Extreme Agenda Setting Power in Dynamic Bargaining Games

John Duggan
Dept. of Political Science
and Dept. of Economics
University of Rochester

Zizhen Ma
Dept. of Economics
University of Rochester

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Abstract

In a general bargaining model with a fixed proposer, we show that the agenda setter's equilibrium payoff is never lower than her payoff in the static model. There is a cutoff level of voter patience such that below the cutoff, the static equilibrium (possibly along with others) obtains; and above the cutoff, all equilibria are in mixed proposal strategies, and the agenda setter does strictly better than the static payoff. When the dimensionality of the set of alternatives is high, the power of the agenda setter typically becomes extreme as voters become patient: equilibrium outcomes converge to the ideal point of the agenda setter. Voters accept outcomes worse than the status quo because they anticipate the possibility of even worse outcomes in the future; and as voters become patient, this threat looms large, conferring increasing leverage to the agenda setter. In the majority rule case, for example, if the set of alternatives has dimension three or more, then for generic profiles of utilities, agenda setting power becomes extreme as voters become patient.

1 Introduction

This paper studies the power of the agenda setter in dynamic bargaining games. We provide a general framework that captures the case of a simple take-it-or-leave-it offer, in which case the agenda setter must offer a winning coalition of voters something weakly preferable to the status quo, and in equilibrium, the agenda setter simply maximizes her payoff subject to this acceptance constraint. In this static setting, if a proposal is rejected, then the game ends with the status quo outcome, but when follow-up proposals are possible, dynamic incentives enter the equilibrium analysis:

if a proposal is rejected, then the status quo is maintained in the current period, but future outcomes will be determined by proposals of the agenda setter in the future. This means that a voter may accept a proposal that is worse than the status quo, if it is possible that a follow-up proposal will lead to an even worse outcome in the event that the initial proposal is rejected. In case the set of alternatives is one-dimensional, it is known that the equilibrium outcomes of the bargaining game are not affected by dynamic incentives, but when the set of alternatives is multidimensional, we show that the threat of "even worse" outcomes in the future confers leverage on the agenda setter, and the equilibrium analysis changes substantially. Not only do dynamic incentives affect equilibrium outcomes, but the threat of even worse future outcomes looms larger as voters place more weight on the future, conferring greater leverage on the agenda setter.

In the dynamic model, equilibrium outcomes are always weakly better for the agenda setter than the static equilibrium. There is a cutoff level of voter patience such that below the cutoff, the static equilibrium (possibly along with others) obtains; and above the cutoff, all equilibria are in mixed proposal strategies, and the agenda setter's equilibrium payoff in the dynamic model is strictly higher than in the static model. When the set of alternatives is multidimensional, this cutoff is typically strictly less than one; that is, voter patience decreases the inertia of the status quo, allowing the agenda setter to impose outcomes that are strictly worse than the status quo for voters and to attain a payoff above the static equilibrium level. Surprisingly, as voters become more patient, the power of the agenda setter typically becomes extreme: equilibrium outcomes converge to the ideal point of the agenda setter as the voters' discount factor goes to one. To be more precise, we show that as voters become patient, the agenda setter's equilibrium proposals either converge to her ideal point or to a *constrained core point.* The latter refers to an alternative x such that in the hyperplane orthogonal the agenda setter's gradient at x, there is no other alternative strictly preferred to x by all members of any decisive coalition. We show that at such an alternative, voter utilities must satisfy a gradient condition that becomes restrictive as the number of dimensions increases. In particular, when the number of agents is odd and the dimensionality of the set of alternatives is three or more, we show that for generic profiles of utility functions, the constrained core for majority rule is empty—implying that equilibrium outcomes converge to the agenda setter's ideal point. For any voting rule based on a quota less than unanimity, there is a dimensionality cutoff such that generic emptiness of the constrained core holds, implying that agenda setting power becomes extreme.

We assume that bargaining takes place in discrete time over an infinite horizon, and that a fixed agenda setter can propose any alternative, and that this proposal is then subject to consideration by voters. If a winning coalition of voters accept the proposal, then the game ends with this outcome; otherwise, the proposal is rejected, the status quo remains in place for the current period, the agenda setter makes a follow-up proposal in the next period, which is subject to consideration by voters, and so on. We impose minimal structure on the set of alternatives and preferences of the agents: the set of alternatives is a nonempty, compact, and convex subset of Euclidean space, and stage utilities are continuous, concave, and satisfy a weakening of strict quasi-concavity. In particular, we admit the multidimensional spatial model of politics, as well as a large class of economic environments of interest. Thus, our bargaining protocol is the protocol used by Baron and Ferejohn (1989) in their model of legislative bargaining, as generalized by Banks and Duggan (2000,2006). We assume, as in Banks and Duggan (2006), that the status quo is an arbitrary alternative that may be desirable to voters, rather than setting it equal to the zero allocation, as in Baron and Ferejohn (1989), or assuming a bad status quo, as in Banks and Duggan (2000).

The topic of agenda setting power has been broached before in the formal political science literature. Romer and Rosenthal (1978) take up the question assuming an agenda setter can make a take-it-or-leave-it offer of a one-dimensional policy, which is then subject to a majority vote. They show that the equilibrium proposal makes a pivotal voter indifferent between accepting and rejecting the proposal, and that if the median voter is between the status quo and the agenda setter, then the agenda setter obtains an outcome strictly preferred to the median ideal point in equilibrium, despite the fact that the median defeats all other policies (i.e., it is a Condorcet winner) in pairwise votes. Whereas the Romer-Rosenthal model consists of a single period, Primo (2002) examines the power of the agenda setter in a one-dimensional model with a single voter using the same bargaining protocol as we do here: if a proposal is rejected by the voter, then the game continues to the next period, and the agenda setter makes a follow-up proposal to the voter, and this process can continue ad infinitum. He shows that the outcome of all pure strategy subgame perfect equilibria coincides with that of the Romer-Rosenthal game; that is, dynamic incentives do not affect the equilibrium outcome of the static game. Banks and Duggan (2006) assume majority voting and quadratic utilities, and they similarly conclude that the outcome of every stationary equilibrium is the same as in the static game.

These findings may suggest that for the widely used protocol of Baron and Ferejohn (1989), considerations of the future do not affect the power of the agenda setter. To our knowledge, the question has not been further explored in other environments using this standard protocol. In particular, it was not previously known whether the flexibility afforded by multiple dimensions could allow the agenda setter to play different winning coalitions off against each other, using proposals to different conditions to threaten voters with "even worse" outcomes in the future to obtain outcomes better for the agenda setter than in the static game. Nevertheless, it has long been well known that the properties of majority voting in one dimension are starkly different from the properties in two or more dimensions: assuming an odd number of voters, the median ideal point is a Condorcet winner in one dimension, but in two or more dimensions, a Condorcet winner exists only in knife-edge cases. Typically, there is no Condorcet winner, because even if it is not possible to make all members of a majority coalition better off by moving along a given line, it is generally possible, in two or more dimensions, to move in a *different* direction that *is* preferred by all members of a majority coalition. We find that the presence of multiple dimensions indeed leads to a qualitative difference in the equilibrium analysis: when voters are sufficiently patient, the agenda setter obtains outcomes strictly better than in the static setting, and as voters become arbitrarily patient, equilibrium outcomes converge to the agenda setter's ideal point.

In earlier work, McKelvey (1976,1979) takes a social choice approach to agenda setting, and he shows in a general spatial model that majority preference cycles fill the set of alternatives, so that from any given status quo, a sophisticated agenda setter can design a (possibly long) sequence of binary votes that lead from the status quo to her ideal policy. This analysis is highly suggestive, but it relies on the assumption that voters are naive—they are unaware of the agenda setter's motives and cannot anticipate (or do not care about) the future consequences of a vote.

The topic of agenda setting power has been examined in another branch of the game-theoretic literature on dynamic bargaining with an endogenous status quo. Here, (i) an initial status quo is given, and the agenda setter makes a proposal that is subject to a vote; (ii) if the proposal passes, then it is the outcome in the current period, and it becomes the status quo in the next; (iii) if the proposal passes, then the status quo remains in place and carries over to the next period. and (iv) in the next period (and in all subsequent ones), the protocol is repeated. This class of games is technically difficult to work with, and most work has focused on relatively specialized environments. Diermeier and Fong (2011) consider the case of a finite set of alternatives with a fixed (or "persistent") agenda setter, and after verifying existence of a pure strategy Markovian equilibrium, they examine an example in which an agenda setter divides a discretized dollar among herself and two voters. They find that when voters are patient relative to the size of the grid, there exist equilibria in which the agenda setter's allocation of the dollar is bounded away from one—so that agenda setting power does not become extreme as players become patient.[1] Moreover, they find that the equilibrium payoff of the agenda setter can decrease relative to her static payoff. The restriction of proposals to a finite grid limits the flexibility of the agenda setter to exploit the multidimensionality of the set of alternatives in their model, and the concurrent assumption of patient voters means that the "gaps" between alternatives become significant for voters. The spirit of Diermeier and Fong's result diverges significantly from ours, but differences between the models make the results of the

[1]Diermeier and Fong (2012) continue the previous analysis of the fixed agenda setter, and for the case sufficiently patient players, they characterize the set of absorbing points of pure strategy Markovian equilibria as the unique von Neumann-Morgenstern stable set. Anesi and Duggan (2016) focus on bargaining with veto players in the finite framework, but they show that the latter characterization carries over when mixed strategies are allowed.

papers difficult to compare.

Kalandrakis (2010) considers the model with a continuously divisible dollar, and he constructs a class of Markovian equilibria for arbitrary recognition probabilities. In the case of a fixed agenda setter, his equilibrium exhibits a form of extreme agenda setter power, consonant with the results of the current paper: in his equilibrium, regardless of the status quo, the agenda setter consumes the entire dollar from the second period onward. His result differs from ours in that the equilibrium path of play in his model is a sequence of alternatives, so that the agenda setter's ideal point is reached after two periods (rather than immediately), and it holds for any rate of discount by the voters. In contrast to McKelvey's (1976,1979) findings, but similar to ours, voters are strategic and place positive weight on the future, but the agenda setter is nevertheless able to leverage the threat of "even worse" outcomes for voters in the future to obtain desirable outcomes for herself in the present.

The remainder of the paper is organized as follows. In Section 2, we describe the bargaining framework, and in Section 3, we explore four special cases of interest, highlighting the possibility of extreme agenda setting power in two tractable examples: the divide the dollar model and a symmetric spatial model in two dimensions. We establish existence of stationary bargaining equilibria in Section 4, and we show that under very general conditions, all equilibria are no-delay, i.e., the agenda setter's proposal is accepted in the first period. In Section 5, we compare equilibria in the dynamic and static models, showing that the agenda setter's equilibrium payoff in the dynamic model is no lower than the static equilibrium payoff; and we establish a cutoff level of voter patience such that above that level, all equilibria are in mixed strategies, and the agenda setter does strictly better than the static equilibrium. Section 6 is the heart of the analysis of extreme agenda setting power: we show that if the equilibrium proposals of the agenda setter do not converge to her ideal point as voters become patient, then a set of restrictive conditions must be met, and in particular, the constrained core must be nonempty. In Section 7, we derive strong gradient conditions that must be satisfied at any constrained core point, and in Section 8, we show that when the dimensionality of the set of alternatives is high, these gradient conditions become prohibitive, and for generic profiles of utility functions, the constrained core is empty. Combined with the necessary conditions of Section 6, we conclude that the equilibrium proposals converge to the ideal point of the agenda setter: as voters become patient, the power of the agenda setter becomes extreme.

2 Dynamic bargaining model

Assume a set of agents, indexed $0, 1, \ldots, n$, must choose collectively from a nonempty, compact, convex set $X \subseteq \Re^d$ of alternatives. We consider a bargaining protocol in which agent 0 is a fixed agenda setter, agents $1, \ldots, n$ are voters, and $q \in X$ is a fixed status quo alternative. Let $N = \{1, \ldots, n\}$ denote the set of voters, and let $\mathcal{D} \subseteq 2^N$ be a *voting rule,* i.e., a collection of nonempty coalitions, termed *decisive,* that have the authority to pass an alternative proposed by the agenda setter. In each period, the agenda setter proposes any alternative x, and then voters simultaneously decide to accept or reject the proposal. If the coalition of voters who accept is decisive, then the game ends with outcome x; otherwise, the status quo q remains in place for the current period, the game continues to the next period, and the process is repeated.

Assume that the voting rule $\mathcal{D}$ is *monotonic,* in the sense that if a coalition is decisive, then every coalition containing it is decisive as well, i.e., for all $C \in \mathcal{D}$ and all $C' \supseteq C$, we have $C' \in \mathcal{D}$. We say $\mathcal{D}$ is *proper* if there do not exist disjoint decisive coalitions, i.e., for all $C, C' \in \mathcal{D}$, we have $C \cap C' \neq \emptyset$. We say the voting rule is *collegial* if there is some voter who belongs to every decisive coalition, i.e., if $\bigcap \mathcal{D} \neq \emptyset$, in which case such a voter is a *veto player;* and otherwise, if there exists $C \in \mathcal{D}$ with $i \notin C$, then i is a *rank and file* voter. We say it is *oligarchic* if the coalition of veto players is itself decisive, i.e., $\bigcap \mathcal{D} \in \mathcal{D}$. A special case of the general model is majority rule or any voting rule with quota m, where a coalition C of voters is decisive if and only if $|C| \geq m$. This representation of voting is quite general, and it captures many special cases of interest. We technically assume that decisive coalitions consist only of voters, an assumption which suits the equilibrium analysis, below, and which is without loss of generality: because the agenda setter can always vote for her own proposals, we can also capture settings in which the agenda setter has voting power. When n is even, the separate roles of the agenda setter and voters leads to two versions of majority rule, depending on whether the agenda setter's vote counts, in which it is necessary and sufficient that at least $\frac{n}{2}$ voters accept, or she cannot vote, in which case $m = \frac{n}{2} + 1$ are needed. We define *inclusive majority rule* as the quota rule with $m = \frac{n}{2}$; and we define *exclusive majority rule* as the quota rule with $m = \frac{n}{2} + 1$. This distinction does not arise when n is odd.

Each agent i evaluates alternatives according to a continuous, concave stage utility function $u_i \colon X \to \Re$, and we assume that each u_i has a unique maximizer $\hat{x}^i$, the *ideal point* of agent i. For some results, we assume that each u_i is continuously differentiable, by which we mean there is an open set $V \subseteq \Re^d$ containing X such that u_i can be extended to a continuously differentiable function on V.[2] We say u_i is *Euclidean* if the agent's preferences are a function of distance to the ideal point: for all $x, y \in X$, $\|x - \hat{x}^i\| = \|y - \hat{x}^i\|$ implies $u_i(x) = u_i(y)$. To rule out trivial cases, we at times assume that the agenda setter has at least some scope to change the status quo. Formally, we say *no gridlock* holds if there exist an alternative $x \in X$ with $u_0(x) > u_0(q)$ and a decisive coalition $C \in \mathcal{D}$ such that for all $i \in C$, we have $u_i(x) > u_i(q)$. Let $\delta \in [0, 1)$ be the common discount factor of the voters, and let $\delta_0 \in [0, 1)$ be the discount factor of the agenda setter. If the game ends in period t with outcome x, then each voter i receives a payoff of $(1 - \delta^{t-1})u_i(q) + \delta^{t-1}u_i(x)$,

[2]This convention follows Mas-Colell (1985).

and if no alternative is ever passed, then i receives payoff $u_i(q)$; the agenda setter's payoffs are defined analogously using discount factor δ_0.

In addition to continuity and concavity, we take a weakening of strict quasi-concavity as a maintained assumption: if an alternative y is weakly preferred to an alternative $x \neq y$ by all members of a coalition, then x and y can be approximated by alternatives that all coalition members strictly prefer to x. Formally, define the weak and strict upper contour sets of agent $i \in N \cup \{0\}$ at $x \in X$, respectively, by

$$\begin{aligned} R_i(x) &= \{y \in X \mid u_i(y) \geq u_i(x)\} \\ P_i(x) &= \{y \in X \mid u_i(y) > u_i(x)\}. \end{aligned}$$

Define coalitional upper contour sets by

$$R_C(x) = \bigcap_{i \in C} R_i(x) \quad \text{and} \quad P_C(x) = \bigcap_{i \in C} P_i(x)$$

for each $C \subseteq N \cup \{0\}$. Then *limited shared weak preference* (LSWP) holds if for all $C \subseteq N \cup \{0\}$ and all $x \in X$,

$$|R_C(x)| > 1 \quad \text{implies} \quad R_C(x) \subseteq \text{clos}(P_C(x)).$$

To see that strict quasi-concavity of utilities implies LSWP, consider any coalition C, any $x \in X$, and any $y \in R_C(x) \setminus \{x\}$. Then for all $\alpha \in (0,1)$, we have $x(\alpha) \equiv (1-\alpha)x + \alpha y \in P_C(x)$, and $\lim_{\alpha \downarrow 0} x(\alpha) = x$, as required. Under our background conditions, LSWP is equivalent to strict quasi-concavity when $d = 1$. In fact, an implication of LSWP is that each u_i has a unique maximizer, which is just the ideal point of the agent.

Our assumptions on preferences capture important classes of environments with private goods that would be excluded by strict quasi-concavity. Several familiar examples of environments satisfying LSWP are as follows:

- *Classical spatial model/Pure public goods.* Alternatives are vectors of ideological policies or public good levels. Each u_i is strictly quasi-concave, e.g., $u_i(x) = -\|x - \hat{x}^i\|^2$ or $u_i(x) = -\|x - \hat{x}^i\|$.

- *Public decisions with transfers.* The set X of alternatives is a subset of $Z \times T$, where Z is a set of public decisions and $T \subseteq \mathbb{R}^{n+1}_+$ is a set of allocations of private good satisfying a weak transferability condition: for all $x = (z,t) \in X$ and all $t' \in \mathbb{R}^{n+1}_+$ such that $\sum_{i=0}^n t_i = \sum_{i=0}^n t'_i$, we have $(z,t') \in X$. Each u_i is quasi-linear, i.e., $u_i(z,t) = \phi_i(z) + t_i$, with valuation function ϕ_i strictly quasi-concave.

- *Exchange economy.* Alternatives are allocations of a fixed endowment of private goods, and each u_i is strictly quasi-concave and strictly increasing in i's consumption.

A special case of the second and third examples above is the simple *divide the dollar* environment, in which alternatives are vectors $x = (x_1, \ldots, x_n)$ of allocations of a dollar, with $\sum_{i=0}^{n} x_i \leq 1$, and $u_i(x)$ is constant in x_{-i} and strictly increasing and concave in x_i for each player. The condition of LSWP is purely ordinal, but Lemma 1, in Appendix A.1, shows that the assumption of continuous stage utilities permits a cardinal reformulation.

To obtain sharp lower bounds on the agenda setter's equilibrium payoffs, we later consider environments such that stage utility functions are strictly concave, a condition that is clearly consistent with applications to the classical spatial model or problems or pure public good provision. But assuming at least two voters, strict concavity is violated in exchange economies and in models of public decisions with transfers. In the latter models, linearity of utility in the private good is immediately inconsistent with strict concavity. A more general issue that affects exchange economies as well is the fact that even if an agent's utility is strictly concave in her own consumption of a private good, she is indifferent between reallocations of private good between other agents. For example, if $x = (x_0, x_1, \ldots, x_n)$ is an allocation with $x_1 \neq 0$, then the agenda setter is indifferent between x and any convex combination with the alternative $x' = (x_0, x_1 + x_2, x_3, \ldots, x_n)$ that reallocates voter 1's bundle to voter 2. In particular, $u_0(\frac{1}{2}x + \frac{1}{2}x') = \frac{1}{2}u_0(x) + \frac{1}{2}u_0(x')$, violating strict concavity. To capture environments with private good components, we establish that our characterization results also hold under the assumption of *minimal transferability,* i.e., for all $x \in X$ and all $j \in N$ with $u_j(x) > \min_{z \in X} u_j(z)$, there exists $y \in X$ such that $u_0(y) > u_0(x)$ and such that for all $i \in N \setminus \{j\}$, we have $u_i(y) > u_i(x)$.[3] Intuitively, if x does not minimize voter j's stage utility, then her consumption of private good must be positive, and we reallocate goods from her to all other agents, making them better off. This assumption is weak and is satisfied in models of public decisions with transfers, exchange economies, and other environments with a transferable private good.

The *core* of the voting rule $\mathcal{D}$ consists of the alternatives that are maximal with respect to the social preference relation induced by individual preferences. Formally, we define this social preference relation as follows: given any $x, y \in X$, we have $x \succ y$ if and only if there is a decisive coalition $C \in \mathcal{D}$ such that for all $i \in C$, $u_i(y) > u_i(x)$. Then alternative x belongs to the core if and only if there is no alternative socially preferred to it, i.e., there does not exist $y \in X$ satisfying $y \succ x$. Assuming $d = 1$, our assumptions imply that voter preferences are single-peaked, and if n is odd, then the median voter theorem implies that the core of exclusive majority rule consists of the unique median ideal point; and if n is even, then the core of exclusive majority rule consists of two median ideal points and the alternatives between them. In general, regardless of the dimensionality of the alternatives, Plott's theorem (Plott (1967)) implies that if n is odd and an exclusive majority core alternative exists, say x, then

[3]Banks and Duggan (2006) refer to minimal transferability as "limited transferability."

it is unique and is the ideal point of a "core voter." An if no other voter shares this ideal point (which is the generic case), then x satisfies *radial symmetry* in the sense that for every direction p with norm one, the number of voters with gradients pointing in the p direction equals the number of voters with gradients pointing in the $-p$ direction:

$$|\{i \in N \mid \nabla u_i(x) = \|\nabla u_i(x)\| p\}| \quad = \quad |\{i \in N \mid \nabla u_i(x) = -\|\nabla u_i(x)\| p\}|.$$

This condition is highly restrictive, and Schofield (1983) shows that when n is odd and $d \geq 2$, for generic profiles of utility functions, there is no alternative satisfying radial symmetry; thus, the exclusive majority core is generically empty. When $n \geq 4$ is even, an exclusive majority core x alternative either satisfies radial symmetry, or it is the ideal point of some voter, and there are three other voters, say $i, j, k \in N$ with linearly dependent gradients. This necessary condition is weaker but still restrictive: Schofield (1983) shows that when $n \geq 4$ is even and $d \geq 3$, for generic profiles of utility functions, the exclusive majority core is empty.

