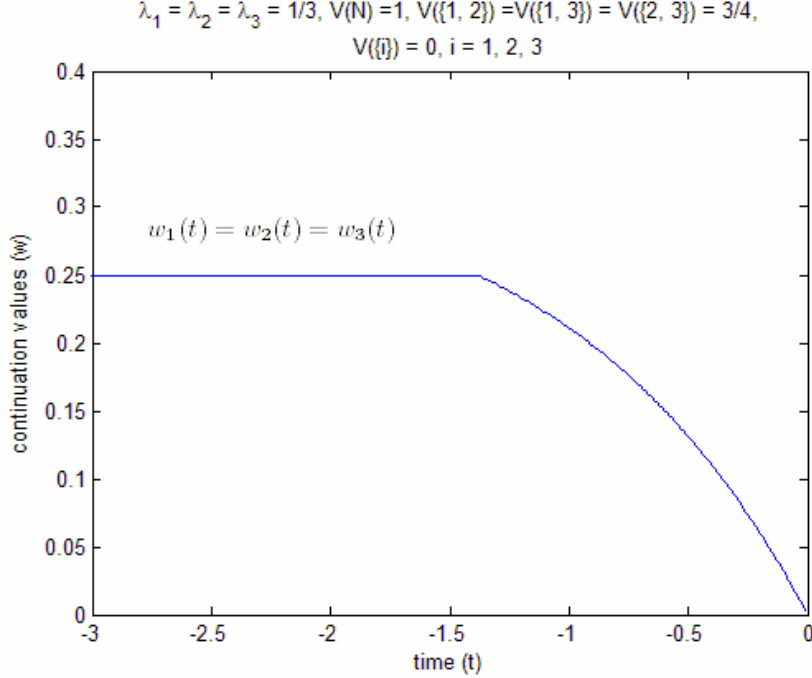


Figure 3.

4 Basic properties of Markov-perfect equilibrium

In this section, we first connect MPE of coalitional bargaining games in the continuous-time framework to SPNE of nearby discrete-time coalitional bargaining games. A corollary of this result is the existence of MPE in our model. Then we establish uniqueness of expected payoffs in MPE, for every game in our framework. That is, while strategies in our model might not be uniquely determined in MPE, they can only vary in a payoff-irrelevant way.¹⁴ We also show, by example, that the uniqueness result in the general case does not extend to all SPNE in the continuous-time framework, even generically.

Let $k \in \mathbb{Z}_{++}$, and fix a set of players N with $|N| = n$, a characteristic function V , and a vector of arrival rates $\lambda \in \mathbb{R}_{++}^n$. As usual, we abuse notation and denote $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$. In what follows, we assume a fixed discount rate r .

First we formally define discrete coalitional bargaining games, and such games being close to a continuous-time game.

Definition: A k -period *discrete random arrival coalitional bargaining game* with time horizon T , denoted $G^k(N, V, \lambda, T)$, is a k -period random-arrival discrete game in which $\frac{T}{k}$

¹⁴For example, if there are three players with equal arrival rates, and all players approach two-player coalitions in a way that every player is approached by others with the same probability, then it is payoff-irrelevant whether all players approach each of the other two players with probability $1/2-1/2$, or whether player 1 always approaches player 2, player 2 always approaches player 3, and player 3 always approaches player 1.

units of time lapse between subsequent periods, and in each period player i gets the chance to make a proposal with probability $\frac{\lambda_i}{\lambda}(1 - e^{-\lambda \frac{T}{k}})$, while with probability $e^{-\lambda \frac{T}{k}}$ no one gets to propose. The periods are denoted $\{1, 2, \dots, k\}$.

Definition: A sequence of discrete coalitional bargaining games $\{G_j^{k(j)}(N, V, \lambda^j, T)\}_{j=1}^{\infty}$ converges to continuous-time bargaining game $G(N, V, \lambda, T)$ if $k(j) \rightarrow \infty$ and $\lambda^j \rightarrow \lambda$ as $j \rightarrow \infty$.

If $\{G_j^{k(j)}(N, V, \lambda^j, T)\}_{j=1}^{\infty}$ converge to $G(N, V, \lambda, T)$, then the arrival process indeed converges to Poisson process defined for the continuous game: see Billingsley (1995), Theorem 23.2 (p302). From now on, for notational simplicity, we omit the superscript $k(j)$ for discrete games indexed by j .

Next, we focus on a particularly useful class of discrete games.

Definition: $G^k(N, V, \lambda, T)$ is *regular* if it has a unique SPNE payoff vector.

If the game is regular, then expected continuation payoffs in SPNE are Markovian: they depend only on the time remaining before the deadline. It is to these payoffs that we wish to relate MPE payoffs in the continuous-time game.

Next we show that regularity is a generic property for discrete-time coalitional bargaining games in the arrival rates, for any fixed characteristic function.¹⁵ This implies that for any continuous-time game, we can pick a sequence of regular discrete-time games that converges to it.

Claim 1 (Genericity): $U = \{\lambda \in \mathbb{R}_{++}^n | G^k(N, V, \lambda, T) \text{ is regular}\}$ is open and dense.

To show openness, we rely on the fact that when no player is indifferent between proposing to different coalitions, payoffs in each period are continuous in the payoffs of the following period. Thus, a slight perturbation in the latter implies a slight perturbation in the former, and the lack of indifference is preserved throughout the game for small enough changes. For density, we first show that small changes in $\lambda \in \mathbb{R}_{++}^n$ lead to full-dimensional changes each period's payoff until (going back in time) a period where some proposer is indifferent. Full dimensionality follows from the facts that: (i) an increase in λ_i , holding λ_j constant for all $j \neq i$, increases i 's payoff in each period and weakly decreases all other players' as long as (going back in time) no proposer is indifferent, and (ii) reservation value functions are infinitely differentiable in λ and the following period's values, and therefore linear approximations can be used for small changes. Then if $\lambda \notin U$, it is possible to make an arbitrarily small perturbation breaking the latest point of proposer indifference while preserving the lack thereof in subsequent periods. Iterating this argument yields a λ' arbitrarily close to λ such that all indifferences are broken.

Next, we establish that if a sequence of regular discrete coalitional bargaining games converges to a continuous-time bargaining game, then the following holds. The sequence

¹⁵Norman (2002) established an analogous result in the context of discrete time legislative bargaining games.

has a subsequence such that the associated (unique) SPNE collections of continuation payoff functions, extended to continuous time in a natural way, converge uniformly to an MPE collection of continuation payoff functions of the limit game. To show this, we construct strategy profiles in continuous time such that the associated continuation value functions approximate the SPNE continuation payoff functions arbitrarily well as $k \rightarrow \infty$. These generated functions are Lipschitz-continuous, with a uniform Lipschitz constant given by the discount rate, the arrival rates, and $V(N)$. Hence, by the Ascoli-Arzelà theorem, there is a subsequence of the games such that the associated continuation payoffs uniformly converge to a limit function (which is Lipschitz-continuous with the same constant) for each player.

To establish that these limit functions constitute the continuation payoff functions of an MPE of the limit game, we first prove a mathematical theorem, that at points t where both the continuation payoff functions along the sequence and the limit functions are differentiable (which holds for almost all points of time), the derivatives of the limit functions are in the convex hull of points that can be achieved as limit points of derivatives at points t_1, t_2, \dots along the sequence, where $t_k \rightarrow t$ as $k \rightarrow \infty$. Each of these limit points correspond to (proposer) strategies that are played arbitrarily close to t arbitrarily high along the sequence. It follows that these corresponding strategies are optimal in the limit game, assuming that players play Markovian strategies and the limit functions are indeed the continuation payoff functions. The same holds for mixtures of these strategies. We can use this fact to define strategies that are optimal if the continuation payoff functions are given by the limit functions, and at the same time generate the limit functions as the continuation payoff functions of the game. Put simply, we can create an MPE strategy profile such that the equilibrium continuation payoff functions are exactly the limit functions.

Given regular discrete coalitional bargaining game G_j , let $w^{G_j} : [-T, 0] \rightarrow \mathbb{R}^n$ denote the SPNE continuation payoff function of G_j , embedded in the continuous time framework as a step function. See Appendix B for the formal definition of w^{G_j} .

Claim 2: Suppose that the sequence of regular discrete coalitional bargaining games $\{G_j(N, V, \lambda^j, T)\}_{j=1}^{\infty}$ (G_1, G_2, \dots for short) converges to continuous-time bargaining game $G(N, V, \lambda, T)$. Then the sequence has a subsequence $\{G_{j_h}\}_{h=1}^{\infty}$ such that $\{w^{G_{j_h}}(\cdot)\}_{h=1}^{\infty}$, the sequence of SPNE payoff functions, converges uniformly. Moreover, for any such subsequence, the limit of $\{w^{G_{j_h}}(\cdot)\}_{h=1}^{\infty}$ corresponds to the continuation payoff functions of an MPE of $G(N, V, \lambda, T)$.

A straightforward consequence of Claim 2 is existence of MPE.

Corollary 1 (Existence): Every continuous-time random arrival coalitional bargaining game has an MPE.

We proceed by showing a simple result that reveals an important feature of MPE in our model, and will be used in the subsequent uniqueness proof. It states that at any point of time in an MPE, any player with an arrival only approaches coalitions that maximize the difference between the value of the coalition and the sum of continuation payoffs of players

in the coalition. Intuitively, players only approach coalitions that are the cheapest to buy relative to the value they can generate.

Claim 3: In any MPE, at any $t \leq 0$ where $i \in N$ receives an arrival, she approaches a coalition $C \in \arg \max_{D \ni i} V(D) - \sum_{j \in D \setminus \{i\}} w_j(t)$ and offers exactly $w_j(t)$ to every $j \in C \setminus \{i\}$. Furthermore, the offer is accepted with probability 1.

We now establish the uniqueness of MPE payoffs. The intuitive summary of the proof is as follows. Suppose that there are two Markov-perfect equilibria, A and B , with different continuation payoff functions. Suppose that t is the earliest time such that continuation payoffs in the two equilibria are equal for all times on the interval $[t, 0]$ (note that such time exists, as equilibrium continuation functions are continuous, and at $t = 0$, all players' continuation payoffs are 0 in all equilibria).

In the first part of the proof, we show that close to t , for any player, continuation values depend primarily on the probability of being approached. To see why this is the case, note that in general, holding arrival rates and future expected payoffs fixed, one expects that both the probability of being approached and the share obtained when proposing influence continuation values. However, close to t , the difference between the two equilibria for the former, a jump variable, is of a greater order of magnitude than for the latter, which is Lipschitz-continuous. The second part of the proof notes that optimality imposes that a coalition can take proposals away from another as we switch equilibria only if it has become relatively "cheaper". Therefore, loosely speaking, for any coalition, having a strictly higher continuation value in A than in B is generally associated with being approached less often under A than under B . Combining the two parts of the proof produces a contradiction.

In more detail, the proof proceeds as follows: define $f_i(\tau)$ as the difference between player i 's payoff in equilibrium A (w_i^A) and her payoff in equilibrium B (w_i^B), at time τ ; let $g_i(\tau)$ be the analogous difference in the density of being proposed to by another player. We wish to show that we can find τ arbitrarily close to t such that $\sum_{j \in N} \left[f_j(\tau) \int_{\tau}^t g_j(t') dt' \right] > 0$: summing over all players, continuation values and probability of being approached change in the same direction. To do so, we argue that for τ close enough to t , $\int_{\tau}^t g_j(t') dt'$ is at least sometimes of the same sign as $f_j(\tau)$, and cannot be too strongly of the opposite sign. We start by showing that $f_i(\tau) - \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} w_i^B(t') g_i(t') dt'$ is bounded by a quantity on the order of $(t - \tau) f_i$, when (roughly speaking) $|f_i|$ is large enough compared to $|f_j|$ for $j \neq i$. A technical result (see Lemma 2 in the proof) allows us to eliminate the $e^{-(\lambda+r)(t'-\tau)} w_i^B(t')$ term inside the integral. We also show that the right endpoint of the interval $[\tau, t]$ can be taken to be different from t , as long as it is close enough to t . In particular, this allows us to pick an interval I where, for any pair of coalitions (C, C') , the sign of the change (as we move from B to A) in the difference of continuation values between the two coalitions ($f_{C,C'} \equiv \sum_{i \in C} f_i - \sum_{i \in C'} f_i$) remains the same for all times within I .

For the second part of the proof, denote the change (from equilibrium A to equilibrium B) in the relative probability of the two coalitions being approached by $g_{C,C'}$. Then we have $f_{C,C'} g_{C,C'} \leq 0$. Since within I , $f_{C,C'}$ retains the same sign, then so does $g_{C,C'}$, so that for any

$\tau \in I$, $f_{C,C'}(\tau) \int_I g_{C,C'} \leq 0$. This observation is the opposite of the one made in the first part of the proof ($\sum_{j \in N} [f_j(\tau) \int_{\tau}^t g_j(t') dt'] > 0$), but for coalitions instead of individual players. Finally, using simple calculations, we derive a contradiction with the result obtained in the first part.

Theorem 2 (Uniqueness): In every continuous-time coalitional bargaining game, MPE payoff functions are unique.

Claim 2 and Theorem 2 together establish that given a sequence of regular discrete-time coalitional bargaining games converging to a continuous-time coalitional bargaining game, any convergent subsequence of the SPNE continuation payoff functions converges to the unique MPE payoff function of the limit game. This can be used to show that the original sequence of SPNE continuation payoff functions has to be convergent, with the same limit.

Theorem 3: Suppose that the sequence of regular discrete coalitional bargaining games G_1, G_2, \dots converges to continuous-time bargaining game $G(N, V, \lambda, T)$. Then the corresponding sequence of SPNE collection of continuation value functions converges to the unique MPE collection of continuation value functions of $G(N, V, \lambda, T)$.

