

# Markov Voting Equilibria: Theory and Applications\*

Daron Acemoglu

MIT

Georgy Egorov

Northwestern University

Konstantin Sonin

New Economic School

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## Abstract

We provide a general framework for the analysis of dynamics of institutional change (e.g., democratization, extension of political rights or repression), and how this dynamics interacts with (anticipated and unanticipated) changes in the distribution of political power and changes in economic structure (e.g., social mobility or other changes affecting individuals' preferences over different types of economic policies and allocations). We focus on the Markov voting equilibria, which require that economic and political changes should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected continuation utility by doing so. Under the assumption that different economic and social institutions/policies as well as individual types can be ordered, and preferences and the distribution of political power satisfy “single crossing” condition, we prove the existence of pure-strategy equilibrium and provide conditions for its uniqueness. Despite its generality, we show that the framework yields a number of comparative static results. For example, we show that if there is a change from one environment to another (with different economic payoffs and distribution of political power) but the two environments coincide up to a certain state  $s'$  and before the change the steady state of equilibrium was that some state  $x \leq s'$ , then the new steady state that emerges after the change in environment can be no smaller than  $x$ . We also show how this framework can be applied to the study of the dynamics of political rights and repression, and derive a range of additional comparative statics for this more specific application.

Very Preliminary and Incomplete. Please Do Not Circulate.

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**JEL Classification:**

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# 1 Introduction

An idea going back at least to De Tocqueville (1835) relates the emergence of a stable democratic system to an economic structure with relatively high rates of social mobility and limited inequality.<sup>1</sup> When different groups in society have similar preferences and also expect their preferences to change in the future, it is intuitive to presume that each will be more willing to recognize the political rights of others and less willing to repress them or exclude them from political decision-making. The reasoning in this case is that an individual would not like to support institutions that restrict the political or social rights of another social group because she expects that she may transition to that social group at some point in the near future.<sup>2</sup> In contrast to this intuition, dynamic political economy models also tend to imply that anticipation of future changes in economic or political conditions can often trigger conflict and even undermine democracy (Acemoglu and Robinson, 2000, 2001). This is because the expectation of a shift in political power tomorrow often motivates costly actions to undermine this shift or repress or disenfranchise the groups that would be its beneficiaries. We next illustrate how both of these possibilities can arise in the context of a simple example.

**Example 1** Consider a society consisting of two groups, the poor and the middle class. Suppose that currently the middle class hold political power and implement their preferred policy, giving them per period  $u_m^*$ , but with probability  $q$ , power may shift to the poor (for example because they will be able to solve the collective action problem and exercise power commensurate with their numbers). Suppose that in this case political power stays with the poor forever and they will implement their preferred policy which will give per period utility  $u'_m < u_m^*$  to the middle class. The middle class can prevent this outcome by using repression against the poor, which essentially takes away their political rights and prevents the power ever shifting to them. Then it is straightforward to see that if such repression is not very costly for the middle class, they would choose such repression. The underlying reason for repression is the one discussed in the previous paragraph: the middle class holds political power today but anticipates (and fears)

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<sup>1</sup>This is also consistent with Barrington Moore (1966) and Lipset (1960). See also Erikson and Goldthorpe (1992).

<sup>2</sup>This might, for example, contribute to an explanation for why members of the landed aristocracy have typically been less likely to support the extension of political rights the poorer segments of society than have merchants and professional classes.

change in the balance of power in the future, away from itself towards the poor.

Now let us augment this model with social mobility. In particular, suppose that each middle-class agent also expects to become poor with some probability  $q'$ . To simplify this example, suppose that the repression decision is irreversible. Then it is again evident that if  $q'$  is sufficiently high, the middle class would not favor repression even if they expect political power to shift away from themselves towards the poor, because they themselves expect to be in the poor's shoes in the future.

A systematic study of both intuitions discussed in the first paragraph and illustrated in Example 1 requires a dynamic and stochastic framework where individuals make political decisions anticipating how current decisions will impact the future distribution of political power and future choices. Such a framework would be useful not only for an in-depth analysis of the relationship between social mobility, changes in the balance of political power and repression (or more generally, the willingness of different individuals to recognize the political rights of others, and the emergence of democracy) but also to understand the dynamics of political regimes in general and the stability of democracy in particular. In fact, our above description already reveals that dynamic considerations complicate the reasoning isolated in Example 1. For example, it may well be that if there is also a third social group, the rich, the middle class may refrain from repressing the poor, because this is a “slippery slope”: once repression is used against the poor, the middle class itself becomes weakened against the rich, which may then use repression against the middle class.<sup>3</sup> The same considerations are relevant in thinking about social mobility, since such mobility not only changes the economic interests of the individuals who have political power today but the identity of who will have political power tomorrow and thus how this power will be exercised. As a result, we will see that, in contrast to the intuition above, social mobility might also increase incentives for repression under certain circumstances.

In this paper, we develop and systematically analyze such a framework, and then apply it to the study of social mobility, changes in the balance of political power and repression. The first part of the paper introduces this framework and provides general characterization and comparative static results. In particular, we consider a society consisting of  $i = 1, 2, \dots, n$  types

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<sup>3</sup>Such “slippery slope” arguments are advanced in the literature by Schauer (1985), and have been formalized by Acemoglu, Egorov and Sonin (2008, 2010) in different contexts. See also Schwarz and Sonin (2008) and Hirschleifer, Boldrin, and Levine (2009).

of individuals and  $s = 1, 2, \dots, m$  states, which represent both different economic arrangements, with varying payoffs for different types of individuals, and different political arrangements and institutional choices. Stochastic shocks may both change preferences and the distribution of political power (e.g., by increasing the number of people of a certain type or by resolving their collective action problem) and potentially shuffle individuals across different types in a way that corresponds to social mobility in society. Individuals care about the expected discounted sum of their utility, and may also suffer some costs when there are political transitions. We impose “single crossing” in economic preferences and political power. In particular, we assume that the types of individuals and states are “ordered,” and assume that higher-indexed individuals relatively prefer higher-indexed states and also tend to have greater political power in such states. We then define a notion of Markov voting equilibrium, capturing two natural requirements of equilibria in this environment: (1) that changes in states should take place if there exists a subset of players with the power to implement them and will obtain higher continuation utility (along the equilibrium path) by doing so; (2) that strategies and continuation utilities should only depend on payoff-relevant variables and states. Under these assumptions, we establish the existence of pure strategy equilibria. Furthermore, we show that the stochastic path of states in any Markov voting equilibrium ultimately converges to a steady state—i.e., to a state that does not induce further changes once reached (Theorems 1 and 3). Although Markov voting equilibria are not necessarily unique, we provide sufficient conditions that ensure uniqueness (Theorems 2 and 4). We also establish a close correspondence between these Markov voting equilibria and the pure-strategy Markov perfect equilibria of the dynamic game described above (Theorem 5).

Despite the generality of the framework described here, we also show that some strong comparative static results always hold. Most notably, we establish the following results. First, suppose that when the environment changes in anticipated or unanticipated manner (either because preferences or political power changes or because there is social mobility) but this does not change preferences or the allocation of political power in any of the states  $s = 1, \dots, s'$ , but potentially changes them in states  $s = s' + 1, \dots, m$ . The result is that if the steady state of equilibrium dynamics described above,  $x$ , was at a state that did not experience change (i.e.,  $x \leq s'$ ), then the new steady state emerging after the change in environment can be no smaller than this steady state (Theorem 6). Intuitively, a transition to any of the smaller states,  $s \leq x$ , could have been chosen, but was not, before the change. Now, given that preferences and political

power did not change for these states, they have not become more attractive. In contrast, some of the higher-ranked states may have become more attractive, thus inducing a transition to a higher state. In fact, perhaps somewhat surprisingly, transition to a state  $s \geq s' + 1$  can take place even if all states  $s = s' + 1, \dots, m$  become less attractive for all agents in society. An interesting and novel implication of this result is that in some environments, there may exist critical states such that if they are reached before certain major shocks or changes take place, then there will be no turning back. One application would be to a situation in which this critical state can be interpreted as “stable democracy” (see Corollary 1). Then, if it is reached before some major shocks take place, it will never collapse (though the entire democratic system might have collapsed if such a shock arrived before stable democracy was reached).

Second, our framework also implies a related result on dynamic equilibrium trajectories (Theorem 7). Consider a similar change to that discussed in the previous paragraph, leaving preferences and the distribution of the power the same in states  $s = 1, \dots, s'$ . However, suppose that this change arrives before the steady state  $x \leq s'$  is reached. The result is that when all agents in society have discount factor sufficiently small (smaller than some threshold  $\beta_0$ ), then the direction of changes states will remain the same as before (i.e., if there were transitions towards higher states before, this will continue, and vice versa). Intuitively, this happens because, with sufficiently small discount factor, all agents care about the payoffs in the next period most, and by assumption, these payoffs have not changed (though payoffs of states to the right of  $s'$  may have changed very significantly). This result again has a range of important and novel implications. Consider an application to a model of democratization, where those currently holding power were considering extending the franchise to some of the poorer segments of society cards excluded from voting rights. The shock may now correspond to a greater likelihood of more radical policies or even revolution in the future if enfranchisement takes place. This result states that such a change will not deter enfranchisement provided that agents are not very forward-looking, but may do so if they are sufficiently forward-looking (have very high discount factors).

Third, suppose that a change in environment makes extreme states “sticky,” for example, high-indexed individuals, which prefer the highest-indexed states, increase their political powers (but preferences remain unchanged). Our next comparative static shows that if the shock happened when the society was away from these extreme states (in this example, in the suffi-

ciently low-indexed state), then the equilibrium trajectory is not affected (Theorem 8). This once again has interesting implications. For example, suppose that in a democracy the poor become sufficiently more powerful that any move away from democracy becomes impossible if the poor oppose it. Then our result implies that this change will only impact the equilibrium if we were currently in democracy and considering a move away from it. If the equilibrium, before the change, involved a transition from limited democracy to a more democratic state, then this change in environment does not affect the equilibrium path (Corollary 2).

It is notable that the comparative static results mentioned in the previous three paragraphs hold despite the generality of the framework we consider. This highlights that they do indeed reflect quite powerful forces in this class of dynamic political economy models (with single-crossing properties).

The second part of the paper applies the general framework and these ideas to the study of the relationship between social mobility, changes in the balance of power and repression. The model we adopt for this analysis is summarized in the next example.

**Example 2** *We now give a brief overview of our model of repression, which is described in greater detail in Section 4. The society consists of  $N = \{1, 2, \dots, n\}$  types of individuals (with individuals of each type sharing the same economic interests and preferences). We represent these by*

$$u_i = -(p - b_i)^2 - \text{cost of repression}, \quad (1)$$

*where  $b_i$  is the political bliss point of type  $i$  agents, and the cost of repression represents the costs that agents encounter due to repressive activities or other choices that restrict the political rights of some subgroup of agents. Let us order types such that  $b_i$ 's are an increasing sequence (and we assume it to be strictly increasing). Social mobility is captured by allowing agents' types to change over time. The number of agents of type  $i$  is denoted by  $\gamma_i$ , and this is allowed to change over time as well. We model repression by allowing those who hold political power at the moment from banning the political participation of some subset of types. Policies and repression decisions are voted over, and are decided by a majority (the results can also be extended to supermajority rules). In particular, letting  $\alpha_i$  be an indicator for whether a group has political rights (thus  $\alpha_i = 0$  stands for that group being repressed), a policy can be implemented if it is supported by*

collection of groups  $X$  such that

$$\sum_{i \in X} \alpha_i \gamma_i > \sum_{i \in N \setminus X} \alpha_i \gamma_i. \quad (2)$$

After showing that all of the general results of our framework can be applied in the context of the model described in the previous example (and thus to the study of the interplay between social mobility, changes in the balance of power and repression), we derive further results that shed light on the questions raised above, including why both of the perspectives about the effects of anticipated future changes on democracy and distribution of political rights may be true, but they both require further refinement. More specifically, we establish the following results. First, we clarify the conditions under which social mobility, captured by shuffling of individuals across different economic positions and preferences, reduces repression and makes democracy more stable. In particular, Proposition 2 shows that if currently powerful groups are sufficiently forward-looking, expect to change their economic interests due to social mobility, and median preferences are close to mean preferences, then starting with any distribution of political rights, society will transition to full democracy. This is because full democracy is the best guarantee that future policies will not be too far from the preferences of individuals that are currently powerful, regardless of how their preferences change. This result thus provides a simple formalization of the de Tocqueville hypothesis outlined in the first paragraph.

Second, and in contrast to the first, we also show that anticipation of social mobility can be a force towards greater repression. This happens when the current median voter expects social mobility to change its status in one direction (in expectation) and may wish to change the identity of future median voters (Proposition 3). For example, a median voter who expects to become richer may wish to increase repression so as to shift future median voters to the right and have more pro-rich policies in the future (which would not otherwise be chosen by a median voter in the same status as the current one).

Third, we also provide a partial characterization of which groups benefit from the shift in policies due to social mobility. In particular, we show that the political response to social mobility tends to shift policies in favor of those who are likely to move up due to social mobility and also make those that are not part of this mobility process worse off (Proposition 4).

Fourth, we also establish that, in line with the intuition of the first paragraph, anticipated changes in environments that will alter the distribution of political power in society will indeed

induce more repression and tend to undermine democratic institutions. In particular, Proposition 5 shows that, with sufficiently forward-looking agents, the anticipation that political power will shift at some point in the future is sufficient to induce repression. Moreover, both lower costs of repression and a higher likelihood of future changes in the balance of political power make such repression more likely.

Fifth, we complement the previous result by showing that early revelation of news about the realization of the distribution of future political power tends to reduce repression (Proposition 1). Intuitively, part of the reason why repression takes place is to prevent the political adverse effects of changes in the distribution of political power. Once this change is realized, there is less need for such repression.

Finally, we identify a new source of strategic complementarity in repression: an increase in the cost of repression of a group tends to discourage repression by other groups (Proposition 7). The reason for this is that repression is partly driven by the fear that other groups will come to power and will engage in repression to preserve their newly-gained political position. When repression becomes more costly for them, this fear is diminished, and the reason for the initial repression becomes less strong. This result is more broadly interesting, because it also provides a perspective on why repression differs markedly across societies. For example, Russia before the Bolshevik Revolution repressed leftist views, and after the Bolshevik Revolution systematically repressed rightist and centrist views, while the extent of repression of either extreme has been more limited in the United Kingdom. Such differences are often treated to differences in “political culture”. Proposition 7 suggests that small differences in economic interests or political costs of repression (which were of course quite different between Russia and the United Kingdom) can lead to significantly different repression outcomes: in one of two similar societies, very different parts of the political spectrum may be repressed at different points in time, while the experiences much more limited repression of similar positions because of the strategic complementarities in the incentives to repress of different groups.

Our paper is related to a large political economy literature. First, our previous work, in particular Acemoglu, Egorov and Sonin (2011), takes one step in this direction by introducing a model for the analysis of the dynamics and stability of different political rules and constitutions. However, that approach not only heavily relies on deterministic and stationary environments (thus ruling out both social mobility and anticipated changes in political power) but also focuses



on environments in which the discount factor is sufficiently close to 1 that all agents just care about the payoff from a stable state (that will emerge and persists) if such a state exists. Here, in contrast, it is crucial that political change and choices are motivated by potentially short-term gains.<sup>4</sup>

Second, a diverse range of papers in dynamic political economy and in dynamics of clubs emerge as special cases of our paper. Among these, Roberts (1999) deserves special mention as important precursors of our analysis.<sup>5</sup> Roberts studies a dynamic model of club formation in which current members of the club vote about whether to admit new members and whether to contract the club. Roberts also makes single-crossing type assumptions and focuses on non-stochastic environments and majoritarian voting (see also Barberà, Maschler, and Shalev, 2001, for a related setup). Both our framework and characterization results are more general, not only because they incorporate stochastic elements but also because we provide conditions for uniqueness, convergence to steady states and general comparative static results. Furthermore, in the context of our leading application, we isolate the different effects of social mobility and changes in future political power, which do not have equivalents in Roberts. In addition, Gomes and Jehiel's (2005) paper, which studies dynamics in a related environment with side transfers, is also noteworthy. This paper, however, does not include stochastic elements or similar general characterization results either.<sup>6</sup> Strulovici (2010) studies a voting model with stochastic arrival of new information. In that paper, the focus is on information leading to inefficient dynamics, and changes in political institutions or voting rules are not part of the model.

Third, our motivation is also related to the literature on political transitions. Acemoglu and Robinson (2000a, 2001) considered environments in which institutional change is partly

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<sup>4</sup>In Acemoglu, Egorov and Sonin (2010), we study political selection and government formation in a population with heterogeneous abilities and allow stochastic changes in the competencies of politicians. Nevertheless, this is done under two assumptions, which significantly simplify the analysis and make it inapplicable to the general sets of issues we are interested in here: stochastic shocks are assumed to be very infrequent and the discount factor is again taken to be large (close to 1).

<sup>5</sup>Other important contributions here include Barberà and Jackson (2004), Burkart and Wallner (2000), Jehiel and Scotchmer (2001), Alesina, Angeloni, and Etro (2005), Bordignon and Brusco (2003), Lizzeri and Persico (2004), and Lagunoff (2006).

<sup>6</sup>We also differ from Roberts (1999) as we look at (Markov perfect) equilibria of a fully specified dynamic game, while Roberts's analysis imposes that, under the assumption of majoritarian voting, at each stage the choices of the "median voter" will be implemented.

motivated by a desire to reallocate political power in the future to match the current distribution of power. Acemoglu and Robinson's analysis is simplified by focusing on a society consisting of two social groups (and in Acemoglu and Robinson, 2006, with three social groups). Social mobility and issues related to how current changes pave the way for future changes are not studied.

Fourth, in Acemoglu and Robinson (2001), Fearon (2005), Powell (2005), and Acemoglu, Ticchi and Vindigni (2010) anticipation of future changes in political power leads to inefficient policies, civil war or collapse of democracy. There is a growing literature that focuses on situations where decisions of the current policy makers affect the future allocation of political power. Some of these issues are discussed in Besley and Coate (1998). In Acemoglu and Robinson (2000a), the current elite decides whether to extend the franchise to change the future distribution of political power as a commitment to future policies (and thus potentially staving off costly social unrest or political revolution). In Bourguignon and Verdier (2000), the choice of educational policy today affects political participation in the future. This might result in an inferior growth path: the elite deny the poor majority a proper education as it fears their increased political participation in the future. In Dolmas and Huffman (2004), immigration policy plays a similar role. Glaeser and Shleifer (2005) consider a politician who chooses policy to trim the electorate to increase his re-election prospects. Wright (1986), Piketty (1995) and Benabou and Ok (2001) discuss the relationship between social mobility and redistribution. For example, Benabou and Ok provide an explanation for why a poor median voter may not necessarily support high redistribution (because he expects to suffer from redistribution in the next period in case he becomes richer). None of these papers studied the general theoretical issues that we focus on here.

Fifth, there is a small literature on strategic use of repression, which includes Acemoglu and Robinson (2000b), Gregory, Schroeder, and Sonin (2010) and Wolitzky (2011). In Wolitzky (2011), different political positions (rather than different types of individuals) are repressed in order to shift the political equilibrium in the context of a two-period model of political economy. In Acemoglu and Robinson (2000b), repression arises because political concessions can be interpreted as a sign of weakness. Issues related to social mobility, changes in the future distribution of political power and dynamics of institutions do not arise in these models.

The rest of the paper is organized as follows. In Section 2, we formulate a general framework of political economy with institutional changes and shocks: the environment, assumptions and definitions we will use throughout the paper, and the concept of Markov Voting Equilibrium. Section 3 contains the analysis of Markov Voting Equilibria. We start with the stationary case (without shocks), then extend the analysis to the general case where shocks are possible, and then compare the concepts of Markov Voting Equilibrium to Markov Perfect Equilibrium in a properly defined dynamic game. We establish several comparative statics results that hold even at this level of generality; this allows us to study the society's reactions to shocks in applied models. Section 4 applies the general model to issues of social mobility and dynamic (dis)enfranchisement. Section 5 discusses possible extensions and limitations of the general framework. Section 6 concludes.

## 2 General Framework

Time is discrete and infinite, indexed by  $t \geq 1$ . The society consists of  $n$  agents,  $N = \{1, \dots, n\}$ . The set of agents is ordered, and the order reflects the initial distribution of some variable of interest: agents with lower numbers may be the elite (and pro-authoritarian rule), while those with higher numbers may be workers or peasants favoring democracy; other possible scales include rich-vs.-poor or secular-vs.-religious. In each period, the society may find itself in one of the  $h$  environments  $E^1, \dots, E^h$ ; we denote the set of environments by  $\mathcal{E}$ . The environment that the society finds itself in encapsulates agent's economic payoffs and political rules, which are described below in detail. Most importantly, the transitions between environments are stochastic and follows a Markov chain: the probability that the society which lived period  $t$  in environment  $E$  will find itself in environment  $E'$  equals

$$\pi(E, E'). \tag{3}$$

Naturally,

$$\sum_{E' \in \mathcal{E}} \pi(E, E') = 1.$$

Importantly, changes between environments are beyond the control of agents, and transitions between them are intended to capture the stochastic processes in the world, to which the society would respond within an environment. For example, Nature cannot abolish a constitution in

favor of another one, nor it can depose a tyrant, but it may change the economic payoffs or reallocate political or military power so that powerful agents in the society decide to undertake these acts.

We want the model to capture the possibility that poor become rich and vice versa, i.e., that the relative order of individuals changes over time as a result of a shock. To do this, with each shock we associate a probability distribution over permutations  $\eta \in S^n$  (symmetric group). Namely, for any pair  $E, E' \in \mathcal{E}$  we take a probability distribution  $\mu_{E,E'}(\eta)$ , interpreting this as the probability that if the environment changes from  $E$  to  $E'$ , agents are reordered according to  $\eta$  (so, if before the shock player  $i \in N$  was ranked  $\rho(i)$ , after the shock he is ranked  $\eta \circ \rho(i)$  with probability  $\mu_{E,E'}(\eta)$ ). By definition of probability distribution, for all  $E, E' \in \mathcal{E}$ ,

$$\sum_{\eta \in S^n} \mu_{E,E'}(\eta) = 1.$$

Let us denote the probability that an agent ranked  $i$  will become ranked  $j$  as a result of a shock that changes  $E$  to  $E'$  by

$$\lambda_{E,E'}(i, j) = \sum_{\eta \in S^n} \mu_{E,E'}(\eta) \mathbf{I}_{\eta(i)=j};$$

then  $\lambda_{E,E'}(i, \cdot)$  is a probability distribution on  $N$  for any  $i \in N$ . The following assumption is assumed to hold throughout the paper.