The analysis of the dynamic bargaining model focuses on a class of subgame perfect equilibria in stationary strategies. Thus, a strategy profile is $\sigma = (\pi, \alpha)$, where π is a Borel probability measure on X representing the agenda setter's proposal strategy, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that each $\alpha_i \colon X \to [0, 1]$ is a Borel measurable mapping representing voter i's acceptance strategy. Here, $\alpha_i(x)$ denotes the probability that i accepts the proposal x, and we use the shorthand $\alpha(x)$ to denote the probability that all members of at least one decisive coalition accept x if proposed. Say the proposal strategy π is *pure* if it is degenerate on a single alternative, i.e., $\pi(\{x\}) = 1$ for some $x \in X$. We consider subgame perfect profiles σ such that the agenda setter's proposal strategy π is optimal given acceptance strategies α, and acceptance strategies are stage-undominated given π. To formalize these ideas, note that each strategy profile σ defines in an obvious (if notationally dense) manner a probability distribution over sequences of outcomes and, with it, an expected payoff $v_i(\sigma)$ for each agent $i \in N \cup \{0\}$ as evaluated at the beginning of the game. By stationarity, this is also agent i's continuation value throughout the game, i.e., i's expected payoff evaluated at the beginning of next period if the current period's proposal is rejected.

Formally, σ is a *stationary bargaining equilibrium* if two conditions hold. First, we require that the voters' acceptance strategies satisfy *stage dominance,* i.e., voter i accepts x if the stage utility from x strictly exceeds the expected payoff from rejection, and only if it weakly exceeds the expected payoff from rejection: $\alpha_i(x) = 1$ if

$$u_i(x) \quad > \quad (1 - \delta) u_i(q) + \delta v_i(\sigma)$$

and $\alpha_i(x) = 0$ if

$$u_i(x) \quad < \quad (1 - \delta) u_i(q) + \delta v_i(\sigma),$$

where we may refer to the rejection payoff $(1-\delta)u_i(q)+\delta v_i(\sigma)$ as the *reservation value* of the voter. This condition eliminates implausible equilibria in which, given a quota rule with $m<n$, all voters accept every proposal independently of preferences: the problem in such situations is that no individual's vote will change the outcome of the game, and hence everyone's vote is a best response, despite the fact that some voters may be accepting undesirable policies. Second, we require that agenda setter's proposals are *sequentially rational,* in the sense that with probability one, proposals maximize the expected payoff of the agenda setter: every $x\in\text{supp}(\pi)$ solves

$$\max_{y\in X}\alpha(y)u_0(y)+(1-\alpha(y))[(1-\delta_0)u_0(q)+\delta_0 v_0(\sigma)].$$

Note that the agenda setter is free to propose q, and it follows that in equilibrium, the agenda setter's expected payoff from every $x\in\text{supp}(\pi)$ is at least equal to

$$\alpha(q)u_0(q)+(1-\alpha(q))[(1-\delta_0)u_0(q)+\delta_0 v_0(\sigma)].$$

Integrating with respect to π, we obtain the inequality $v_0(\sigma)\geq u_0(q)$, so that the agenda setter's equilibrium expected payoff is at least equal to the stage utility from the status quo.

A stationary bargaining equilibrium σ is *no-delay* if the agenda setter's proposals are accepted with probability one, i.e., $\int\alpha(x)\pi(dx)=1$. In this case, the continuation value of each agent i takes the especially simple form,

$$v_i(\sigma) \;\;=\;\; \int u_i(z)\,\pi(dz),$$

and as this depends on the proposal strategy alone, we henceforth write $v_i(\pi)$ for the continuation value of agent i in a no-delay stationary equilibrium. We say σ is *gridlocked* if the status quo is maintained with probability one, either because no proposal is ever passed, or because the status quo is the only proposal passed with positive probability: $\int_{X\setminus\{q\}}\alpha(z)\pi(dz)=0$.[4] Such equilibria may exhibit delay or may not, as is the case if q is proposed and accepted with positive probability.

3 Special cases of interest

3.1 Static model

In this application, we consider the simple take-it-or-leave-it offer model of Romer and Rosenthal (1978), extending their one-dimensional setting to a general set of

[4]Such equilibria are termed "static" by Banks and Duggan (2006). We use this term differently in our analysis of the static model in Section 3.

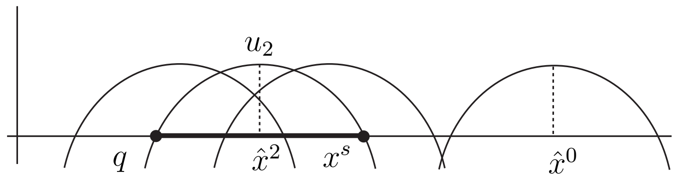


Figure 1: Static equilibrium in one-dimensional model

alternatives and general preferences. We obtain this as a special case of the general bargaining framework by setting $\delta = 0$ and focusing on pure proposal strategies (since mixing is no longer needed for existence). Thus, we say (x, α) is a *static equilibrium* if x solves

$$\max_{y \in X} \alpha(y)u_0(y) + (1 - \alpha(y))u_0(q),$$

and acceptance strategies satisfy $\alpha_i(x) = 1$ if $u_i(x) > u_i(q)$ and $\alpha_i(x) = 0$ if $u_i(x) < u_i(q)$ for all voters i. The static equilibrium (x, α) is no delay if and only if $\alpha(x) = 1$, and a no-delay equilibrium is gridlocked if and only if $x = q$. Assuming one dimension with three voters, and majority rule, Figure 1 depicts a static equilibrium in which the agenda setter moves the outcome from the status quo, which is on the far side of the median voter, to an alternative closer to her ideal point. Static equilibria are depicted for the two-dimensional spatial model in Figure 2 for inclusive and exclusive majority rule, where each voter accepts the alternatives in the upper contour set through the status quo, and the shaded regions consist of alternatives that will pass if proposed.

It is useful to define the *static acceptance set* of voter i as

$$A_i^s \quad = \quad \{x \in X \mid u_i(x) \geq u_i(q)\},$$

and then we define

$$A_C^s \;=\; \bigcap_{i \in C} A_i^s \quad \text{and} \quad A^s \;=\; \bigcup_{C \in \mathcal{D}} A_C^s,$$

where the set A^s is the *static social acceptance set,* which consists of every alternative such that a decisive coalition weakly prefers it to the status quo. Note that each A_i^s contains q; and by continuity and concavity of u_i, the set is compact and convex, and these properties are inherited by the set A_C^s. Nonemptiness and compactness (though not generally convexity) carry over to A^s. In Figure 1, the static social acceptance set is the dark interval, and i Figure 2, it is the gray region in the left and right panels.

We will characterize outcomes of no-delay equilibria in the static model as solutions to the following simple maximization problem,

$$\max_{x \in A^s} u_0(x).$$

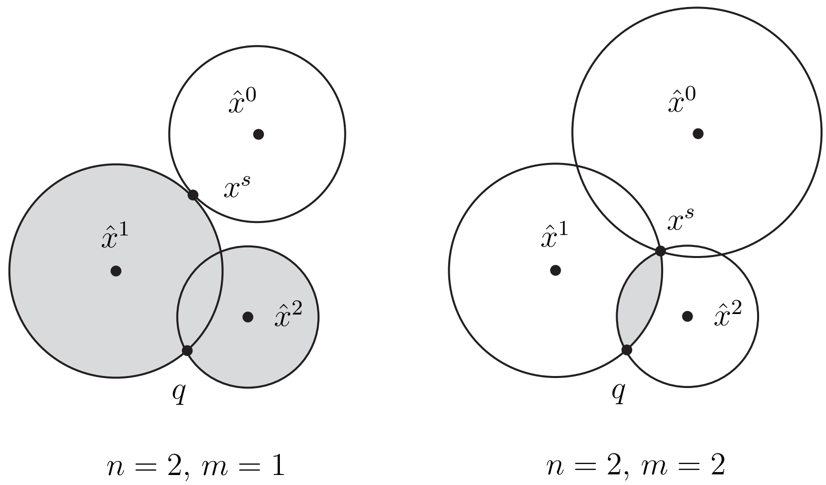


Figure 2: Static equilibria in two dimensions

removing voters from the analysis. Since no-delay static equilibrium proposals can be determined separately from voting strategies, we can then designate such an alternative x^s as a *static equilibrium,* without reference to acceptance strategies. For future use, we denote the maximized value of the agenda setter's objective function by

$$u_0^s \quad = \quad \max_{x \in A^s} u_0(x).$$

Moreover, we can decompose the above problem into separate coalitional optimization problems. By concavity of u_i, each A_C^s is convex, so there is a unique solution to

$$\max_{x \in A_C^s} u_0(x),$$

and we denote this by x^C. There is a finite number of such *candidate static equilibria,* and we show that x^s is a static equilibrium if and only if it is a candidate static equilibrium and maximizes the agenda setter's stage utility over these candidates. Finally, we note that in one dimension or when the voting rule is oligarchical, the no-delay static equilibrium is unique; thus, uniqueness in Figure 1 and in the right-hand panel of Figure 2, is not a coincidence.

Proposition 1: *For every alternative x, there exist acceptance strategies α such that (x, α) is a no-delay static equilibrium if and only if*

$$x \in \arg\max_{y \in A^s} u_0(y),$$

and thus, there is at least one no-delay static equilibrium. For each coalition $C \subseteq N$, the problem

$$\max_{x \in A_C^s} u_0(x)$$

has a unique solution, denoted by x^C, and x^s is a no-delay static equilibrium if and only if

$$x^s \in \arg\max\{u_0(x) \mid x = x^C \text{ for some } C \in \mathcal{D}\}.$$

The agenda setter's static equilibrium payoff is at least equal to the stage utility from the status quo, i.e., $u_0^s \geq u_0(q)$, and if there is a no-delay static equilibrium $x \neq q$, then $u_0^s > u_0(q)$. Moreover, if $d = 1$ or if $\mathcal{D}$ is oligarchic, then there is a unique no-delay static equilibrium.

3.2 One-dimensional model

In this application, we assume $X = [\underline{x}, \overline{x}] \subseteq \Re$ is a compact interval. Primo (2002) considers a special case of this model with a single voter and such that the voter and agenda setter have Euclidean utilities. He shows that the unique outcome of every (possibly non-stationary) pure-strategy subgame perfect equilibrium is the static equilibrium. A strength of the analysis is that it allows for non-stationary equilibria, but it is limited by the assumption of a single voter. Banks and Duggan (2006) allow multiple voters and assume quadratic stage utility and majority rule, and they show that, again, the unique no-delay stationary bargaining equilibrium outcome is the static equilibrium. Here, we carry forward the assumption of quadratic stage utility, and we allow for any number of voters and any quota rule with quota $m > \frac{n}{2}$. Let x^s be the unique no-delay static equilibrium, and assume without loss of generality that $q \leq x^s \leq \hat{x}^0$. To support x^s as a stationary bargaining equilibrium outcome in the dynamic game with $\delta > 0$, specify that π is degenerate on x^s and that each voter accepts a proposal if and only if it offers at least the corresponding reservation payoff, i.e., $\alpha_i(x) = 1$ if $u_i(x) \geq (1-\delta)u_i(q) + \delta u_i(x^s)$, and $\alpha_i(x) = 0$ if $u_i(x) < (1-\delta)u_i(q) + \delta u_i(x^s)$. The strategy profile $\sigma = (\pi, \alpha)$ specified thusly is a stationary bargaining equilibrium, independent of the discount factor of the voters.

To see this, we first claim that x^s gains the support of a decisive coalition. Indeed, note that for all voters i, we have $u_i(x^s) \geq (1-\delta)u_i(q) + \delta u_i(x^s)$ if and only if $u_i(x^s) \geq u_i(q)$. Define the coalition $C = \{i \in N \mid u_i(x^s) \geq u_i(q)\}$ of voters who are willing to accept x^s in the static model. Then $x^s \in A_C^s$, and thus for all $i \in C$, we have $\alpha_i(x^s) = 1$. Since $x^s \in A^s$, we have $C \in \mathcal{D}$, as claimed. To verify sequential rationality, we must argue that no other proposal yields a greater expected payoff to the agenda setter. This is clearly true if the static outcome is ideal for the agenda setter, so assume $x^s < \hat{x}^0$. Consider any other alternative y, and since proposing an alternative that is rejected or is less than x^s cannot be better than x^s, assume that $\alpha(y) = 1$ and $y > x^s$. By the above arguments, we have $x^s = x^C$, so there must be some voter $i \notin C$ such that $\alpha_i(y) = 1$, but for such a voter, we have $u_i(q) > u_i(x^s)$, so strict concavity of u_i implies $\hat{x}^i < x^s < y$, and this implies $(1-\delta)u_i(q) + \delta u_i(x^s) > u_i(y)$, a contradiction. We conclude that (x, α) is a

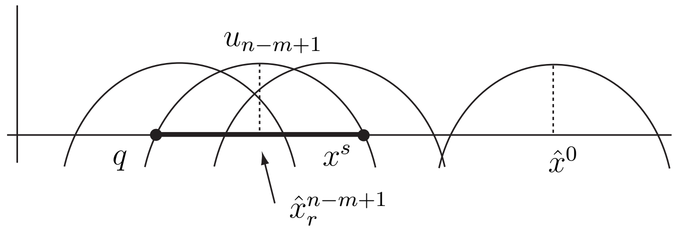


Figure 3: Static equilibrium maintained in one-dimensional model

static equilibrium. This observation leaves open, however, the possibility that other equilibrium outcomes are possible in the dynamic model.

It is instructive to delve further into the structure of the equilibrium constructed above. Assuming that $q < \hat{x}^0$, for each voter i, we let x_r^i denote the alternative weakly greater than q such that $(1-\delta)u_i(q) + \delta u_i(x^s) = u_i(x_r^i)$, if such an alternative exists; otherwise, let $x_r^i = \overline{x}$. Assume that voters are indexed in the order of their ideal points, i.e., $\hat{x}^1 \leq \hat{x}^2 \leq \cdots \leq \hat{x}^n$. In addition, assume that $q < x_r^{n-m+1}$, so that in the equilibrium constructed, if $x \in [q, x_r^{n-m+1}]$, then x is accepted by all voters $i \geq n-m+1$, i.e., the coalition $C = \{n-m+1, \ldots, n\} \in \mathcal{D}$ accepts x if proposed, and since this coalition has m members, x passes. On the other hand, a proposal outside the interval fails to be accepted by m voters. Thus, the set of alternatives that pass if proposed in this equilibrium is just $[q, x_r^{n-m+1}]$, and if $x_r^{n-m+1} < \hat{x}^0$, then the equilibrium outcome is just $x^s = x_r^{n-m+1}$, the outcome that makes the "pivotal voter" $n-m+1$ indifferent between acceptance and rejection.

We have argued that the static equilibrium carries over to the dynamic bargaining model with $\delta > 0$, and the next proposition establishes that the static equilibrium is actually the unique stationary bargaining equilibrium outcome, independent of the voters' rate of discounting. That is, in the one-dimensional model, with a quota rule and quadratic utilities, dynamic incentives do not alter the static outcome. We will establish later, in Theorem 4, that the result does not depend on any assumptions on the voting rule or stage utilities, so the equilibrium characterization in Proposition 2 holds for the general one-dimensional model; but we will see that the one-dimensional model is exceptional in this regard, as the agenda setter's increased latitude in the multidimensional model quite generally permits her to improve on the static outcome.

Proposition 2: *Assume $d = 1$, $\mathcal{D}$ is a quota rule with $m > \frac{n}{2}$, and each u_i is quadratic. There is a no-delay stationary bargaining equilibrium, and every no-delay stationary bargaining equilibrium proposal strategy π is degenerate on the unique static equilibrium.*

3.3 Divide the dollar

In this application, we establish that the stability of static equilibria in the one-dimensional model is overturned in the divide the dollar model, where transfers are possible. Let $X = \{x \in \Re^{n+1}_{+} \mid \sum_{i=0}^{n} x_i = 1\}$, and assume $u_i(x)$ is constant in x_{-i} and strictly increasing and concave in x_i. Normalize utility from zero consumption by setting $u_i(0, x_{-i}) = 0$. Assume a quota rule with quota m. If $m = n$, then all voters receive the status quo, regardless of discount factor, so we assume $m < n$ for the remainder of the subsection.

In a symmetric version of the model, with $q_i = \overline{q}$ for every voter i and common, strictly increasing stage utility $u(z)$ from consumption, it is straightforward to compute a symmetric equilibrium. In this case, we specify that the agenda setter mixes uniformly over coalitions of size m, so that the probability that a voter is proposed to is $\frac{m}{n}$, and that she offers an amount z of the dollar to "in voters" and nothing to "out voters." Letting v be the common continuation value of the voters, we can solve two equations in two unknowns to obtain a stationary bargaining equilibrium:

$$\begin{aligned} u(z) &= (1-\delta)u(\overline{q}) + \delta v \\ v &= \left(\frac{m}{n}\right) u(z) + \left(\frac{n-m}{n}\right) u(0). \end{aligned}$$

We find that the equilibrium continuation value of the voters is

$$v = \frac{(\frac{m}{n})(1-\delta)u(\overline{q}) + (\frac{n-m}{n})u(0)}{1 - \frac{m}{n}\delta}.$$

In contrast to the one-dimensional application, now dynamic incentives matter: the equilibrium continuation value varies non-trivially with the discount factor, and the power of the agenda setter increases (v decreases) with the discount factor. We depict the equilibrium with two voters and inclusive majority rule in Figure 4, where proposals in the shaded area are those that will pass if proposed, and the agenda setter randomizes between allocations $(1-z, z, 0)$ and $(1-z, 0, z)$, which give voters 1 and 2, respectively, the amount z that gives each her reservation value.

Of note is the fact that when $\delta > 0$, the voters' continuation value is strictly less than the status quo payoff: $v < u(\overline{q})$. As a consequence, each voter is willing to accept alternatives worse than the status quo, because when they place some weight on the future, they must account for the possibility of even worse outcomes following rejection. Moreover, the weight on even worse outcomes increases as voters become patient, and we see that, in contrast to the one-dimensional model, agenda setting power becomes extreme in the limit: as $\delta \to 1$, we have $v \to u(0) = 0$, so that the agenda setter's consumption of the dollar goes to one. Informally, in the dynamic model, the agenda setter is able to achieve superior outcomes by playing off different

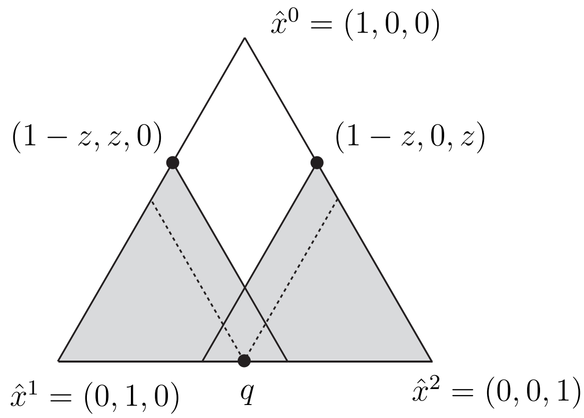


Figure 4: Symmetric equilibrium in divide the dollar

majority coalitions against each other. By this logic, the inertia of the status quo is decreased, and the threat of the worse prospect of receiving zero is increased, as the voters become patient—allowing the agenda setter to consume nearly the entire dollar.

The next proposition establishes that in the divide the dollar model with any quota rule short of unanimity, the extreme agenda setting power exhibited in the simple symmetric equilibrium above is a general property of equilibria: the agenda setter's consumption of the dollar goes to one as voters become patient. We will see in the sequel that the spirit of this result does not depend on the distributive structure of the model, but rather on the ability of the agenda setter to mix between multiple optimal proposals—something that is impossible in the one-dimensional model—and the fact that as voters put less weight on the present, the impact of the status quo goes to zero, and the threat of undesirable outcomes in the future becomes large.

PROPOSITION 3: *In the divide the dollar model with quota $m < n$, if π^δ is a stationary bargaining equilibrium for δ, then as $\delta \to 1$, π^δ converges weak* to the unit mass on x^* with $x_0^* = 1$ and for all $i \in N$, $x_i^* = 0$.*

3.4 Symmetric spatial model

In this application, we consider a spatial example with two voters and inclusive majority rule, i.e., $n = 2$ and $m = 1$, so that the agenda setter requires the acceptance of one voter in order to pass a proposal. Here, we explicitly construct a stationary bargaining equilibrium for arbitrary discount factors assuming specific functional forms, but see Appendix A for a more general construction that does not depend on such assumptions. Assume that each player has quadratic stage utility, i.e., $u_i(x) = -\|x - \hat{x}^i\|^2$, with ideal point $\hat{x}^i$. To simplify the analysis, we place the ideal points at corners of

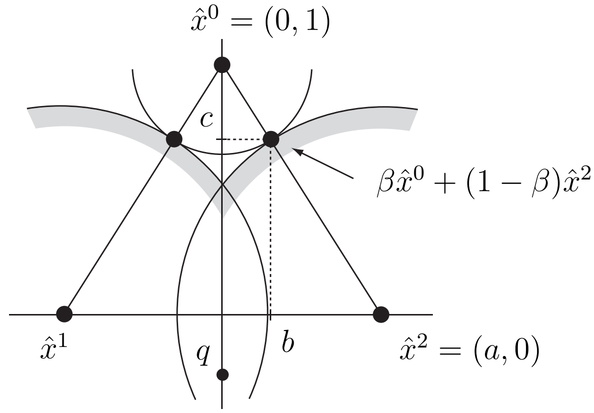


Figure 5: Symmetric equilibrium in spatial model

an Isosceles triangle,

$$\hat{x}^0 = (0,1), \;\; \hat{x}^1 = (-a,0), \;\; \hat{x}^2 = (a,0),$$

and we assume $q_1 = 0$ and $-1 < q_2 < 1$. Thus, the position of voter 1 is symmetric to that of voter 2, relative to the agenda setter and status quo. To illustrate our equilibrium concept and to further develop the themes of the paper, we consider stationary bargaining equilibria in which the agenda setter treats the voters symmetrically: she proposes $x^1 = (-b, c)$ to voter 1 with probability one half, and she proposes $x^2 = (b, c)$ to voter 2 with probability one half, where $0 \le b \le a$ and $0 \le c \le 1$. The environment and structure of equilibrium is depicted in Figure 5.