We conclude the section by pointing out that Theorem 2 (uniqueness) does not extend to SPNE. To see this, consider the second example from Subsection 3.2, with small r . Let t^* denote the time at which players' continuation payoffs in MPE reach 0.25. In any MPE, before t^* , players always approach coalitions such that all players are approached with equal probabilities, and players' continuation payoffs are constantly 0.25 before t^* . Consider now the following non-Markovian strategy profile. After t^* , play is according to an MPE. Before t^* , if no offer was rejected so far in the game, a player who gets the chance to propose approaches the grand coalition and offers 0.25 to each of the other players (and keeps 0.5 for herself). In this phase, any approached player accepts an offer iff she is offered at least 0.25. However, once an offer is rejected, players switch to an MPE. Note that in the above profile, players' continuation values, provided that no rejection occurred so far, increase strictly above 0.25 before t^* . Nevertheless, they are willing to accept an offer of 0.25, because rejecting an offer moves play to a different phase, in which players' continuation payoffs are exactly 0.25. Such discontinuous change in continuation payoffs is not possible in MPE. Note that the above non-Markovian SPNE strictly payoff-dominates the MPE if the time horizon is longer than $-t^*$, and payoffs converge to the efficient division $(1/3, 1/3, 1/3)$ as $T \rightarrow \infty$ and $r \rightarrow 0$.

The qualitative conclusions from the previous game carry through to an ε -neighborhood of arrival vectors around $(1/3, 1/3, 1/3)$. In particular, for small enough $\varepsilon > 0$, there exists $\delta \geq 0$ such that before time $t^* - \delta$, the MPE continuation payoff of all three players is 0.25. Nevertheless, using exactly the same construction as above (with $t^* - \delta$ instead of t^* as the switching point between the history-dependent and the history-independent phases of the game), one can create an SPNE in which all players' payoffs converge to $(1/3, 1/3, 1/3)$ as T goes to infinity. This shows that there is an open set of arrival rates, for the given

characteristic function, for which there are multiple SPNE with distinct expected payoff vectors.¹⁶

5 Limit of MPE payoffs in convex games

In this section, we investigate MPE payoffs as the time horizon of the game goes to infinity, in games with convex characteristic functions. We first show that for any low enough discount rate $r > 0$, the MPE payoff vector converges, as $T \rightarrow -\infty$, to a payoff vector corresponding to a stationary SPNE payoff of the infinite-horizon bargaining game with the same discount rate. Given the justification for focusing on MPE in finite-horizon coalitional bargaining games, this result provides a justification for selecting stationary SPNE in infinite-horizon coalitional bargaining games, provided that the game is convex and players are patient enough.

Second, we establish an exact equivalence result in convex games between the core of the underlying characteristic function and the set of payoffs that can be achieved in the limit as the time horizon goes to infinity and players become more patient. In particular, any such limit MPE payoff, for any vector of arrival rates, corresponds to a point in the core. Conversely, for any point of the core, there is a vector of arrival rates such that the limit MPE payoff of the game with the given arrival rates corresponds to the selected point. In fact, the second result holds for all games in our framework. On the other hand, we show by example that the first result does not extend to all games with a nonempty core: in non-convex games, it is possible that the limit MPE payoff vector is outside the core of the underlying characteristic function.

Throughout this section, we let $v_C = V(N) - V(N \setminus C)$ for any $C \in 2^N$.

Definition: A bargaining game is convex if $V(C \cup A) - V(C) \geq V(C' \cup A) - V(C')$, whenever $C \supset C'$ and $C \cap A = C' \cap A = \emptyset$.

We start by establishing a lower bound for each coalition's continuation value far enough from the deadline. Intuitively, since a coalition C 's marginal contribution to any other coalition is at least $V(C)$ in a convex game, as long as C 's continuation value is below $V(C)$, it should be approached by any player (the proof of the claim uses induction to rule out only part of C being approached). When this player is part of C , then the proposal will give a total payoff of at least $V(C)$ to players in C , since otherwise the proposer can do better by approaching C only.

Claim 4: Let $\lambda_M = \min_{i \in N} \lambda_i$. If V is convex, then for any $\varepsilon > 0$, there exists T^* such that in any MPE of a game with $T > T^*$, continuation values satisfy $\sum_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M + r} V(C) - \varepsilon$, $\forall C \subseteq N$ and $t \leq -T^*$.

Claim 4 implies that far away from the deadline, there is an upper bound on the extra surplus one gets from proposing, and it tends to at most $\frac{r}{\lambda_M + r} V(N)$ as the deadline recedes.

¹⁶Slight changes in the characteristic function do not alter these conclusions either.

Thus, in a convex game where players are very patient and the deadline is far away, a player's payoff from proposing is only a little higher than her payoff from being approached. This implies that a player i 's continuation value, going back in time, would fall if no other player approaches her, which occurs if it is above $v_{\{i\}}$. The proof for Claim 5 shows that the value must in fact fall to $v_{\{i\}}$, and inductively extends the argument to coalitions. Thus, for low discount rates and far enough away from the deadline, it is optimal to approach N .

Claim 5: If V is convex, there exists \hat{r} such that whenever $r \in (0, \hat{r})$, $\exists t$ such that $\forall t' \leq t$, $N \in \arg \max_{C \subseteq N} \{V(C) - \sum_{i \in C} w_i(t')\}$.

For the rest of this section, we only consider $r < \hat{r}$ and early enough times so that N is an optimal coalition to approach for all players. Let $S(t)$ be the set of all coalitions that are as attractive as N at time t , i.e. $C \in S(t)$ iff $V(C) - w_C(t) = V(N) - w_N(t)$ (note that this must be positive).

Due to the convexity of the game, $S(t)$ must have an element that is contained in all others ("minimal optimal coalition"). To see this, suppose that were not the case. Then there must be two non-nested coalitions $D, E \in S(t)$ with $V(D) - w_D(t) = V(E) - w_E(t) = V(N) - w_N(t)$ and $V(D \cap E) - w_{D \cap E}(t) < V(N) - w_N(t)$, implying that $V(E) - V(D \cap E) > w_{E \setminus (D \cap E)}(t)$. By convexity, we have $w_{E \setminus (D \cap E)}(t) < V(D \cup E) - V(D)$, which implies that it is strictly better to approach $D \cup E$ than D , a contradiction. Note that this argument also implies that for any $D, E \in S(t)$, $D \cap E \in S(t)$ and $D \cup E \in S(t)$.

Let $C(t)$ be the minimal optimal coalition at t , and let $P(t)$ be the coarsest partition of N such that each element of $S(t)$ can be expressed as a union of cells defined by $P(t)$. Note that $C(t) \in P(t)$. Since, as shown above, for any $D, E \in S(t)$, $D \cap E \in S(t)$ and $D \cup E \in S(t)$, it follows that the union of $C(t)$ and any collection of the other cells in $P(t)$ is an element of $S(t)$. In particular, this means that $w_D(t) = V(N) - V(N \setminus D)$ for all $D \in P(t)$, $D \neq C(t)$.

Using the above structure, we show that the proposer surplus $V(N) - w_N(\cdot)$ is weakly monotonic far enough away from the deadline. Intuitively, we first note that the function must be monotonic whenever the minimal optimal coalition $C(\cdot)$ does not change. But, going back in time, if $C(\cdot)$ grows at t , it must be that the proposer surplus is decreasing both to the right and to the left of t , while if $C(\cdot)$ shrinks at t , then because the continuation value of the group of players leaving $C(\cdot)$ goes from rising to being flat, the proposer surplus cannot attain a maximum at t . If the proposer surplus increases going back in time, then no player can join $C(\cdot)$, while if it decreases, then no group of players can join at two different times. Either way, due to the finiteness of players, $C(\cdot)$ must be constant far enough from the deadline; we then argue that $P(\cdot)$ must be so as well to establish Claim 6.

Claim 6: For small enough r , there exists \hat{t} such that $\forall t \leq \hat{t}$, $P(t)$ remains constant.

Because the structure of proposals remains constant before a certain time, we can compute each player's limit payoffs (Theorem 4).

Theorem 4: For a convex game V and small enough r , let P be the earliest partition and C be the earliest minimal optimal coalition. Then: (i) $\lim_{\tau \rightarrow -\infty} w_i(\tau) = \frac{\lambda_i}{r + \lambda_C} V(C)$ for all $i \in C$, and (ii) $\lim_{\tau \rightarrow -\infty} w_i(\tau) = \frac{\lambda_i}{\lambda_D} v_D$ for all $i \in D \in P \setminus \{C\}$.

Because the set of optimal coalitions to approach is upper-hemicontinuous in continuation values, the union of C and any set of the other cells of P is an optimal coalition to approach if the values correspond to the limits in Theorem 4. Using this fact, we check that the limit payoffs from Theorem 4 constitute stationary equilibrium payoffs.

Corollary 2: In convex games, for small enough r , MPE limit payoffs ($\lim_{\tau \rightarrow \infty} w_i(\tau)$) constitute stationary equilibrium payoffs.

For any P and C , let $w_{P,C} = (w_1, \dots, w_n)$ where $w_i = \frac{\lambda_i}{\lambda_C} V(C)$ for all $i \in C$ and $w_i = \frac{\lambda_i}{\lambda_D} v_D$ for all $i \in D \in P \setminus \{C\}$. If $w_{P,C} \notin \mathcal{C}(V)$ (the core of V), then again by upper-hemicontinuity, for low enough r , P and C cannot describe the earliest structure of proposals in an MPE. Thus, for low enough r , whenever some P and C describe the earliest proposal structure, it must be that $w_{P,C} \in \mathcal{C}(V)$. This directly implies Claim 7.

Claim 7: If V is convex, then for low enough r , the Euclidean distance between the limit of MPE payoffs when taking T to infinity and $\mathcal{C}(V)$ is bounded above by $\frac{nr}{r+\lambda_M} V(N)$.

The converse of the above result can be established for all games with nonempty core, even nonconvex ones. The central idea is that with relative arrival rates proportional to relative payoffs in a core allocation, if every player always proposes to the grand coalition (offering the appropriate continuation payoff to every other player), then expected payoffs as $T \rightarrow \infty$ converge to the core allocation at hand, from below. This means that the sum of continuation payoffs of players of any coalition at no time exceeds the marginal contribution of the coalition to the grand coalition, confirming the optimality of proposing to the grand coalition.

Claim 8: For every $x \in \mathcal{C}(V)$, there exist arrival rates $\{\lambda_i\}_{i \in N}$ such that the expected MPE payoffs converge to $\frac{x}{1+r}$ as $T \rightarrow \infty$.

Claim 8 is similar to the main result in Yan (2002) on *ex ante* expected payoffs in stationary SPNE of infinite-horizon random-proposer discrete-time games, with the caveat that in our model, the discount rate has to converge to 0 to achieve the core convergence result because in our model, the expected time before the first proposal is positive. Therefore, the limit case of our model with infinitely patient players can be compared to the model in Yan (2002). In addition, we establish in Claim 7 that in our context, the unique MPE payoff of a convex game with any vector of arrival rates converges to a core allocation if the deadline goes to infinity and the discount rate goes to zero. The analog of this result is absent from Yan (2002).¹⁷

Combining Claims 7 and 8 immediately yields the equivalence between the core of a convex game and the limit set of possible MPE payoffs as $T \rightarrow \infty$ and $r \rightarrow 0$.

¹⁷Yan shows that if players are sufficiently patient and the vector of arrival rates is outside the core, the resulting stationary SPNE allocation is inefficient, but does not examine whether the inefficiency vanishes as players become patient.

Theorem 5: Let $\mathcal{S}(r, V)$ be the set of possible limit MPE payoffs (as $T \rightarrow \infty$) obtained by varying $\lambda \in \mathbb{R}_{++}^n$, and let $\mathcal{S}(V)$ be the limit of $\mathcal{S}(r, V)$ as $r \rightarrow 0$. If V is convex, then $\mathcal{S}(V) = \mathcal{C}(V)$.

We conclude the section by providing an example of a 4-player game with a nonempty core, in which payoffs converge to a point outside the core, as the deadline gets infinitely far away in the limit $r \rightarrow 0$. In this example, the underlying game is not totally balanced; that is, although the game has a nonempty core, it is not true that all subgames have a nonempty core. We do not know whether the core convergence result can be extended to all totally balanced games, a superset of both convex games and 3-player games with nonempty core.

Example. Let $N = \{1, 2, 3, 4\}$, $V(N) = 1$, $V(\{1, 2, 3\}) = \frac{1}{2}$, $V(\{1, 2\}) = V(\{2, 3\}) = V(\{3, 1\}) = \frac{3}{8}$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{15}$, $\lambda_4 = \frac{4}{5}$. The game is not totally balanced since the core in the subgame with players 1, 2 and 3 is empty. However, the complete game has a nonempty core; for example, $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in \mathcal{C}(V)$. It can be shown that in the limit as the deadline gets infinitely far away, expected payoffs converge to the inefficient allocation $(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$.

6 Discussion: extensions

Our model can be extended in many directions. Some of them, like incorporating asymmetric information, are beyond the scope of this paper. Others are relatively straightforward; we discuss two of these below.

6.1 Infinite horizon

Without a deadline, our model yields very similar results to a discrete-time model in which a proposer is selected randomly at every period (with perhaps a positive probability of no one being selected). As previously noted, in the group bargaining case with infinite horizon and constant discount rate r , any given allocation of the surplus can be supported in SPNE for r low enough. A further similarity with the discrete model is that there is only one stationary SPNE in discounted infinite-horizon group bargaining, which is characterized by:

$$w_i = \int_0^{\infty} [\lambda_i e^{-(\lambda+r)\tau} (1 - \sum_{j \in N \setminus \{i\}} w_j) + \sum_{j \in N \setminus \{i\}} \lambda_j e^{-(\lambda+r)\tau} w_j] d\tau.$$

The solution of this system is $w_i = \frac{\lambda_i}{r+\lambda}$, the same as the limit in the finite-horizon model as the horizon goes to infinity.¹⁸

6.2 Gradually disappearing pies

Our model assumes that the surplus generated by any coalition stays constant until a certain point of time (the deadline) and then discontinuously drops to zero. Although there are

¹⁸For small r this is a special case of Theorem 4, since group bargaining games are convex.

many situations in which there is such a highlighted point of time that makes subsequent agreements infeasible, in other cases, it is more realistic to assume that the surpluses start decreasing at some point, but only go to zero gradually. For example, agreeing upon broadcasting the games of a sports season yields diminishing payoffs once the season started, but if there are still remaining games in the season, a fraction of the original surplus can still be attained.

