Prizes, Groups and Pivotal Voting in a Poisson Voting Game

Alastair Smith Tom LaGatta Bruce Bueno de Mesquita

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Abstract

We model elections between two parties in a Poisson random population of voters (Myerson 1998, 2000). In addition to offering different policy benefits, parties offer contingent prizes to those identifiable groups of voters that offer the highest level of political support. In large populations, voters are only likely to influence the electoral outcome when the vote share between two parties is perfectly equal and even then their influence on the outcome is small. In contrast voters retain significant influence over the distribution of prizes even in lopsided elections. Equilibrium behavior is driven by voters competing to win preferential treatment for their group and not by policy concerns. The results address variance in turnout in elections, political rewards and the persistence of dominant parties even when they are popularly perceived as inferior.

1 Introduction

We present a random population model of voting and examine the consequences of parties targeting selective rewards to those groups of voters that offer them the highest level of electoral support. The model explains voting behavior in both competitive electoral systems, such as those in established democracies, and patronage styled democracies in which a dominant party persists in the presence of free and fair elections even when it is widely regarded by citizens as inferior to other parties. The model resolves turnout problems seen in standard accounts of rational voting. In particular, we explain variance in voter turnout, including high turnout, in large electorates in both competitive and non-competitive elections. The model also resolves credibility issues in vote buying accounts within patronage democracies and explains the persistence of dominant parties.

The basic setup integrates Smith and Bueno de Mesquita's (2012) concept of Contingent Prize Allocation Rules (CPAR) and Myerson's (1998, 2000) Poisson random population of voters model. In the simple CPAR examined here, parties provide selective rewards (a.k.a. prizes) to identifiable groups of voters based on their level of electoral support. One simple illustration of such a grouping is geographical districts. In established democracies parties rarely observe the vote choice of individuals. Yet, they readily observe the level of electoral support at the ward or precinct level. Under the CPAR applied here, parties reward the most supportive group. For instance, in the geographical context they might allocate new infrastructure projects such as a new school or hospital to the most supportive precinct within a larger electoral district. Alternatively they could reward a particular ward or precinct through the provision of superior services, such as better trash pickup or more reliable street plowing. Parties might also disproportionately hire public employees from the most supportive group. This was a standard practice within the party machines which dominated many large US cities (Allen 1993). Richard J. Daley, the long term mayor of Chicago, was notorious in this respect (Rakove 1975). This is perhaps unsurprising since the internal rules of the Democratic Party of Cook County (which contains Chicago) specifies that on committees, ward representatives are given voting rights in proportion to the level of democratic votes their ward delivered in previous elections (Gosnell 1937). US national parties also structure rules to reward their loyalists. For instance, both parties skew Presidential nomination procedures in favor of states that gave them high levels of support in previous elections (see for instance, Democratic Party Headquarters 2007).

Although the model is applicable to any set of groups whose electoral support is observable, we focus on geographical groupings for ease of exposition. Others, in a different context, have shown that non-geographic group voting can also be observable (Chandra 2004). Groups might, therefore, be based on religion, ethnicity or profession instead of geographical voting districts. What matters is that parties can, in the aggregate, observe who supports them and can selectively reward groups.

The analysis takes the form of a rational choice vote calculation in which voters individually compare the benefits of voting for each party with the costs of casting a vote (Downs 1957; Riker and Ordeshook 1968). A classic criticism of such an approach, and one of Green and Shapiro's (1996) main arguments against rational choice modeling, is that in a large electorate voters are extremely unlikely to be pivotal. Hence, if voting is costly and their vote is extremely unlikely to matter, then critics questions why voters turn out. We provide a rational choice model in which voters are unlikely to be pivotal in altering who wins and yet still turn out to vote. Rather than concerns only over which party wins, at least some voters in our model are motivated by the influence they have over the distribution of contingent prizes. The model provides a solution to the voter-turnout puzzle.

The model considers different form of pivots. Consistent with standard models, we examine a voter's influence on which party wins the election. We refer to this as the outcome pivot. As Schwartz (1987) argues, vote choice might matter at a sub-constituency level. If parties reward supportive groups, then an individual voter's vote choice affects the distribution of prizes. The extent to which an additional vote changes the distribution of prizes is referred to as the prize pivot.

Smith and Bueno de Mesquita (2012) illustrate how voters retain their prize pivotalness even in large electorates with a stylized example of three villages each with n voters. The victorious party offers to build a hospital, or other project, in the village that gives it the most votes. There is an equilibrium in which all voters support one party and the pivotalness of the vote choice is 1/3 even if the electorate is large. In this illustration, no single voter is influential in affecting which party wins. Yet, by voting for the dominant party each voter gives their village a 1/3 chance of receiving the prize. If they abstain or vote for another party, then their village has one fewer votes than the other villages and so their village has no chance of receiving the prize. Provided that the cost of voting is less than 1/3 of the value of the prize, all voters strictly want to support the dominant party.

Smith and Bueno de Mesquita's (2012) example is highly stylized and their model is of limited generality. It considers only three groups and everyone is assumed to turn out to vote. Here we develop a more general model with broader implications. We introduce two types of uncertainty and vary both the number of voter blocs and the nature of contingent prizes. In doing so, we draw out new results regarding endogenous voter-bloc formation, the optimal number of blocs from an office-seeker's perspective, the distribution of rival and non-rival prizes across blocs and variations in the level of voter turnout within each group and across the electorate.

Rather than assume perfectly informed voters and politicians, we model uncertainty about bloc sizes and voter preferences. We treat the population size (and therefore the population of each group) as a Poisson random variable (to be explained below). Given this assumption, no one is quite certain how many voters there are in each group. Second, each voter has a personal -and private-evaluation of one party relative to the other, which we also model as a random variable. We impose minimal assumptions on the probability distribution describing voters' preferences.

Myerson (1998, 2000) shows that treating population size as a Poisson random variable creates a flexible framework within which it is straightforward to analyze pivotal voting decisions. For instance, he shows how the Poisson approach avoids the messy combinatoric calculations involved in large fixed population voting models (Palfrey and Rosenthal 1983, 1985; Ledyard 1984). Consistent with these models he finds that in large electorates, voters are only outcome pivotal in very close elections and even then their influence is small; so even a small cost of voting discourages significant turnout. The key to the analyses presented here is that while outcome pivotalness quickly approaches zero as the electorate grows, prize pivotalness goes to zero much more slowly. In particular if n is expected group size then prize pivotalness is proportional to $\frac{1}{\sqrt{n}}$, so it decays relatively slowly in terms of population size. In contrast, outcome pivot is proportional to $\frac{1}{\sqrt{n_T}}e^{-n_T(\sqrt{p}-\sqrt{q})^2}$, where n_T is expected population size and p and q are the probabilities that voters support parties $\mathcal{A}$ and $\mathcal{B}$ respectively. Except in the case of perfect electoral balance ($p = q$), outcome pivotalness declines at an exponential rate as the expected population size increases. Therefore, except in extremely close elections or very small electorates, prize pivots dominate outcome pivot. The larger the electorate the greater the importance of the prize pivot. We formally develop the concept of prize pivotalness and derive approximations for the extent to which voters influence electoral outcomes and the distribution of prizes in large electorates of uncertain size. A central result is that as the electorate increases in size, the motivation to vote is increasingly focused on competition for selective rewards rather than on the policy differences between parties. We explore these ideas both in competitive elections, in which parties are anticipated to receive roughly similar numbers of votes, and in non-competitive elections, where one party dominates. Contingent prizes increase turnout in both settings.

In non-competitive elections, as we will see, dispensing prizes can result in stable patronage-style arrangements in which one party virtually always wins. In this setting, voters do not vote solely in the hope of influencing the electoral outcome. Rather, they also vote to increase their group's chance of receiving a prize. Indeed, for some voters in non-competitive elections, this can be their only reason for turning out. We examine the equilibria with a dominant party in two settings. First, we consider symmetric equilibria in which all groups support parties at the same rate. Following that, we explore equilibria in which different groups support the parties at different rates. Effectively, in this latter setting the groups polarize. Some groups support party $\mathcal{A}$ while other groups predominantly direct their votes to the alternative party. Such polarization can result in a stable electoral arrangement where one party virtually always wins. This results offers an alternative to a Schelling (1971) style segregation model where polarization occurs via migration. The coordination of votes into loyalist and opposition groups is an equilibrium feature of the choices of individual voters who are maximizing their welfare. The alignment of groups affects the turnout rate and vote choices of members of those groups.

In competitive elections groups are more diverse in their motivation for voting. Policy preferences, pursuit of contingent prizes and general party affiliation all influence the decision to vote. Whereas turnout in non-competitive elections is strongly influenced by the probability of receiving the contingent prize and also by size of the prize, this is less true in competitive elections. Specifically, independent voters – those whose groups do not clearly identify with either party – are less likely in our model to turn out in comparison to groups of party loyalists.

Our analyses provide insight into the optimal arrangement of groups from the perspective of parties. When rewards are non-rival so that one member's enjoyment of a prize does not diminish another members enjoyment, parties prefer national level competition between a relatively small number of roughly even sized groups, five being the optimal number of groups. In contrast, if each member of the group needs to be individually rewarded because the prize is rival in nature, such as transfer payment or better services, then parties prefer the political system to be broken down into a large number of small competitions, within each of which many groups compete.

The final section contrasts the winner-take-all style of CPAR considered throughout the paper and shows that, from the perspective of parties, its is superior to other intuitively appealing CPAR.

2 Literature Review

Pivotality lies at the heart of rational choice models of voting behavior (Downs, 1957; Aldrich 1993; Riker and Ordeshook 1968; Ferejohn and Fiorina 1974). Voters not only assess their expected rewards under each party, they also factor in the likelihood that their vote matters. Formally, a vote only matters if it breaks a tie or turns defeat into a tie. Under all other circumstances, an additional vote is immaterial in determining which party wins. In a large electorate, even if the outcome is expected to be close, the probability that a voter's vote matters is extremely small (Myerson 1998, 2000). Although Myatt (2011), building off an earlier result by Good and Mayer (1975), suggests that in the presence of uncertainty about the relative popularity of the parties, pivot probabilities do not go to zero as quickly as in perfect information models, the likelihood of influencing an electoral outcome is small. Other scholars argue that electoral influence is best assessed using statistical predictions based upon forecasts of vote shares (Gelman, King and Boscardin 1998). For instance, Gelman, Silver and Edlin (2010) suggest that in the 2008 US Presidential election the average voter had about a one in 60 million chance of influencing the outcome although this figure varied greatly by state.

However pivotality is calculated, voters are expected to have only a minuscule influence on outcomes. Voters might therefore be expected to abstain.[1] That electoral turnout is much higher than anticipated by such approaches is a central critique of rational choice (Green and Shapiro 1996; see Geys 2006 for a survey of this literature). In electoral systems with a dominant party that continually wins this critique is especially pertinent. We propose a solution to this puzzle of variation in rational voter turnout both in competitive and non-competitive elections.

Many branches of the voting literature consider factors beyond pure policy comparisons of parties. Castanheira (2003) and Razin (2003), for instance, suggest the signaling value of voting is important because the margin of victory influences policy implementation. Others point to voters being motivated by personal or local benefits, such as patronage and pork (Ferejohn 1974; Fenno 1978; Schwartz 1987; Stokes 2005, 2007). Most relevant to the discussion here is Schwartz's (1987) expected utility model in which he argues that voters care about how their precinct votes in terms of potentially courting favor from the victorious party. He shifts the focus of turnout from the global policy difference between parties to the selective provision of local public goods or club goods to sub-electorates. Although he maintains a focus on pivotality with regards to which party wins, his decision-theoretic approach concentrates on a smaller,

[1] Edlin, Gelman and Kaplan (2005) argue that if each voter cares about the welfare of other voters then as the number of voters increases pivotality declines but, to counteract this, the stakes rise. They argue these offsetting factors mean the incentives to turnout and to vote are relatively constant for all electorate sizes.

local level of analysis where voters are likely to be more influential than at the macro level. Our strategic analysis focuses on pivotality with regard to the distribution of prizes.