In contrast to the one-dimensional model, but like the divide the dollar example of the preceding subsection, equilibrium proposals will depend on the discount factor of the voters. When $\delta = 0$, the equilibrium proposals are solutions to the static problem, as alternative x^i makes voter i indifferent between acceptance, with payoff $u_i(x^i)$, and rejection, with payoff $u_i(q)$. When the voters place positive weight on the future, however, these static equilibria do not persist: for example, if voter 2 expects the agenda setter to offer $(-b, c)$ to voter 1 with positive probability, then she will be willing to accept alternatives slightly worse than (b, c), but then $(-b, c)$ would not be optimal for the agenda setter. Instead, when $\delta > 0$, equilibrium proposals respond by moving closer to the ideal point of the agenda setter, and as voters become patient, the respite offered by the status quo declines: equilibrium proposals converge to the ideal point of the agenda setter, so that the power of the agenda setter becomes extreme.

At work is the fact that voter i compares the agenda setter's proposal x^i, which is worse than the status quo, with the continuation value of rejection; in the next period, the agenda setter proposes an alternative to the other voter with probability one half, and that alternative will be even worse than x^i for voter i. As voters become patient and proposals converge to $\hat{x}^0$, the wedge between x^1 and x^2 decreases, but it is magnified by the increase in patience, forcing each voter i to accept proposals further from the status quo. The threat of even worse outcomes becomes more salient, and the equilibrating adjustment is that the voter becomes more accommodating while the agenda setter's proposals converge to her ideal point. These intuitions are formalized in the next proposition.

PROPOSITION 4: *There exists a unique no-delay stationary bargaining equilibrium of the form* $\pi(\{(b,c)\}) = \pi(\{(-b,c)\}) = \frac{1}{2}$*, and in equilibrium,* $(b,c) = \beta(0,1) + (1-\beta)(a,0)$*, where*

$$\beta \;=\; \frac{-\delta a^2 + \sqrt{\delta^2 a^4 - (1-\delta)(1+a^2)[(1-\delta)u_2(q) - 2\delta a^2]}}{(1-\delta)(1+a^2)}. \tag{1}$$

Moreover, $\beta \to 1$ *as* $\delta \to 1$.

4 Equilibrium existence and the no-delay property

In this section, we establish existence of no-delay stationary bargaining equilibrium, and we provide general conditions under which all equilibria are no-delay. Equilibria in pure proposal strategies do not exist generally, but we give conditions under which they do exist and are, in fact, unique: when the set of alternatives is one-dimensional or the voting rule is oligarchical, the unique static equilibrium from Proposition 1 is the unique no-delay stationary bargaining equilibrium; thus, we extend Proposition 2 to a general voting rule and utilities. Existence of no-delay stationary equilibria follows from Banks and Duggan (2006) when the agenda setter discounts future payoffs at the same rate as voters. To cover the case in which the discount rate of the agenda setter differs from that of voters, perhaps due to institutional features (e.g., if the agents are politicians, then the agenda setter's term of office may differ from that of other agents') and to facilitate the subsequent analysis, we show that their existence result can be applied, despite this difference in the models.[5]

The existence argument relies on a correspondence of solutions to a certain constrained optimization problem, where the constraints reflect incentives of voters in no-delay equilibria. Given a proposal strategy π, let $v_i(\pi) = \int u_i(z)\pi(dz)$ be the imputed continuation value for voter i. Then the *acceptance set* of voter i is

$$A_i(\pi) \;=\; \{x \in X \mid u_i(x) \geq (1-\delta)u_i(q) + \delta v_i(\pi)\},$$

which consists of alternatives that meet or exceed the voter's reservation payoff, calculated assuming π is no-delay. We define the *coalitional acceptance set* for C and

[5]For an alternative route to existence when $\delta_0 \neq \delta$, Theorem 3.1 of Duggan (2017) yields a stationary bargaining equilibrium that may exhibit delay, and then we can apply Theorem 3, below, to conclude that the equilibrium is in fact no-delay.

the *social acceptance set* as

$$A_C(\pi) = \bigcap_{i \in C} A_i(\pi) \quad \text{and} \quad A(\pi) = \bigcup_{C \in \mathcal{D}} A_C(\pi),$$

respectively. Thus, the social acceptance set $A(\pi)$ consists of the alternatives that would receive the support of all members of at least one decisive coalition, and would therefore pass, if proposed by the agenda setter.

For each $\pi \in \Delta(X)$, we define $B(\pi)$ be the set of probability measures with support on optimal proposals of the agenda setter, i.e.,

$$B(\pi) = \Delta\left(\arg\max_{x \in A(\pi)} u_0(x)\right).$$

The interest in the correspondence $B\colon \Delta(X) \rightrightarrows \Delta(X)$ so-defined derives from the properties of its fixed points: if $\pi \in B(\pi)$ for some proposal strategy, then the agenda setter places probability one on alternatives that maximize her utility, subject to the constraint that any proposal garners the support of a decisive coalition of voters. It is not immediate that such a fixed point translates to a stationary bargaining equilibrium, because the correspondence restricts the agenda setter to proposals that will pass: it is possible that the status quo does not belong to the social acceptance set, raising the possibility that the agenda setter can deviate to the status quo (or another rejected alternative) and thereby increase her expected payoff. Lemmas 2 and 3, in Appendix A.1, show that this possibility is not realized: a mixed proposal strategy π is a fixed point of B if and only if there exist acceptance strategies α such that (π, α) is a no-delay stationary bargaining equilibrium.

Thus, existence of no-delay equilibrium reduces to confirming existence of a fixed point of the correspondence B. For this, we can apply Theorem 1 of Banks and Duggan (2006), which establishes that B has a fixed point. Since discount factors enter this correspondence only through the social acceptance set $A(\pi)$, the set of fixed points is independent of δ_0 in the model with a fixed agenda setter. Thus, the fixed points of B remain even if $\delta_0 \neq \delta$, as allowed in the current framework.

THEOREM 1: *A no-delay stationary bargaining equilibrium exists.*

By the same argument, we can parameterize stage utilities by the elements λ of a metric space Λ, i.e., we view the stage utility of agent i as a mapping $u_i\colon X \times \Lambda \to \Re$. Assume that for all $i \in N \cup \{0\}$, $u_i(x, \lambda)$ is jointly continuous in (x, λ), that $u_i(x, \lambda)$ is concave in x, and that LSWP is satisfied for all λ. When we fix λ at some value, the implied stage utilities $u_i(\cdot, \lambda)$ satisfy our maintained assumptions, and for parameters $(q, \delta_0, \delta, \lambda)$, we can let $E(q, \delta_0, \delta, \lambda)$ denote the set of proposal strategies π for which there exists α such that (π, α) is a no-delay stationary bargaining equilibrium given $(q, \delta_0, \delta, \lambda)$. Lemma 3, Theorem 3 of Banks and Duggan (2006) implies that the correspondence E of no-delay stationary bargaining equilibrium proposal strategies is upper hemi-continuous.

THEOREM 2: *The correspondence E of no-delay stationary bargaining equilibrium proposal strategies is upper hemi-continuous in the parameters of the model.*

Next, we establish that all stationary bargaining equilibria are no-delay, a result that allows us to focus our characterization results on no-delay equilibria without loss of generality. Banks and Duggan (2006) establish for arbitrary recognition probabilities that if stage utilities are strictly concave or minimal transferability holds, then every stationary bargaining equilibria is either no-delay or gridlocked. Moreover, the maintained assumptions of concavity and LSWP are not sufficient for the result in that paper: Model 6 of Banks and Duggan (2006) contains a one-dimensional example in which voters are risk neutral, three agents have equal recognition probabilities, and there is a stationary bargaining equilibrium that is not gridlocked and that exhibits delay. Our theorem shows that when agenda setting power rests with a single agent, the additional concavity and transferability assumptions are not needed to obtain the no-delay result.

THEOREM 3: *Every stationary bargaining equilibrium is either no-delay or gridlocked.*

The next result establishes existence of pure strategy equilibria when the set of alternatives is one-dimensional or the voting rule is oligarchical; in fact, under these conditions, the stationary bargaining equilibrium is unique. It extends Primo's (2002) result, which assumes a single voter and symmetric stage utility, and it generalizes Proposition 2, which is restricted to quota rules and quadratic utility in one dimension.

THEOREM 4: *Assume $d = 1$ or $\mathcal{D}$ is oligarchical. There is a unique no-delay stationary bargaining equilibrium proposal strategy, and it is degenerate on the unique static equilibrium.*

We remark that in the oligarchical case, with $C = \bigcap \mathcal{D}$, if utilities are continuously differentiable and $\hat{x}^0 \neq x^C$, then it is not possible to move in a direction orthogonal to the agenda setter's gradient to an alternative that is strictly preferred by all members of a decisive coalition. Formally, there is no alternative y such that $\nabla u_0(x^C)\cdot(y-x^C) = 0$ and such that for all $i \in C$, we have $u_i(y) > u_i(x^C)$. In other words, letting H denote the $(d-1)$-dimensional subspace orthogonal to $\nabla u_0(x^C)$, the static equilibrium x^C belongs to the core when alternatives are restricted to $x^C + H$. To see this, note that the agenda setter solves the convex problem

$$\begin{array}{c} \max_{x \in X} u_0(x) \\ \text{s.t. } u_i(x) \geq u_i(q), i \in C. \end{array}$$

Letting $C' = \{i \in C \mid u_i(x) = u_i(q)\}$ consist of the binding voter constraints, it can be seen that x^C also solves the reduced problem with constraints $u_i(x) \geq u_i(q)$, $i \in C'$. By LSWP, there is an alternative x' such that for all $i \in C'$, we have $u_i(x') > u_i(x)$, and by concavity this implies $\nabla u_i(x) \cdot (x' - x) > 0$ for all $i \in C'$, so that the constraint qualification for the reduced problem holds. By the Kuhn-Tucker theorem, there are non-negative multipliers $\lambda_i \geq 0$ such that

$$\nabla u_0(x^C) \;=\; -\sum_{i \in C'} \lambda_i \nabla u_i(x^C).$$

We then have for all $r \in H$,

$$r \cdot \sum_{i \in C} \lambda_i \nabla u_i(x^C) \;=\; 0.$$

Thus, we cannot separate zero from the convex hull $\text{conv}\{\nabla u_i(x^C) \mid i \in C'\}$, and by the separating hyperplane theorem, we have $0 \in \text{conv}\{\nabla u_i(x^C) \mid i \in C'\}$. Therefore, x^C is Pareto optimal for C when alternatives are restricted to $x^C + H$, as claimed. We return to this point in Section 6.

5 Static lower bounds on agenda setting power

In this section, we provide lower bounds on the agenda setter's equilibrium payoff in the dynamic bargaining game, and we establish a cutoff level of voter patience such that below the cutoff, the static equilibrium (possibly along with other equilibria) obtains; and above the cutoff, all equilibria are in mixed proposal strategies, and the agenda setter does strictly better than the static payoff. Along with these results, we draw several important implications. In particular, if a stationary bargaining equilibrium is non-degenerate or there are multiple static equilibria, then the agenda setter's equilibrium payoff in the dynamic game strictly exceeds her static payoff; and a stationary bargaining equilibrium in which the agenda setter uses a pure proposal strategy is only possible if the equilibrium is essentially static: in this case, the static equilibrium must be unique, and the agenda setter must propose that alternative with probability one.

We begin by showing that the agenda setter's static equilibrium payoff provides a general lower bound on her payoff from stationary bargaining equilibria. The proof follows immediately from Lemmas 2 and 3 in Appendix A.1: given any no-delay stationary bargaining equilibrium π, Lemma 3 implies that it is a fixed point of B, and then Lemma 2 implies that for all $x \in \text{supp}(\pi)$, we have $u_0(x) \geq u_0^s$

Theorem 5: *For every no-delay stationary bargaining equilibrium proposal strategy π and for all $x \in supp(\pi)$, we have $u_0(x) \geq u_0^s$.*

Under quite general assumptions on preferences, the weak bound on the agenda setter's equilibrium payoff from Theorem 5 in fact holds strictly. For this characterization, it is sufficient to assume that the agenda setter's stage utility is strictly concave, or that all voters' utilities are strictly concave, or that the environment has a minimal private good component.

THEOREM 6: *Assume $\delta > 0$. Assume either (i) u_0 is strictly concave, or (ii) for all $i \in N$, u_i is strictly concave, or (iii) minimal transferability holds. For every no-delay stationary bargaining equilibrium proposal strategy π, if there is a static equilibrium x^s such that π is not degenerate on x^s, then for all $x \in supp(\pi)$, we have $u_0(x) > u_0^s$.*

To gain some insight into the lower bound under condition (iii) of Theorem 6, we note that result actually holds even if the status quo minimizes the stage utility of every voter. In this case, the status quo is the worst alternative for every voter, so the unique static equilibrium is the ideal point of the agenda setter, and thus her payoff in the dynamic game cannot possibly exceed her static payoff. As a consequence, not surprisingly, Theorem 6 implies that every stationary bargaining equilibrium will also be degenerate on the agenda setter's ideal point in the dynamic bargaining game.

Theorem 6 has several important implications that we record next. First, under the conditions of the theorem, if a stationary bargaining equilibrium proposal strategy is non-degenerate, then given any static equilibrium, the proposal strategy is not degenerate on it —and this implies that the agenda setter's equilibrium payoff strictly exceeds her static equilibrium payoff.

COROLLARY 1: *Assume $\delta > 0$. Assume either (i), (ii), or (iii) from Theorem 6. If a stationary bargaining equilibrium proposal strategy π is non-degenerate, then for all $x \in supp(\pi)$, we have $u_0(x) > u_0^s$.*

Second, we characterize stationary bargaining equilibria in pure proposal strategies and show that the possibilities for such equilibria are substantially limited: if a stationary bargaining equilibrium proposal strategy is pure, then the static equilibrium must be unique, and the agenda setter proposes this alternative with probability one.[6] Thus, it is no coincidence that in Theorem 4, our sufficient conditions for existence of equilibria in pure proposal strategies also imply that the equilibrium proposal strategy is degenerate on the unique static equilibrium.

COROLLARY 2: *Assume $\delta > 0$. Assume either (i), (ii), or (iii) from Theorem 6. If a stationary bargaining equilibrium proposal strategy π is degenerate, then there is a*

[6]To see the result, suppose toward a contradiction that there are a static equilibrium x^s and a no-delay stationary equilibrium π that is degenerate on some $y \neq x^s$. Then Theorem 6 implies $u_0(y) > u_0(x^s)$. Since π is no-delay, however, there is some decisive coalition $C \in \mathcal{D}$ such that for all $i \in C$, we have $u_i(y) \geq (1-\delta)u_i(q) + \delta v_i(\pi)$. But $v_i(\pi) = u_i(y)$ every voter i, and thus $u_i(y) \geq u_i(q)$ for all $i \in C$, implying $y \in A^s$, contradicting the fact that x^s maximizes the agenda setter's stage utility over A^s.

unique static equilibrium x^s, and π is degenerate on x^s.

Third, if there are multiple static equilibria, then by Corollary 2, all stationary bargaining equilibrium proposal strategies are non-degenerate, implying that the agenda setter's payoff in the dynamic game strictly exceeds her static payoff.

Corollary 3: *Assume $\delta > 0$. Assume either (i), (ii), or (iii) from Theorem 6. If there are multiple static equilibria, then every stationary bargaining equilibrium proposal strategy π is non-degenerate, and so for all $x \in supp(\pi)$, we have $u_0(x) > u_0^s$.*

Finally, we establish the existence of a cutoff discount factor such that below the cutoff, the agenda setter receives the static payoff in some equilibrium; and above the cutoff, all equilibria are in mixed proposal strategies, and the agenda setter's payoff is strictly higher than the static payoff in all equilibria. In the statement of the following theorem, let $\underline{v}_0^\delta = \min\left\{v_0(\pi) \mid \pi \in B^\delta(\pi)\right\}$ be the lowest equilibrium payoff to the agenda setter when voters' discount factor is δ.

Theorem 7: *Assume either (i), (ii), or (iii) from Theorem 6. There is a unique cutoff discount factor $\underline{\delta} \in [0, 1]$ such that for every $\delta \leq \underline{\delta}$, we have $\underline{v}_0^\delta = u_0^s$, and for every $\delta > \underline{\delta}$, we have $\underline{v}_0^\delta > u_0^s$.*

The preceding theorem establishes that the agenda setter's minimum equilibrium payoff for discount factors $\delta \leq \underline{\delta}$ is equal to the static payoff, but it leaves open the possibility that multiple equilibria exist and that the agenda setter's maximum equilibrium payoff strictly exceeds the static payoff. Figure 6 shows that this possibility can be realized in some cases. Here, we depict a case $\delta > 0$ in which the static equilibrium x^s persists as a stationary bargaining equilibrium outcome. We can specify the curvature of Euclidean utilities so that the threat of y for voter 1 and the threat of x for voter 2 can be be arbitrarily great. In particular, we can specify utilities so that the mixed proposal strategy with equal probability on x and y generates the reservation values indicated by the level sets through x and y. Given these acceptance sets, the proposals x and y are optimal for the agenda setter, and we have an additional equilibrium in mixed proposal strategies, in which the agenda setter's expected payoff is strictly higher than the static payoff.

6 Conditions for extreme agenda setting power

In this section, we establish conditions under which the agenda setter has extreme power as the voters become patient. The source of this power is the agenda setter's ability—when the set of alternatives is multidimensional—to obtain desirable outcomes by playing off decisive coalitions against each other. For such a coalition, there is the implicit threat that if a proposal is rejected in the current period, then the

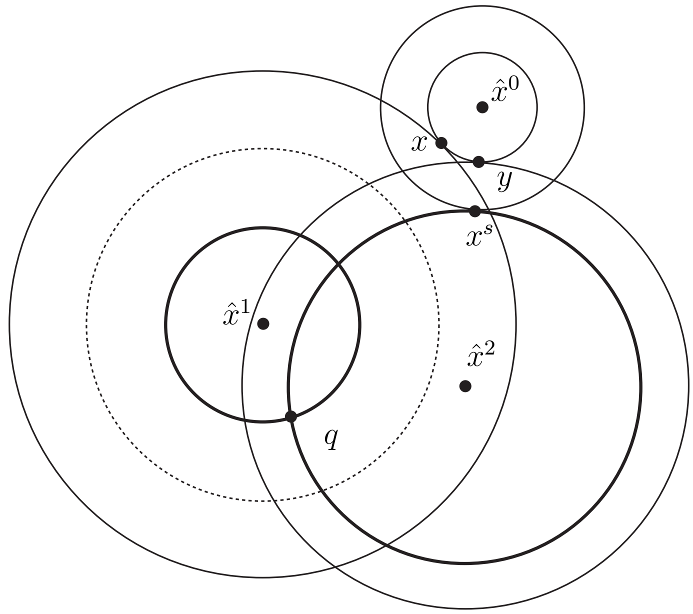


Figure 6: Multiple equilibria

agenda setter will approach a competing coalition in the future. This threat leads voters to become more accommodating, and it looms larger as voters become more patient. We show that as voters become patient, the agenda setter's equilibrium proposals approach a set of limit points in a strong sense. Given such a limit point, say x^*, there cannot be another alternative that is preferred to x^* by the agenda setter and all members of a decisive coalition, for it that were the case, then a profitable deviation would be available to the agenda setter when voters are sufficiently patient. Moreover, if the limiting proposal is not the agenda setter's ideal point and utilities are continuously differentiable, then it must satisfy a stringent necessary condition: x^* must belong to the core when alternatives are restricted to the hyperplane through x^* orthogonal to the agenda setter's gradient $\nabla u_0(x^*)$. This necessary condition becomes more restrictive when the dimensionality of the set of alternatives is high, as discussed in the next two sections, for such a "constrained core" point must satisfy a version of Plott's radial symmetry condition. As a consequence, in the limit, except in rare circumstances, the agenda setter has extreme power: equilibrium outcomes converge, in a strong sense, to the agenda setter's ideal point. Our results are summarized in Corollary 5 at the end of this section.

The first step in our analysis is to establish that as voters become patient, the equilibrium proposals approach a set limit points in a strong sense. Given a weak* convergent sequence $\{\pi^\delta\}$ of equilibrium proposal strategies as $\delta \to 1$, the proposals converge to a single alternative, say x^*, in the following sense: for every open set $G \subseteq X$ containing x^*, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$, the support of π^δ

is contained in G. We then refer to x^* as a *limit proposal.* Uniqueness of this limit proposal does not hold in general: Appendix A.4 shows that when $n = 2$ and $m = 1$, we can support multiple limit proposals if the contract curve of the agenda setter and voter 1 intersects the contract curve of the agenda setter and voter 2 multiple times. Our first result also adds an immediate restriction on preferences that holds at any limit proposal: there cannot be an alternative $y \neq x^*$ that is weakly preferred to x^* by the agenda setter and all members of a decisive coalition.