Some of our results can be extended to this framework. For example, the case of group bargaining remains tractable when $V(N)$ is time-dependent, even without assuming specific functional forms. Indeed, if $V(N)(t)$ is continuous and nonincreasing, and there is some time t^* at which $V(N)$ becomes zero, our argument for the uniqueness of SPNE payoffs applies with minor modifications. Continuation payoff functions are then $w_i(t) = \lambda_i \int_t^\infty e^{-(\lambda+r)(\tau-t)} V(N)(\tau) d\tau$, so payoffs remain proportional to arrival rates at all times, and since the grand coalition always forms at an arrival, the sum of expected payoffs across all players is simply the expected size of the pie at the next arrival (0 after t^*). Even if we do not assume that there is a time t^* as above, but instead only that $V(N)(t)$ is nonincreasing and $\lim_{t \rightarrow \infty} V(N)(t) = 0$, it is possible to show uniqueness of MPE payoffs. It is an open question whether this uniqueness result for gradually disappearing pies extends to general coalitional bargaining.

7 Conclusion

In this paper, we propose a tractable noncooperative framework for coalitional bargaining, which can be used to derive sharp predictions with respect to the division of the surplus. In subsequent research, we plan to extend the framework to settings with asymmetric information, as well as situations in which a successful agreement by a proper subcoalition does not end the game, and the remaining players can continue bargaining with each other.

We also plan to use our framework in various applications. In a companion paper (Ambrus and Lu (2010)) we apply our model to legislative bargaining, where there is a natural upper bound for negotiations: the end of the legislature's mandate. We characterize limit payoffs when the time horizon for negotiations goes to infinity, and show that there is a discontinuity between long finite-horizon legislative bargaining and infinite-horizon legislative bargaining. In particular, even in the limit the model with deadline puts restrictions on the distributions of surplus that can be achieved by varying recognition probabilities of different players, leading to a lower-dimensional subset of all feasible distributions.¹⁹

8 Appendix A: Formal definition of strategies

First, we define the set of possible histories of the game formally. We need to consider two types of histories.

For any $t \in [-T, 0]$, a time- t *proposer-history* consists of:

¹⁹In contrast, Kalandrakis (2006) shows that in the infinite-horizon Baron and Ferejohn legislative bargaining model, any division of the surplus can be achieved as an expected stationary SPNE payoff if recognition probabilities can be freely specified.

- (i) arrival times $-T \leq t_1, \dots, t_k < t$ for $k \in \mathbb{Z}_+$;
- (ii) proposers assigned to the above arrival times $i_{t_1}, \dots, i_{t_k} \in N$;
- (iii) feasible proposals (C_{t_m}, x_{t_m}) by i_{t_m} , where $C_{t_m} \subseteq N$ and $x_{t_m} \in \mathbb{R}_+^n$, for every $m \in \{1, \dots, k\}$; and
- (iv) acceptance-rejection responses $(y_{j_{m,1}}^{t_m}, \dots, y_{j_{m,n_m}}^{t_m})$ such that, for every $m \in \{1, \dots, k\}$:
 $n_m = |C_{t_m}| - 1$, $j_{m,1}, \dots, j_{m,n_m} \in C_{t_m}$, $j_{m,k'} \neq j_{m,k''}$ if $k' \neq k''$, $y_{j_{m,k'}}^{t_m} \in \{accept, reject\}$ for every $k' \in \{1, \dots, n_m\}$, and $y_{j_{m,k'}}^{t_m} = reject$ for some $k' \in \{1, \dots, n_m\}$.

Let \mathcal{H}_t^p denote the set of all time- t proposer-histories, and let $\mathcal{H}^p = \bigcup_{t \in [-T, 0]} \mathcal{H}_t^p$.

For any $t \in [-T, 0]$, a time- t responder-history of $i \in N$ consists of:

- (i) a time- t proposer history $h_t^p \in \mathcal{H}_t^p$;
- (ii) a time- t proposer $j \in N \setminus \{i\}$;
- (iii) a feasible proposal at time t : (C_j, x_j) by j , where $C_j \subseteq N$ and $x_j \in \mathbb{R}_+^n$, such that $i \in C_j$;
- (iv) previous acceptance-rejection decisions at time t : $(y_{j_1}, \dots, y_{j_m})$ such that $j_1, \dots, j_m \in C_j \setminus \{i, j\}$, $j_{k'} \neq j_{k''}$ if $k' \neq k''$, and $y_{j_{k'}} \in \{accept, reject\}$ for every $k' \in \{1, \dots, m\}$.

Let $\mathcal{H}_t^{r_i}$ denote all time- t responder-histories of i , and let $\mathcal{H}^{r_i} = \bigcup_{t \in [-T, 0]} \mathcal{H}_t^{r_i}$.

Next we construct metrics on the spaces of different types of histories, which we will use to impose a measurability condition on strategies.

Define a metric d^p on \mathcal{H}^p such that $d^p(h_t^p, \tilde{h}_{t'}^p) < \varepsilon$ for $\varepsilon > 0$ iff (i) $|t - t'| < \varepsilon$; (ii) h_t^p and $\tilde{h}_{t'}^p$ have the same number of arrival times $k \in \mathbb{Z}_+$; (iii) denoting the arrival times of h_t^p and $\tilde{h}_{t'}^p$ by t_1, \dots, t_k and t'_1, \dots, t'_k , $|t_l - t'_l| < \varepsilon \forall l \in \{1, \dots, k\}$; (iv) $i_{t_l} = i'_{t'_l} \forall l \in \{1, \dots, k\}$, where i_{t_l} is the proposer assigned at t_l by h_t^p and $i'_{t'_l}$ is the proposer assigned at t'_l by $\tilde{h}_{t'}^p$; (v) $C_{t_l} = C'_{t'_l} \forall l \in \{1, \dots, k\}$, where C_{t_l} is the approached coalition at t_l in h_t^p and $C'_{t'_l}$ is the approached coalition at t'_l in $\tilde{h}_{t'}^p$; (vi) $\|x_{t_l} - x'_{t'_l}\| < \varepsilon$, where $\|\cdot\|$ stands for the Euclidean norm in \mathbb{R}^n , and x_{t_l} and $x'_{t'_l}$ are the proposed allocations at t_l in h_t^p and at t'_l in $\tilde{h}_{t'}^p$; (vii) $y^{t_l} = \tilde{y}'_{t'_l} \forall l \in \{1, \dots, k\}$, where y^{t_l} and $\tilde{y}'_{t'_l}$ are the vector of acceptance-rejection responses at t_l in h_t^p and at t'_l in $\tilde{h}_{t'}^p$.

Define a metric d^{r_i} on \mathcal{H}^{r_i} such that $d^{r_i}(h_t^{r_i}, \tilde{h}_{t'}^{r_i}) < \varepsilon$ for $\varepsilon > 0$ iff (i) $|t - t'| < \varepsilon$; (ii) $d^p(h_t^p, \tilde{h}_{t'}^p) < \varepsilon$, where h_t^p and $\tilde{h}_{t'}^p$ are the proposer-histories belonging to $h_t^{r_i}$ and $\tilde{h}_{t'}^{r_i}$; (iii) the time- t proposer in $h_t^{r_i}$ and the time- t' proposer in $\tilde{h}_{t'}^{r_i}$ is the same player $j \in N \setminus \{i\}$; (iv) $C_j = C'_j$ and $\|x_j - x'_j\| < \varepsilon$, where (C_j, x_j) is the time- t proposal in $h_t^{r_i}$ and (C'_j, x'_j) is the time- t' proposal in $\tilde{h}_{t'}^{r_i}$; (v) $(y_{j_1}, \dots, y_{j_m}) = (y'_{j_1}, \dots, y'_{j_m})$, where $(y_{j_1}, \dots, y_{j_m})$ is the previous acceptance-rejection responses in $h_t^{r_i}$ and $(y'_{j_1}, \dots, y'_{j_m})$ is the previous acceptance-rejection responses in $\tilde{h}_{t'}^{r_i}$.

The set of proposer action choices of player i at any $t \in [-T, 0]$, denoted by A_i^p , is defined as $\{(C_i, x_i) | i \in C_i \subseteq N, \sum_{j \in N} x_j^i \leq V(C_i), x_i \geq 0\}$. Define a metric $d^{a_{p,i}}$ on A_i^p such that $d^{a_{p,i}}((C_i, x_i), (C'_i, x'_i)) < \varepsilon$ for $\varepsilon > 0$ iff $C_i = C'_i$ and $\|x_i - x'_i\| < \varepsilon$.

Definition: A pure strategy of player i in game G is a pair of functions: a proposal function $\mathcal{H}^p \rightarrow A_i^p$ that is measurable with respect to the σ -algebras generated by d^p and $d^{a_{p,i}}$, and a responder function $\mathcal{H}^{r_i} \rightarrow \{accept, reject\}$ that is measurable with respect to

the σ -algebra generated by d^T and the σ -algebra belonging to the discrete topology on $\{\text{accept}, \text{reject}\}$.

The measurability requirement on pure strategies is imposed to ensure that the expected payoffs of players are well-defined after any history.²⁰

9 Appendix B: embedding continuation payoff functions of discrete games into continuous time

For regular discrete coalitional bargaining game $G^k(N, V, \lambda, T)$, let $w_i^k(m)$ be player i 's SPNE continuation value before the random arrival in period m , for $m \in \{1, 2, \dots, k\}$ (thus, $w_i^k(k) = \frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}\lambda_i$). Let $w_i^k(k+1) = 0$. We extend these regular SPNE continuation payoff functions to continuous time.

Definition: For all $t \in [-T, 0]$, let $w_i^{G^k}(t) = e^{-r\Delta}w_i^k(\lfloor \frac{T+t}{T}k \rfloor + 1)$, where Δ satisfies $e^{-r\Delta}(1 - e^{-\lambda\frac{T}{k}}) = \frac{\lambda}{r+\lambda}(1 - e^{-(r+\lambda)\frac{T}{k}})$. Thus, $w_i^{G^k}(t)$ is a step-function derived from the discrete game payoffs.

Note that for k high enough, $\Delta \in (0, \frac{T}{k})$. Also note that as $k \rightarrow \infty$, $\Delta \rightarrow 0$.

The definition is consistent with the following setup: place the m^{th} period of the discrete game at time $-\frac{k-(m-1)}{k}T + \Delta$. For $t \in [-\frac{k-(m-1)}{k}T, -\frac{k-m}{k}T)$ (which corresponds to the m^{th} of the $k\frac{T}{k}$ -sized interval in $[-T, 0]$), $w_i^{G^k}(t)$ is simply $w_i^k(m)$ discounted from the perspective of time $-\frac{k-(m-1)}{k}T$. At time $-\frac{k-(m-1)}{k}T$, a player receiving an expected payoff x with density $\lambda e^{-\lambda(\tau + \frac{k-(m-1)}{k}T)}$ throughout $[-\frac{k-(m-1)}{k}T, -\frac{k-m}{k}T)$ has value $x \int_0^{\frac{T}{k}} \lambda e^{-(\lambda+r)\tau} d\tau = x \frac{\lambda(1-e^{-(\lambda+r)\frac{T}{k}})}{r+\lambda}$, while a player receiving the same expected payoff x at time $-\frac{k-(m-1)}{k}T + \Delta$ with probability $\int_0^{\frac{T}{k}} \lambda e^{-\lambda\tau} d\tau = 1 - e^{-\lambda\frac{T}{k}}$ has value $x e^{-r\Delta}(1 - e^{-\lambda\frac{T}{k}})$. Thus, our definition of Δ implies that any player will be indifferent between the continuous and the discrete arrivals specified above.

10 Appendix C: Proofs

Proof of Theorem 1: Let $\bar{v}_i(t)$ and $\underline{v}_i(t)$ be the supremum and the infimum, over all SPNE and all histories preceding t , of player i 's share when she makes an offer at time t . Let $\bar{w}_i(t)$ and $\underline{w}_i(t)$ be the supremum and the infimum of player i 's share when no player is making an offer, over all SPNE, histories and $j \neq i$.

Note that the density of i getting the next arrival of any player, at x time units from the current time, is $\lambda_i e^{-\lambda x}$, and payoffs received at that point are discounted by a factor e^{-rx} .

²⁰We do not need the measurability assumption to make sure that strategies lead to well-defined outcomes for any realization of the Poisson arrival processes. In contrast with differential games, the conceptual problems pointed out in Alós-Ferrer and Ritzberger (2008) do not arise in our context.

First, note that $\overline{v}_i(t) + \sum_{j \neq i} \underline{w}_j(t) = 1$, since this will be true in an SPNE where i offers everyone $\underline{w}_j(t)$ and takes the rest, and where, if any such offer by i is rejected, we move to a SPNE giving a continuation value of $\underline{w}_j(t)$ to the first rejector.²¹

Consider the following profile:

1. When any player $k \neq i$ makes an offer, the offer to player i must be $\overline{w}_i(t)$, and the offer to all $j \neq i, k$ is $\underline{w}_j(t)$. If k offers less to any player, the offer is rejected by that player; if player $j \neq i, k$ is the rejector, we move to an SPNE giving player k an expected payoff of $\underline{w}_k(t)$, and if player i is the rejector, we move to an SPNE giving player i an expected payoff of $\overline{w}_i(t)$. If k makes the correct offer and player j is the first rejecting the offer, then we move to an equilibrium giving $\underline{w}_j(t)$ to j .

2. When i makes an offer, she gives herself $\overline{v}_i(t)$ and gives $\underline{w}_j(t)$ to all $j \neq i$, as specified above.