The vote buying literature assesses where parties can most effectively buy electoral support (Ansolabehere and Synder. 2006; Myerson 1993, Dekel et al 2008, Kovenock and Roberson 2009, Cox and McCubbins 1986, Lindbeck and Weibull 1987, and Dixit and Londregan 1995, 1996). One common question is whether parties increase their vote share more by offering turnout-inducing rewards to party loyalists or to marginal voters, a swing in whose vote might be critical. Such approaches treat the parties as strategic competitors while the voters respond to rewards in a non-strategic manner. Consistent with the critique of pivotal voting, although these vote buying tactics increase the attractiveness of one party relative to another, they do not mitigate the problem that individual voters have little influence over electoral outcomes. Given the low probability of influencing the outcome, making a party more attractive only minimally increases the incentive to vote for it. Further, such approaches fail to explain voter turnout in non-competitive elections in which one party is virtually certain to win.

Pork or patronage rewards are often proffered for voter support (Ferejohn 1974; Fenno 1978; Kitschelt and Wilkinson 2007; Stokes 2005). By offering upfront bribes and the prospects of rewards, such as jobs or better services, after the election, patronage based parties directly influence voters. Even if such targeted rewards might be economically inefficient relative to public goods (Lizzeri and Persico 2001), vote buying is politically valuable because it obfuscates the pivotality issue. Since the quid pro quo of benefits for votes is carried out at the individual level, pivotality is not relevant for patronage models. Parties buy individual votes rather than make themselves electorally more attractive. As such, voters don't discount the value of the party by the likelihood that their vote is influential. Yet a number of credibility issues surround patronage vote buying (Stokes 2005, 2007).

At least in established democracies, all of which have a secret ballot, once a voter enters the voting booth, parties can not observe whether the voter delivers the promised vote (although see Gerber et al 2009). Neither can the voter be certain that a party will deliver its promised rewards after the election. Norms and reciprocity are often offered as solutions to these credibility issues (see Kitschelt and Wilson 2007 for a reviews) and scholars such as Stokes have developed repeated play models to explicitly address these concerns (Stokes 2005). However, other problematic issues remain. For example, relatively few voters receive goods from the party. Stokes (2005 p. 315) illustrates the problem with the example of an Argentinean party worker given ten tiny bags of food with which to buy the 40 voters in her neighborhood. Further, survey evidence by Brusco, Nazareno and Stokes (2004) suggests that the receipt of bribes does not guarantee that voters support the party. Similarly, Guterbock (1980) found that Chicago residents who received party service were no more likely to vote Democratic than those receiving no favors. The contingent prize allocation perspective resolves these issues. It provides an equilibrium mechanism for credibly rewarding voters. Even though individual votes cannot be observed, voters support parties in the hope of winning prizes for their group.

3 Basic Setup

An election takes place between two parties, $\mathcal{A}$ and $\mathcal{B}$, for a single office. All voters have the option of voting for party $\mathcal{A}$, voting for party $\mathcal{B}$ or abstaining. Each voter pays a cost c to vote; abstention is free. The outcome of the election is determined by a plurality of the votes cast, with ties decided by a coin flip.

There is a large number of voters who are divided into K roughly equal sized groups. These groups are indexed $1, 2, .., K$. Although these groups might be based on any underlying societal cleavage, they can simply be thought of as geographically based wards within an electoral district.

Group k has size N_k which we treat as an unknown Poisson random variable with mean n_k. Therefore, $\Pr(N_k = x) = f_{n_k}(x) = \frac{n_k^x}{x!}e^{-n_k}$ and $\Pr(N_K \leq x) = F_{n_k}(x) = \frac{\Gamma(x+1,n_k)}{x!} = \sum_{z=0}^{x} f_{n_k}(z)$ where Γ is the incomplete gamma function. The total number of voters is $N_T = \sum_{k=1}^{K} N_k$, which, by the aggregation property of the Poisson distribution (Johnson and Kotz 1993), is also a Poisson random variable with mean $n_T = \sum_{k=1}^{K} n_k$. Note that to avoid confusion, we emphasize the expected size of the entire population with subscript T.

Let p_k represent the average probability that members of group k vote for $\mathcal{A}$. Let q_k represent the probability that members of k vote for party $\mathcal{B}$. By the decomposition property of the Poisson distribution (Johnson and Kotz 1993), A_k, the number of votes for party $\mathcal{A}$ in group k, is a Poisson random variable with mean $p_k n_k$. We use the notation $(p, q) = ((p_1, q_1), ..., (p_K, q_K))$ as the profile of vote probabilities and $(A, B) = ((A_1, B_1), ..., (A_K, B_K))$ as the profile of actual votes. Party $\mathcal{A}$ wins the election if it receives more votes than party $\mathcal{B}$ ($\sum_{k=1}^{K} A_k > \sum_{k=1}^{K} B_k$). Ties are resolved by a coin flip. The goal of this paper is to characterize profiles of vote probabilities that can be supported under Nash equilibrium and show how these equilibria vary with the mechanisms that parties use to distribute rewards.

Voters care both about policy benefits and any potential prizes the parties distribute. With regard to policy benefits, voter i receives a policy reward of $\gamma + \varepsilon_i$ if party $\mathcal{A}$ wins the election and a policy payoff of 0 if $\mathcal{B}$ wins. The γ term represents the average evaluation of party $\mathcal{A}$ relative to party $\mathcal{B}$. The random variable ε_i represents individual i's private evaluation of party $\mathcal{A}$ relative to party $\mathcal{B}$. We assume the individual evaluations are independently identically distributed with distribution $\Pr(\varepsilon_i < r) = G(r)$. To avoid the need to introduce additional notation for mixed strategies, we assume $G(r)$ is smooth, strictly increasing and continuous. For illustration, all the examples are constructed assuming a Gaussian distribution.

In addition to policy benefits, individuals care about the benefits and rewards that parties might provide to specific groups of voters. The nature of these benefits can vary widely and often depends upon the nature of groups. For instance, if groups are defined upon occupational categories, then a party can reward one group relative to another with favorable regulatory or trade policies. If groups are based on religion, then a party can privilege a particular group with legislation that favors a particular faith or by grants to organizations associated with that faith. Similarly, a party could adopted preferential hiring practices to reward a particular ethnic group. When groups are geographically defined, parties can reward the people in one locale relative to people in another by basing pork projects in one area or by providing superior services there.

This essay focuses on allocation mechanisms rather than on what is being allocated. Hence rather than work with the litany for titles for benefits we simply refer to all preferential rewards and benefits as *prizes*. What is essential for our basic model is that parties can observe the level of political support from each group and that there exists a means of preferentially rewarding groups.

Later in the paper we explore the non-rival versus rival nature of prizes, as it turns out this factor influences the optimal division of society into groups from the perspective of political parties. Although in practice all policies have private and public goods components, we contrast the limiting cases. We treat a prize as a non-rival local good or a pure club good if its provision is unrelated to the size of the group that benefits from it. An example of such a prize might be the granting of primacy to a specific language. At the other extreme, prizes can be more completely rival in nature and so the more people who need to receive the benefit, the more expensive its provision becomes. For instance, if a party simply gave money to each member of a group, then the cost of providing the prize linearly increases in the size of the group. While in reality there is great variation in the marginal cost of prizes, we focus on the limiting cases. We refer to the first case, where the marginal cost of increasing group size is zero, as a non-rival prize. Private goods based prizes are rival in nature and have a constant marginal cost of providing the prize as group size increases. However, until we examine the relative cost of prize provisions under different arrangements of groups, the essential point is that members of the group allocated the prize get benefits worth Ω relative to members of other groups.

Next we explore how parties can condition their distribution of prizes on the vote outcome (A, B). Smith and Bueno de Mesquita (2012) refer to such mechanisms as Contingent Prize Allocation Rules.

3.1 Contingent Prize Allocation Rules (CPAR)

Parties can allocate prizes to the various groups in many ways. Let $GA(k, (A, B), r)$ be the expected value of the prize that party $\mathcal{A}$ provides group k under rule r if the vote profile is (A, B). Although we develop the logic of our arguments with respect to party $\mathcal{A}$, throughout there are parallel considerations with respect to party $\mathcal{B}$. For instance, $GB(k, (A, B), r)$ is the prize distribution rule for party $\mathcal{B}$. Although in principal parties can punish groups, the focus here is on positive inducements only: $GA(k, (A, B), r) \geq 0$. Additionally, we restrict attention to monotonic rules so $GA(k, (A, B), r)$ is weakly increasing in A_k. Since budgets are finite, prizes are bounded: $\max_{(A,B)} GA(k, (A, B), r) \leq \Upsilon$. As a final restriction, for notational convenience we restrict party $\mathcal{A}$'s prize allocation to depend only on the distribution of votes for $\mathcal{A}$.

Although there are many plausible CPARs, our primary focus is on a specific rule:

Winner-Takes-All Rule: $r = WTA$. Under this rule party $\mathcal{A}$ rewards the most supportive group (or groups). Other groups receive nothing.

$$GA(k,(A,B),WTA) = \begin{cases} \Omega & if \quad A_k = \max\{A_1,...,A_K\} \\ 0 & \quad otherwise \end{cases}$$

Under the WTA rule, party $\mathcal{A}$ gives a prize to the most supportive group (or groups). This rule creates a race between the groups to get the prize. Members of each group have an impetus to support party $\mathcal{A}$ in the hope that their additional vote tips the balance and wins the prize for their group.

We contrast the WTA with other CPAR that are defined on simple statistics of the vote profile (A, B). Specifically:

1) Group Specific Rule ($r = S$). Under this scheme, party $\mathcal{A}$ rewards a specific group or set of groups. For instance, A might reward a specific occupational category, such as by granting price subsidies for farmers. Alternatively, $\mathcal{A}$ could privilege a specific ethnic, religious or language group or groups.

$$G(k,(A,B),S) = \begin{cases} \Omega & if \quad k \in S \\ 0 & \quad otherwise, \end{cases}$$

where S is a sub-set of groups. This rule violates anonymity as the group labels matter. Party $\mathcal{A}$ treats parties in S differently from parties that are not in S. The rest of the rules we consider are anonymous in the sense that the labels don't matter.

2) Threshold Rule ($r = T$). The reward is given to group k if and only if k delivers at least t votes.

$$GA(k,(A,B),T) = \begin{cases} \Omega & if \quad A_k \geq t \\ 0 & \quad otherwise \end{cases}$$

3) Proportionate Reward Rule:$r = R$.

$$GA(k,(A,B),R) = \min\{\rho A_k, \Upsilon\}$$

where ρ is a constant. So up to the bound Υ, each additional vote for party $\mathcal{A}$ increases group k's reward by ρ. Provided that $\Upsilon >> n_K\rho$, then the bound has little impact.

4 Pivotality and Voting

In the voting game, each voter simultaneously decides whether to vote $\mathcal{A}$, vote $\mathcal{B}$ or abstain. Voting is costly. Any voter who votes, whether for $\mathcal{A}$ or $\mathcal{B}$, pays a cost c. The concept of pivotality lies at the heart of rational choice analyses of voting. Voters weigh the costs and benefits of voting: they vote for the alternative they prefer (at least in two-party competition), but they only vote when their expected influence on the outcome outweighs the cost of voting. The standard concept of this influence is the likelihood of shifting the outcome from one party to another. We refer to this as the outcome pivot, OP_A.

Voters can also be pivotal in terms of the distribution of the prize. That is, by voting for party $\mathcal{A}$, a voter not only increases the likelihood that party $\mathcal{A}$ wins, she also increases the probability that her group will be the most supportive group and so receive selective benefits from party $\mathcal{A}$. We refer to the likelihood of being pivotal in terms of prize allocation as the Prize Pivot, PP_A. In all cases we define analogous definitions with respect to party $\mathcal{B}$.

As Myerson (1998) demonstrates, the Poisson model provides a convenient framework for modeling pivotality. The approach assumes that the size of a group is a Poisson distributed random variable. In the case of the current model, group k has N_k members, where N_k is an unknown random variable that is Poisson distributed with mean n_k. Given this Poisson assumption, from the perspective of each member of group k, the votes of the other N_k-1 members of k (excluding themselves) can also be assumed to be Poisson distributed with mean n_k. This result, which Myerson (1998, Theorem 2 p384) refers to as *environmental equivalence*, means that each voter's calculation about the other members of the group is mathematically equivalent to an external analyst's perspective of the whole group.