THEOREM 8: *If π^δ is a no-delay stationary bargaining equilibrium proposal strategy for δ such that π^δ is weak* convergent as $\delta \to 1$, then π^δ converges strongly to a limit proposal, i.e., there exists $x^* \in X$ such that supp$(\pi^\delta) \to \{x^*\}$ in the Hausdorff metric. Moreover, for all $y \in X$ such that $y \neq x^*$, if $u_0(y) \geq u_0(x^*)$, then for all $C \in \mathcal{D}$, there exists $i \in C$ such that $u_i(y) < u_i(x^*)$.*

The preference restriction deduced in the preceding theorem, while simple, has immediate application to any environment with a transferable private good: if the voting rule is non-collegial and minimal transferability is satisfied, then every limit proposal x^* must minimize the stage utility of each voter. Indeed, if there were some voter i such that $u_i(x^*) > \min_{z \in X} u_i(z)$, then minimal transferability would yield an alternative y such that $u_0(y) > u_0(x^*)$ and for all $j \in N \setminus \{i\}$, $u_j(y) > u_j(x^*)$; but since i is not a veto player, the coalition $N \setminus \{i\}$ is decisive, contradicting Theorem 8. It immediately follows that as voters become patient, the equilibrium payoff of each voter converges to the minimum stage utility. Given this observation, stated next, the result of Proposition 3 for the divide the dollar environment follows easily as a special case.

COROLLARY 4: *Assume $\mathcal{D}$ is non-collegial and minimal transferability is satisfied. For all limit proposals x^* and all voters $i \in N$, $u_i(x^*) = \min_{z \in X} u_i(z)$, and therefore $v_i(\pi^\delta) \to \min_{z \in X} u_i(z)$.*

The main conclusion of this paper is that when the set of alternatives is multidimensional, there is a unique limit proposal x^*, and this is equal to the ideal point of the agenda setter—except in rare circumstances. The remainder of this section deduces necessary conditions for $x^* \neq \hat{x}^0$ to hold; in the following two sections, we establish the restrictiveness of these necessary conditions when the voting rule is non-collegial and the set of alternatives is multidimensional, allowing us to conclude that the cases for which $x^* \neq \hat{x}^0$ are indeed "rare." To this end, we say an alternative x is a *constrained core point* with respect to a non-zero vector $p \in \Re^d$ if, letting H be the hyperplane orthogonal to p, there does not exist $y \in X \cap (H + x)$ such that $y \succ x$.[7] Assuming stage utilities are continuously differentiable, the *constrained core* consists

[7]This concept is indirectly employed by Schofield (1978,1983) in his analysis of the local cycle set; see, e.g., Lemma 7 of Schofield (1983), where he effectively shows that every alternative outside the set $IC(\sigma)$ is a constrained core point with respect to some vector v.

of every alternative $x \neq \hat{x}^0$ such that x is a constrained core point with respect to the agenda setter's gradient $\nabla u_0(x)$.

Figure 7 illustrates this concept in four cases. In the upper panels, we assume two voters and inclusive majority rule, i.e., $n = 2$ and $m = 1$, with the left-hand panel depicting Euclidean preferences, and the right-hand panel depicting non-Euclidean preferences. Since x^* is a constrained core point with respect to the agenda setter's gradient, there cannot be an alternative on $H + x^*$ that is strictly preferred to x^* by either voter, so the projected gradients $p_1 = \text{proj}_H \nabla u_1(x^*)$ and $p_2 = \text{proj}_H \nabla u_2(x^*)$ must both equal zero. In case preferences are Euclidean, this implies that the ideal points of the agenda setter and voters are collinear, as in the left-hand panel. For general preferences, it implies that the contract curves for $\{0, 1\}$ and for $\{0, 2\}$ cross at x^*, as in the right-hand panel. In the lower panels, we assume three voters and majority rule, i.e., $n = 3$ and $m = 2$, with the left panel depicting Euclidean preferences and the right non-Euclidean. In the Euclidean case, if we project the ideal points of the voters to the hyperplane $H + x^*$, then the alternative x^* is the median of these projections. For general preferences, x^* must belong to the contract curve for $\{0, i\}$ for some voter i, but in two dimensions, further restrictions on voter gradients are less stark: all that is required is that we cannot have two voters whose gradients point to the same side of x^*.

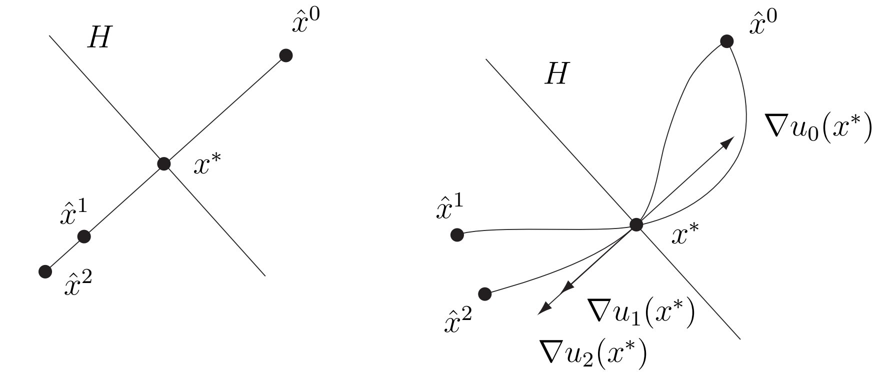
Next, we derive the central necessary condition of the analysis: if a limiting proposal is not equal to the agenda setter's ideal point, then it is a constrained core point with respect to the agenda setter's gradient, $\nabla u_0(x^*)$, at x^*.[8]

THEOREM 9: *Assume each u_i is continuously differentiable. For all limit proposals x^*, if $x^* \neq \hat{x}^0$, then x^* is a constrained core point with respect to $\nabla u_0(x^*)$.*

Although Theorem 9 has strong ramifications for agenda setting power when the voting rule is non-collegial, this is not the case when some voter is a veto player. When $\mathcal{D}$ is oligarchical, for example, Theorem 4 implies that the unique stationary bargaining equilibrium is the static equilibrium $x^s \neq \hat{x}^0$ for all discount factors, regardless of the dimensionality of the set of alternatives. Theorem 9 does apply in this case, and indeed, we argued following the proof of Theorem 4 that the static equilibrium is a constrained core point with respect to $\nabla u_0(x^s)$, i.e., x is Pareto optimal for the voters in $H + x^*$. This is depicted in Figure 8 for the case $n = m = 3$, where there is no move from x^* on the hyperplane H that is strictly preferred by all three voters. Thus, the sharp implications drawn in the sequel necessarily concern non-collegial rules.

[8] Because we will argue that constrained core points are exceptional, we do not provide a general analysis of sufficiency, but in the example of Appendix A, we construct a sequence of stationary bargaining equilibria converging to any alternative $\tilde{x}$ that is a constrained core point with respect to $\nabla u_0(\tilde{x})$.

$n = 2$ and $m = 1$:



$n = 3$ and $m = 2$:

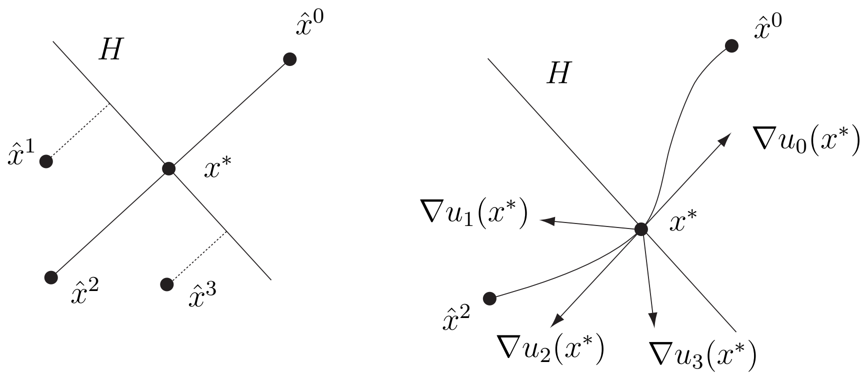


Figure 7: Constrained core point under majority rule

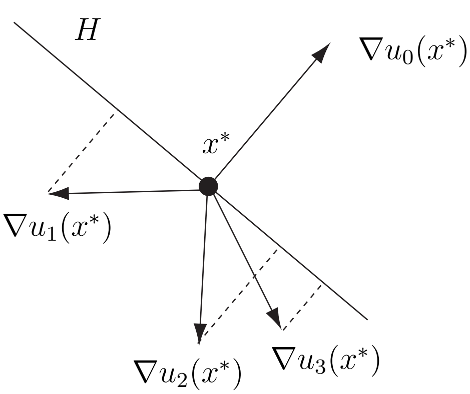


Figure 8: Constrained core point, oligarchical rule

Next, we establish a final restriction on voter preferences between a limit proposal and the status quo: q must be weakly socially preferred to x^*, in the sense that there does not exist a decisive coalition, all the members of which strictly prefer x^* to q.

THEOREM 10: *For all limit proposals x^*, if $x^* \neq \hat{x}^0$, then for every decisive coalition $C \in \mathcal{D}$, there exists $i \in C$ such that $u_i(q) \geq u_i(x^*)$.*

Of course, Theorem 5 implies that the agenda setter's equilibrium payoff is at least equal to her static equilibrium payoff, and this bound holds at a limit proposal x^*. With Theorems 8–10, we then have an immediate corollary that summarizes our sufficient conditions for the power of the agenda setter to become extreme in the limit. The corollary adds an important insight about the necessity of mixing in equilibrium: 1) equilibrium outcomes converge to the agenda setter's ideal point, while 2) Corollary 2 establishes that pure equilibrium proposal strategies must be degenerate on a unique static equilibrium, so 3) if the agenda setter does not obtain her ideal point in the static equilibrium, then 4) all stationary bargaining equilibria rely on non-trivial mixing when voters are sufficiently patient.

COROLLARY 5: *Assume each u_i is continuously differentiable. Then the unique limit proposal is $x^* = \hat{x}^0$ if there is no alternative x such that all of the following hold:*

- *for all $y \in X \setminus \{x\}$ with $u_0(y) \geq u_0(x)$ and for all $C \in \mathcal{D}$, there exists $i \in C$ such that $u_i(y) < u_i(x)$,*
- *x is a constrained core point with respect to $\nabla u_0(x)$,*
- *for all $C \in \mathcal{D}$, there exists $i \in C$ such that $u_i(q) \geq u_i(x)$,*
- *$u_0(x) \geq u_0^s$,*

and in this case, if $u_0^s < u_0(\hat{x}^0)$, then the stationary bargaining equilibrium proposal strategy π^δ is non-degenerate, when δ is close enough to one.

The implications of Corollary 5 for the power of the agenda setter hinge on the assumption that there is no alternative possessing three properties, the key being that no x is a constrained core point with respect to $\nabla u_0(x)$. The scope of these implications is large if the existence of such an alternative is exceptional. The focus of the following two sections is to draw restrictive necessary conditions that must be satisfied at any constrained core point, and to demonstrate that when the voting rule is non-collegial and the set of alternatives is high dimensional, the constrained core is almost always empty. Thus, existence of an alternative satisfying the three properties of Corollary 5 is indeed the exception, and under these general conditions the power of the agenda setter becomes extreme as voters become patient.

Before proceeding, we briefly adapt the concept of constrained core to extract further implications for boundary alternatives $x \in \mathrm{bd}(X)$. We say the set X of alternatives is *piecewise smooth* if it is cut out by a finite number of continuously differentiable, quasi-concave mappings, $f^\ell \colon \Re^d \to \Re$, $\ell = 1, \ldots, k$, so that

$$X \;=\; \{x \in \Re^d \mid \text{for all } \ell = 1, \ldots, k,\ f^\ell(x) \geq 0\}.$$

In this case, given boundary alternative $x \in \mathrm{bd}(X)$, there is a finite set $K(x) = \{\ell \mid f^\ell(x) = 0\}$ of binding feasibility constraints, and we let $L(x) = \{y \in \Re^d \mid \text{for all } \ell \in K(x),\ y \cdot \nabla f^\ell(x) = 0\}$ be the subspace orthogonal to the gradients of the binding constraints; in case $x \in \mathrm{int}(X)$, we let $L(x) = \Re^d$. We say X is *regular* if for all $x \in \mathrm{bd}(X)$, the gradients $\{\nabla f^\ell(x) \mid \ell \in K(x)\}$ of the binding constraints are linearly independent. Assuming $x \neq \hat{x}^0$, define $H(x) = \{y \in \Re^d \mid y \cdot \nabla u_0(x) = 0\}$ be the hyperplane through the origin orthogonal to the agenda setter's gradient. Then the *tangent core* consists of every $x \in X \setminus \{\hat{x}^0\}$ such that there is no direction $r \in H(x) \cap L(x)$ such that the coalition of voters with positive derivative at x in direction r, namely

$$\{i \in N \mid r \cdot \nabla u_i(x) > 0\},$$

is decisive. For interior alternatives, this concept preserves the original definition of constrained core, but it imposes different restrictions on boundary alternatives.

Theorem 11: *Assume X is regular and each u_i is continuously differentiable. For all limit proposals x^*, if $x^* \neq \hat{x}^0$, then x^* belongs to the tangent core.*

7 Implications of the constrained core condition

In this section, assuming a non-collegial voting rule, we derive restrictions on voter gradients that must be satisfied at any alternative that is a constrained core point with respect to the agenda setter's gradient. We focus initially on the case of majority rule, where the restrictions are sharpest. First, assuming inclusive majority rule with n even, we establish a very stringent restriction that generalizes our observations of Figure 7: if x is a constrained core point with respect to $\nabla u_0(x)$, then there are at least two voters whose gradients are collinear with the agenda setter's at x. The gradient restriction applies only to interior constrained core points. In many environments of interest, alternatives on the boundary $\mathrm{bd}(X)$ are Pareto inefficient and cannot belong to the constrained core; and in environments with private goods, where it is Pareto efficient for the agenda setter to consume the endowment, the only possible boundary point belonging to the constrained core is the ideal point of the agenda setter. We provide gradient restrictions for boundary constrained core points at the end of the section.

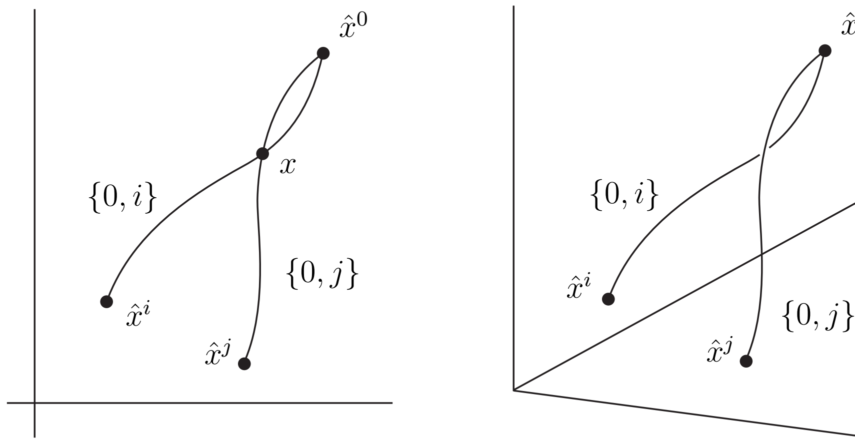


Figure 9: Gradient restriction in two and three dimensions

Theorem 12: *Assume $n \geq 2$ is even, inclusive majority rule with $m = \frac{n}{2}$, and each u_i is continuously differentiable. Let $x \in int(X)$ belong to the constrained core. There exist distinct voters $i, j \in N$ and scalars $\alpha_i, \alpha_j \in \Re$ such that $\alpha_i \nabla u_0(x) = \nabla u_i(x)$ and $\alpha_j \nabla u_0(x) = \nabla u_j(x)$.*

Geometrically, given a constrained core point x and voters i and j from the above theorem, the contract curves for $\{0, i\}$ and for $\{0, j\}$ must intersect at x, as in the top panels of Figure 7, where $i = 1$ and $j = 2$. For generic utilities, these curves will be one-dimensional manifolds, and when utilities are Euclidean, they are in fact straight lines. In the latter case, the gradient restriction of Theorem 12 actually implies that the ideal points of the agenda setter and voters i and j are collinear, a non-generic situation in two or more dimensions. For non-Euclidean utilities, when the set of alternatives is two-dimensional, the two contract curves can intersect transversally at a constrained core point, such as x in the left-hand panel of Figure 9, in which case the constrained core condition is stable, i.e., small perturbations of utilities will determine a new constrained core point close to the original. When the set of alternatives is dimension three or higher, as in the right-hand panel of Figure 9, it is impossible for the contract curves to intersect transversally; thus, if there is a constrained core point for one specification of utilities, then small perturbations can (and typically will) lead the constrained core to be empty. We return to the genericity analysis more formally in the next section.

Moving to the case of n odd, we can apply Plott's (1967) theorem, as stated in Section 2, to the set of alternatives restricted to $H + x$ to conclude that the projected gradient of some voter equals zero, i.e., their gradient is collinear with the agenda setter's, and the projected gradients of the other voters satisfy radial symmetry.

Theorem 13: *Assume $n \geq 3$ is odd, majority rule with $m = \frac{n+1}{2}$, and each u_i is continuously differentiable. Let $x \in int(X)$ belong to the constrained core, let H be*

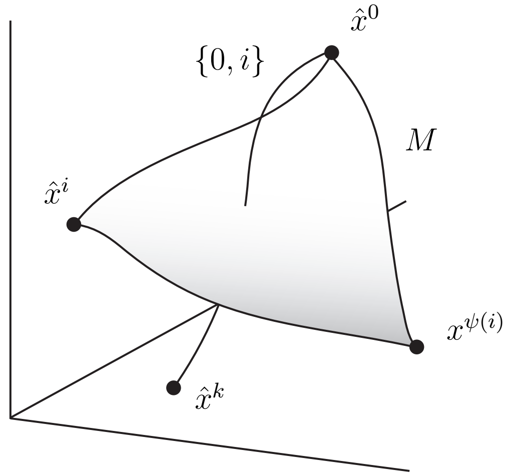


Figure 10: Gradient restriction in three dimensions, $n = 3$

the hyperplane through the origin orthogonal to $\nabla u_0(x)$*, and for each voter* i*, let* $p_i = proj_H \nabla u_i(x)$ *be* i*'s gradient projected onto* H*. There exists* $k \in N$ *with* $p_k = 0$*, and if there does not exist* $j \in N \setminus \{k\}$ *with* $p_j = 0$*, then for every unit vector* $r \in H$*, we have*

$$\left|\left\{i \in N \setminus \{k\} \mid \frac{1}{\|p_i\|} p_i = r\right\}\right| \quad = \quad \left|\left\{i \in N \setminus \{k\} \mid \frac{1}{\|p_i\|} p_i = -r\right\}\right|.$$

When the number n of voters is odd, there is again at least one voter with $p_k = 0$. In contrast to Theorem 12, we cannot argue that a second voter also has projected gradient equal to zero; but if no other voter has a zero projected gradient, then radial symmetry yields a bijection $\psi\colon N \setminus \{i\} \to N \setminus \{i\}$ with no fixed points such that for all $i \in N \setminus \{k\}$, p_i and $p_{\psi(i)}$ point in opposite directions. This means that x lies on the contract curve for $\{0, k\}$, and the gradients of $0, i, \psi(i)$ at x are coplanar. For generic preferences, the set of alternatives such that the gradients of these three agents are coplanar will be a manifold, say M, of dimension two, and the contract curve for $\{0, k\}$ may intersect the manifold M transversally in three dimensions; see Figure 10. But in four or more dimensions, the contract curve will typically "miss" the manifold M, i.e., the intersection is non-transversal. Again, the constrained core is generically empty when the dimensionality of the set of alternatives is high enough, the critical dimensionality now being four, rather than three.

Next, we return to n even for the case of exclusive majority rule. In this case, it is known that if an alternative belongs to the core, then either the gradients of the voters satisfy radial symmetry, or some voter k has a zero gradient, and for every two voters, i and j, other than k, there is a fourth such that the gradients of the four voters are linearly dependent (see, e.g., Lemmas 3–5 of Schofield (1983)).[9] The following

[9]Our statement of the necessary condition is somewhat stronger than Schofield's, which states

theorem applies this observation to the set of alternatives restricted to $H+x$, stating the necessary condition in terms of projected gradients.

THEOREM 14: *Assume $n \geq 4$ is even, exclusive majority rule with $m = \frac{n}{2}+1$, and each u_i is continuously differentiable. Let $x \in int(X)$ belong to the constrained core, let H be the hyperplane through the origin orthogonal to $\nabla u_0(x)$, and for each voter i, let $p_i = proj_H \nabla u_i(x)$ be i's gradient projected onto H. Either (i) there is no $j \in N$ with $p_j = 0$, and for every unit vector $r \in H$, we have*

$$\left|\left\{i \in N \mid \frac{1}{\|p_i\|}p_i = r\right\}\right| \quad = \quad \left|\left\{i \in N \mid \frac{1}{\|p_i\|}p_i = -r\right\}\right|,$$

or (ii) there is a voter k such that $p_k = 0$, and for all distinct $i, j \in N \setminus \{k\}$, there exists $h \in N \setminus \{i,j,k\}$ for which $\{p_h, p_i, p_j\}$ is linearly dependent.

Finally, for a general voting rule, it is clear that an alternative x belongs to the core if and only if for every decisive coalition $C \in \mathcal{D}$, x is Pareto optimal for C, in the sense that for all $y \in X$, there exists $i \in C$ such that $u_i(x) \geq u_i(y)$. Assuming continuously differentiable utilities, Smale (1973) shows that the zero vector then belongs to the convex hull of the gradients of the members of C. Applied to the set of alternatives restricted to $H+x$, we conclude that zero belongs to the convex hull of projected gradients of members of every decisive coalition. Define D to consist of coalition sizes ℓ such that $|C| \geq \ell$ implies C is decisive:

$$D \quad = \quad \{\ell \in N \mid \text{for all } C,\, |C| \geq \ell \text{ implies } C \in \mathcal{D}\},$$

and then define $\underline{m} = \min D$. That is, $\underline{m}-1$ is the size of the largest coalition that is not decisive. In particular, for a quota rule, we have $\underline{m} = m$; and if the voting rule is non-collegial, then we have $\underline{m} \leq n-1$. Using Smale's result, Lemma 3 of Banks (1995) implies that the projected gradients of the voters have rank less than $\underline{m}$.