**Assumption 1** *The probability distributions  $\{\mu_{E,E'}\}_{E,E' \in \mathcal{E}}$  satisfy the following:*

1. *For any  $E \in \mathcal{E}$  and any  $i \in N$ ,  $\lambda_{E,E}(i, i) = 1$  (equivalently,  $\mu_{E,E}(\text{id}) = 1$ , where  $\text{id} \in S^n$  is the identity permutation).*
2. *For  $E, E' \in \mathcal{E}$  and any two agents  $i, j \in N$  such that  $i < j$ , the distribution  $\lambda_{E,E'}(i, \cdot)$  is (weakly) first-order stochastically dominated by  $\lambda_{E,E'}(j, \cdot)$ .*

Clearly, Assumption 1 is satisfied if  $\lambda_{E,E'}(i, i) \equiv 1$ , i.e., if there is no reshuffling ever. More generally, part 1 requires that reshuffling cannot happen when there is no shock, while part 2 means that there is no reversal on average: a poor person is more likely to stay poor than a rich person is to become poor. Both parts are very natural.

Most of the time we will make the following assumption on the probabilities of transitions between environments:

**Assumption 2** *If  $1 \leq x < y \leq h$ , then*

$$\pi(E^y, E^x) = 0. \quad (4)$$

In other words, Assumption 2 stipulates only a finite number of shocks.<sup>7</sup> Moreover, it assumes that environments are numbered so that only transitions to higher-numbered environments are possible; this, however, is without loss of generality.

Fix an environment  $E$ . In this environment, there is a finite set of states  $S = S_E = \{1, \dots, m\}$  (the number of states is  $m = m_E$ ). By states we mean political or social arrangements, distribution of political power or of means of production, over which the society, in principle, has control, at least if it gets support of sufficiently many powerful agents. The set of states may be the same in all environments  $E \in \mathcal{E}$ , or, possibly, shocks may make new states available or change the set of states altogether. More importantly, the set of states is *ordered*: this may be interpreted as a sequence of political arrangements which gives less and less power to the poor and more and more power to the elite as  $s \in S$  increases. To each state we assign stage payoff  $u_i(s) = u_{E,i}(s)$ , which individual  $i$  gets in a period which ends at state  $s$  if the current environment is  $E$ .

**Definition 1 (*Increasing Differences*)** *Vector  $\{w_i(s)\}_{i \in A}^{s \in B}$ , where  $A, B \subset \mathbb{R}$ , satisfies Weak Increasing Differences condition (WID), if for any agents  $i, j \in A$  such that  $i > j$  and any states  $x, y \in B$  such that  $x > y$ ,*

$$w_i(x) - w_i(y) \geq w_j(x) - w_j(y). \quad (5)$$

*It satisfies Strict Increasing Differences condition (SID), if for any agents  $i, j \in N'$  such that  $i > j$  and any states  $x, y \in S'$  such that  $x > y$ ,*

$$w_i(x) - w_i(y) > w_j(x) - w_j(y). \quad (6)$$

We assume that the stage payoffs, with properly ordered individuals, satisfy the *WID* property.

**Assumption 3** *In every environment  $E \in \mathcal{E}$ , the vector of utility functions,  $\{u_{E,j}(s)\}_{j \in N}^{s \in S}$ , satisfies *SID* property.*

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<sup>7</sup>Notice that Assumption 2 does not preclude the possibility that the environment returns to the state where it was before, but requires that it happens a finite number of times. Indeed, to model the possibility of  $q$  transitions between  $E^1$  and  $E^2$ , we can define  $E^3 = E^1$ ,  $E^4 = E^2$ , etc.

In the model, payoffs  $\{u_i(s)\}$  are assigned to combinations of environments, states and individuals rather than endogenously determined; this is made to simplify notation and the game. Implicitly, we think that in every state there is some economic interaction that results in (expected) payoffs  $\{u_i(s)\}$ . Any such interaction is permissible in our model, as long as Assumption 3 is satisfied. Most of the results would hold if we required that only WID property held.

Apart from stage payoffs, states are characterized by political power. We capture this by the set of winning coalitions,  $W_s = W_{E,s}$ . As standard, we make the following assumption:

**Assumption 4 (*Winning Coalitions*)** For environment  $E \in \mathcal{E}$  and state  $s \in S$ , the set of winning coalitions  $W_s = W_{E,s}$  satisfies:

1. (*monotonicity*) if  $X \subset Y \subset N$  and  $X \in W_s$ , then  $Y \in W_s$ ;
2. (*properness*) if  $X \in W_s$ , then  $N \setminus X \notin W_s$ ;
3. (*decisiveness*)  $W_s \neq \emptyset$ .

The first part of Assumption 4 states that if some coalition has the capacity to implement (social or political) change, then a larger coalition also does. The second part ensures that if some coalition has the capacity to implement change, then the coalition of the remaining players (its complement) does not have one. Finally, the third part, in the light of monotonicity property, is equivalent to  $N \in W_s$ , and it thus states that if all players want to implement a change, they can do so.

Assumption 4 puts minimal and natural restrictions on the set of winning coalitions  $W_s$  in each given state  $s \in S$ . We next impose a between-state (albeit still within-environment) restriction on the sets of winning coalitions. We do so to capture the idea that states ranked higher are also likely to be governed by people ranked higher. More formally, we adopt the following definition of quasi-median voter from Acemoglu, Egorov, and Sonin (2011, forthcoming).

**Definition 2 (*Quasi-Median Voter*)** Player ranked  $i$  is a quasi-median voter (QMV) in state  $s$  (in environment  $E$ ) if for any winning coalition  $X \in W_s$ ,  $\min X \leq i \leq \max X$ .

Equivalently, player ranked  $i$  is a QMV if for any  $X \in W_s$ ,  $\{j \in N : j < i\} \notin W_s$  and  $\{j \in N : j > i\} \notin W_s$ , or if  $i$  belongs to any “connected” winning coalition (we say  $X$  is connected

if  $X$  contains individuals ranked from  $a$  to  $b$  for some  $a \leq b$ . If we let  $M_s = M_{E,s}$  denote the ranks of QMV in state  $s$  in environment  $E$  then, by Assumption 4,  $M_s \neq \emptyset$  for any  $s \in S$  and  $E \in \mathcal{E}$ ; moreover, the set  $M_s$  is itself connected. In many cases, the set of quasi-median voters is a singleton,  $|M_s| = 1$ . This will hold whenever one individual is the dictator, i.e.,  $X \in W_s$  if and only if  $i \in X$  (and then  $M_s = \{i\}$ ), but it will also be true in other remarkable cases such as majority voting with odd number of players, or even majority voting among a subset of players. If this holds, we would be able to prove stronger uniqueness results. An example when  $M_s$  is not a singleton is unanimity rule, provided that there are at least two players.

The monotonicity assumption we impose is the following.

**Assumption 5 (*Monotone Quasi-Median Voter Property, MQMV*)** *The sequences  $\{\min M_s\}_{s \in S}$  and  $\{\max M_s\}_{s \in S}$  are non-decreasing in  $S$ .<sup>8</sup>*

Assumption 5 is mild and very intuitive; it ensures that states are ordered consistently with agents' power. It suggests that if a certain number of higher-ranked agents is sufficient to implement a change in some state, then they are enough to implement a change in an even higher state. As we will show, it holds in a wide variety of examples. Trivially, if  $M_s$  is a singleton in every state, it is equivalent to  $M_s$  being nondecreasing (where  $M_s$  is treated as the single element).

While we allow the players to change their relative positions, e.g., for poor to become rich or vice versa, we assume that it does not happen too often.

Each environment is also characterized by transition costs. We stress that all results hold if all transition costs are zero. However, we want to be able to capture situations where only transitions to adjacent states are possible, or other transitions are possible but quite costly, or where all transitions involve some cost the the members of the society. Fortunately, we can study all these cases together. To do so, we denote the cost of transition from state  $x$  to state  $y$  (in environment  $E$ ) by  $c(x, y) = c_E(x, y)$  and impose the following simple assumption.

**Assumption 6** *The costs of transition satisfy the following assumptions:*

1. *For any  $x, y \in S$ ,  $c(x, y) \geq 0$ ;*

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<sup>8</sup>Equivalently, the set-valued function  $M_s$  is monotone nondecreasing on  $S$  (with respect to the strong set order); see Milgrom and Shannon (1994).

2. For  $x \in S$ ,  $c(x, x) = 0$ ;<sup>9</sup>

3. Vector  $\{-c(x, y)\}_{\substack{y \in S \\ x \in S}}$  satisfies WID property.

Both nonnegativity (part 1) and normalization (part 2) of Assumption 6 are natural, given the interpretation of  $c(x, y)$  as costs. Part 3 is a compact way of formulating the idea that longer transitions are costlier than shorter ones. It does so in two ways. First, notice that it implies the “reverse triangle inequality”: if either  $x < y < z$  or  $x > y > z$ , then

$$c(x, y) + c(y, z) \leq c(x, z).^{10} \quad (7)$$

Reverse triangle inequality (7) implies, given nonnegativity of costs, that transitions to more distant states are more costly than transitions to closer states, and that moving to a distant state at once costs at least as much as the cumulative cost of a more gradual move.

Second, part (3) implies that for any  $y < z$ , the difference

$$c(x) \equiv c(x, z) - c(x, y). \quad (8)$$

is a decreasing function of state  $x$ . Consider this statement on the following three intervals separately. If  $x \leq y$ , then  $c(x) \geq 0$  as follows from the reverse triangle inequality (7), and  $c(x)$  decreasing means that the longer it takes to travel from  $x$  to  $y$ , the (weakly) more costly is the additional segment from  $y$  to  $z$ , which is natural. If  $y \leq x \leq z$ , then  $c(x)$  decreasing is a trivial assumption in the light of (7), as on this interval,  $c(x, z)$  is decreasing and  $c(x, y)$  is increasing in  $x$ . Finally, if  $x \geq z$ ,  $c(x) \leq 0$ , and the assumption means the same as in the first case: the greater the distance from  $x$  to  $z$ , the (weakly) more costly it is to travel the remaining segment from  $z$  to  $y$ . In other words, part (3) is equivalent to two natural requirements: reverse triangle inequality (7) and the requirement that transition cost is “convex” in the sense that additional segments are more costly if the overall length of transition is larger.<sup>1112</sup>

<sup>9</sup>We could allow for  $c(x, x) > 0$ ; in fact, we could reduce that to the current case by setting  $\tilde{u}_i(x) = u_i(x) - c(x, x)$  and  $\tilde{c}(x, x) = 0$ ; we avoid doing this as we interpret  $c(x, y)$  as the costs of transitions.

<sup>10</sup>The WID property means that for  $x_1, x_2, y_1, y_2 \in S$  such that  $x_1 < x_2$  and  $y_1 < y_2$ ,  $c(x_2, y_2) - c(x_2, y_1) \leq c(x_1, y_2) - c(x_1, y_1)$ . Now, for the case  $x < y < z$  we plug  $x_1 = x$ ,  $x_2 = y_1 = y$ ,  $y_2 = z$ , and for the case  $x > y > z$ , we plug  $x_2 = x$ ,  $x_1 = y_2 = y$ ,  $y_1 = z$  and use  $c(y, y) = 0$  to get the result.

<sup>11</sup>A broad class of cost functions for which Assumption 6 is satisfied is where cost is a convex function of distance, i.e., if there is a (weakly) convex function  $\tilde{c}(\cdot)$  such that  $c(x, y) = \tilde{c}(|x - y|)$  and  $\tilde{c}(0) = 0$ .

<sup>12</sup>We could allow extra generality by allowing the cost to be different for different individuals, so that  $c_{E,i}(x, y)$



The last part of environment characterization is the discount factor,  $\beta = \beta_E$ , which we assume to be the same for all players. We require that  $\beta \in [0, 1)$  and we assume that  $\beta$  is the same for all environments. To summarize, the full description of each environment  $E \in \mathcal{E}$  is

$$E = \left( N, S, \beta, \{u_i(s)\}_{i \in N}^{s \in S}, \{c(x, y)\}_{x, y \in S}, \{W_s\}_{s \in S} \right). \quad (9)$$

In the game, each period  $t$  starts with environment  $E_{t-1} \in \mathcal{E}$  and with state  $s_{t-1}$  inherited from the previous period; then Nature determines  $E_t$  according to the Markov chain rule (3) and perhaps reshuffles players according to permutation  $\eta^t$  and the society decides on  $s_t$ . The game starts with the initial environment  $E_0 \in \mathcal{E}$  and with state  $s_0 \in S$  exogenously given. The society may face a shock (change of the environment) and then decides which state to move to, thereby determining state  $s_t$ . At the end of period  $t$ , an individual ranked  $i$  gets instantaneous payoff

$$v_i^t = u_{E_t, i}(s_t) - c_{E_t}(s_{t-1}, s_t). \quad (10)$$

Denoting the expectation at time  $t$  by  $\mathbb{E}_t$ , the expected discounted payoff of individual ranked  $i$  by the end of period  $t$  can be written as

$$V_i^t = \mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left( u_{E_{t+k}, \eta^{t+k} \circ \dots \circ \eta^{t+1}(i)}(s_{t+k}) - c_{E_{t+k}}(s_{t+k-1}, s_{t+k}) \right). \quad (11)$$

The following sums up the within-period timing in period  $t$ .

1. The environment  $E_{t-1}$  and state  $s_{t-1}$  are inherited from period  $t-1$ .
2. Shock which determines  $E_t$  may occur:  $E_t = E \in \mathcal{E}$  with probability  $\pi(E_{t-1}, E)$ , and the permutation of ranks  $\eta^t$  is chosen from  $\mu_{E, E'}$ .
- 3.-5. The society (collectively) decides on state  $s^t$ .
6. Each individual gets instantaneous payoff given by (10).

We deliberately do not describe in detail how the society makes collective decisions as this is not required for the Markov Voting Equilibrium concept; we introduce the detailed steps when we study the noncooperative foundations of MVE.

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could be different for different  $i$ . Our results may be proved under the following additional assumption: for any  $s \in S$ , vectors  $\{-c_i(s, x)\}_{i \in N}^{s \in S}$  and  $\{-c_i(x, s)\}_{i \in N}^{s \in S}$  satisfy WID property. In words, this would suggest that transitions between higher states is less costly for higher-ranked agents, and transitions between lower states is less costly for lower-ranked ones.

The equilibria will be characterized by a family of transition mappings  $\phi = \{\phi_E : S \rightarrow S\}_{E \in \mathcal{E}}$ . We let  $\phi_E^k$  be  $k$ th iteration of  $\phi_E$ , and we denote throughout  $\phi_E^0(s) = s$  for any  $s \in S$ . With each family of transition mappings we can associate continuation payoffs  $V_{E,i}^\phi(s)$  for individual ranked  $i$  if the state is  $s$ , which are recursively given by

$$V_{E,i}^\phi(s) = u_{E,i}(s) + \beta_E \sum_{E' \in \mathcal{E}} \sum_{j \in N} \pi(E, E') \lambda_{E,E'}(i, j) \left( -c_{E'}(s, \phi_{E'}(s)) + V_{E',j}^\phi(\phi_{E'}(s)) \right). \quad (12)$$

(as  $0 < \beta_E < 1$ , the values  $V_{E,i}^\phi(s)$  are uniquely defined by (12)). We can incorporate the cost of transition from  $x$  to  $s$ :

$$V_{E,i}^\phi(s | x) = V_{E,i}^\phi(s) - c_E(x, s). \quad (13)$$

**Definition 3 (Markov Voting Equilibrium, MVE)** *A set of transition mappings  $\phi = \{\phi_E : S \rightarrow S\}_{E \in \mathcal{E}}$  is a Markov Voting Equilibrium if the two properties hold:*

1. (core) for any environment  $E \in \mathcal{E}$  and for any states  $x, y \in S$ ,

$$\left\{ i \in N : V_{E,i}^\phi(y | x) > V_{E,i}^\phi(\phi_E(x) | x) \right\} \notin W_{E,x}. \quad (14)$$

2. (persistence) for any environment  $E \in \mathcal{E}$  and for any state  $x \in S$ ,

$$\left\{ i \in N : V_{E,i}^\phi(\phi_E(x)) - c_E(x, \phi_E(x)) \geq V_{E,i}^\phi(x) \right\} \in W_{E,x}. \quad (15)$$

Property 1 is satisfied if no alternative  $y \neq \phi(x)$  is supported by a winning coalition in  $x$  over  $\phi(x)$  prescribed by the transition mapping  $\phi^E$ . This is analogous to a “core” property: no alternative should be both preferred to the proposed transition by some coalition of players and at the same time enforceable by this coalition. Property 2 requires that it takes a winning coalition to move from any state to some alternative. This requirement singles out the status quo if there is no alternative which some winning coalition would prefer. To put it another way, it takes a winning coalition to move away from a status quo. Both properties will be required for Markov Perfect Equilibria of noncooperative game that we study below.

Throughout the paper, we focus on monotone MVE, i.e., MVE with monotone transition mappings for each  $E \in \mathcal{E}$ . (Theorem 9 states sufficient conditions for when all MVEs are monotone, and Example 7 shows that a nonmonotone MVE may exist if these conditions fail.)

### 3 Analysis

In this section, we analyze the structure of equilibria and a general framework introduced in Section 2. We first (Subsection 3.1) prove existence of monotone MVE in a stationary (deterministic) environment. We then (Subsection 3.2) extend these results to situations in which there are stochastic shocks and non-stationary elements. In Subsection 3.3, we study the relation between MVE and Markov Perfect Equilibria (MPE) of a dynamic game representing the framework of Section 2. We then derive a number of comparative static results for the general model in Subsection 3.4. After that, in Subsection 3.5, we formulate the (simple and relatively mild) conditions under which all MVE are monotone. This justifies our focus on monotone MVE in the first place. At the end of this section, in Subsection 3.6, we show how this paper generalizes Roberts (1999) on voting in clubs, which would suggest that this framework may be useful for club theory with dynamic collective decision-making and stochastic changes in power and/or preferences.

#### 3.1 Stationary environment

We first study the case of only one environment ( $|\mathcal{E}| = 1$ ); this will form the induction base later. For this part, we drop the index for the environment, and thus the only environment persists.

For any mapping  $\phi : S \rightarrow S$ , the continuation utility of player  $i$  after a transition to  $s$  has taken place (and the transition cost  $c(\cdot, s)$ , if any, has been deducted) is given by

$$V_i^\phi(s) = u_i(s) + \sum_{k=1}^{\infty} \beta^k \left( u_i(\phi^k(s)) - c(\phi^{k-1}(s), \phi^k(s)) \right). \quad (16)$$

The continuation utility that takes into account the cost of transition to state  $s$  from  $x$  is

$$V_i^\phi(s | x) = V_i^\phi(s) - c(x, s). \quad (17)$$

We start our analysis with some preliminary lemmas which we think are of independent interest. The next lemma emphasizes the critical role of quasi-median voters (QMV) in our theory.

**Lemma 1** *Suppose that vector  $\{w_i(s)\}$  satisfies Weak Increasing Differences property for  $S' \subset S$ . Take  $x, y \in S'$ ,  $s \in S$  and  $i \in N$  and let*

$$P = \{i \in N : w_i(y) > w_i(x)\}.$$

Then  $P \in W_s$  if and only if  $M_s \subset P$ . A similar statement is true for relations  $\geq, <, \leq$ .

Lemma 1 is an immediate consequence of WID property. If  $w_i(y) > w_i(x)$  for members of  $W_s$ , then this holds for all  $i \leq \max M_s$  if  $y < x$  and for all  $i \geq \min M_s$  if  $y > x$ . In either case, this holds for members of some winning coalition. The ‘‘only if’’ part also follows from WID property: it implies that  $w_i(y) > w_i(x)$  must hold for a connected coalition, and therefore it holds for all members of  $M_s$  by the Definition 2 of quasi-median voter.

For each  $s \in S$ , let us introduce the binary relation  $>_s$  on the set of  $n$ -dimensional vectors:

$$w^1 >_s w^2 \Leftrightarrow \{i \in N : w_i^1 > w_i^2\} \in W_s,$$

and let us introduce notation  $\geq_s$  in a similar way. Lemma 1 now implies that if a vector  $\{w_i(x)\}_{x \in S}$  satisfies WID, then for any  $s \in S$ , the relations  $>_s$  and  $\geq_s$  are transitive on  $\{w_i(x)\}_{x \in S}$ .

The next result shows that if  $\phi$  is monotone, then continuation utilities  $\left\{V_i^\phi(s)\right\}_{i \in N}^{s \in S}$  and  $\left\{V_i^\phi(s|x)\right\}_{i \in N}^{s \in S}$  satisfy SID property, provided that Assumption 3 and Assumption 6 are satisfied.

**Lemma 2** *For a mapping  $\phi : S \rightarrow S$ , vectors  $\left\{V_i^\phi(s)\right\}_{i \in N}^{s \in S}$  and  $\left\{V_i^\phi(s|x)\right\}_{i \in N}^{s \in S}$  (for any fixed  $x$ ), given by (16) and (17), respectively, satisfy WID, if at least one of the two properties hold:*

1.  $\phi$  is monotone;
2. for all  $x \in S$ ,  $|\phi(x) - x| \leq 1$ .

*Moreover, if the utilities satisfied SID, the same is true for the continuation utilities.*

This result is trivial but critical for what follows. Part 1 will allow us to establish that continuation utilities are monotone. Part 2 is helpful in studying the properties of one-stage transitions.

The next Lemma is of great separate interest. It says that if a monotone mapping is not MVE because the ‘core’ property (1) of Definition 3 is violated, then there must exist a *monotone deviation* (i.e., a deviation such that the resulting mapping is also monotone).

**Lemma 3 (Monotone Deviation Principle)** *Suppose that mapping  $\phi : S \rightarrow S$  is monotone, but property 1 of the Definition of MVE is violated, i.e., for some  $x, y \in S$ ,*

$$V^\phi(y) - c(x, y) >_x V^\phi(\phi(x)) - c(x, \phi(x)). \quad (18)$$

Then one can pick  $x, y \in S$  such that (18) holds and the mapping  $\phi' : S \rightarrow S$  given by

$$\phi'(s) = \begin{cases} \phi(s) & \text{if } s \neq x \\ y & \text{if } s = x \end{cases} \quad (19)$$

is monotone.

