Policy Contests[1]

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[1]VERY PRELIMINARY AND INCOMPLETE

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Abstract

In many political environments, individuals or groups compete to have their preferred spatial policies enacted by exerting costly up-front effort; examples include valence competition in elections (Wiseman 2006, Meirowitz 2008, Ashworth & Bueno de Mesquita 2009), expenditures in lobbying contests (Jordan & Meirowitz 2012), and productive investments in quality (Lax & Cameron 2007, Hirsch & Shotts 2012). Because such contests typically exhibit mixed strategy equilibria, previous analyses have gained tractability by assuming sequential movers (Wiseman 2006, Lax & Cameron 2007), a binary policy space (Meirowitz 2008), or sequentially separate choices of policy and effort (Ashworth & Bueno de Mesquita 2009). We propose a general model for analyzing simultaneous-move all-pay policy contests with a continuous policy space.

1 Introduction

In politics, individuals or groups often compete to have their preferred spatial policies enacted by exerting costly up-front effort. In electoral contests, competing candidates or parties make costly up-front investments in campaign spending or in governing ability in order to gain the support of the electorate (Wiseman 2006, Meirowitz 2008, Ashworth & Bueno de Mesquita 2009). In contests for policy influence, interest groups make costly up-front expenditures in contributions or persuasion to improve the chances that a legislator, executive, or bureaucrat will choose their policy proposal over that of a competitor (Jordan & Meirowitz 2012). And within many political organizations including legislatures, cabinets, and courts, competing actors make costly investments in the quality of their proposals (or their clarity and persuasiveness in the case of court opinions) in order to persuade a decision-maker to choose it (Lax & Cameron 2007, Hirsch & Shotts 2012, Hirsch & Shotts 2015*a*).

Because such contests are all-pay, they have generally proved difficult to analyze. Previous scholars have thus gained tractability in a variety of ways. Some models assume that the "proposers" move sequentially, but that each makes their choice of policy and effort simultaneously (Wiseman 2006, Lax & Cameron 2007). This approach eliminates mixed strategy equilibria but can be cumbersome to analyze, and makes results dependent on an arbitrarily-chosen ordering of the players. Other models assume that the proposers choose simultaneously, but sequentially separate their choice of policy and effort (Ashworth & Bueno de Mesquita 2009, Serra 2010). This choice is natural in many substantive applications, but yields strategic incentives that depend heavily on the game sequence. For example, in Ashworth & Bueno de Mesquita (2009) the proposers effectively collude through the game sequence, moderating their ideological platforms in order to reduce the intensity of the anticipated spending contest. Other models eliminate the proposers' ability to choose their ideological platforms, reducing the game to a more-standard all pay contest with flexible payoffs for winning and losing (Meirowitz 2008). Finally, the all pay contest literature commonly assumes a fixed payoff for losing (Che & Gale 2003, Siegel 2009), which arguably eliminates the core feature of a political contest: that a policy-motivated actor cares specifically about which *policy* they lose to, and not simply whether they lose.

In this paper, we develop a model of policy contests using an alternative but arguably-more straightforward approach than these papers. Building on the competitive policy development models of Hirsch & Shotts (2015*a*) and Hirsch (2015), we simply assume that the players in the contest both move simultaneously, and choose policy and effort simultaneously.[1] This approach eliminates the need to make arbitrary decisions about the sequence of play. Somewhat surprisingly, it also can yield simpler equilibria. In many substantively-interesting cases (including ones in which the proposers have asymmetric preferences and costs), the approach yields unique equilibria with strategies that can be characterized analytically. It is also often straightforward to analytically characterize the equilibrium distribution over spatial policy proposals and outcomes, the utilities of the proposers, and their probabilities of victory. Finally, in cases in which strategies cannot be characterized analytically, it is simple to do so computationally.

The sequence of the game is as follows. Two competing policy proposers simultaneously each choose spatial ideological locations at which to propose policies (from a unidimensional space), as well as how much up-front costly effort to invest in helping their proposal get chosen by a "decisionmaker." The decisionmaker, who has preferences over both ideology and effort, then chooses between these two proposals; in the baseline variant of the model he may also choose "none of the above."[2] The decisionmaker may be thought of as a single individual, or as a reduced-form representation of a collective decisionmaking process. Because proposers can gain the decisionmaker's support both by exerting more effort and/or by making ideological concessions, in the model they effectively try to exploit the former to reduce their need to do the latter.

Unlike standard all-pay contests in which the participants have a fixed payoff from losing, each proposer's final utility depends upon the ideology and effort of the *winning proposal,* as well as *whether* they are the winner (and any costs incurred). The model thus features what Baye, Kovenock & Vries (2012) term "second order rank order spillovers": the strategy of the winning player "spills over" onto the utility of the "second ranked" losing player. In addition, the players' final payoffs over the winning proposal may depend only on the ideology of that proposal, the effort expended to help it get chosen, or both. The framework thus accommodates electoral contests with unproductive effort (including or excluding pure "office-holding" benefits), policy-development contests with productive effort, and combinations (e.g. if effort on an opponent's proposal is discounted or even disliked).

[1] Hirsch & Shotts's (2015*b*) policy development model with veto players, however, does not fit in the framework in this paper – while similar techniques can be used, the inclusion of veto players substantially complicates the contest.

[2] Specifically, in the present draft we develop necessary and sufficient conditions for equilibrium assuming a "none of the above" choice. These equilibria are unique up to a particular aspect of the strategies that does not affect payoffs or equilibrium. We then show that choosing this aspect in a simple way yields an equilibrium of the model without a "none of the above" choice.

We first provide necessary and sufficient conditions for equilibrium in the general model. Equilibria are in two-dimensional mixed strategies. These strategies can be characterized by a univariate probability distribution over the decisionmaker's utility, and simple functions that associate each utility with a specific combination of ideology and effort. The proposers' probability distributions over the decisionmaker's utility have common support, and participation in the contest is generically asymmetric: one proposer always exerts effort, while the other sometimes does not. Finally, the equilibrium probability distributions are simply characterized by a system of differential equations and boundary conditions that is straightforward to compute numerically.

Next we consider a symmetric version of the model. We show that this yields a simpler symmetric equilibrium in which both proposers are always active. In addition, we provide a condition on the cost and payoff functions of the proposers that dramatically simplifies a partial equilibrium characterization. We show how when this condition is satisfied, the distribution over ideological proposals and outcomes is almost trivially characterized analytically without need to fully derive the proposers equilibrium strategies, and even when the latter does not yield an analytical characterization.

Last, we consider a parametric version of the model that is sufficiently flexible to accommodate many applications of interest. In the parametric model, each player's utility over the winning proposal is a linear function of two parameters – the squared distance of the player's ideal point from the proposal's ideology, and the effort exerted on the proposal. The functions may differ depending on the source of the winning proposal. As a special case the parametric model includes spending contests (with and without office holding benefits) and policy development contests (Hirsch & Shotts 2015*a*, Hirsch 2015).

We first explore a symmetric version of the parametric model. We provide an analytical characterization of ideological proposals and outcomes, and perform comparative statics on the decisionmaker's utility. Substantively, we show some surprising and subtle effects. Decisionmakers who value ideology more elicit more moderate proposals closer to their ideal points, but those who are more malleable and responsive to effort elicit more extreme proposals as a by-product. Depending on the application effect this effect may be interpreted as a negative welfare effect of decisionmakers who are more easily "bought."

Proposers who unconditionally value victory more, who discount or intrinsically dislike effort expended to promote their opponents' proposals, or who place greater weight on ideological losses when they lose, yield unambiguous benefits for the decisionmaker; these factors have no effect on the ideological extremism of their proposals, but increase the effort they expend to encourage the decisionmaker to choose them. Proposers who are more ideologically extreme or intense, who have lower costs of exerting effort, or who place a greater intrinsic value on their own effort, make proposals that are first-order stochastically better for the decisionmaker, but that are also more ideologically extreme as a by-product. The welfare effects of such comparative statics thus depends on whether effort is interpreted literally as something that improves the decisionmaker's welfare (e.g. a transfer or an investment in quality), or as behavior that influences his choices without actually making him better off (e.g. campaign advertising on the electorate).

Next we explore asymmetric variants of the parametric model. We provide analytical solutions for a special case that generalizes the pure policy-policy development model of Hirsch (2015), in which effort is productive and common value. Key results are similar; we show that a proposer becoming more ideologically extreme or effective at exerting effort increases the ideological extremism of her own proposals, decreases the ideological extremism of her opponent's proposals, and makes her opponent worse off. We also show that the decisionmaker ambiguously benefits when the proposers have higher unconditional victory payoffs, i.e. "office holding benefits"; this increases the intensity of competition without affecting the ideology of proposals.

Last, we explore numerical results for the fully general parametric model, and consider the effects of unilateral changes in a proposers' parameters. We find that magnifying a proposer's incentive to participate in the contest in any capacity – be it making her more ideologically extreme, more efficient, more opposed to her opponent's policies, or more desirous of victory irrespective of the policy benefits, has similar effects. Her proposals become first order stochastically better for the decisionmaker, while those of her opponent either *also* become first-order stochastically better (if her opponent was initially more motivated), or respond in a mixed fashion. Somewhat surprisingly, on average the decisionmaker is usually made better off, even if the change results in the contest becoming increasingly imbalanced, and the likelihood of direct competition is low. The weaker proposer remains sufficiently active to prevent the stronger one from becoming too ideologically aggressive.

Finally, magnifying a proposer's incentive to participate in the contest in any capacity appears to first order stochastically increases the ideological extremism of her proposals, while first-order stochastically decreasing the extremism of her opponent's proposals. Interestingly, this is true even for factors that increase a proposer's cost of losing the contest *without* influencing her desire for policy gains, such as her "office-holding" benefits and her dislike of her opponent's policies. Overall, these results indicate that factors in the political environment that seem orthogonal to policy preferences can have subtle effects on ideological proposals and outcomes when considered jointly in a contest structure.

Related Literature The model is a direct generalization of the policy development model of Hirsch & Shotts (2015*a*) and (Hirsch 2015) – its place in the literature is therefore similar but with applicability to a wider range of substantive settings.

As previously described, our model is closely related to political economy models of "valence competition," in which competing individuals or groups make costly all-pay investments (both productive and unproductive) that improve the attractiveness of their spatial proposals to other actors. Such models have been applied to study (among other things) electoral competition, policy development in legislatures, and judicial opinion writing. In a subset of this work, the "all pay" choices of competing groups are made simultaneously, and therefore yield all-pay like contests with mixed strategy equilibria (Meirowitz 2008, Ashworth & Bueno de Mesquita 2009).

Within the study of contests, our model marries three strands of the literature.[3] One strand considers a variety of contest structures in which the players' strategies have a "spatial" component (Esteban & Ray 1999, Epstein & Nitzan 2004, Konrad 2000, Mnster 2006), and with both deterministic and Tullock-like outcome rules. A second strand of the literature considers players who may make multidimensional proposals, and in which the outcome is determined by a "scoring rule" (Che & Gale 2003, Siegel 2009). Finally, a small but growing strand of the literature considers contest models with "rank order spillovers" – that is, in which there are direct effects of the players strategies on their opponents that depend on the final ordering of the players (Baye, Kovenock & Vries 2012, Chowdhury & Sheremeta 2010) – for example, in an R&D contest the research conducted by the winning player may yield some smaller direct benefit for the loser. The existing literature suggests that in the presence of rank order spillovers, it is not possible to use techniques as in Siegel (2009) to calculate equilibrium payoffs without also fully characterizing equilibrium strategies.

At a broad leveler, the model provides a flexible approach to studying competition for intra- and inter-organizational influence. One strand of this existing literature analyzes competitive informational lobbying with either general policy-relevant information (Gilligan & Krehbiel 1989, Battaglini 2002) or information specific to an exogenous binary set of alternatives (Dewatripont & Tirole 1999, Gul & Pesendorfer 2012). Another strand considers influence via transfers to a decisionmaker or decisionmakers that are either contractible

[3] I thank Dan Kovenock for bringing this broader literature to our attention.

(Grossman & Helpman 1994) or non-contractible (Groseclose & Snyder 1996). Finally, intra-firm influence of various forms is analyzed in several models, including Milgrom & Roberts's (1988) model of self-promotion by subordinates, and Rotemberg & Saloner's (1994) model of competitive project investments by divisions within a firm. The model herein is distinct from each of these literatures–it is non-informational, the set of available alternatives is a continuum, and influence-generating investments are all-pay.

2 The Model

We analyze a contest over policies played by two competing proposers and a decisionmaker.

In the first stage, each of two proposers $i \in \{-1, 1\}$ makes two simultaneous choices – a choice of the *ideology* $y \in \mathbb{R}$ of their proposal, and a choice of costly up-front *effort* $q \in [0, \infty) = \mathbb{R}^+$ to help the proposal get chosen. Intuitively, proposer $i = -1$ (also referred to as the *left* proposer) prefers ideologies to the left of the decisionmaker's ideal, while $i = 1$ (i.e. the *right* proposer) prefers those to the right. Proposer i's up-front cost of effort q_i is $c_i(q_i)$, which is continuously differentiable, strictly increasing, and satisfies $c_i(0) = 0$ (i.e., there are no "fixed costs" to making a proposal).

In the second stage the decisionmaker chooses one of the two proposals or none of the above (which we term the "reservation proposal") and the game ends. (However, our necessary and sufficient conditions for equilibrium will, with a minor tweak, yield sufficient conditions for equilibrium absent the reservation proposal).[4] The decisionmaker's utility for a proposal (y, q) is $s(y, q)$. We assume that $s(y, q)$ is continuously differentiable and strictly quasi-concave in both parameters. We furthermore assume that the decisionmaker's unique ideal ideology is at $y = 0$, that his utility is strictly increasing in q $\forall y$, and impose the normalization that $s(0, 0) = 0$. Finally, the reservation proposal is assumed to be $(0, 0)$, or the decisionmaker's ideal with no effort.[5] A proposer can thus achieve exactly the decisionmaker's reservation value by proposing the decisionmaker's ideal point with no effort, and effort increases the value that the decisionmaker places on any proposal. Depending on the application, the effect of effort can be interpreted as a "black box" that increases the decisionmaker's chance of choosing a proposal, persuasion, or an investment in the "quality"

[4] Necessary and sufficient conditions for this variant are probably straightforward as well, time permitting.

[5] For notational purposes we place the reservation outcome in proposal "space"; but what matters is just that the proposers can offer equal utility by proposing the decisionmaker's ideal with no effort, and that this outcome would be weakly better for a proposer than the decisionmaker choosing the reservation outcome.

of the proposal such as increased efficacy or cost effectiveness. More abstractly, the decision-maker's utility is the scoring rule for the contest, and treating him as a player is equivalent to allowing an endogenous tie-breaking rule.

Each proposer's final utility in the game is based on (i) any up-front effort costs $c_i(q)$ incurred, (i) the *winning proposal* (y^w, q^w), and (iii) whether she is the winner. Cost and proposal utilities are assumed to be additively separable. Let $v_i^o(y^w, q^w)$ denote i's proposal utility (henceforth just utility) when her *own* proposal is the winner, $v_i^c(y^w, q^w) \leq v_i^o(y^w, q^w)$ denote i's utility when her *competitor's* proposal is the winner, and $V_i^r \leq v_i^o(0, 0)$ her utility when the "reservation proposal" is the winner. In the process of characterizing equilibria we will add assumptions that imply additional constraints on these functions, but for now we simply assume that they are all continuously differentiable in all parameters.