To show that the exhibited profile is an SPNE, we need to verify that it indeed exists, i.e. that offers are feasible. Note that player k 's offer is feasible if $\overline{w}_i(t) + \sum_{j \neq i} \underline{w}_j(t) \leq 1$. But this must be true since the sum of all continuation values in any SPNE must be less than 1, and the SPNE where $\overline{w}_i(t)$ is attained has a sum of continuation values at t of at least $\overline{w}_i(t) + \sum_{j \neq i} \underline{w}_j(t)$. As established above, player i 's offer is feasible. We also need to check that players' actions are optimal. The only case where this is not trivial is that when k makes an offer, she may prefer to make one that is rejected by i . However, this will not be the case in an interval close to 0 where the probability of any future arrival $\leq \frac{1}{n}$, since then $\overline{w}_k(t) \leq \frac{1}{n} = 1 - \frac{n-1}{n} \leq 1 - \sum_{i \neq k} \overline{w}_i(t)$, so k will want the offer to be accepted. Denote this interval $[s, 0]$ (so $s = \frac{1}{\lambda} \ln(\frac{n-1}{n})$).

The above profile is of course the best possible one for i , so on $[s, 0]$ we have:

$$\begin{aligned} \overline{w}_i(t) &= \int_t^0 \left[\lambda_i e^{-(\lambda+r)(\tau-t)} \overline{v}_i(\tau) + \sum_{j \neq i} \lambda_j e^{-(\lambda+r)(\tau-t)} \overline{w}_i(\tau) \right] d\tau \\ &= \int_t^0 \left[\lambda_i e^{-(\lambda+r)(\tau-t)} (1 - \sum_{j \neq i} \underline{w}_j(\tau)) + \sum_{j \neq i} \lambda_j e^{-(\lambda+r)(\tau-t)} \overline{w}_i(\tau) \right] d\tau \end{aligned}$$

²¹Strictly speaking, at this point in the argument, it is possible that $\underline{w}_j(t)$ is not attained in any SPNE. However, since values arbitrarily close to it are attained in some SPNE, it follows that $v_i(t)$ can be arbitrarily close to $1 - \sum_{j \neq i} \underline{w}_j(t)$, which implies that $\overline{v}_i(t) + \sum_{j \neq i} \underline{w}_j(t) = 1$. To simplify exposition, we proceed in the proof as if all suprema and infima are attained, keeping in mind that we are actually referring to arguments analogous to the one presented in this footnote.

Since $\overline{w}_i(t)$ is the integral of a continuous function, its derivative exists, so:

$$\begin{aligned}
\overline{w}_i'(t) &= -\lambda_i(1 - \sum_{j \neq i} \underline{w}_j(t)) - \sum_{j \neq i} \lambda_j \overline{w}_i(t) \\
&\quad + (\lambda + r) \int_t^0 \left[\lambda_i e^{-(\lambda+r)(\tau-t)} (1 - \sum_{j \neq i} \underline{w}_j(\tau)) + \sum_{j \neq i} \lambda_j e^{-(\lambda+r)(\tau-t)} \overline{w}_i(\tau) \right] d\tau \\
&= -\lambda_i(1 - \sum_{j \neq i} \underline{w}_j(t)) - \sum_{j \neq i} \lambda_j \overline{w}_i(t) + (\lambda + r) \overline{w}_i(t) \\
&= (\lambda_i + r) \overline{w}_i(t) - \lambda_i(1 - \sum_{j \neq i} \underline{w}_j(t)).
\end{aligned}$$

Similarly, we note that $\underline{v}_i(t) + \sum_{j \neq i} \overline{w}_j(t) = 1$ on $[s, 0]$, since this occurs when i offers everyone $\overline{w}_j(t)$ and takes the rest, and where, if i gives any less than $\overline{w}_j(t)$ to a player, we move to a SPNE giving a continuation value of $\overline{w}_j(t)$ to the first rejector. On $[s, 0]$, $\overline{w}_j(t)$ and the probability of a future arrival are close to 0, so it will be optimal for i to make such an offer. By a similar argument as above, we can show that:

$$\underline{w}_i'(t) = (\lambda_i + r) \underline{w}_i(t) - \lambda_i(1 - \sum_{j \neq i} \overline{w}_j(t))$$

Thus on a nontrivial interval $[s, 0]$, we have a system of $2n$ differential equations continuous in t , and Lipschitz continuous in $2n$ unknown functions with initial values $\overline{w}_i(0) = \underline{w}_i(0) = 0$. By the Picard-Lindelof theorem, this initial value problem has a unique solution. It is easy to check that the following functions constitute the solution:

$$\overline{w}_i(t) = \underline{w}_i(t) = \frac{\lambda_i}{\lambda + r} (1 - e^{-(\lambda+r)t}) \equiv w_i(t)$$

The above argument can be iterated for $[2s, s]$ since the game ending at s with payoffs $w_i(s)$ is simply a scaled version of the original game, and so on. QED

Proof of Claim 1: Let $S = \{v \in \mathbb{R}_+^n \mid \exists C_1, C_2 \in 2^N \text{ s.t. } C_1, C_2 \in \arg \max_{C \ni i} (V(C) - \sum_{j \in C} v_j) \text{ for some } i \in N\}$. This is the set of reservation payoff vectors for which at least one player has at least two different optimal coalitions to approach.

Let $v^k(m)$ denote an SPNE reservation value vector in period $m \in \{1, \dots, k\}$ in $G^k(N, V, \lambda, T)$. Since k is fixed in the following proof, we abbreviate by writing $v(m)$. Note that in any $G^k(N, V, \lambda, T)$, $v(k-1) = e^{-r \frac{T}{k}} V(N) \left[\frac{1 - e^{-\lambda \frac{T}{k}}}{\lambda} (\lambda_1, \lambda_2, \dots, \lambda_n) \right]$. When we vary λ , we will write $v^\lambda(m)$.

Suppose that the reservation value is arbitrarily given to be v in period m , and all players play optimally in that period. Then, we denote the set of reservation value vectors attainable in period $m-1$ as $F(v, \lambda)$, where F is a correspondence. Note that $v \notin S \Leftrightarrow F(v, \lambda)$ is single-valued, in which case we denote its unique element as $f(v, \lambda)$. Since S is a finite collection of $(n-1)$ -dimensional hyperplanes, the set on which F is single-valued is open and dense (call this set W) within $\mathbb{R}_+^n \times \mathbb{R}_+^{22}$; within W , f is clearly continuous.

²²Obviously, if $(v, \lambda) \in W$, then $(v, \lambda') \in W$ as well.

Openness: Suppose $\lambda \in U$. By definition, for all $m \in \{1, 2, \dots, k-1\}$, $(v^\lambda(m), \lambda) \in W$, with $v^\lambda(k-1) = e^{-r\frac{T}{k}}V(N) \left[\frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1, \lambda_2, \dots, \lambda_n) \right]$ and $v^\lambda(m) = f(v^\lambda(m+1), \lambda)$. Now note that because f is continuous and W is open, for any λ' close enough to λ , we have that for all $m \in \{1, 2, \dots, k-1\}$, $v^{\lambda'}(m)$ is close to $v^\lambda(m)$, where $v^{\lambda'}(k-1) = e^{-r\frac{T}{k}}V(N) \left[\frac{1-e^{-\lambda'\frac{T}{k}}}{\lambda'}(\lambda'_1, \lambda'_2, \dots, \lambda'_n) \right]$ and $v^{\lambda'}(m) = f(v^{\lambda'}(m+1), \lambda')$. Again due to the openness of W , this implies that $\lambda' \in U$.

Density: We show that a payoff $v^\lambda(t)$ can be changed in "any direction" in \mathbb{R}^n by perturbing λ . To do so, we argue that in the linear approximation of changes in $v^\lambda(t)$ with respect to changes in λ , the transformation has full rank. This will allow us to break any indifferences at t using infinitesimal changes in λ .

When $v(m) \notin S$, we can write:

$$v_i(m-1) = \left[\frac{\lambda_i}{\lambda} (1 - e^{-\lambda\frac{T}{k}}) \max_{C \ni i} (V(C) - \sum_{j \in C} v_j(m)) + \left[1 - (1 - p_i(m))(1 - e^{-\lambda\frac{T}{k}}) \right] v_i(m) \right] e^{-r\frac{T}{k}}$$

where $p_i(m)$ is the probability that i is included in period m 's proposal given that there is one. Note that in a neighborhood of $v(m) \notin S$, $\arg \max_{C \ni i} (V(C) - \sum_{j \in C} v_j(m))$ is single-valued and constant. Fixing λ and in such a neighborhood, $v_i(m-1)$ is linear in each $v_j(m)$, so we can write $f(v(m) + \delta, \lambda) - f(v(m), \lambda) = A_m \delta$, where A_m is an $n \times n$ matrix. Note that the i th column of A_m must have a strictly positive i th element and have all other elements weakly negative. Similarly, fixing $v(m) \notin S$, we note that each $v_i(m-1)$ is infinitely differentiable in each component of λ , so we have the linear approximation $f(v(m), \lambda + \gamma) - f(v(m), \lambda) \approx B_m \gamma$, where B_m is an $n \times n$ matrix. Just like A_m , the i th column of B_m must have a strictly positive i th element and have all other elements weakly negative. We have $f(v(m) + \delta, \lambda + \gamma) - f(v(m), \lambda) \approx A_m \delta + B_m \gamma$. Define $B_k = D_\lambda [e^{-r\frac{T}{k}}V(N) \frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1, \lambda_2, \dots, \lambda_n)]$.

Fix $\varepsilon > 0$, and suppose $\lambda \notin U$. Then $\exists \tau$ such that $v(\tau) \in S$. Let t be the largest such τ . We still have that all $m \in \{t+1, t+2, \dots, k-1\}$, $F(\cdot, \cdot)$ is single-valued in a neighborhood of $(v(m), \lambda)$, with $v(k-1) = e^{-r\frac{T}{k}}V(N) \left[\frac{1-e^{-\lambda\frac{T}{k}}}{\lambda}(\lambda_1, \lambda_2, \dots, \lambda_n) \right]$ and $v(m) = f(v(m+1), \lambda)$.

Let $\lambda^1 = \lambda + \gamma$ be in a neighborhood of λ . Then we have the linear approximation $v^{\lambda^1}(t) \approx v^\lambda(t) + (A_{t+1}A_{t+2}\dots A_{k-1}B_k + A_{t+1}A_{t+2}\dots A_{k-2}B_{k-1} + \dots + A_{t+1}B_{t+2} + B_{t+1})\gamma \equiv v^\lambda(t) + M\gamma$. Since the set of matrices with strictly positive diagonal entries and weakly negative entries elsewhere is closed under addition and multiplication, M must retain that property. Thus, M has full rank. Therefore, $\exists \lambda^1$ within distance $\frac{\varepsilon}{2}$ of λ such that $v^{\lambda^1}(\tau) \notin S$ for all $\tau \geq t$.

Now with λ^1 , go back in time until the next indifference point, and iterate the argument with $\frac{\varepsilon}{4}, \frac{\varepsilon}{8}$, etc. Since there is a finite number of periods $k+1$, there is a finite number, say q , of indifferences to be broken. So λ^q , which is by construction within ε of λ , ensures that $G^k(N, V, \lambda^q, T)$ is regular. QED

Lemma 1: Suppose $f^1 \equiv (f_1^1, \dots, f_n^1), f^2 \equiv (f_1^2, \dots, f_n^2), \dots$ is a sequence of collections of functions, where $f_j^k : [0, T] \rightarrow \mathbb{R}$ are Lipschitz-continuous with Lipschitz-constant L , for every $k \in \mathbb{Z}_{++}$ and $j \in \{1, \dots, n\}$. Moreover, suppose that the sequence converges uniformly

to $f \equiv (f_1, \dots, f_n)$, where each f_j is also Lipschitz-continuous with Lipschitz constant L . Let Ξ be the set of all subsequences of f^1, f^2, \dots . For any $t \in [0, T]$, let $D(t) = \{x \in \mathbb{R}^n \mid \exists (f^{j_1}, f^{j_2}, \dots) \in \Xi \text{ and } t_1, t_2, \dots \rightarrow t \text{ s.t. } \nabla f^{j_i}(t_i) \rightarrow x \text{ as } i \rightarrow \infty\}$. Then f differentiable at t implies $\nabla f(t) \in co(D(t))$, where co stands for the convex hull operator.

Proof: First we show that $co(D(t))$ is closed. Consider a sequence of points in $D(t)$, x_1, x_2, \dots , converging to $x \in \mathbb{R}^n$. This means there are $(f^{j_1^m}, f^{j_2^m}, \dots) \in \Xi$ and $t_1^m, t_2^m, \dots \rightarrow t$ s.t. $\nabla f^{j_i^m}(t_i^m) \rightarrow x_m$ as $i \rightarrow \infty$, for every $m \in \mathbb{Z}_{++}$. Let $k(\cdot)$ be such that $|\nabla f^{j_{k(m)}}(t_{k(m)}^m) - x_m| < \frac{\varepsilon}{m}$ and $|t_{k(m+1)}^{m+1} - t| < |t_{k(m)}^m - t|$. Then the sequence $\nabla f^{j_{k(1)}}(t_{k(1)}^1), \nabla f^{j_{k(2)}}(t_{k(2)}^2), \dots$ converges to x , and $t_{k(1)}^1, t_{k(2)}^2, \dots \rightarrow t$, hence $x \in D(t)$. This implies that $D(t)$ is closed. Since $-L \leq D_i(t) \leq L$ for every $i \in N$, $D(t)$ is compact. Hence, $co(D(t))$ is compact.

For every $\delta \geq 0$, let $co^\delta(D(t)) = \{x \in \mathbb{R}^n \mid d(x, co(D(t))) \leq \delta\}$, where $d(x, co(D(t)))$ is the Hausdorff-distance between point x and set $co(D(t))$. Suppose the statement does not hold. Then, since $co(D(t))$ is closed, there is $\delta > 0$ such that $\nabla f(t) \notin co^\delta(D(t))$. By definition of $D(t)$, there exist $n_{\varepsilon(\delta)}(t)$, a relative $\varepsilon(\delta)$ -neighborhood of t in $[0, T]$, and $k \in \mathbb{Z}_+$ such that for any $k' \geq k$ and for any $t' \in n_{\varepsilon}(t)$ at which $f^{k'}$ is differentiable, $\nabla f^{k'}(t') \in co^\delta(D(t))$. Then for any $t' \in n_{\varepsilon}(t)$ and any $k' \geq k$, $f^{k'}(t') - f^{k'}(t) \in (t' - t)co^\delta(D(t))$. However, $\nabla f(t) \notin co^\delta(D(t))$ implies that there is $t' \in n_{\varepsilon}(t)$ such that $f(t') - f(t) \notin (t' - t)co^\delta(D(t))$. This contradicts that f^1, f^2, \dots converges uniformly to f . QED

Proof of Claim 2: We first consider an arbitrary k -period regular discrete game of the form $G^k(N, V, \lambda^k, T)$.