Monotone Deviation Principle suggests that if  $\phi$  is monotone but not a MVE because the ‘core’ property is violated, then there is a “deviation” which preserves monotonicity of mapping. This result is extremely helpful: to show that some  $\phi$  is a MVE, one has to consider only (relatively few) monotone deviations. Throughout the proofs, the result of Lemma 3 will be heavily used. The idea of proof is to take the “shortest” deviation, i.e., a pair  $(x, y)$  with minimal  $|y - \phi(x)|$  such that (18) holds. Without loss of generality,  $y > \phi(x)$ . Since  $\phi$  is monotone and  $\phi'$ , given by (19), is not, there must be some  $z < x$  such that  $y < \phi(z) \leq \phi(x)$ ; take the minimal of such  $z$ . A deviation at  $z$  from  $\phi(z)$  to  $y$  is monotone, and by assertion must hurt at least one QMV at  $z$ , and thus by Assumption 3 it would hurt individual  $i = \max M_x$  as  $z < x$ . If so, this individual  $i$  should benefit not only from deviation from  $\phi(x)$  to  $y$  at state  $x$ , but also from  $\phi(x)$  to  $\phi(z)$ . Now,  $\phi(z) \leq \phi(x)$  implies that a winning coalition at  $x$  benefits from such deviation. But  $|\phi(z) - \phi(x)| < |y - \phi(x)|$ , which contradicts that we took the shortest deviation.

This result, Monotone Deviation Principle, makes the following result easy to prove.

**Lemma 4 (No Double Deviation)** *Let  $a \in [1, m - 1]$ , and let  $\phi_1 : [1, a] \rightarrow [1, a]$  and  $\phi_2 : [a + 1, m] \rightarrow [a + 1, m]$  be two monotone mappings which are MVE on their respective domains.*

*Let  $\phi : S \rightarrow S$  be defined by*

$$\phi(s) = \begin{cases} \phi_1(s) & \text{if } s \leq a \\ \phi_2(s) & \text{if } s > a \end{cases} \quad (20)$$

*Then exactly one of the following is true:*

1.  $\phi$  is a MVE on  $S$ ;
2. there is  $z \in [a + 1, \phi(a + 1)]$  such that  $V^\phi(z | a) >_a V^\phi(\phi(a) | a)$ ;
3. there is  $z \in [\phi(a), a]$  such that  $V^\phi(z | a + 1) >_{a+1} V^\phi(\phi(a + 1) | a + 1)$ .

The question addressed here is whether or not mapping  $\phi : S \rightarrow S$ , obtained by combining monotone mappings on smaller domains, will be a MVE. If  $\phi$  satisfies the ‘core’ property in the definition of MVE, then one immediately gets that it is MVE, because the second property (‘persistence’) is satisfied for  $\phi_1$  and  $\phi_2$  on  $S_1$  and  $S_2$  as those are MVE. Otherwise, then Lemma 3 tells us to look for a monotone deviation, and since  $\phi_1$  and  $\phi_2$  are MVE on these domains, there may only be a deviation in state  $a$  to some state  $z \in S_2$  (more precisely, to the segment  $[a + 1, \phi(a + 1)]$ ), or a deviation in state  $a + 1$  to some state  $z \in S_1$ . It remains to prove that the last two possibilities are mutually exclusive, and we do so in the Appendix.

The last Lemma allows for a simple corollary, when either  $S_1$  or  $S_2$  is a singleton.

**Lemma 5 (*Extension of Equilibrium*)** *Let  $S_1 = [1, m - 1]$  and  $S_2 = \{m\}$ . Suppose that  $\phi : S_1 \rightarrow S_1$  is a monotone MVE. Let*

$$a = \min \left( \arg \max_{b \in [\phi(m-1), m-1]} V_{\max M_m}^\phi (b | m) \right). \quad (21)$$

*If*

$$V^\phi (a | m) >_m u(m) / (1 - \beta), \quad (22)$$

*then mapping  $\phi' : S \rightarrow S$  defined by*

$$\phi' (s) = \begin{cases} \phi(s) & \text{if } s < m \\ a & \text{if } s = m \end{cases} \quad (23)$$

*is MVE.*

*Similarly, let  $S_1 = \{1\}$  and  $S_2 = [2, m]$  and suppose that  $\phi : S_2 \rightarrow S_2$  is a monotone MVE. Let*

$$a = \min \left( \arg \max_{b \in [2, \phi(2)]} V_{\min M_1}^\phi (b | 1) \right). \quad (24)$$

*If*

$$V^\phi (a | 1) >_1 u(1) / (1 - \beta), \quad (25)$$

*then mapping  $\phi' : S \rightarrow S$  defined by*

$$\phi' (s) = \begin{cases} a & \text{if } s = 1 \\ \phi(s) & \text{if } s > 1 \end{cases} \quad (26)$$

*is MVE.*

Essentially, Lemma 5 assumes that one of  $S_1$  and  $S_2$  is a singleton. If  $S_1$  is a singleton  $\{1\}$ , then the only mapping  $\phi_1 : S_1 \rightarrow S_1$  is MVE. If there is a deviation from state 1 to  $S_2$ , then, as Lemma 4 suggests, there is no deviation from  $S_2$  to  $\{1\}$ , and therefore it is easy to verify that the ‘core’ property holds. By taking the appropriate  $\phi'(1)$ , one can ensure that  $\phi'$  is MVE.

We are now ready to prove the existence result.

**Theorem 1 (Existence)** *There exists a monotone MVE. Moreover, if  $\phi$  is a monotone MVE, then evolution  $s_0, s_1 = \phi(s_1), s_2 = \phi(s_2), \dots$  is monotone, and there exists a limit state  $s_\tau = s_{\tau+1} = \dots = s_\infty$ .*

The idea of the proof is to use induction by the number of states. If  $m = 1$ , then  $\phi : S \rightarrow S$  given by  $\phi(1) = 1$  is MVE for trivial reasons. For  $m > 1$ , we assume, to obtain a contradiction, that there is no MVE. Take any of  $m - 1$  possible splits of  $S$  into nonempty  $C_a = \{1, \dots, a\}$  and  $D_a = \{a + 1, \dots, m\}$ , where  $a \in \{1, \dots, m - 1\}$ , and any equilibrium  $\phi_1^a$  on  $C_a$  and  $\phi_2^a$  on  $D_a$ . Lemma 4 suggests that, since  $\phi^a$  obtained by combining  $\phi_1^a$  and  $\phi_2^a$  is not MVE, then only case 2 – deviation from  $a$  to  $[a + 1, \phi_2^a(a + 1)]$  or case 3 – deviation from  $a + 1$  to  $[\phi_1^a(a), a]$  is possible. Let us say that  $g(a) = r$  (from “right”) if case 2 holds, and that  $g(a) = l$  otherwise. (For function  $g$  to be well-defined, we need to specify a particular way of choosing  $\phi_1^a$  and  $\phi_2^a$  if there are multiple equilibria. We do so in the formal proof in the Appendix; for now, let us assume that equilibria on  $C_a$  and  $D_a$  were unique.) We then have the following possibilities.

If  $g(1) = r$ , then we can use Lemma 5 to construct MVE  $\phi$  on  $S$ . The same is true if  $g(m - 1) = l$ . In the remaining case, if  $g(1) = l$  and  $g(m - 1) = r$ , there must exist  $a \in \{2, \dots, m - 1\}$  such that  $g(a - 1) = l$  and  $g(a) = r$ . We take equilibria  $\phi_1^{a-1}$  on  $[1, a - 1]$  and  $\phi_2^a$  on  $[a + 1, m]$ . Since there were deviations from  $a$  both to  $[\phi_1^{a-1}(a - 1), a - 1]$  and to  $[a + 1, \phi_2^a(a + 1)]$ , let us define  $\phi : S \rightarrow S$  by

$$\phi(s) = \begin{cases} \phi_1^{a-1}(s) & \text{if } s < a \\ b & \text{if } s = a \\ \phi_2^a(s) & \text{if } s > a \end{cases}, \quad (27)$$

where  $b \in [\phi_1^{a-1}(a - 1), a - 1] \cup [a + 1, \phi_2^a(a + 1)]$  is picked in a way that MVE conditions are satisfied at  $s = a$ . Suppose, without loss of generality, that  $b < a$ , then  $\phi|_{[1, a]}$  is a MVE on  $[1, a]$ . By Lemma 3, it suffices to check that there is no deviation from  $a + 1$  to  $[b, a]$ , but this is

implied by  $g(a) = r$ . Thus, with the help of Lemma 3, we establish that there are no monotone deviations. By construction the second part (‘persistence’) also holds, and thus  $\phi(s)$  is MVE.

We now study, under which conditions there is only one monotone MVE. To formulate the uniqueness result, we need the following definitions.

**Definition 4** *Individual preferences are single-peaked if for every  $i \in N$  there is  $x \in S$  such that for states  $y, z \in S$  such that  $z < y < x$  or  $z > y > x$ ,  $u_i(z) < u_i(y) < u_i(x)$ .*

This definition is standard. The next definition defines precisely what we mean by one-step transitions.

We say that *only one-step transitions are possible* if

$$\forall E \in \mathcal{E}, x, y \in S : |x - y| > 1 \Rightarrow c_E(x, y) > \frac{2}{1 - \beta} \max_{E' \in \mathcal{E}, i \in N, s \in S} |u_{E', i}(s)|. \quad (28)$$

The next examples shows that equilibrium is not always unique.

**Example 3** (Example with two MVE) There are three states  $A, B, C$ , and two players 1 and 2. The decision-making rule is unanimity in all states. Payoffs are given by

<i>id</i>	$A$	$B$	$C$
1	20	5	10
2	10	5	20

Suppose that  $\beta$  is sufficiently close to 1, e.g.,  $\beta = 0.9$ . Then there are two MVE. In one,  $\phi_1(A) = \phi_2(B) = A$  and  $\phi_1(C) = C$ . In another,  $\phi_2(A) = A$ ,  $\phi_2(B) = \phi_2(C) = C$ . This is possible because preferences are not single-peaked, and there are more than one quasi-median voters in all states.

However, single-peakedness alone is not enough, as the next example shows.

**Example 4** (Example with single-peaked preferences and two MVE) There are three states  $A, B, C$ , and two players 1 and 2. The decision-making rule is unanimity in state  $A$  and dictatorship of player 2 in states  $B$  and  $C$ . Payoffs are given by

<i>id</i>	$A$	$B$	$C$
1	2	25	20
2	1	20	25



Then  $\phi_1$  given by  $\phi_1(A, B, C) = (B, C, C)$  and  $\phi_2$  given by  $\phi_2(A, B, C) = (C, C, C)$  are both MVE when the discount factor is any  $\beta \in [0, 1)$ .

The following theorem presents cases where equilibrium is (generically) unique.

**Theorem 2 (Uniqueness)** *The monotone MVE is (generically) unique if at least one of the following conditions holds:*

1. *for every  $s \in S$ ,  $M_s$  is a singleton;*
2. *only one-step transitions are possible and preferences are single-peaked.*

In other words, we can prove uniqueness essentially for the same set of assumptions for which we can establish that any MVE is monotone (Theorem 9 below), and in the second case we require, in addition, that preferences are single-peaked. This means that if either of the conditions in Theorem 2 holds, then there is a unique MVE, and this MVE is monotone.

### 3.2 Stochastic environment

We now extend our analysis to the case in which there are stochastic shocks. As our analysis will clarify, this also enables us to deal with potentially “nonstationary” problems where the distribution of political power or economic preferences will change in a specific direction in the future. By Assumption 2, environments are ordered as  $E^1, E^2, \dots, E^h$  so that  $\pi(E^x, E^y) = 0$  if  $x > y$ . This means that if we reached environment  $E^h$ , there are no further shocks, and the analysis from Section 3.1 is applicable. In particular, we get the same conditions for existence and uniqueness of MVE. We now use backward induction to find equilibrium transition mappings in earlier environments.

The following Lemma is crucial for the analysis.

**Lemma 6** *Suppose  $\phi$  is a monotone MVE in a stationary environment. Then continuation payoff vectors  $\{V_i(s)\}_{i \in N}^{s \in S}$  and  $\{V_i(\phi(s) | s)\}_{i \in N}^{s \in S}$  satisfy WID condition, and so does  $\left\{ \sum_{j \in N} \lambda_{E, E'}(i, j) V_j^\phi(\phi(s) | s) \right\}_{i \in N}^{s \in S}$  for any  $E, E' \in \mathcal{E}$ .*

Lemma 6 is the cornerstone in our study of stochastic environments. It suggests that if utility functions satisfied WID, then for any monotone MVE  $\phi$ , certain continuation payoffs

under do as well. This makes the backward induction argument in the proof of Theorem 3 and 4 possible: we will know that after a shock, continuation utilities satisfy the same properties as ones we required from instantaneous utilities, which will make much of the argument we made in the stationary case applicable. This is not true for single-peakedness, so uniqueness is slightly more difficult to prove. In the case without reshuffling, however, this holds nevertheless.

To proceed by backward induction, let us take MVE  $\phi_{E^h}$  in the environment  $E^h$ ; its existence is guaranteed by Theorem 1. Suppose that we have found MVE  $\{\phi_E\}_{E \in \{E^{k+1}, \dots, E^h\}}$  for  $k = 1, \dots, h-1$ ; let us construct  $\phi_{E^k}$  which would make  $\{\phi_E\}_{E \in \{E^h, \dots, E^k\}}$  MVE in the environments  $\{E^k, \dots, E^h\}$ . Suppose that as long as the environment is  $E^k$ , transition mappings are given by  $\phi_{E^k}$ . In this case, continuation utilities of agent  $i$  are given by

$$\begin{aligned} V_{E^k, i}^\phi(s) &= u_{E^k, i}(s) + \beta_{E^k} \sum_{E' \in \{E^k, \dots, E^h\}} \sum_{j \in N} \pi(E^k, E') \lambda_{E^k, E'}(i, j) V_{E', j}^\phi(\phi_{E'}(s) | s) \\ &= u_{E^k, i}(s) + \beta_{E^k} \sum_{E' \in \{E^{k+1}, \dots, E^h\}} \pi(E^k, E') \lambda_{E^k, E'}(i, j) V_{E', j}^\phi(\phi_{E'}(s) | s) \\ &\quad + \beta_{E^k} \pi(E^k, E^k) V_{E^k, i}^\phi(\phi_{E^k}(s) | s), \end{aligned} \quad (29)$$

where

$$V_{E', i}^\phi(\phi_{E'}(s) | s) = V_{E', i}^\phi(\phi_{E'}(s)) - c_{E'}(s, \phi_{E'}(s)); \quad (30)$$

we also used part 1 of Assumption 1. By induction, we know  $\phi_{E'}$ ,  $c_{E'}(s, \phi_{E'}(s))$ ,  $V_{E'}^\phi(\phi_{E'}(s))$  for  $E' \in \{E^{k+1}, \dots, E^h\}$ . It suffices, therefore, to show that there exists mapping  $\phi_{E^k}$  such that continuation values  $\{V_{E^k, i}^\phi(s)\}_{s \in S}$ , determined from (29), would make  $\phi_{E^k}$  a MVE. Denote

$$\tilde{u}_{E^k, i}(s) = u_{E^k, i}(s) + \beta_{E^k} \sum_{E' \in \{E^{k+1}, \dots, E^h\}} \sum_{j \in N} \pi(E^k, E') \lambda_{E^k, E'}(i, j) V_{E', j}^\phi(\phi_{E'}(s) | s) \quad (31)$$

$$\tilde{\beta}_{E^k} = \beta_{E^k} \pi(E^k, E^k) \quad (32)$$

Then equation (29) may be rewritten as

$$V_{E^k, i}^\phi(s) = \tilde{u}_{E^k, i}(s) + \tilde{\beta}_{E^k} \left( V_{E^k, i}^\phi(\phi_{E^k}(s)) - c_{E^k}(s, \phi_{E^k}(s)) \right). \quad (33)$$

Notice that  $\{\tilde{u}_{E^k, i}(s)\}_{i \in N}^{s \in S}$  satisfy WID; this follows from Lemma 6 and from the additivity of the WID property. We also have that  $\tilde{\beta}_{E^k} \in [0, 1)$ . This means that if we consider a game without shocks, with the environment given by

$$E = \left( N, S, \tilde{\beta}_{E^k}, \{\tilde{u}_{E^k, i}(s)\}_{i \in N}^{s \in S}, \{c_{E^k}(x, y)\}_{x, y \in S}, \{W_{E^k, s}\}_{s \in S} \right), \quad (34)$$

then the recursive equation for continuation values under the transition mapping  $\phi_{E^k}$  would be given precisely by (33). This makes Theorem 1 applicable; therefore, there is a transition mapping  $\phi_{E^k}$  which constitutes a MVE in the environment  $E$ . But then by definition of MVE, since  $\{\phi_E\}_{E \in \{E^{k+1}, \dots, E^h\}}$  was MVE, we have that  $\{\phi_E\}_{E \in \{E^k, \dots, E^h\}}$  is MVE in the environments  $\{E^k, \dots, E^h\}$ . This proves the induction step, and proceeding likewise, we can obtain the entire MVE  $\phi = \{\phi_E\}_{E \in \{E^1, \dots, E^h\}}$ .

Notice that this reasoning used backward induction, and thus Assumption 2 was indispensable. In Section 5 below, we study the possibility of an infinite number of shocks. For now, we formally state the result we have just proved.

**Theorem 3 (Existence)** *Suppose that all environments  $E \in \mathcal{E}$  satisfy the assumptions of the paper, and Assumption 2 holds. Then there is a MVE  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ . The evolution  $s_0, s_1 = \phi_{E^1}(s_0), s_2 = \phi_{E^2}(s_1), \dots$  results in a limit state  $s_\tau = s_{\tau+1} = \dots = s_\infty$ , but need not be monotone. The limit state may depend on the time of arrival of shocks.*

Now that we have established existence of MVE in a stochastic environment, a natural question is whether or not it is unique. Our approach to this question is similar: using backward induction, we reduce the problem to studying uniqueness of MVE in the environment  $E$  given by (34), where the utilities are given by (31) and the discount factor is given by (32). However, this is not straightforward, because single-crossing condition need not be preserved for continuation utilities, as the next example shows (and it also need not be additive).

**Example 5** (Continuation utilities need not satisfy single-peakedness) There are four states and three players, player 1 is the dictator in state  $A$ , player 2 is the dictator in state  $B$ , and player 3 is the dictator in states  $C$  and  $D$ . The payoffs are given by the following matrix:

<i>id</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
1	20	30	90	30
2	5	20	85	90
3	5	25	92	99

All payoffs are single-peaked. Suppose  $\beta = 0.5$ ; then the unique equilibrium has  $\phi(A) = C$ ,  $\phi(B) = \phi(C) = \phi(D) = D$ . Let us compute the continuation payoffs of player 1. We have:  $V_1(A) = 40$ ,  $V_1(B) = 30$ ,  $V_1(C) = 50$ ,  $V_1(D) = 30$ ; the continuation utility of player 1 is thus not single-peaked.

Nevertheless, we can establish uniqueness under one of the following conditions.

**Theorem 4 (*Uniqueness*)** *The monotone MVE is (generically) unique if at least one of the following conditions holds:*

1. *for every environment  $E \in \mathcal{E}$  and any state  $s \in S$ ,  $M_s$  is a singleton;*
2. *there is no reshuffling after any transition; in each environment, only one-step transitions are possible; each player's preferences are single-peaked; and, moreover, for each state  $s$  there is a player  $i$  such that  $i \in M_{E,s}$  for all  $E \in \mathcal{E}$  and the peaks (for all  $E \in \mathcal{E}$ ) of  $i$ 's preferences do not lie on different sides on  $s$ .*

The first case is the same as in the stationary environment studied above. The second is more demanding, but nevertheless worth stating. The last complex condition holds automatically if political rules do not change as a result of shocks, and neither do players' ideal states under each environment.

The 'no reshuffling' condition in the second part is important, as the following example shows.

**Example 6** (Example with multiple equilibria with reshuffling) There are two environments,  $E^1$  and  $E^2$ , with identical payoffs and ruling coalitions. We consider  $\pi(E^1, E^2)$  a parameter. If a shock arrives, assume that there is full reshuffling, so the chance for each player to occupy each position is 0.25. There are three states and two players, and the rule is unanimity. The payoffs are given by the following matrix:

<i>id</i>	<i>A</i>	<i>B</i>	<i>C</i>
1	90	30	10
2	10	30	90

Then the weak regularity property holds. The equilibrium is unique if either  $\beta$  is close to 0 or  $\pi(E^1, E^2)$  is close to 0, and this equilibrium is given by  $\phi_{E^1}(x) = \phi_{E^2}(x) = x$  for each state  $x$ .

Suppose, however, that  $\beta = 0.9$  and  $\pi(E^1, E^2) = 0.9$ . Then in environment  $E^2$  the equilibrium is still unique and given by  $\phi_{E^2}(x) = x$  for each state  $x$ . One can check that then there are two mappings  $\phi_{E^1}$  that would form a monotone MVE with  $\phi_{E^2}$ :  $\phi_{E^1}^1(A) = \phi_{E^1}^1(B) = A$ ,  $\phi_{E^1}^1(C) = C$ , and  $\phi_{E^1}^2(A) = A$ ,  $\phi_{E^1}^2(B) = \phi_{E^1}^2(C) = C$ . The reason for multiplicity is that

both players expect reshuffling, and this justifies the risk-taking behavior, i.e., choosing  $C$  over  $B$  by player 1 or choosing  $A$  over  $B$  by player 2.

### 3.3 Noncooperative game

So far, we have not specified a noncooperative game which would substantiate MVE. We do so in this section, and first we describe the game fully. For all environments  $E \in \mathcal{E}$  and states  $s \in S$ , we introduce a protocol  $\theta_{E,s}$ , which is a finite sequence of all states in  $S \setminus \{s\}$ .

1. The environment  $E_{t-1}$  and state  $s_{t-1}$  are inherited from period  $t - 1$ .
2. Shock which determines  $E_t$  may occur:  $E_t = E \in \mathcal{E}$  with probability  $\pi(E_{t-1}, E)$ , and the permutation of ranks  $\eta^t$  is chosen from  $\mu_{E,E'}$ .
3. Let  $b_1 = s_{t-1}$ . In the subsequent stages, alternative  $b_j$ ,  $j = 1, \dots, m - 1$ , is voted against  $\theta_{E_t, s_{t-1}}(j)$ . That is, all agents are ordered in a sequence and must support either  $b_j$  or  $\theta_{E_t, s_{t-1}}(j)$ . If the set of those who supported  $\theta_{E_t, s_{t-1}}(j)$  is a winning coalition, i.e., is in  $W_{E_t, s_{t-1}}$ , then  $b_{j+1} = \theta_{E_t, s_{t-1}}(j)$ ; otherwise,  $b_{j+1} = b_j$ . When all alternatives have been voted, the new state is  $s_t = b_m$ .
4. Each individual gets instantaneous payoff given by (10).

We study Markov Perfect equilibria of this game.<sup>13</sup> Naturally, with every MPE, a set of transition mappings  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$  is associated:  $\phi_E(s)$  is the state with which period which started with state  $s_{t-1}$  and where there was a shock leading to state  $E$  ends. We can get the following results.