3 Equilibrium Characterization

The game is effectively a multidimensional all-pay contest (Che & Gale 2003, Siegel 2009), with some distinctive properties that we shortly highlight. As in Che & Gale (2003) the contest is multidimensional because proposals (y, q) are two-dimensional, and there is a "score function" $s(y, q)$ that determines the winner. In our model, a proposal's score is the utility it provides to the decisionmaker; the reason is that the decisionmaker cannot commit in advance to which proposal he will choose. Thus, he must always choose the proposal with the strictly highest score (and may randomize arbitrarily in the event of ties). Also like Che & Gale (2003), there are a *continuum* of proposals with different ideologies that have the same score s – specifically, the set (y, q) such that $s(y, q) = s$. These proposals have different costs, and are valued differently by each player.

The model has two features that distinguish it from previous multidimensional contests. First, the proposers are "policy motivated" rather than rent seeking (as in Tullock (1980) and Baye, Kovenock & Vries (1993)). This is captured by the fact that both players' utilities are based on the winning proposal and not just the identity of the winner. In particular, the winner's proposal has a direct effect on the loser's utility, a property that Baye, Kovenock & Vries (2012) term "second order rank order spillovers." Second, in the model the efforts made to gain influence over the decisionmaker are not simply transfers to the decisionmaker. Instead, the winner's effort q^w has a direct effect on both players' final utilities. Thus, effort may be productive and valued by both players, directly costly to both players, or some mixture. These differences stem from the fact that the model is designed to apply to policy-motivated actors in political environments.

Transforming the Problem Given the above, it is helpful to transform proposals (y, q) to be expressed in terms of score and ideology (s, y). This allow us to characterize a proposer's problem as the choice of a *score curve* s and an ideology y to propose along that score curve. First let $q(s, y)$ denote the function that solves $s(y, q(s, y)) = s$ (that is, the inverse of $s(y, q)$ in q). Thus, to make a proposal with score and ideology (s, y), a proposer must invest effort $q(s, y)$. It is easily verified that $q(s, y)$ is strictly increasing in s and strictly quasi-convex in y with minimum at $y = 0$, and that $q(0, 0) = 0$.

In the transformed problem, a proposer's pure strategy (s_i, y_i) is then a two-dimensional element of $\mathbb{B} \equiv \{(s, y) \in R^2 \,|\, q(s, y) \geq 0\}$, or the set of scores and ideologies that imply positive-effort proposals. A mixed strategy σ_i is a probability measure over the Borel subsets of $\mathbb{B}$, and let $F_i(s)$ denote the CDF over scores induced by i's mixed strategy σ_i. For technical convenience we restrict attention to strategies generating score CDFs that can be written as the sum of an absolutely continuous and a discrete distribution. The decisionmaker is the last mover, so equilibrium requires that he choose a proposal (s, y) with the maximum score. While a complete description of a strategy profile also requires specifying his tie-breaking rules, equilibria are invariant to this decision so we omit the additional notation.

We now introduce the following essential notation that rewrites the players' utilities and cost functions in terms of (s, y).

Definition 1 *Player i's utility if her own proposal (s_i, y_i) wins is*

$$V_i^o(s_i, y_i) = v_i^o(y_i, q(s_i, y_i))$$

Her utility if her opponent's proposal (s_{-i}, y_{-i}) wins is

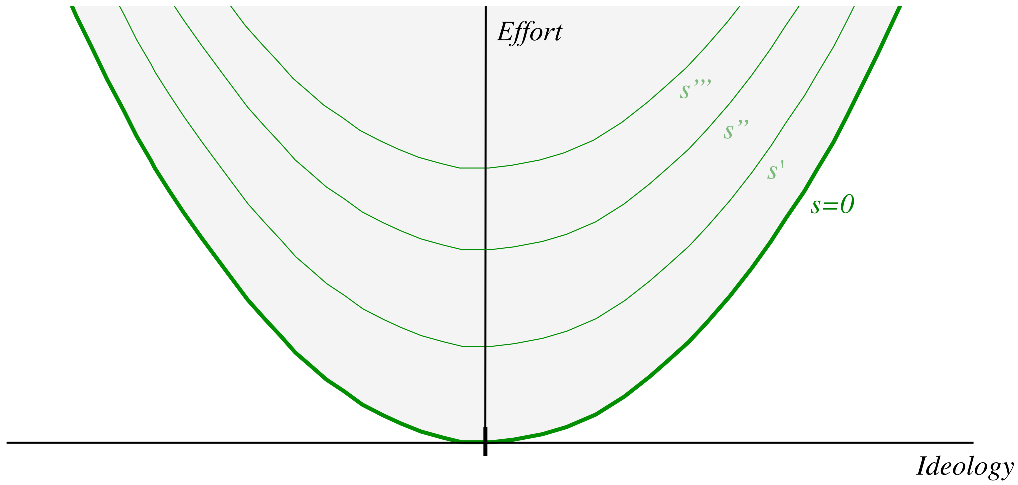
$$V_i^c(s_{-i}, y_{-i}) = v_i^c(y_{-i}, q(s_{-i}, y_{-i}))$$

Her up-front cost of proposing (s_i, y_i) is

$$C_i(s_i, y_i) = c_i(q(s_i, y_i))$$

Figure 1 depicts the basic structure of game in ideology-effort space. The decisionmaker's indifference curves, i.e., the policies with equal score, are depicted by green lines.

Figure 1: Setup of the Policy Contest



Optimal Ideologies and Model Assumptions Once the problem is transformed, the first step of deriving equilibrium is to characterize the optimal ideology $y_i^* (s)$ for a proposer to choose at every score s. To do this, let $\Pi_i (s_i, y_i; \sigma_{-i})$ denote i's expected utility from proposing (s_i, y_i) (suppressing the dependence on the DM's tie-breaking rules). At any score $s_i > 0$ where $-i$ has no atom,

$$\Pi_i (s_i, y_i; \sigma_{-i}) = -C_i (s_i, y_i) + F_{-i} (s_i) \cdot V_i^o (s_i, y_i) + \int_{s_{-i} > s_i} (V_i^c (s_{-i}, y_{-i})) \, d\sigma_{-i}. \quad (1)$$

The key property of equation 1 is that a proposer's choice of ideology *at any given score* s_i (where her opponent has no atom) has no effect on the probability she wins the contest, or on her utility in the event that she loses. Consequently, the optimal ideology to propose does not depend on the specific proposals that her opponent makes; only on the score s and the probability $F_{-i} (s)$ that her opponent makes a lower-score proposal. To ensure that the proposers' ideological maximization problem is well behaved at each score, we henceforth assume that the transformed cost and payoff functions yield a strictly concave objective with optima on the side of the decisionmaker associated with the proposer.

Assumption 1 *For all $F \in (0, 1]$, the function $-C_i (s, y) + F \cdot V_i^o (s, y)$ is strictly concave in y for all $s \geq 0$, with strictly positive (negative) maximum for proposer 1 (-1).*

Proposer i's unique optimal ideology $y_i^* (s)$ at almost every score is thus simply characterized with the first order condition, as stated in the following Lemma.

Lemma 1 *At any score $s > 0$ where $-i$ has no atom, proposing $y_i^* (s) = \hat{y}_i (s; F_{-i} (s)))$ is strictly better than proposing any other ideology y_i, where $\hat{y}_i(s;F)$ is the unique solution to*

$$-\frac{\partial C_i (s_i, y_i)}{\partial y_i} + F \cdot \frac{\partial V_i^o (s_i, y_i)}{\partial y_i}\bigg|_{y_i = \hat{y}_i(s;F)} = 0$$

Note that assumption 1 further implies that $sign\left(\frac{\partial \hat{y}_i(s;F)}{\partial F}\right) = sign(i)$ since $sign\left(\frac{\partial C_i(s_i,y_i)}{\partial y_i}\right) = sign(y_i)$ – so ceteris paribus, a higher probability F of winning with a score-s proposal yields a more ideologically-aggressive proposal.

With a characterization of the optimal ideologies in hand, we now make two additional assumptions that are necessary to derive simple conditions for equilibrium – these assumptions can be easily checked for any particular functional forms in the model.

The first assumption pertains to the marginal costs of making a higher score proposal.

Assumption 2 *For all $F \in (0, 1]$ and $s \geq 0$,*

$$-\frac{\partial C_i (s, y)}{\partial s} + F \cdot \frac{\partial V_i^o (s, y)}{\partial s}\bigg|_{y = \hat{y}_i(s;F)} < 0$$

Assumption 2 effectively states that at any weakly positive score, the marginal benefit of a higher-score proposal is strictly negative when the impact of the score on the probability of victory or the optimal ideology are ignored. In short, we assume that the proposers do not want to leave the decisionmaker with higher utility than they need to.

The second assumption pertains to the policy benefits of winning the contest.

Assumption 3 *For any mixture $\sigma_s (y)$ over score-s proposals (with $s \geq 0$),*

$$\int V_i^o (s, y)\, d\sigma_s (y) > V_i^o (s, 0) \rightarrow V_{-i}^o (s, 0) \geq \int V_{-i}^o (s, y)\, d\sigma_s (y)$$

Assumption 3 effectively states that despite the potential for some shared interests (e.g. if effort is partially productive) the proposers are always in a state of ideological conflict at a given score, with the decisionmaker between them.[6] If at any given score s a proposer i strictly prefers a mixture of proposals $\sigma_s (y)$ to the decisionmaker's ideal $(s, 0)$, then her opponent $-i$ at least weakly prefers the decisionmaker's ideal.[7]

[6]An implication of the preceding assumptions is that $V_i^o (\hat{y}_i (F_{-i} (s) ; s)) \geq V_i^c (\hat{y}_{-i} (F_i (s) ; s))$ with strict inequality if $F_k (s) > 0\ \forall k \in \{-1, 1\}$. That is, for any profile of score CDFs and at any score, it is better to win with one's optimal proposal than to lose to one's opponent's optimal proposal.

[7]This assumption is almost surely too strong, but concessions must be made for deadlines.

Necessary and Sufficient Conditions We now derive necessary and sufficient conditions for equilibrium in a series of lemmas.

We first establish that in equilibrium there is 0 probability of a tie at a strictly positive score. The absence of "score ties" is an intuitive consequence of the fact that effort is all-pay–if an proposer knew that her proposal might tie with her opponent's proposal, she could invest a bit more effort up front to break the tie.[8]

Lemma 2 *In equilibrium there is 0-probability of a tie at scores $s > 0$.*

Next, having ruled out ties at strictly positive scores, we show that one of the proposers must always be *active*, in the sense of making a proposal with score strictly higher than 0 (the score that can be achieved with no effort by proposing $(0, 0)$). In other words, $F_k(0) > 0$ for *at most* one $k \in \{-1, 1\}$, implying that k's opponent $-k$ is always active ($F_{-k}(0) = 0$).

Lemma 3 *In equilibrium $F_k(0) = 0$ for some $k \in \{-1, 1\}$.*

Lemmas 1 – 3 jointly imply that in equilibrium, proposer i can compute her expected utility *as if* her opponent always makes proposals of the form $(s_{-i}, \hat{y}_{-i}(s_{-i}; F_i(s_{-i})))$. Thus, proposer i's utility from any proposal (s_i, y_i) with $s_i > 0$ where her opponent's score CDF F_{-i} has no atom (or if a tie would be broken in her favor) can be written as

$$\Pi_i^*(s_i, y_i; F) = \underbrace{-C_i(s_i, y_i)}_{\text{effort cost}} + \underbrace{F_{-i}(s_i) \cdot V_i^o(s_i, y_i)}_{\text{Pr win} \cdot \text{utility if win}} + \underbrace{\int_{s_i}^{\infty} V_i^c(s_{-i}, \hat{y}_{-i}(s_{-i}; F_i(s_{-i})))\, dF_{-i}}_{\text{utility when lose}}, \quad (2)$$

where $F = (F_i, F_{-i})$ denotes the profile of score CDFs. Her utility from making the *best* proposal with score s_i is $\Pi_i^*(s_i, \hat{y}_i(s_i; F_{-i}(s_i)); F)$, which we henceforth denote $\Pi_i^*(s_i; F)$.

Finally we show that equilibrium score CDFs must satisfy the following natural properties that arise from the all pay nature of the contest.

Lemma 4 *The support of the equilibrium score CDFs over $s \geq 0$ is common, convex, and includes 0. In addition, both CDFs are atomless $\forall s > 0$.*

To conclude, we combine the preceding lemmas to state a preliminary characterization of all equilibria in the form of necessary and sufficient conditions.

[8] Proving this property is more complex than in all-pay contests without spillovers, because the utility from tying can be a complicated function of the opponent's policies and the decisionmaker's decision rule.

Proposition 1 *A strategy profile σ is a SPNE i.f.f. it satisfies the following conditions.*

1. ***(Ideological Optimality)*** *With Pr. 1, each i makes proposals with either*

   (a) *negative score $s_i \leq 0$ and no effort $(q(s_i, y_i) = 0)$, or*

   (b) *positive score $s_i > 0$ with ideology $y_i = \hat{y}_i(s_i; F_{-i}(s_i))$.*

2. ***(Score Optimality)*** *The profile of score CDFs F satisfy the following boundary conditions and differential equations.*

   - ***(Boundary Conditions)*** *$F_k(0) > 0$ for at most one proposer $k \in \{-1, 1\}$, and there $\exists \bar{s} > 0$ such that $\lim_{s \to \bar{s}} \{F_i(s)\} = 1 \ \forall i$.*
   - ***(Differential Equations)*** *For all $i \in \{-1, 1\}$ and $s \in [0, \bar{s}]$,*

$$\frac{\partial C_i(s, y)}{\partial s} - F_{-i}(s) \cdot \left.\frac{\partial V_i^o(s, y)}{\partial s}\right|_{y = \hat{y}_i(s; F_{-i}(s))}$$
$$= \quad f_{-i}(s) \cdot \left(V_i^o(s, \hat{y}_i(s; F_{-i}(s))) - V_i^c(s, \hat{y}_{-i}(s; F_i(s)))\right)$$

In equilibrium, proposer i's expected utility is $-C_i(\bar{s}, \hat{y}_i(\bar{s}; 1)) + V_i^o(\bar{s}, \hat{y}_i(\bar{s}; 1))$ and the decisionmaker's expected utility is $\int_0^{\bar{s}} \frac{\partial}{\partial s}(F_1(s) \cdot F_{-1}(s))\, ds$.

This completes our general characterization of equilibrium.

Observations about equilibria Proposition 1 implies that equilibria have a simple form. First, at least one proposer $-k$ is *always* active – thus, competition always strictly benefits the decisionmaker. The other proposer k may also always be active ($F_k(0) = 0$) or be inactive with strictly positive probability ($F_k(0) > 0$). Second, when either proposer i is active, she mixes smoothly over the ideologically-optimal policies $(s, \hat{y}_i(s; F_{-i}(s)))$ with scores in a common mixing interval $[0, \bar{s}]$ according to the CDF $F_i(s)$.[9]

The differential equations that must be satisfied by equilibrium score CDFs arise intuitively from the requirement that both proposers be indifferent over making all ideologically-optimal proposals with scores in $[0, \bar{s}]$. The left hand side of each differential equation is $i's$ net marginal cost of producing a higher-score proposal, given a fixed probability $F_{-i}(s)$ of winning the contest; the proposer pays marginal cost $\frac{\partial C_i(s,y)}{\partial s}$ for sure, but with probability

[9]Technically this wording is a bit sloppy since the proposition does not state that the support interval is also bounded ($\bar{s} < \infty$), but this is true in our parametric applications.

$F_{-i}(s)$ her proposal is chosen and she enjoys a marginal benefit of $\frac{\partial V_i^o(s,y)}{\partial s}$ (if she places some intrinsic value on the products of effort). The right hand side represents $i's$ marginal policy benefit of producing a higher score. Doing so increases by $f_{-i}(s)$ the probability that her proposal wins, which changes the ideological outcome from her opponent's optimal proposal $(s, \hat{y}_{-i}(s; F_i(s)))$ at score s to her own optimal proposal $(s, \hat{y}_i(s; F_{-i}(s)))$.