THEOREM 15: *Let $x \in int(X)$ belong to the constrained core, let H be the hyperplane through the origin orthogonal to $\nabla u_0(x)$, and for each voter i, let $p_i = proj_H \nabla u_i(x)$ be i's gradient projected onto H. For all $C \in \mathcal{D}$, we have $0 \in conv(\{p_i \mid i \in C\})$. Furthermore, the projected gradients $\{p_i \mid i \in N\}$ of the voters have rank strictly less than $\underline{m}$.*

The necessary condition stated above is not restrictive if voting is by unanimity rule, so that $\mathcal{D} = \{N\}$, for then every alternative that is Pareto optimal in $H + x^*$ satisfies $0 \in \text{conv}(\{p_i | i \in N\})$, and the rank of the voters' projected gradients is less than $n = \underline{m}$. But for quota rules with $m < n$, it becomes restrictive when the dimensionality of the set of alternatives is high.

only that there exist some pair i and j of voters for which this holds.

We can extend the above gradient restrictions to boundary alternatives, now focusing on tangent core alternatives and assuming the set of alternatives is regular. The result follows from Theorem 11 and, once again, Lemma 3 of Banks (1995), now applied to voters' gradients projected onto the set $H \cap L$ of directions orthogonal to the agenda setter's gradient and to the gradients of binding constraints.

THEOREM 16: *Assume X is regular. Let $x \in bd(X)$ belong to the tangent core, let H be the hyperplane through the origin orthogonal to $\nabla u_0(x)$, let L be the linear subspace orthogonal to the gradients $\{\nabla f^\ell(x) \mid \ell \in K(x)\}$ of binding constraints, and for each voter i, let $p_i = proj_{H\cap L} \nabla u_i(x)$ be i's gradient projected onto the linear subspace $H \cap L$. For all $C \in \mathcal{D}$, we have $0 \in conv(\{p_i \mid i \in C\})$. Furthermore, the projected gradients $\{p_i \mid i \in N\}$ of the voters have rank strictly less than $\underline{m}$.*

8 Generic emptiness of the constrained core

In this section, we formalize the discussion above by showing that when the set of alternatives is of high dimension, the constrained core is empty for generic specifications of preferences. Given any set $Z \subseteq \Re^d$, we say a mapping $f\colon Z \to \Re^{n+1}$ is twice continuously differentiable if it can be extended to a twice continuously differentiable mapping on an open set containing Z, and we denote by $C^2(Z, \Re^{n+1})$ the set of all such mappings. Such a mapping $f = (f_0, \ldots, f_n)$ consists of $n+1$ components. Let $d^1 f\colon Z \to \Re^{(n+1)d}$ be the mapping of first derivatives, i.e., if we view the gradient $\nabla f_i(x)$ as a row vector, then

$$d^1 f(x) \quad = \quad \begin{bmatrix} \nabla f_0(x) \\ \vdots \\ \nabla f_n(x) \end{bmatrix}$$

is the $n \times d$ matrix of gradients. In the remainder of the section, we assume without loss of generality that X has non-empty interior.

We begin the analysis by considering the status of constrained core points that are interior to the set of alternatives, and since we are concerned only with properties of stage utilities on interior alternatives, we define $\mathcal{U}_{\mathrm{int}(X)} = C^2(\mathrm{int}(X), \Re^{n+1})$ as the space of all twice continuously differentiable mappings $u\colon \mathrm{int}(X) \to \Re^{n+1}$, with the component u_i representing the preferences of agent $i = 0, 1, \ldots, n$; thus, we term such a mapping u a *vector utility function.* We endow $\mathcal{U}_{\mathrm{int}(X)}$ with the Whitney (or strong) topology (Hirsch, 1976, p.34). To capture the subspace of mappings satisfying the concavity assumptions of the bargaining model, let $\hat{\mathcal{U}}_{\mathrm{int}(X)}$ denote the mappings u such that the Hessian of each component, denoted $d^2 u_i$, is everywhere negative definite:

$$\hat{\mathcal{U}}_{\mathrm{int}(X)} \quad = \quad \left\{ u \in \mathcal{U}_{\mathrm{int}(X)} \mid \begin{array}{c} \text{for all } i = 0, \ldots, n \text{ and all } x \in \mathrm{int}(X), \\ d^2 u_i(x) \text{ is negative definite} \end{array} \right\}.$$

The set $\hat{\mathcal{U}}_{\text{int}(X)}$ is open in $\mathcal{U}_{\text{int}(X)}$ with the Whitney topology, and we give it the relative Whitney topology, making $\hat{\mathcal{U}}_{\text{int}(X)}$ a Baire space (Hirsch, 1976, Theorem 4.4). In this setting, our notion of genericity is that of an open, dense set, but because the Whitney topology is extremely fine, openness is a correspondingly weak property.

For this reason, we also consider vector utility functions defined on the full set of alternatives. Let $\mathcal{U}_X = C^2(X, \Re^{n+1})$ be the space of such vector utility functions, endowed with the topology of C^2-uniform convergence.[10] Again, we focus on the set of mappings that are differentiably concave, denoted

$$\hat{\mathcal{U}}_X \quad = \quad \left\{ u \in \mathcal{U}_X \mid \begin{array}{c} \text{for all } i = 0, \ldots, n \text{ and all } x \in X, \\ d^2 u_i(x) \text{ is negative definite} \end{array} \right\}.$$

We give $\hat{\mathcal{U}}_X$ the relative topology inherited from $\mathcal{U}_X$, making $\hat{\mathcal{U}}_X$ a complete metrizable space, and thus a Baire space. Because X is compact, the set $\hat{\mathcal{U}}_X$ is an open subset of $\mathcal{U}_X$ in the C^2-uniform convergence topology.[11] We use $\mathcal{B}_\epsilon(u)$ to denote the open ball of radius ϵ around u for a compatible metric; for the open ball around x in the Euclidean metric, we write $B_\epsilon(x)$. A subset of $\hat{\mathcal{U}}_X$ is *residual* if it contains the countable intersection of sets that are open and dense; because $\hat{\mathcal{U}}$ is Baire, it follows that a residual subset is also dense itself. This formalizes our notion of genericity for vector utilities defined on the full set of alternatives.

For u belonging to $\hat{\mathcal{U}}_{\text{int}(X)}$ or $\hat{\mathcal{U}}_X$, define the relation $x \succ_u y$ over alternatives to hold if and only if there is a coalition $C \in \mathcal{D}$ such that for all $i \in C$, we have $u_i(x) > u_i(y)$. Given $u \in \hat{\mathcal{U}}_{\text{int}(X)}$, let $CC_{\text{int}(X)}(u)$ denote the interior constrained core points determined by stage utilities u_i, $i = 0, 1, \ldots, n$. That is, $CC_{\text{int}(X)}(u)$ consists of $x \in \text{int}(X)$ such that x does not maximize u_0 over $\text{int}(X)$, and letting H be the hyperplane through the origin orthogonal to $\nabla u_0(x)$, there is no $y \in (\text{int}(X)) \cap (H + x)$ such that $y \succ_u x$. Given $u \in \hat{\mathcal{U}}_X$, let $CC_X(u)$ denote the constrained core points, i.e., $CC_X(u)$ consists of $x \in X$ such that x does not maximize u_0 over X, and there is no $y \in X \cap (H + x)$ such that $y \succ_u x$.

Our first theorem focuses on the generic impossibility of constrained core points belonging to the interior of the set of alternatives. We show that for a dimensionality above a critical level, which depends on the voting rule, the interior constrained core is generically empty. Of note, the theorem imposes no differentiable structure on the set of alternatives. The first part of the genericity result holds on an open and dense (not merely residual) set of vector utility functions in the Whitney topology, while the

[10] Because X is compact, the Whitney topology and the topology of C^2-uniform convergence on compacta (or weak topology) coincide on $C^2(X, \Re^n)$. See Mas-Colell (1985), Section K for further details.

[11] Indeed, let $\{u^k\}$ be a sequence in $\mathcal{U} \setminus \hat{\mathcal{U}}$ with limit u. Then for all k, there exist i^k and x^k such that $d^2 u_{i^k}(x^k)$ is not negative definite, i.e, there is a unit vector t^k such that $t^k d^2 u_{i^k}(x^k) t^k \geq 0$. Going to convergence subsequences, with limits say i, x, and t, continuity of the Hessian implies $t d^2 u_i(x) t \geq 0$, and thus $u \in \mathcal{U} \setminus \hat{\mathcal{U}}$.

second part of the result employs a weaker notion of genericity in terms of a residual set, but uses the more familiar topology of C^2-uniform convergence.

THEOREM 17: *Assume one of the following holds:*

(i) $n \geq 2$ *is even, inclusive majority rule with* $m = \frac{n}{2}$, *and* $d \geq 3$,

(ii) $n \geq 3$ *is odd, majority rule with* $m = \frac{n+1}{2}$, *and* $d \geq 4$,

(iii) $n \geq 4$ *is even, exclusive majority rule with* $m = \frac{n}{2} + 1$, *and* $d \geq 5$,

(iv) $\mathcal{D}$ *non-collegial and* $d > \underline{m} + \frac{m}{n-\underline{m}}$.

The set of vector utility functions defined on the interior of X *for which there is no constrained core point, i.e.,*

$$\hat{\mathcal{U}}^0_{\mathrm{int}(X)} \quad = \quad \{u \in \hat{\mathcal{U}}_{\mathrm{int}(X)} \mid CC_{\mathrm{int}(X)}(u) = \emptyset\},$$

contains an open and dense subset of $\hat{\mathcal{U}}_{\mathrm{int}(X)}$ *with the relative Whitney topology; and the set of vector utility functions defined on* X *for which there is no interior constrained core point, i.e.,*

$$\hat{\mathcal{U}}^0_X \quad = \quad \{u \in \hat{\mathcal{U}}_X \mid CC_X(u) \cap (\mathrm{int}(X)) = \emptyset\},$$

is residual in $\hat{\mathcal{U}}_X$ *with the relative topology of* C^2*-uniform convergence.*

In some environments, the constrained core must belong to the interior of the set of alternatives; if, for example, the environment is spatial and the voters' ideal points belong to the interior of the set of alternatives, then the constrained core always belongs to $\mathrm{int}(X)$. In general, however, we cannot rule out the possibility that constrained core points exist in the boundary of X. To draw further implications for agenda setting power, we switch focus to the tangent core and show that we can rule out tangent core points in parts of the boundary of X that are not too "thin." Assuming X is regular, let $L \subseteq \{1, 2, \ldots, k\}$ represent a subset of binding constraints, and define the *L-face* of X, denoted $F(L)$, by

$$F(L) \quad = \quad \{x \in X \mid \text{for all } \ell = 1, \ldots, k,\ f^\ell(x) = 0 \text{ iff } \ell \in L\}.$$

By regularity, the face $F(L)$ is a manifold of dimension $d - |L|$. Note that we allow $L = \emptyset$, in which case $F(\emptyset)$ is the interior of X. Let d^* denote the critical level of dimensionality identified in Theorem 17:

$$d^* \quad = \quad \begin{cases} 2 & \text{if } n \geq 2 \text{ even, } m = \frac{n}{2}, \\ 3 & \text{if } n \geq 3 \text{ odd, } m = \frac{n+1}{2}, \\ 4 & \text{if } n \geq 4 \text{ even, } m = \frac{n}{2} + 1, \\ \underline{m} + \frac{d}{n-\underline{m}+1} & \text{else, } \mathcal{D} \text{ non-collegial.} \end{cases}$$

For $u \in \hat{\mathcal{U}}_X$, let $TC_X(u)$ be the set of tangent core alternatives at u. Our final result establishes that for generic vector utility functions, tangent core alternatives are possible only in faces of low dimension relative to the set of alternatives. In many environments, this conclusion is enough to preclude any tangent core alternatives, so that equilibrium outcomes of bargaining must converge to the agenda setter's ideal point. More generally, our analysis implies that if agenda setting power does not become extreme, then equilibrium outcomes must converge to lower dimensional faces.

THEOREM 18: *Assume X is regular. Let $L \subseteq K$ satisfy $|L| < d - d^*$. The set of utility vectors $u \in \hat{\mathcal{U}}_X$ for which there is no tangent core point on the face $F(L)$, i.e.,*

$$\hat{\mathcal{U}}_X^L \;=\; \{u \in \hat{\mathcal{U}}_X \mid TC_X(u) \cap F(L) = \emptyset\},$$

is residual in $\hat{\mathcal{U}}_X$ with the relative topology of C^2-uniform convergence.

To understand the applicability of Theorem 18, assume for simplicity that X is cut out by a single constraint, so that $X = \{x \in \Re^d \mid f^1(x) \geq 0\}$, and the boundary of X is the level set of the constraint function f^1 at zero. The interior alternatives are such that the set of binding constraints is empty, i.e., $L = \emptyset$, and Theorem 18 implies that generically, there is no interior tangent core alternative when $0 < d - d^*$, consistent with Theorem 17. Furthermore, the boundary alternatives are such that the constraint is binding, i.e., $L = \{1\}$, and Theorem 18 implies that generically, there is no boundary tangent core alternative when $1 < d - d^*$. Thus, the tangent core is generically empty when the dimensionality of the set of alternatives exceeds $d^* + 1$, incrementing the critical dimensionality from Theorem 17 by one. In general, we obtain generic emptiness of the tangent core when the dimensionality of the set of alternatives exceeds d^* plus the total number of constraint functions describing X.

A Technical material

A.1 Supporting lemmas

We begin by reformulating LSWP in cardinal terms. For each $i \in N \cup \{0\}$ and each utility vector $\tilde{u} \in \Re^n$, let

$$\begin{aligned} \tilde{R}_i(\tilde{u}) &= \{x \in X \mid u_i(x) \geq \tilde{u}_i\} \\ \tilde{P}_i(\tilde{u}) &= \{x \in X \mid u_i(x) > \tilde{u}_i\}. \end{aligned}$$

Then for each $C \subseteq N \cup \{0\}$, define the upper contour sets at utility vector $\tilde{u}$ by

$$\tilde{R}_C(\tilde{u}) \;=\; \bigcap_{i \in C} \tilde{R}_i(\tilde{u}) \quad \text{and} \quad \tilde{P}_C(\tilde{u}) \;=\; \bigcap_{i \in C} \tilde{P}_i(\tilde{u}).$$

We say that *LSWP** holds if for all $C \subseteq N \cup \{0\}$ and all $\tilde{u} \in \Re^n$, $|\tilde{R}_C(\tilde{u})| > 1$ implies $\tilde{R}_C(\tilde{u}) \subseteq \text{clos}(\tilde{P}_C(\tilde{u}))$. It is easy to see that LSWP* implies LSWP by setting $\tilde{u} = (u_0(x), u_1(x), \ldots, u_n(x))$. In fact, the conditions are equivalent under our maintained assumption that utility functions are continuous.

LEMMA 1: *LSWP holds if and only if LSWP* holds.*

PROOF: We prove the necessity direction. Let $C \subseteq N \cup \{0\}$ and $\tilde{u} \in \Re^n$ be such that $|\tilde{R}_C(\tilde{u})| > 1$. Given any $x \in \tilde{R}_C(\tilde{u})$, choose $y \in \tilde{R}_C(\tilde{u}) \setminus \{x\}$. Partition C into two groups:

$$\begin{aligned} I &= \{i \in C \mid u_i(x) = \tilde{u}_i\} \\ J &= \{j \in C \mid u_j(x) > \tilde{u}_j\}. \end{aligned}$$

For each $i \in I$, we have $u_i(y) \geq \tilde{u}_i = u_i(x)$. Then $|R_I(x)| \geq |\{x, y\}| > 1$, and LSWP implies that $x \in \text{clos}(P_I(x)) = \text{clos}(\tilde{P}_I(\tilde{u}))$. For each $j \in J$, continuity implies that if an alternative x' is close enough to x, then $u_j(x') > \tilde{u}_j$. Thus, $x \in \text{clos}(\tilde{P}_C(\tilde{u}))$, and since x is an arbitrary element of $\tilde{R}_C(\tilde{u})$, we conclude that $\tilde{R}_C(\tilde{u}) \subseteq \text{clos}(\tilde{P}_C(\tilde{u}))$. □

Next, we show that at a fixed point of the correspondence B, the agenda setter's payoff is at least equal to her static equilibrium payoff, and therefore exceeds the stage utility from the status quo.

LEMMA 2: *For all $\pi \in \Delta(X)$, if π is a fixed point of the correspondence B, then for all $x \in supp(\pi)$, we have $u_0(x) \geq u_0^s > u_0(q)$.*

PROOF: Let π be a fixed point of B, let x^s be any static equilibrium, and suppose toward a contradiction that for all $x \in \text{supp}(\pi)$, we have $u_0(x) < u_0(x^s)$. Let $\overline{x} = \int z\pi(dz)$ be the mean of π, and define $\tilde{x} = (1-\delta)x^s + \delta\overline{x}$. Given any $x \in \text{supp}(\pi)$, concavity of u_0 implies

$$u_0(\tilde{x}) \;\; \geq \;\; (1-\delta)u_0(x^s) + \delta u_0(x) \;\; > \;\; u_0(x).$$

Let $C^s \in \mathcal{D}$ be such that $x^s = x^{C^s}$, and note that for all $i \in C^s$, concavity of u_i and $u_i(x^s) \geq u_i(q)$ implies

$$\begin{aligned} u_i(\tilde{x}) &\geq (1-\delta)u_i(x^s) + \delta v_i(\pi) \\ &\geq (1-\delta)u_i(q) + \delta v_i(\pi), \end{aligned}$$

which implies $\tilde{x} \in A_{C^s}(\pi) \subseteq A(\pi)$, contradicting sequential rationality of π. Thus, we have $u_0(x) \geq u_0^s$, and the inequality $u_0^s > u_0(q)$ follows from Proposition 1. □

Finally, we verify the connection between fixed points of the correspondence B and the no-delay stationary equilibrium proposal strategies.

LEMMA 3: *For all $\pi \in \Delta(X)$, there exist acceptance strategies α such (π, α) is a no-delay stationary bargaining equilibrium if and only if π is a fixed point of the correspondence B.*

Proof: First, assume π is a no-delay equilibrium, and suppose toward a contradiction that it is not a fixed point of B, i.e., there exist $y \in A(\pi)$ and $x \in \mathrm{supp}(\pi)$ such that $u_0(x) < u_0(y)$. Let $\overline{x} = \int z\pi(dz)$ be the mean of π, and define $\tilde{x} = (1-\delta)q + \delta\overline{x}$. Define the utility vector $\tilde{u}$ such that for all $i \in N \cup \{0\}$,

$$\tilde{u}_i \;=\; (1-\delta)u_i(q) + \delta v_i(\sigma).$$

We consider two cases.

Case 1: $y \neq \tilde{x}$. Let $C \in \mathcal{D}$ be such that $y \in A_C(\pi)$. We then have $y \in \tilde{R}_C(\tilde{u})$ by construction, and by concavity of stage utilities, we also have $\tilde{x} \in \tilde{R}_C(\tilde{u})$. Then $|\tilde{R}_C(\tilde{u})| \geq |\{\tilde{x}, y\}| > 1$. Using Lemma 1, LSWP* implies that $\tilde{R}_C(\tilde{u}) \subseteq \mathrm{clos}(\tilde{P}_C(\tilde{u}))$. Then there exists $x' \in X$ arbitrarily close to y such that for all $i \in C$, we have

$$u_i(x') \;>\; \tilde{u}_i \;=\; (1-\delta)u_i(q) + v_i(\pi),$$

and by continuity, we can choose x' close enough to y such that $u_0(x') > u_0(x)$. By stage dominance, we have $\alpha_i(x') = 1$ for all $i \in C$, contradicting sequential rationality.

Case 2: $y = \tilde{x}$. Since π is a no-delay equilibrium, there exists $C' \in \mathcal{D}$ such that $x \in A_{C'}(\pi)$. Again, by concavity, we have $x, y \in \tilde{R}_{C'}(\tilde{u})$, and since $u_0(y) > u_0(x)$, this implies $|\tilde{R}_{C'}(\tilde{u})| > 1$. Then LSWP* implies that $\tilde{R}_{C'}(\tilde{u}) \subseteq \mathrm{clos}(\tilde{P}_{C'}(\tilde{u}))$. Then there exists $x' \in X$ arbitrarily close to y such that for all $i \in C'$, we have

$$u_i(x') \;>\; \tilde{u}_i \;=\; (1-\delta)u_i(q) + v_i(\pi),$$

and by continuity, we can choose x' close enough to y such that $u_0(x') > u_0(x)$. By stage dominance, we have $\alpha_i(x') = 1$ for all $i \in C$, contradicting sequential rationality of π.

Second, assume π is a fixed point of B. Then we can specify acceptance strategies α_i such that for all $x \in X$,

$$\alpha_i(x) \;=\; \begin{cases} 1 & \text{if } u_i(x) \geq (1-\delta)u_i(q) + \delta v_i(\pi), \\ 0 & \text{else,} \end{cases}$$

automatically satisfying stage dominance. Since $\pi \in B(\pi)$, it follows that π is an optimal proposal strategy if the agenda setter's stage utility from $x \in \mathrm{supp}(\pi)$ weakly exceeds the expected payoff from proposing an alternative that is rejected, i.e.,

$$u_0(x) \;\geq\; (1-\delta_0)u_0(q) + \delta_0 v_0(\pi).$$

Using $u_0(x) = v_0(\pi)$, this holds if and only if $u_0(x) \geq u_0(q)$, which follows from Lemma 2. $\square$

A.2 Proofs of propositions

PROOF OF PROPOSITION 1: Assume (x, α) is a no-delay static equilibrium. Then $\alpha(x) = 1$, and by stage dominance, we have $x \in A^s$. Suppose toward a contradiction that there exists $y \in A^s$ such that $u_0(y) > u_0(x)$, so there is some $C \in \mathcal{D}$ with $y \in A^s_C$. Then for all $i \in C$, we have $u_i(y) \geq u_i(q)$, so that $y \in R_C(q)$. By LSWP, there exists $z \in P_C(q)$ arbitrarily close to y, and thus $\alpha(z) = 1$. Choosing z close enough to y that $u_0(z) > u_0(x)$, the agenda setter can increase her payoff from x by proposing z, contradicting sequential rationality. Thus, x maximizes the agenda setter's stage utility over A^s. Now consider any alternative x that maximizes the agenda setter's stage utility over A^s. Defining acceptance strategies as $\alpha_i(y) = 1$ if $y \in A^s_i$ and $\alpha_i(y) = 0$ otherwise, the profile (x, α) is a no-delay static equilibrium.