**Theorem 5** *The following is true:*

1. For any MVE  $\phi$  (monotone or not) there exists a set of protocols  $\{\theta_{E,s}\}_{E \in \mathcal{E}}^{s \in S}$  such that there exists a Markov Perfect equilibrium of the game above which implements  $\phi$ . Moreover, if  $\phi$  is the unique MVE, then protocol

$$\{\theta_{E,s}(j)\}_{j=1}^{m-1} = (s + 1, s + 2, \dots, m, s - 1, s - 2, \dots, 1) \quad (35)$$

---

<sup>13</sup>To avoid the usual problems with equilibria in voting games, we assume sequential voting for some fixed sequence of players. See Acemoglu, Egorov, and Sonin (2009) for a solution concept which would refine out unnatural equilibria in voting games with simultaneous voting.

may be used;

2. Conversely, if for some set of protocols  $\{\theta_{E,s}\}_{E \in \mathcal{E}}^{s \in S}$  and some MPE  $\sigma$ , the corresponding transition mapping  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$  is monotone, then it is MVE.
3. Under either of the assumptions of Theorem 9, a nonmonotone MPE cannot exist for any set of protocols.

This theorem establishes the relation between the cooperative and noncooperative approaches. On the one hand, any MVE may be made an MPE of the game, if a protocol is taken appropriately. If the equilibrium is unique, such protocol is easy to describe, and one possible variant is given by (35). In fact, a stronger result is true: the protocol given by (35) always has a monotone MPE. On the other hand, an MPE gives rise to an MVE, provided that the transition mapping is monotone. Part 3 of Theorem 5 gives sufficient conditions which ensure that any MPE is monotone.

### 3.4 Comparative statics

In this section, we compare different environments, and study properties that must hold. Comparative statics results are strongest when equilibrium is unique; hence, throughout this section, we assume that either of Theorem 4, which guarantee uniqueness of MVE, holds. Assume, furthermore, that parameter values are generic.

**Definition 5** *We say that environments  $E^1$  and  $E^2$ , defined for the same set of players and set of states, coincide on  $S' \subset S$ , if for each  $i \in N$  and for any states  $x, y \in S'$ ,  $u_{E^1,i}(x) = u_{E^2,i}(x)$ ,  $c_{E^1}(x, y) = c_{E^2}(x, y)$ , and  $W_{E^1,x} = W_{E^2,x}$ .*

Consider two environments,  $E^1$  and  $E^2$  that coincide on a subset of states (and differ arbitrarily on other states). We next show that there is on him because relationship between the MVEs in these environments represented by the transition functions  $\phi_{E^1}$  and  $\phi_{E^2}$ .

**Theorem 6** *Suppose that environments  $E^1$  and  $E^2$  coincide on  $S' = [1, s] \subset S$  and  $\beta_{E^1} = \beta_{E^2}$ ,  $\phi_1$  and  $\phi_2$  are MVE in these environments. Suppose  $x \in S'$  is such that  $\phi_1(x) = x$ . Then  $\phi_2(x) \geq x$ .*

The idea of the proof is simple. To explain it, let us introduce the notation  $\phi|_{S'}$  to represent the transition function  $\phi$  restricted to the subset of states  $S'$ . Now, if we had that  $\phi_2(x) < x$ , then  $\phi_1|_{S'}$  and  $\phi_2|_{S'}$  would be two different mappings both of which would be MVEs on  $S'$ . But this would contradict the uniqueness of MVE. Of course, if  $y \in S'$  is such that  $\phi_2(y) = y$ , then  $\phi_1(y) \geq y$ . This proposition does not say anything about the existence of a stable point in  $S'$  for either of the mappings, however, if it does, then it must either be a stable point for the other mapping, or the other mapping should move it right. Obviously, these results generalize for the case where  $S' = [s, m]$  rather than  $[1, s]$ .

Theorem 6 implies the following simple corollary.

**Corollary 1** *Suppose that  $\mathcal{E} = \{E^1, E^2\}$  and there is no reshuffling. Suppose, furthermore, that  $E^1$  and  $E^2$  coincide on  $S' = [1, s] \subset S$ , and  $\beta_{E^1} = \beta_{E^2}$ . Suppose that, in the unique MVE  $\phi$ ,  $\phi_{E^1}(x) = x$  for some  $x \in S'$ . Suppose also that this state  $x$  is reached before the shock arise. Suppose the shock arrives at time  $t$ . Then for all  $\tau \geq t$ ,  $s_\tau \geq x$ .*

This result seems particularly interesting, because it is so general. Suppose that before the shock, the society had found itself in a stable point, and as a result of the shock, only higher states were affected (agents' utilities, costs of transition, or sets of winning coalitions could change). Corollary 1 implies that this could only make the society move towards the direction where shock happened or stay where it was. In other words, the only possibility for the society to stay in the region  $[1, x - 1]$  is not to leave it before the shock arrives.

Interestingly, this result is not affected by the nature of the shock. Nevertheless, the implications of the result become clearer when we focus on a particular type of shock. Suppose, for example, that preferences change such that utilities of all agents become higher in some state on the right of where the society is before the shock. Then it would be intuitive that transitions take place towards that state. But in contrast, the corollary implies that even when the utilities of all agents become lower in one of the states on the right, the society may still decide to move to the right. Intuitively, it is possible that some transition to the right would benefit the current decision-makers, but the possibility of further transitions to the right made them prefer the status quo. The shock removed this last threat by making it empty, and now the society may be willing to make a transition to the right. Of course, it is possible that the society will stay where it was; this would be the case, for example, if the shocks was minor.

Corollary 1 was formulated under the assumption that stable state  $x$  was reached before the shock occurred. The next result removes this constraint, but only under the assumption that the discount factor is low enough, i.e., that players are sufficiently myopic.

**Theorem 7** *Suppose that  $\mathcal{E} = \{E^1, E^2\}$ ,  $0 < \pi(E^1, E^2) < 1$ ,  $\pi(E^2, E^1) = 0$ , and  $E^1$  and  $E^2$  coincide on  $S' = [1, s] \subset S$ . Then there exists  $\beta_0 > 0$  such that if  $\beta_{E^1} < \beta_0$  and  $\beta_{E^2} < \beta_0$ , then in the unique MVE  $\phi$ , if the initial state is  $s_0 \in S'$  such that  $\phi_{E^1}(s_0) \geq s_0$ , then the entire path  $s_0, s_1, s_2, \dots$  (induced both under environment  $E^1$  and after the switch to  $E^2$ ) is monotone. Moreover, if the shock arrives at time  $t$ , then for all  $\tau \geq t$ ,  $s_\tau \geq \tilde{s}_\tau$ , where  $\tilde{s}_\tau$  is the hypothetical path if the shock never arrives.*

If we lay some restrictions on the nature of shock, then we can make even stronger conclusions about the effects of shocks. In the next Theorem, we allow for the set of quasi-median voters to change while keeping instantaneous payoffs fixed.

**Theorem 8** *Suppose that environments  $E^1$  and  $E^2$  have the same payoffs,  $u_{E^1,i}(x) = u_{E^2,i}(x)$ , the same transition costs,  $c_{E^1}(x, y) = c_{E^2}(x, y)$ , the same discount factors  $\beta_{E^1} = \beta_{E^2}$ , and suppose that  $M_{E^1,x} = M_{E^2,x}$  for  $x \in [1, s]$  and  $\min M_{E^1,x} = \min M_{E^2,x}$  for  $x \in [s+1, m]$ . Let  $\phi_1$  and  $\phi_2$  be MVE in these environments. Then  $\phi_1(x) = \phi_2(x)$  for any  $x \in [1, s]$ .*

This result suggests that if in some right states the sets of winning coalition change in a way that the sets of quasi-median voters change on the right without changes on the left, then the mapping is unaffected for states on the left (i.e., those states that are not directly affected by the change). For example, applied to the dynamics of democratization, this theorem implies that an absolute monarch's decision of whether to move to a constitutional monarchy is not affected by the power that the poor will be able to secure—provided that the middle class is still a powerful player.

This theorem also has an interesting corollary, which is taken next.

**Corollary 2** *Let  $\mathcal{E} = \{E^1, \dots, E^h\}$ . Suppose that:*

1. *for all environments  $E, E' \in \mathcal{E}$  and for all states  $x, y \in S$  and individuals  $i \in N$ , we have  $u_{E,i}(x) = u_{E',i}(x)$ ,  $c_E(x, y) = c_{E'}(x, y)$ ,  $\beta_E = \beta_{E'}$  and  $\min M_{E,x} = \min M_{E',x}$ ;*
2. *if  $x \in [1, s]$ , then  $\max M_{E,x} = \max M_{E',x}$ ;*



3. *there is no reshuffling.*

*Then if  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$  is the unique MVE, we have that  $\phi_{E^1}(x) = \dots = \phi_{E^h}(x)$  for all  $x \in [1, s]$ . Thus if  $s_0 \in [1, s]$  and there is a stable  $x \in [s_0, s]$ , then arrival of shocks does not alter the equilibrium paths.*

We thus have identified a class of shocks which do not change the evolution of the game. A priori, one could imagine, for example, that if the poor lose the ability to protect democracy, and instead the elite will be able to stage a coup, then this consideration may affect the desire of the elite to extend the franchise and move to democracy, or that it may affect the desire of the monarch to grant more power to the broad elite in the first place. Corollary 2 suggests, however, that unless something else is going on, these shocks and these considerations alone are not sufficient to change the equilibrium path (unless, of course, the society starts the game in democracy with the elite dreaming of staging a coup).

### 3.5 Monotone vs nonmonotone MVE

So far, we focused on monotone MVE. In many interesting cases this is without loss of generality, as the following theorem establishes.

**Theorem 9** *The following are sufficient conditions for any MVE  $\phi$  to be (generically) monotone:*

1. *In all environments, the sets of quasi-median voters in two different states have either none or exactly one individual in common:  $\forall E \in \mathcal{E}, x, y \in S : x \neq y \Rightarrow |M_{E,x} \cap M_{E,y}| \leq 1$ .*
2. *In all environments, only one-step transitions are possible.*

This theorem is quite general. Part 1 covers, in particular, all situations where the sets of quasi-median voters are singletons in all states. This implies that whenever in each state there is a dictator (which may be the same for several states), or there is majority voting among sets of odd numbers of players, any MVE is monotone, and thus all results in the paper are applicable to all MVEs. The second part suggests that if only one-step transitions, i.e., transitions to adjacent states are possible, then again any MVE is monotone. This means that our focus on monotone MVE is without any loss of generality for many interesting and relevant cases. Also,

coupled with the result that monotone MVE always exist, this justifies our focus on monotone MVE even if the conditions of Theorem 9 fail.

In addition, inspection of the conditions in Theorem 9 reveals that they are weaker than conditions in Theorem 2 and 4. Consequently, when these theorems guarantee the uniqueness of a monotone MVE, they, in fact, guarantee that it is unique in the class of all possible MVEs. Moreover, if MVE is unique, it is monotone.

The next example shows that nonmonotone MVE are possible, if both conditions in Theorem 9 fail.

**Example 7** There are three states  $A, B, C$ , and two players 1 and 2. The decision-making rule is unanimity in all states. Payoffs are given by

<i>id</i>	$A$	$B$	$C$
1	30	50	40
2	10	40	50

Suppose  $\beta$  is relatively close to 1, e.g.,  $\beta = 0.9$ . Then there is a nonmonotone MVE  $\phi(A) = \phi(C) = C$ ,  $\phi(B) = B$ . (There is also a monotone equilibrium with  $\phi(A) = \phi(B) = B$ ,  $\phi(C) = C$ .)

The next example shows that genericity is also an important requirement.

**Example 8** (Example with nonmonotone equilibrium due to nongeneric preferences.) There are two states  $A$  and  $B$  and two players 1 and 2. Player 1 is the dictator in both states. Payoffs are given by

<i>id</i>	$A$	$B$
1	20	20
2	15	25

Take any discount factor  $\beta$ , e.g.,  $\beta = 0.5$ , and any protocol. Then  $\phi$  given by  $\phi(A) = B$  and  $\phi(B) = A$  is nonmonotone (in fact, cyclic). This equilibrium is only possible for measure 0 of preferences.

However, even if nonmonotone MVE exist, they still possess a certain degree of monotonicity, namely, “monotone paths”, as the next results show.

**Definition 6** A mapping  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$  has monotone paths if for any  $E \in \mathcal{E}$  and  $x \in S$ ,  $\phi(x) \geq x$  implies  $\phi_E^2(x) \geq \phi_E(x)$ .

In other words, all equilibrium paths that this mapping generates, as long as the environment does not change, are weakly monotone. We have the following result:

**Theorem 10** Any MVE  $\phi$  (not necessarily monotone) has, generically, monotone paths.

### 3.6 Relationship to Roberts's model

As discussed in the Introduction, our paper is most closely related to Roberts (1999). Our notion of MVE extends Roberts's notion of equilibrium, who also looks at a dynamic equilibrium in an environment that satisfies single-crossing type restrictions. More specifically, in Roberts's model, the society consists of  $n$  individuals, and there are  $n$  possible states  $s_k = \{1, \dots, k\}$ ,  $1 \leq k \leq n$ . Each state  $s_k$  describes the situation where individuals  $\{1, \dots, k\}$  are members of the club, while others are not. There is the following condition on individual payoffs:

$$\text{for all } l > k \text{ and } j > i, \quad u_j(s_l) - u_j(s_k) > u_i(s_l) - u_i(s_k), \quad (36)$$

which is the same as the strict increasing differences condition we imposed above (Definition 1).

Roberts (1999) focuses on deterministic environments with majoritarian voting among club members. He then looks at a notion of Markov Voting Equilibrium (defined as an equilibrium path where there is a transition to a new club whenever there is an absolute majority in favor of it) and a median voter rule (defined as an equilibrium path where at each point current median voter chooses the transition for the next step). Roberts proves existence for mixed-strategy equilibria for each of the voting rules; they define the same set of clubs that are stable under these rules.

Roberts's notion of Markov Voting Equilibrium is closely related to ours, only differing from ours in the way that he treats clubs that have even numbers of members and thus might have ties. In any case, the two notions coincide for generic preferences.

This description clarifies that Roberts's setup is essentially a special case of what we have considered so far. The dimensions in which our framework is more general are several: first, Roberts focuses on the deterministic and stationary environment, whereas we allow for nonstationary elements and arbitrary stochastic shocks. Second, we allow for a much richer set of states

and richer distributions of political power across states (e.g., instead of majority rule, we allow weighted supermajority rule, which could be different across states, or dictatorial rule). Third, we prove existence of pure strategy equilibria and provide conditions for uniqueness. Fourth, we provide general conditions for equilibria to be characterized by monotone transition maps and to exhibit monotone sample paths. Fifth, we show the relationship between this equilibrium concept and MPE of a fully specified dynamic game.

Most importantly, however, we provide a fairly complete characterization of the structure of transitions and steady states, and show that at this level of generality, the range of comparative static results can be derived. Such comparative statics are not only new but, thanks to their generality, quite widely applicable. We also show how the framework can be applied in a somewhat more specific but still general analysis of dynamics of political rights and repression, and derive additional results in this context.

## 4 Application: Repression and Social Mobility

In this section, we apply our general framework and the results derived so far to the study of the dynamics of the political rights and repression.

### 4.1 Preferences and distribution of political power

In what follows, we use the language and formalism of Section 2. There is a fixed set of players  $N = \{1, \dots, n\}$ , which we interpret as groups of (potentially large numbers of) individuals with the same preferences (e.g., ethnicities or classes). When there is “reshuffling,” this will capture social mobility, and we interpret this as each one of the individual’s within a group facing an independent probability of moving up or down to another group. The order of groups here can be interpreted as preferences on some social policy dimension (e.g., from religiosity to secularism) or as representing some economic interests (e.g., from rich to poor). The weight of each group  $i \in N$  is denoted by  $\gamma_i$  and represents the number of people within the group, and thus the group’s political power. This may change over time due to social mobility or other factors, also potentially changing the balance of political power.

Preferences are represented as follows. Individuals of group  $i$  have bliss point  $b_i$  and thus

the utility over policies of the form when policy  $p$  is implemented

$$-(p - b_i)^2. \tag{37}$$

Our single-crossing assumptions here implies that the sequence  $\{b_i\}$  is increasing in  $i$ .

States here correspond to different combinations of political rights. We think of repression as a way of reducing the political rights of certain groups. In particular, there are  $2n - 1$  states corresponding to different sets of individuals who are repressed, and only those not repressed are eligible to vote.<sup>14</sup> We assume that repression is costly. In particular, repressing agents of type  $j$  costs  $C_j$ , and that this cost is incurred by all agents within society (e.g., economic efficiency is reduced or taxes have to be raised to support repression).

Let us denote the set of players who are not repressed in state  $s$  by  $H_s$ ; then  $H_s = \{1, \dots, s\}$  for  $s \leq n$  and  $H_s = \{s - n + 1, \dots, n\}$  for  $s > n$ . In other words, the first  $n - 1$  states correspond to repressing—taking away the political rights of—players with have right-wing views, and the last  $n + 1$  states correspond to repressing players with left-wing views; the middle state  $s = n$  involves no repression. States effectively differ by voting rules: in each state, the voting rule is majoritarian among those who are not repressed or who have been granted clinical right. (This implies that the share of population within each group acts like a weight in a weighted majoritarian system, with each player’s weight being proportional to his share in the population). More specifically, in state  $s$ , coalition  $X$  is winning if and only if

$$\sum_{i \in H_s \cap X} \gamma_i > \frac{1}{2} \sum_{i \in H_s} \gamma_i. \tag{38}$$

For generic sequences of proportions across groups,  $\{\gamma_i\}$ , the quasi-median voter will be in a single group, and this is equivalent to having a singleton set of quasi-median voters in terms of the analysis so far. In this section, we will, interchangeably, refer to a quasi-median voter (QMV), median voter or quasi-median group.

There are two decisions made in each period. First, the current period state  $s$  (i.e., the repression policy) is decided. Second, in the end of the period, the payoffs are determined. Since decision-making is by weighted majority, the policy which determines the payoffs is given, in state  $s$ , by  $b_{M_s}$ , where  $M_s$  is the (generically unique) quasi-median voter.

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<sup>14</sup>We can allow for “partial” repression, where the votes of players who are repressed are discounted with some factor. This would correspond to, say, repressing only a certain fraction of some population group. Ultimately, this would complicate the notation without delivering major insights.

Given the above description, the stage payoff of individuals in group  $i$  in state  $s$  can be written as:

$$u_i = -(b_{M_s} - b_i)^2 - \sum_{j \notin H_s} \gamma_j C_j. \quad (39)$$

Until the last result we present in this section, the reader may focus on the case in which  $C_j = C$  for all  $j \in N$ .

## 4.2 Characterization and comparative statics

For most results below, we assume there are two environments  $E^1$  and  $E^2$ , and the society starts in  $E^1$ . The results can be extended to more environments. Focusing on two environments simplifies and clarifies the results. We assume that the environments coincide in everything that we do not mention explicitly to be different. A shock is then interpreted as changes of only those parameters that need to be changed. Also for clarification, we will often focus on shocks that only change the distribution of political power and shocks that correspond to “pure” social mobility, i.e., reshuffle individuals but do not change the distribution of political power across different groups. Moreover, we will also take pure social mobility shocks to take a simple form: in particular, for some interval of groups  $X = [a, b] \subset [1, n]$ , each individual within this interval is interpreted as equally likely to move to one of the other groups within the interval (once again, it is possible to allow a richer structure of transitions across groups that still correspond to social mobility, but our choice here simplifies the exposition and notation).<sup>15</sup>

Our first proposition characterizes the structure of stationary equilibrium (without stochastic shocks).

**Proposition 1** *There exists a unique MVE. In this equilibrium, the state  $s = n$  (corresponding to no repression) is stable, and for any  $s$ ,  $|\phi(s) - n| \leq |s - n|$ .*

In other words, in the absence of shocks, repression will not arise if it was not present from the beginning, and if it was, then there will not be more repression and, perhaps, there will be less. Intuitively, the quasi-median voter can choose the preferred policy anyway, and since repression

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<sup>15</sup>Naturally, the general comparative static results contained in Theorems 6-8 and their corollaries also apply in this more specific environment without restricting the type of shocks any further. Since the insights that they imply are in line with our discussion within the general framework, we do not spell those out here.

is costly, choosing more of it makes no sense. On the other hand, having less repression may change the balance of power, but it may be worth it if the savings are sufficient.

Our next result formalizes one version of the De Tocqueville hypothesis mentioned in the first paragraph of the paper, that anticipated social mobility makes democracy more likely (and the limiting of political rights and repression less likely). It shows that under some additional conditions, social mobility (anticipation of social mobility) reduces repression.

**Proposition 2** *Let  $x$  be the quasi-median group (QMV) under full democracy, and suppose that the most preferred policies  $\{b_i\}$  by different groups are such that*

$$b_x = \sum_{i \in N} \gamma_i b_i.$$

*In other words, the average weighted policy coincides with the median one. Consider a change from  $E^1$  to  $E^2$  that leaves the distribution of political power unchanged and thus corresponds to pure social mobility across all groups in society. Then, for any  $\pi(E^1, E^2) \in (0, 1)$ , there exists  $\bar{\beta} < 1$  such that for  $\beta \geq \bar{\beta}$ , society will transit from any initial state to full democracy (no repression) immediately (i.e., at  $t = 1$ ).*

The intuition for this result is simple: the current (quasi-)median voter, with preferences  $b_{E^1}$  (which is not necessarily equal to  $b_x$ ), realizes that in the long run (which is what he cares about given the high discount factor), his preferences may change, and thus removing repression and transiting into full democracy is preferable to repressing some groups—such repression would have led to some policy different than  $b_x$ , but social mobility makes the current median voter prefer, in expectation,  $b_x$ . It is worth noting that the condition that  $\beta$  is above a certain threshold  $\bar{\beta}$  is important. Otherwise, the currently powerful group would put a high weight on the policies that will be implemented before the shock arrives, and this may lead to perverse results.

The key to this proposition is that social mobility not only leaves the distribution of put the power across groups unchanged, but also does not create a situation in which the current median voter expects to relinquish that position to somebody else and would like to influence the future median voter's identity. Notably, this can happen even if there are no shocks to change the distribution of political power across different social groups, i.e., simply as a result of anticipated social mobility. The next proposition studies the stability of perfect democracy in the presence of different kinds of shocks.