Notation: Let s^k denote a pure strategy SPNE strategy profile in $G^k(N, V, \lambda^k, T)$, for every $k \in \mathbb{Z}_{++}$. Let $C_i^k(m)$ denote the coalition that player i approaches in s^k in period m , for $m \in \{1, \dots, k\}$.

Based on s^k , for every $i \in N$, define as follows strategy \hat{s}_i^k of i in the continuous-time game $G(N, V, \lambda^k, T)$:

Divide $[-T, 0]$ into m equal intervals. If i gets an arrival in the m^{th} interval (*i.e.* at $t = -T + (m - \alpha)\frac{T}{k}$ for $\alpha \in [0, 1)$ and $m \in \{1, \dots, k\}$), she approaches $C_i^k(m)$ and offers $e^{-r\frac{T}{k}}w_j^k(m+1)$ to every $j \in C_i^k(m) \setminus \{i\}$. If player i is approached in the m^{th} interval by any player, then she accepts the offer if and only if it is at least $e^{-r\frac{T}{k}}w_j^k(m+1)$.

Let $\hat{w}_i^k(t)$ be player i 's continuation value in the continuous-time game generated by the profile $\hat{s}^k = (\hat{s}_1^k, \hat{s}_2^k, \dots, \hat{s}_n^k)$, and let $\hat{w}^k(t) = (\hat{w}_1^k(t), \hat{w}_2^k(t), \dots, \hat{w}_n^k(t))$.

Fact 1: For any $\varepsilon > 0$, there is a $k^\varepsilon \in \mathbb{Z}_{++}$ such that for any $k > k^\varepsilon$, $\hat{s}^k = (\hat{s}_1^k, \hat{s}_2^k, \dots, \hat{s}_n^k)$ is an ε -perfect equilibrium of G^k .

Note that by construction, whenever $t = -T + m\frac{T}{k}$, we have $\hat{w}^k(t) = w^{G^k}(t)$, for every $m \in \{1, \dots, k\}$. (Recall that $w_i^{G^k}(t)$ is a step-function derived from the discrete game payoffs.)

Second, note that given \hat{s}_{-i}^k , strategy \hat{s}_i^k specifies an optimal action for i if she has an arrival, at every $t \in [-T, 0]$.

Next, we bound the suboptimality of \hat{s}_i^k when i considers an offer. Observe that as we approach the end of the m^{th} interval (*i.e.* for $t = -T + (m - \alpha)\frac{T}{k}$, as $\alpha \searrow 0$), $\hat{w}_i^k(t) \rightarrow w_i^{G^k}(-T + m\frac{T}{k}) = e^{-r\Delta}w_i^k(m+1)$. Given that \hat{s}^k is Markovian, the optimal action for i in G^k when she is approached by any other player at $t = -T + (m - \alpha)\frac{T}{k}$ for $\alpha \in [0, 1)$ and $m \in \{1, \dots, k\}$ is, independently of payoff-irrelevant history, to accept the offer if it

is at least $\widehat{w}_i^k(t)$, and reject it otherwise. Instead, strategy \widehat{s}_i^k specifies that i accepts the offer if and only if it is at least $e^{-r\frac{T}{k}}w_i^k(m+1)$; hence, after some histories, \widehat{s}_i^k specifies a suboptimal action for i . However, since $\widehat{w}_i^k(t)$ is between $e^{-r\Delta}w_i^k(m)$ and $e^{-r\Delta}w_i^k(m+1)$, the difference between the expected payoff resulting from following \widehat{s}_i^k versus choosing the optimal action at t is bounded by $|w_i^k(m) - w_i^k(m+1)| + w_i^k(m+1)(e^{-r\Delta} - e^{-r\frac{T}{k}})$. Given that the probability of an arrival between $t = -T + (m-1)\frac{T}{k}$ and $t = -T + m\frac{T}{k}$ is $1 - e^{-\lambda^k\frac{T}{k}}$, $|w_i^k(m) - w_i^k(m+1)| \leq V(N)(1 - e^{-(\lambda^k+r)\frac{T}{k}})$. Thus, as $k \rightarrow \infty$, since $\Delta, \frac{T}{k} \rightarrow 0$, we have $|w_i^k(m) - w_i^k(m+1)| + w_i^k(m+1)(e^{-r\Delta} - e^{-r\frac{T}{k}}) \rightarrow 0$. This means that for any $\varepsilon > 0$, there is a $k^\varepsilon \in \mathbb{Z}_{++}$ such that for any $k > k^\varepsilon$, \widehat{s}^k specifies an ε -perfect equilibrium of G^k (which is also Markovian, by construction).

We now return to our original sequence $\{G_j^{k(j)}(N, V, \lambda^j, T)\}_{j=1}^\infty \equiv G_1, G_2, \dots$

Fact 2: Uniform convergence of $\widehat{w}^k(\cdot)$ along a subsequence of G_1, G_2, \dots

Define $\widehat{t}^k(\tau) = \lceil (T + \tau)\frac{k}{T} \rceil$. By construction,

$$\begin{aligned} \widehat{w}_i^k(t) &= \int_t^0 e^{-(r+\lambda^k)(\tau-t)} [\lambda_i^k \left(V[C_i^k(\widehat{t}^k(\tau))] - \sum_{j \in C_i^k(\widehat{t}^k(\tau)) \setminus \{i\}} e^{-r\frac{T}{k}} w_j^k(\widehat{t}^k(\tau) + 1) \right) \\ &\quad + \sum_{j \neq i: i \in C_j^k(\widehat{t}^k(\tau))} \lambda_j^k e^{-r\frac{T}{k}} w_i^k(\widehat{t}^k(\tau) + 1)] d\tau. \end{aligned}$$

It is easy to see that for every $i \in N$ and $k \in \mathbb{Z}_+$, $\widehat{w}_i^k(\cdot)$ is Lipschitz-continuous with Lipschitz constant $(r + \lambda^k)V(N)$. Moreover, all $\widehat{w}_i^k(\cdot)$ are uniformly bounded by 0 below and $V(N)$ above. Therefore, returning to our sequence G_1, G_2, \dots (and, for simplicity, now indexing our continuation value functions by the index of the corresponding game rather than the number of periods), by the Ascoli-Arzelà theorem (see Royden (1988), p169), the sequence of functions $\{\widehat{w}^j(\cdot)\}_{j=1}^\infty$ has a subsequence $\{\widehat{w}^{j_h}(\cdot)\}_{h=1}^\infty$ that converges uniformly to functions $\widehat{w}^*(\cdot) = (\widehat{w}_1^*(\cdot), \dots, \widehat{w}_n^*(\cdot))$, as $h \rightarrow \infty$. Moreover, because $\lambda^{j_h} \rightarrow \lambda$ as $h \rightarrow \infty$, each $\widehat{w}_i^*(\cdot)$ is also Lipschitz-continuous with constant $(r + \lambda)V(N)$. Without loss of generality, assume that the original sequence G_1, G_2, \dots is convergent.

Facts 1 and 2 taken together establish that if strategies are history-independent and continuation payoff functions are given by $\widehat{w}^*(\cdot)$, then when approached at t , an optimal strategy for j' is accepting the offer iff it gives her at least $\widehat{w}_{j'}^*(t)$. Below, we complete the proof by constructing optimal strategies for proposers that generate these payoff functions

Let \mathcal{T} stand for the set of points in $[-T, 0]$ where $\widehat{w}_i^*(\cdot)$ and $\widehat{w}_i^j(\cdot)$ are differentiable, for every $i \in N$ and $j \in \mathbb{Z}_{++}$. Since the above functions are all Lipschitz-continuous, $[-T, 0] \setminus \mathcal{T}$ is a null set.

By Lemma 1, for any $t \in \mathcal{T}$, $\nabla \widehat{w}^*(t) \in \text{co}(D(t))$. By Caratheodory's theorem, there exist points $x_1, \dots, x_{n+1} \in \text{co}(D(t))$ such that $\nabla \widehat{w}^*(t) = \alpha_1 x_1 + \dots + \alpha_{n+1} x_{n+1}$ for $\alpha_1, \dots, \alpha_{n+1} \geq 0$ such that $\sum_{i=1}^{n+1} \alpha_i = 1$. For every $m \in \{1, \dots, n+1\}$, let G_{m_1}, G_{m_2}, \dots be a subsequence of G_1, G_2, \dots and t^{m_1}, t^{m_2}, \dots be a sequence of points in $[-T, 0]$ converging to t such that

$\nabla \widehat{w}^{m_h}(t^{m_h}) \rightarrow x_m$. Because there are only a finite number of coalitions, G_{m_1}, G_{m_2}, \dots has a subsequence $G_{\widehat{m}_1}, G_{\widehat{m}_2}, \dots$ such that for every $i \in N$ and $\widehat{m}_h \in \mathbb{Z}_{++}$, $C_i^{m_h}(t^{\widehat{m}_h}) = C_i^{m_*}$ for some $C_i^{m_*} \in 2^N$. If approaching $C_i^{m_*}$ and offering $\widehat{w}_p^{m_h}(t^{\widehat{m}_h})$ to every player $p \in C_i^{m_*}$ is an optimal strategy for i in $G_{\widehat{m}_h}$ at $t^{\widehat{m}_h}$ given $\widehat{s}^{\widehat{m}_h}$, then by upper hemicontinuity of the best-response correspondence, approaching $C_i^{m_*}$ and offering $\widehat{w}_p^*(t)$ to every $p \in C_i^{m_*}$ is an optimal strategy for i in $G(N, V, \lambda, T)$ at t , provided that any approached player p' at any point of time t' accepts an offer iff the offer to her is at least $\widehat{w}_{p'}^*(t')$.

Since the above holds for all $m \in \{1, \dots, n+1\}$, the strategy of approaching $C_i^{m_*}$ with probability α_m (and offering \widehat{w}_p^* to every $p \in C_i^{m_*}$) is an optimal strategy for i in $G(N, V, \lambda, T)$ at t , provided that any approached player p' at any point of time t' accepts an offer iff the offer to her is at least $\widehat{w}_{p'}^*(t')$.

Consider now the following Markovian strategy profile s^* in $G(N, V, \lambda, T)$: (i) For any $i \in N$ and any $t \in [-T, 0]$, if i is approached at t , she accepts the offer iff it gives her at least $\widehat{w}_i^*(t)$; (ii) For any $i \in N$ and any $t \in \mathcal{T}$, if i gets an arrival at t , she approaches $C_i^{m_*}$ with probability α_m and offers $\widehat{w}_p^*(t)$ to every $p \in C_i^{m_*} \setminus \{i\}$, for every $m \in \{1, \dots, n+1\}$; (iii) For any $i \in N$ and any $t \in [-T, 0] \setminus \mathcal{T}$, if i gets an arrival at t , she approaches some coalition $C \in \arg \max_{C' \in 2^N: i \in C'} (V(C') - \sum_{p \in C' \setminus \{i\}} \widehat{w}_p^*(t))$ and offers $\widehat{w}_p^*(t)$ to every $p \in C \setminus \{i\}$.

By construction, if all players follow the above Markovian strategies, then the gradient of the continuation payoff function at t is exactly $\nabla \widehat{w}^*(t)$ at every $t \in \mathcal{T}$. Note also that $\widehat{w}^*(0) = 0$ and continuation payoffs at 0 are also equal to 0. Given that both the continuation payoff functions given the above strategies, and $\widehat{w}_i^*(\cdot)$ for all $i \in N$ are Lipschitz-continuous, this implies that the continuation payoff functions generated by the above strategies are exactly $\widehat{w}^*(\cdot)$. Since we established above the optimality of these strategies given that continuation payoffs are $\widehat{w}^*(\cdot)$, we constructed an MPE of $G(N, V, \lambda, T)$ such that the continuation payoffs defined by the MPE are given by $\widehat{w}^*(\cdot)$.

Finally, note that $\sup_{t \in [-T, 0]} |w_i^{G_j}(t) - \widehat{w}_i^j(t)| \leq (r + \lambda^j)V(N)\frac{T}{k(j)}$, where the right-hand sides goes to 0 as $j \rightarrow \infty$. Hence, the sequence of SPNE continuation payoff functions $\{w_{j_h}^{G^{k(j_h)}}(\cdot)\}_{h=1}^\infty$ converges to the same limit as any convergent subsequence $\widehat{w}^{j_1}(\cdot), \widehat{w}^{j_2}(\cdot), \dots$
QED

Proof of Claim 3: Note that $\sum_{j \in N} w_j(t) \leq V(N) - e^{\lambda t}$, where $e^{\lambda t} > 0$ is the probability that no one has the chance to make an offer during $[t, 0]$. Furthermore, in any MPE, if $C \subset N$ is approached by i at t , and every $j \in N \setminus \{i\}$ is offered strictly more than $w_j(t)$, then the offer has to be accepted by everyone with probability 1. Therefore, player i can guarantee a payoff strictly larger than $w_i(t)$ by approaching N and offering $w_j(t) + \varepsilon$ to every $j \in N \setminus \{i\}$ for small enough $\varepsilon > 0$. On the other hand, a rejected offer results in continuation payoff $w_i(t)$ for i . Next, note that approaching a coalition C and offering strictly less than $w_j(t)$ to some $j \in C$ results in rejection of the offer with probability 1, and is therefore not optimal. Approaching a coalition C and offering $w_j(t) + \varepsilon$ for $\varepsilon > 0$ to some $j \in C$ is also suboptimal, because offering instead $w_j(t) + \varepsilon/n$ to every $j \in C \setminus \{i\}$ results in acceptance of the offer with probability 1 and strictly higher payoff. Therefore, whatever coalition C is approached, player i has to offer exactly $w_j(t)$ to every $j \in C \setminus \{i\}$. It cannot be that this offer is accepted with probability less than 1, since then player i could strictly improve

her payoff by offering slightly more than $w_j(t)$ to every $j \in C \setminus \{i\}$, and that offer would be accepted with probability 1. Finally, it cannot be that $C \notin \arg \max_{D \subset N} V(D) - \sum_{j \in D \setminus \{i\}} w_j$, since then approaching some $C' \in \arg \max_{D \subset N} V(D) - \sum_{j \in D \setminus \{i\}} w_j$ instead, and offering slightly more than $w_j(t)$ to every $j \in C' \setminus \{i\}$ would result in a strictly higher payoff. QED

Lemma 2: Let $g(\cdot)$ be an integrable function taking values between $-K$ and K , and let $h(\cdot) \geq 0$ be Lipschitz continuous with Lipschitz bound L . Then $\left| \int_{\tau}^t g(t')h(t')dt' - h(t) \int_{\tau}^t g(t')dt' \right| < 2L(t - \tau) \max_{\tau' \in [\tau, t]} \left| \int_{\tau'}^t g(t')dt' \right|$.