Existence of a solution to the coalitional problem follows from compactness of A^s_C and continuity of u_0. To prove uniqueness, suppose toward a contradiction that there exist distinct $x, y \in A^s_C$ that maximize the agenda setter's stage utility. In particular, $u_0(x) = u_0(y)$, and thus $y \in R_{C\cup\{0\}}(x)$. By LSWP, we can approximate y by alternatives $z \in P_{C\cup\{0\}}(x)$, but then we have $z \in A^s_C$ and $u_0(z) > u_0(x)$, a contradiction. If x^s is a no-delay static equilibrium, then it belongs to A^s, so there is some $C \in \mathcal{D}$ such that $x^s \in A^s_C$, and by the first part of the proposition, it follows that $x^s = x^C$. The opposite direction also follows directly from the first part of the proposition.

The inequality $u^s_0 \geq u_0(q)$ holds because $q \in A^s$. If there is a no-delay static equilibrium $x^s \neq q$, then there is a coalition $C \in \mathcal{D}$ such that $x^s \in A^s_C$, and thus $x^s \in R_{C\cup\{0\}}(q)$. Then LSWP yields $y \in P_{C\cup\{0\}}(q)$, and stage dominance implies $\alpha(y) = 1$. It follows that the agenda setter's equilibrium payoff is at least equal to the stage utility from y, i.e., $u^s_0 \geq u_0(y) > u_0(q)$.

For the last part, when $d = 1$ or $\mathcal{D}$ is oligarchic, we claim that A^s is convex. In the first case, Lemma 1 in Cho and Duggan (2003) establishes that regardless of the discount factor, the social acceptance set $A(\pi)$ is a nonempty compact interval. When $\delta = 0$, this result implies that A^s is convex. In the second case, letting $C = \bigcap \mathcal{D}$, we have $A^s = A^s_C$, which is convex. This establishes the claim, and it follows that u_0 has a unique maximizer over A^s, and thus, by the first part of the proposition, this is the unique no-delay static equilibrium. □

PROOF OF PROPOSITION 2: Existence follows from the above discussion. Now, consider any no-delay stationary bargaining equilibrium $\sigma = (\pi, \alpha)$. Let $\tilde{x} = (1 - \delta)q + \int z\pi(dz)$, and note that by concavity, we have

$$u_i(\tilde{x}) \quad \geq \quad (1 - \delta)u_i(q) + \delta v_i(\pi),$$

and thus $\alpha_i(\tilde{x}) = 1$, for every voter. Assume that voters are indexed in order of their ideal points, and that $\tilde{x} < \hat{x}^0$. For each voter i, let y^i_r denote the alternative weakly greater than $\tilde{x}$ such that $u_i(y_r^i) = (1-\delta)u_i(q) + \delta v_i(\pi)$, if such an alternative exists; otherwise, let $y_r^i = \overline{x}$. Similarly, let y_ℓ^i denote the alternative weakly less than $\tilde{x}$ such that $u_i(y_\ell^i)$ is equal to the voter's reservation payoff. By Proposition 3 of Duggan (2014), voter preferences over lotteries are order restricted, and it follows that for any x, we have $u_i(x) \geq (1-\delta)u_i(q) + \delta v_i(\pi)$ if and only if $y_\ell^m \leq x \leq y_r^{n-m+1}$. Thus, the set of alternatives that pass if proposed is an interval containing (y_ℓ^m, y_r^{n-m+1}) and contained in $[y_\ell^m, y_r^{n-m+1}]$. In particular, this is a convex set, and thus the agenda setter has a unique optimal proposal in equilibrium, so that π is degenerate on some alternative x.

For each voter i, we then have $v_i(\pi) = u_i(x)$, so that $u_i(x)$ weakly exceeds voter i's reservation value if and only if $u_i(x) \geq u_i(q)$. Since the equilibrium is no-delay, x passes with probability one, and thus the coalition of voters i such that $u_i(x) \geq u_i(q)$ is decisive, i.e., $x \in A^s$. If $x = \hat{x}^0$, then it is the static equilibrium. Otherwise, if $x \neq \hat{x}^0$, then we can assume without loss of generality that $y^{n-m+1} < \hat{x}^0$, so that $x = y_r^{n-m+1} < \hat{x}^0$. Since $x \in A^s$, we conclude that $u_0(x) \leq u_0(x^s)$, where x^s is the unique no-delay static equilibrium from Proposition 1, and thus $x \leq x^s < \hat{x}^0$. Now, suppose toward a contradiction that $x < x^s$, define $x' = (1-\delta)x^s + \delta x$, and note that $u_0(x') > u_0(x)$. Let $C \in \mathcal{D}$ be a decisive coalition such that $x^s \in A_C^s$, and note that for all $i \in C$, strict concavity of u_i implies

$$u_i(x') \quad > \quad (1-\delta)u_i(x^s) + \delta u_i(x).$$

Then for all $i \in C$, from $u_i(x^s) \geq u_i(q)$, we conclude that

$$u_i(x') \quad > \quad (1-\delta)u_i(q) + \delta u_i(x),$$

but then every member of C accepts x' if proposed, so that $\alpha(x') = 1$, contradicting sequential rationality of π. Therefore, $x = x^s$, as required. □

Proof of Proposition 3: In equilibrium, a voter i accepts a proposal x if and only if it meets or exceeds the reservation value $r_i(\pi) = (1-\delta)u_i(q) + \delta v_i(\pi)$. Let $\xi_i(\pi) = u_i^{-1}(r_i(\pi))$ be the present value of rejection for voter i, i.e., the amount of the dollar needed to buy i's vote. Going to a subsequence if needed, assume that $\xi_i(\pi) \to \overline{x}_i$ for each voter i, and assume without loss of generality that $\xi_1(\pi) \leq \cdots \leq \xi_n(\pi)$. This implies $\overline{x}_1 \leq \cdots \leq \overline{x}_n$. Suppose toward a contradiction that

$$\max_{i=1,\ldots,n} \overline{x}_i = \overline{x}_n > 0.$$

For each voter i with $\overline{x}_i = \overline{x}_n$, let $\rho_i(\pi)$ denote the probability that the agenda setter makes a proposal that voter i accepts. Then $v_i(\pi) = \rho_i(\pi) r_i(\pi)$. As $\delta \to 1$, we have

$$\lim v_i(\pi) = \lim r_i(\pi) = \lim u_i(\xi_i(\pi)) = u_i(\overline{x}_i) > 0.$$

Therefore, we have $\lim v_i(\pi) = (\lim \rho_i(\pi))(\lim v_i(\pi)) > 0$, and this implies $\rho_i(\pi) \to 1$. Thus, for δ close to one, there is positive probability that the agenda setter proposes to all of the most expensive voters, and these voters have positive present value of rejection. Let $x \in \text{supp}(\pi)$ be such that for each voter i with $\overline{x}_i = \overline{x}_n$, we have $x_i \geq \xi_i(\pi) > 0$. If every voter accepts x, i.e., $x_j \geq \xi_j(\pi)$ for all j, then the agenda setter can deviate to $\tilde{x}$ by retaining the amount $x_n > 0$ of the dollar offered to voter n; since $m < n$, the deviation is still accepted. Otherwise, we have $x_j < \xi_j(\pi)$ for some voter j with $\overline{x}_j < \overline{x}_n$. For δ close to one, the agenda setter can deviate to $\tilde{x}$ by transferring $\xi_j(\pi) - x_j$ units of the dollar from voter n to voter j and retaining the residual, $x_n - \xi_j(\pi)$, which is positive by $x_n - \xi_j(\pi) \geq \xi_n(\pi) - \xi_j(\pi) > 0$. We conclude that $\overline{x}_n = 0$, which implies that $v_i(\pi) \to 0$ for every voter i, and thus $\pi \to \hat{x}^0$. □

Proof of Proposition 4: In an equilibrium of the form described in the proposition, the continuation value of voter 2 is

$$\begin{aligned} v_2(b,c) &= -\frac{1}{2}[(a+b)^2 + c^2] - \frac{1}{2}[(a-b)^2 + c^2] \\ &= a^2 + b^2 + c^2, \end{aligned}$$

and voter 1's is the same. In equilibrium, the proposal (b, c) gives voter 2 exactly her reservation value, i.e.,

$$u_2(b,c) = (1-\delta)u_2(q) + \delta v_2(b,c). \tag{2}$$

Moreover, sequential rationality of the proposal (b, c) implies that it lies on the contract curve for the agenda setter and voter 2, so there exists $\beta \in [0, 1]$ such that

$$(b,c) = \beta(0,1) + (1-\beta)(a,0). \tag{3}$$

Then (2) and (3) give us three equations in three unknowns, b, c, and β. Clearly, the optimality equations immediately imply that $b = (1 - \beta)a$ and $c = \beta$, and thus the problem reduces to solving

$$-(\beta a)^2 - \beta^2 = (1-\delta)u_2(q) + \delta(a^2 + [(1-\beta)a]^2 + \beta^2).$$

After manipulating, the quadratic formula yields the solution for β in (1), which lies strictly between zero and one. Thus, the unique proposal to voter 2 in a symmetric stationary bargaining equilibrium is $x^2 = \beta(0, 1) + (1 - \beta)(a, 0)$, and the unique proposal to voter 1 is the symmetric alternative $x^1 = \beta(0, 1) + (1 - \beta)(-a, 0)$, with β given above. Finally, L'Hopital's rule implies that $\beta \to 1$ as $\delta \to 1$, as required. □

A.3 Proofs of theorems

Proof of Theorem 3: Let $\sigma = (\pi, \alpha)$ be a stationary bargaining equilibrium that is not gridlocked, and suppose toward a contradiction that π is not no-delay, so there exists $y \in \mathrm{supp}(\pi)$ such that $\alpha(y) < 1$. The explicit formula for voter i's continuation value, given profile σ, is straightforward to derive: it is

$$v_i(\sigma) = \frac{\int [\alpha(z)u_i(z) + (1-\alpha(z))(1-\delta)u_i(q)]\pi(dz)}{1-\delta\int(1-\alpha(z))\pi(dz)}, \tag{4}$$

and similarly for the agenda setter. Since the discount factor δ is common across voters, we may write each $v_i(\sigma)$ as the expectation of u_i with respect to a single probability measure, say ν, that is independent of i. Specifically, for any Borel measurable Y, we specify that

$$\nu(Y) = \frac{\int_Y [\alpha(z) + I_Y(q)(1-\delta)(1-\alpha(z))]\pi(dz)}{1-\delta\int(1-\alpha(z))\pi(dz)},$$

where $I_Y(\cdot)$ is the indicator function for Y. Letting μ denote the unit mass on q, now define the probability measure $\gamma = (1-\delta)\mu + \delta\nu$, so that the expectation $\int u_i(z)\gamma(dz)$ is just $(1-\delta)u_i(q) + \delta v_i(\sigma)$. Following Banks and Duggan (2006), we refer to γ as the *continuation distribution* corresponding to σ. Letting

$$x(\gamma) = \int z\gamma(dz)$$

denote the mean of the continuation distribution, concavity of u_i implies that

$$u_i(x(\gamma)) \geq \int u_i(z)\gamma(dz) = (1-\delta)u_i(q) + \delta v_i(\sigma)$$

for every voter i.

Since σ is not gridlocked, it follows that π is not degenerate on q, and thus there exists $x \in \mathrm{supp}(\pi)$ that attains the agenda setter's equilibrium payoff and such that $\alpha(x) < 1$. The agenda setter's expected payoff from proposing x is equal to her continuation value, i.e.,

$$v_0(\sigma) = \alpha(x)u_0(x) + (1-\alpha(x))[(1-\delta_0)u_0(q) + \delta_0 v_0(\sigma)], \tag{5}$$

and thus we observe that $v_0(\sigma)$ is a convex combination of $u_0(x)$ and $u_0(q)$. We claim that the equilibrium payoff of the agenda setter equals the stage utility from the status quo: $v_0(\sigma) = u_0(q)$. This follows directly from (5) if x is rejected with probability one, i.e., $\alpha(x) = 0$. To prove the claim, we consider the case $\alpha(x) > 0$. By (5), then $v_0(\sigma)$ is actually a strict convex combination of $u_0(x)$ and $u_0(q)$. We have already argued that $v_0(\sigma) \geq u_0(q)$, and we conclude that $u_0(q) \leq v_0(\sigma) \leq u_0(x)$. We claim that, in fact, the opposite inequalities also hold. The argument proceeds in two cases. In the remainder of the proof, we define the utility vector $\tilde{u}$ so that for all $i \in N \cup \{0\}$,

$$\tilde{u}_i = (1-\delta)u_i(q) + \delta v_i(\sigma).$$

By the above remarks, concavity implies $x(\gamma) \in \tilde{R}_N(\tilde{u})$.

Case 1: $x \neq x(\gamma)$. Note that since $\alpha(x) > 0$, there is a decisive coalition $C^x \in \mathcal{D}$ that accepts x with positive probability. Thus, we have $x \in \tilde{R}_{C^x}(\tilde{u})$, which implies $|\tilde{R}_{C^x}(\tilde{u})| \geq |\{x, x(\gamma)\}| > 1$. Using Lemma 1, LSWP* yields an alternative $x' \in \tilde{P}_{C^x}(\tilde{u})$ arbitrarily close to x, and stage dominance implies that for all $i \in C^x$, we have $\alpha_i(x') = 1$. Thus, the agenda setter's expected payoff from proposing x' is $u_0(x')$. Since x' may be chosen arbitrarily close to x, sequential rationality of π implies that the agenda setter's continuation value is at least equal to the stage utility from x, i.e., $v_0(\sigma) \geq u_0(x)$. Since $v_0(\sigma)$ is a strict convex combination of $u_0(q)$ and $u_0(x)$, we conclude that $u_0(q) = v_0(\sigma) = u_0(x)$.

Case 2: $x = x(\gamma)$. Since σ is not gridlocked, and in particular γ is not degenerate, there is an alternative $y \neq x(\gamma)$ such that $\alpha(y) > 0$. Then there is a decisive coalition $C^y \in \mathcal{D}$ that accepts y with positive probability. Using concavity, we then have $x, y \in \tilde{R}_{C^y}(\tilde{u})$, so that $|\tilde{R}_{C^y}(\tilde{u})| > 1$. By Lemma 1, LSWP* yields an alternative $y' \in \tilde{P}_{C^y}(\tilde{u})$ arbitrarily close to x, and stage dominance implies that for all $i \in C^y$, we have $\alpha_i(y') = 1$. Since y' may be chosen arbitrarily close to x, sequential rationality implies $v_0(\sigma) \geq u_0(x)$. Once again, we conclude that $u_0(q) = v_0(\sigma) = u_0(x)$.

The above arguments establish the claim that $v_0(\sigma) = u_0(q)$. Finally, since σ is not gridlocked, there exists $z \in \text{supp}(\pi)$ that attains the agenda setter's equilibrium payoff such that $z \neq x(\gamma)$ and $\alpha(z) > 0$. Since z attains the agenda setter's equilibrium payoff, we have $u_0(z) = u_0(q) = \tilde{u}_0$. In addition, the above claim and concavity of u_0 imply that

$$\begin{aligned} u_0(x(\gamma)) &\geq (1-\delta_0)u_0(q) + \delta_0 v_0(\sigma) \\ &= (1-\delta)u_0(q) + \delta v_0(\sigma) \\ &= \tilde{u}_0. \end{aligned}$$

Since $\alpha(z) > 0$, there is a decisive coalition $C^z \in \mathcal{D}$ that accepts z with positive probability. We then have $z, x(\gamma) \in \tilde{R}_{C^z \cup \{0\}}(\tilde{u})$, so that $|\tilde{R}_{C^z \cup \{0\}}(\tilde{u})| > 1$. Then LSWP* yields an alternative $z' \in \tilde{P}_{C^z \cup \{0\}}(\tilde{u})$. In particular, stage dominance implies that for all $i \in C^z$, we have $\alpha_i(z') = 1$, and $u_0(z') > \tilde{u}_0 = u_0(z)$, contradicting sequential rationality. □

Proof of Theorem 4: Assume $d = 1$ or $\mathcal{D}$ is oligarchical, and let π be a no-delay stationary bargaining equilibrium. Existence follows from Theorem 1. If $X \subseteq \Re$, then by Lemma 1 in Cho and Duggan (2003), the social acceptance set $A(\pi)$ is an interval. If $\mathcal{D}$ is oligarchic, letting $C = \bigcap \mathcal{D}$, then the social acceptance set is just $A_C(\pi)$, which is again convex. Then concavity and LSWP imply that the agenda setter has a unique maximizer over $A_C(\pi)$, so in both cases π is degenerate on some alternative $y \in X$. We claim that y is a candidate static equilibrium. Indeed, let $C \in \mathcal{D}$ be the coalition of voters who accept y in equilibrium, so that for all $i \in C$,

$$u_i(y) \;\geq\; (1-\delta)u_i(q) + \delta v_i(\pi) \;=\; (1-\delta)\, u_i(q) + \delta u_i(y),$$

which implies $u_i(y) \geq u_i(q)$. Thus, $y \in A^s_C \subseteq A^s$, which implies $u_0(y) \leq u^s_0$. By Lemma 2, we have $u_0(y) = u^s_0$, and we conclude that y is a static equilibrium. $\square$

PROOF OF THEOREM 6: Assume $\delta > 0$, let π be a no-delay stationary bargaining equilibrium, and let x^s be a static equilibrium such that π is not degenerate on x^s. Suppose toward a contradiction that the agenda setter's payoff from π is no greater than her static equilibrium payoff, i.e., for all $x \in \text{supp}(\pi)$, we have $u_0(x) \leq u^s_0$. Let $C^s \in \mathcal{D}$ be such that for all $i \in C^s$, we have $u_i(x^s) \geq u_i(q)$. Let $\overline{x} = \int z\pi(z)$ be the mean of π, and define $\tilde{x} = (1-\delta)\,x^s + \delta\overline{x}$. Note that for all $i \in C^s$, concavity of u_i and $u_i(x^s) \geq u_i(q)$ imply

$$\begin{aligned} u_i(\tilde{x}) &\geq (1-\delta)u_i(x^s) + \delta u_i(\overline{x}) && (6)\\ &\geq (1-\delta)u_i(x^s) + \delta v_i(\pi) && (7)\\ &\geq (1-\delta)u_i(q) + \delta v_i(\pi), && (8) \end{aligned}$$

and therefore $\tilde{x} \in A_{C^s}(\pi) \subseteq A(\pi)$. Moreover, for all $x \in \text{supp}(\pi)$, we have

$$\begin{aligned} u_0(x) &\geq u_0(\tilde{x}) && (9)\\ &\geq (1-\delta)u_0(x^s) + \delta u_0(\overline{x}) && (10)\\ &\geq (1-\delta)u_0(x^s) + \delta u_0(x) && (11)\\ &\geq u_0(x), && (12) \end{aligned}$$

where the first inequality follows from the fact that x maximizes the agenda setter's stage utility over the social acceptance set; the second follows from concavity; the third follows from the fact that the agenda setter is indifferent across alternatives in the support of π; and the fourth follows from $u^s_0 \geq u_0(x)$.

We deduce a contradiction in three cases. First, assume (i) holds. We consider two subcases. First, assume π is non-degenerate. By inequalities (9)–(12), we have $u_0(x) = u_0(\overline{x})$ for all $x \in \text{supp}(\pi)$, contradicting strict concavity of u_0. Second, assume π is degenerate. Then π is degenerate on $\overline{x}$, and we have $\overline{x} \neq x^s$ by assumption. But (9)–(12) imply $u_0(\tilde{x}) = (1-\delta)u_0(x^s) + \delta u_0(\overline{x})$, and with $\delta > 0$, this again contradicts strict concavity of u_0.

Next, assume (ii) holds. Note that $\tilde{x} \neq \hat{x}^0$, for otherwise, we would have $\hat{x}^0 = \tilde{x} \in A(\pi)$, and sequential rationality of π would imply that π is degenerate on $\hat{x}^0$. Then $u^s_0 \geq u_0(\hat{x}^0)$ would imply that $x^s = \hat{x}^0$, contradicting the assumption that π is not degenerate on x^s. Now, using (6)–(8), we can partition C^s into two groups:

$$\begin{aligned} I &= \{i \in C^s \mid u_i(\tilde{x}) = (1-\delta)u_i(q) + \delta v_i(\pi)\}\\ J &= \{i \in C^s \mid u_i(\tilde{x}) > (1-\delta)u_i(q) + \delta v_i(\pi)\}. \end{aligned}$$

For all $i \in I$, with (6)–(8), we in fact have

$$\begin{aligned} u_i(\tilde{x}) &= (1-\delta)u_i(x^s) + \delta u_i(\overline{x}) \\ &= (1-\delta)u_i(x^s) + \delta v_i(\pi) \\ &= (1-\delta)u_i(q) + \delta v_i(\pi). \end{aligned}$$

In particular, since $\delta > 0$, we have $u_i(\overline{x}) = v_i(\pi)$. Since u_i is strictly concave, this implies that π is degenerate, in which case it is degenerate on $\overline{x} \neq x^s$. But with $\delta > 0$, the equality $u_i(\tilde{x}) = (1-\delta)u_i(x^s) + \delta u_i(\overline{x})$ again contradicts strict concavity. We conclude that $I = \emptyset$, and $C^s = J$. Defining $z = (1-\alpha)\tilde{x} + \alpha\hat{x}^0$, we can choose $\alpha > 0$ small enough that for all $i \in C^s$, we have $u_i(z) > (1-\delta)u_i(q) + \delta v_i(\pi)$. Thus, $z \in A(\pi)$. But then for all $x \in \mathrm{supp}(\pi)$, concavity of u_0, $\tilde{x} \neq \hat{x}^0$, and (9)–(12) imply

$$u_0(z) \;>\; u_0(\tilde{x}) \;=\; u_0(x),$$

contradicting sequential rationality of π.