Environmental equivalence results from two factors perfectly offsetting each other. The first factor is a signal about group size. Given that an individual is a member of the group provides the signal that the expected group size is larger than the prior mean. The second factor is that when formulating her optimal actions, a voter considers only the $N_k - 1$ other members of the group. These two factors result in each voter's perception of the other members of her group being identical to the analyst's perception of the whole group. This feature makes the Poisson framework especially attractive for modeling pivotality.

The proposition below provides a definition and calculation of outcome pivot. If i is a representative voter in group k, then the probability that by voting for A rather than abstaining she alters the electoral outcome is referred to as the Outcome Pivot, OP_A. Given vote probability profile (p, q), the number of votes for party $\mathcal{A}$ in district k is a Poisson random variable with mean $p_k n_k$ and the total number of votes for $\mathcal{A}$ in district k is a Poisson random variable A with mean $n_T p = \sum_{k=1}^{K} p_k n_k$, where $n_T = \sum\limits_{k=1}^{K} n_k$. Analogously, the total number of votes for party $\mathcal{B}$ is B, a Poisson random variable with mean $n_T q = \sum_{k=1}^{K} q_k n_k$. Given the well-known result that an individual's vote only influences who wins if it breaks a tie or turns a loss into a draw (see Riker and Ordeshook 1968 for instance), the proposition below defines and characterizes OP_A.

Proposition 1 *Given the vote probability profile* (p, q),

$$\begin{aligned} OP_A &= \Pr(\mathcal{A} \text{ wins} \mid \text{voter } i \text{ votes } \mathcal{A}) - \Pr(\mathcal{A} \text{ wins} \mid \text{voter } i \text{ abstains}) \quad (1) \\ &= \frac{1}{2}\Pr(A = B) + \frac{1}{2}\Pr(A = B - 1) \\ &= e^{-n_T(p+q)}((\frac{p}{q})^{\frac{m}{2}}\frac{1}{2}(I_0(2n_T\sqrt{pq}) + (\frac{q}{p})^{\frac{1}{2}}I_1(2n_T\sqrt{pq})) \quad (2) \end{aligned}$$

where $A = \sum_{k=1}^{K} A_k$, $B = \sum_{k=1}^{K} B_k$ and $I_m(x)$ is the modified Bessel function of the first kind.

Proof. From Skellam (1946), if A and B are Poisson random variables with means $n_T p$ and $n_T q$, respectively, then $S(n_T p, n_T q, m) = \Pr(A - B = m) = e^{-(n_T p + n_T q)}(\frac{n_T p}{n_T q})^{\frac{m}{2}} I_{|m|}(2n_T\sqrt{pq})$, where I_m is the modified Bessel function of the first kind. The function S is called the Skellam distribution with parameters $n_T p$ and $n_T q$. Therefore OP_A is simply the average of the Skellam distribution evaluated at $m = 0$ and $m = -1$. So $OP_A = e^{-n_T(p+q)}((\frac{p}{q})^{\frac{m}{2}}\frac{1}{2}(I_0(2n_T\sqrt{pq}) + (\frac{q}{p})^{\frac{1}{2}}I_1(2n_T\sqrt{pq}))$. The Outcome Pivot with respect to voting for party $\mathcal{B}$ is analogously defined, OP_B. ■

Voters not only affect which party wins but also the distribution of prizes. Prize Pivot, $PP_A(k,(p,q),r)$, refers to the expected change in the prize distribution from party $\mathcal{A}$ under CPAR r for group k if a member of k votes for $\mathcal{A}$ rather than abstains if the vote probability profile is (p,q): $PP_A(k,(p,q),r) = E[\text{Prize}|\text{vote}\mathcal{A}] - E[\text{Prize}|\text{abstain}]$

Lemma 2 *$PP_A(k,(p,q),r)$ is continuous in all components of (p,q).*

Proof. In full generality, PP_A depends upon the full distribution of votes. Given that votes are Poisson distributed, $\Pr((A,B) = (a,b)) = \Pr(A_1 = a_1)\Pr(A_2 = a_2)...\Pr(A_K = a_K)\Pr(B_1 = b_1)\Pr(B_2 = b_2)...\Pr(B_K = b_K) = \prod_{k=1}^{K}\Pr(A_k = a_k)\Pr(B_k = b_k) = \prod_{k=1}^{K} f_{n_k p_k}(a_k) f_{n_k q_k}(b_k) \in [0,1)$ and $\sum_{(a,b)}\prod_{k=1}^{K} f_{n_k p_k}(a_k) f_{n_k q_k}(b_k) = 1$. Therefore,

$$PP_A = \sum_{(a,b)}\prod_{k=1}^{K} f_{n_k p_k}(a_k) f_{n_k q_k}(b_k)(GA(k,((a_1,b_1),...,(a_k{+}1,a_k),...),r) - GA(k,((a_1,b_1),...,(a_k,b_k),...),r))$$

Now, let (p',q') be another voting probability profile, and let PP'_A be the corresponding prize pivot. Then

$$|PP_A - PP'_A| \leq \sum_{(a,b)}\left|\prod_{k=1}^{K} f_{n_k p_k}(a_k) f_{n_k q_k}(b_k) - \prod_{k=1}^{K} f_{n_k p'_k}(a_k) f_{n_k q'_k}(b_k)\right|\Upsilon,$$

since the change in prize is bounded above uniformly by Υ.

We next show that $|PP_A - PP'_A| \to 0$ as $(p',q') \to (p,q)$, using the Dominated Convergence Theorem (Folland 1999). Since the pmf's are Poisson, the summand is bounded above by $2\left|\prod_k n_k^{a_k} a_k! n_k^{b_k} b_k!\right|$. This is summable, so the DCT applies, and we interchange the limit and the sum. Clearly,

$$\left|\prod_k f_{n_k p_k}(a_k) f_{n_k q_k}(b_k) - \prod_k f_{n_k p_k'}(a_k) f_{n_k q_k'}(b_k)\right| \to 0,$$

which implies that $|PP_A - PP_A'| \to 0$, and this completes the proof. ■

While in full generality, the prize pivot of CPAR require the calculation of infinite sums over all possible vote outcomes, as characterized in lemma, in practice, for CPARs based on simple statistics of (A, B) calculation is much simpler.

4.1 Voting Calculus

Suppose we consider any fixed vote profile $(p, q) = ((p_1, q_1), (p_2, q_2), ..., (p_K, q_K))$ that describes the probability with which members of each group support $\mathcal{A}$ and $\mathcal{B}$ respectively. Given this profile, the following equations characterize the private evaluation of party $\mathcal{A}$ relative to $\mathcal{B}$ (that is value of ε_i) that would make an individual in group k indifferent between her various vote choices.

$$U_k(vote\mathcal{A}) - U_k(abstain) = (\gamma + \tau_{Ak})OP_A + PP_A(k, (p, q), r) - c = 0 \quad (3)$$

$$U_k(vote\mathcal{B}) - U_k(abstain) = (\gamma + \tau_{Bk})OP_B + PP_B(k, (p, q), r) - c = 0 \quad (4)$$

$$\begin{gathered} U_k(voteA) - U_k(voteB) = (\gamma + \tau_{ABk})(OP_A - OP_B) + \\ PP_A(k, (p, q), r) - PP_B(k, (p, q), r) = 0 \end{gathered} \quad (5)$$

The thresholds, τ_{Ak}, τ_{Bk} and τ_{ABk} that solve these equations characterize Nash equilibria.

Theorem 3 *There exist vote probability profiles (p, q) supported by Nash equilibrium voting behavior: voter i in group k votes for party $\mathcal{A}$ if $\varepsilon_i > \max\{\tau_{Ak}, \tau_{ABk}\}$; votes for $\mathcal{B}$ if $\varepsilon_i < \min\{\tau_{Bk}, \tau_{ABk}\}$ and abstains otherwise. The thresholds, τ_{Ak}, τ_{Bk} and τ_{ABk}, solve equations 3, 4, and 5 for each group and $p_k = 1 - G(\max\{\tau_{Ak}, \tau_{ABk}\})$ and $q_k = G(\min\{\tau_{Bk}, \tau_{ABk}\})$.*

Proof. Given the Poisson population assumption, there is always some, be it very small, probability that i is the only voter. In such a setting, her vote would determine the outcome. This ensures that $OP_A > 0$ and $OP_B < 0$. Therefore equation 3 is an increasing linear functions of τ_{Ak}. Therefore for any given vote profile (p, q), there is a unique threshold that solves the equations (and the same for equations 4, and 5). As annotated, these three equations correspond to differences in expected value from each of the voter's actions. If $\varepsilon_i > \max\{\tau_{Ak}, \tau_{ABk}\}$ then, i votes for A, since $U_k(vote\mathcal{A}) > U_k(abstain)$ and $U_k(vote\mathcal{A}) > U_k(vote\mathcal{B})$. Similarly if $\varepsilon_i < \min\{\tau_{Bk}, \tau_{ABk}\}$, then i votes for $\mathcal{B}$. Given the thresholds, an individual in group k votes for $\mathcal{A}$ with probability $\widetilde{p}_k(p, q) = 1 - F(\max\{\tau_{Ak}, \tau_{ABk}\})$ and votes for $\mathcal{B}$ with probability $\widetilde{q}_k(p, q) =$

$F(\min\{\tau_{Bk},\tau_{ABk}\})$. Since both outcome and prize pivots are continuous in all components of the vote profile (p,q), the τ thresholds, and hence $\widetilde{p}_k(p,q)$ and $\widetilde{q}_k(p,q)$, are continuous in all components of the vote profile. Let $M : [0,1]^{2K} \to [0,1]^{2K}$ be this best response function for all the groups. That is to say, M maps (p,q) into simultaneous best responses for all groups $(\widetilde{p},\widetilde{q}) = ((\widetilde{p}_1(p,q),\widetilde{q}_1(p,q)),...,(\widetilde{p}_K(p,q),\widetilde{q}_K(p,q)))$. As M is continuous and maps a compact set back into itself, by Brouwer's fixed point theorem (1912), a fixed point exists.[2] ■

Not only do Nash equilibria exist, we can approximate them well in large electorates using asymptotic approximations for outcome and prize pivots. Having derived these approximations, we show how groups and CPAR affects equilibrium voting. Political parties want to gain and retain office. To do so they need to incentivize voters to support them. CPAR affect the expected prize gains a voter gains from her political support. Although stated informally here, the key to increasing vote share is to increase pivotality. The greater the extent to which a voter's vote matters, the more likely she is to turnout.

5 Asymptotic Approximations of Pivots

As the number of voters grows large, there are simple approximations for the pivots. To derive these approximations we assume the expected number of voters, $n_k p_k$, is relatively large and the number of groups K is relatively small.

In what follows, the symbol $\approx$ should be treated as meaning approximately equal. More formally, $u(x) \approx v(x)$ if the difference $|u(x) - v(x)| \to 0$ as $x \to \infty$. As the expected number of voters converges towards infinity, the approximations converge to their true value. We indicate the accuracy of these approximation in finite populations.