Proof: Suppose $\int_{t_1}^{t_2} g(t')dt' = 0$ and $\left| \int_{\tau}^{t_2} g(t')dt' \right| \leq C$ for all $\tau \in [t_1, t_2]$. Then $\left| \int_{t_1}^{t_2} g(t')h(t')dt' \right| < LC(t_2 - t_1)$, as the maximum corresponds to the case where $h(\tau)$ follows a Lipschitz bound, $g(\tau) = K$ for $\tau \in [\max\{t_2 - \frac{C}{K}, \frac{t_1+t_2}{2}\}, t_2]$, $g(\tau) = -K$ for $\tau \in [t_1, \min\{t_1 + \frac{C}{K}, \frac{t_1+t_2}{2}\}]$, and $g(\tau) = 0$ for $\tau \in (\min\{t_1 + \frac{C}{K}, \frac{t_1+t_2}{2}\}, \max\{t_2 - \frac{C}{K}, \frac{t_1+t_2}{2}\})$.

Now partition $[\tau, t]$ into measurable sets A and B , where $\int_S g(t')dt' = 0$ for any connected set $S \subset A$ and $\int_U g(t')dt'$ has the same sign as $\int_{\tau}^t g(t')dt'$ for any connected set $U \subset B$. Then we have $\left| \int_{\tau}^t g(t')h(t')dt' - h(t) \int_{\tau}^t g(t')dt' \right| \leq \left| \int_A g(t')h(t')dt' \right| + \left| \int_B g(t')h(t')dt' - h(t) \int_B g(t')dt' \right| < LC(t - \tau) + L(t - \tau) \left| \int_{\tau}^t g(t')dt' \right|$, where $C = \max_{\tau' \in [\tau, t]} \left| \int_{\tau'}^t g(t')dt' \right|$. QED

Proof of Theorem 2: First note that if $\lambda_i = 0$, the only possible MPE continuation value for i is 0 at all times. The game is then equivalent to an alternative game with players $N \setminus \{i\}$ and characteristic function $V'(C) = V(C \cup \{i\}), \forall C \subseteq N \setminus \{i\}$. So we assume without loss of generality that $\lambda_i > 0, \forall i \in N$.

The proof requires the introduction of some extra notation.

Let $v_i(t) = \max_{C \ni i} \left\{ v(C) - \sum_{j \in C \setminus \{i\}} w_j(t) \right\}$. Also, let $p_{ij}(t)$ be the probability of j receiving an offer at time t given that i gets an arrival at that time, and let $p_{iC}(t)$ be the probability of C (and only C) receiving an offer at time t given that i gets an arrival at that time.

We proceed by contradiction. Suppose two MPE, A and B , of the same bargaining game with characteristic function V and arrival rates $(\lambda_1, \dots, \lambda_n)$ do not have the same continuation value functions.

Define $f_i(\tau) = w_i^A(\tau) - w_i^B(\tau)$, and note that $f_i(\tau)$ is Lipschitz continuous.

Let $t = \min\{\tau | f_i(t') = 0, \forall t' \in [\tau, 0], \forall i \in N\}$. Note that $t < 0$ since there must be some nontrivial interval just before time 0 where proposing to a coalition of value $V(N)$ is strictly optimal for everyone. When the only such coalition is N , MPE payoffs are clearly unique within this interval; the same can be shown if multiple coalitions have value $V(N)$.²³ Thus we have $w_i(t) > 0$.

²³Note that at $t = 0$, the left derivative of continuation value functions must exist since $w_i(0) = 0$ (so the probability of being proposed to given an arrival, which may be discontinuous, does not affect the rate of change of w_i). In fact, we have $w_i'(0) = -\lambda_i V(N)$. So when there are coalitions $C_1, \dots, C_m \neq N$ with $i \in C_j$

Define $g_j(\tau) = \sum_{i \neq j} \lambda_i p_{ij}^A(\tau) - \sum_{i \neq j} \lambda_i p_{ij}^B(\tau)$. Note that by our definition of strategies, continuation value functions exist, which implies that $\int_{\tau}^t g_j(t') dt'$ exists for all $\tau < t$.

Step 1: We establish that:

$$\forall \varepsilon > 0, \exists \tau \in [t - \varepsilon, t) \text{ s.t. } \sum_{j \in N} \left[f_j(\tau) \int_{\tau}^t g_j(t') dt' \right] > 0. \quad (1)$$

Please refer to the main text for an explanation of our procedure.

Note that:

$$\begin{aligned} f_j(\tau) &= \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} \left[\begin{aligned} &\lambda_j (v_j^A(t') - v_j^B(t')) + (w_j^A(t') - w_j^B(t')) \left(\sum_{i \neq j} \lambda_i p_{ij}^A(t') \right) \\ &+ w_j^B(t') \left(\sum_{i \neq j} \lambda_i p_{ij}^A(t') - \sum_{i \neq j} \lambda_i p_{ij}^B(t') \right) \end{aligned} \right] dt' \\ &= \lambda_j \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} [v_j^A(t') - v_j^B(t')] dt' + \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} f_j(t') \left(\sum_{i \neq j} \lambda_i p_{ij}^A(t') \right) dt' \\ &\quad + \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} w_j^B(t') g_j(t') dt' \end{aligned}$$

For $\tau < t$, let $F_i(\tau) = \max_{t' \in [\tau, t]} |f_i(t')|$, so $F_i(\tau)$ is Lipschitz continuous and nonincreasing. Using the fact that on $[\tau, t]$, $e^{-(\lambda+r)(t'-\tau)} < 1$ and $\sum_{i \neq j} \lambda_i p_{ij}^A(t') < 1$, we have the following

bound:

$$\begin{aligned} \left| f_j(\tau) - \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} w_j^B(t') g_j(t') dt' \right| &\leq (\lambda+r)(t-\tau) \sum_{i \in N \setminus \{j\}} F_i(\tau) + (\lambda+r)(t-\tau) F_j(\tau) \\ &= (\lambda+r)(t-\tau) \sum_{i \in N} F_i(\tau) \end{aligned}$$

Now we relate $\int_{\tau}^t g_j(t') dt'$ to $\int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} w_j^B(t') g_j(t') dt'$. By Lemma 2,

$$\begin{aligned} &\left| \int_{\tau}^t g_j(t') dt' - \frac{e^{(\lambda+r)(t-\tau)}}{w_j^B(t)} \int_{\tau}^t e^{-(\lambda+r)(t'-\tau)} w_j^B(t') g_j(t') dt' \right| \\ &< 2L_j(t-\tau) \max_{t' \in [\tau, t]} \left| \int_{\tau'}^t e^{-(\lambda+r)(t'-\tau)} w_j^B(t') g_j(t') dt' \right| \\ &\leq 2L_j(t-\tau) (F_j(\tau) + (\lambda+r)(t-\tau) \sum_{i \in N} F_i(\tau)) \end{aligned}$$

where L_j , the Lipschitz constant for $\frac{e^{(\lambda+r)(t-\tau)}}{w_j^B(t)}$, is finite when $w_j^B(t) \neq 0$. Combining this with the above yields:

and $V(C_j) = V(N)$ for $j = 1, \dots, m$, it must be true that in a neighborhood of 0, i only proposes to C_k with positive probability if $\sum_{l \in C_k} \lambda_l \leq \sum_{i \in C_j} \lambda_i$ for $j = 1, \dots, m$. When there are multiple such coalitions, if feasible, they will be approached such that their continuation values are equalized; if equalization cannot be achieved, those having the choice between many such coalitions will propose to the cheapest one.

$$\begin{aligned}
& \left| f_j(\tau) - \frac{w_j^B(t)}{e^{(\lambda+r)(t-\tau)}} \int_{\tau}^t g_j(t') dt' \right| \\
& < (\lambda+r)(t-\tau) \sum_{i \in N} F_i(\tau) + 2 \frac{w_j^B(t)}{e^{(\lambda+r)(t-\tau)}} L_j(t-\tau) (F_j(\tau) + (\lambda+r)(t-\tau) \sum_{i \in N} F_i(\tau)) \\
& \leq (\lambda+r + 2L_j w_j^B(t)) \sum_{i \in N} F_i(\tau)(t-\tau) + 2(\lambda+r)L_j w_j^B(t) \sum_{i \in N} F_i(\tau)(t-\tau)^2 \\
& \equiv \sum_{i \in N} F_i(\tau)(t-\tau)(k_j + q_j(t-\tau))
\end{aligned}$$

For $(t-\tau)$ small enough, $q_j(t-\tau)$ is negligible, so we omit it below. Thus:

$$f_j(\tau) \frac{w_j^B(t)}{e^{\lambda(t-\tau)}} \int_{\tau}^t g_j(t') dt' \geq [f_j(\tau)]^2 - k_j(t-\tau) \sum_{i \in N} F_i(\tau) |f_j(\tau)|$$

We are now ready to provide bounds for $f_j(\tau) \int_{\tau}^t g_j(t') dt'$. Let $I(\tau) = \arg \max_i F_i(\tau)$, and let $S = \{t' | F_{I(t')}(t') = |f_{I(t')}(t')| \text{ and } t' < t\}$. Note that for all $\tau < t$, $S \cap [\tau, t) \neq \emptyset$. We show that for all $\tau \in S$, $f_j(\tau) \int_{\tau}^t g_j(t') dt'$ cannot be more than second-order negative for any j , and that $f_{I(\tau)}(\tau) \int_{\tau}^t g_{I(\tau)}(t') dt'$ must be first order positive. The intuition is that just before t , the payoff effect of differences in other players' payoffs between A and B is generally small relative to the payoff effect of differences in the probability of being approached. While for an individual player, the two might be roughly equal, there must be a player, namely $I(\tau)$, for whom this holds strongly, since all players other than $I(\tau)$ have payoff differences smaller than $I(\tau)$'s.

We have that for all $\tau \in S$:

$$f_j(\tau) \int_{\tau}^t g_j(t') dt' \geq \frac{1}{w_j^B(t)} ([f_j(\tau)]^2 - k_j(t-\tau)n |f_{I(\tau)}(\tau)| |f_j(\tau)|)$$

This expression is minimized when $|f_j(\tau)| = \frac{k_j(t-\tau)n |f_{I(\tau)}(\tau)|}{2}$, so:

$$f_j(\tau) \int_{\tau}^t g_j(t') dt' \geq \frac{-k_j^2(t-\tau)^2 n^2 |f_{I(\tau)}(\tau)|^2}{4w_j^B(t)}$$

Moreover,

$$f_{I(\tau)}(\tau) \int_{\tau}^t g_{I(\tau)}(t') dt' \geq \frac{(1 - k_{I(\tau)}(t-\tau)n) |f_{I(\tau)}(\tau)|^2}{w_{I(\tau)}^B(t)}$$

Thus, for small enough $(t-\tau)$, we have $\sum_{j \in N} [f_j(\tau) \int_{\tau}^t g_j(t') dt'] > 0$. This establishes (1), since for all $\tau < t$, $S \cap [\tau, t) \neq \emptyset$.

Intermediate Observation: Now let $f_{C,C'}(\tau) \equiv f_C(\tau) - f_{C'}(\tau) \equiv \sum_{i \in C} f_i(\tau) - \sum_{i \in C'} f_i(\tau)$, and $Z_{\nu'}(\tau) \equiv \sum_{j \in N} [f_j(t') \int_{\tau}^t g_j(s) ds]$. So we know that $\forall \varepsilon > 0$, $\exists t' \in [t-\varepsilon, t)$ such that $Z_{\nu'}(t') > 0$.

Since $Z_{\nu'}(\tau)$ and $f_{C,C'}(\tau)$ are Lipschitz continuous in τ , and $Z_{\nu'}(t) = 0$, we have that for all $\varepsilon > 0$, there must exist a nontrivial interval $[t', t''] \in [t-\varepsilon, t]$ where $f_{C,C'}(\tau)$ do not change sign for all $(C, C') \in 2^N \times 2^N$, and $0 < Z_{\nu'}(t') - Z_{\nu'}(t'') = \sum_{j \in N} [f_j(t') \int_{t''}^{t'} g_j(s) ds]$.

Step 2: We show that $\sum_{j \in N} \left[f_j(t') \int_{t'}^{t''} g_j(s) ds \right] \leq 0$, which gives us the desired contradiction.