**Proposition 3** *Suppose that we start with full democracy and let  $x$  be the initial QMV, and the change from  $E^1$  to  $E^2$  leaves the distribution of political power unchanged and thus corresponds to pure social mobility. Then:*

1. *If  $x \notin [a, b]$ , then repression will not occur under either  $E^1$  or  $E^2$ .*
2. *On the other hand, if  $x = a$  or  $x = b$ , repression in environment  $E^1$  (in anticipation of future social mobility, i.e., the switch to  $E^2$ ) is possible and this repression may also continue after the switch to  $E^2$ . In particular, if  $x = a$ , it is left wing groups and if  $x = b$ , it is right wing groups that will be repressed.*
3. *Repression is more likely (in part 2) if it is less costly (cost of repression  $C$  is low) and if the shock is expected to happen soon ( $\pi(E^1, E^2)$  is high).*

This result thus shows the limits of the De Tocqueville hypothesis. Social mobility creates a force towards tolerance to other groups because individuals expect to switch to those other groups in the future. However, it also creates a desire for repression because the current (quasi-)median voter  $x$  anticipates that there is a chance that he will not be one in the future, and would like to ensure that the new QMV is closer, on average, to his expected future preferences. As a result, if  $x$  expects to move to the left, he may opt to repress right-wing voters, and if he expects to move to the right, he may opt to repress left-wing. On the other hand, there are several countervailing forces. First, repression is costly. Second, the point of repression is to make another group responsible for the decisions, but when it happens, this group may decide to delegate decision making even further, which need not be in the original QMV's best interests. Third and relatedly, delegation of political decision-making has a direct cost as well, as the new group will make economic decisions as well. Perhaps the most important result contained in Proposition 3 is part 3, which shows that repression is more likely if it is less costly (as this mitigates the first countervailing force), and if the shock is expected to happen soon (this mitigates the second and the third, and also perhaps the first forces).

The next proposition shows that social mobility shifts policies towards the groups that currently have little political power but are part of the social mobility process.

**Proposition 4** *Let  $x$  be the initial QMV, and suppose that change from  $E^1$  to  $E^2$  leaves the distribution of political power unchanged and thus corresponds to pure social mobility among*



players  $[1, x]$ . Then the anticipation of social mobility shifts policy from  $b_x$  (weakly) to the left. Individuals in groups  $x + 1, \dots, n$ , who are not part of social mobility, are made worse off by the prospect of social mobility.

We next show that the second hypothesis discussed in the first paragraph, that anticipation of future changes in the distribution of political power can induce repression and moves away from democracy, receives support from this environment (under certain conditions).

**Proposition 5** *Suppose that we start with full democracy and change from  $E^1$  to  $E^2$  potentially changes the distribution of political power but does not involve reshuffling (social mobility). Let  $M_{E^k}$  be the quasi-median voter in environment  $E^k$  (under full democracy), and assume that  $b_{M_{E^1}} \neq b_{M_{E^2}}$ . Suppose also that  $\beta$  is greater than some  $\bar{\beta}$ . If the costs of repression, the  $C_j$ 's, are sufficiently low, then the median voter  $M_{E^1}$  will choose to repress the same side of the political spectrum as  $M_{E^2}$ . Moreover, lower costs of repression and higher  $\pi(E^1, E^2)$  make repression more likely.*

This proposition shows that anticipation of a change that will alter the distribution of political power may lead to repression of groups that would get more powerful from this shift if this change is viewed as sufficiently likely and the costs of repression are not too high. It therefore formalizes the idea that expectation of future changes in the distribution of political power is a force towards repression and move away from democracy. The next proposition shows that, for related reasons, once change happens, repression never increases. Intuitively, repression was largely driven by the anticipation of future change, and thus once change occurs, there is less reason for repression

**Proposition 6** *Suppose that change from  $E^1$  to  $E^2$  potentially changes the distribution of political power but does not involve reshuffling (social mobility). Then repression never increases after the shock has happened, but it may decrease.*

Finally, we show that repression decisions are “strategic complements”. To state this result, recall that  $C_j$  is the cost of repressing group  $j$ .

**Proposition 7** *Suppose that group  $x$  is the (quasi-)median voter in environment  $E^1$  and change from  $E^1$  to  $E^2$  potentially changes the distribution of political power and may bring groups to*

*the right of  $x$  to power. If the cost of repressing group  $x$ ,  $C_x$ , increases, then repression on all groups decreases or remains the same (does not increase).*

The intuition for this result is instructive. Repression is partly driven by the fear that some other group will come to power in the future and will use repression against those who are now (quasi-)median voters or support their positions. If the cost of repressing this group increases, then it fears future repression less, and this reduces its incentives to use repression in order to maintain its position. This result suggests that repression against all groups may become more likely when political, economic or social factors make repressing a particular social group or viewpoint easier. It might also contribute to an understanding for why, in some societies, very different parts of the political spectrum are repressed at different points in time, while other societies, with only small differences, experience much more limited repression of similar positions.

## 5 Extensions

In this section, we relax some of the assumption made in Section 2. In particular, we study the possibility that there is an infinite number (a continuum) of states and/or agents and establish an existence of MVE. We allow for the possibility of an infinite number of shocks and establish the existence of a “mixed-strategy” MVE. Finally, we show that our approach to stochastic shocks allows for studying situations where the probabilities of transitions,  $\pi(E, E')$ , may depend on the state of the world. This would be realistic, for example, when studying political experimentation.

### 5.1 Continuous space

Here, we assume that states, and perhaps individuals, are taken from a continuous set. We study Markov Voting equilibria in such environments. Namely, we study Markov Voting equilibria in discrete environments obtained by sampling a sufficiently dense but finite set of points.

More precisely, assume that the set of states is  $S = [s_l, s_h]$ , and the set of individuals is given by a unit continuum  $N = [i_l, i_h]$ . (The construction and reasoning below are easily extendable to the case where they are a finite number of individuals but a continuum of states, or vice versa.) We assume that each individual has a utility function  $u_i(s) : S \rightarrow \mathbb{R}$ , which is continuous as a function of  $(i, s) \in N \times S$  and satisfies the SID condition: for all  $i > j$ ,  $x > y$ ,

$$u_i(x) - u_i(y) > u_j(x) - u_j(y). \quad (40)$$

Cost function is given by  $c(x, y)$  and is assumed to be continuous on  $S \times S$  and to satisfy Assumption 6. Finally, for each state  $s$  there is a set of winning coalitions  $W_s$ , which are assumed to satisfy Assumption 4. As before, for each state  $s$ , we have a non-empty set of quasi-median voters  $M_s$  (which may nevertheless be a singleton). We make the following monotonicity of quasi-median voters assumption: functions  $\inf M_s$  and  $\sup M_s$  are continuous and increasing functions of  $s$ .

For simplicity, we assume there are no shocks, so the environment is fixed. Time is discrete as before. We are interested in monotone transition functions  $\phi : S \rightarrow S$ ; however, we do not impose additional restrictions, e.g., continuity (it may be possible that there is no equilibrium with a continuous transition function). The notions of MVE is the same as before, i.e., it is given by Definition 3.

In this environment, we can establish the following existence result.

**Theorem 11 (*Existence*)** *In the environment with continuous set of states and/or continuous set of individuals, there exists a MVE  $\phi$ . Moreover, take any sequence of sets of states  $S_1 \subset S_2 \subset \dots$  and any sequence of individuals  $N_1 \subset N_2 \subset \dots$  such that  $\bigcup_{j=1}^{\infty} S_j$  is dense in  $S$  and  $\bigcup_{j=1}^{\infty} N_j$  is dense in  $N$ . Consider any sequence of monotone functions  $\{\phi_j : S_j \rightarrow S_j\}_{j=1}^{\infty}$  which are MVE (not necessarily unique) in the environment*

$$E_j = \left( N, S, \beta, \{u_i(s)\}_{i \in N_j}^{s \in S_j}, \{c(x, y)\}_{i \in N_j}^{x, y \in S_j}, \{W_s\}_{s \in S_j} \right) \quad (41)$$

*(existence of such MVE is guaranteed by Theorem 1, as all assumptions are satisfied). Then there is a subsequence  $\{j_k\}_{k=1}^{\infty}$  such that  $\{\phi_{j_k}\}_{k=1}^{\infty}$  converges, pointwisely on  $\bigcup_{j=1}^{\infty} S_j$ , to some MVE  $\phi : S \rightarrow S$ .*

While this existence result does not characterize the set of equilibria in full, it guarantees existence, and also shows that a MVE may be found as a limit of equilibria for finite sets of states and individuals. The idea of the proof is simple. Take an increasing sequence of sets of states,  $S_1 \subset S_2 \subset \dots$  and an increasing sequence of sets of individuals  $N_1 \subset N_2 \subset \dots$  such that  $S_{\infty} = \bigcup_{j=1}^{\infty} S_j$  is dense in  $S$  and  $N_{\infty} = \bigcup_{j=1}^{\infty} N_j$  is dense in  $N$ . For each  $S_j$ , take MVE  $\phi_j$ . We know that  $\phi_j$  is a monotone function on  $S_j$ ; Since let us complement it to a monotone

(not necessarily continuous) function on  $S$  which we denote by  $\tilde{\phi}_i$  for each  $i$ . Since  $S_\infty$  and  $N_\infty$  are countable, there is a subsequence  $\phi_{j_k}$  which converges to some  $\phi : S_\infty \rightarrow S_\infty$  pointwisely. (Indeed, we can pick a subsequence which converges on  $S_1$ , then a subsequence converging on  $S_2$  etc; then use a diagonal process.) We then complete it to a function  $\phi : S \rightarrow S$  by demanding that  $\phi$  is either left-continuous or right-continuous at any point; in the Appendix, we show that we can do that so that the continuation values are either left-continuous or right-continuous as well). Then this continuity of continuation values will ensure that  $\phi$  is MVE.

## 5.2 Infinitely many shocks

Suppose that there is a finite set of states  $\mathcal{E}$ , but we relax Assumption 2. This allows for the possibility that an infinite number of shocks happens during the game. In this case, the question about existence of MVE as defined earlier is open. Nevertheless, we can prove existence of a “mixed” equilibrium. Suppose, indeed, that for each environment  $E \in \mathcal{E}$  we have several mappings  $\{\phi_E\}$  which may be played with different probabilities, and the particular transition mapping is picked anew each time a shock leading to environment  $E$  happens.

More formally, let  $\Phi$  be the (finite) set of monotone mappings from  $S$  to  $S$ , and let  $\Delta(\Phi)$  be the set of convex combinations. Let  $\phi_E \in \Delta(\Phi)$ , and for each  $\chi \in \Phi$ , let  $\alpha_{E,\chi} = \alpha_\chi(\phi_E)$  be the weight of  $\chi$  in the combination  $\phi_E$ . As said earlier, we assume that every time there is a shock that changes the environment from some  $E'$  to  $E$ , with probability  $\alpha_{E,\phi}$ , transition mapping  $\chi$  is used until the next shock. Therefore, the continuation payoffs are recursively given as follows:

$$\begin{aligned} V_{E,\chi,i}^\phi(s) &= u_{E,i}(s) + \beta_E \pi(E, E) V_{E,\chi,i}^\phi(\chi(s) | s) \\ &\quad + \beta_E \sum_{E' \neq E} \pi(E, E') \sum_{\chi' \in \Phi} \alpha_{\chi'}(\phi_{E'}) V_{E',\chi',i}^\phi(\chi'(s) | s) \end{aligned} \quad (42)$$

where for every  $\chi \in \Phi$ ,

$$V_{E,\chi,i}^\phi(\chi(s) | s) = V_{E,\chi,i}^\phi(\chi(s)) - c_E(s, \chi(s)). \quad (43)$$

Clearly, continuation payoffs are well-defined for every  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$  with  $\phi_E \in \Delta(\Phi)$ .

The definition of MVE is as follows.

**Definition 7 (Mixed MVE)** *A set  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$ , where  $\phi_E \in \Delta(\Phi)$  for all  $E$ , is a Mixed MVE if the following two properties hold:*

1. for any environment  $E \in \mathcal{E}$  and any  $\chi \in \Delta(\Phi)$ , if  $\alpha_\chi(\phi_E) > 0$ , then for any  $x, y \in S$ ,

$$\left\{ i \in N : V_{E,\chi,i}^\phi(y | x) > V_{E,\chi,i}^\phi(\chi(x) | x) \right\} \notin W_{E,x}; \quad (44)$$

2. for any environment  $E \in \mathcal{E}$  and any  $\chi \in \Delta(\Phi)$ , if  $\alpha_\chi(\phi_E) > 0$ , then for any state  $x \in S$ ,

$$\left\{ i \in N : V_{E,\chi,i}^\phi(\chi(x) | x) - c_E(x, \chi(x)) \geq V_{E,i}^\phi(x) \right\} \in W_{E,x}$$

We can establish the following existence result.

**Theorem 12** *If the assumptions above, except Assumption 2, are satisfied, then there exists a Mixed MVE.*

The idea of the proof of Theorem 12 is straightforward. We for any candidate equilibrium  $\phi$  we compute the continuation payoffs, and then for every environment  $E$  we find, as before, the set of monotone MVE (to do this, we need to verify that the usual assumptions, like single-crossing conditions, are satisfied). Let  $\mu(\phi, E)$  be the set of convex combinations of MVE, and let  $\mu(\phi)$  be the collection of  $\mu(\phi, E)$  for all  $E \in \mathcal{E}$ . It then suffices to verify that function  $\mu$  satisfies the hypotheses of Kakutani theorem.

However, a pure-strategy MVE need not exist, as the next example shows.

**Example 9 (No pure-strategy MVE with infinite number of shocks)** Below is an example of with finite number of states and players and finite number of environments such that all assumptions, except for the assumption that the number of shocks is finite, are satisfied, but there is no Markov Voting Equilibrium in pure strategies (but as Theorem 12 suggests, a mixed-strategy equilibrium exists). One could also construct a “Non-Markovian Voting Equilibrium”.

There are three environments  $E^1, E^2, E^3$ , three states  $A = 1, B = 2, C = 3$ , and three players 1, 2, 3. The history of environments follows a simple Markov chain; in fact, in each period the environment is drawn separately. More precisely,

$$\begin{aligned} \pi(E^1) & : = \pi(E^1, E^1) = \pi(E^2, E^1) = \pi(E^3, E^1) = \frac{1}{2}; \\ \pi(E^2) & : = \pi(E^1, E^2) = \pi(E^2, E^2) = \pi(E^3, E^2) = \frac{2}{5}; \\ \pi(E^3) & : = \pi(E^1, E^3) = \pi(E^2, E^3) = \pi(E^3, E^3) = \frac{1}{10}. \end{aligned}$$

In each environment, transition costs are either 0 or prohibitively large, so we describe these in terms of feasible/infeasible transitions. The discount factor is  $\frac{1}{2}$ .

The following matrices describe instantaneous payoffs, winning coalitions, and feasible transitions.

<i>Environment <math>E^1</math></i>	<i>State A</i>	<i>State B</i>	<i>State C</i>
<i>Winning coalition</i>	<i>Dictatorship of Player 1</i>		
<i>Feasible transitions</i>	<i>to A, B</i>	<i>to B</i>	<i>to C</i>
<i>Player 1</i>	60	150	-800
<i>Player 2</i>	30	130	60
<i>Player 3</i>	-100	60	50

<i>Environment <math>E^2</math></i>	<i>State A</i>	<i>State B</i>	<i>State C</i>
<i>Winning coalition</i>	<i>Dictatorship of Player 2</i>		
<i>Feasible transitions</i>	<i>to A</i>	<i>to A, B</i>	<i>to C</i>
<i>Player 1</i>	100	80	-800
<i>Player 2</i>	80	70	60
<i>Player 3</i>	-100	60	50

<i>Environment <math>E^3</math></i>	<i>State A</i>	<i>State B</i>	<i>State C</i>
<i>Winning coalition</i>	<i>Dictatorship of Player 3</i>		
<i>Feasible transitions</i>	<i>to A</i>	<i>to B, C</i>	<i>to C</i>
<i>Player 1</i>	100	80	-800
<i>Player 2</i>	80	70	60
<i>Player 3</i>	-100	60	50

It is straightforward to see that SID property holds; moreover, payoffs are single-peaked, and in each environment and each state, the set of quasi-median voters is a singleton.

The intuition behind the example is the following. The payoff matrices in environment  $E^2$  and  $E^3$  coincide, so “essentially”, there are two equally likely environments  $E^1$  and “ $E^2 \cup E^3$ ”. Both player 1 and 2 prefer state  $B$  when the environment is  $E^2$  and state  $A$  when the environment is  $E^1$ ; given the payoff matrix and the discount factor, player 1 would prefer to move from  $A$  to  $B$  when in  $E^1$ , and knowing this, player 2 would be willing to move to  $A$  when in  $E^2$ . However, there is a chance that the environment becomes  $E^3$  rather than  $E^2$ , in which case a “maniac”

player 3 will become able to move from state  $B$  (but not from  $A$ !) to state  $C$ ; the reason for him to do so is that although he likes state  $B$  (in all environments), he strongly dislikes  $A$ , and thus if players 1 and 2 are expected to move between these states, player 3 would rather lock the society in state  $C$ , which is only slightly worse for him than  $B$ .

State  $C$ , however, is really hated by player 1, who would not risk the slightest chance of getting there. So, if player 3 is expected to move to  $C$  when given such chance, player 1 would not move from  $A$  to  $B$  when the environment is  $E_1$ , because player 3 is only able to move to  $C$  from  $B$ . Now player 2, anticipating that if he decides to move from  $B$  to  $A$  when the environment is  $E^2$ , the society will end up in state  $A$  forever; this is something player 2 would like to avoid, because state  $A$  is very bad for him when the environment is  $E^1$ . In short, if player 3 is expected to move to  $C$  when given this chance, then the logic of the previous paragraph breaks down, and neither player 1 nor player 2 will be willing to move when they are in power. But in this case, player 3 is better off staying in state  $B$  even when given a chance to move to  $C$ , as he trades off staying in  $B$  forever versus staying in  $C$  forever. These considerations should prove that there is no MVE.

More formally, the reasoning goes as follows. Notice that there are only eight candidate mappings to consider (some transitions are made infeasible precisely to simplify the argument; alternatively, we could allow any transitions and make player 1 the dictator in state  $A$  when the environment is  $E^3$ ). We consider these eight mappings separately, and point out the deviation. Obviously, the only values of the transition mappings to be specified are  $\phi_{E^1}(A)$ ,  $\phi_{E^2}(B)$ , and  $\phi_{E^3}(B)$ .

1.  $\phi_{E^1}(A) = A$ ,  $\phi_{E^2}(B) = A$ ,  $\phi_{E^3}(B) = B$ . Then  $\phi'_{E^3}(B) = C$  is a profitable deviation.
2.  $\phi_{E^1}(A) = B$ ,  $\phi_{E^2}(B) = A$ ,  $\phi_{E^3}(B) = B$ . Then  $\phi'_{E^3}(B) = C$  is a profitable deviation.
3.  $\phi_{E^1}(A) = A$ ,  $\phi_{E^2}(B) = B$ ,  $\phi_{E^3}(B) = B$ . Then  $\phi'_{E^1}(A) = B$  is a profitable deviation.
4.  $\phi_{E^1}(A) = B$ ,  $\phi_{E^2}(B) = B$ ,  $\phi_{E^3}(B) = B$ . Then  $\phi'_{E^2}(B) = A$  is a profitable deviation.
5.  $\phi_{E^1}(A) = A$ ,  $\phi_{E^2}(B) = A$ ,  $\phi_{E^3}(B) = C$ . Then  $\phi'_{E^2}(B) = B$  is a profitable deviation.
6.  $\phi_{E^1}(A) = B$ ,  $\phi_{E^2}(B) = A$ ,  $\phi_{E^3}(B) = C$ . Then  $\phi'_{E^1}(A) = A$  is a profitable deviation.
7.  $\phi_{E^1}(A) = A$ ,  $\phi_{E^2}(B) = B$ ,  $\phi_{E^3}(B) = C$ . Then  $\phi'_{E^3}(B) = B$  is a profitable deviation.

8.  $\phi_{E^1}(A) = B$ ,  $\phi_{E^2}(B) = B$ ,  $\phi_{E^3}(B) = C$ . Then  $\phi'_{E^3}(B) = B$  is a profitable deviation.

This proves that there is no MVE in pure strategies (i.e., in the sense of Definition 3).

### 5.3 Other types of shocks

Here, we show that the framework is by and large applicable to situations where shocks that do not follow the Markov chain rule. We give two simple examples: experimentation and exogenous transitions.

Suppose that there are two environments  $E^1$  and  $E^2$ . They coincide on  $S' = [1, m - 1]$ , have the same discount factor, and, moreover, for  $s \in S'$ ,  $W_{E^1,s} = W_{E^2,s}$ . Assume, for simplicity, that all costs are the same as well. The only difference that  $E^1$  and  $E^2$  have is instantaneous payoffs of players in state  $m$ .

The initial state is  $s_0 < m$  and the initial environment is  $E^1$ . Assume that the environment is  $E^1$  as long as  $s_t \neq m$ . If  $s_t = m$  for the first time, then the environment switches to  $E^2$  with probability  $\rho$  and stays to be  $E^1$  with probability  $1 - \rho$ ; after that, there are no further changes. This setting could be used to model political experimentation. Indeed, suppose that players are initially aware of payoffs in states  $s < m$ , but are ignorant whether the payoffs in state  $m$  are  $\{u_{E^1,i}(m)\}_{i \in N}$  or  $\{u_{E^2,i}(m)\}_{i \in N}$  (although they are aware of the probability  $\rho$ ). The only way to find out, which of the environments they are in, is to make a transition to state  $m$ . (It might be natural to introduce the initial environment  $E_0$  and assume the agents do not get any payoff in state  $m$ , but if they transit to state  $m$ , then there is an immediate transition to either  $E^1$  or  $E^2$ . However, this would be redundant, and assuming that the society starts in environment  $E^1$  which may switch to  $E^2$  immediately after they experiment with state  $m$  should be enough.)

We do not formulate a formal existence or uniqueness result here. Clearly, we can apply a backward induction argument in a way similar to Subsection 3.2. Moreover, we can allow for the possibility that agents get some signal about whether they end up in  $E^1$  or  $E^2$  after trying  $m$ . We leave these possibilities for future research.

One other type of shock which we have not studied before is the possibility of exogenous transitions. Suppose, for example, that there are two identical environments  $E^1$  and  $E^2$ , and the probabilities of transition satisfy  $0 < \pi(E^1, E^2) < 1$ , and  $\pi(E^2, E^1) = 0$ . The difference is that when there is a shock that switches the environment from  $E^1$  to  $E^2$ , the state automatically moves to, say,  $\tilde{s}$ . With some single-crossing assumptions on the nature of the shock, a similar



backward induction reasoning would allow us to show existence and, under stronger assumptions, uniqueness of a Markov Voting equilibrium.