Proposition 4 *Outcome Pivot: OP_A: If $n_T = \sum_{k=1}^{K} n_k$, $p = \frac{1}{n_T}\sum_{k=1}^{K} p_k n_k$ and $q = \frac{1}{n_T}\sum_{k=1}^{K} q_k n_k$ then*

$$OP_A \approx \widetilde{OP}_A = \frac{1}{2\sqrt[4]{pq}\sqrt{\pi n_T}} \frac{\sqrt{p}+\sqrt{q}}{2\sqrt{p}} \cdot e^{-n_T\left(\sqrt{p}-\sqrt{q}\right)^2} \tag{6}$$

This is the same approximation used by Myerson (1998), so we provide only a brief sketch. As derived above the difference between the vote for A and B is Skellam distributed: $OP_A = e^{-n_T(p+q)}((\frac{p}{q})^{\frac{m}{2}}\frac{1}{2}(I_0(2n\sqrt{pq}) + (\frac{q}{p})^{\frac{1}{2}} I_1(2n_T\sqrt{pq}))$. The modified Bessel function of the first kind, $I_m(x)$ is a well known mathematical function that for fixed m and large x is well approximated (Abramowitz and Stegun 1965, p. 377):

$$I_{|m|}(x) \quad \approx \quad \frac{e^x}{\sqrt{2\pi x}}(1 - \frac{4m^2-1}{8x} + \frac{(4m^2-1)(4m^2-9)}{2!(8x)^2}$$

[2] If $G()$ was discontinuous then we would need to introduce mixed strategies for types at the discontinuities and use Kakutani's fixed point theorem.

$$-\frac{(4m^2-1)(4m^2-9)(4m^2-25)}{3!(8x)^3}+...)$$

We use the first term of this approximation $I_{|m|}(x) \approx \frac{e^x}{\sqrt{2\pi x}}$ and equation 6 follows directly. To check the accuracy we evaluate $(I_m(x) - \frac{e^x}{\sqrt{2\pi x}})/I_m(x)$ for $m = 0, 1$. Ninety nine percent accuracy is attained when $x > 38.2$. The approximation become better as x increases.

For large populations the outcome pivot estimates are accurate. For instance, if the population mean is $n_T = 100,000$ and voters support parties $\mathcal{A}$ and $\mathcal{B}$ with probability $p = .5$ and $q = .5$, then the approximation error for the outcome pivot is around .0001%.

Proposition 5 *Winner-Take-All Prize Pivot. :*

$$PP_A(k,(p,q),WTA) = \Omega \sum_{a=0}^{\infty} f_{n_k p_k}(a)(\prod_{j\neq k} F_{n_j p_j}(a+1) - \prod_{j\neq k} F_{n_j p_j}(a))$$

For the symmetric case of $n_j p_j = np$ for all j and $K \geq 3$, as the expected number of votes for $\mathcal{A}$ is large ($np \to \infty$), then PP_A is well approximated by[3]

$$PP_A \approx \widetilde{PP}_A = \frac{\Omega(K-1)}{2\pi p} e^{nh(\alpha_0)} \sqrt{\frac{2\pi}{n|h''(\alpha_0)|}} \tag{7}$$

where

$$h(\alpha) = -(\frac{\alpha - p}{\sqrt{p}})^2 + \frac{1}{n}(K-2)\log \Phi(\sqrt{n}\frac{\alpha - p}{\sqrt{p}})$$

a_0 *solves*

$$h'(\alpha_0) = -\frac{2}{\sqrt{p}}(\frac{\alpha_0 - p}{\sqrt{p}}) + \frac{1}{\sqrt{n}\sqrt{p}}(K-2)\frac{\phi(\sqrt{n}\frac{\alpha_0 - p}{\sqrt{p}})}{\Phi(\sqrt{n}\frac{\alpha_0 - p}{\sqrt{p}})} = 0$$

and

$$h''(\alpha_0) = -\frac{2}{p} - \frac{2n}{p}(\frac{\alpha_0 - p}{\sqrt{p}})^2(1 + \frac{2}{(K-2)})$$

Proof.

Suppose $A_k = a_k$. If voter i in group k abstains then her group receives the prize Ω if $a_k \geq \max\{A_{j\neq k}\}$. Since A_j is Poisson distributed with mean $n_j p_j$, $\Pr(a_k \leq A_j) = F_{n_j p_j}(a_k)$ and the probability that a_k is the maximum of all groups' support for $\mathcal{A}$ is $\prod_{j\neq k} \Pr(A_j \leq a_k) = \prod_{j\neq k} F_{n_j p_j}(a_k)$. Since A_k is Poisson distributed, group k's expected prize if i abstains is $\sum_{a=0}^{\infty} f_{n_k p_k}(a) \prod_{j\neq k} F_{n_j p_j}(a)$.

[3]If $K = 2$, then there is a simpler approximation for PP_A based on the Skellam distribution.

If i votes for $\mathcal{A}$, then $\Pr(a_k + 1 \geq \max\{A_{j\neq k}\}) = \prod_{j\neq k} \Pr(A_j \leq 1 + a_k) = \prod_{j\neq k} F_{n_j p_j}(a_k+1)$ and the expected prize for k is $\Omega \sum_{a=0}^{\infty} f_{n_k p_k}(a) \prod_{j\neq k} F_{n_j p_j}(a+1)$. Therefore $PP_A = \Omega \sum_{a=0}^{\infty} f_{n_k p_k}(a)(\prod_{j\neq k} F_{n_j p_j}(a+1) - \prod_{j\neq k} F_{n_j p_j}(a))$.

Next we derive a simple approximation based on using the normal distribution as an asymptotic approximation for the Poisson distribution and integration using Laplace's method. When the expected number of votes is large, the normal distribution is an asymptotic approximation of the Poisson: $f_\lambda(a) \approx \phi(\frac{a-\lambda}{\sqrt{\lambda}}) = \frac{1}{\sqrt{2\pi}\sqrt{\lambda}} e^{-\frac{1}{2}(\frac{a-\lambda}{\sqrt{\lambda}})^2}$ and $F_\lambda(a) \approx \Phi(\frac{a-\lambda}{\sqrt{\lambda}}) = \int_{-\infty}^{a} \phi(\frac{x-\lambda}{\sqrt{\lambda}})dx$.

Therefore, for large $n_j p_j$, $PP_A \approx \Omega \sum_{a=0}^{\infty} \phi(\frac{a-\lambda_k}{\sqrt{\lambda_k}})(\prod_{j\neq k} \Phi(\frac{a-\lambda_j+1}{\sqrt{\lambda_j}}) - \prod_{j\neq k} \Phi(\frac{a-\lambda_k}{\sqrt{\lambda_k}}))$.

We focus on the symmetric case, $n_j p_j = np$ for all j, and use a change of variable, $a = \alpha n$.

$$PP_A \approx \Omega \sum_{\alpha=0}^{\infty} \phi(\frac{\alpha n - np}{\sqrt{np}})(\Phi(\frac{\alpha n + 1 - np}{\sqrt{np}})^{(K-1)} - \Phi(\frac{\alpha n - np}{\sqrt{np}})^{(K-1)}),$$

with the summation now being over $\alpha = 0, \frac{1}{n}, \frac{2}{n}, \ldots$ and let $\delta = 1/\sqrt{np}$.

Next we approximate $\Phi(\frac{\alpha n+1-np}{\sqrt{np}})$ as a Taylor Series about $\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p)$:

$$\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p) + \delta) \approx (\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p)) + \delta \frac{d\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))}{d\alpha} + \frac{1}{2}\delta^2 \frac{d^2\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))}{d\alpha^2} + \ldots)$$

We use the first two terms since $\delta = 1/\sqrt{np}$ is small. Therefore

$$\begin{aligned}
& \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p) + \delta)^{(K-1)} - \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-1)} \\
\approx \; & (\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p)) + \delta \frac{\sqrt{n}}{\sqrt{p}} \phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p)))^{K-1} - \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-1)} \\
= \; & \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-1)} + (K-1)\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-2)} \delta \frac{\sqrt{n}}{\sqrt{p}} \phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p)) + \ldots \\
& \ldots + \frac{(K-1)!}{j!(K-1-j)!} \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-1-j)} \delta^j (\frac{\sqrt{n}}{\sqrt{p}})^j \phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^j + \ldots - \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-1)} \\
\approx \; & (K-1)\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-2)} \delta \frac{\sqrt{n}}{\sqrt{p}} \phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p)) \text{ since } \delta \text{ is small.}
\end{aligned}$$

Thus

$$PP_A \approx \Omega \sum_{\alpha=0}^{\infty} \phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^2 (K-1) \Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^{(K-2)} \delta \frac{\sqrt{n}}{\sqrt{p}}$$

As $n \to \infty$, this Riemann sum converges to an integral so

$$\begin{aligned} PP_A &\approx \Omega \int_0^\infty \phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha - p))^2 (K-1)\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha-p))^{(K-2)}\delta\frac{\sqrt{n}}{\sqrt{p}}d\alpha \\ &= \frac{\Omega(K-1)}{2\pi p}\int_0^\infty e^{-n(\frac{\sqrt{n}}{\sqrt{p}}(\alpha-p))^2}\Phi(\frac{\sqrt{n}}{\sqrt{p}}(\alpha-p))^{(K-2)}d\alpha \end{aligned}$$

This integral can be rewritten as

$$PP_A \approx \frac{\Omega(K-1)}{2\pi p}\int_0^\infty e^{n(-(\frac{\alpha-p}{\sqrt{p}})^2+\frac{1}{n}(K-2)\log\Phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}}))}d\alpha = \frac{\Omega(K-1)}{2\pi p}\int_0^\infty e^{n(h(\alpha))}d\alpha$$

Note that $\Phi(x)$ is bounded between 0 and 1 so its logarithm is negative. Therefore

$$h(\alpha) = -(\frac{\alpha-p}{\sqrt{p}})^2 + \frac{1}{n}(K-2)\log\Phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})$$

has a unique maximum:

$$h'(\alpha) = -\frac{2}{\sqrt{p}}(\frac{\alpha-p}{\sqrt{p}}) + \frac{1}{n}(K-2)\frac{\sqrt{n}}{\sqrt{p}}\frac{\phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})}{\Phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})}$$

and

$$h''(\alpha) = -\frac{2}{p} - \frac{1}{n}(K-2)\frac{n}{p}\frac{\Phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})\phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}}) + \phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})^2}{\Phi(\sqrt{n}\frac{\alpha-p}{\sqrt{p}})^2} < 0$$

Let α_0 solve $h'(\alpha_0) = 0$ so

$$\begin{aligned} h''(\alpha_0) &= -\frac{2}{p} - \frac{1}{p}(K-2)\frac{\Phi(\sqrt{n}\frac{\alpha_0-p}{\sqrt{p}})(\sqrt{n}\frac{\alpha_0-p}{\sqrt{p}})\phi(\sqrt{n}\frac{\alpha_0-p}{\sqrt{p}}) + \phi(\sqrt{n}\frac{\alpha_0-p}{\sqrt{p}})^2}{\Phi(\sqrt{n}\frac{\alpha_0-p}{\sqrt{p}})^2} \\ &= -\frac{2}{p} - \frac{2}{p}(K-2)\frac{1}{(K-2)}n(\frac{\alpha_0-p}{\sqrt{p}})^2(1+\frac{2}{(K-2)}) \end{aligned}$$

The integral $\int_0^\infty e^{nh(\alpha)}d\alpha$ is a standard form for a Laplace integral: by Taylor series about a_0,

$$\begin{aligned} \int_0^\infty e^{nh(\alpha)}d\alpha &\approx \int_0^\infty e^{n(h(\alpha_0)+(\alpha-\alpha_0)h'(\alpha_0)+\frac{1}{2}(\alpha-\alpha_0)^2h''(\alpha_0)+\dots)}d\alpha \\ &\approx \int_0^\infty e^{nh(\alpha_0)}e^{n(\alpha-\alpha_0)nh'(\alpha_0)}e^{\frac{1}{2}n(\alpha-\alpha_0)^2h''(\alpha_0)}d\alpha \end{aligned}$$

Since at α_0, $h(\alpha_0) = 0$ the second term in the integrand is 0. The third term $(e^{\frac{1}{2}(\alpha-\alpha_0)^2h''(\alpha_0)})$ has the form of a Gaussian distribution. Therefore, as $n \to \infty$,

$$PP_A \approx \widetilde{PP}_A = \frac{\Omega(K-1)}{2\pi p}e^{nh(\alpha_0)}\int_0^\infty e^{\frac{1}{2}(\alpha-\alpha_0)^2h''(\alpha_0)}d\alpha = \frac{\Omega(K-1)}{2\pi p}e^{nh(\alpha_0)}\sqrt{\frac{2\pi}{n|h''(\alpha_0)|}}$$

Thus as $n \to \infty$, $\widetilde{PP}_A$ approximates PP_A.