Last, assume (iii) holds. For every voter $i \in N$, let $\underline{u}_i = \min_{z \in X} u_i(z)$ be the lowest possible payoff for voter i. Note that π is not degenerate on $\tilde{x}$, for otherwise, we would have $\tilde{x} = \overline{x} = x^s$, a contradiction. Thus, we can choose $x \in \mathrm{supp}(\pi)\backslash\{\tilde{x}\}$, and since π is no-delay, the coalition $C^x = \{i \in N \mid u_i(x) \geq (1-\delta)u_i(q) + \delta v_i(\pi)\}$ is decisive. Next, we claim that for every $i \in C^x \backslash C^s$, we have $u_i(\tilde{x}) = u_i(x) = \underline{u}_i$. Indeed, suppose toward a contradiction that for some $i \in C^x \setminus C^s$, we have $u_i(\tilde{x}) > \underline{u}_i$. Then minimal transferability yields $x' \in X$ such that $u_0(x') > u_0(\tilde{x})$ and for all $j \in N \setminus \{i\}$, we have $u_j(x') > u_j(\tilde{x})$. By (6)–(8), we have $\tilde{x} \in A_{C^s}(\pi)$, and by (9)–(12), we have $u_0(\tilde{x}) = u_0(x)$. We conclude that $u_0(x') > u_0(x)$ and for all $j \in C^s$, $u_j(x') > (1-\delta)u_j(q) + \delta v_j(\pi)$. By stage dominance, it follows that $\alpha_j(x') = 1$ for all $j \in C^s$, but then the agenda setter can increase her payoff by proposing x', contradicting sequential rationality of π. Thus, $u_i(\tilde{x}) = \underline{u}_i$. Now, since $\delta > 0$ and

$$\underline{u}_i \;=\; u_i(\tilde{x}) \;\geq\; (1-\delta)u_i(x^s) + \delta v_i(\pi),$$

it follows that $v_i(\pi) = \underline{u}_i$, and in particular $u_i(x) = \underline{u}_i$. This establishes the claim. Next, we claim that $\tilde{x} \in A_{C^x}(\pi)$. To show this, consider any $i \in C^x$. If $i \notin C^s$, then

$$u_i(\tilde{x}) \;=\; \underline{u}_i \;=\; u_i(x) \;\geq\; (1-\delta)u_i(q) + \delta v_i(\pi).$$

If $i \in C^s$, then (6)–(8) imply $\tilde{x} \in A_i(\pi)$, as claimed. Define the utility vector $\tilde{u}$ such that $\tilde{u}_0 = u_0(\tilde{x}) = u_0(x)$ and for all $i \in N$, we have

$$\tilde{u}_i \;=\; (1-\delta)u_i(q) + \delta v_i(\pi).$$

We have shown that $|\tilde{R}_{C^x \cup \{0\}}(\tilde{u})| \geq |\{x, \tilde{x}\}| > 1$, and using Lemma 1, LSWP* yields $x'' \in \tilde{P}_{C^x \cup \{0\}}(\tilde{u})$. Thus, $u_0(x'') > u_0(x)$ and $x \in A_{C^x}(\pi)$, contradicting sequential rationality of π. □

Proof of Theorem 7: By Corollaries 1 and 2, it suffices to show that there exists $\underline{\delta} \in [0, 1]$ such that for every $\delta \leq \underline{\delta}$, there is a pure strategy equilibrium, and for every $\delta > \underline{\delta}$, there is no pure strategy equilibrium. By Corollary 3, if there are multiple static equilibria, then $\underline{\delta} = 0$ serves as the cutoff. Then it remains to consider the case in which there is a unique static equilibrium x^s. Let $\pi^s \in \Delta(X)$ be the unit mass on x^s. We write $\pi^s \in B^\delta(\pi^s)$ if π^s is a stationary bargaining equilibrium proposal strategy for δ, and $\pi^s \notin B^\delta(\pi^s)$ if not. Let $C^s \in \mathcal{D}$ be such that for all $i \in C^s$, we have $u_i(x^s) \geq u_i(q)$. Then for all $i \in C^s$ and all $\delta > 0$, we have

$$u_i(x^s) \;\; \geq \;\; (1-\delta)u_i(q) + \delta u_i(x^s),$$

and thus $x^s \in A^\delta(\pi^s)$. Therefore, $\pi^s \notin B^\delta(\pi^s)$ if and only if there exists $y \in A^\delta(\pi^s)$ such that $u_0(y) > u_0^s$.

We claim that if $\pi^s \notin B^\delta(\pi^s)$, then for every $\delta' > \delta$, we have $\pi^s \notin B^{\delta'}(\pi^s)$. Indeed, let $y \in X$ and $C^y \in \mathcal{D}$ be such that for all $i \in C^y$,

$$u_i(y) \;\; \geq \;\; (1-\delta)u_i(q) + \delta v_i(\pi^s),$$

and

$$u_0(y) \;\; > \;\; u_0^s.$$

For every $\delta' > \delta$, let

$$\tilde{x}^{\delta'} \;\; = \;\; \left(\frac{1-\delta'}{1-\delta}\right) y + \left(1 - \frac{1-\delta'}{1-\delta}\right) x^s.$$

Note that $\delta' \in (\delta, 1)$ implies that $\frac{1-\delta'}{1-\delta} \in (0, 1)$. By concavity, for all $i \in C^y$,

$$\begin{aligned} u_i(\tilde{x}^{\delta'}) \;\; &\geq \;\; \left(\frac{1-\delta'}{1-\delta}\right) u_i(y) + \left(1 - \frac{1-\delta'}{1-\delta}\right) u_i(x^s) \\ &\geq \;\; \left(\frac{1-\delta'}{1-\delta}\right) [(1-\delta)u_i(q) + \delta v_i(\pi^s)] + \left(1 - \frac{1-\delta'}{1-\delta}\right) u_i(x^s) \\ &= \;\; (1-\delta')u_i(q) + \delta' v_i(\pi^s), \end{aligned}$$

and

$$u_0(\tilde{x}^{\delta'}) \;\; \geq \;\; \left(\frac{1-\delta'}{1-\delta}\right) u_0(y) + \left(1 - \frac{1-\delta'}{1-\delta}\right) u_0(x^s) \;\; > \;\; u_0^s.$$

Therefore, $\pi^s \notin B^{\delta'}(\pi^s)$, as claimed.

Let $D = \{\delta \in (0,1) \mid \pi^s \notin B^\delta(\pi^s)\}$. If $D \neq \emptyset$, let $\underline{\delta} = \inf D$; otherwise, let $\underline{\delta} = 1$. By definition of $\underline{\delta}$ and the previous claim, for every $\delta > \underline{\delta}$, we have $\pi^s \notin B^\delta(\pi^s)$, and for every $\delta < \underline{\delta}$, we have $\pi^s \in B^\delta(\pi^s)$. By Theorem 2, the correspondence of no-delay stationary bargaining equilibrium proposal strategies is upper hemi-continuous in the voters' discount factor, and thus $\pi^s \in B^{\underline{\delta}}(\pi^s)$, as required. □

Proof of Theorem 8: For the first part of the theorem, let π^δ be a weak* convergent sequence of equilibrium proposal strategies as $\delta \to 1$, and let π^* denote the limit of this subsequence. Let $\overline{x} = \int z\pi^*(dz)$ be the mean of π^*, and suppose toward a contradiction that there is a subsequence $y^\delta \in \text{supp}\left(\pi^\delta\right)$ such that $y^\delta \to \tilde{y} \neq \overline{x}$. Furthermore, we can choose y^δ so that for each δ, there is a decisive coalition $C^\delta \in \mathcal{D}$ such that for each $i \in C^\delta$, we have

$$u_i(y^\delta) \;\; \geq \;\; (1-\delta)u_i(q) + \delta \int u_i(z)\pi^\delta(dz),$$

and since N is finite, we can go to a subsequence along which this coalition is fixed, i.e., $C^\delta = C$. Taking limits, for each $i \in C$, we have

$$u_i(\tilde{y}) \;\; \geq \;\; \int u_i(z)\pi^*(dz).$$

Moreover, $u_0(y^\delta) = \int u_0(z)\pi^\delta(dz)$ and thus

$$u_0(\tilde{y}) \;\; = \;\; \int u_0(z)\pi^*(dz).$$

Define the utility vector $\tilde{u}$ so that for all $i \in N \cup \{0\}$, we have $\tilde{u}_i = \int u_i(z)\pi^*(dz)$. We then have $\tilde{y} \in \tilde{R}_{C\cup\{0\}}(\tilde{u})$ by the above continuity argument, and concavity implies $\overline{x} \in \tilde{R}_{C\cup\{0\}}(\tilde{u})$. Using Lemma 1, LSWP* yields an alternative x' arbitrarily close to $\tilde{y}$ such that $u_0(x') > u_0(\tilde{y})$ and for all $i \in C$, we have $u_i(x') > u_i(\tilde{y})$. Thus, when δ is close to one, we have for all $i \in C$,

$$u_i(x') \;\; > \;\; (1-\delta)u_i(q) + \delta \int u_i(z)\pi^\delta(dz),$$

which implies $x' \in A_C(\pi^\delta) \subseteq A(\pi^\delta)$, and we have

$$u_0(x') \;\; > \;\; \int u_0(z)\pi^\delta(dz),$$

contradicting sequential rationality of π. We conclude that π^δ converges strongly to a degenerate proposal.

For the second part of the theorem, suppose toward a contradiction that there exist $y \in X$ and $C \in \mathcal{D}$ such that $y \neq x^*$, we have $u_0(y) \geq u_0(x^*)$, and for all $i \in \mathcal{C}$, we have $u_i(y) \geq u_i(x^*)$. By LSWP, there exists $x' \in X$ such that $u_0(x') > u_0(x^*)$ and for all $i \in C$, $u_i(x') > u_i(x^*)$. Since π^δ converges to x^*, we have $v_i(\pi^\delta) \to u_i(x^*)$ for all $i \in C$, and then for δ close enough to one, we have

$$u_i(x') \;\; > \;\; (1-\delta)u_i(q) + \delta v_i(\pi^\delta)$$

for all $i \in C$, which implies $x' \in A(\pi^\delta)$. Moreover, for δ close enough to one, we have $u_0(x') > u_0(x)$ for all $x \in \text{supp}(\pi^\delta)$, contradicting sequential rationality of π^δ, as required. $\square$

PROOF OF THEOREM 9: Consider a limit proposal $x^* \neq \hat{x}^0$, and suppose toward a contradiction that there is some alternative $y \in H + x$ such that $y \succ x^*$. Let $C \in \mathcal{D}$ be a decisive coalition such that for all $i \in C$, we have $u_i(y) > u_i(x^*)$. For $\epsilon \in (0,1)$, define $\tilde{x} = y + \epsilon(\hat{x}^0 - y) \in X$, and using continuity of u_i, choose $\epsilon > 0$ small enough that for all $i \in C$, we have $u_i(\tilde{x}) > u_i(x)$. For $\alpha \in (0,1)$, define $x' = (1-\alpha)x^* + \alpha\tilde{x} \in X$, and note that for all $i \in C$, concavity implies $u_i(x') > u_i(x^*)$. Moreover, define $p = x' - x^* = (1-\alpha)x^* + \alpha\tilde{x} - x^* = \alpha(\tilde{x} - x^*)$. Then the derivative of u_0 at x^* in direction p is

$$\begin{aligned}
\nabla u_0(x^*) \cdot p &= \alpha \nabla u_0(x^*) \cdot (\tilde{x} - x^*) \\
&= \alpha \nabla u_0(x^*) \cdot (y + \epsilon(\hat{x}^0 - y) - x^*) \\
&= \alpha(\nabla u_0(x^*) \cdot (y - x^*)) + \alpha\epsilon \nabla u_0(x^*) \cdot (\hat{x}^0 - y) \\
&= \alpha\epsilon \nabla u_0(x^*) \cdot (\hat{x}^0 - x^* + x^* - y) \\
&= \alpha\epsilon \nabla u_0(x^*) \cdot (\hat{x}^0 - x^*) \\
&> 0.
\end{aligned}$$

where the last two equalities use $\nabla u_0(x^*)(y - x^*) = 0$, and the strict inequality follows from concavity of u_0. Thus, for small enough α, we have $u_0(x') > u_0(x^*)$ and for all $i \in C$, $u_i(x') > u_i(x^*)$, contradicting Theorem 8, as required. $\square$

PROOF OF THEOREM 10: Consider a limit proposal $x^* \neq \hat{x}^0$, and suppose toward a contradiction that there is some $C \in \mathcal{D}$ such that for all $i \in C$, we have $u_i(x^*) > u_i(q)$. By Theorem 8, it follows that for δ close to one: for all $y \in \text{supp}(\pi^\delta)$ and all $i \in C$, $u_i(y) > u_i(q)$. Let $\overline{x}^\delta = \int z\pi^\delta(dz)$ be the mean of π^δ, and note that $\overline{x}^\delta \to x^*$. For all $i \in C$, concavity of u_i implies

$$u_i(\overline{x}^\delta) \;\geq\; \int u_i(z)\pi^\delta(dz) \;>\; u_i(q),$$

which implies

$$u_i(\overline{x}^\delta) \;>\; (1-\delta)\,u_i(q) + \delta \int u_i(z)\pi^\delta(dz),$$

for δ close to one. Furthermore, $x^* \neq \hat{x}^0$ implies that for δ close to one, we have $\overline{x}^\delta \neq \hat{x}^0$. Given such δ, define $\tilde{x} = (1-\epsilon)\,\overline{x}^\delta + \epsilon\hat{x}^0$ for $\epsilon \in (0,1)$. By concavity of u_0, we have $u_0(\tilde{x}) > u_0(\overline{x}^\delta)$, and we can choose $\epsilon > 0$ small enough such that $\tilde{x} \in A_C(\pi^\delta)$, contradicting Theorem 8, as required. $\square$

PROOF OF THEOREM 11: Consider a limit proposal $x^* \neq \hat{x}^0$, and suppose toward a contradiction that there exists $r \in H(x^*) \cap L(x^*)$ such that $C = \{i \in N \mid r \cdot \nabla u_i(x) >$

$0\} \in \mathcal{D}$. We claim that $0 \notin \mathrm{conv}(\{\nabla u_0(x^*)\} \cup \{\nabla f^\ell(x^*) \mid \ell \in K(x^*)\})$, for suppose otherwise; then there exist coefficients β_0 and β_ℓ, $\ell \in K(x^*)$, such that

$$0 \;=\; \beta_0 \nabla u_0(x^*) + \sum_{\ell \in K(x^*)} \beta_\ell \nabla f^\ell(x^*).$$

Since X is regular, it follows that $\beta_0 \neq 0$, but then defining $\lambda_\ell = -\beta_\ell/\beta_0$, we can write

$$\nabla u_0(x^*) \;=\; \sum_{\ell \in K(x^*)} \lambda_\ell \nabla f^\ell(x^*).$$

Because u_0 is concave and satisfies the first order condition, it solves the maximization problem

$$\begin{array}{c} \max_{y \in \Re^d} u_0(y) \\ \text{s.t. } f^\ell(y) \geq 0,\ \ell = 1, \ldots, m, \end{array}$$

but this would imply $x^* = \hat{x}^0$. Thus, zero is not in the convex hull of $\{\nabla u_0(x^*)\} \cup \{\nabla f^\ell(x^*) \mid \ell \in K(x^*)\}$. By the separating hyperplane theorem, there is a vector $s \in \Re^d$ such that $s \cdot \nabla u_0(x^*) > 0$ and for all $\ell \in K(x^*)$, $s \cdot \nabla f^\ell(x^*) > 0$. Defining $t = r + \epsilon s$, from $r \in H(x^*) \cap L(x^*)$, we have

$$t \cdot \nabla u_0(x^*) \;=\; s \cdot \nabla u_0(x^*) \;>\; 0$$

and for all $\ell \in K(x^*)$,

$$t \cdot \nabla f^\ell(x^*) \;=\; s \cdot \nabla f^\ell(x^*) \;>\; 0.$$

In addition, for $\epsilon > 0$ small enough, we have for all $i \in C$,

$$t \cdot \nabla u_i(x) \;=\; r \cdot \nabla u_i(x) + \epsilon(s \cdot u_i(x)) \;>\; 0.$$

Now define $x' = x^* + \eta t$ for $\eta > 0$ small. Then we have $u_0(x') > u_0(x^*)$ and for all $\ell \in K(x^*)$, $f^\ell(x') > 0$. By continuity, we have for all $\ell \notin K(x^*)$, $f^\ell(x') > 0$ as well, implying $x' \in X$. Finally, we have $u_i(x') > u_i(x^*)$ for all $i \in C$, contradicting Theorem 8. □

Proof of Theorem 12: Suppose toward a contradiction that there is at most one voter whose gradient is collinear with the agenda setter's. Let H be the hyperplane through the origin orthogonal to $\nabla u_0(x)$. For each voter h, let $p_h = \mathrm{proj}_H \nabla u_h(x)$ be the projection of the voter's gradient onto H. By assumption, we have $p_h = 0$ for at most one voter. Let $\tilde{N} = \{h \in N \mid p_h \neq 0\}$, so that $\tilde{n} \equiv |\tilde{N}| \geq n - 1$. Let $r \in H$ be any vector such that for all $h \in \tilde{N}$, we have $\nabla u_h(x) \cdot r \neq 0$, and without loss of generality, assume

$$\tilde{N}_r \;\equiv\; \{h \in \tilde{N} \mid \nabla u_h(x) \cdot r > 0\} \;\geq\; \{h \in \tilde{N} \mid \nabla u_h(x) \cdot r < 0\}.$$

Since

$$|\{h \in N \mid \nabla u_h(x) \cdot r > 0\}| + |\{h \in N \mid \nabla u_h(x) \cdot r < 0\}| \;=\; \tilde{n} \;\geq\; n-1,$$

this implies

$$2|\tilde{N}_r| \;\geq\; n-1,$$

which implies $|\tilde{N}_r| \geq \frac{n-1}{2}$. Since n is assumed even, this in fact implies $|\tilde{N}_r| \geq \frac{n}{2}$, so that $\tilde{N}_r \in \mathcal{D}$. Since the derivative of u_i at x in direction r is positive for all $i \in \tilde{N}_r$, we can choose $\epsilon > 0$ small enough that $y = x + \epsilon r \in X$ and for all $i \in \tilde{N}_r$, we have $u_i(y) > u_i(x)$. But then $\nabla u_0(x) \cdot (y - x) = \epsilon r$, so that $y \in X \cap (H + x)$, and $y \succ x$, a contradiction. □

Proof of Theorem 17: Let $\mathcal{M}$ be the set of all $(n+1) \times d$ matrices, denoted M, with rows indexed $0, 1, \ldots, n$. We employ the following notational convention for subsets of $\mathcal{M}$ throughout the proof: given any groups $G_1, \ldots, G_k \subseteq N \cup \{0\}$ and any natural numbers $r_1, \ldots, r_k$, let $\mathcal{M}_{r_1, r_2, \ldots, r_k}[G_1; G_2; \ldots; G_k]$ denote the set of matrices such that for all $\ell = 1, \ldots, k$, the rows of M corresponding to members of G_ℓ have rank r_ℓ in $\Re^d$. To ease notation, we may omit braces around the elements of G_ℓ in the sequel.

First, assume (i). Given distinct voters $i, j \in N$, let $\mathcal{M}_{1,1,1}[0; 0i; 0j]$ be the set of $(n+1) \times d$ matrices M such that: row zero has at least one non-zero entry; row zero and row i have rank one; and row zero and row j have rank one. Because row zero is non-zero, this means that row i is a scalar multiple of row zero, as is row j. Thus, the set $\mathcal{M}_{1,1,1}[0; 0i; 0j]$ is a manifold of dimension $d+1+1+(n+1-3)d = (n+1)d+2-2d$, and it has codimension $(n+1)d-(n+1)d-2+2d = 2d-2$. Moreover, it is straightforward to verify that $\mathcal{M}_{1,1,1}[0; 0i; 0j]$ is semialgebraic. Indeed, let $A = \{(z, \alpha, \beta) \in \Re^{d+2} \mid z \neq 0\}$, and define the polynomial function $\phi\colon \Re^2 \to \Re^{(n+1)d}$ by $\phi(z, \alpha, \beta) = (z, \alpha z, \beta z)$. Since $\mathcal{M}_{1,1,1}[0; 0i; 0j]$ is the image of the semialgebraic set A under ϕ, result (2.1) of Gibson et al. (1976) implies that $\mathcal{M}_{1,1,1}[0; 0i; 0j]$ is semialgebraic, as claimed. Furthermore, result (2.3) of Gibson et al. (1976) implies that the closure $\mathcal{M}^{i,j} = \mathrm{clos}(\mathcal{M}_{1,1,1}[0; 0i; 0j])$ is also semialgebraic. By result (2.7) of Gibson et al. (1976), it follows that $\mathcal{M}^{i,j}$ is in fact a Whitney stratified subset of $\Re^d$ (cf. p.11 of the latter reference), and by their result (1.2), the product $\mathrm{int}(X) \times \Re^{n+1} \times \mathcal{M}^{i,j}$ is a Whitney stratified subset of $\Re^{d+n+1+(n+1)d}$.