Worth noting is that certain aspects of the strategies are irrelevant for equilibrium. The first of these is the decisionmaker's tie-breaking rule; this is a consequence both of the absence of ties at positive scores, as well as the fact that one proposer is always active. The second of these is the exact way that a sometimes inactive proposer k (if one exists) is inactive, since Proposition 1 places no conditions whatsoever on the score CDFs below 0. Thus, the sometimes-inactive proposer can be inactive by having an atom at score 0. However, she may also have an arbitrary CDF over scores $s \leq 0$ generated by 0-quality proposals. The key thing is that she be "sometimes inactive" with 0-quality proposals that are costless to make and always lose the contest.

Finally, it is simple to generate sufficient conditions for equilibrium of the model without the reservation proposal (that is, in which the decisionmaker is constrained to choose one of the proposals made by the policy-motivated players) by simply adding the additional constraint that the sometimes-inactive proposer k uses an atom at exactly score 0, which we state in the following corollary.

Lemma 5 *The conditions in Proposition 1 and also* $F_i(s) = 0\ \forall i, s < 0$ *are sufficient for equilibrium in the model without the reservation proposal.*

3.1 Symmetric Variant

Considering a symmetric version of the model yields the potential for an equilibrium in symmetric strategies. In a symmetric equilibrium, both proposers use a common score CDF $F(s)$, and at each score propose ideologies that are equidistant from the decisionmaker, i.e. $y_i^*(s) = -y_{-i}^*(s)$. This simplifies solving for equilibrium in two ways. First, we need only solve for the single equilibrium score CDF $F(s)$ rather than a coupled system $(F_1(s), F_{-1}(s))$. Second, we no longer need search for the atom $F_{-k}(0) > 0$ that the "sometimes inactive" proposer uses that yields a pair of CDFs $(F_1(s), F_{-1}(s))$ with common support. Instead, symmetry and Lemma 3 imply that both proposers are always active ($F_1(0) = F_{-1}(0) = F(0) = 0$). This substantially simplifies deriving numerical solutions when analytical ones are not available.

Formally, to generate a symmetric game we add three additional assumptions:

Assumption 4 *The symmetric model satisfies the following additional assumptions:*

1. *The decisionmaker's utility is symmetric about his ideal* ($s(y,q) = s(-y,q)\,\forall q$).

2. *The proposers have identical cost functions* ($c_i(\cdot) = c_{-i}(\cdot)$).

3. *The proposers' utility functions are mirror images of each other about the decisionmaker's ideal* ($v_i^o(y,q) = v_{-i}^o(-y,q)$ *and* $v_i^c(y,q) = v_{-i}^c(-y,q)$).

Observe that these assumptions jointly imply that the transformed game is symmetric. Assumption 4.1 implies that each proposer must develop the same level of quality to get an equally extreme ideology on their "side" ($s(y,q) = s(-y,q) \to q(s,y) = q(s,-y)$). Combined with assumption 4.2, this implies that the proposers' cost functions are both identical ($C_i(s,y)) = C_{-i}(s,y) = C(s,y)$) and symmetric ($C(s,y) = C(s,-y)$).

Finally, Assumptions 4.1 and 4.3 together imply that in the transformed game the proposers' utility functions are also mirror images of each other, i.e.,

$$\begin{aligned} V_i^o(s,y) &= v_i^o(y,q(s,y)) = v_i^o(y,q(s,-y)) \text{ (by 4.1)} \\ &= v_{-i}^o(-y,q(s,-y)) = V_{-i}^o(s,-y) \text{ (by 4.2)}. \end{aligned}$$

In the symmetric game we therefore denote $V_1^o(s,y)$ as $V^o(s,y)$ and $V_1^c(s,y)$ as $V^c(s,y)$, so that $V^o(s,y) = V_{-1}^o(s,-y)$. The interpretation is that $V^o(s,y)$ (or $V^c(s,y)$) denotes a proposer's utility for winning with (or losing to) a proposal of score s and ideological distance y *in their direction* from the decisionmaker.

With the above notation in hand, we may now provide a characterization of symmetric equilibria in the symmetric model, which is a simplification of Proposition 1.

Corollary 1 *In the symmetric model, a symmetric strategy profile* $\left(F(s), y_i^*(s) = -y_{-i}^*(s)\right)$ *is an equilibrium if and only if it satisfies the following conditions:*

1. ***(Ideological Optimality)*** *With Pr.* 1*, each* i *makes proposals of the form* $(s, i \cdot \hat{y}(s, F(s)))$*, where*

$$\left.\frac{\partial C(s,y)}{\partial y} + F\frac{\partial V^o(s,y)}{\partial y}\right|_{y=\hat{y}(s;F)} = 0$$

2. ***(Score Optimality)*** *The proposers' common score CDF F satisfies $F(0) = 0$ and $\forall s \in [0, \bar{s}]$,*

$$\begin{aligned} & \left. \frac{\partial C(s,y)}{\partial s} - F(s) \cdot \frac{\partial V^o(s,y)}{\partial s} \right|_{y=\hat{y}(s;F(s))} \\ = & \; f(s) \cdot \left(V^o(s, \hat{y}(s; F(s))) - V^c(s, -\hat{y}(s; F(s))) \right) \end{aligned}$$

Note that we do not assert that all equilibria in the symmetric model are symmetric – only the weaker claim that any symmetric equilibrium is characterized as above.[10] In addition, since in a symmetric equilibrium both proposers are always active, a further implication is that any symmetric equilibrium of the baseline model remains an equilibrium when the reservation proposal is removed from the game (see Lemma 5).

3.2 Special Case: Separable Utilities and Costs

Of interest in the model are the equilibrium distribution over proposals, over the decisionmaker's utility, and over the ideology of the resulting proposals and outcomes. Generically, characterizing any one of these quantities for a particular set of parametric utilities and costs requires fully characterizing an equilibrium profile satisfying Proposition 1.

However, in the special case that the assumption below is satisfied, it can sometimes be more straightforward to derive the equilibrium ideological proposals and outcomes without reference to the full equilibrium distribution over proposals inclusive of effort.

Assumption 5 *For all (i, s, y), $\frac{\partial^2 C_i(s,y)}{\partial y \partial s} = \frac{\partial^2 V_i^o(s,y)}{\partial y \partial s} = 0$*

The assumption states that the effect of a proposal's score and ideology on *both* the proposers' effort costs and victory benefits are separable. When assumption 5 holds, it is straightforward to see from Lemma 1 that the optimal ideology at each score $\hat{y}_i(s, F_{-i}(s))$ sheds its direct dependence on the score, and depends only *probability* $F_{-i}(s)$ that a score-s proposal from i win will the contest. When applicable we will therefore write this function as $\hat{y}_i(F_{-i})$.

Before preceding with a discussion of how this simplifies deriving equilibrium ideologies, we note that this property also has a strong substantive implication: a strict positive association between the *ideological extremism* of a proposal and the utility it gives the decisionmaker (since the CDFs are strictly increasing). This yields the following.

[10] Conditions for uniqueness are work in progress.

Corollary 2 *When own proposal utility and effort costs are separable in score and ideology:*

- *in any equilibrium with $F_i(s) = 0\ \forall s < 0$, the decisionmaker prefers each proposer's more ideologically extreme proposals to her more ideologically moderate ones.*
- *in any symmetric equilibrium, the decisionmaker always chooses the most ideologically-extreme proposal presented to her.*

Thus, a moderate decisionmaker's "endogenous extremism" exhibited in the parametric policy-development contests of Hirsch & Shotts (2015*a*) and (Hirsch 2015) is a generic property of policy contests with this form of separability.

Figure 2 depicts a typical equilibrium of the asymmetric model with these properties, while Figure 3 depicts a typical equilibrium of the symmetric model with them. In each figure, the top panel depicts proposers' score CDFs. The bottom panel depicts the ideological locations and effort of the proposals over which each proposer mixes – that is, a parametric plot of $\left(\hat{y}_i\left(F_{-i}\left(s\right)\right), s + \left(\hat{y}_i\left(F_{-i}\left(s\right)\right)\right)^2\right)$ for $s \in [0, \bar{s}]$. The ideological locations each i's proposals extend out to $\hat{y}_i(1)$, which is the ideological proposal she would make absent competition.[11]

Asymmetric Model To see how assumption 5 can sometimes simplify the derivation of equilibrium ideological proposals, first recall that in any equilibrium there is at most one proposer who is sometimes inactive ($F_k(0) > 0$) who we denote with k – hence, k's opponent $-k$ satisfies $F_{-k}(0) = 0$. Now let $s_i(F_i)$ denote the inverse of a proposer's equilibrium score CDF $F_i(s_i)$, and observe that in any equilibrium it is possible to define the function:

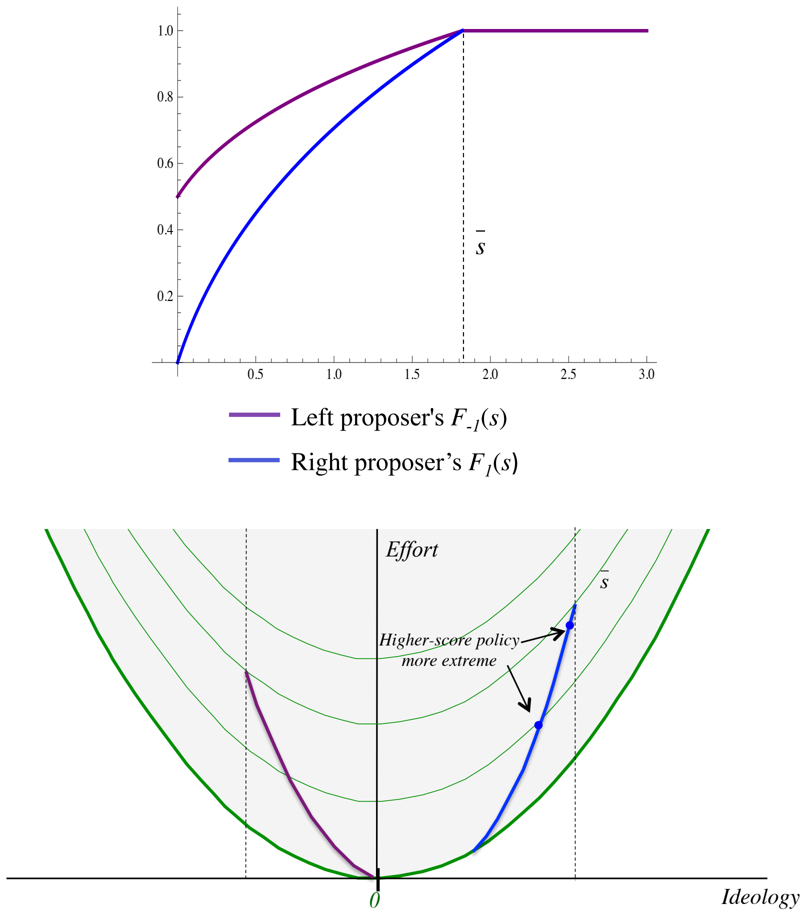
$$\phi\left(F_{-k}\right) = F_k\left(s_{-k}\left(F_{-k}\right)\right)$$

mapping the probability $-k$ makes a proposal with score lower than $s_{-k}(F_{-k})$ to the probability $F_k(s_{-k}(F_{-k}))$ her opponent k makes a proposal with score less than that same score.

We now provide expressions for the equilibrium ideological distributions when assumption 5 is satisfied that only use $\phi(F_{-k})$ and the optimal ideologies $\hat{y}_i(F_{-i})$ defined in Lemma 1 – recall that these are simply defined directly from the payoffs. This characterization is useful because for some parametric forms (solved later in this manuscript) it is possible to directly derive $\phi(F_{-k})$ without deriving the full equilibrium.

[11] The separability condition is sufficient for this property, but by no means necessary. For example, it is exhibited in numerical solutions of the symmetric model with quadratic costs (contact author for details).

Figure 2: Equilibrium Score CDFs and Policies, Asymmetric Model with Separability

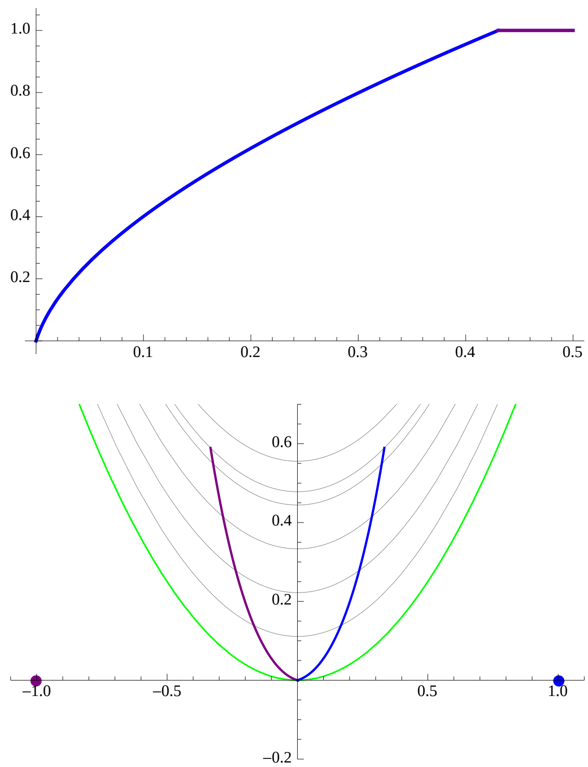


Proposition 2 *Let $\hat{F}_{-i}(y_i)$ denote the well-defined inverse of the* ***absolute value*** *of $\hat{y}_i(F_{-i})$, and consider an equilibrium with $F_i(s) = 0\ \forall s < 0$. Then:*

- *The probability $G_k(y_k)$ that the sometimes-inactive proposer k makes a proposal less ideologically extreme than y_k is $\phi\left(\hat{F}_{-k}(y_k)\right)$.*
- *The probability $G_{-k}(y_k)$ that the always-active proposer $-k$ makes a proposal less ideologically extreme than y_{-k} is $\phi^{-1}\left(\hat{F}_k(y_{-k})\right)$.*

The proof of Proposition 2 is relatively simple, although somewhat confusing due to the abundance of notation. Consider first the sometimes-inactive proposer k. Since assumption 5 implies a positive association between extremism and score, the probability that k makes a proposal less extreme than y_k is equal to the probability $F_k(s_k(y_k))$ that she makes a proposal

Figure 3: Equilibrium Score CDFs and Policies, Symmetric Model with Separability



with score less than $s_k(y_k)$, where $s_k(y_k)$ is the well-defined inverse of $y_k(s_k)$. Again applying assumption 5, in equilibrium $y_k(s_k) = \hat{y}_k(F_{-k}(s_k))$; we may thus immediately extract her *opponent* $-k$'s probability $F_{-k}(s_k(y_k))$ of making a lower-score proposal than $s_k(y_k)$ by inverting, which is $\hat{F}_{-k}(y_k)$. Finally, applying the transformation $\phi(\cdot)$ yields the desired probability $\phi\left(\hat{F}_{-k}(y_k)\right)$. The argument is identical to derive the always-active proposer k's distribution over ideologies except in reverse, using that $F_{-k}(s) = \phi^{-1}(F_k(s))$.

Finally, we note that it is also simple to use the function $\phi(\cdot)$ to express the probability that each proposer wins the contest. The probability that the always-active proposer $-k$ wins is $\int_0^{\bar{s}} F_k(s) f_{-k}(s)\, ds$; applying a change of variables (and using that $F_{-k}(0) = 0$ and $F_{-k}(\bar{s}) = 1$) yields that $-k$'s probability of victory is equal to

$$\int_0^1 \phi(F_{-k})\, dF_{-k}$$

The probability that k wins the contest is then $1 - \int_0^1 \phi(F_{-k})\, dF_{-k}$.