■

We now turn to an assessment of the approximation. If the electoral turnout is low (np) and there are numerous groups, then $\widetilde{PP}_A$ is a relatively poor approximation of PP_A. However, as the expected number of voters increases, the approximation becomes increasingly accurate. This convergence occurs quicker for a smaller number of groups. Figure 1 plots the PP_A and $\widetilde{PP}_A$ as a function of mean group size for the cases of 3 and 9 groups. The horizontal axis plots mean group size, n_k, when $p = .3$. As the number of voters increases, the approximations converge to the prize pivots.

Figure 1 about here

6 Motivations to Vote

Using the results above we explore the substantive topics of turnout, vote choice, voter motivation and the rewards parties offer their supporters. Before turning to equilibrium analysis, figure 2 graphs the outcome and prize pivots, $\widetilde{OP}_A$ and $\widetilde{PP}_A$, as a function of p when the voters are divided into $K = 3$ groups. This figure assumes $n_T = 100,000$ and $q = .3$. The solid line represents $\widetilde{OP}_A$. As expected it takes its maximal value at $p = q$, which in this case is at 30%. At this point approximately $30,000$ voters vote for $\mathcal{A}$ and $30,000$ vote for $\mathcal{B}$. The chance that an additional vote for either party is influential in determining the electoral outcome is approximately 0.16%. However, except around this point where expected voter support is almost perfectly matched, voters have virtually no influence on the electoral outcome. For instance, in this example if $p = .31$, that is one percent higher than q, then $OP_A \approx 4.4 \cdot 10^{-7}$.

Figure 2 about here.

In contrast, the prize pivot, $\widetilde{PP}_A$, shown by the dotted line, is a smoothly declining function of p. When turnout is low (small p), casting an additional vote for party $\mathcal{A}$ has a strong influence on the allocation of the prize. As the number of expected voters for party $\mathcal{A}$ increases, the influence of an additional vote diminishes. However, as the figure makes clear, prize pivotalness does not go to zero in the manner that outcome pivotalness does. Referring back to equations 6 and 7 we can see the reason for this. The approximation for $\widetilde{OP}_A$ contains the terms $\frac{e^{-n_T(\sqrt{p}-\sqrt{q})^2}}{\sqrt[4]{pq}\sqrt{n_T}}$. As n_T gets large, $\frac{1}{\sqrt{n_T}}$term converges to zero; however it does so at a slow rate. However, the outcome pivot also contains an exponential term, $e^{-n_T\left(\sqrt{p}-\sqrt{q}\right)^2}$, which converges rapidly to zero as n_T gets large unless $p = q$. Thus, except around the precise balance position of $p = q$, outcome pivotalness is extremely tiny for even modest sized n_T.

The prize pivot exhibits important differences from the outcome pivot. The radical in the exponential component of $\widetilde{PP}_A$ is evaluated close to zero so that this term does not drive the prize pivot rapidly to zero, as was the case in the outcome pivot. The net effect is that $\widetilde{PP}_A$ converges to zero at the rate of $\frac{1}{\sqrt{n}}$.

Although we have not yet moved to an equilibrium analysis, figure 2 provides much of the intuition about voter motivations. The desire to influence which party wins is only a significant motivation for voters when elections are close. Rewards that a party gives to their most supportive groups provide voters with a motive to turnout even when electoral competition is not close. Given these intuitions we now examine a series of equilibria to see how these incentives play out in influencing equilibrium voting behavior.

6.1 Fully Symmetric Equilibria

If there is no policy bias in favor of either party ($\gamma = 0$) and both parties offer the same prizes ($\Omega_A = \Omega_B = \Omega$) under the WTA rule, then there is a symmetric equilibrium where all groups vote for parties A and B at the same rate ($p_k = q_k = p$ for all k) and, assuming a sufficiently high cost of voting, p solves

$$U_k(vote\mathcal{A}) - U_k(abstain) = G^{-1}(1-p)OP_A + PP_A(k,(p,q),WTA) - c = 0 \quad (8)$$

These claims follow directly from the propositions above. If the cost of voting is very low then $p = q = 1/2$ solves equation 5 instead. Figure 3 plots voter turnout for each party as a function of the size of the prizes (Ω) for this equilibrium assuming the expected number of voters is $n_T = 100,000$, the cost of voting is $c = 1$ and the variance in the assessments of individual assessments of party $\mathcal{A}$ is 100. When the parties offer small prizes, turnout is low. For instance when there are no prizes, turnout for each party is about 80 voters, that is around .08%. This low level of turnout forms the basis of the standard critique of rational choice voting models. Since voters have little influence on the outcome, even in this evenly balanced situation, only those voters who have a strong (either positive or negative) evaluation of party $\mathcal{A}$ vote. However, if voters are also competing to win prizes for their groups, then turnout increases as the value of these prizes rises.

Figure 3 about here

If the parties offer prizes worth $\Omega = 100$, that is on the scale of the variance of policy differences between voters, then about 300 voters support each party, about 3%. As prizes increase so does turnout. About 38% of voters support each party once the prize size is $\Omega = 400$.

Although it is always difficult to accurately parameterize models it is worth pausing to compare the scale of rewards in the final case. This example yielded about a 76% turnout when the size of prizes what 400 times the cost of voting. As a comparison, to induce such a turnout based on policy differences alone (that is outcome pivot) would require that the variance in policy differences between voters to be on the scale of 5 million times the cost of voting. While caution should always be shown when comparing the scale of costs and benefits, it is clear that when parties hand out rewards based upon vote outcomes rather than just offering a policy platform, they can induce much higher levels of support.

6.2 Dominant Party Equilibria

Magaloni (2006) argues Mexico's PRI remained a dominant party long after it was generally acknowledged by virtually all voters to be the inferior choice. This is not an isolated example. Numerous books have been dedicated to characterizing dominant parties (see, for example, Simkins 1999; Sartori 1976). Obviously in some cases there is substantial voter intimidation and fraud, but even excluding these, in many countries the incumbent party appears to win election after election even though it fails to provide effective public policy. Patronage is often cited as the reason for such party dominance. Yet, as reviewed above, this classic vote buying approach is problematic.

Contingent prize allocations can incentivize voters to turnout and vote even for a party they dislike. The contingent prize model describes a patronage-like system, but without encountering credibility problems or requiring many people to receive rewards. Once one party is perceived as dominant and virtually certain to win the election, policy preferences have little influence on voting behavior. To see why this is so, we consider the case where p is larger than q and first examine the decision to vote for A rather than abstain as characterized by equation 3. Then we consider the decision to vote for B or abstain when A is virtually certain to win.

In the case in which A is the dominant party, so that $p > q$ for large n_T, $OP_A \approx \frac{1}{2}\left(1 + \frac{q^{1/2}}{p^{1/2}}\right) \cdot \frac{e^{-n_T(\sqrt{p}-\sqrt{q})^2}}{2\sqrt[4]{pq}\sqrt{\pi n_T}} \approx 0$ so a voter's preference for party $\mathcal{A}$ over party $\mathcal{B}$ has virtually no influence in her voting calculus. As shown by figure 2, voters have a negligible influence on the electoral outcome unless the expected party vote shares are equal. Equally, figure 2 shows that even if party $\mathcal{A}$ vastly out performs party $\mathcal{B}$, voters are still pivotal in the allocation of prizes.

We now examine a symmetric equilibrium ($n_i p_i = n_j p_j >> n_i q_i = n_j q_j$ for all i, j) in which disproportionately more voters support $\mathcal{A}$ than $\mathcal{B}$. Although this condition requires that a similar number of voters support party $\mathcal{A}$ in each group, it does not require the expected size of groups to be exactly the same. If the expected size of one group is slightly smaller than that of other groups, then the symmetry condition is met if that group supports party $\mathcal{A}$ at a slightly higher rate. Below we relax the symmetry requirement further and show that high turnout equilibria can be supported when only a subset of the groups delivering high support for party $\mathcal{A}$.

When $\mathcal{A}$ is the dominant party, party $\mathcal{B}$ virtually never wins: $OP_A \approx 0$. Given that $\mathcal{A}$ is dominant it is perhaps reasonable to assume that it has more resources with which to reward supporters than party $\mathcal{B}$. Figure 4 is constructed on that basis. We assume that party $\mathcal{A}$ allocates a prize worth $\Omega_A = 300$ while party $\mathcal{B}$'s prize is only $\Omega_B = 100$. The vote probabilities (p, q) solve $\widetilde{PP}_A = c$ and $\widetilde{PP}_B = c$, which are plotted in figure 4 as a function of the cost of voting. The figure show that voters support $\mathcal{A}$ at about 9 times the rate they support $\mathcal{B}$. As is to be expected, as the cost of voting declines, a greater proportion of voters vote. When the voting cost drops below about 0.5 there is full turnout, and party $\mathcal{A}$ beats $\mathcal{B}$ by a ratio of nine votes to one. It is worth noting that some voters vote for $\mathcal{A}$ even though they prefer party $\mathcal{B}$, $\varepsilon_i < 0$. In the presence of a dominant party, voters have virtually no influence on which party wins and their voting decisions are dominated by the competition for prizes alone.

Figure 4 about here

6.3 Group Polarization

Huckfeldt and Sprague (1995) claim socialization is an important component of how people vote. They argue people adopt the values of their neighbors and so eventually neighbors end up voting for similar candidates. While Schelling's (1971) model of segregation by migration offers a means for group polarization to occur, Huckfeldt and Sprague argue convergence of political support does not require such migration. Our model provides an explanation for the convergence of vote choices within groups. Equilibrium behavior supports the endogenous polarization of groups into pro-government and pro-opposition supporters.

Suppose that voters in $K_A \geq 2$ of the K groups predominately vote for party $\mathcal{A}$, while the voters in the remaining K_B groups predominately support party $\mathcal{B}$. In the pro-$\mathcal{A}$ groups, voters support party $\mathcal{A}$ with probability p_A and virtually no voters from these groups support party $\mathcal{B}$. In the pro-$\mathcal{B}$ groups, voters support $\mathcal{B}$ with probability q_B and virtually never vote for $\mathcal{A}$. The technology above used for examining fully symmetric equilibria is readily adapted to this situation. For instance, we can define the prize pivot PP_A for the two different types of groups. In particular, let $\widetilde{PP}_A|_{K_A}$ and $\widetilde{PP}_A|_{K_B}$ be the prize pivot for voters in the pro-$\mathcal{A}$ and pro-$\mathcal{B}$ groups respectively. Since the few voters in the pro-$\mathcal{B}$ groups support $\mathcal{A}$, these groups have virtually no chance of ever being the most supportive groups for party $\mathcal{A}$, with or without an additional vote. With virtually no influence over the competition to be $\mathcal{A}$'s most ardent supporters, for individuals in pro-$\mathcal{B}$ groups $PP_A|_{K_B} \approx 0$. Competition for the prizes allocated by party $\mathcal{A}$ is effectively restricted to the pro-$\mathcal{A}$ groups and the prize pivot for voters in these groups is readily approximated by calculating $\widetilde{PP}_A$ but restricting the calculation to the K_A pro-$\mathcal{A}$ group: $\widetilde{PP}_A|_{K_A} = \frac{\Omega(K_A-1)}{2\pi\sqrt{np_A}} e^{nh(\alpha_0,K_A)} \sqrt{\frac{2\pi}{n|h''(\alpha_0,K_A)|}}$ where a_0 solves $h'(\alpha_0, K_A) = -\frac{2}{\sqrt{p_A}}(\frac{\alpha_0-p_A}{\sqrt{p_A}}) + \frac{1}{\sqrt{n}\sqrt{p_A}}(K_A-2)\frac{\phi(\sqrt{n}\frac{\alpha_0-p_A}{\sqrt{p_A}})}{\Phi(\sqrt{n}\frac{\alpha_0-p_A}{\sqrt{p_A}})} = 0$. We can make analogous calculations with respect to the prize pivots for party $\mathcal{B}$.