Let $O : 2^N \setminus \{\emptyset\} \rightarrow \{1, 2, \dots, 2^n - 1\}$. We will use this ordering of coalitions to define shifts of proposals from a coalition to another in an intuitive (but cumbersome) way. [Readers may wish to skip the tedium and proceed to the word description of $g_{C,C'}(\tau)$, below its definition.] Define $g_{iC}(\tau) = p_{iC}^A(\tau) - p_{iC}^B(\tau)$. Let the sequences (ordered according to O) $A_i^0(t) = \{C \in 2^N \setminus \{\emptyset\} | g_{iC}(\tau) > 0\}$ and $B_i^0(t) = \{C \in 2^N \setminus \{\emptyset\} | g_{iC}(\tau) < 0\}$. Let $g_{iC}^0(\tau) = |g_{iC}(\tau)|$. Denote the k^{th} element of $A_i^0(t)$ as $A_i^0(t)_k$, and similarly for the other sequence. If $g_{iA_i^k(t)_1}^k(\tau) > g_{iB_i^k(t)_1}^k(\tau)$, let $A_i^{k+1}(t) = A_i^k(t)$, and $B_i^{k+1}(t)$ be such that $B_i^{k+1}(t)_m = B_i^k(t)_{m+1}$ (so $B_i^{k+1}(t)$ is one element shorter than $B_i^k(t)$); this will be referred to as "shifting $B_i^k(t)$ "; if $g_{iA_i^k(t)_1}^k(\tau) < g_{iB_i^k(t)_1}^k(\tau)$, shift $A_i^k(t)$, but leave $B_i^k(t)$ unchanged; if $g_{iA_i^k(t)_1}^k(\tau) = g_{iB_i^k(t)_1}^k(\tau)$, shift both sequences. Then define:

$$g_{iC}^{k+1}(\tau) = \begin{cases} g_{iC}^k(\tau) - g_{iB_i^k(t)_1}^k(\tau) & \text{if } C = A_i^k(t)_1 \text{ and } g_{iA_i^k(t)_1}^k(\tau) > g_{iB_i^k(t)_1}^k(\tau) \\ g_{iC}^k(\tau) - g_{iA_i^k(t)_1}^k(\tau) & \text{if } C = B_i^k(t)_1 \text{ and } g_{iA_i^k(t)_1}^k(\tau) < g_{iB_i^k(t)_1}^k(\tau) \\ g_{iC}^k(\tau) & \text{otherwise} \end{cases}$$

Now define:

$$g_{iC,C'}(\tau) = \begin{cases} \min\{g_{iC}^k(\tau), g_{iC'}^k(\tau)\} & \text{if } C = A_i^k(t)_1 \text{ and } C' = B_i^k(t)_1 \text{ for some } k \\ -\min\{g_{iC}^k(\tau), g_{iC'}^k(\tau)\} & \text{if } C = B_i^k(t)_1 \text{ and } C' = A_i^k(t)_1 \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

Finally, let:

$$g_{C,C'}(\tau) = \sum_{i \in N} \lambda_i g_{iC,C'}(\tau)$$

Observe that $g_{C,C'}(\tau)$ has a simple interpretation: it measures the frequency of proposals gained by coalition C from C' in equilibrium A relative to equilibrium B . It is easy to verify that $\sum_{C' \in 2^N} g_{C,C'}(\tau) = -\sum_{C' \in 2^N} g_{C',C}(\tau) \equiv g_C(\tau)$, and $g_i(\tau) = \sum_{C \ni i} g_C(\tau)$.

By optimality, it is clear that $f_{C,C'}(\tau) g_{C,C'}(\tau) \leq 0$, for all $C, C' \in 2^N$ and $\tau < 0$. Since $f_{C,C'}(\tau)$ maintain their sign in $[t', t'']$, we have: $f_{C,C'}(t') \int_{t'}^{t''} g_{C,C'}(s) ds \leq 0$. Now note that:

$$\begin{aligned} \sum_{(C,C') \in 2^N \times 2^N} f_{C,C'}(t') \int_{t'}^{t''} g_{C,C'}(s) ds &= \sum_{(C,C') \in 2^N \times 2^N} f_C(t') \int_{t'}^{t''} g_{C,C'}(s) ds \\ &\quad - \sum_{(C,C') \in 2^N \times 2^N} f_{C'}(t') \int_{t'}^{t''} g_{C,C'}(s) ds \\ &= \sum_{C \in 2^N} f_C(t') \int_{t'}^{t''} g_C(s) ds + \sum_{C' \in 2^N} f_{C'}(t') \int_{t'}^{t''} g_{C'}(s) ds \\ &= 2 \sum_{C \in 2^N} \left(\sum_{i \in C} f_i(t') \int_{t'}^{t''} g_C(s) ds \right) \\ &= 2 \sum_{i \in N} f_i(t') \left(\sum_{C \ni i} \int_{t'}^{t''} g_C(s) ds \right) \\ &= 2 \sum_{i \in N} f_i(t') \int_{t'}^{t''} g_i(s) ds \end{aligned}$$

Thus, $\sum_{i \in N} f_i(t') \int_{t'}^{t''} g_i(s) ds \leq 0$, which completes the contradiction. QED

Proof of Theorem 3: Note that the functions $\widehat{w}_i^j(\cdot)$ defined in the proof of Claim 2 are Lipschitz-continuous with Lipschitz constant $(r + \lambda^j)V(N)$, with $\lambda^j \rightarrow \lambda$. Moreover, they are uniformly bounded, by 0 from below, and $V(N)$ from above. Combining Claim 2 and Theorem 2 implies that all convergent subsequences of continuation functions of G_1, G_2, \dots (with respect to the uniform topology) converge to the same limit, namely the unique MPE continuation payoff functions of $G(N, V, \lambda, T)$. The set of uniformly bounded Lipschitz-continuous functions on $[-T, 0]$ with Lipschitz constant $(r + \lambda + \varepsilon)V(N)$ is compact with respect to the uniform topology for any $\varepsilon > 0$, from which it follows that the sequence $\widehat{w}^1(\cdot), \widehat{w}^2(\cdot), \dots$ itself converges uniformly to the unique MPE continuation payoff functions of $G(N, V, \lambda, T)$. Again, note that $\sup_{t \in [-T, 0]} |w_i^{G_j}(t) - \widehat{w}_i^j(t)| \leq (r + \lambda^j)V(N) \frac{T}{k(j)}$, where the right-hand sides goes to 0 as $j \rightarrow \infty$. Hence, the SPNE continuation payoff functions of games in the sequence G_1, G_2, \dots converge to the same limit as the sequence $\widehat{w}^1(\cdot), \widehat{w}^2(\cdot), \dots$ QED

Proof of Claim 4: First, note that for any $i \in N$ and any $t \leq 0$, $w_i(t) < V(\{i\})$ implies that $V(C \cup \{i\}) - w_i(t) > V(C)$ for any $i \notin C$. This and Claim 3 imply that at any time where $w_i(t) < V(\{i\})$ in a Markov perfect equilibrium, any player $j \in N$ will include player i in the approached coalition at an arrival, and offer her exactly $w_i(t)$. Furthermore, note that if player i has the chance to make an offer at t , then she can guarantee a payoff of at least $V(\{i\})$ by approaching herself. This implies that $w_i(t)$ is bounded below by $\int_t^0 [\lambda_i e^{-(\lambda+r)(\tau-t)} V(\{i\}) + \sum_{j \neq i} \lambda_j e^{-(\lambda+r)(\tau-t)} w_i(\tau)] d\tau$. It is easy to check that this implies $w_i(t) \geq \frac{\lambda_i}{\lambda_i+r} V(\{i\})(1 - e^{(\lambda_i+r)t})$ in every MPE. Therefore, if $T_1(\varepsilon) = \min_{i \in N} \frac{1}{\lambda_i+r} \log \frac{\varepsilon(\lambda_i+r)}{\lambda_i V(\{i\})}$, then for any $t \leq T_1(\varepsilon)$ and $i \in N$, $w_i(t) \geq \frac{\lambda_i}{\lambda_i+r} V(\{i\}) - \varepsilon$, for every $\varepsilon > 0$.

Assume now that for some $K \in \{1, \dots, n-1\}$, there exists a finite $T_K(\varepsilon)$ for any $\varepsilon > 0$ such that for every $C \subset N$ with $|C| \leq K$, it holds that $\sum_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M+r} V(C) - \varepsilon$, $\forall t \leq T_K(\varepsilon)$. Below we show that this implies that for any $\varepsilon > 0$, there exists a finite $T_{K+1}(\varepsilon)$ such that for every $C \subset N$ with $|C| \leq K+1$, it holds that $\sum_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M+r} V(C) - \varepsilon$, $\forall t \leq T_{K+1}(\varepsilon)$. Fix any $\varepsilon > 0$ and any C with $|C| = K+1$. From the induction assumption, $\sum_{i \in C'} w_i(t) \geq \frac{\lambda_M}{\lambda_M+r} V(C') - \varepsilon$, $\forall t \leq T_K(\varepsilon)$ and $C' \subsetneq C$. Consider now any such t , and assume that $\sum_{i \in C} w_i(t) < \frac{\lambda_M}{\lambda_M+r} V(C) - \varepsilon$. Suppose that there is $i \in N$ such that i does not approach everyone in C with probability 1 at t . Let D be such that there is a positive probability that D is approached at t by i , and $C \not\subseteq D$. Since $t \leq T_K(\varepsilon)$, $\sum_{i \in C \cap D} w_i(t) \geq \frac{\lambda_M}{\lambda_M+r} V(C \cap D) - \varepsilon$. Then $\sum_{i \in C} w_i(t) < \frac{\lambda_M}{\lambda_M+r} V(C) - \varepsilon$ implies $\sum_{i \in C \setminus D} w_i(t) < \frac{\lambda_M}{\lambda_M+r} V(C) - \frac{\lambda_M}{\lambda_M+r} V(C \cap D)$. Convexity of V then implies $\sum_{i \in C \setminus D} w_i(t) < \frac{\lambda_M}{\lambda_M+r} [V(D \cup C) - V(D)]$. By Claim 3, i could strictly improve her payoff by approaching $D \cup C$ instead of D , a contradiction. Therefore, for any $C \subset N$ for which $|C| \leq K+1$, $\sum_{i \in C} w_i(t) < \frac{\lambda_M}{\lambda_M+r} V(C) - \varepsilon$ and $t \leq T_K(\varepsilon)$

imply that everyone in C is approached by every player at t with probability 1. Therefore, for $t \leq T_K(\varepsilon)$, $\sum_{i \in C} w_i(t)$ is bounded below by $\min(\frac{\lambda_M}{\lambda_M+r}V(C) - \varepsilon, k(t))$, where $k(t) =$

$\int_t^{T_K(\varepsilon/2)} \left[\left(\sum_{i \in C} \lambda_i \right) e^{-(\lambda+r)(\tau-t)} V(C) + \left(\sum_{j \notin C} \lambda_j \right) e^{-(\lambda+r)(\tau-t)} \left(\sum_{i \in C} w_i(\tau) \right) \right] d\tau$. Thus, there exists $T_{K+1}^C(\varepsilon)$ such that $\sum_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M+r}V(C) - \varepsilon, \forall t \leq T_{K+1}^C(\varepsilon)$. Then for $T_{K+1}(\varepsilon) = \min \left\{ \min_{C:|C|=K+1} T_{K+1}^C(\varepsilon), T_K(\varepsilon) \right\}$, for every $C \subset N$ with $|C| \leq K+1$, it holds that $\sum_{i \in C} w_i(t) \geq \frac{\lambda_M}{\lambda_M+r}V(C) - \varepsilon, \forall t \leq T_{K+1}(\varepsilon)$. QED

Proof of Claim 5: It is sufficient to show that $\exists t$ such that $\sum_{i \in C} w_i(t') \leq v_C$, for every $t' < t$ and $C \subset N$. We proceed by induction.

First, suppose $|C| = 1$. Note that given an arrival at t' , player i 's expected payoff is $\frac{\lambda_i}{\lambda} \max_{D \ni i} \{V(D) - \sum_{j \in D} w_j(t')\} + p_i(t') w_i(t')$, where $p_i(t')$ is the probability that i is included in the proposal (which may or may not be her own) at t' . If $w_i(t') > v_{\{i\}}$, then no one in $N \setminus \{i\}$ would include i in a proposal at t' , and player i 's expected payoff becomes $\frac{\lambda_i}{\lambda} \max_{D \ni i} \{V(D) - \sum_{j \in D} w_j(t')\} + \frac{\lambda_i}{\lambda} w_i(t')$. By Claim 4, for r low enough and some $\delta < 1$, $\exists t_i$ s.t. $\max_{D \ni i} \{V(D) - \sum_{j \in D} w_j(t')\}$ is small enough that player i 's expected payoff at all $t' < t_i$ is less than $\max\{\delta w_i(t'), \delta v_{\{i\}}\}$ as long as no one in $N \setminus \{i\}$ includes i in a proposal. This implies that going back in time, w_i will eventually reach $v_{\{i\}}$, and can then never increase from that value. Thus, with low enough r , $\exists t^1$ s.t. our claim holds for all $|C| = 1$.

Now let t^m be such that our claim holds for all coalitions of size at most m . Let $|C| = m+1$, and $t' < t^m$. Define $\lambda_C = \sum_{i \in C} \lambda_i$, $w_C(t') = \sum_{i \in C} w_i(t')$. Note that if $w_C(t') > v_C$, we have $w_C(t') > V(D) - V(D \setminus C)$ for all $D \supset C$. Thus C cannot be entirely included in the proposal of anyone outside C . Moreover, for any $E \subset C$, $w_{C \setminus E}(t') \leq v_{C \setminus E}$ by the induction hypothesis, so $w_E(t') > v_C - v_{C \setminus E} = V(N \setminus (C \setminus E)) - V(N \setminus C) \geq V(D \cup E) - V(D)$ for all $D \subseteq N \setminus C$. Thus no part of C can be included in the proposal of anyone outside C . Due to Claim 4, by an argument similar to the $|C| = 1$ case, it follows that $\exists t^{m+1}$ s.t. our claim holds for $|C| = m+1$. QED

Proof of Claim 6: First, we establish the following lemma:

Lemma 3: There exists a time to the left of which $V(N) - w_N(\cdot)$ is weakly monotonic.

Proof: We will show that within any time interval $[A, B]$ early enough so that N is always the optimal coalition to approach, the proposer surplus $V(N) - w_N(\cdot)$ attains its rightmost maximum at A or at B .

First, we note that since $V(N) - w_N(\cdot)$ is continuous, the maximum on $[A, B]$ indeed exists, and there exists a rightmost point where it occurs. We proceed by contradiction: assume that the rightmost maximum is attained at $t \in (A, B)$.