Symmetric Model In the symmetric model, assumption 5 immediately implies a very simple characterization of equilibrium ideological proposals and outcomes in any symmetric equilibrium. Again $\hat{y}(s;F)$ may be written as simply $\hat{y}(F)$, and denote the inverse as $\hat{F}(y)$. We then have the following.

Corollary 3 *In a symmetric equilibrium of the symmetric game, the* ***ideological extremism*** *of each player's proposals is distributed according to the CDF*

$$G(y)=\hat{F}(y)$$

In addition, since the decisionmaker always chooses the most extreme proposal, the CDF describing the ideological extremism of the final outcome is just $\left[\hat{F}(y)\right]^2$.

A direct proof of Corollary 3 is simple (although it also follows immediately from Proposition 2). Let $y(s)$ denote the equilibrium ideological extremism of a proposal of score s and $s(y)$ its inverse, which is well defined because of the strict positive association between extremism and score. In a symmetric equilibrium, $y(s)=\hat{y}(F(s))$ which implies that $y=\hat{y}(F(s(y)))$ and that $\hat{F}(y)=F(s(y))$. But the right hand side is exactly the probability $G(y)$ of a proposal less extreme than y, since there is a positive association between extremism and score and it is the probability of proposing a score $\leq s(y)$.

4 Parametric Model

Last we examine a flexible parametric model which can cover many political applications, including the pure "policy development" model (Hirsch & Shotts 2015*a*, Hirsch 2015).

Suppose that the decisionmaker's utility is $s(y,q)=\gamma_d q-\lambda_d y^2$ with $\gamma_d,\lambda_d>0$. He thus values quality linearly with weight γ_d, and has a quadratic loss function over ideology with weight λ_d and an ideal point at 0.

For the proposers, suppose that their cost functions are $c_i(q)=\alpha_i q$ – that is, the cost effort is linear. For their utility functions, suppose that

$$v_i^o(y^w,q^w)=\theta_i+\gamma_i^o q^w-\lambda_i^o(x_i-y)^2 \quad \text{and} \quad v_i^c(y^w,q^w)=\gamma_i^c q^w-\lambda_i^c(x_i-y^w)^2.$$

We make the following assumptions on the above parameters. First, $\theta_i\geq 0$ – the fixed benefit of winning the contest is weakly positive. Second, $x_{-1}<0<x_1$ – the proposers have ideal ideological outcomes on either side of the decisionmaker. Third, $\lambda_i^c\geq\lambda_i^o>0$ – ideological losses are weakly worse when a proposer loses than when she wins. Fourth, $\gamma_i^o \geq \max\{0, \gamma_i^c\}$ – effort on one's own proposal is not intrinsically harmful, and it is at least as beneficial as an opponent's effort on their proposal. Finally, $\frac{\lambda_i^o}{\gamma_i^o} > \frac{\lambda_d}{\gamma_d}$ – the proposers (relatively) value ideological outcomes at least as much as the decisionmaker.[12] [13]

The interpretation of the parameters is as follows. First, the parameters λ_d and γ_d capture the decisionmaker's responsiveness to ideological losses and to effort, respectively. The ratio $\frac{\lambda_d}{\gamma_d}$ therefore reflects how much effort is required to compensate the decisionmaker for a fixed amount of ideological concessions.

Second, the parameter θ_i captures proposer i's fixed payoff from winning the contest independent of the specific proposal – in an electoral context this would capture pure officeholding benefits, while in an organizational decisionmaking contest this may capture the benefits of increased future influence or a higher salary. In standard models of political contests θ_i is the only parameter (Meirowitz 2008).

Third, the parameter γ_i captures the extent to which effort expended to help a proposal get passed is *productive* rather than wasteful – γ_i^o captures the value of one own's effort should their proposal get chosen, and γ_i^c captures the value of one's opponent's effort should that proposal instead get chosen. If $\gamma_i^o = \gamma_i^c = 0$ then the proposers place no intrinsic value on their own or each other's efforts; in an electoral context effort may be thought of campaign advertising to influence the voters, whereas in a decisionmaking context it may represent upfront transfers or effort expended at persuasion. The case of $\gamma_i^o = \gamma_i^c = 1$ is the pure "policy development" model of Hirsch & Shotts (2015*a*) and Hirsch (2015) – effort is expended to improve the "quality" of a proposal, and that quality benefits the decisionmaker and the proposers equally. The difference $\gamma_i^o - \gamma_i^c$ captures the extent to which the intrinsic value of an opponent's effort is discounted. For example, in an electoral context a higher quality policy platform from one's opponent may be better for one own's welfare if implemented ($\gamma_i^c > 0$), but is also costly because it damages one's own future political and electoral prospects (so $\gamma_i^o - \gamma_i^c > 0$). We allow the case in which an opponent's effort is intrinsically valuable but discounted ($\gamma_i^o > \gamma_i^c > 0$), as well as when an opponent's effort is actually directly costly ($0 > \gamma_i^c$).

Last, the parameter λ_i captures the extent to which the proposer cares about ideological outcomes. If $\lambda_i^c = \lambda_i^o$ are ≈ 0 (and $\theta_i > 0$) then the proposers care mostly about office-

[12]This last assumption is stronger than necessary to derive equilibria, but used to ensure that the equilibrium conditions in Proposition 1 are necessary as well as sufficient.

[13]Recall also the implicit assumption on a proposer's utility from the reservation policy that $v_i^r \leq v_i^o(0, 0)$.

holding benefits. The difference $\lambda_i^c - \lambda_i^o \geq 0$ captures the extent to which ideological losses are felt more greatly when an opponent's proposal wins than when one's own proposal wins.

Transforming the Problem We first derive the necessary functions to solve the transformed problem, i.e., $q(s,y)$, $C_i(s,y)$, $V_i^o(s,y)$, and $V_i^c(s,y)$. This is straightforward from Definition 1. We first have that the level of effort $q(s,y)$ needed to propose a policy with score s and ideology y is :

$$s = \gamma_d \cdot q(s,y) - \lambda_d y^2 \rightarrow q(s,y) = \frac{s + \lambda_d y^2}{\gamma_d}.$$

This furthermore implies a cost function in the transformed problem of,

$$C_i(s,y) = c_i(q(s,y)) = \alpha_i \left(\frac{s + \lambda_d y^2}{\gamma_d} \right),$$

and finally utility functions of,

$$\begin{aligned} V_i^r(s,y) &= \mathbf{1}_{r=o} \cdot \theta_i + \gamma_i^r \left(\frac{s + \lambda_d y^2}{\gamma_d} \right) - \lambda_i^r (x_i - y)^2 \text{ where } r \in \{o, c\} \\ &= \mathbf{1}_{r=o} \cdot \theta_i + \gamma_i^r \left(\frac{s}{\gamma_d} - \left(\frac{\lambda_i^r}{\gamma_i^r} - \frac{\lambda_d}{\gamma_d} \right) y^2 \right) + \lambda_i^r \left(2x_i y_i - x_i^2 \right) \end{aligned}$$

Conditions for Equilibrium We now verify that transformed problem in the parametric model satisfies Assumptions 1–3; we may thus directly apply the equilibrium characterization in Proposition 1. Assumption 1 follows from the fact that the portion of $C_i(s,y) + F \cdot V_i^o(s,y)$ depending on y may be rewritten as $F \cdot \lambda_i^o 2x_i y - \left(F\lambda_i^o + \left(\frac{\lambda_d}{\gamma_d}\right)(\alpha_i - F\gamma_i^o) \right) y^2$ and also $\alpha_i > \gamma_i^o$ and $x_1 > 0 > x_{-1}$. Assumption 2 follows immediately from $\alpha_i > \gamma_i^o$. To see assumption 3, observe that for any mixture of proposals $\sigma_s(y)$ with score s, we have

$$\int (V_i^o(s,y) - V_i^o(s,0)) \, d\sigma_s(y) = \lambda_i^o \cdot 2x_i E[y] - \left(\frac{\lambda_i^o}{\gamma_i^o} - \frac{\lambda_d}{\gamma_d} \right) E\left[y^2\right]$$

Since $\frac{\lambda_i^o}{\gamma_i^o} \geq \frac{\lambda_d}{\gamma_d}$ the second term is always weakly negative, so for the entire expression to be strictly positive for some i requires $E[y_i] \neq 0$ and $sign(E[y_i]) = sign(i)$. But then the expression must be strictly negative for $-i$.

Last we observe that the transformed problem in the parametric model also satisfies assumption 5 (separability) due to the linearity of effort costs. Thus, the optimal ideology at every score $\hat{y}_i(s; F_{-i})$ sheds its direct dependence on s and is equal to:

$$\hat{y}_i(F_{-i}) = \frac{\lambda_i^o x_i \cdot F_{-i}}{\left(\frac{\lambda_d}{\gamma_d}\right)(\alpha_i - \gamma_i^o F_{-i}) + \lambda_i^o F_{-i}} \tag{3}$$

In addition, the inverse of the absolute value of $\hat{y}_i(F_{-i})$ is $\hat{F}_{-i}(y_i) = \frac{\alpha_i y}{\frac{\lambda_i^o}{\lambda_d/\gamma_d}(x_i - y) + \gamma_i^o \cdot y}$. We may therefore also apply the techniques in Section 3.2 to simply derive the equilibrium distribution over ideologies.

With the necessary assumptions verified we apply Proposition 1 to generate necessary and sufficient conditions for equilibrium in the parametric model.

Corollary 4 *A strategy profile σ of the parametric model is a SPNE i.f.f. it satisfies Proposition 1, where $\hat{y}_i(F_{-i}) = \frac{\lambda_i^o x_i \cdot F_{-i}}{\left(\frac{\lambda_d}{\gamma_d}\right)\left(\alpha_i - \gamma_i^o F_{-i}\right) + \lambda_i^o F_{-i}}$ and the differential equations are*

$$\frac{\alpha_i - F_{-i}(s) \cdot \gamma_i^o}{\gamma_d} = f_{-i}(s) \cdot \left(\begin{array}{c} \theta_i + \left(\gamma_i^o \cdot q\left(s, \hat{y}_i\left(F_{-i}(s)\right)\right) - \gamma_i^c \cdot q\left(s, \hat{y}_{-i}\left(F_i(s)\right)\right)\right) \\ + \left(\lambda_i^c \left(x_i - \hat{y}_{-i}\left(F_i(s)\right)\right)^2 - \lambda_i^o \left(x_i - \hat{y}_i\left(F_{-i}(s)\right)\right)^2\right) \end{array} \right) \ \forall i$$

A proposer's equilibrium utility is $\theta_i - \lambda_i^o \left(x_i - \hat{y}_i(1)\right)^2 - \left(\alpha_i - \gamma_i^o\right)\left(\frac{\bar{s} + \lambda_d [\hat{y}_i(1)]^2}{\gamma_d}\right)$

4.1 Symmetric Game

To generate a symmetric game in the parametric model by assumption 4 it is only necessary to impose symmetry on the proposers' parameters, i.e. $\forall i$, $|x_i| = x$, $\alpha_i = \alpha$, $\theta_i = \theta$, $\lambda_i^o = \lambda_p^o$, $\lambda_i^c = \lambda_p^c$, $\gamma_i^o = \gamma_p^o$, and $\gamma_i^c = \gamma_p^c$. (Symmetry of the decisionmaker's score score function is already assumed). In general the symmetric model only admits analytical solutions of the full equilibrium strategies in special cases (although several of these are substantively interesting). Despite this quite a lot can be said about equilibria without imposing further restrictions. First, it is trivial to analytically derive equilibrium *ideologies* using Corollary 3. Second, it is possible to show that the symmetric equilibrium is unique. Third, it is relatively straightforward to derive comparative statics results directly from the differential equation.

We first apply Corollary 1 to state necessary and sufficient conditions in the symmetric parametric model.

Corollary 5 *In the symmetric parametric model, a symmetric strategy profile is an equilibrium if and only if it satisfies Corollary 1, where $\hat{y}(F) = \frac{F \cdot \lambda_p^o x}{F \cdot \gamma_p^o \left(\frac{\lambda_p^o}{\gamma_p^o} - \frac{\lambda_d}{\gamma_d}\right) + \alpha\left(\frac{\lambda_d}{\gamma_d}\right)}$ and the differential equation is*

$$\frac{\alpha - F(s) \cdot \gamma_p^o}{\gamma_d} = f(s) \cdot \left(\begin{array}{c} \theta + \lambda_p^o \cdot 4x\hat{y}\left(F(s)\right) + \left(\gamma_p^o - \gamma_p^c\right) \cdot q\left(s, \hat{y}\left(F(s)\right)\right) \\ + \left(\lambda_p^c - \lambda_p^o\right)\left(x + \hat{y}\left(F(s)\right)\right)^2 \end{array} \right)$$

Equilibrium distribution over ideologies Because the parametric symmetric model satisfies assumption 5, the model inherits the "endogenous extremism" of Hirsch and Shotts (2015) and we may also apply Corollary 3 to simply characterize the equilibrium distribution over ideological proposals and outcomes as follows.

Corollary 6 *In any symmetric equilibrium of the parametric symmetric model, the ideological extremism of each player's proposals is distributed according to the CDF*

$$G(y) = \hat{F}(y) = \frac{\alpha y}{\frac{\lambda_p^o}{\lambda_d/\gamma_d}(x-y) + \gamma_p^o \cdot y}.$$

The CDF describing the ideological extremism of the final outcome is $[G(y)]^2$. *In addition, both proposals and outcomes become first-order stochastically more extreme when* x *increases,* λ_p^o *increases,* γ_p^o *increases,* α *decreases, or* $\frac{\lambda_d}{\gamma_d}$ *decreases.*

Symmetry thus yields a simple analytical characterization of the equilibrium ideologies in the parametric model. The distribution not only exhibits the endogenous extremism of Hirsch & Shotts (2015*a*), but also obeys similar comparative statics despite the inclusion of additional parameters.

These comparative statics results are as follows. First and unsuprisingly, proposers who have either more extreme ideal points x or place a higher value λ_p^o on ideology make more ideologically-extreme proposals, and result in more polarized outcomes. Second and somewhat more counterintuitively, proposers who either have a lower marginal cost of effort α, *or* who find their effort to be intrinsically more "productive" (higher γ_p^o) also make more extreme proposals and generate more polarized outcomes as a by-product – both factors reduce the proposers' *effective* cost of using effort to convince the decisionmaker to make ideological concessions.

Next, a *decisionmaker* who cares more about ideology (high λ_d) elicits more ideologically moderate proposals, while one who is more responsive to effort (γ_d) elicits more ideologically extreme proposals. Several possible interpretations are possible depending on the parameters of the model. In the pure policy-development model, an implication is that empowering decisionmakers who value "quality" more will yield ideological extremism as a by-product. Conversely, in variants where effort is unproductive, an interpretation is that decisionmakers who are more easily bought off with transfers, or more malleable to advertising, will result in more extreme outcomes.

Last, several parameters have no effect on the equilibrium distribution of ideological proposals – a proposer's "office-holding benefits" θ_i, and the values she places on ideology λ_p^c and effort γ_p^c should her opponent's proposal win the contest. Instead, these parameters only affect equilibrium through the amount of effort expended to help a given ideological proposal get selected.

Equilibrium distribution over scores We next discuss the equilibrium distribution over scores, uniqueness of symmetric equilibria, and comparative statics. To do this, it is helpful to transform the differential equation characterizing the equilibrium score CDF $F(s)$ in Corollary 5 to one characterizing the *inverse* $s(F)$ (which must satisfy the boundary condition $s(0)=0$). To do so simply substitute in $s(F)$ for F and observe that $f(s(F)) = \frac{1}{s'(F)}$. This yields the following.