Given that $\widetilde{PP}_A|_{K_B}$ is virtually zero, the only incentive for voters in strongly $\mathcal{B}$ leaning groups to vote for $\mathcal{A}$ is to influence the electoral outcome. Yet, since OP_A is small, only the most extremist supporters of $\mathcal{A}$ (very high ε) will support $\mathcal{A}$ when they are in a pro-$\mathcal{B}$ group. Similarly, in pro-$\mathcal{A}$ groups, there is little incentive to support party $\mathcal{B}$ and so only extremists ($\varepsilon << 0$) do so. Once groups have polarized, individuals within those groups can do little better than go along with their fellow group members. Figure 5 illustrates just such equilibrium behavior in the context of polarized groups and a dominant party.

Figure 5 about here

Suppose there are an expected $n_T = 100,000$ voters divided into nine groups, three of which are pro-$\mathcal{A}$ in alignment and the remaining 6 of which predominantly support $\mathcal{B}$. Further, suppose, as a dominant party, A has greater resources to distribute as prizes. In particular, figure 5 is constructed assuming $\Omega_A = 300$ and $\Omega_B = 100$. The equilibrium probabilities with which the voters support the party that their group is aligned with is plotted against the cost of voting, c. Provided the cost of voting, c, is above about 0.3, then such a setting is plausible. If the cost of voter were lower, then the pro-$\mathcal{B}$ groups would deliver more votes for $\mathcal{B}$ than pro-$\mathcal{A}$ groups can deliver for $\mathcal{A}$, and party $\mathcal{A}$ would lose the election and no longer be a dominant party able to dispense larger prizes than $\mathcal{B}$.

Before moving on, it is worth exploring some important insights about the number of pro-$\mathcal{A}$ and pro-$\mathcal{B}$ groups. Perhaps surprisingly the number of pro-$\mathcal{A}$ groups can be less than the number of pro-$\mathcal{B}$ groups. That is dominant parties can maintain their status based on support from a minority of the population. Since party $\mathcal{A}$ is anticipated to win, voters in groups that support $\mathcal{A}$ compete for a larger prize than voters in groups that align with $\mathcal{B}$. With greater prizes to allocate, party $\mathcal{A}$ can elicit greater turnout from supporters in its smaller number of aligned groups than can party $\mathcal{B}$ from a greater number of groups.

Another interesting point is that the dominant party must have at least two groups vying for the prizes it offers. To see why, consider the contradiction and suppose that party $\mathcal{A}$ had only one group of supporters. Voters in this group know there is no rival group competing for the prize so they have no incentive to vote for $\mathcal{A}$ ($PP_A|_{K_A} \approx 0$). This results in a low turnout within this group, which contradicts $\mathcal{A}$ being a dominant party. Parties garner greater support when there are factions competing for any prize they can offer. Therefore, parties have an incentive to create divisions even within their own support base.

6.3.1 Differential Motivation and Turnout

Voters have different electoral incentives depending upon how their group is aligned. A voter in a group strongly aligned with party $\mathcal{A}$ has little influence over the distribution of the prize allocated by party $\mathcal{B}$. As such her incentives to vote are driven by their ability to capture a prize from $\mathcal{A}$ for their group and to influence the outcome of the election. Likewise, voters in pro-$\mathcal{B}$ groups have little influence of the distribution of prizes from party $\mathcal{A}$. Their motivation to vote is to capture prizes from party $\mathcal{B}$ and to influence the electoral outcome. If there are strongly aligned groups, then voters belonging to moderate, non-aligned groups have little chance of capturing prizes from either party. Instead, such voters are motivated to turnout only by the prospect of influencing the electoral outcome.

Voters have different motivations to vote depending upon the alignment of their group. Voters in groups aligned with a dominant party incumbent have the greatest incentive to turnout as they are competing for large prizes. Voters in non-aligned groups are predicted to have the lowest turnout rates. Between these extremes are voters in groups aligned with an opposition party in a dominant party system and voters in groups aligned with parties in a competitive system. Although voters make vote choices based solely on what is best for them, context matters. The same voter turnouts at different rates and votes for different parties under different circumstances.

7 The Nature of Prizes, Optimal Groups and Competition

Throughout this paper, the number of groups is taken as given. However, cities are not divided into wards without taking political considerations into account. Neither are religious or ethnic cleavages activated by chance. In this section we approach the questions of politically activated groups by looking at how the number of groups and the rival versus non-rival nature of prizes affects the ease with which a party can elicit political support. We characterize which configurations enable parties to secure a given level of support for the minimum expenditure. It is from the perspective of a dominant political party that we examine what constitutes optimal.

A simple way to operationalize comparisons is to setup a challenge for a dominant political party. Specifically we ask how much the party would need to spend in order to secure votes from 60% of the voters under different configurations of prizes, groups and competitions. We explain each of these dimensions.

In the context of the model, groups need to have identifiable vote totals and to be rewardable. Without these properties parties could not know how many votes each group delivered or be able to reward them for their support. The division of society into geographic units satisfies these criteria readily. For ease of language, we describe the difference group configurations in terms of a single electoral district that is divided into wards. We then examine the cost of buying 60% political support with rival prizes and non-rival prizes and examine whether parties can lower these costs by organizing the groups into several competitions rather than just one. We introduce this concept of different competitions next.

Suppose the expected number of voters is n_T =100,000 and that they are divided into 27 wards. The dominant political party might offer a single prize to the single ward that produces the great political support. But this is not the only way it could organize the political competition. A party might aggregate wards into larger units, which for clarity we will refer to as counties. For instance, the party might aggregate wards to form three counties each with nine wards in them, or alternatively nine counties each with three wards. Once wards are aggregated into counties, a party can structure the competition for prizes in two very different ways. First, the party could have a single competition between the counties for a single prize. Second, the party could run a separate competition between the wards within each county. This could mean either 3 competitions each between 9 wards or nine competitions each between three wards. The question we pose is which configuration is the most efficient for parties to buy political support when prizes are non-rival in nature and rival in nature.

The rival nature of prizes affects the marginal cost of providing the good. We take limiting cases and say the cost of a non-rival prize is fixed however many members there are in the group. The marginal cost of providing local public goods benefits or club goods to additional group members is zero. However, if the prizes are rival, then as the group gets larger there are more people who need to be rewarded so the cost of prize provision increases. In this case, the marginal cost of increasing group size is constant.

The nature of the prizes affects the optimal configuration of competitions and groups. When prizes are non-rival in nature, then a dominant party most efficiently buys political support with a single competition between a limited number of groups, five groups being the best. In contrast, when prizes are rival -so the cost of providing them increases as more people need to be rewarded– then parties reduce the cost of buying political support by creating numerous competitions, each between a large number of groups. Figures 6 and 7 demonstrate these results graphical.

Figure 6 and 7 look at the symmetric equilibrium in the case of a dominant party, $n_T = 100,000$, $c = 1$ and party $\mathcal{A}$ eliciting 60% support, $p = .6$. Figure 6 examines the effect of the number of competitions and the number of groups on the overall cost of prize distribution. The lower line corresponds to a single competition, and the upper line represents the cost of providing prizes in three distinct competitions. As is unsurprizing, when the cost of providing a prize is fixed independent of the number of people benefiting from it, then parties lower the cost of political support by having a single large competition. The size of the prize needed to gain a 60% vote share depends upon the number of groups in a nonlinear way. The size of prize needed to buy political support is minimized by having five groups.

Rival prizes create different incentive for parties. When the cost of providing a prize linearly increases in the size of the prize and the number of people who receive it, parties prefer a configuration with many competition each between many groups, as seen in figure 7. The logic behind this result stems from the number of people being rewarded under different configuration. If there were a single competition between three groups then obtaining 60% support requires a prize worth about 503 times the cost of voting and this reward is given to one third of the expected 100,000 voters. If we setting the cost of voting at one dollar and treat the rival prize as a simple cash reward, then this would cost the party around $\frac{100,000}{3} * \$503 \approx \$17$ million. If instead the parties configured nine competitions, each between nine groups, then the value of the prize in each of the competition needs to be \$500. Given the nine competitions and nine groups in each, the number of people that receive a reward is $100,000/9$ so the total cost is about \$5.6 million. Figure 7 shows these trends. Parties lower the cost of buying political support with rival goods by increasing the number of competition and increasing the number of groups competing for the prize.

The cleavages in society present constraints on how politicians can define groups. Yet, the analysis above provides powerful intuitions. A dominant party minimizes its cost of buying political support by having competition at the national level be between a small number of groups for non-rival goods. State promotion of a language, religion or ethnic good from a small number of groups constitutes such a configuration. However, handing out cash, jobs or services under such configurations is an expensive means of buying support. Such rewards are better dispensed locally in competitions between large numbers of wards.

8 Alternative CPAR

The winner-takes-all rule way of allocating prizes provides parties with a means to incentivize voters to support them. Of course, this rule is not the only means by which parties might allocate rewards. In this section we examine alternative rules. Unfortunately such a survey of alternatives can not be comprehensive of the full range of CPAR. Rather it looks at several intuitively plausible alternatives.

Proposition 6 *The prize pivots for alternative CPAR are as follows:.*
1) Specific Groups) $PP_A(k,(A,B),r=S)=0$
2) Threshold Rule) $PP_A(k,(A,B),r=T)=f_{n_kp_k}(t-1)=\frac{(n_kp_k)^{t-1}}{(t-1)!}e^{-n_kp_k}$
3) Proportionate Reward Rule) $PP_A(k,(A,B),r=P)=\rho$

The proofs are straightforward and discussed informally below.

In the specific groups rule, party $\mathcal{A}$ provides prizes to some specific groups and not other. The allocation of these prizes does not depend upon votes. Therefore, voting one way or another does not alter the distribution of prizes. The only motivation to vote under this scheme is to influence which party wins. If we extend the model such that the party could only give out the prize if it won, then such an allocation increases the desire of members of particular groups to see a particular party win, but only to the extent that they are outcome pivotal. As seen here and argued widely in critiques of rational choice voting, such outcome pivotality provides little incentive to vote. Rather than commit to rewarding specific groups, our analysis suggests parties can more effectively use their resources to incentivize voters by conditioning which groups receive benefits on relative support.

Under the threshold rule, a party gives a group a prize if and only if that group delivers at least t votes. For example, along the lines of Schwartz (1987) the party might promise to effectively collect the trash in a ward only if half the voters in the ward support it: $t=n_k/2$. While intuitively we might suspect this encourages groups member to vote to gain a prize, each voter is pivotal only if $t-1$ others in the group vote for $\mathcal{A}$. If more do so, then the group wins the prize anyway. If less do so, then the contribution of another vote is still insufficient to gain the prize for the group. Hence the chance of being pivotal is $f_{n_kp_k}(t-1)=\frac{(n_kp_k)^{t-1}}{(t-1)!}e^{-n_kp_k}$, which is small for large n_k. For instance, if $t=n_k/2$, and the group contains $n_k=10,000$ expected voters, then $PP_A(k,(A,B),r=T)$ has a maximum value of .0056 when $p=.5$, and is nearly zero virtually everywhere else.

Finally we consider a proportionate rule. Under anonymous voting rules, parties can not observe voter behavior directly. Without information on how voters actually behave in the voting booth, direct voting buying is problematic. However, parties can indirectly buy votes by rewarding a group in proportion to the aggregate number of votes it produces. Unfortunately for parties such indirect vote buying is an extremely expensive way of buying support. It is useful to compare the cost of providing a proportionate reward to the cost of the WTA rule shown in the example in figure 3. In that example, party A needed a prize worth $\Omega_A = 400$ to elicit votes from 38% of the voters. To elicit the same vote rate, the marginal returns of another vote would need to be approximately the cost of voting, which in the example is $c = 1$. Therefore the total rewards to elicit 38% support are on the magnitude of $0.38n_T$, which is about 95 times larger that prize needed under the WTA rule.

Although our survey of alternative CPAR is far from comprehensive, it does illustrate the power of the WTA relative to a range of other rules.