## 6 Conclusion

This paper has provided a general framework for the analysis of dynamics of institutional change (e.g., democratization, extension of political rights or repression), and how this interacts with (anticipated and unanticipated) changes in the distribution of political power and changes in economic structure (e.g., social mobility or other changes affecting individuals' preferences over different types of economic policies and allocations). We have focused on the Markov voting equilibria, which require that economic and political changes should take place if there exists a subset of players with the power to implement such changes and who will obtain higher expected continuation utility by doing so. Under the assumption that different economic and social institutions/policies as well as individual types can be ordered, and preferences and the distribution of political power satisfy “single crossing,” we prove the existence of pure-strategy equilibria and provide conditions for their uniqueness.

Despite its generality, we have shown that the framework yields a number of comparative static results. For example, if there is a change from one environment to another (with different economic payoffs and distribution of political power) but the two environments coincide up to a certain state  $s'$  and before the change the steady state of equilibrium was that some state  $x \leq s'$ , then the new steady state that emerges after the change in environment can be no smaller than  $x$ . Another comparative static result is the following: again consider a change leaving preferences and the distribution of the power the same in states  $s \leq s'$ , but now arriving before the steady state  $x \leq s'$  is reached. Then when all agents in society have discount factor sufficiently small (smaller than some threshold  $\bar{\beta}$ ), the direction of changes states will remain the same as before (i.e., if there were transitions towards higher states before, this will continue, and vice versa). Finally, we have also shown that a change in environment makes extreme states “sticky” takes place away from these extreme states, then the equilibrium trajectory is not affected.

We have also shown how this framework can be applied to the study of the dynamics of political rights and repression, and derived a range of additional comparative statics for this more specific application.

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## Appendix

**Proof of Lemma 1.** “If”: Suppose  $M_s \subset P$ , so for each  $i \in M_s$ ,  $w_i(y) > w_i(x)$ . Consider two cases. If  $y > x$ , then WID implies that  $w_j(y) > w_j(x)$  for all  $j \geq \min M_s$ , and such players  $j$  form a winning coalition by definition of QMV. If  $y < x$ , then, similarly,  $w_j(y) > w_j(x)$  for all  $j \leq \min M_s$ , and thus for a winning coalition. In either case,  $P$  contains a subset (either  $[\min M_s, n]$  or  $[1, \max M_s]$ ) which is a winning coalition, and thus  $P \in W_s$ .

“Only if”: Suppose  $P \in W_s$ . First, consider the case  $y > x$ . Let  $i = \min P$ ; then for all  $j \geq i$ ,  $w_j(y) > w_j(x)$ . This means that  $P = [i, n]$ , and is thus a connected coalition. Since  $P$  is winning, we must have  $j \in P$  for any  $j \in M_s$ , so  $M_s \subset P$ . The case where  $y < x$  is analogous, so  $M_s \subset P$ .

The proofs for relations  $\geq, <, \leq$  are similar and are omitted. ■

**Proof of Lemma 2. Part 1.** We prove this part if for cost  $c_i(\cdot, \cdot)$  depends on  $i$  (provided that the extra properties required in Footnote 12 hold), as this comes at no extra cost. Take  $y > x$  and any  $i \in N$ . We have:

$$\begin{aligned} V_i^\phi(y) - V_i^\phi(x) &= u_i(y) + \sum_{k=1}^{\infty} \beta^k \left( u_i(\phi^k(y)) - c_i(\phi^{k-1}(y), \phi^k(y)) \right) \\ &\quad - u_i(x) - \sum_{k=1}^{\infty} \beta^k \left( u_i(\phi^k(x)) - c_i(\phi^{k-1}(x), \phi^k(x)) \right) \\ &= (u_i(y) - u_i(x)) + \sum_{k=1}^{\infty} \beta^k \left( u_i(\phi^k(y)) - u_i(\phi^k(x)) \right) - \sum_{k=1}^{\infty} \beta^k \left( c_i(\phi^{k-1}(y), \phi^k(y)) - \right. \end{aligned}$$

The first term is weakly (strictly) increasing in  $i$  if  $\{u_i(s)\}_{i \in N}^{s \in S}$  satisfies WID (SID, respectively), the second is weakly increasing in  $i$ , as  $\phi^k(y) \geq \phi^k(x)$  for  $k \geq 1$  due to monotonicity, and for the last term, it follows from

$$\begin{aligned} - \left( c_i(\phi^{k-1}(y), \phi^k(y)) - c_i(\phi^{k-1}(x), \phi^k(x)) \right) &= \\ &= - \left( c_i(\phi^{k-1}(y), \phi^k(x)) - c_i(\phi^{k-1}(x), \phi^k(x)) \right) \\ &\quad - \left( c_i(\phi^{k-1}(y), \phi^k(y)) - c_i(\phi^{k-1}(y), \phi^k(x)) \right) \end{aligned} \tag{46}$$

which is (weakly) increasing in  $i$ . (Notice that if  $c_i(\cdot, \cdot)$  does not depend on  $i$ , then (46) is a constant.) Consequently, (16) is weakly (strictly) increasing in  $i$ . Now,

$$\begin{aligned} V_i^\phi(y | s) - V_i^\phi(x | s) &= \left( V_i^\phi(y) - c_i(s, y) \right) - \left( V_i^\phi(x) - c_i(s, x) \right) \\ &= \left( V_i^\phi(y) - V_i^\phi(x) \right) - (c_i(s, y) - c_i(s, x)). \end{aligned} \tag{47}$$

We proved that the first term is weakly (strictly) increasing in  $i$ , and the last one is weakly increasing in  $i$  by assumption. Hence, both  $\left\{V_i^\phi(x)\right\}_{i \in N}^{x \in S}$  and  $\left\{V_i^\phi(x|s)\right\}_{i \in N}^{x \in S}$  satisfy WID (SID) for any  $s$ .

**Part 2.** If  $\phi$  is monotone, then Part 1 applies. Otherwise, for some  $x < y$  we have  $\phi(x) > \phi(y)$ , and this means that  $y = x + 1$ ; there may be one or more such pairs. Notice that for such  $x$  and  $y$ , we have

$$\begin{aligned} V_i^\phi(y) - V_i^\phi(x) &= \\ & \left( u_i(y) + \sum_{k=1}^{\infty} \beta^{2k-1} (u_i(x) - c(y, x)) + \sum_{k=1}^{\infty} \beta^{2k} (u_i(y) - c(x, y)) \right) \\ & - \left( u_i(x) + \sum_{k=1}^{\infty} \beta^{2k-1} (u_i(y) - c(x, y)) + \sum_{k=1}^{\infty} \beta^{2k} (u_i(x) - c(y, x)) \right) \\ & = \frac{1}{1+\beta} (u_i(y) - u_i(x)) - \frac{\beta}{1+\beta} (c(x, y) - c(y, x)). \end{aligned} \quad (48)$$

The first term is weakly (strictly) increasing in  $i$  and the second does not depend on it.

Let us now modify instantaneous payoffs and define

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ (1-\beta)V_i(x) & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases} \quad (49)$$

Consider mapping  $\tilde{\phi}$  given by

$$\tilde{\phi}(s) = \begin{cases} \phi(x) & \text{if } \phi(x) = x \text{ or } \phi^2(x) \neq x; \\ x & \text{if } \phi(x) \neq x = \phi^2(x). \end{cases} \quad (50)$$

Then  $\tilde{\phi}$  is monotone and  $\{\tilde{u}_i(x)\}_{i \in N}^{x \in S}$  satisfies WID (SID). By Part 1, the continuation values  $\left\{\tilde{V}_i^{\tilde{\phi}}(x)\right\}_{i \in N}^{x \in S}$  computed for  $\tilde{\phi}$  and  $\{\tilde{u}_i(x)\}_{i \in N}^{x \in S}$  using (16) and (17) satisfy WID (SID) as well. But by construction,  $\tilde{V}_i^{\tilde{\phi}}(x) = V_i^\phi(x)$  for each  $i$  and  $s$ , and thus  $\left\{V_i^\phi(x)\right\}_{i \in N}^{x \in S}$  satisfy WID (SID). The argument for  $\left\{V_i^\phi(x|s)\right\}_{i \in N}^{x \in S}$  is the same as before and is omitted. ■

**Proof of Lemma 3.** Suppose, to obtain a contradiction, that for each  $x, y \in S$  such that (18) holds (such pair of  $x$  and  $y$  exists because property (1) from Definition 3 is violated for  $\phi$ ),  $\phi'$  given by (19) is not monotone.

Take  $x, y \in S$  such that  $|y - \phi(x)|$  is minimal among all pairs  $x, y \in S$  that satisfy (18) (informally, we consider the shortest deviation). By our assertion,  $\phi'$  is not monotone. Since  $\phi$  is monotone and  $\phi$  and  $\phi'$  differ by the value at  $x$  only, there are two possibilities: either for

some  $z < x$ ,  $y = \phi'(x) < \phi(z) \leq \phi(x)$  or for some  $z > x$ ,  $\phi(x) \leq \phi(z) < \phi'(x) = y$ . Assume the former (the latter case may be considered similarly). Let  $s$  be defined by

$$s = \min(z \in S : \phi(z) > y); \quad (51)$$

in the case under consideration, the set of such  $z$  is nonempty (e.g.,  $x$  is its member, and  $z$  found earlier is one as well), and hence state  $s$  is well-defined. We have  $s < x$ ; since  $\phi$  is monotone,  $\phi(s) \leq \phi(x)$ .

Notice that a deviation in state  $s$  from  $\phi(s)$  to  $y$  is monotone: indeed, there is no state  $\tilde{z}$  such that  $\tilde{z} < s$  and  $y < \phi(\tilde{z}) \leq \phi(s)$  by construction of  $s$ , and there is no state  $\tilde{z} > s$  such that  $\phi(s) \leq \phi(\tilde{z}) < y$  as this would contradict  $\phi(s) > y$ . By assertion, this deviation cannot be profitable, i.e.,

$$V^\phi(y) - c(s, y) \not\geq_s V^\phi(\phi(s)) - c(s, \phi(s)). \quad (52)$$

By Lemma 2, since  $y < \phi(s)$ ,

$$V_{\max M_s}^\phi(y) - c_{\max M_s}(s, y) \leq V_{\max M_s}^\phi(\phi(s)) - c_{\max M_s}(s, \phi(s)). \quad (53)$$

Since  $s < x$ , Assumption 5 implies (for  $i = \max M_x$ )

$$V_i^\phi(y) - c_i(s, y) \leq V_i^\phi(\phi(s)) - c_i(s, \phi(s)). \quad (54)$$

On the other hand, (18) implies

$$V_i^\phi(y) - c_i(x, y) > V_i^\phi(\phi(x)) - c_i(x, \phi(x)). \quad (55)$$

We therefore have

$$\begin{aligned} & \left( V_i^\phi(\phi(s)) - c_i(x, \phi(s)) \right) - \left( V_i^\phi(\phi(x)) - c_i(x, \phi(x)) \right) \\ & > \left( V_i^\phi(y) - c_i(s, y) + c_i(s, \phi(s)) - c_i(x, \phi(s)) \right) - \left( V_i^\phi(y) - c_i(x, y) \right) \\ & = (c_i(s, \phi(s)) - c_i(s, y)) - (c_i(x, \phi(s)) - c_i(x, y)) \geq 0, \end{aligned} \quad (56)$$

since  $s < x$  and  $y < \phi(s)$ . Consequently,

$$V_i^\phi(\phi(s)) - c_i(x, \phi(s)) > V_i^\phi(\phi(x)) - c_i(x, \phi(x)) \quad (57)$$

and thus, by Lemma 2, since  $\phi(s) < \phi(x)$  (we know  $\phi(s) \leq \phi(x)$ , but  $\phi(s) = \phi(x)$  would contradict (57)),

$$V^\phi(\phi(s)) - c(x, \phi(s)) >_x V^\phi(\phi(x)) - c(x, \phi(x)). \quad (58)$$

Notice, however, that  $y < \phi(s) < \phi(x)$  implies that  $|\phi(s) - \phi(x)| < |y - \phi(x)|$ . This contradicts the choice of  $y$  such that  $|y - \phi(x)|$  is minimal among pairs  $x, y \in S$  such that (18) is satisfied. This contradiction proves that our initial assertion that for any  $x, y \in S$ , (18) implies that  $\phi'$  is nonmonotone, is incorrect. Consequently, there are  $x, y \in S$  such that (18) holds and  $\phi'$  given by (19) is monotone. ■

**Proof of Lemma 4.** We show first that if (1) is the case, then (2) and (3) are impossible. We then show that if (1) does not hold, then either (2) or (3) are satisfied. We finish the proof by showing that (2) and (3) are mutually exclusive.

First, suppose, to obtain a contradiction, that both (1) and (2) hold. Then (2) implies that for some  $z \in [a + 1, \phi(a + 1)]$

$$V_{\min M_a}^\phi(z) - c_{\min M_a}(a, z) > V_{\min M_a}^\phi(\phi(a)) - c_{\min M_a}(a, \phi(a)). \quad (59)$$

By Lemma 2,

$$V^\phi(z) - c(a, z) >_a V^\phi(\phi(a)) - c(a, \phi(a)), \quad (60)$$

which contradicts that  $\phi$  is MVE, so (1) cannot hold. This contradiction shows that if (1) holds, (2) is impossible. We can similarly prove that if (1) holds, (3) cannot be the case.

Second, suppose that (1) does not hold. Notice that for any  $x \in S$ ,

$$V^\phi(\phi(x)) - c(x, \phi(x)) \geq_x V^\phi(x) \quad (61)$$

because this holds for  $\phi_1$  if  $x \in [1, a]$  and for  $\phi_2$  if  $x \in [a + 1, m]$ . Consequently, if  $\phi$  is not MVE, then it is because property (1) of Definition 3 is violated. Lemma 3 then implies existence of a monotone deviation, i.e.,  $x, y \in S$  such that

$$V^\phi(y) - c(x, y) >_x V^\phi(\phi(x)) - c(x, \phi(x)). \quad (62)$$

Since  $\phi_1$  and  $\phi_2$  are MVE on their respective domains, we must have that either  $x \in [1, a]$  and  $y \in [a + 1, m]$  or  $x \in [a + 1, m]$  and  $y \in [1, m]$ ; assume the former. Since the deviation is monotone, we must have  $x = a$  and  $a + 1 \leq y \leq \phi(a + 1)$ . Hence, (62) may be rewritten as

$$V^\phi(y) - c(a, y) >_a V^\phi(\phi(a)) - c(a, \phi(a)). \quad (63)$$

Suppose, to obtain a contradiction, that (2) does not hold. This means that there is some  $z \in [\phi(a), a]$  such that

$$V_{\min M_a}^\phi(z) - c_{\min M_a}(a, z) \geq V_{\min M_a}^\phi(y) - c_{\min M_a}(a, y). \quad (64)$$



But (63) and (64) together imply

$$V_{\min M_a}^\phi(z) - c_{\min M_a}(a, z) > V_{\min M_a}^\phi(\phi(a)) - c_{\min M_a}(a, \phi(a)). \quad (65)$$

Since  $z > \phi(a)$  ( $z = \phi(a)$  is impossible since then (65) would not hold), by Lemma 2, we have

$$V^\phi(z) - c(a, z) >_a V^\phi(\phi(a)) - c(a, \phi(a)).$$

This contradicts that  $\phi_1$  is MVE on  $[1, a]$ , which proves that in the case where  $x \in [1, a]$  and  $y \in [a + 1, m]$ , (2) must hold. Now, if, instead of  $x \in [1, a]$  and  $y \in [a + 1, m]$ , we have  $x \in [a + 1, m]$  and  $y \in [1, m]$ , we can similarly prove that (3) holds. Therefore, if (1) does not hold, then either (2) or (3) does.

Third, suppose that both (2) and (3) hold. Let

$$x \in \arg \max_{z \in [\phi(a), \phi(a+1)]} \left( V_{\min M_a}^\phi(z) - c_{\min M_a}(a, z) \right), \quad (66)$$

$$y \in \arg \max_{z \in [\phi(a), \phi(a+1)]} \left( V_{\max M_{a+1}}^\phi(z) - c_{\max M_{a+1}}(a + 1, z) \right); \quad (67)$$

then  $x \geq a + 1 > a \geq y$ . Moreover,

$$V_{\min M_a}^\phi(x) - c_{\min M_a}(a, x) > V_{\min M_a}^\phi(y) - c_{\min M_a}(a, y), \quad (68)$$

$$V_{\max M_{a+1}}^\phi(y) - c_{\max M_{a+1}}(a + 1, y) > V_{\max M_{a+1}}^\phi(x) - c_{\max M_{a+1}}(a + 1, x). \quad (69)$$

However,  $x > y$ , and Assumption 5 implies  $\min M_a \leq \max M_{a+1}$ , so (68) implies

$$V_{\max M_{a+1}}^\phi(x) - c_{\max M_{a+1}}(a, x) > V_{\max M_{a+1}}^\phi(y) - c_{\max M_{a+1}}(a, y), \quad (70)$$

and this contradicts (69). This contradiction proves that (2) and (3) are mutually exclusive, which completes the proof. ■

**Proof of Lemma 5.** We prove the first part of Lemma only (the second part is completely analogous). Let us first prove that  $\phi'$  satisfies property (1) of Definition 3. Suppose, to obtain a contradiction, that this is not the case. By Lemma 3, there is a monotone deviation at state  $x \in [1, m]$  to state  $y$ , i.e.,

$$V^{\phi'}(y) - c(x, y) >_x V^{\phi'}(\phi'(x)) - c(x, \phi'(x)). \quad (71)$$

If  $x < m$  then, since deviation is monotone,  $y \leq \phi(m) = a \leq m - 1$ . For any  $z \leq m - 1$ ,  $(\phi')^k(z) = \phi^k(z)$  for any  $k \geq 0$ , and thus  $V^{\phi'}(z) = V^\phi(z)$ ; therefore,

$$V^\phi(y) - c(x, y) >_x V^\phi(\phi(x)) - c(x, \phi(x)). \quad (72)$$

However, this would contradict that  $\phi$  is a MVE on  $S$ . Consequently,  $x = m$ . If  $y < m$ , then (71) implies, given  $a = \phi'(m)$ ,

$$V^\phi(y) - c(m, y) >_m V^\phi(a) - c(m, a). \quad (73)$$

Since the deviation is monotone,  $y \in [\phi(m-1), m-1]$ , but then (73) contradicts the choice of  $a$  in (21). This implies that  $x = y = m$ , so (71) may be rewritten as

$$V^{\phi'}(m) >_m V^\phi(a) - c(m, a). \quad (74)$$

But since

$$V^{\phi'}(m) = u(m) - \beta c(m, a) + \beta V^\phi(a), \quad (75)$$

(74) implies

$$u(m) >_m (1 - \beta) \left( V^\phi(a) - c(m, a) \right). \quad (76)$$

This, however, contradicts (22), which proves that  $\phi'$  satisfies property (1) of Definition 3.

To prove that  $\phi'$  is MVE, we need to establish that

$$V^{\phi'}(\phi'(x)) - c(x, \phi'(x)) \geq_x V^{\phi'}(x) \quad (77)$$

for each  $x \in S'$ . If  $x \in S$  (i.e.,  $x < m$ ), then  $(\phi')^k(x) = \phi^k(x)$  for any  $k \geq 0$ , so (77) is equivalent to

$$V^\phi(\phi(x)) - c(x, \phi(x)) \geq_x V^\phi(x). \quad (78)$$

Since  $\phi$  is MVE on  $S$ , (78) holds for  $x < m$ . It remains to prove that (77) is satisfied for  $x = m$ . In this case, (77) may be rewritten as

$$V^\phi(a) - c(x, a) \geq_m V^{\phi'}(m). \quad (79)$$

Taking (75) into account, (79) is equivalent to

$$(1 - \beta) \left( V^\phi(a) - c(x, a) \right) \geq_m u(m), \quad (80)$$

which is true, provided that (22) is satisfied. We have thus proved that  $\phi'$  is MVE on  $S'$ , which completes the proof. ■

**Proof of Theorem 1.** We prove this result by induction by the number of states. For any set  $X$ , let  $\Phi^X$  be the set of monotone MVE, so we have to prove that  $\Phi^X \neq \emptyset$ .

Base: If  $m = 1$ , then  $\phi : S \rightarrow S$  given by  $\phi(1) = 1$  is MVE for trivial reasons.

Step: Suppose that if  $|S| < m$ , then MVE exists. Let us prove this if  $|S| = m$ . Consider the set  $A = [1, m - 1]$ , and for any  $a \in A$ , consider two monotone MVE  $\phi_1^a : [1, a] \rightarrow [1, a]$  and  $\phi_2^a : [a + 1, m] \rightarrow [a + 1, m]$ . Without loss of generality, we may assume that

$$\phi_1^a \in \arg \max_{\phi \in \Phi^{[1, a]}, z \in [\phi(a), a]} \left( V_{\max M_{a+1}}^\phi(z) - c_{\max M_{a+1}}(a + 1, z) \right), \quad (81)$$

$$\phi_2^a \in \arg \max_{\phi \in \Phi^{[a+1, m]}, z \in [a+1, \phi(a+1)]} \left( V_{\min M_a}^\phi(z) - c_{\min M_a}(a, z) \right). \quad (82)$$

Define  $\phi^a : S \rightarrow S$  by

$$\phi^a(s) = \begin{cases} \phi_1^a(s) & \text{if } s \leq a \\ \phi_2^a(s) & \text{if } s > a \end{cases}. \quad (83)$$

Let us define function  $f : A \rightarrow \{1, 2, 3\}$  as follows. By Lemma 4, for every split  $S = [1, a] \cup [a + 1, m]$  given by  $a \in A$  and for MVE  $\phi_1^a$  and  $\phi_2^a$ , exactly one of three properties hold; let  $f(a)$  be the number of the property. Then, clearly, if for some  $a \in A$ ,  $f(a) = 1$ , then  $\phi^a$  is a monotone MVE by construction of function  $f$ .

Now let us consider the case where for every  $a \in A$ ,  $f(a) \in \{2, 3\}$ . We have the following possibilities.