Proposition 3 *In the symmetric parametric model there is a unique symmetric equilibrium. The inverse $s(F)$ of the common score CDF $F(s)$ is the unique solution to the boundary condition $s(0)=0$ and differential equation*

$$s'(F) = \left(\frac{\alpha - F \cdot \gamma_p^o}{\gamma_d}\right)^{-1} \cdot \left(\begin{array}{c} \theta + \lambda_p^o \cdot 4x\hat{y}(F) + \left(\gamma_p^o - \gamma_p^c\right) \cdot q\left(s(F), \hat{y}(F)\right) \\ + \left(\lambda_p^c - \lambda_p^o\right)\left(x + \hat{y}(F)\right)^2 \end{array}\right)$$

The equilibrium score CDF (and thus the decisionmaker's utility) is first-order stochastically increasing in θ, x, γ_p^o, λ_p^c, and λ_p^o holding $\lambda_p^o - \lambda_p^c$ constant. It is first-order stochastically decreasing in α and γ_p^c.

Uniqueness of symmetric equilibria can be seen by observing that the characterization of $s(F)$ is an explicit first-order ODE taking the form $s'(F) = t(F, s(F))$ with boundary condition $s(0)=0$. Since $t(\cdot)$ only depends on $s(F)$ through $q(s(F), \hat{y}(F))$, the numerator is linear in $q(s(F), \hat{y}(F))$, and $\frac{\partial q(\cdot)}{\partial s} = \frac{1}{\gamma_d}$, the differential equation satisfies a Lipschitz condition. A rough sketch of the comparative statics proofs is as follows; to show that $F(s;p)$ is decreasing in some parameter p (so the score distribution is first-order stochastically) it suffices to show that $\frac{\partial t(s,F;p)}{\partial s} \geq 0$ and $\frac{\partial t(s,F;p)}{\partial p} > 0$.

Interpretation of the comparative statics results is as follows. First, higher officeholding benefits θ result in first-order stochastically better proposals for the decisionmaker with greater effort, with *no effect* on ideological extremism. Greater officeholding benefits are thus unambiguously beneficial for the decisionmaker. Second, a lower intrinsic value on one's opponent's effort γ_p^c or a higher weight λ_p^c on ideological losses when one's opponent wins also unambiguously benefits the decisionmaker; higher score proposals with no effect on ideological outcomes. Thus, proposers who discount or even dislike effort on their opponent's proposals, or who more strongly dislike ideological losses when opponents win, yield positive benefits for the decisionmaker by increasing the intensity of competition.

Finally, increasing the proposers' ideological extremism x, increasing the strength of their ideological preferences λ_p^o (holding fixed $\lambda_p^o - \lambda_p^c$), increasing the intrinsic value on their own effort γ_p^o, and decreasing their costs of effort α, all have similar effects. They first-order stochastically increase the scores associated with proposals, but also generate first-order stochastically more ideologically extreme proposals as a byproduct. The welfare effect of such changes thus hinges on whether the decisionmaker's welfare is evaluated with respect only to ideological outcomes – if effort is interpreted as behavior that changes the decisionmaker's choices without actually making him better off – or with respect to his entire utility function. Overall, these results illustrate that changing the parameters of political competition in policy contests can have surprising and subtle effects.

Equilibrium distribution over effort We last discuss the equilibrium level of effort. Using the preceding results, the inverse $q(H)$ of a proposer's CDF over effort $H(q)$ is

$$q(H) = q(s(H), \hat{y}(H)) = \frac{s(H) + \lambda_d [\hat{y}(H)]^2}{\gamma_d}$$

Comparative statics on the effort CDF for any parameter p that is *not* λ_d or γ_d are thus identical to the comparative statics on the score CDF discussed in Proposition 3; the reason is that *all* such parameters affect $s(H)$ and $\hat{y}(H)$ in (weakly) the same way.

For the proposers' weight on ideology λ_d and effort γ_d however, effects are ambiguous. Increasing the decisionmaker's weight on ideology λ_d increases the effort needed to make any given proposal (s, y), but in equilibrium also makes proposals less ideologically aggressive and potentially lower score as well. The same effects in reverse are true for γ_d. This ambiguity yields the following results.

Proposition 4 *Increasing the decisionmaker's relative value $\frac{\lambda_d}{\gamma_d}$ of ideology to effort first-order stochastically decreases effort if $\frac{\lambda_d}{\gamma_d}$ is already sufficiently high $\left(\frac{\lambda_d}{\gamma_d} \geq \frac{\lambda_p^o}{\alpha - \gamma_p^o}\right)$, but otherwise has ambiguous effects.*

Thus, as intuition suggests there is no simple answer to the question of whether a more ideological decisionmaker magnifies or diminishes the incentive of ideologically-motivated proposers to participate in the contest. However, once the decisionmaker become very ideological, then the answer is that it unambiguously diminishes participation.

Analytical Solutions We conclude discussion of the symmetric parametric model by noting that it yields a closed form characterization for a number of interesting parameter values. In the interest of brevity we provide a partial list of these variants but delegate the characterizations to the Appendix. An exploration of possible additional results is reserved for future work.

Proposition 5 *The following variants of the parametric symmetric model without office-holding benefits* ($\theta = 0$) *admit complete closed form solutions*

- *The pure "policy development" model* ($\lambda_d = \lambda_p^o = \lambda_p^c$ *and* $\gamma_d = \gamma_p^o = \gamma_p^c$)
- *A partial "policy development" model where the decisionmaker intrinsically values effort more than the proposers* ($\lambda_d = \lambda_p^o = \lambda_p^c$ *and* $\gamma_d > \gamma_p^o = \gamma_p^c > 0$)
- *A pure "lobbying" model* ($\lambda_d = \lambda_p^o = \lambda_p^c$ *and* $\gamma_d > \gamma_p^o = \gamma_p^c = 0$)
- *A "policy development" model where an opponent's effort is discounted, not valued at all, or disliked* ($\lambda_d = \lambda_p^o = \lambda_p^c$ *and* $\gamma_d = \gamma_p^o > \gamma_p^c$). *An alternative interpretation of* $\gamma_p^c = 0$ *is that each proposer makes up-front investments in "spoils" that are shared between the decisionmaker and the winner.*

4.2 Asymmetric Game

In general, asymmetric variants of the parametric model to not appear to admit analytical solutions, nor are comparative statics straightforward to perform directly on the coupled system of differential equations. However, in a few special cases it is possible to derive a unique analytical solution using Proposition 2 and techniques similar to Hirsch (2015). We first examine one such case, which is a generalized version of the asymmetric "policy development" model therein. The generalization allows for variation in the players' relative weight of ideology vs. effort, and also allows the proposers to have "office-holding" benefits in addition to policy-motives for participation. After characterizing this variant, we conclude by exploring comparative statics in the general parametric model using Mathematica.

4.2.1 Special Case: Generalized Policy-Development Model

Hirsch & Shotts (2015*a*) and Hirsch (2015) consider a "policy development" model in which the interpretation of effort is as an investment in quality by all players, all players place an equal weight of 1 on both ideological and effort, and there are no "office holding" benefits. We now briefly solve a generalization of this model.

Each proposer places the same weight on ideology and effort whether they win or lose ($\lambda_i^o = \lambda_i^c = \lambda_i$ and $\gamma_i^o = \gamma_i^c = \gamma_i$), and these weights are also identical across players ($\lambda_i = \lambda_{-i} = \lambda_p$ and $\gamma_i^o = \gamma_i^c = \gamma_i$). The decisionmaker may have different weights on ideology and effort than the proposers, but values them in the same *ratio* $\frac{\lambda_d}{\gamma_d} = \frac{\lambda_p}{\lambda_p} = \rho$, so that $\lambda_d = \rho\gamma_d$ and $\lambda_p = \rho\gamma_p$. The generalization is thus also interpreted as a policy-development model – everyone intrinsically values effort on the winning proposal, but the players' collective relative valuation of ideology to effort may vary.

The proposers may have positive "office holding benefits" θ_i, but are constrained to have the same ratio of ideology to office-holding benefits, i.e. $\frac{\theta_i}{|x_i|} = \frac{\theta_{-i}}{|x_{-i}|} = \pi$ so that $\theta_i = \pi\,|x_i|$. We do not have an obvious substantive interpretation of this constraint – it simply allows for analytical solutions. Summarizing, there are eight free parameters of the model – x_i and α_i for $i \in \{-1, 1\}$, γ_p, γ_d, ρ, and π.

Equilibrium distribution over ideologies Since the parametric symmetric model satisfies assumption 5, a simple characterization of the equilibrium distribution over ideologies can be found provided that the function $\phi\left(F_{-k}\right)$ can be derived, which maps $-k$'s probability of developing a policy below some score to k's probability of developing a policy below that same score. To derive this function and the distributions, first observe that with the additional parameter constraints, optimal ideologies reduce to a linear function $\hat{y}_i\left(F_{-i}\right) = \gamma_p\left(\frac{x_i}{\alpha_i}F_{-i}\right)$. In addition, the system of differential equations that must be satisfied over the common score interval $[0, \bar{s}]$ reduces to

$$\frac{\alpha_i - F_{-i}\left(s\right)\cdot\gamma_p}{\gamma_d\cdot f_{-i}\left(s\right)} = \theta_i + \lambda_p 2x_i\cdot\left(\hat{y}_i\left(F_{-i}\left(s\right)\right) - \hat{y}_{-i}\left(F_{-i}\left(s\right)\right)\right)$$

With some manipulation this is equal to:

$$\frac{\alpha_i - F_{-i}\left(s\right)\cdot\gamma_p}{\gamma_d\cdot f_{-i}\left(s\right)|x_i|} = \pi + 2\lambda_p\left(\left|\hat{y}_i\left(F_{-i}\left(s\right)\right)\right| + \left|\hat{y}_{-i}\left(F_{-i}\left(s\right)\right)\right|\right) \qquad \forall i$$

Now since the right hand side is identical $\forall i$, the following crucial equality must be satisfied in equilibrium:

$$\frac{\alpha_i - F_{-i}(s) \cdot \gamma_p}{f_{-i}(s)|x_i|} = \frac{\alpha_{-i} - F_i(s) \cdot \gamma_p}{f_i(s)|x_{-i}|}$$

Solving this differential equation and applying the boundary condition that $F_i(\bar{s}) = F_{-i}(\bar{s}) = 1$ in turn implies that at every score $s \in [0, \bar{s}]$ we must have

$$\epsilon_i(F_{-i}(s)) = \epsilon_{-i}(F_i(s)), \text{ where } \epsilon_i(F_{-i}) = \left(\frac{\alpha_i - \gamma_p F_{-i}}{\alpha_i - \gamma_p}\right)^{|x_i|}$$

This crucial equilibrium relationship allows us to map from $F_{-i}(s)$ to $F_i(s)$ at a given score s without knowing the other characteristics of the equilibrium, since $F_i(s) = \epsilon_{-i}^{-1}(\epsilon_i(F_{-i}(s)))$. Applying the boundary condition at the bottom that $F_i(0) = 0$ for at most one i also allows us to identify the sometimes-inactive proposer k, who is simply $k = \arg\min_i \{\epsilon_i(0)\}$, as well as her probability of being inactive, which is $F_k(0) = \epsilon_{-k}^{-1}(\epsilon_k(0))$. We henceforth denote the value $\epsilon_k(0)$ as simply ϵ_k since it is used repeatedly in the characterization.

Using the preceding observations we then have $\phi(F_{-k}) = \epsilon_{-k}^{-1}(\epsilon_k(F_{-k}))$ and $\phi^{-1}(F_k) = \epsilon_k^{-1}(\epsilon_{-k}(F_k))$. Applying Proposition 2 yields a characterization of equilibrium ideologies.

Corollary 7 *In an equilibrium of the generalized policy development model with* $F_i(s) = 0$ $\forall s < 0$, *the probability* $G_i(y)$ *that* i *makes a proposal less ideologically extreme than* y *is*

$$\epsilon_{-i}^{-1}\left(\epsilon_i\left(\hat{F}_{-i}(y_i)\right)\right) = \frac{\alpha_{-i}}{\gamma_p} - \left(\frac{\alpha_{-i}}{\gamma_p} - 1\right)\left(\frac{|x_i| - y}{|x_i| - \frac{|x_i|}{\alpha_i}\gamma_p}\right)^{\frac{|x_i|}{|x_{-i}|}}$$

The ideological extremism of i*'s proposals is thus first-order stochastically increasing in her own ideological extremism* $|x_i|$ *and her opponent's costs of effort* α_i. *It is first-order stochastically decreasing in her own costs* α_i *and her opponent's ideological extremism* $|x_i|$.

The generalized policy-development model thus exhibits "own" effects and "cross" effects on the ideology of proposals identical to Hirsch (2015). A proposer's own ideology and costs affects her ideological proposals in the natural way. More subtly, her opponent's ideology and costs influence her own ideological proposals – a more ideological opponent forces her to moderate by becoming more active in the contest, while a less able opponent is less active in the contest and allows her to become more extreme.

The proposers' intrinsic valuation of effort γ_p has differential and complex effects on the players depending on who is advantaged in the contest. If the players are relatively even it can make both more ideologically extreme, as in the symmetric model. However, it can also magnify an imbalance and make an advantaged proposer more extreme in a first-order stochastic sense (and her opponent less extreme on average, although not unambiguously). Finally, the value of office-holding benefits π and the players' relative valuation of ideology to effort ρ has *no effect* on the ideology of equilibrium proposals.

Equilibrium distribution over scores We next discuss the equilibrium distribution over scores, uniqueness, and comparative statics. The equilibrium equality $\epsilon_i (F_{-i}(s)) = \epsilon_{-i}(F_i(s))$ implies that both quantities are equal to some common function $\epsilon(s)$ in equilibrium. If this function can be derived, then it yields both score CDFs since $F_i(s) = \epsilon_{-i}^{-1}(\epsilon(s))$.

Now observe that the system of differential equations characterizing the score CDFs may be rewritten into a single differential equation using this function since the left hand side of each equation is equal to $-\left(\frac{\gamma_p}{\gamma_d}\right)\frac{\epsilon(s)}{\epsilon'(s)}$ (this can be seen by beginning with $\epsilon(s) = \epsilon_i(F_{-i}(s))$ and then taking logs and differentiating both sides). Substituting in we have:

$$-\left(\frac{\gamma_p}{\gamma_d}\right)\frac{\epsilon(s)}{\epsilon'(s)} = \pi + 2\lambda_p\left(|\hat{y}_i(F_{-i}(s))| + |\hat{y}_{-i}(F_{-i}(s))|\right) \qquad \forall s \in [0, \bar{s}]$$

which may be rewritten as

$$\frac{1}{\epsilon'(s)} = -\left(\frac{\gamma_d}{\gamma_p}\right)\frac{1}{\epsilon(s)}\left(\pi + 2\lambda_p \sum_i \left|\hat{y}_i\left(\epsilon_i^{-1}(\epsilon(s))\right)\right|\right) \; \forall s \in [0, \bar{s}]$$

We rewrite this as an easily-solved differential equation on the inverse $s(\epsilon)$, i.e.

$$s'(\epsilon) = -\left(\frac{\gamma_d}{\gamma_p}\right)\frac{1}{\epsilon}\left(\pi + 2\lambda_p \sum_i \left|\hat{y}_i\left(\epsilon_i^{-1}(\epsilon)\right)\right|\right) \; \forall \epsilon \in [1, \epsilon_k],$$

where $s(\epsilon)$ is a decreasing function. Solving this and applying the boundary condition that $s(\epsilon_k) = 0$ yields a characterization of the equilibrium score CDFs.