9 Conclusions

A central puzzle of democratic politics is why anyone votes when the electorate is large and, therefore, each individual voter's chance of influencing the outcome is near zero. Critics of rational choice theory point to voter turnout as a contradiction of rational action when voting is even slightly costly. We present a model in which some prospective voters abstain, others turnout to vote, and we show how turnout varies with changes in the costs of voting, the size of pork-like rewards, and how groups of voters are organized in society. The model examines the incentives to vote on the basis of policy preferences, partisanship, and in pursuit of what we call contingent prizes.

If contingent prizes are in the form of non-rival good that cost a similar amount to provide independent of the number of group members who benefit from them, then parties prefer political competition to be between a relatively small number of large groups. The promotion of one group's language over other would be such an example. In contrast, when prizes are rival in nature so that as more people benefit from them the more expensive they become to provide, parties prefer that voters are divided into lots of small groups competing in lots of competition. A central result of the model demonstrates that even though voters in large electorates have little influence over which party wins, they retain substantial influence over the distribution of rewards and benefits if parties disproportionately reward groups that provide them with high levels of political support. Our analysis suggests group-level rewards can provide a stronger impetus for voting than do policy differences across parties or other sources of partisanship. What is more, since voters are shown generally to compete over prizes, our model predicts non-trivial turnout even in cases where one party is almost certain to win. Further, it predicts lower turnout among non-partisan voters than partisans in competitive two-party contests.

Pork barrel rewards to local constituencies or preferential benefits, like tax breaks or government jobs, to specific groups are a persistent feature of politics. The model endogenizes the use of these political tools to disproportionately reward identifiable groups of voters. In doing so it provides an explanation for the aggregation, for instance, of Congressional election votes by precinct even though the electoral outcome is unrelated to winning any number of individual precincts. Of course, if votes were put in a common pool and added up at the district level then it would not be possible for parties to identify their most supportive precinct-based voter groups, eradicating the contingent prize incentive for voting based on neighborhood. That incentive serves the interests both of parties and of voters. According to the model, when parties offer few prizes, turnout is low, with only voters with a strong preference for one or the other party voting. Once contingent prizes are thrown into the mix, then voters compete to win benefits for their group and turnout increases as the value of the prizes increases. Indeed, even when the electoral outcome is a forgone conclusion, the value of the contingent prize can induce voters to turnout even to support a party whose policies they oppose. Hence, we show that the strong incentive to pursue local group benefits is an explanation for the persistence of dominant parties even when the electorate acknowledges that an alternative party's policies are preferable.

The model shows that when a party is interested in winning with high turnout, it is better off having even its own supporters divided into at least two discernible groups rather than being one hegemonic bloc. Two groups foster competition over contingent benefits and, therefore, induce higher turnout which can then be used to claim an electoral mandate. Voters share the desire to organize into groups as it increases their access to rewards.q

As with any model, there are, of course, limitations. The model points to the necessity of developing a theory to explain when prizes take the form of non-rival goods and when they take the form of rival goods. Group formation may be driven by political elites or by local voter-entrepreneurs. Depending on which plays the main role in bloc formation there will be variation in the number of groups, the size of prizes, and probably the form of prizes. We have not addressed these issues except by assumption or by example. Then, of course, there is the empirical challenge of testing the observable conditions of turnout, pork allocation and the like under the model's equilibrium conditions. All of these issues remain for future research.

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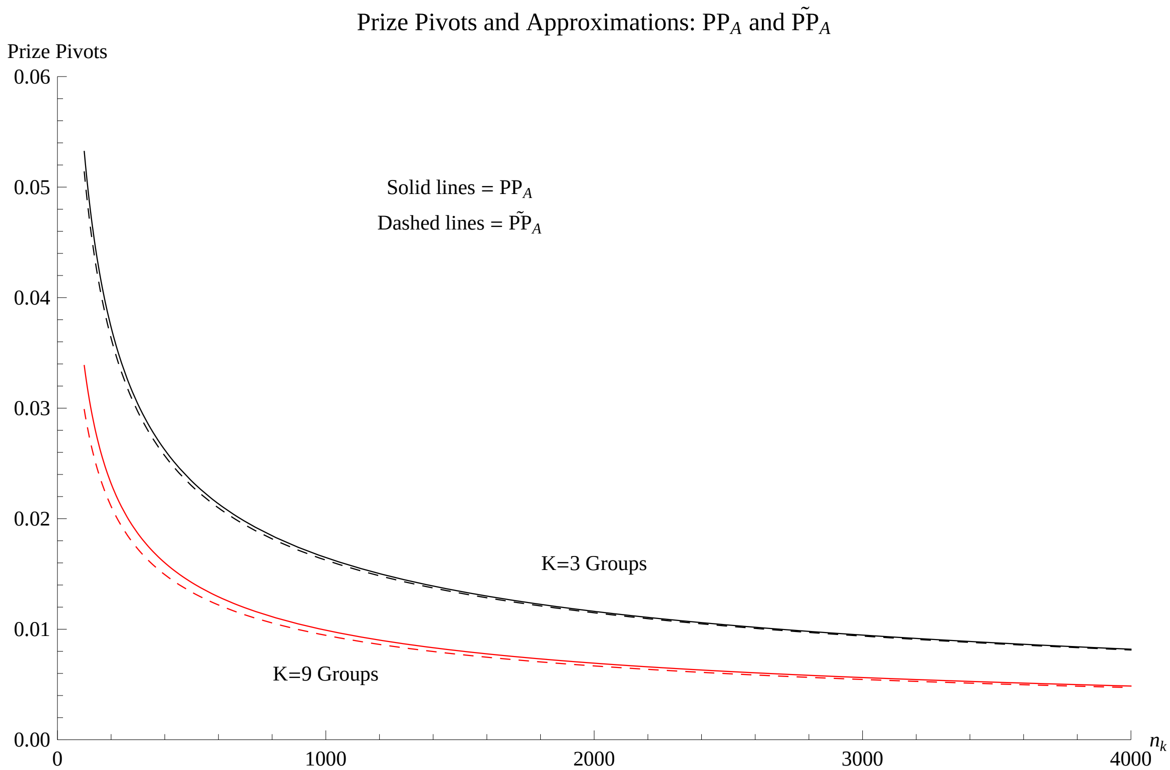