First, suppose that  $f(1) = 2$ . This means that (since  $\phi_1^a(1) = 1$  for  $a = 1$ )

$$\arg \max_{z \in [1, \phi(2)]} \left( V_{\min M_1}^{\phi^1}(z) - c_{\min M_1}(1, z) \right) \subset [2, \phi^1(2)]. \quad (84)$$

Let

$$b \in \arg \max_{z \in [2, \phi(2)]} \left( V_{\min M_1}^{\phi^1}(z) - c_{\min M_1}(1, z) \right) \quad (85)$$

and define  $\phi' : S \rightarrow S$  by

$$\phi'(s) = \begin{cases} b & \text{if } s = 1 \\ \phi^1(s) & \text{if } s > 1 \end{cases}; \quad (86)$$

let us prove that  $\phi'$  is a MVE. Notice that (84) and (85) imply

$$V_{\min M_1}^{\phi^1}(b) - c_{\min M_1}(1, b) > V_{\min M_1}^{\phi^1}(1) - c_{\min M_1}(1, 1) = V_{\min M_1}^{\phi^1}(1). \quad (87)$$

By Lemma 2, since  $b > 1$ ,

$$V^{\phi^1}(b) - c(1, b) >_1 V^{\phi^1}(1). \quad (88)$$

Notice, however, that

$$V^{\phi^1}(1) = u(1) / (1 - \beta), \quad (89)$$

and also  $V^{\phi^1}(b) = V^{\phi_2^1}(b)$ ; therefore, (88) may be rewritten as

$$V^{\phi_2^1}(b) - c(1, b) >_1 u(1) / (1 - \beta). \quad (90)$$

By Lemma 5,  $\phi' : S \rightarrow S$  defined by (86), is a MVE.

Second, suppose that  $f(m-1) = 3$ . In this case, using the first part of Lemma 5, we can prove that there is a MVE similarly to the previous case.

Finally, suppose that  $f(1) = 3$  and  $f(m-1) = 2$  (this already implies  $m \geq 3$ ), then there is  $a \in [2, m-1]$  such that  $f(a-1) = 3$  and  $f(a) = 2$ . Define, for  $s \in S \setminus \{a\}$  and  $i \in N$ ,

$$V_i^*(s) = \begin{cases} V_i^{\phi_1^{a-1}}(s) & \text{if } s < a \\ V_i^{\phi_2^a}(s) & \text{if } s > a \end{cases}. \quad (91)$$

Let us first prove that there exists  $b \in [\phi_1^{a-1}(a-1), a-1] \cup [a+1, \phi_2^a(a+1)]$  such that

$$V^*(b) - c(a, b) >_a u(a) / (1 - \beta), \quad (92)$$

and let  $B$  be the set of such  $b$  (so  $B \subset [\phi_1^{a-1}(a-1), a-1] \cup [a+1, \phi_2^a(a+1)]$ ). Indeed, since  $f(a-1) = 3$ ,

$$\arg \max_{z \in [\phi^{a-1}(a-1), \phi^{a-1}(a)]} \left( V_{\max M_a}^{\phi^{a-1}}(z) - c_{\max M_a}(a, z) \right) \subset [\phi^{a-1}(a-1), a-1]. \quad (93)$$

Let

$$b \in \arg \max_{z \in [\phi^{a-1}(a-1), a-1]} \left( V_{\max M_a}^{\phi^{a-1}}(z) - c_{\max M_a}(a, z) \right), \quad (94)$$

then (93) and (94) imply

$$V_{\max M_a}^{\phi^{a-1}}(b) - c_{\max M_a}(a, b) > V_{\max M_a}^{\phi^{a-1}}(a) - c_{\max M_a}(a, a) = V_{\max M_a}^{\phi^{a-1}}(a). \quad (95)$$

By Lemma 2, since  $b < a$ ,

$$V^{\phi^{a-1}}(b) - c(a, b) >_a V^{\phi^{a-1}}(a). \quad (96)$$

We have, however,

$$V^{\phi^{a-1}}(a) = V^{\phi_2^{a-1}}(a) = u(a) - \beta c(a, \phi_2^{a-1}(a)) + \beta V^{\phi_2^{a-1}}(\phi_2^{a-1}(a)) \geq_a u(a) + \beta V^{\phi_2^{a-1}}(a) = u(a) + \beta V^{\phi^{a-1}}(a) \quad (97)$$

( $V^{\phi^{a-1}}(a) = V^{\phi_2^{a-1}}(a)$  by definition of  $\phi^{a-1}$ , and the inequality holds because  $\phi_2^{a-1}$  is MVE on  $[a, m]$ ). Consequently, (95) and (96) imply (92). (Notice that using  $f(a) = 2$ , we could similarly prove that there is  $b \in [a+1, \phi^a(a+1)]$  such that (92) holds.)

Let us now take state some quasi-median voter in state  $a$ ,  $j \in M_a$ , and state  $d \in [\phi_1^{a-1}(a-1), a-1] \cup [a+1, \phi_2^a(a+1)]$  such that

$$d = \arg \max_{b \in B} (V_j^*(b) - c_j(a, b)), \quad (98)$$

and define monotone mapping  $\phi : S \rightarrow S$  as

$$\phi(s) = \begin{cases} \phi_1^{a-1}(s) & \text{if } s < a \\ d & \text{if } s = a \\ \phi_2^a(s) & \text{if } s > a \end{cases} \quad (99)$$

(note that  $V^\phi(s) = V^*(s)$  for  $x \neq a$ ). Let us prove that  $\phi(s)$  is MVE.

By construction of  $d$  (98) that for any  $b \in [\phi_1^{a-1}(a-1), \phi_2^a(a+1)]$ ,

$$V^\phi(b) - c(a, b) \not>_a V^\phi(d) - c(a, d). \quad (100)$$

This is automatically true for  $b \in B$ , whereas if  $b \notin B$  and  $b \neq a$ , the opposite would imply

$$V^\phi(b) - c(a, b) >_a u(a) / (1 - \beta), \quad (101)$$

which would contradict  $b \notin a$ ; finally, if  $b = a$ ,

$$V^\phi(a) >_a V^\phi(d) - c(a, d) \quad (102)$$

is impossible, as this would imply

$$u(a) >_a (1 - \beta) (V^\phi(d) - c(a, d)) \quad (103)$$

contradicting (92), given the definition of  $d$  (98). Now, Lemma 5 implies that  $\phi' = \phi|_{[1, a]}$  is a MVE on  $[1, a]$ .

Suppose, to obtain a contradiction, that  $\phi$  is not MVE. Since  $\phi$  is made from MVE  $\phi'$  on  $[1, a]$  and MVE  $\phi_2^a$  on  $[a+1, m]$ , there are only two possible monotone deviations that may prevent  $\phi$  from being MVE. First, suppose that for some  $y \in [a+1, \phi_2^a(a+1)]$ ,

$$V^\phi(y) - c(a, y) >_a V^\phi(d) - c(a, d). \quad (104)$$

However, this would contradict (98) (and if  $y \notin B$ , then (104) is impossible as  $d \in B$ ). The second possibility is that for some  $y \in [d, a]$ ,

$$V^\phi(y) - c(a+1, y) >_{a+1} V^\phi(\phi_2^a(a+1)) - c(a+1, \phi_2^a(a+1)). \quad (105)$$

This means that

$$V_{\max M_{a+1}}^\phi(y) - c_{\max M_{a+1}}(a+1, y) > V_{\max M_{a+1}}^\phi(\phi_2^a(a+1)) - c_{\max M_{a+1}}(a+1, \phi_2^a(a+1)). \quad (106)$$

At the same time, for any  $x \in [a+1, \phi_2^a(a+1)]$ , we have

$$V_{\max M_{a+1}}^\phi(x) - c_{\max M_{a+1}}(a+1, x) \leq V_{\max M_{a+1}}^\phi(\phi_2^a(a+1)) - c_{\max M_{a+1}}(a+1, \phi_2^a(a+1)) \quad (107)$$

(otherwise Lemma 2 would imply a profitable deviation to  $x$ ). This implies that for any such  $x$ ,

$$V_{\max M_{a+1}}^\phi(y) - c_{\max M_{a+1}}(a+1, y) > V_{\max M_{a+1}}^\phi(x) - c_{\max M_{a+1}}(a+1, x). \quad (108)$$

Now, recall that

$$\phi_1^a \in \arg \max_{\phi \in \Phi^{[1, a]}, z \in [\phi(a), a]} \left( V_{\max M_{a+1}}^\phi(z) - c_{\max M_{a+1}}(a+1, z) \right). \quad (109)$$

This means that there is  $z \in [\phi_1^a(a), a]$  such that

$$V_{\max M_{a+1}}^{\phi_1^a}(z) - c_{\max M_{a+1}}(a+1, z) \geq V_{\max M_{a+1}}^\phi(y) - c_{\max M_{a+1}}(a+1, y), \quad (110)$$

and thus for any  $x \in [a+1, \phi_2^a(a+1)]$ ,

$$V_{\max M_{a+1}}^{\phi_1^a}(z) - c_{\max M_{a+1}}(a+1, z) > V_{\max M_{a+1}}^\phi(x) - c_{\max M_{a+1}}(a+1, x). \quad (111)$$

But  $\phi_1^a = \phi^a$  on the left-hand side, and  $\phi = \phi^a$  on the right-hand side. We therefore have that the following maximum is achieved on  $[\phi^a(a), a]$ :

$$\arg \max_{z \in [\phi^a(a), \phi^a(a+1)]} \left( V_{\max M_{a+1}}^{\phi^a}(z) - c_{\max M_{a+1}}(a+1, z) \right) \subset [\phi^a(a), a], \quad (112)$$

i.e., that (3) in Lemma 4 holds. But this contradicts that  $f(a) = 2$ . This contradiction completes the induction step, which proves existence of MVE.

The last statement follows from that any MVE has monotone paths, and any monotone sequence converges. ■

**Proof of Theorem 2. Part 1.** Suppose that there are two MVEs  $\phi_1$  and  $\phi_2$ . Without loss of generality, assume that  $m$  is the minimal number of states for which this is possible, i.e., if  $|S| < m$ , then transition mapping is unique. Obviously,  $m \geq 2$ .

Let us first prove that if  $\phi_1(x) = x$ , then  $x = 1$  or  $x = m$ . Indeed, suppose the opposite, and consider  $\phi_2(x)$ . If  $\phi_2(x) < x$ , then  $\phi_1|_{[1, x]}$  and  $\phi_2|_{[1, x]}$  are two different mappings, both of

which are MVEs in the same static environment restricted on the set of states  $S' = [1, x]$ . This would contradict the assertion that  $m$  is the minimal number of states for which this is possible. If  $\phi_2(x) > x$ , we get a similar contradiction by considering the subset of states  $[x, m]$ , and if  $\phi_2(x) = x$ , we get a contradiction by considering  $[1, x]$  or  $[x, m]$  depending on where  $\phi_1$  and  $\phi_2$  differ. We similarly prove that if  $\phi_2(x) = x$ , then either  $x = 1$  or  $x = m$ .

Next, let us prove that  $|x \in S : \phi_1(x) \neq \phi_2(x)| \geq 2$ . Indeed, if there was a single point  $x$  where  $\phi_1(x) \neq \phi_2(x)$ , then consider the following cases. If  $1 < x < m$ , then we must have  $\phi_1(x) \neq x$  and  $\phi_2(x) \neq x$ . Now, if  $\phi_1(x) < x$  and  $\phi_2(x) < x$ , then we get two different MVEs on  $[1, x]$ , and if  $\phi_1(x) > x$  and  $\phi_2(x) > x$ , then there are two different MVEs on  $[x, n]$ . This is impossible, and thus without loss of generality  $\phi_1(x) < x$  and  $\phi_2(x) > x$ . By monotonicity,  $\phi_1(y) < x$  for  $y < x$ , and  $\phi_2(y) > x$  for  $y > x$ . Therefore,  $\phi_1^k(y) = \phi_2^k(y)$  for any  $y \neq x$  and  $k \geq 0$  and any  $y \neq x$ , and thus  $V_i^{\phi_1}(y) = V_i^{\phi_2}(y)$  for all  $i \in N$  and  $y \neq x$ . Since the  $M_x$  is a singleton, it is (generically) impossible that  $\phi_1(x) \neq \phi_2(x)$ , as property (1) of Definition 3 would be violated for at least one of the two mappings. We get a contradiction, and the only remaining case to consider is  $x = 1$  or  $x = m$ . Suppose  $x = 1$  (the case  $x = m$  is analogous). Either  $\phi_1(1) > 1$  or  $\phi_2(1) > 1$  (or both); in either case, we have, by monotonicity, that  $\phi_1(y) \geq 2$  for  $y \geq 2$  and therefore  $V_i^{\phi_1}(y) = V_i^{\phi_2}(y)$  for all  $i \in N$  and  $y > 1$ . If both  $\phi_1(1) > 1$  and  $\phi_2(1) > 1$ , then, again, property (1) of Definition 3 would be violated for either  $\phi_1$  or  $\phi_2$ . Otherwise, without loss of generality,  $\phi_1(1) = 1$  and  $\phi_2(1) = z > 1$ . Let  $M_1 = \{j\}$ ; then  $\phi_1(1) = 1$  implies  $V_j^{\phi_1}(z | 1) \leq V_j^{\phi_1}(1) = \frac{1}{1-\beta} u_j(1)$ , and  $\phi_2(1) = z$  implies  $V_j^{\phi_2}(z | 1) \geq V_j^{\phi_2}(1) = u_j(1) + \beta V_j^{\phi_2}(z | 1)$ , which is equivalent to  $V_j^{\phi_2}(z | 1) \geq \frac{1}{1-\beta} u_j(1)$ . But  $V_j^{\phi_1}(z | 1) = V_j^{\phi_2}(z | 1)$ , and thus  $V_j^{\phi_1}(z | 1) = \frac{1}{1-\beta} u_j(1)$  which cannot hold generically. We thus proved that  $|x \in S : \phi_1(x) \neq \phi_2(x)| \geq 2$ .

We can now prove the following. Let  $\text{Im } \phi_1$  and  $\text{Im } \phi_2$  be the images of  $\phi_1$  and  $\phi_2$ , respectively. Then  $\text{Im } \phi_1 \cup \text{Im } \phi_2 = S$ . Indeed, if there was some  $s \notin \text{Im } \phi_1 \cup \text{Im } \phi_2$ , we could drop this state without changing continuation payoffs  $\left\{ V_i^{\phi_1}(x) \right\}_{i \in N}^{x \in S}$  and  $\left\{ V_i^{\phi_2}(x) \right\}_{i \in N}^{x \in S}$  for  $x \neq s$ . Then  $\phi_1|_{S \setminus \{s\}}$  and  $\phi_2|_{S \setminus \{s\}}$  would be MVE, and they would be different, as  $\phi_1(x)$  and  $\phi_2(x)$  differ for at least two states.

In what follows, let  $b(x)$  be the state that maximizes  $u_{M_x}(y)$  on  $S$  (generically, it is unique). By Assumptions 5 and 3, the sequence  $\{b(x)\}_{x=1}^m$  is nondecreasing. Therefore, it has a fixed point.

Suppose, to obtain a contradiction, that  $b(2) = 1$ . By monotonicity,  $b(1) = 1$ , and then by property (2) of Definition 3,  $\phi_1(1) = \phi_2(1) = 1$ . For  $j \in \{1, 2\}$ , denote  $a_j = \max \{s \in S : \forall x \leq s, \phi_j(x) \leq x\}$ . Then  $a_1$  and  $a_2$  are well-defined and different: indeed, if  $a_1 = a_2 = a$ , then  $\phi_1$  and  $\phi_2$  would map  $[1, a]$  to  $[1, a]$  and, unless  $a = n$ , would map  $[a + 1, n]$  to  $[a + 1, n]$ . This would lead to an immediate contradiction unless  $a = n$ , as we would again get different MVEs on either  $[1, a]$  or  $[a + 1, n]$ . If  $a = n$  and  $\phi_1$  and  $\phi_2$  differ on  $[1, n - 1]$ , we get the same contradiction. If  $a = n$  and  $\phi_1|_{[1, n-1]} = \phi_2|_{[1, n-1]}$ , then we have  $|\{x \in S : \phi_1(x) \neq \phi_2(x)\}| = 1$ , which we proved to be impossible. This proves that  $a_1 \neq a_2$ , and without loss of generality, assume  $a_1 < a_2$ .

It is easy to show that  $\phi_1|_{[1, a_1]} = \phi_2|_{[1, a_1]}$  and  $\phi_1|_{[a_2+1, n]} = \phi_2|_{[a_2+1, n]}$ . The first is true as otherwise we would have two different MVEs on  $[1, a_1]$  and  $a_1 < n$  as  $a_1 < a_2 \leq n$ . For the second, observe that if  $s > a_1$  then  $\phi_1(s) > s$  for  $s < n$  and  $\phi_1(s) = s$  for  $s = n$ . If this were not the case, then, since  $\phi_1(a_1 + 1) > a_1 + 1$ , we would have a state  $x \in [a_1 + 1, s]$  with  $\phi_1(x) = x$ , which we already ruled out. Similarly, if  $s > a_2$ , then  $\phi_2(s) > s$  for  $s < n$  and  $\phi_2(s) = s$  for  $s = n$ . This implies that if  $a_2 < n$ , then  $\phi_1|_{[a_2+1, n]} = \phi_2|_{[a_2+1, n]}$ : otherwise we would get two different MVEs on  $[a_2 + 1, n]$ .

We have proved that we must have  $a_2 - a_1 \geq 2$ : otherwise  $\phi_1$  and  $\phi_2$  would differ for only one value of  $a$ . Without loss of generality, suppose that  $a_2 - a_1$  is minimal (for given  $m$ ) among all possible environments and pairs of different MVE  $\phi_1$  and  $\phi_2$ .

We have that  $a_1 + 2 \leq a_2$ , so  $\phi_2(a_1 + 2) \leq a_1 + 2$  (with equality possible only if  $a_1 + 2 = n$ ). Consider the following cases.

Case 1:  $\phi_2(a_1 + 2) = a_1 + 1$ . Then we must have  $\phi_2(a_1 + 1) = a_1$  (indeed, if  $\phi_2(a_1 + 1) < a_1$  then  $a_1 \notin \text{Im } \phi_1$  as  $\phi(a_1) = a_1$  is impossible since then  $a_1 > 1$ , and clearly  $a_1 \notin \text{Im } \phi_2$ , but we proved that this is impossible). Likewise, we have  $\phi_2(a_1) = a_1 - 1$  (provided that  $a_1 \geq 2$ ) etc; in other words, in this case  $\phi_2(x) = x - 1$  for all  $x \in [2, a_1 + 2]$  (and  $a_1 + 2 \geq 3$ ). We must also have  $\phi_1(x) \geq 2$  for  $x > 2$ : if  $x \leq a_1$  then  $\phi_1(x) = \phi_2(x) = x - 1 \geq 2$ , and if  $x > a_1$ , then  $\phi_1(x) \geq x > 2$ . Consider state space  $\tilde{S} = [2, m]$  and define utilities by

$$\tilde{u}_i(s) = \begin{cases} (1 - \beta) u_i(2) - (1 - \beta) \beta c(2, 1) + \beta u_i(1) & \text{if } s = 2; \\ u_i(s) & \text{if } s \geq 3. \end{cases} \quad (113)$$

Consider two mappings  $\tilde{\phi}_1$  and  $\tilde{\phi}_2$  given by  $\tilde{\phi}_1(x) = \phi_1(x)$  for  $s \geq 3$ ,  $\tilde{\phi}_1(2) = \max(\phi_1(1), 2)$ , and  $\tilde{\phi}_2(x) = \phi_2(x)$  for  $s \geq 3$ ,  $\tilde{\phi}_2(2) = 2$ . Notice that  $\tilde{V}_i^{\tilde{\phi}_2}(s) = V_i^{\phi_2}(s)$  for all  $s \in \tilde{S}$ . This means



that property (2) of Definition 3 holds. If property (1) is violated then, by Lemma 3 there must be a monotone deviation. Any deviation from at  $x > 2$  from  $\tilde{\phi}_2(x)$  to some  $y$  would imply that there was a deviation for  $\phi_2$ , so  $\phi_2$  was not a MVE. At the same time, deviation at  $x = 2$  from 2 to some  $y > 2$  is not monotone, since  $\tilde{\phi}_2(3) = \phi_2(3) = 2$  in the case under consideration. Hence,  $\tilde{\phi}_2$  is a MVE. Now, consider  $\tilde{\phi}_1$ . We have  $\tilde{V}_i^{\tilde{\phi}_1}(s) = V_i^{\phi_1}(s)$ , except for  $s = 2$  if  $\phi_1(2) = 2$ . Since we picked  $\tilde{\phi}_1(2) = 2$ , we anyway conclude that property (2) of Definition 3 holds. If property (1) is violated then there is a monotone deviation. As before, any deviation at  $x > 2$  from  $\tilde{\phi}_2(x)$  to some  $y$  would lead to an immediate contradiction. However, a deviation at  $x = 2$  from 2 to some  $y > 2$  may be monotone if  $\tilde{\phi}_1(3) > 2$ . In this case, we notice that  $\tilde{\phi}_1|_{[3,n]}$  is a MVE on  $[3, n]$ , and by Lemma 5 we can have a MVE  $\tilde{\phi}'_1$  on  $[2, n]$  which coincides with  $\tilde{\phi}_1$  on  $[3, n]$ . We must have  $\tilde{\phi}'_1 = \tilde{\phi}_2$ , but then  $\phi_1$  and  $\phi_2$  coincide on  $\{1\} \cup [3, n]$ , and thus may differ at most one point. This is a contradiction.

Case 2:  $\phi_2(a_1 + 2) \leq a_1$ . Then let us modify the costs in the following way:

$$\tilde{c}(x, y) = \begin{cases} c(x, y) & \text{if } x > a_1 + 1 \text{ or } y < a_1 + 2; \\ c(x, y) + X & \text{if } x \leq a_1 + 1 \text{ and } y \geq a_1 + 2, \end{cases} \quad (114)$$

where  $X$  is a sufficiently large number; in doing so, we make transitions from states  $x \leq a_1 + 1$  to states  $y \geq a_1 + 2$  prohibitively costly. Notice that for the new environment, mapping  $\phi_2$  is still MVE. Consider mapping  $\tilde{\phi}$  given by

$$\tilde{\phi}_1(s) = \begin{cases} \phi_2(s) & \text{if } s \leq a_1 + 1; \\ \phi_1(s) & \text{if } s \geq a_1 + 2. \end{cases} \quad (115)$$

Property (2) of Definition 3 then holds because it held for  $\phi_1$  and  $\phi_2$ , and the extra cost  $X$  is not involved on equilibrium path. If property (1) is violated, there is a monotone deviation. It is clear, however, that the only candidate monotone deviation is one at  $a_1 + 2$  from  $\tilde{\phi}_1(a_1 + 2) = \phi_1(a_1 + 2)$  to some  $y \leq a_1 + 1$ . If  $y \leq a_1$ , then existence of such deviation would imply that  $\phi_2$  is not MVE, as  $\phi_1|_{[1, a_1]} = \phi_2|_{[1, a_1]}$ . The only deviation to consider is thus to  $y = a_1 + 1$ ; it would be a profitable deviation if  $V_j(a_1 + 1 | a_1 + 2) > V_j(\tilde{\phi}_1(a_1 + 2) | a_1 + 2)$ . However, since  $\phi_2(a_1 + 2) \leq a_1$ , we have (for  $z = \phi_2(a_1 + 2)$  and  $j = M_{a_1+2}$ ) that  $V_j(z | a_1 + 2) \geq V_j(a_1 + 1 | a_1 + 2)$ , and thus  $V_j(z | a_1 + 2) > V_j(\tilde{\phi}_1(a_1 + 2) | a_1 + 2)$ ; this would again imply that  $\phi_2$  is not MVE, thus giving us a contradiction. Consequently, both  $\tilde{\phi}_1$  and  $\phi_2$  are MVE in the modified environment. However, if we compute  $\tilde{a}_1$ , we would get  $\tilde{a}_1 = a_1 + 1$  (as  $\tilde{\phi}_1(a_1 + 1) =$

$\phi_2(a_1 + 1) \leq a_1 + 1$ ), and thus  $a_2 - \tilde{a}_1 < a_2 - a_1$ . This contradicts our choice of mappings so as to minimize  $a_2 - a_1$ . This contradiction proves that  $b(2) \geq 2$ .