Proposition 6 *In the generalized policy development model the equilibrium score CDFs over $s \geq 0$ are uniquely equal to $F_i(s) = \epsilon_{-i}^{-1}(\epsilon(s))$, where $\epsilon(s)$ is the inverse*

$$s(\epsilon) = \frac{\gamma_d}{\gamma_p}\pi \ln\left(\frac{\epsilon_k}{\epsilon}\right) + 2\gamma_d\rho\sum_i |x_i|\left(\ln\left(\frac{\epsilon_k}{\epsilon}\right) + \gamma_p\left|\frac{x_i}{\alpha_i}\right| \cdot \left(\epsilon_i^{-1}(\epsilon_k) - \epsilon_i^{-1}(\epsilon)\right)\right)$$

which is a decreasing function satisfying $s(1) = \bar{s}$ and $s(\epsilon_k) = 0$.
A proposer's equilibrium utility is

$$\pi x_i - \rho\gamma_p(x_i - \hat{y}_i(1))^2 - (\alpha_i - \gamma_p)\left(\frac{s(1)}{\gamma_d} + \rho[\hat{y}_i(1)]^2\right)$$

The decisionmaker's equilibrium utility is $\int_{\epsilon_k}^1 s(\epsilon) \cdot \frac{\partial}{\partial \epsilon}\left(\epsilon_i^{-1}(\varepsilon) \cdot \epsilon_{-i}^{-1}(\varepsilon)\right) d\varepsilon$

The generalized model thus yields analytical characterizations of equilibrium and key quantities of interest, including payoffs and probabilities of victory (see Section 3.2). It is easily observed that a proposer i's equilibrium utility is decreasing in her opponent's extremism $|x_{-i}|$ and increasing in her opponent's costs α_{-i} through the effects on the maximum score $\bar{s} = s(1)$. In addition, higher office-holding benefits π unambiguously benefit the decisionmaker by magnifying competition via effort with no corresponding effect on ideologies, as in the symmetric model. The effect of other parameters on both the welfare of the proposers' and of the decisionmaker are mixed.

4.2.2 Numerical Analysis

The generalized policy development model sharply restricts possible asymmetries in the model, and also imposes a particular interpretation on effort as productive that may not be suitable for some applications such as lobbying or campaign spending. The more-general asymmetric model fits these applications but does not admit neat analytical solutions.

Nevertheless, it is simple to compute equilibria using Mathematica with the necessary and sufficient conditions for equilibrium in Proposition 1. The only subtlety in computing equilibria is in identifying the sometimes inactive proposer k, as well as the size of her atom $F_k(0) > 0$ at the bottom score $s = 0$. The computational procedure involves searching for the value of this quantity that solves the boundary condition at the "top" that $F_i(\bar{s}) = F_{-i}(\bar{s}) = 1$ at some common $\bar{s}$. It is potentially possible that multiple solutions to the system of differential equations exists, but the procedure employed appears to always deliver a unique solution. Moreover, we verify that this solution coincides with, and converges continuously to, the analytical solutions previously discussed when available.

We now discuss comparative statics in the general parametric model derived from computational analysis; *this discussion should be taken as preliminary* and is not the product of a fully comprehensive search of the parameter space. First, we discuss the effect of unilateral changes in a proposer's parameters on her own score CDF, her opponent's score CDF, and the decisionmaker's welfare. Second, we discuss the effect of unilateral changes in a proposer's parameters on her own ideologies and her opponent's ideologies. Last, we discuss how the decisionmaker's parameters affect the proposer's ideologies and efforts.

Proposer Effects on Score In general, unilaterally increasing a proposer's motivation to participate in the contest in any number of ways has similar effects on her own score CDF, her opponent's score CDF, and the decisionmaker's welfare. A proposer's incentive to participate in the contest is enhanced by increasing her extremism $|x_i|$, decreasing her costs α_i, increasing her weight on ideology λ_i^o (holding $\lambda_i^c - \lambda_i^o$ fixed), increasing her extra ideological losses $\lambda_i^c - \lambda_i^o$ when she loses, increasing her intrinsic valuation of effort γ_i^o (holding $\gamma_i^o - \gamma_i^c$), increasing her discounting of opponent's effort $\gamma_i^o - \gamma_i^c$, and increasing her unconditional victory benefits θ_i.

The effect of any of these comparative statics is roughly as follows. Regardless of where the parameters start, any one of these changes increases proposer i's participation in the contest, in the sense of first-order stochastically increasing her scores. However, the effect on her opponent's participation and on the decisionmaker's welfare depends on whether i is initially the sometimes-inactive proposer k, or the always active proposer $-k$.

If proposer i begins sometimes-inactive ($i = k$) and some factor increases her willingness to participate in the contest, then this also necessarily increases the *parity* between i and her opponent $-i$. Consequently, it also increases her opponent's score CDF in a first order stochastic sense. Moreover, by increasing the activity of both proposers, it unambiguously increases the decisionmaker's welfare in a first-order stochastic sense.

However, if proposer i begins always active ($i = -k$) and some factor *further* increases her willingness to participate in the contest, then this necessarily increases the *imbalance* between her and her opponent. The effects here are more subtle. In classical contests without spillovers, increasing the imbalance between players tends to make an opponent less active and reduce the decisionmaker's welfare. Here, the ideological spillovers mitigate against this effect when the opponent's weight on ideology λ_{-i} is sufficiently high.

Specifically, in the policy contest it remains true that as proposer i becomes increasingly motivated her opponent is less likely to participate, as in a classical contest. However, conditional on participating, the opponent will mix over a wider range of scores. Her score CDF thus does *not* unambiguously decrease in a first-order stochastic sense – rather it spreads out. The reason is effectively the policy spillovers. As proposer i becomes further motivated and the contest becomes increasingly imbalanced, she proposes increasingly extreme ideologies that are increasingly costly for her opponent to lose to. Even worse, the more her opponent drops out of the contest, the more ideologically aggressive she becomes (recalling that $\hat{y}_i(s) = \hat{y}_i(F_{-i}(s))$). These effect keep the weaker player active in the contest.

Finally, the effect of making a stronger player stronger tends to actually be beneficial for the decisionmaker (sometimes first-order stochastically and sometimes just on average). The decreasing likelihood of the weaker player participating in the contest is outweighed by her greater average activity conditional on participation, as well as the unambiguously greater activity of the stronger player.

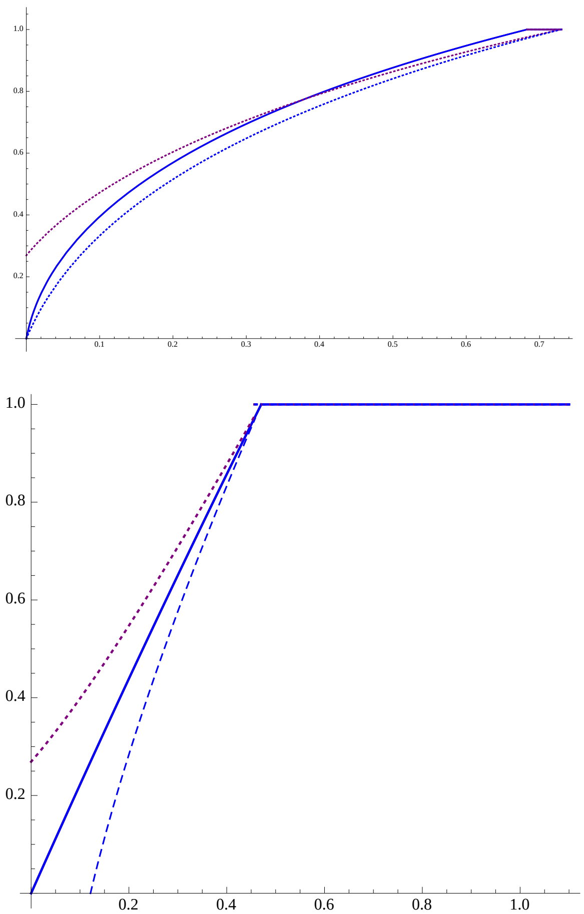
Proposer Effects on Ideology As in the analysis regarding score, unilaterally increasing a proposer's motivation to participate in the contest in any number of ways has similar effects on the ideological extremism of her proposals and of her opponent's proposals – it appears to increase the extremism of her proposals (in a first-order stochastic sense) while decreasing the extremism of her opponent's proposals.

Unlike the discussion about score, however, the fact that these parameters have similar effects on the ideological extremism of a proposer's own ideologies is actually somewhat surprising. As in Hirsch (2015) it is logical that increasing a proposer's extremism $|x_i|$ or decreasing her costs α_i increases the extremism of her proposals first-order stochastically; the former makes her more willing to invest in effort to realize ideological gains, while the latter makes her better able. The same logic holds for her own weight on ideology λ_i^o and intrinsic valuation of effort γ_i^o. However, it is much less obvious that increasing her additional ideological costs when losing $\lambda_i^c - \lambda_i^o$, her discounting of opponent's effort $\gamma_i^c - \gamma_i^o$, or her unconditional victory benefits θ_i, would have these effects. The reason is that increasing each of these parameters magnifies a proposer's desire to *avoid losing* ceteris paribus, but has *no effect* on her desire to realize ideological gains when winning. A reasonable intuition would therefore be that such changes would induce a proposer to *moderate* the ideology of her proposals, in order to increase her chance of winning the contest.

What this intuition fails to account for is the interplay between activity and ideology in the model. Any factor that magnifies a proposer's desire to participate in the contest naturally first-order stochastically increases her activity in the contest, and generally decreases her opponent's. But a proposer who is more active must *necessarily* make more ideologically-extreme proposals even if her desire for ideological gains or her ability to invest in effort are unchanged – being more likely to win, she necessarily finds it a better bet to invest up-front effort to realize ideological gains in the event of victory. This logic is exactly why increasing a proposer's desire to win in any capacity also moderates the ideological proposals of her *opponent,* even though that opponent's ideological motives have remained unchanged.

Figure 4 depicts an example comparing two profiles of parameter values – one in which the players are symmetric, and a second in which the "right" proposer enjoys a higher unconditional benefit $\theta_1 > \theta_{-1}$ from winning the contest irrespective of ideological gains. The top panel depicts score CDFs, while the bottom panel depicts CDFs over ideological extremism; in both panels the right proposer's CDF is blue while the left proposer's CDF is purple. The solid lines depict the CDFs where victory benefits are equal, and the dotted line depicts CDF when "right" has a higher victory benefit. As can be seen, a higher fixed payoff from winning makes "right" more ideologically extreme and first-order stochastically more active, and "left" more ideologically moderate while less likely to participate.

Figure 4: Score Effect of Unilateral Change in θ_i



Decisionmaker Effects on Proposers In the parametric model, the decisionmaker's parameters λ_d and γ_p only affect the proposers' equilibrium ideologies and efforts through the ratio $\frac{\lambda_d}{\gamma_d}$ (changing one of these parameters without changing the ratio changes the score CDFs only because the relationship between score and effort changes). The numerical results appear to indicate that increasing $\frac{\lambda_d}{\gamma_d}$ – effectively, how much effort the decisionmaker demands in exchange for ideological concessions – unambiguously decreases both equilibrium effort and the extremism of proposals. One interesting avenue to explore for future work is whether an "ideological" decisionmaker would prefer to delegate her decisionmaking authority to another actor who cares more about effort; this could be beneficial by increasing the proposers' effort in the contest, even though it would also increase the extremism of their proposals. An implication is that ideological decisionmakers could be "too principled" at the expense of their own interests.[14]

5 Conclusion

This paper develops a general framework for analyzing contests in which ideology-motivated actors expend costly-up front effort to have their proposals chosen by an ideological decisionmaker. Some potential applications of the model are electoral contests, lobbying contests, and organizational decisionmaking more generally. We provide a general necessary and sufficient conditions for equilibrium, and then explore symmetric and asymmetric versions of a flexible parametrized version of the model. We uncover subtle effects regarding the impact of factors in the political environment on ideological outcomes, the power of ideologically-motivated actors, and the welfare of decisionmakers.

[14] In the symmetric pure policy development model, the decisionmaker would indeed be better off delegating her authority to another actor who cares only about "quality."

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Policy Contests

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This Appendix is organized as follows. Appendix A proves Proposition 1 providing a characterization of equilibrium and necessary Lemmas. Appendix B proves other results.

A Proof of Equilibrium Conditions

Lemma 1 Taking the first order condition with respect to y_i yields the result. ■

Lemma 2 Suppose not, i.e., in equilibrium each proposer's strategy generates an atom of size p_i at some common $s > 0$. Let $F_{-i}^{-}(s_i) = \lim_{s \to s_i^{-}} \{F_{-i}(s)\}$, and let $w_i(y_i, y_{-i})$ denote the probability the DM chooses i's policy when (s, y_i) and (s, y_{-i}) are proposed. Proposer i's utility from playing according to her strategy conditional on a tie (which may involve mixing over ideologies) is then her equilibrium utility, which is equal to:

$$\int\limits_{s_i=s} \left(\begin{array}{c} -C_i(s, y_i) + F_{-i}^{-}(s) \cdot V_i^o(s, y_i) + \int\limits_{s_{-i}>s} (V_i^c(s_{-i}, y_{-i}))\, d\sigma_{-i} + \\ \int\limits_{s_{-i}=s} (w_i(y_i, y_{-i}) V_i^o(s, y_i) + w_{-i}(y_i, y_{-i}) V_i^c(s, y_i)) \frac{d\sigma_{-i}}{p_{-i}} \end{array} \right) \frac{d\sigma_i}{p_i}$$
$$\leq \int\limits_{s_i=s} \left(-C_i(s, y_i) + F_{-i}^{-}(s) \cdot V_i^o(s, y_i)\right) \frac{d\sigma_i}{p_i} + \int\limits_{s_{-i}>s} (V_i^c(s_{-i}, y_{-i}))\, d\sigma_{-i}$$
$$+ p_{-i} \int\limits_{s_i=s} \int\limits_{s_{-i}=s} (w_i(y_i, y_{-i}) V_i^o(s, y_i) + w_{-i}(y_i, y_{-i}) V_i^o(s, y_i)) \frac{d\sigma_i}{p_i} \frac{d\sigma_{-i}}{p_{-i}},$$

where the second inequality follows from $V_i^o(s, y) \geq V_i^c(s, y)\ \forall (s, y)$ and re-arranging.

Now the last term is just each player's expected proposal utility were they to always win with the mixture of proposals resulting from a tie and the decisionmaker's tie-breaking rule. Hence by assumption 3 it is $\leq V_k(s, 0)$ for at least some $k \in \{-1, 1\}$, so the above is $\leq$

$$\int\limits_{s_k=s} \left(-C_k(s, y_k) + F_{-k}^{-}(s) \cdot V_k^o(s, y_k)\right) \frac{d\sigma_k}{p_k} + p_{-k} \cdot V_k(s, 0) + \int\limits_{s_{-k}>s} (V_k^c(s_{-k}, y_{-k}))\, d\sigma_{-k}. \quad (4)$$

Finally, by assumption 1 (and $p_{-k} > 0$) observe that

$$\begin{aligned}
&\int\limits_{s_k=s} \left(-C_k\left(s, y_k\right) + F_{-k}^{-}\left(s\right) \cdot V_k^o\left(s, y_k\right)\right) \frac{d\sigma_k}{p_k} + p_{-k} \cdot V_k\left(s, 0\right) \\
\leq\; & -C_k\left(s, \hat{y}_k\left(F_{-k}^{-}\left(s\right); s\right)\right) + F_{-k}^{-}\left(s\right) \cdot V_k^o\left(s, \hat{y}_k\left(F_{-k}^{-}\left(s\right); s\right)\right) + p_{-k} \cdot V_k\left(s, 0\right) \\
\leq\; & -C_k\left(s, \hat{y}_k\left(F_{-k}^{-}\left(s\right); s\right)\right) + F_{-k}^{-}\left(s\right) \cdot V_k^o\left(s, \hat{y}_k\left(F_{-k}^{-}\left(s\right); s\right)\right) + p_{-k} \cdot V_k\left(s, \hat{y}_k\left(F_{-k}^{-}\left(s\right); s\right)\right) \\
<\; & -C_k\left(s, \hat{y}_k\left(F_{-k}\left(s\right); s\right)\right) + F_{-k}\left(s\right) \cdot V_k^o\left(s, \hat{y}_k\left(F_{-k}\left(s\right); s\right)\right)
\end{aligned}$$

which implies that eqn. 4 is $<$

$$-C_k\left(s, \hat{y}_k\left(F_{-k}\left(s\right); s\right)\right) + F_{-k}\left(s\right) \cdot V_k^o\left(s, \hat{y}_k\left(F_{-k}\left(s\right); s\right)\right) + \int\limits_{s_{-k}>s} \left(V_k^c\left(s_{-k}, y_{-k}\right)\right) d\sigma_{-k}$$

which is k's expected utility from just barely beating score s rather than tie-ing with the ideologically-optimal proposal $\hat{y}_k\left(F_{-k}\left(s\right); s\right)$. Hence k has a strictly profitable deviation. ■

Lemma 3 The argument is essentially identical as the proof of Lemma 2 with slightly different notation, but note that the effective equivalence is due to the fact that the decisionmaker has access to the reservation proposal $(0, 0)$.