Figure 1: Prize Pivot Approximations

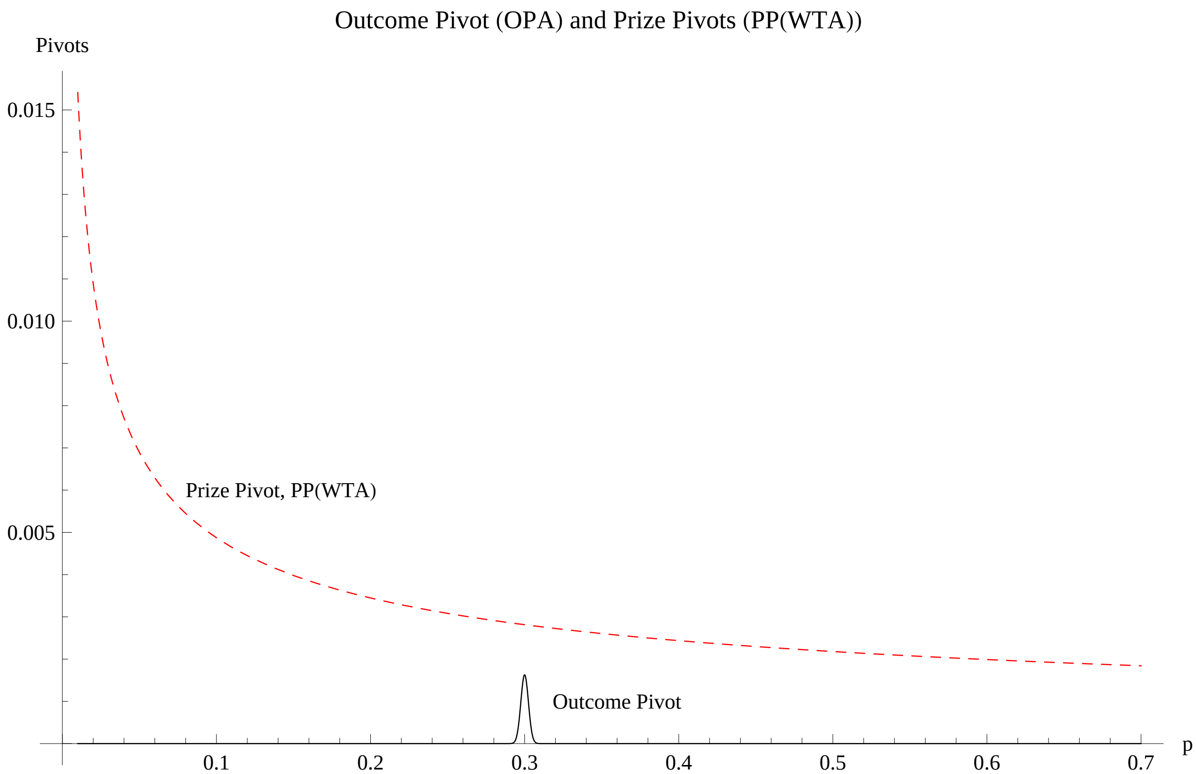

Figure 2: Outcome Pivot and Prize Pivot

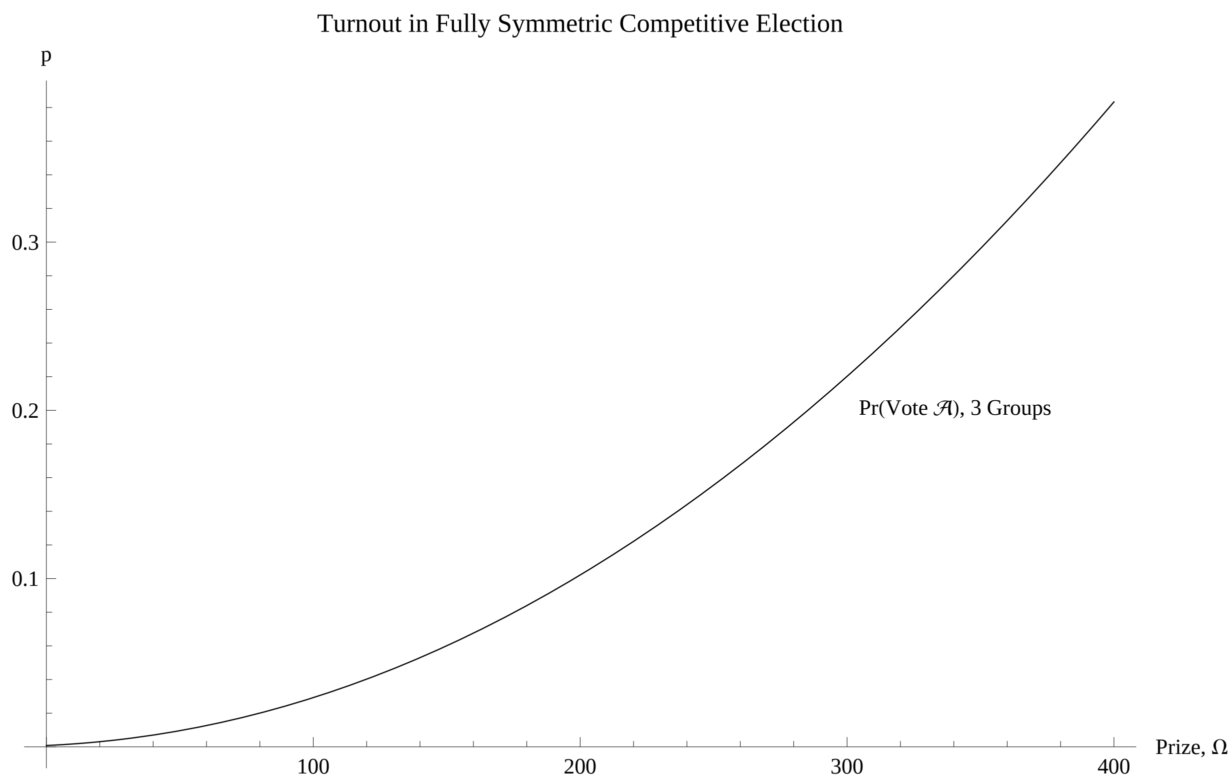

Figure 3: Turnout and Prizes in Fully Symmetric Competitive Voting Equilibrium

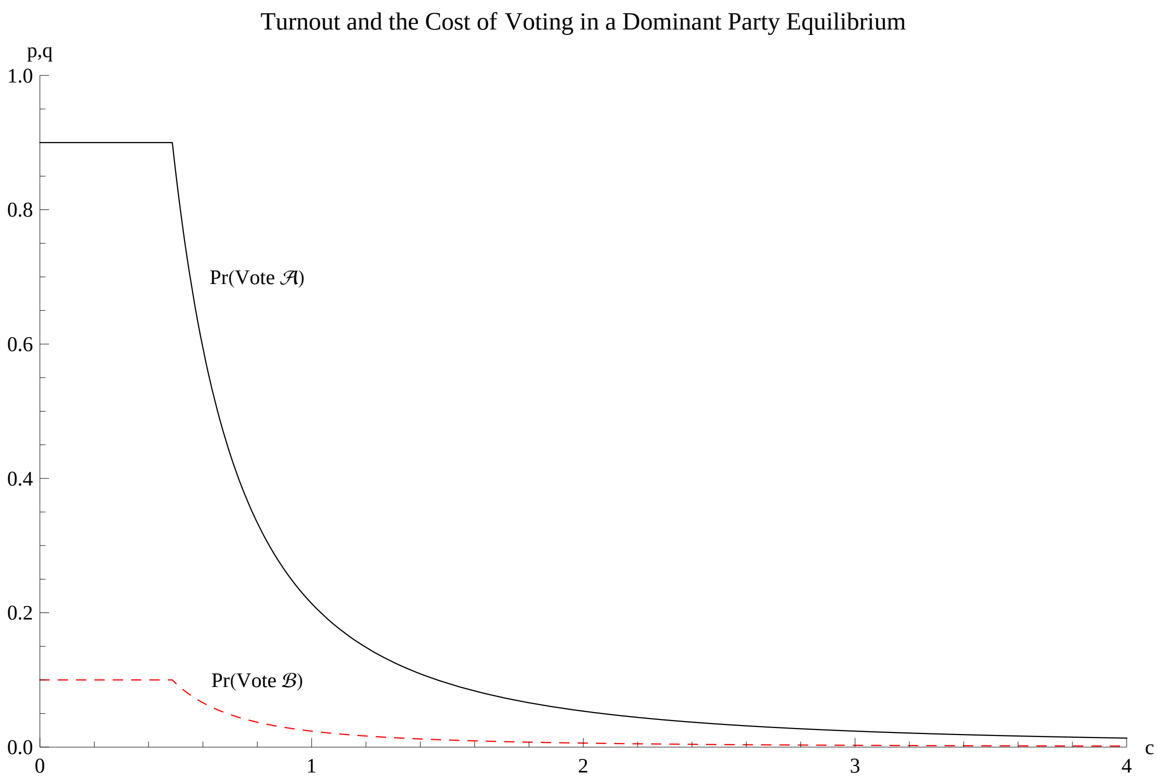

Figure 4: Dominant Party Equilibrium

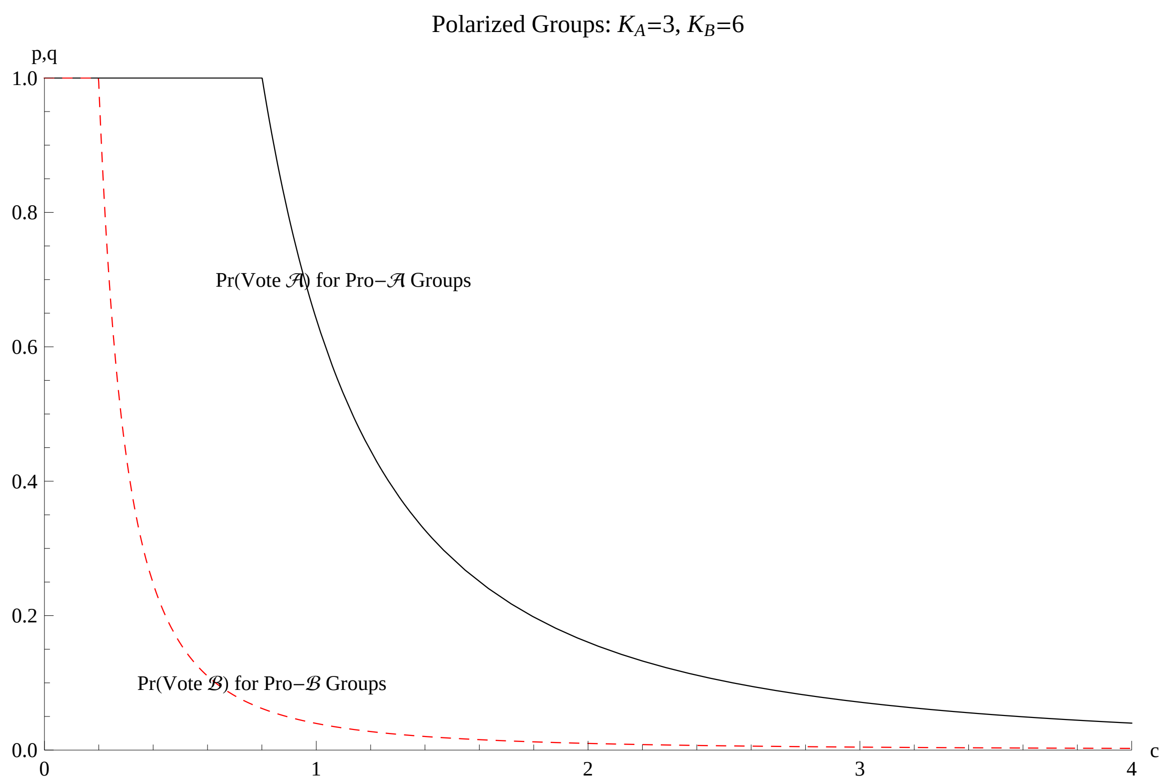

Figure 5: Group Polarization in Dominant Party Equilibrium

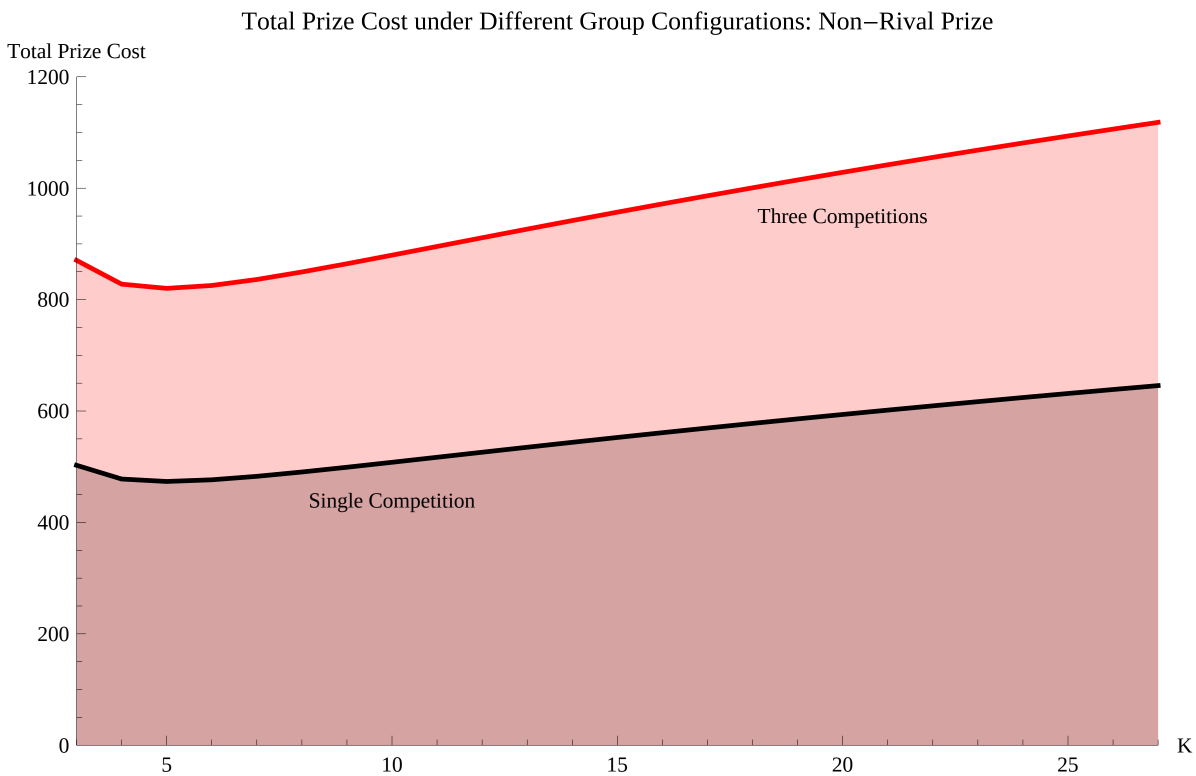

Figure 6: Total Prize Cost under Different Group Configurations: Non-Rival Prize

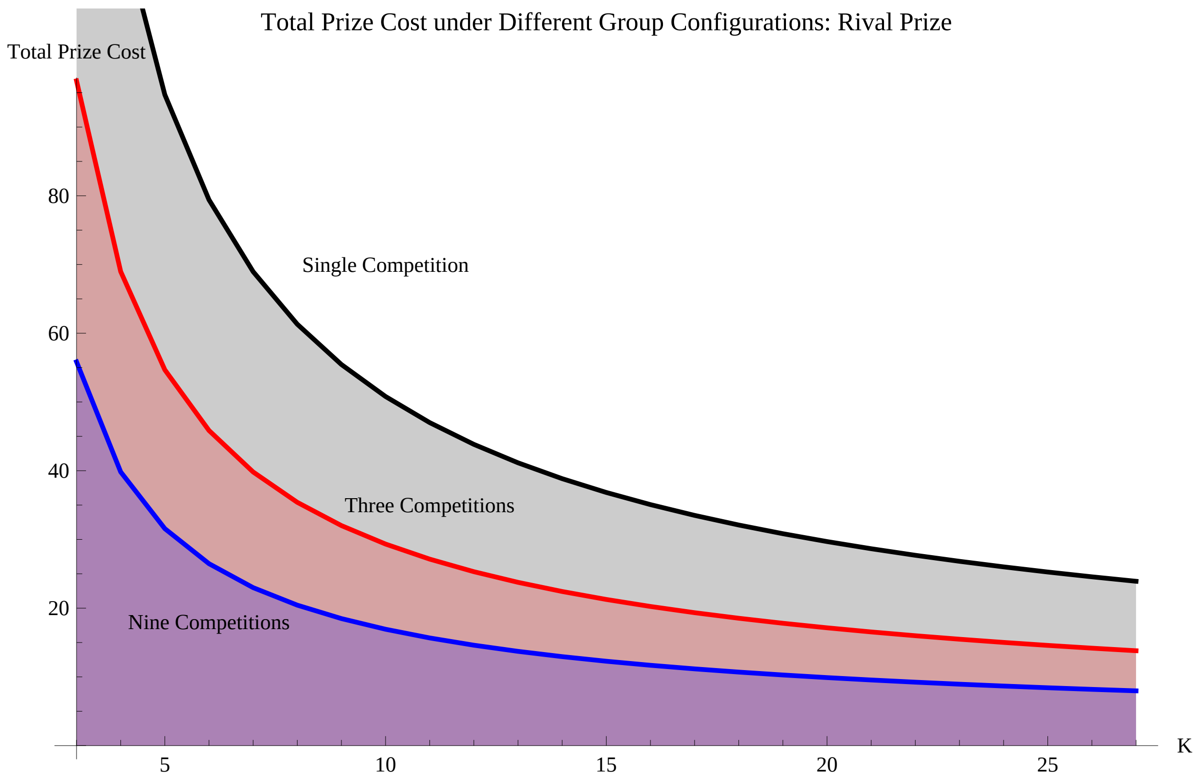

Figure 7: Total Prize Cost under Different Group Configurations: Rival Prize