Suppose $C(t) = C$, so $\max_{D: C(t) \not\subseteq D} \{V(D) - w_D(t)\} < V(C) - w_C(t)$. Because continuation value functions are Lipschitz continuous, $\exists \delta > 0$ such that $C \subseteq C(t')$ for all $t' \in (t - \delta, t + \delta)$. We say that, going back in time, $C(\cdot)$ "changes at t " if:

- a) for all $\varepsilon > 0$, $C(t) \subsetneq C(t')$ for some $t' \in (t, t + \varepsilon)$ [$C(\cdot)$ shrinks at t]; and/or
- b) for all $\varepsilon > 0$, $C(t) \supsetneq C(t')$ for some $t' \in (t - \varepsilon, t)$ [$C(\cdot)$ grows at t].

Note that within a time interval $[t_1, t_2]$ where $C(\cdot) = C$, we have $w_C(\tau) = e^{-(\lambda+r)(t_2-\tau)}w_C(t_2) + \int_{\tau}^{t_2} [\lambda_C e^{-(\lambda+r)(\tau'-\tau)}(V(C) - w_C(\tau')) + \lambda e^{-(\lambda+r)(\tau'-\tau)}w_C(\tau')] d\tau'$. Thus $w'_C(\tau) = rw_C(\tau) - \lambda_C(V(C) - w_C(\tau))$, which implies that $w_C(\cdot)$ is monotonic (increasing iff $w_C(\cdot) > \frac{\lambda_C}{r+\lambda_C}V(C)$), so the proposer surplus $V(N) - w_N(\cdot)$ is monotonic. Thus, it must be that $C(\cdot)$ changes at t .

In each of the scenarios a) and b), let $C(t') = C(t) \cup D$. Scenario b) can only occur if for any $\varepsilon > 0$, $\exists t'' \in (t, t + \varepsilon)$ such that $V(N) - w_N(t'') \geq \frac{rv_D}{\lambda_D}$ (otherwise, D either would have joined the minimal optimal coalition to the right of t), and $V(N) - w_N(t) = \frac{rv_D}{\lambda_D}$ (can't be strictly less due to Lipschitz continuity, or strictly more because then D couldn't join $C(\cdot)$). This violates our assumption, so we rule out scenario b).

Scenario a) can be further subdivided into two cases:

1) $V(N) - w_N(t) = \frac{rv_D}{\lambda_D}$: this can be ruled out since for any $\varepsilon > 0$, $\exists t'' \in (t, t + \varepsilon)$ such that $V(N) - w_N(t'') > \frac{rv_D}{\lambda_D}$ (otherwise, $w_D(\cdot)$ could not go from being strictly less than v_D immediately to the right of t to equaling v_D at t);

2) $V(N) - w_N(t) > \frac{rv_D}{\lambda_D}$: in this case, in an interval to the right of t , it must be that $w_D(\cdot)$ is strictly increasing going back in time. Thus, it is always approached within that interval, and has strictly negative derivative bounded away from 0. To the left of t , $w_D(\cdot) = v_D$. Also, we know that $w_{C(t)}(\cdot)$'s derivative exists and is continuous in an interval around t . Finally, the continuation value of any cell of $P(t)$ not included in this case (i.e., either not joining or leaving $C(\cdot)$) is constant in some interval around t . Combining the above facts implies that $V(N) - w_N(\cdot) = V(C(\cdot)) - w_{C(\cdot)}(\cdot)$ cannot attain a maximum at t .

Having exhausted all possible cases, we conclude that the lemma holds. QED

Proof of Claim: We distinguish two cases:

a) Going back in time, $V(N) - w_N(\cdot)$ is weakly increasing (has weakly negative derivative where it exists). This means that no new players can ever join $C(\cdot)$ (since this happening at some time t requires $V(N) - w_N(t) = \frac{rv_D}{\lambda_D}$ and $V(N) - w_N(t') < \frac{rv_D}{\lambda_D}$ at some t' to the left of, and arbitrarily close to, t), which implies that there's a time before which $C(\cdot)$ remains constant.

b) Otherwise, going back in time, $V(N) - w_N(\cdot)$ is weakly decreasing forever (has a weakly positive derivative where it exists). As above, if a group of player $D \in P(t)$ joins $C(t) = E$ at some time t , it must be that $V(N) - w_N(t) < \frac{rv_D}{\lambda_D}$ at some t' to the left of, and arbitrarily close to, t . Thus, at any point $\tau < t$, $V(N) - w_N(\tau) < \frac{rv_D}{\lambda_D}$, so D cannot once again join $C(\cdot)$ at any point to the left of t . Since there is a finite number of coalitions, this implies again that there's a time before which $C(\cdot)$ remains constant.

Given that there's a time before which $C(\cdot)$ remains constant ($\equiv C$), we just need to show that it is impossible for the other cells of $P(\cdot)$ to change indefinitely. Suppose that at t , $D \neq C$ is a cell of $P(t)$, so $w_D(t) = v_D = V(C \cup D) - V(C)$. But from previous arguments, we know that for early enough times, $w_D(\cdot)$ can never exceed v_D or drop below $V(C \cup D) - V(C)$, so it must in fact hold constant. This implies that at all times prior to t , D can be expressed as the union of cells of $P(\cdot)$. Thus, going back in time, $P(\cdot)$ can only get finer. The finiteness of the number of players then implies Claim 6. QED

Proof of Theorem 4: (i) For each $i \in C$ and $t < \hat{t}$, we have $w_i(t) = e^{-(\lambda+r)(\hat{t}-t)}w_i(\hat{t}) +$

$\int_t^{\hat{t}} [\lambda_i e^{-(\lambda+r)(\tau-t)} (V(C) - w_C(\tau)) + \lambda e^{-(\lambda+r)(\tau-t)} w_i(\tau)] d\tau$, so

$w_C(t) = e^{-(\lambda+r)(\hat{t}-t)} w_C(\hat{t}) + \int_t^{\hat{t}} [\lambda_C e^{-(\lambda+r)(\tau-t)} (V(C) - w_C(\tau)) + \lambda e^{-(\lambda+r)(\tau-t)} w_C(\tau)] d\tau$. Thus $w'_i(t) = r w_i(t) - \lambda_i (V(C) - w_C(t))$ and $w'_C(t) = r w_C(t) - \lambda_C (V(C) - w_C(t))$. Therefore, $w_C(t) = (w_C(\hat{t}) - \frac{\lambda_C}{r+\lambda_C} V(C)) e^{(r+\lambda_C)(t-\hat{t})} + \frac{\lambda_C}{r+\lambda_C} V(C)$, and algebraic manipulations show that $w_i(t) = [w_i(\hat{t}) - \frac{\lambda_i}{\lambda_C} w_C(\hat{t})] e^{r(t-\hat{t})} + \frac{\lambda_i}{\lambda_C} [w_C(\hat{t}) - \frac{\lambda_C}{r+\lambda_C} V(C)] e^{(r+\lambda_C)(t-\hat{t})} + \frac{\lambda_i}{r+\lambda_C} V(C)$, which implies our result.

(ii) At all times $t < \hat{t}$ and for all $D \in P \setminus \{C\}$, we must have $w_D(\tau) = v_D$, so it must be that $\frac{\lambda_D}{\lambda} (V(C) - w_C(t)) + p_D(t) v_D = \frac{\lambda+r}{\lambda} v_D$, where $p_D(t)$ is the common value of $p_i(t)$ for all $i \in D$.

We have $w_i(t) = e^{-(\lambda+r)(\hat{t}-t)} w_i(\hat{t}) + \int_t^{\hat{t}} [\lambda_i e^{-(\lambda+r)(\tau-t)} (V(C) - w_C(\tau)) + \lambda p_D(\tau) e^{-(\lambda+r)(\tau-t)} w_i(\tau)] d\tau$,

so $w'_i(t) = (V(C) - w_C(t)) (\frac{\lambda_D}{v_D} w_i(t) - \lambda_i)$ (we can differentiate since $p_D(\cdot)$ is continuous). There must be a time before which $V(C) - w_C(t) > 0$, which implies that $w_i(\cdot)$ is monotonic as $t \rightarrow -\infty$, so it converges. Since $\lim_{t \rightarrow -\infty} (V(C) - w_C(t)) > 0$, we must have $\lim_{t \rightarrow -\infty} \frac{\lambda_D}{v_D} w_i(t) - \lambda_i = 0$, as desired. QED

Proof of Corollary 2: Take the earliest partition P from the game with deadline. Suppose that limit stationary payoffs are as follows: $w_i = \frac{\lambda_i}{\lambda_D} v_D$ for all $i \in D \in P \setminus \{C\}$, and $w_i = \frac{\lambda_i}{r+\lambda_C} V(C)$ for all $i \in C$. Let q_D be the stationary probability that $D \in P$ is approached given an arrival. Note that $\int_0^\infty e^{-(\lambda+r)\tau} d\tau = \frac{1}{\lambda+r}$. We need to verify that the following condition is satisfied: $w_i = \frac{1}{\lambda+r} (\lambda_i \frac{r}{r+\lambda_C} V(C) + \lambda q_D w_i)$, where D is the cell of P containing i , with $q_C = 1$ and $q_D \in [0, 1]$ for all $D \in P \setminus \{C\}$.

It is easy to check that our condition is satisfied for $i \in C$. For $i \notin C$, the condition becomes: $q_D = \frac{\lambda+r}{\lambda} - \frac{\lambda_D}{\lambda v_D} \frac{r}{r+\lambda_C} V(C)$. This expression is clearly continuous and approaches 1 as $r \rightarrow 0$. We need only check that it is decreasing in r in a right neighborhood of 0 $\iff 1 < \frac{\lambda_D}{v_D} V(C) \frac{\lambda_C}{(r+\lambda_C)^2}$ for small enough $r \iff \frac{V(C)}{\lambda_C} > \frac{v_D}{\lambda_D}$.

In the game with deadline, note that for $i \in D$,

$w_i(t) < e^{-(\lambda+r)(\hat{t}-t)} w_i(\hat{t}) + \int_t^{\hat{t}} [\lambda_i e^{-(\lambda+r)(\tau-t)} (V(C) - w_C(\tau)) + \lambda e^{-(\lambda+r)(\tau-t)} w_i(\tau)] d\tau = [w_i(\hat{t}) - \frac{\lambda_i}{\lambda_C} w_C(\hat{t})] e^{r(t-\hat{t})} + \frac{\lambda_i}{\lambda_C} [w_C(\hat{t}) - \frac{\lambda_C}{r+\lambda_C} V(C)] e^{(r+\lambda_C)(t-\hat{t})} + \frac{\lambda_i}{r+\lambda_C} V(C)$, which converges to $\frac{\lambda_i}{r+\lambda_C} V(C)$. Thus, $\lim_{\tau \rightarrow -\infty} w_i(\tau) = \frac{\lambda_i}{\lambda_D} v_D \leq \frac{\lambda_i}{r+\lambda_C} V(C) < \frac{\lambda_i}{\lambda_C} V(C)$, which implies $\frac{V(C)}{\lambda_C} > \frac{v_D}{\lambda_D}$, as desired. QED

Proof of Claim 7: The distance in dimension $i \in C$ is $(\frac{\lambda_i}{\lambda_C} - \frac{\lambda_i}{r+\lambda_C}) V(C) < \frac{r}{r+\lambda_M} V(N)$, while it is 0 in all other dimensions. QED

Proof of Claim 8: The statement holds vacuously if $\mathcal{C}(V) = \emptyset$, so we assume $\mathcal{C}(V) \neq \emptyset$. Normalize payoffs with $V(N) = 1$. Let $x \in \mathcal{C}(V)$, and set $\lambda_i = x_i$. For any $T > 0$, consider continuation value functions $w_i(t) = \frac{\lambda_i}{1+r} (1 - e^{(1+r)t})$, $\forall t \in [-T, 0]$, $i \in N$, and specify strategies as follows:

For every $i \in N$, if player i gets the chance to make an offer at $t \in [-T, 0]$, she approaches the grand coalition and offers exactly $w_j(t)$ to every $j \in N \setminus \{i\}$. If player i gets approached

at t , then independently of who approached her and what coalition was approached, she accepts the offer if and only if she is offered at least $w_i(t)$.

We will show that the strategies specified above comprise a Markov perfect equilibrium, in which expected payoffs are given by $\frac{x_i}{1+r}(1 - e^{-(1+r)t})$ for every $i \in N$. First, note that if no offer is accepted at t , and everyone subsequently plays according to the prescribed strategies, then the expected continuation payoff of player j is:

$$\int_t^0 [\lambda_j e^{-(1+r)(\tau-t)} (1 - \sum_{k \neq j} \frac{\lambda_k}{1+r} (1 - e^{-(1+r)\tau})) + \left(\sum_{k \neq j} \lambda_k \right) e^{-(1+r)(\tau-t)} \frac{\lambda_j}{1+r} (1 - e^{-(1+r)\tau})] d\tau = w_j(t).$$

In particular, the expected payoff of player j at the beginning of the game is $w_j(-T) = \frac{x_j}{1+r}(1 - e^{-(1+r)T})$.

Second, note that given other players' strategies, the best offer player i can give to the grand coalition is the one specified above, and since $1 - \sum_{j \neq i} w_j(t) > w_i(t)$, it yields a higher payoff than making an unacceptable offer.

Next, note that $w_j(t) < x_j \forall t \in [-T, 0]$ and $j \in N$. Since $x \in \mathcal{C}(V)$, this implies that $\sum_{j \in C} w_j < V(N) - V(N \setminus C)$ for any $C \subset N$. Given others' strategies, this means that there is no $C \subset N$ such that player i could give an acceptable offer to coalition $N \setminus C$ and get a strictly higher payoff than what she obtains when following the strategy prescribed above.

We conclude that no player can profitably deviate, given the above profile, at any point where it is her turn to make an offer.

If player $i \in N$ is approached by another player at t , then rejecting the offer results in continuation payoff $w_i(t)$, which means that it is optimal to reject the offer when it is not above $w_i(t)$, and it is optimal to accept the offer when it is not below $w_i(t)$. Hence, the strategy prescribed above is optimal for i .

Thus, the MPE expected payoffs are given by $\frac{x_j}{1+r}(1 - e^{-(1+r)T})$ for every $i \in N$.

Finally, note that as $T \rightarrow \infty$, the expected MPE payoff of player i converges to $\frac{x_i}{1+r}$.
QED

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