Suppose not, i.e., $F_i\left(0\right) > 0\ \forall i$; this could be due to atoms at 0, developing scores lower than 0, or both. If both players make proposals with scores ≤ 0 then the chosen proposal will have score $= 0$ with probability 1 since the decisionmaker has access to the reservation proposal. Let $\sigma_o\left(y\right)$ denote the resulting probability distribution over ideological outcomes conditional on both players making a proposal of score ≤ 0. Now using that $V_i^c\left(0, y\right) \leq V_i^o\left(0, y\right)$ and $V_i^r \leq V_i^o\left(0, 0\right)$, we have that i's utility from mixing according to her strategy conditional on generating score ≤ 0 (which yields her equilibrium utility), is $\leq \int\limits_{s_i\leq 0} - C_i\left(s_i, y_i\right) \frac{d\sigma_i}{F_i(0)} + F_{-i}\left(0\right) \int V_i^o\left(0, y\right) d\sigma_o\left(y\right) + \int\limits_{s_{-i}>0} \left(V_i^c\left(s_{-i}, y_{-i}\right)\right) d\sigma_{-i}$. Now by assumption 3, $\int V_i^o\left(0, y\right) d\sigma_o\left(y\right) \leq V_i^o\left(0, 0\right)$ for at least some k. In addition, $-C_k\left(s_k, y_k\right) \leq -C_k\left(0, 0\right) = 0$. So for k the above is $\leq$

$$-C_k\left(0, 0\right) + F_{-k}\left(0\right) \cdot V_k^o\left(0, 0\right) + \int\limits_{s_{-k}>0} \left(V_k^c\left(s_{-k}, y_{-k}\right)\right) d\sigma_{-k}$$

which then by assumption 1 (and $F_{-k}\left(0\right) > 0$) is $<$

$$-C_k\left(0, \hat{y}_k\left(F_{-k}\left(0\right); 0\right)\right) + F_{-k}\left(0\right) \cdot V_k^o\left(0, \hat{y}_k\left(F_{-k}\left(0\right); 0\right)\right) + \int\limits_{s_{-k}>0} \left(V_k^c\left(s_{-k}, y_{-k}\right)\right) d\sigma_{-k}$$

which is k's expected utility from just barely beating score 0 with the ideologically-optimal proposal $\hat{y}_k\left(F_{-k}\left(0\right);0\right)$ rather than mixing over proposals with scores ≤ 0. Hence k has a strictly profitable deviation. ■

Lemma 4 We first show that if $\hat{s} > 0$ is in the support of F_i then $F_{-i}\left(\hat{s}\right) - F_{-i}\left(\hat{s} - \varepsilon\right) > 0$ $\forall \varepsilon > 0$. Suppose not; then $\exists \varepsilon > 0$ such that $F_{-i}\left(s\right)$ is constant over $\left[\hat{s} - \varepsilon, \hat{s}\right]$ and $-i$ has no atom at $\hat{s} - \varepsilon$ or $\hat{s}$. Furthermore since $\hat{s}$ is in i's support and $-i$ has no atom there, i can achieve her equilibrium utility U_i^* by proposing $\left(\hat{s}, \hat{y}_i\left(\hat{s}, F_{-i}\left(\hat{s}\right)\right)\right)$.

Now let F denote the value of $F_{-i}\left(s\right)$ over this range; applying the envelope theorem, it is easily verified that

$$\frac{\partial}{\partial s}\left(\Pi_i\left(s, \hat{y}_i\left(s, F\right); \sigma_{-i}\right)\right) = -\frac{\partial C_i\left(s, y\right)}{\partial s} + F \cdot \left.\frac{\partial V_i^o\left(s, y\right)}{\partial s}\right|_{y = \hat{y}_i\left(s; F\right)} < 0$$

by assumption 2. So $\Pi_i\left(\hat{s} - \varepsilon, \hat{y}_i\left(\hat{s} - \varepsilon, F\right); \sigma_{-i}\right) > \Pi_i\left(\hat{s}, \hat{y}_i\left(\hat{s}, F\right); \sigma_{-i}\right)$, a contradiction.

Now the preceding argument implies several of the desired properties. If the players' score CDFs did not have common support over $s > 0$, then one player would have support at a score where the other player's CDF is constant below, violating the condition. If the common support did not include 0 or were not convex, then there would exist a score $s'' > 0$ in the common support and a strictly lower score $s' \geq 0$ such that $F_i\left(s\right)$ was constant $\forall i$ over $\left[s', s''\right)$, at least one k had $F_k\left(s''\right) = F_k\left(s'\right)$ (since both cannot have atoms at s'' by Lemma 2), and the condition would again be violated.

Finally we show that no proposer has an atom above 0 by contradiction. Suppose $-i$ has an atom at $\hat{s} > 0$ of size p_{-i}; then i does not (by Lemma 2) which implies that when $-i$ makes a score-$\hat{s}$ proposal she proposes the unique ideology $y_{-i}\left(\hat{s}; F_i\left(\hat{s}\right)\right)$ with probability 1. We henceforth denote this $\hat{y}_{-i}$. Now by the argument in the preceding paragraph, i's support includes $\left[0, \hat{s}\right]$. This implies that $F_i\left(\hat{s}\right) > 0$ and $\hat{y}_{-i} \neq 0$ – that is, at the atom $-i$ proposes something distinct from the decisionmaker's ideal. In addition, i's utility $\lim_{s_i \to \hat{s}^-}\left\{\Pi_i\left(s_i, \hat{y}_i\left(s_i; F_{-i}\left(s_i\right)\right); \sigma_{-i}\right)\right\}$ from just barely losing at score $\hat{s}$ must be equal to her equilibrium utility U_i^*.

Now let $\hat{y}_i = \lim_{s_i \to \hat{s}^-}\left\{\hat{y}_i\left(s_i; F_{-i}\left(s_i\right)\right)\right\}$ (the optimal ideology to propose conditional on just losing at score $\hat{s}$), and observe that $\hat{y}_i \neq 0$ since $-i$ must also have support below $\hat{s}$. We now argue that i would achieve utility strictly higher than her equilibrium utility by just winning at score $\hat{s}$ with ideology $\hat{y}_i$, a contradiction. Her utility gain from doing so is,

$$\begin{aligned} & \lim_{s_i \to \hat{s}^+} \{\Pi_i(s_i, \hat{y}_i; \sigma_{-i})\} - \lim_{s_i \to \hat{s}^-} \{\Pi_i(s_i, \hat{y}_i(s_i; F_{-i}(s_i)); \sigma_{-i})\} \\ = \; & p_{-i}(V_i^o(\hat{s}, \hat{y}_i) - V_i^c(\hat{s}, \hat{y}_{-i})) > 0 \end{aligned}$$

since $V_i^o(\hat{s}, \hat{y}_i) > V_i^o(\hat{s}, \hat{y}_{-i})$ (by assumptions 1 and 3 and $\hat{y}_i, \hat{y}_{-i} \neq 0$), and $V_i^o(\hat{s}, \hat{y}_{-i}) \geq V_i^c(\hat{s}, \hat{y}_{-i})$. ■

Proposition 1 *Necessity:* We first argue necessity of ideological optimality. Observe that since the score CDFs are atomless over $(0, \infty)$ and at such scores ideology $\hat{y}_i(s_i; F_{-i}(s_i))$ is strictly better than any other ideology (by Lemma 1), in equilibrium the probability a proposal (s_i, y_i) with $s_i > 0$ satisfies $y_i = \hat{y}_i(s_i; F_{-i}(s_i))$ must be 1. Ideological optimality then immediately follows: if $F_i(0) = 0$ then i's proposals are strictly positive-score with probability 1, and if $F_i(0) > 0$ then $F_{-i}(0) = 0$ (by Lemma 3), proposals with score $s_i \leq 0$ both lose for sure and never affect a tie, and therefore must be 0-effort with probability 1.

We next argue necessity of score optimality by deriving the system of differential equations and boundary conditions. Observe that Lemma 4 (combined with our technical restriction that score CDFs be the sum of a discrete and absolutely continuous distribution) immediately imply (i) score CDFs are absolutely continuous over $[0, \bar{s}]$, where $\bar{s} > 0$ is the maximum score and may be $= \infty$, (ii) $F_k(0) = 0$ for some $k \in \{L, R\}$, and (iii) $\lim_{s \to \bar{s}} \{F_i(s)\} = 1 \; \forall i$.

Now recall that the function $\Pi_i^*(s_i; F)$ over all scores $s_i \geq 0$ is equal to a proposer's expected utility were she always to win at score-ties, and with the optimal ideologies $\hat{y}_i(s_i; F_{-i}(s_i))$ substituted in for both proposers. Let σ^* denote an equilibrium strategy profile, and U_i^* denote i's equilibrium utility in that profile. At an equilibrium profile there are no atoms above 0, so i's utility from proposing the optimal ideology at any score $s_i > 0$ is exactly equal to $\Pi_i^*(s_i; F)$. The statement is not necessarily true at $s_i = 0$, but i can achieve utility arbitrarily close to $\Pi_i^*(0; F)$ by proposing a score ε above. Consequently $U_i^* \geq \Pi_i^*(s_i; F)$ for all $s_i \geq 0$. In addition, $\Pi_i^*(s_i; F) \geq U_i^*$ and thus $= U_i^* \; \forall s_i \in [0, \bar{s}]$; if instead for some $s_i \in [0, \bar{s}]$ we had $U_i^* > \Pi_i^*(s_i; F)$ then by continuity of (F_i, F_{-i}) over $s_i > 0$ (and right continuity at 0) i would be developing scores with positive probability that yield strictly lower utility than her equilibrium utility, a contradiction.

Finally, since (F_i, F_{-i}) are also absolutely continuous and strictly increasing $\forall s \in [0, \bar{s}]$ by full support, $U_i^* = \Pi_i^*(s_i; F) \; \forall s_i \in [0, \bar{s}] \iff \frac{\partial}{\partial s_i}(\Pi_i^*(s_i; F)) = 0$ for almost all $s \in [0, \bar{s}]$, which yields the system of differential equations stated in the proposition.

Sufficiency: Suppose the conditions hold. First observe that all proposals $(s_i, y_i^*(s_i))$ s.t. $s_i \in (0, \bar{s}]$ yield utility equal to a constant $\hat{U}_i^*$, since by score optimality $\Pi_i^*(s_i; F)$ is constant over $[0, \bar{s}]$. We argue that this is i's utility from playing her strategy. If i is always active ($F_i(0) = 0$), then this follows immediately from ideological optimality. If instead i is sometimes inactive ($F_i(0) > 0$) then the score optimality boundary conditions imply that her opponent is always active ($F_{-i}(0) = 0$). In this case proposals (s_i, y_i) s.t. $s_i \leq 0$ and $q(s_i, y_i) = 0$ are free to propose and always lose. They therefore yield the same utility as proposing $(0, \hat{y}_i^*(0)) = (0, 0)$, so again by ideological optimality $\hat{U}_i^*$ is i's utility from playing her strategy.

We now argue that neither proposer has a profitable deviation in two steps. We first argue that $\hat{U}_i^*$ is higher than the utility from any other proposal (s_i, y_i) with strictly positive score $s_i > 0$. For any $s_i \in (0, \bar{s}]$ in the common mixing interval we have $U_i^* = \Pi_i^*(s_i; F) \geq \Pi_i^*(s_i, y_i; F)$ by Lemma 1. Next for all $s \geq \bar{s}$ observe that

$$\frac{\partial}{\partial s}\left(\Pi_i^*(s; F)\right) = -\frac{\partial C_i(s, y)}{\partial s} + \left.\frac{\partial V_i^o(s, y)}{\partial s}\right|_{y = \hat{y}_i(s;1)} < 0,$$

so for any (s_i, y_i) with $s_i > \bar{s}$ we have $\Pi_i^*(s_i, y_i; F) \leq \Pi_i^*(s_i; F) < \Pi_i^*(\bar{s}; F) = U_i^*$.

We second argue that $\hat{U}_i^*$ is higher than the utility from any other proposal (s_i, y_i) with weakly negative score $s_i \leq 0$. Suppose first that proposer i is sometimes inactive ($F_i(0) > 0$) – then by the boundary conditions her opponent $-i$ is always active ($F_{-i}(0) = 0$). As previously argued, if i makes a negative-score proposal it always loses, so 0-effort negative score proposals yield exactly her equilibrium utility $\hat{U}_i^*$, while positive-effort negative score proposals are strictly worse.

Last suppose that proposer i is always active ($F_i(0) = 0$). If $-i$ is also always active, then the same argument applies, so suppose that her opponent is sometimes inactive ($F_{-i}(0) > 0$). By ideological optimality, whenever $-i$ makes a $0-$score proposal it is exactly $(0, 0)$.[15] Thus, were i to deviate to developing a strictly-negative score proposal $s_i < 0$, the outcome when $s_{-i} \leq 0$ would be weakly worse than the outcome of winning with proposal $(0, 0)$. Proposer i's utility from making any a strictly-negative score proposal is thus weakly worse than proposing $(0, 0)$. We last argue that U_i^* is $\geq$ i's utility for making any proposal $(0, y_i)$ including $(0, 0)$. Let $w_i(y_i)$ denote the probability proposal $(0, y_i)$ wins conditional on $-i$ being inactive ($s_{-i} \leq 0$), which depends on the full profile of strategies (including the DM's

[15] For the model without the reservation proposal, substitute the preceding with the following – "By ideological optimality and $F_{-i}(s) = 0$ $\forall s < 0$, whenever $-i$ makes a $\leq 0-$score proposal it is exactly $(0, 0)$."

tie-breaking rule). With the remaining probability either $-i$ wins with $(0,0)$ or the DM chooses the reservation proposal.[16] Thus i's utility from making this proposal is $\leq$

$$-C_i(0,y_i)+F_{-i}(0)\cdot(w_i(y_i)\cdot V_i^o(0,y_i)+(1-w_i(y_i))\cdot V_i^o(0,0))+\int\limits_{s_{-i}>0} V_i\left(s_{-i},y_{-i}^*(s_{-i})\right)dF_{-i}$$

By assumption 1 this is $\leq$

$$\begin{aligned}&-C_i(0,\hat{y}_i(F_{-i}(0)\cdot w_i(y_i);0))+F_{-i}(0)\cdot w_i(y_i)\cdot V_i^o(0,\hat{y}_i(F_{-i}(0)\cdot w_i(y_i);0))\\&+F_{-i}(0)\cdot(1-w_i(y_i))\cdot V_i^o(0,\hat{y}_i(F_{-i}(0)\cdot w_i(y_i);0))+\int\limits_{s_{-i}>0} V_i\left(s_{-i},y_{-i}^*(s_{-i})\right)dF_{-i}\end{aligned}$$

which in turn is $<$

$$-C_i(0,\hat{y}_i(F_{-i}(0);0))+F_{-i}(0)\cdot V_i^o(0,\hat{y}_i(F_{-i}(0);0))+\int\limits_{s_{-i}>0} V_i\left(s_{-i},y_{-i}^*(s_{-i})\right)dF_{-i}$$

which is $=\Pi_i^*(0;F)$, or i's utility if she were to develop the optimal $0-$score proposal presuming she would always win a tie. Finally CDFs are right continuous, so i's equilibrium utility U_i^* is equal to $\Pi_i^*(0;F)$; thus, no $(0,y_i)$ or (s_i,y_i) with $s_i<0$ yields a profitable deviation. ■

Lemma 5 The sufficiency proof is identical to that in the baseline model except with the two substitutions described in footnotes 15 and 16. ■

[16] For the model without the reservation proposal, substitute the preceding with the following – "With the remaining probability $-i$ wins with (0,0)."