Now, if  $b(m - 1) = m$ , we can obtain a similar contradiction. Therefore,  $b(2) \geq 2$ ,  $b(m - 1) \leq m - 1$ . By monotonicity, this implies  $m - 1 \geq 2$ , i.e.,  $m \geq 3$ . But then there is  $x \in [2, m - 1]$  such that  $b(x) = x$ . Then  $\phi_1(x) = x$ , for otherwise property (2) of Definition 3 would be violated. But, as we proved above, this is impossible. This contradiction completes the proof.

**Part 2.** As in Part 1, we can assume that  $m$  is the minimal number of states for which this is possible. We can then establish that if  $\phi_1(x) = x$ , then  $x = 1$  or  $x = m$ . If  $\phi_1(x) < x < \phi_2(x)$  or vice versa, then for all  $i \in M_x$ , there must be both a state  $x_1 < x$  and a state  $x_2 > x$  such that  $u_i(x_1) > u_i(x)$  and  $u_i(x_2) > u_i(x)$ , which contradicts the assumption in this case. Since for  $1 < x < m$ ,  $\phi(x) \neq x$ , we get that  $\phi_1(x) = \phi_2(x)$  for such  $x$ . Let us prove that  $\phi_1(1) = \phi_2(1)$ . If this is not the case, then  $\phi_1(1) = 1$  and  $\phi_2(1) = 2$  (or vice versa). If  $m = 2$ , then monotonicity implies  $\phi_2(2) = 2$ , and if  $m > 2$ , then, as proved earlier, we must have  $\phi_2(x) = x + 1$  for  $1 < x < m$  and  $\phi_2(m) = m$ . In both cases, we have  $\phi_1(x) = \phi_2(x) > 1$  for  $1 < x \leq m$ . Hence,  $V_i^{\phi_1}(2) = V_i^{\phi_2}(2)$  for all  $i \in N$ . Since  $\phi_1$  is MVE, we must have  $u_i(1) / (1 - \beta) \geq V_i^{\phi_1}(2) - c(1, 2)$  for  $i \in M_1$ , and since  $\phi_2$  is MVE, we must have  $V_i^{\phi_2}(2) \geq u_i(1) / (1 - \beta) - c(1, 2)$ . Generically, this cannot hold, and this proves that  $\phi_1(1) = \phi_2(1)$ . We can likewise prove that  $\phi_1(m) = \phi_2(m)$ , which implies that  $\phi_1 = \phi_2$ . This contradicts the hypothesis of non-uniqueness. ■

**Proof of Lemma 6.** For a monotone MVE, Lemma 2 implies that  $\{V_i(s)\}_{i \in N}^{s \in S}$  and  $\{V_i(\phi(s) | s)\}_{i \in N}^{s \in S}$  satisfy WID condition. Now, consider the difference

$$\begin{aligned} & \sum_{j \in N} \lambda_{E, E'}(i, j) V_j^\phi(\phi(s') | s') - \sum_{j \in N} \lambda_{E, E'}(i, j) V_j^\phi(\phi(s) | s) \\ &= \sum_{j \in N} \lambda_{E, E'}(i, j) \left( V_j^\phi(\phi(s') | s') - V_j^\phi(\phi(s) | s) \right) \end{aligned} \quad (116)$$

for  $s' > s$ . The right-hand-side of (116) is a mathematical expectation, with the distribution  $\lambda_{E, E'}(i, j)$  over  $j \in N$ , of a function  $V_j^\phi(\phi(s') | s') - V_j^\phi(\phi(s) | s)$  which is increasing by  $j$  as already established. By Assumption 1, (116) is (weakly) increasing, thus proving the statement. ■

**Proof of Theorem 3.** The existence is proved in the text. Since, on equilibrium path, there is only a finite number of shocks, then from some period  $t$  on, the environment will be

the same, some  $E^x$ . Since  $\phi_{E^x}$  is monotone, the sequence  $\{s_t\}$  has a limit by Theorem 1. The fact that this limit may depend on the sequence of shocks realization may be shown by a simple example. ■

**Proof of Theorem 4. Part 1.** Without loss of generality, suppose that  $h$  is the minimal number for which two monotone MVE  $\phi = \{\phi_E\}_{E \in \mathcal{E}}$  and  $\phi' = \{\phi'_E\}_{E \in \mathcal{E}}$  exist. If we take  $\tilde{\mathcal{E}} = \{E^2, \dots, E^h\}$  with the same environments  $E^2, \dots, E$  and the same transition probabilities, we will have a unique monotone MVE  $\tilde{\phi} = \{\phi_E\}_{E \in \tilde{\mathcal{E}}} = \{\phi'_E\}_{E \in \tilde{\mathcal{E}}}$  by assumption. Now, with the help of transformation used in the proof of 3 we get that  $\phi_{E^1}$  and  $\phi'_{E^1}$  must be MVE in a certain stationary environment  $E'$ . However, by Theorem 2 such MVE is unique, which leads to a contradiction.

**Part 2.** The proof is similar to that of Part 1. The only step is that we need to verify that we can apply Part 2 of Theorem 2 to the stationary environment  $E'$ . In general, this will not be the case. However, it is easy to notice (by examining the proof of Part 2 of Theorem 2) that instead of single-peakedness, we could require a weaker condition: that for each  $s \in S$  there is  $i \in M_s$  such that there do not exist  $x < s$  and  $y > s$  such that  $u_i(x) \geq u_i(s)$  and  $u_i(y) \geq u_i(s)$ . We can then prove that if  $\{u_i(s)\}_{i \in N}^{s \in S}$  satisfy this property and  $\phi$  is MVE, then  $\{V_i^\phi(s)\}_{i \in N}^{s \in S}$  also does. The rest of the proof follows. ■

**Proof of Theorem 5. Part 1.** It suffices to prove this result for stationary case. For each  $s \in S$  take any protocol such that if  $\phi(s) \neq s$ , then  $\theta_s(m-1) = \phi(s)$  (i.e., the desired transition is considered last). We claim that there is a strategy profile  $\sigma$  such that if for state  $s$ ,  $\phi(s) = s$ , then no proposal is accepted in periods where  $s_t = s$ , and if  $\phi(s) \neq s$ , then no proposal but the last one,  $\phi(s)$ , is accepted, and the last one is accepted. Indeed, under such profile, the continuation strategies are given by (16) and (17). Hence, if the state is  $s$  such that  $\phi(s) = s$ , there is no winning coalition that wants to have any other alternative  $x \neq s$  accepted. If the state is  $s$  such that  $\phi(s) \neq s$ , then, anticipating that  $\phi(s)$  will be accepted over  $s$  in the last voting round, no winning coalition has an incentive to deviate and potentially support another state  $x \neq \phi(s)$ ; at the last round, however,  $\phi(s)$  would be supported because of property (2) of Definition 3.

To prove that protocol (35) will suffice if the equilibrium is unique, we make the following observation. Close inspection of the proof of Theorem 1 reveals that we could actually prove

existence of monotone MVE which satisfies an additional requirement: If  $x < y < \phi(x)$  or  $x > y > \phi(x)$ , then

$$\left\{ i \in N : V_i^\phi(\phi(x)) - c(x, \phi(x)) \geq V_i^\phi(y) - c(x, y) \right\} \in W_{E,x}. \quad (117)$$

If the equilibrium is unique, it satisfy this additional constraint. It is then straightforward to prove that protocol (35) would suffice.

**Part 2.** If the transition mapping is monotone, then continuation utilities  $\left\{ V_{E,i}^\phi(s) \right\}_{i \in N}^{s \in S} = \left\{ V_{E,i}^\sigma(s) \right\}_{i \in N}^{s \in S}$  satisfy WID for any  $E \in \mathcal{E}$ . Again, the proof that  $\phi$  is MVE reduces to the stationary case. For each state  $s$ , let us define a resolute irreflexive binary relation  $\succ_s$  on  $S$  as follows:  $x \succ_s y$  if either  $V^\phi(x | s) >_s V^\phi(y | s)$  or  $(V^\phi(y | s) \not>_s V^\phi(x | s)$  and for some  $a < b$ ,  $\theta_s(a) = x$  and  $\theta_s(b) = y$ ), with the convention that  $\theta_s(0) = s$ . In other words,  $\succ_s$  resolves  $>_s$  for continuation values by giving precedence to states which are voted earlier in the protocol  $\theta_s$ . The theory of amendment agendas (see Shepsle and Weingast, 1984, and Austen-Smith and Banks, 1999) implies that  $\phi(s)$  must be the state that satisfies both properties of Definition 3. The details are omitted to save space.

**Part 3.** The proof uses theory of amendment agendas, but otherwise similar to the proof of Theorem 9 and is omitted. ■

**Proof of Theorem 6.** Suppose, to obtain a contradiction, that  $\phi_2(x) < x$ . Then  $\phi_1|_{S'}$  and  $\phi_2|_{S'}$  are mappings from  $S'$  to  $S'$  such that both are MVE. Moreover, they are different, as  $\phi_1(x) = x > \phi_2(x)$ . However, this would violate the assumed uniqueness (either assumption needed for Theorem 2 continues to hold if the domain is restricted), which completes the proof. ■

**Proof of Corollary 1.** Consider an alternative set of environments  $\mathcal{E}' = \{E^0, E^2\}$ , where  $E^0$  coincides with  $E^2$  on  $S$ , but the transition probabilities are the same as in  $\mathcal{E}$ . Clearly,  $\phi'$  such that  $\phi'_{E^0} = \phi'_{E^2} = \phi_{E^2}$  is a MVE in  $\mathcal{E}'$ . Let us now consider stationary environments  $\tilde{E}^0$  and  $\tilde{E}^1$  obtained from  $\mathcal{E}'$  and  $\mathcal{E}$ , respectively, using the procedure from the proof of Theorem 3. Suppose, to obtain a contradiction, that  $\phi_{E^2}(x) < x$ , then environments  $\tilde{E}^0$  and  $\tilde{E}^1$  coincide on  $[1.s]$  by construction. Theorem 6 then implies that, since  $\phi_{E^1}(x) = x$ , then  $\phi'_{E^0}(x) \geq x$  (since  $\phi'_{E^0}$  and  $\phi_{E^1}$  are the unique MVE in  $\tilde{E}^0$  and  $\tilde{E}^1$ , respectively). But by definition of  $\phi'$ ,  $x < \phi'_{E^0}(x) = \phi_{E^2}(x) \leq x$ , a contradiction. This contradiction completes the proof. ■

**Proof of Theorem 7.** Let us first prove this result for the case where each QMV is a singleton. Both before and after the shock, the mapping that would map any state  $x$  to a state which maximizes the instantaneous payoff  $u_{M_x}(y) - c(x, y)$  would be a monotone MVE for  $\beta < \beta_0$ . By uniqueness,  $\phi_{E^1}$  and  $\phi_{E^2}$  would be these mappings under  $E^1$  and  $E^2$ , respectively. Now it is clear that if the shock arrives at period  $t$ , and the state at the time of shock is  $x = s_{t-1}$ , then  $\phi_{E^2}(x)$  must be either the same as  $\phi_{E^1}(x)$  or must satisfy  $\phi_{E^2}(x) > s$ . In either case, we get a monotone sequence after the shock. Moreover, the sequence is the same if  $s_\tau \leq s$ , and if  $s_\tau > s$ , then we have  $s_\tau > s \geq \tilde{s}_\tau$  automatically.

The general case may be proved by observing that a mapping that maps each state  $x$  to an alternative which maximizes by  $u_{\min M_x}(y) - c(x, y)$  among the states such that  $u_i(y) - c(x, y) \geq u_i(x)$  for all  $i \in M_x$  is a monotone MVE. Such mapping is generically unique, and by the assumption of uniqueness it coincides with the mapping  $\phi_{E^1}$  if the environment is  $E^1$  and it coincides with  $\phi_{E^2}$  if the environment is  $E^2$ . The remainder of the proof is analogous. ■

**Proof of Theorem 8.** It is sufficient, by transitivity, to prove this Theorem for the case where  $\max M_{E^1, x} \neq \max M_{E^2, x}$  for only one state  $x \in [s + 1, m]$ . Moreover, without loss of generality, we can assume that  $\max M_{E^1, x} < \max M_{E^2, x}$ . Notice that if  $\phi_1(x) \geq x$ , then  $\phi_1$  is MVE in environment  $E^2$ , and by uniqueness must coincide with  $\phi_2$ .

Consider the remaining case  $\phi_1(x) < x$ ; it implies  $\phi_1(x - 1) \leq x - 1$ . Consequently,  $\phi_1|_{[1, x-1]}$  is MVE under either environment restricted on  $[1, x - 1]$  (they coincide on this interval). Suppose, to obtain a contradiction, that  $\phi_1|_{[1, s]} \neq \phi_2|_{[1, s]}$ ; since  $x > s$ , we have  $\phi_1|_{[1, x-1]} \neq \phi_2|_{[1, x-1]}$ . We must then have  $\phi_2(x - 1) > x - 1$  (otherwise there would be two MVEs  $\phi_1|_{[1, x-1]}$  and  $\phi_2|_{[1, x-1]}$  on  $[1, x - 1]$ , and therefore  $\phi_2(x) \geq x$ . Consequently,  $\phi_2|_{[x, m]}$  is MVE on  $[x, m]$  under environment  $E^2$  restricted on  $[x, m]$ . Let us prove that  $\phi_2|_{[x, m]}$  is MVE on  $[x, m]$  under environment  $E^1$  restricted on  $[x, m]$  as well. Indeed, if it were not the case, then there must be a monotone deviation, as fewer QMV (in state  $x$ ) imply that only property (1) of Definition 3 may be violated. Since under  $E^1$ , state  $x$  has fewer quasi-median voters than under  $E^2$ , it is only possible if  $\phi_2(x) > x$ , in which case  $\phi_2(x + 1) \geq x + 1$ . Then  $\phi_2|_{[x+1, m]}$  would be MVE on  $[x + 1, m]$ , and by Lemma 5 we could get MVE  $\tilde{\phi}_2$  on  $[x, m]$  under environment  $E^1$ . This MVE  $\tilde{\phi}_2$  would be MVE on  $[x, m]$  under environment  $E^2$ . But then under environment  $E^2$  we have two MVE,  $\tilde{\phi}_2$  and  $\phi_2|_{[x, m]}$  on  $[x, m]$ , which is impossible.

We have thus shown that  $\phi_1|_{[1, x-1]}$  is MVE on  $[1, x - 1]$  under both  $E^1$  and  $E^2$ , and the

same is true for  $\phi_2|_{[x,m]}$  on  $[x, m]$ . Take mapping  $\phi$  given by

$$\phi(y) = \begin{cases} \phi_1(y) & \text{if } y < x \\ \phi_2(y) & \text{if } y > x \end{cases}. \quad (118)$$

Since  $\phi_1|_{[1,x-1]} \neq \phi_2|_{[1,x-1]}$  and  $\phi_1|_{[x,m]} \neq \phi_2|_{[x,m]}$  ( $\phi_1(x-1) \leq x-1$ ,  $\phi_2(x-1) > x-1$ ,  $\phi_1(x) < x$ ,  $\phi_2(x) \geq x$ ),  $\phi$  is not MVE in  $E^1$  nor it is in  $E^2$ . By Lemma 4, in both  $E^1$  and  $E^2$  only one type of monotone deviation (at  $x-1$  to some  $z \in [x, \phi_2(x)]$  or at  $x$  to some  $z \in [\phi_1(x-1), x]$ ) is possible. But the payoffs under the first deviation is the same under both  $E^1$  and  $E^2$ ; hence, in both environments it is the same type of deviation.

Suppose that it is the former deviation, at  $x-1$  to some  $z \in [x, \phi_2(x)]$ . Consider an increase in the cost

$$\tilde{c}(a, b) = \begin{cases} c(a, b) & \text{if } a \geq x \text{ or } b < x; \\ c(a, b) + X & \text{if } a < x \text{ and } b \geq x, \end{cases} \quad (119)$$

where  $X$  is sufficiently large; denote the resulting environments by  $\tilde{E}^1$  and  $\tilde{E}^2$ . This makes the deviation impossible, and thus  $\phi$  is MVE in  $\tilde{E}^1$  (in  $\tilde{E}^2$  as well). However,  $\phi_1$  is also MVE in  $\tilde{E}^1$ , as it is not affected by the increase in cost, and this contradicts uniqueness. Finally, suppose that the deviation is at  $x$  to some  $z \in [\phi_1(x-1), x]$ . Then consider the costs

$$\bar{c}(a, b) = \begin{cases} c(a, b) & \text{if } a < x \text{ or } b \geq x; \\ c(a, b) + X & \text{if } a \geq x \text{ and } b < x, \end{cases} \quad (120)$$

where  $X$  is again sufficiently large; denote the resulting environments by  $\bar{E}^1$  and  $\bar{E}^2$ . This makes the deviation impossible, and thus  $\phi$  is MVE in  $\bar{E}^2$ . However,  $\phi_2$  is also MVE in  $\bar{E}^1$ , as it is not affected by the increase in cost. Again, this contradicts uniqueness, which completes the proof. ■

**Proof of Corollary 2.** The proof is similar to the proof of Corollary 1 and is omitted. ■

**Proof of Theorem 9. Part 1.** It suffices to prove this result in stationary environments. Suppose MVE  $\phi$  is nonmonotone, which means there are states  $x, y \in S$  such that  $x < y$  and  $\phi(x) > \phi(y)$ . By property (1) of Definition 3 applied to state  $x$ , we get

$$V_{\max M_x}(\phi(x)) - c(x, \phi(x)) \geq V_{\max M_x}(\phi(y)) - c(x, \phi(y)), \quad (121)$$

and if we apply it to state  $y$ ,

$$V_{\min M_y}(\phi(y)) - c(y, \phi(y)) \geq V_{\min M_y}(\phi(x)) - c(y, \phi(x)). \quad (122)$$

Since  $\max M_x \leq \min M_y$  by assumption, (121) implies

$$V_{\min M_y}(\phi(x)) - c(x, \phi(x)) \geq V_{\min M_x}(\phi(y)) - c(x, \phi(y)). \quad (123)$$

Since in the generic case inequalities are strict, adding (122) and (123) yields

$$-c(x, \phi(x)) - c(y, \phi(y)) > -c(x, \phi(y)) - c(y, \phi(x)). \quad (124)$$

This, however, is equivalent to

$$-(c(x, \phi(x)) - c(x, \phi(y))) > -(c(y, \phi(x)) - c(y, \phi(y))), \quad (125)$$

thus contradicting Assumption 6 (as  $\phi(x) > \phi(y)$  and  $y < x$ ).

**Part 2.** Again, consider stationary environments only. If  $\phi$  is monotone, then for some  $x, y \in S$  we have  $x < y$  and  $\phi(x) > \phi(y)$ , which in this case implies  $\phi(x) = y = x + 1$  and  $\phi(y) = x$ . Property 2 of Definition 3, when applied to state  $x$ , implies that for all  $i \in M_x$ ,

$$V_i(y) - c(x, y) \geq V_i(x). \quad (126)$$

This means that generically, for all  $i \in M_y$ ,

$$V_i(y) - c(x, y) > V_i(x). \quad (127)$$

The same property 2, when applied to state  $y$ , implies that for all  $i \in M_y$ ,

$$V_i(x) - c(y, x) \geq V_i(y). \quad (128)$$

But (128) contradicts (127) as costs are nonnegative. This contradiction completes the proof.

■

**Proof of Theorem 10.** Take any MVE  $\phi$ . Suppose, to obtain a contradiction, that for some  $x$ ,  $\phi(x) > x$ , but  $\phi^2(x) < \phi(x)$  (the other case is considered similarly). Denote  $y = \phi(x)$  and  $z = \phi(y)$ . By property (2) of Definition 3 applied to state  $y$ , for all  $i \in M_y$ ,

$$V_i(z) - c(y, z) \geq V_i(y). \quad (129)$$

This means that (129) holds for all  $i \in M_x$ . However, property (1) of Definition 3, applied to state  $x$ , implies that, generically, at least for one  $i \in M_x$ ,

$$V_i(y) - c(x, y) > V_i(z) - c(x, z). \quad (130)$$

Together, (129) and (130)

$$-c(y, z) - c(x, y) > -c(x, z). \quad (131)$$

However, Assumption 6 implies (as  $y > z$  and  $x < y$ ) that

$$c(x, y) - c(x, z) \geq c(y, y) - c(y, z), \quad (132)$$

which contradicts (131). This contradiction completes the proof. ■

**Proof of Theorem 11.** Take an increasing sequence of sets of points,  $S_1 \subset S_2 \subset S_3 \subset \dots$ , so that  $\bigcup_{i=1}^{\infty} S_i$  is dense. For each  $S_i$ , take MVE  $\phi_i$ . We know that  $\phi_i$  is a monotone function on  $S_i$ ; let us complement it to a monotone (not necessarily continuous) function on  $S$  which we denote by  $\tilde{\phi}_i$  for each  $i$ . Since  $\tilde{\phi}_i$  are monotone functions from a bounded set to a bounded set, there is a subsequence  $\tilde{\phi}_{i_k}$  which converges to some  $\tilde{\phi}$  pointwisely. (Indeed, we can pick a subsequence which converges on  $S_1$ , then a subsequence converging on  $S_2$  etc; then use a diagonal process. After it ends, the set of points where convergence was not achieved is at most countable, so we can repeat the diagonal procedure.) To show that  $\tilde{\phi}$  is a MVE, suppose not, then there are two points  $x$  and  $y$  such that  $y$  is preferred to  $\tilde{\phi}(x)$  by all members of  $M_x$ . Here, we need to apply a continuity argument and say that it means that the same is true for some points in some  $S_i$ . But this would yield a contradiction. ■

**Proof of Theorem 12.** The proof follows from Kakutani's theorem and is omitted. ■