B Appendix B – Other Results

Proposition 3 The differential equation is $s(0) = 0$ and:

$$s'(F) = \left(\frac{\alpha - F \cdot \gamma_p^o}{\gamma_d}\right)^{-1} \cdot \left(\begin{array}{c} \theta + \lambda_p^o \cdot 4x\hat{y}(F) + \left(\gamma_p^o - \gamma_p^c\right) \cdot q(s(F), \hat{y}(F)) \\ + \left(\lambda_p^c - \lambda_p^o\right)(x + \hat{y}(F))^2 \end{array}\right)$$

We use the following lemma in Hirsch and Shotts (2015).

Lemma 6 *Consider a continuous function $h(x)$ that is almost-everywhere differentiable, and let $h^i(x)$ denote the i'th derivative of h (with $h^0(x) = h(x)$). Then the following two conditions imply that $h(x)$ is increasing in $x \geq 0$.*

1. $h(x) > 0 \rightarrow h'(x) > 0$ *wherever h is differentiable*
2. $h^k(0) = 0 \ \forall$ *integer* $k \in [0, i]$, *and* $h^{i+1}(0) > 0$

To see how to use the lemma to do comparative statics, suppose we wish to show $F(s; p)$ is first-order stochastically increasing in some parameter p. Then we wish to show that $\frac{\partial s(F;p)}{\partial p} > 0 \ \forall F > 0$ (since $\left.\frac{\partial s(F;p)}{\partial p}\right|_{F=0} = 0$ from the boundary condition). Now clearly it suffices to show the yet-stronger property that $\frac{\partial s(F;p)}{\partial p}$ is *increasing* in F; so in Lemma 6 the function $\frac{\partial s(F;p)}{\partial p}$ plays the role of $h(\cdot)$, and the parameter F plays the role of x. We then must show the following two properties:

1. $\frac{\partial s(F;p)}{\partial p} > 0 \rightarrow \frac{\partial^2 s(F;p)}{\partial p \partial F} > 0$
2. $\left.\frac{\partial^k\left(\frac{\partial s(F;p)}{\partial p}\right)}{\partial F^k}\right|_{F=0} = 0 \ \forall$ integer $k \in [0, i]$ and $\left.\frac{\partial^{i+1}\left(\frac{\partial s(F;p)}{\partial p}\right)}{\partial F^{i+1}}\right|_{F=0} > 0$

To show property (1) it suffices to show the simpler property that $\frac{\partial t(F,s,p)}{\partial p} > 0$ for $F > 0$ and $s \geq 0$, recalling that the differential equation takes the form $\frac{\partial s(F;p)}{\partial F} = t(F, s(F;p); p)$. The reason is that this implies $\frac{\partial s^2(F;p)}{\partial p \partial F} = \frac{\partial t(F,s,p)}{\partial s}\frac{\partial s(F;p)}{\partial p} + \left.\frac{\partial t(s,F,p)}{\partial p}\right|_{s=s(F;p)}$ and as already stated $\frac{\partial t(F,s,p)}{\partial s} > 0$. To show property (2) it is helpful to have $\left.\frac{\partial(s(F;p))}{\partial F}\right|_{F=0}$ and $\left.\frac{\partial^2(s(F;p))}{\partial F^2}\right|_{F=0}$ in hand (since $\left.\frac{\partial^k\left(\frac{\partial s(F;p)}{\partial p}\right)}{\partial F^k}\right|_{F=0} = \frac{\partial}{\partial p}\left(\left.\frac{\partial^k(s(F;p))}{\partial F^k}\right|_{F=0}\right)$). Using $\hat{y}(F) = 0$ and $s(0) = 0$ yields that $\left.\frac{\partial(s(F;p))}{\partial F}\right|_{F=0} = \left(\frac{\gamma_d}{\alpha}\right)\cdot\left(\theta + \left(\lambda_p^c - \lambda_p^o\right)x^2\right)$. Next, implicitly differentiating the differential equation and using the aforementioned equalities as well as $\hat{y}'(0) = \frac{\frac{x}{\alpha}\lambda_p^o}{\lambda_d/\gamma_d}$ yields that

$$\left.\frac{\partial^2\left(s\left(F;p\right)\right)}{\partial F^2}\right|_{F=0} = \left(\frac{\gamma_d}{\alpha^2}\right)\cdot\left(\left(\frac{\lambda_p^o}{\lambda_d/\gamma_d}\right)\cdot 2\left(\lambda_p^o+\lambda_p^c\right)x^2 + \left(2\gamma_p^o - \gamma_p^c\right)\cdot\left(\theta + \left(\lambda_p^c - \lambda_p^o\right)x^2\right)\right)$$

We now do the proof for each parameter.

(x) For property (1), it is easily verified that $t\left(F,s;x\right)$ is increasing in x by using that $\hat{y}\left(F;x\right)$ is increasing in x. For property (2) we have that $\left.\frac{\partial\left(\frac{\partial s(F;x)}{\partial x}\right)}{\partial F}\right|_{F=0} = \left(\frac{\gamma_d}{\alpha}\right)2\left(\lambda_p^c - \lambda_p^o\right)x > 0$ if $\lambda_p^c \neq \lambda_p^o$; otherwise $\left.\frac{\partial^2\left(\frac{\partial s(F;x)}{\partial x}\right)}{\partial F^2}\right|_{F=0} = \left(\frac{\gamma_d}{\alpha^2}\right)\left(\frac{\lambda_p^o}{\lambda_d/\gamma_d}\right)\cdot 4\left(\lambda_p^o+\lambda_p^c\right)x > 0$.

(α) For property (1), it is easily verified that $t\left(F,s;\alpha\right)$ is decreasing in α by using that $\hat{y}\left(F;\alpha\right)$ is decreasing in α. For property 2, we have that $\left.\frac{\partial\left(\frac{\partial s(F;\alpha)}{\partial \alpha}\right)}{\partial F}\right|_{F=0} = 0$ and $\left.\frac{\partial^2\left(\frac{\partial s(F;\alpha)}{\partial \alpha}\right)}{\partial F^2}\right|_{F=0} < 0$ is easily verified.

(θ) For property (1), it is easily verified that $t\left(F,s;\theta\right)$ is increasing in θ. For property 2, we have that $\left.\frac{\partial\left(\frac{\partial s(F;\theta)}{\partial \theta}\right)}{\partial F}\right|_{F=0} = \frac{\gamma_d}{\alpha} > 0$.

(γ_p^o) For property (1) it is easily verified that $t\left(F,s;\gamma_p^o\right)$ is increasing in γ_p^o by using that $\hat{y}\left(F;\gamma_p^o\right)$ is increasing in γ_p^o (here we rely on $\frac{\lambda_p^o}{\gamma_p^o} \geq \frac{\lambda_d}{\gamma_d}$). For property (2) we have $\left.\frac{\partial\left(\frac{\partial s\left(F;\gamma_p^o\right)}{\partial \gamma_p^o}\right)}{\partial F}\right|_{F=0} = 0$ and $\left.\frac{\partial^2\left(\frac{\partial s\left(F;\gamma_p^o\right)}{\partial \gamma_p^o}\right)}{\partial F^2}\right|_{F=0} = \left(\frac{\gamma_d}{\alpha^2}\right)\cdot 2\left(\theta + \left(\lambda_p^c - \lambda_p^o\right)x^2\right) > 0$.[17]

(γ_p^c) For property (1) it is easily verified that $t\left(F,s;\gamma_p^c\right)$ is decreasing in γ_p^c. For property (2) we have $\left.\frac{\partial\left(\frac{\partial s\left(F;\gamma_p^c\right)}{\partial \gamma_p^c}\right)}{\partial F}\right|_{F=0} = 0$ and $\left.\frac{\partial^2\left(\frac{\partial s\left(F;\gamma_p^c\right)}{\partial \gamma_p^c}\right)}{\partial F^2}\right|_{F=0} = -\left(\frac{\gamma_d}{\alpha^2}\right)\cdot\left(\theta + \left(\lambda_p^c - \lambda_p^o\right)x^2\right) < 0$.

(λ_p^c) For property (1) it is easily verified that $t\left(F,s;\lambda_p^c\right)$ is increasing in λ_p^c. For property (2) we have that $\left.\frac{\partial\left(\frac{\partial s\left(F;\gamma_p^c\right)}{\partial \gamma_p^c}\right)}{\partial F}\right|_{F=0} = \left(\frac{\gamma_d}{\alpha}\right)\lambda_p^c x^2 > 0$.

[17] If $\theta = 0$ and $\lambda_p^c = \lambda_p^o$ we would actually need to proceed to $s'''(0)$ but this is cumbersome and I am too lazy for this draft.

$\left(\lambda_p^o\right)$ For property (1) it is easily verified that $t\left(F,s;\lambda_p^o\right)$ is increasing in λ_p^o when $\lambda_p^c-\lambda_p^o=\beta$ constant by using that $\hat{y}\left(F;\gamma_p^o\right)$ is increasing in λ_p^o. For property (2) we have that $\left.\frac{\partial\left(\frac{\partial s\left(F;\lambda_p^o\right)}{\partial\lambda_p^o}\right)}{\partial F}\right|_{F=0}=0$ when $\lambda_p^c-\lambda_p^o$ constant and that $\left.\frac{\partial^2\left(\frac{\partial s\left(F;\lambda_p^o\right)}{\partial\lambda_p^o}\right)}{\partial F^2}\right|_{F=0}=\left(\frac{\gamma_d}{\alpha^2}\right)\cdot\frac{x^2}{\lambda_d/\gamma_d}\left(8\lambda_p^o+2\beta\right)>0$.

(λ_d) For property (1) it is easily verified that $t\left(F,s;\lambda_p^o\right)$ is decreasing in λ_d by using that $\hat{y}\left(F;\lambda_d\right)$ is decreasing in λ_d^o. ■

Proposition 4 We wish to show $q\left(H\right)=\frac{\lambda_d}{\gamma_d}\left[\hat{y}\left(H\right)\right]^2+\frac{s(H)}{\gamma_d}$ is decreasing in λ_d and increasing in γ_d when $\frac{\lambda_d}{\gamma_d}\geq\frac{\lambda_p^o}{\alpha-\gamma_p^o}$. We separately show these statics for each term so that the sum inherits them.

The first term only depends on λ_d and γ_d through the ratio $\frac{\lambda_d}{\gamma_d}=r$, so we are interested in showing that $sign\left(\frac{\partial}{\partial r}\left(r\left[\hat{y}\left(H;r\right)\right]^2\right)\right)<0$ when the stated condition holds. This is $=sign\left(\frac{\partial}{\partial r}\left(\log\left(r\left[\hat{y}\left(H;r\right)\right]^2\right)\right)\right)$ for $H>0$, and

$$\begin{aligned}\frac{\partial}{\partial r}\left(\log\left(r\left[\hat{y}\left(H;r\right)\right]^2\right)\right) &= \frac{\partial}{\partial r}\left(\log\left(r\right)-2\log\left(\left(\alpha-H\cdot\gamma_p^o\right)r+H\lambda_p^o\right)\right)\\ &= \frac{1}{r}-\frac{2\left(\alpha-H\cdot\gamma_p^o\right)}{\left(\alpha-H\cdot\gamma_p^o\right)r+H\lambda_p^o}=\frac{H\lambda_p^o-r\left(\alpha-H\cdot\gamma_p^o\right)}{r\left(\left(\alpha-H\cdot\gamma_p^o\right)r+H\lambda_p^o\right)}\end{aligned}$$

The denominator is strictly positive and the numerator is strictly increasing in H; thus when $\lambda_p^o-r\left(\alpha-\gamma_p^o\right)\leq 0$ (which is just the stated condition re-ordered) the sign is negative $\forall H\leq 1$ which is the desired result.

For the second term $\frac{s(H)}{\gamma_d}$, to do comparative statics we use the techniques in the proof of Proposition 3. We know that

$$\frac{\partial}{\partial H}\left(\frac{s\left(H\right)}{\gamma_d}\right)=\left(\frac{1}{\alpha-H\cdot\gamma_p^o}\right)\cdot\left(\begin{array}{c}\theta+\lambda_p^o\cdot 4x\hat{y}\left(H\right)+\left(\gamma_p^o-\gamma_p^c\right)\cdot\left(\frac{s(H)}{\gamma_d}+\frac{\lambda_d[\hat{y}(H)]^2}{\gamma_d}\right)\\ +\left(\lambda_p^c-\lambda_p^o\right)\left(x+\hat{y}\left(H\right)\right)^2\end{array}\right).$$

Thus to show $\frac{s(H)}{\gamma_d}$ increasing (decreasing) in some p it suffices to show (1) the derivative of the r.h.s. w.r.t. p is positive (negative) holding $\frac{s(H)}{\gamma_d}$ fixed $\forall H$ (since the r.h.s. is weakly increasing in $\frac{s(H)}{\gamma_d}$), and (2) the derivative w.r.t. p of $\frac{\partial^2}{\partial H^2}\left(\frac{s(H)}{\lambda_d}\right)$ evaluated at $H=0$ is positive (negative) (since $\left.\frac{\partial}{\partial H}\left(\frac{s(H)}{\lambda_d}\right)\right|_{H=0}$ does not depend on λ_d and γ_d). For (1), it is clear that holding $\frac{s(H)}{\gamma_d}$ fixed the r.h.s. only depends on λ_d and γ_d through the ratio $\frac{\lambda_d}{\gamma_d} = r$. In addition, when $\lambda_p^o - r\left(\alpha - \gamma_p^o\right) \leq 0$ the expression must be be decreasing in r since both $\hat{y}(H;r)$ and $r\left(\hat{y}(H)\right)^2$ are decreasing in r as previously shown. For (2), we have that $\frac{\partial}{\partial H}\left(\frac{s(H)}{\gamma_d}\right)\Big|_{H=0} = \left(\frac{1}{\alpha^2}\right) \cdot \left(\left(\frac{\lambda_p^o}{r}\right) \cdot 2\left(\lambda_p^o + \lambda_p^c\right)x^2 + \left(2\gamma_p^o - \gamma_p^c\right) \cdot \left(\theta + \left(\lambda_p^c - \lambda_p^o\right)x^2\right)\right)$ which is unambigously decreasing in r. Thus $\frac{s(H)}{\gamma_d}$ is also decreasing in λ_d and increasing in γ_d when $\lambda_p^o - r\left(\alpha - \gamma_p^o\right) \leq 0$, proving the result.