This structure allows us to apply a generalized version of the jet transversality theorem. Given any $f \in C^2(\mathrm{int}(X), \Re^{n+1})$, the 1-*jet* of f is the mapping $j^1 f\colon \Re^d \to \Re^{d+(n+1)+(n+1)d}$ defined by

$$j^1 f(x) \;=\; (x, f(x), d^1 f(x)).$$

By Theorem 7.5.11 of Jongen, Jonkers, and Twilt (2000), the set of twice continuously differentiable mappings $f\colon \mathrm{int}(X) \to \Re^{n+1}$ such that $j^1 f$ intersects $\mathrm{int}(X) \times \Re^{n+1} \times$

$\mathcal{M}^{i,j}$ transversally, i.e.,

$$\mathcal{F}^{i,j} = \left\{ f \in \mathcal{U}_{\text{int}(X)} \mid j^1 f \pitchfork \text{int}(X) \times \Re^{n+1} \times \mathcal{M}^{i,j} \right\}$$

is open and dense in the Whitney topology on $\mathcal{U}_{\text{int}(X)}$.[12] By transversality, for all $f \in \mathcal{F}^{i,j}$, it follows that the set

$$\{x \in \text{int}(X) \mid j^1 f(x) \in \text{int}(X) \times \Re^{n+1} \times \mathcal{M}_{1,1,1}[0; 0i; 0j]\}$$

is a manifold with codimension $2d-2$, and thus it has dimension $d-2d+2 = 2-d$. More succinctly, we can write $\mathcal{F}^{i,j}$ as the set of mappings $f \in \mathcal{U}_{\text{int}(X)}$ such that $d^1 f$ is transversal to $\mathcal{M}^{i,j}$, and we conclude that for all $f \in \mathcal{F}^{i,j}$, the set

$$\{x \in \text{int}(X) \mid d^1 f(x) \in \mathcal{M}_{1,1,1}[0; 0i; 0j]\}$$

is a manifold with codimension $2d-2$, and thus it has dimension $d-2d+2 = 2-d$; in the sequel, we will move to the first derivative directly to save space. Since $d \geq 3$ by assumption (i), we conclude that for all $f \in \mathcal{F}^{i,j}$, this set is empty; in particular, there does not exist $x \in \text{int}(X)$ such that $d^1 f(x) \in \mathcal{M}_{1,1,1}[0; 0i; 0j]$.

Since $\mathcal{U}_{\text{int}(X)}$ is a Baire space with the Whitney topology, it follows that the intersection $\mathcal{F}_{(\text{i})} = \bigcap_{i,j \in N : i \neq j} \mathcal{F}^{i,j}$ is also open and dense in $\mathcal{U}_{\text{int}(X)}$. And since $\hat{\mathcal{U}}_{\text{int}(X)}$ is open in $\mathcal{U}_{\text{int}(X)}$, it follows that the set

$$\hat{\mathcal{F}}_{(\text{i})} = \mathcal{F}_{(\text{i})} \cap \hat{\mathcal{U}}_{\text{int}(X)}$$

is open and dense in $\hat{\mathcal{U}}_{\text{int}(X)}$ with the relative Whitney topology. For all $u \in \hat{\mathcal{U}}_{\text{int}(X)}$, Theorem 12 implies that if $x \in CC_{\text{int}(X)}(u)$, then there exist voters $i, j \in N$ such that $d^1 u(x) \in \mathcal{M}_{1,1,1}[0; 0i; 0j]$, which implies $u \notin \mathcal{F}^{i,j}$. Contrapositively, if $u \in \hat{\mathcal{F}}_{(\text{i})}$, then there does not exist $x \in CC_{\text{int}(X)}(u)$, i.e., $CC_{\text{int}(X)}(u) = \emptyset$. This means that $\hat{\mathcal{F}}_{(\text{i})} \subseteq \hat{\mathcal{U}}^0_{\text{int}(X)}$, establishing the first genericity result.

For the second genericity result, we apply the standard jet transversality theorem (Hirsch, 1976, Theorems 2.8 and 2.9) to the space of twice continuously differentiable mappings $f \colon \Re^d \to \Re^{n+1}$ to conclude that

$$\mathcal{G}^{i,j} = \{f \in C^2(\Re^d, \Re^{n+1}) \mid d^1 f \pitchfork \mathcal{M}_{1,1,1}[0; 0i; 0j]\}$$

is a residual subset of $C^2(\Re^d, \Re^{n+1})$ with the Whitney topology, i.e., it contains the countable intersection of sets that are open and dense. Since $C^2(\Re^d, \Re^n)$ is a Baire space with the Whitney topology, it follows that the intersection $\mathcal{G}_{(\text{i})} = \bigcap_{i,j \in N : i \neq j} \mathcal{G}^{i,j}$

[12] Theorem 7.5.11 of Jongen, Jonkers, and Twilt (2000) is written in terms of mappings defined on the entire Euclidean space $\Re^d$; the result can be directly extended to mappings with convex, open domain.

is also residual in this space. Define $\hat{\mathcal{G}}$ to consist of the restriction to X of each function $f \in \mathcal{G}_{(\mathrm{i})}$ such that f is differentiably concave on X, i.e.,

$$\hat{\mathcal{G}}_{(\mathrm{i})} \;=\; \{f|_X \mid f \in \mathcal{G}_{(\mathrm{i})}\} \cap \hat{\mathcal{U}}_X.$$

We claim that $\hat{\mathcal{G}}_{(\mathrm{i})}$ is dense in $\hat{\mathcal{U}}_X$ with the relative topology of C^2-uniform convergence. Indeed, consider any $g \in \hat{\mathcal{U}}_X$ and any $\epsilon > 0$. The set $\mathcal{H} = \{f \in C^2(\Re^d, \Re^n) \mid f|_X \in \hat{\mathcal{U}}_X \cap \mathcal{B}_\epsilon(g)\}$ is open in the Whitney topology. Since $\mathcal{G}_{(\mathrm{i})}$ is dense, there is a mapping $f \in \mathcal{G}_{(\mathrm{i})} \cap \mathcal{H}$. Then the restriction $f|_X$ belongs to $\hat{\mathcal{G}}_{(\mathrm{i})}$ and to the ball $\mathcal{B}_\epsilon(g)$, and we conclude that $\hat{\mathcal{G}}_{(\mathrm{i})}$ is dense in $\hat{\mathcal{U}}_X$, as claimed.

Next, we construct a countably infinite collection of open sets $\{\mathcal{V}_m\}$ the intersection of which consists of the differentiably concave vector utility functions that admit no interior constrained core alternatives. Choose an interior alternative $\overline{x} \in \mathrm{int}(X)$, and for each natural number m, define K_m by shrinking X down to $\overline{x}$ as follows:

$$K_m \;=\; \left(1 - \frac{1}{m}\right)X + \frac{1}{m}\overline{x}.$$

As m increases, the convex and compact sets K_m increase to fill the interior of X, and we have $\mathrm{int}(X) = \bigcup_{m=1}^{\infty} K_m$. For each m, define

$$\mathcal{V}_m \;=\; \{u \in \hat{U}_X \mid CC_X(u) \cap K_m = \emptyset\},$$

and note that $\hat{\mathcal{U}}_X^0 = \bigcap_{m=1}^{\infty} \mathcal{V}_m$. It is straightforward to show that each $\mathcal{V}_m$ is open in the relative topology of C^2-uniform convergence, and thus it remains to show that each set is dense.

We claim that for each m, $\hat{\mathcal{G}}_{(\mathrm{i})} \subseteq \mathcal{V}_m$. To see this, consider any $u \in \hat{\mathcal{G}}_{(\mathrm{i})}$, so that there exists $f \in \mathcal{G}_{(\mathrm{i})}$ with $u = f|_X$. Note that $f \in \mathcal{G}_{(\mathrm{i})}$ implies that for all distinct voters $i, j \in N$, the set $\{x \in \Re^d \mid d^1 f(x) \cap \mathcal{M}_{1,1,1}[0; 0i; 0j]\}$ is a manifold of dimension $2-d$, and thus assumption (i) implies that the set is empty. If there were an alternative $x \in CC_X(u) \cap K_m$, then Theorem 12 would yield voters $i, j \in N$ such that $d^1 f(x) = d^1 u(x) \in \mathcal{M}_{1,1,1}[0; 0i; 0j]$, which is impossible. Therefore, $CC_X(u) \cap K_m = \emptyset$, i.e., $u \in \mathcal{V}_m$. This implies $\hat{\mathcal{G}}_{(\mathrm{i})} \subseteq \mathcal{V}_m$, as claimed. Since $\hat{\mathcal{G}}_{(\mathrm{i})}$ is dense, it follows that $\mathcal{V}_m$ is dense, as well as open. We conclude that $\hat{\mathcal{U}}_X^0$ is the intersection of sets that are open and dense in $\hat{\mathcal{U}}_X$ with the relative topology of C^2-uniform convergence. Therefore, $\hat{\mathcal{U}}_X^0$ is residual, establishing the second genericity result, as required.

Second, assume (ii). The structure of the argument parallels that above, but with different accounting details, which we explicate here. For distinct voters $i, j, k \in N$, define the following class of matrices:

- $\mathcal{M}_{1,1,2,2}[0; 0k; 0i; 0ij]$ is the set of matrices such that: row zero has at least one non-zero entry; row zero and row k have rank one; row zero and row i are linearly independent; and rows zero, i, and j have rank two. Since row zero is non-zero, row k is a scalar multiple of row zero, and since rows zero and i are linearly independent, row j is a linear combination of row zero and row i. The set is a manifold with dimension $d+1+d+2+(n+1-4)d = (n-1)d+3$ and codimension $2d-3$.

By results of Gibson et al. (1976), $\mathrm{clos}(\mathcal{M}_{1,1,2,2}[0;0k;0i;0ij])$ are semialgebraic, and thus the union

$$\mathcal{M}^{i,j,k} \quad = \quad \mathcal{M}^{i,j} \cup \mathrm{clos}(\mathcal{M}_{1,1,2,2}[0;0k;0i;0ij])$$

is semialgebraic and, in fact, $\mathrm{int}(X) \times \Re^{n+1} \times \mathcal{M}^{i,j,k}$ is a Whitney stratified set.

By Theorem 7.5.11 of Jongen, Jonkers, and Twilt (2000), the set $\mathcal{F}^{i,j,k}$ of mappings $f \in C^2(\mathrm{int}(X), \Re^{n+1})$ such that $j^1 f$ intersects $\mathrm{int}(X) \times \Re^{n+1} \times \mathcal{M}^{i,j,k}$ transversally is open and dense in the Whitney topology on $\mathcal{U}_{\mathrm{int}(X)}$. By transversality, for all $f \in \mathcal{F}^{i,j,k}$, the set

$$\{x \in \mathrm{int}(X) \mid d^1 f(x) \in \mathcal{M}_{1,1,2,2}[0;0k;0i;0ij]\}$$

is a manifold with codimension $2d-3$, and thus it has dimension $3-d$. Since $d \geq 4$ by assumption (ii), we conclude that for all $f \in \mathcal{F}^{i,j,k}$, this set is empty; in particular, using $\mathcal{M}^{i,j} \subseteq \mathcal{M}^{i,j,k}$, there does not exist $x \in \mathrm{int}(X)$ such that $d^1 f(x) \in \mathcal{M}_{1,1,1}[0;0k;0j] \cup \mathcal{M}_{1,1,2,2}[0;0k;0i;0ij]$.

Again, the intersection $\mathcal{F}_{(\mathrm{ii})} = \bigcap_{I \subseteq N:|I|=3} \mathcal{F}^I$ is open and dense, and it follows that $\hat{\mathcal{F}}_{(\mathrm{ii})} = \mathcal{F}_{(\mathrm{ii})} \cap \hat{\mathcal{U}}_{\mathrm{int}(X)}$ is open and dense in $\hat{\mathcal{U}}_{\mathrm{int}(X)}$ with the relative Whitney topology. For all $u \in \hat{\mathcal{U}}_{\mathrm{int}(X)}$, Theorem 13 implies that if $x \in CC_{\mathrm{int}(X)}(u)$, then either there exist voters $j,k \in N$ such that $d^1 u(x) \in \mathcal{M}_{1,1,1}[0;0j;0k]$, or there exist $i,j,k \in N$ such that $d^1 u(x) \in \mathcal{M}_{1,1,2,2}[0;0k;0i;0ij]$, both cases implying $u \notin \hat{\mathcal{F}}_{(\mathrm{ii})}$. Contrapositively, if $u \in \hat{\mathcal{F}}_{(\mathrm{ii})}$, then there does not exist $x \in CC_{\mathrm{int}(X)}(u)$, i.e., $CC_{\mathrm{int}(X)}(u) = \emptyset$. This means that $\hat{\mathcal{F}}_{(\mathrm{ii})} \subseteq \hat{\mathcal{U}}^0_{\mathrm{int}(X)}$, establishing the first genericity result.

The second result follows the lines above, defining subsets $\mathcal{G}^{i,j,k} \subseteq C^2(\Re^d, \Re^{n+1})$ and defining $\mathcal{G}_{(\mathrm{ii})}$ as the intersection over them, so that $\mathcal{G}_{(\mathrm{ii})}$ is residual. We then let $\hat{\mathcal{G}}_{(\mathrm{ii})}$ be the differentiably concave restrictions of functions in $\mathcal{G}_{(\mathrm{ii})}$. This set is dense in $\hat{\mathcal{U}}_X$, and thus each $\mathcal{V}_m$ is open and dense, and we obtain that $\hat{\mathcal{U}}^0_X$ is residual in $\hat{\mathcal{U}}_X$, as required.

Third, assume (iii). Again, the only difference is a matter of accounting. For distinct voters $h,i,j,k \in N$, define the following two classes of matrices, in addition to those defined above:

- $\mathcal{M}_{2,2,2,2}[0h;0j;0hi;0jk]$ is the set of matrices such that: row zero and row h are linearly independent; row zero and row j are linearly independent; row zero, row h, and row i have rank two; and row zero, row j, and row k have rank two. Since row zero and row h are linearly independent, this implies that row i is a linear combination of rows zero and row h, and similarly, row k is a linear combination of row zero and row j. Thus, the set is a manifold with dimension $d + d + 2 + d + 2 + (n-4)d = (n-1)d + 4$ and codimension $(n+1)d - (n-1)d - 4 = 2d - 4$.

- $\mathcal{M}_{1,1,3,3}[0; 0k; 0hi; 0hij]$ is the set of matrices such that: row zero has at least one non-zero entry; row zero and row k have rank one; row zero, row h, and row i have rank three; and row zero, row h, row i, and row j have rank three. Since row zero is non-zero, row k is a scalar multiple of row zero, and row j is a linear combination of rows zero, h, and i. The set is a manifold with dimension $d+1+d+d+3+(n-4)d = (n-1)d+4$ and codimension $(n+1)d-(n-1)d-4 = 2d - 4$.

Again, results of Gibson et al. (1976), imply that the closures of the above manifolds, $\text{clos}(\mathcal{M}_{2,2,2,2}[0h; 0j; 0hi; 0jk])$ and $\text{clos}(\mathcal{M}_{1,1,3,3}[0; 0k; 0hi; 0hij])$, are semialgebraic, and thus the union

$$\mathcal{M}^{h,i,j,k} \;=\; \mathcal{M}^{i,j,k} \cup \text{clos}(\mathcal{M}_{2,2,2,2}[0h; 0j; 0hi; 0jk]) \cup \text{clos}(\mathcal{M}_{1,1,3,3}[0; 0k; 0hi; 0hij])$$

is semialgebraic and, in fact, $\text{int}(X) \times \Re^{n+1} \times \mathcal{M}^{h,i,j,k}$ is a Whitney stratified set.

By Theorem 7.5.11 of Jongen, Jonkers, and Twilt (2000), the set $\mathcal{F}^{h,i,j,k}$ of mappings $f \in \mathcal{U}_{\text{int}(X)}$ such that $j^1 f$ intersects $\text{int}(X) \times \Re^{n+1} \times \mathcal{M}^{h,i,j,k}$ transversally is open and dense in the Whitney topology on $\mathcal{U}_{\text{int}(X)}$. By transversality, for all $f \in \mathcal{F}^{h,i,j,k}$, the sets

$$\left\{ x \in \text{int}(X) \mid d^1 f(x) \in \mathcal{M}_{2,2,2,2}[0h; 0j; 0hi; 0jk] \right\}$$

and

$$\left\{ x \in \text{int}(X) \mid d^1 f(x) \in \mathcal{M}_{1,1,3,3}[0; 0k; 0hi; 0hij] \right\}$$

are manifolds with codimension $2d - 4$, and thus they have dimension $4 - d$. Since $d \geq 5$ by assumption (iii), we conclude that for all $f \in \mathcal{F}^{h,i,j,k}$, these sets are empty; in particular, using $\mathcal{M}^{i,j,k} \subseteq \mathcal{M}^{h,i,j,k}$, there does not exist $x \in \text{int}(X)$ such that

$$d^1 f(x) \in \begin{array}{c} \mathcal{M}_{1,1,1}[0; 0k; 0j] \cup \mathcal{M}_{1,1,2,2}[0; 0k; 0i; 0ij] \cup \\ \mathcal{M}_{2,2,2,2}[0h; 0j; 0hi; 0jk] \cup \mathcal{M}_{1,1,3,3}[0; 0k; 0hi; 0hij]. \end{array}$$

It follows that the intersection

$$\mathcal{F}_{(\text{iii})} \;=\; \bigcap_{I \subseteq N : |I| = 4} \mathcal{F}^I.$$

is open and dense, and thus $\hat{\mathcal{F}}_{(iii)} = \mathcal{F}_{(iii)} \cap \hat{\mathcal{U}}_{\text{int}(X)}$ is open and dense in $\hat{\mathcal{U}}_{\text{int}(X)}$ with the relative Whitney topology.

For all $u \in \hat{\mathcal{U}}_{\text{int}(X)}$, Theorem 14 implies that if $x \in CC_{\text{int}(X)}(u)$, then either (i) there exist voters $h, i, j, k \in N$ such that $d^1u(x) \in \mathcal{M}_{2,2,2,2}[0h; 0j; 0hi; 0jk]$, or (ii) there exist $h, i, j, k \in N$ such that $d^1u(x) \in \mathcal{M}_{1,1,1}[0; 0k; 0j] \cup \mathcal{M}_{1,1,2,2}[0; 0k; 0i; 0ij] \cup \mathcal{M}_{1,1,3,3}[0; 0k; 0hi; 0hij]$, both cases implying $u \notin \hat{\mathcal{F}}_{(iii)}$. Contrapositively, if $u \in \hat{\mathcal{F}}_{(iii)}$, then there does not exist $x \in CC_{\text{int}(X)}(u)$, i.e., $CC_{\text{int}(X)}(u) = \emptyset$. This means that $\hat{\mathcal{F}}_{(iii)} \subseteq \hat{\mathcal{U}}^0_{\text{int}(X)}$, establishing the first genericity result. The second result follows along the lines above.

Fourth, assume (iv). Given a group $G \subseteq N \cup \{0\}$ with $|G| = r$, let $\mathcal{M}(G)$ denote the matrices with row rank r such that rows $i \in G$ are linearly independent. This implies that each row $j \notin G$ is a linear combination of rows $i \in G$. Thus, the set $\mathcal{M}(G)$ is a manifold of dimension $rd+(n+1-r)r$ and codimension $(n+1)d-rd-(n+1-r)r = (n+1-r)(d-r)$. It is straightforward to show that $\mathcal{M}(G)$ is semialgebraic, and it follows that the class $\mathcal{M}[r] = \bigcup_{G:|G|=r} \mathcal{M}(G)$ of matrices with row rank r is semialgebraic. Then result (2.3) of Gibson et al. (1976) implies that the closure $\mathcal{M}^r = \text{clos}(\mathcal{M}[r])$ is also semialgebraic, and in fact, $\text{int}(X) \times \Re^{n+1} \times \mathcal{M}^r$ is a Whitney stratified set. By Theorem 7.5.11 of Jongen, Jonkers, and Twilt (2000), the set $\mathcal{F}^r$ of mappings $f \in \mathcal{U}_{\text{int}(X)}$ such that $j^1 f$ intersects $\text{int}(X) \times \Re^{n+1} \times \mathcal{M}^r$ transversally is open and dense in the Whitney topology on $\mathcal{U}_{\text{int}(X)}$. By transversality, for all $r = 1, \ldots, \underline{m}$ and all $f \in \mathcal{F}^r$, the set

$$\left\{ x \in \text{int}(X) \mid d^1 f(x) \in \mathcal{M}[r] \right\}$$

is contained in a union of manifolds, each with codimension greater than or equal to $(n + 1 - \underline{m})(d - \underline{m})$. Note that $(n + 1 - \underline{m})(d - \underline{m}) > d$ holds if and only if $d > \underline{m} + \frac{\underline{m}}{n-\underline{m}}$, which holds by assumption (iv). Thus, we conclude that for all $r = 1, \ldots, \underline{m}$ and all $f \in \mathcal{F}^r$, the above set is empty; in particular, there do not exist $x \in \text{int}(X)$ and $r = 1, \ldots, \underline{m}$ such that $d^1 f(x) \in \mathcal{M}[r]$. It follows that the intersection $\mathcal{F}_{(iv)} = \bigcap_{r=1}^{\underline{m}} \mathcal{F}^r$ is open and dense, and thus $\hat{\mathcal{F}}_{(iv)} = \mathcal{F}_{(iv)} \cap \hat{\mathcal{U}}_{\text{int}(X)}$ is open and dense in $\hat{\mathcal{U}}_{\text{int}(X)}$ with the relative Whitney topology. For all $u \in \hat{\mathcal{U}}_{\text{int}(X)}$, Theorem 15 implies that if $x \in CC_{\text{int}(X)}(u)$, then $d^1u(x) \in \mathcal{M}[r]$ for some $r = 1, 2, \ldots, \underline{m}$. Contrapositively, if $u \in \hat{\mathcal{F}}_{(iv)}$, then there does not exist $x \in CC_{\text{int}(X)}(u)$, i.e., $CC_{\text{int}(X)}(u) = \emptyset$. This means that $\hat{\mathcal{F}}_{(iv)} \subseteq \hat{\mathcal{U}}_{\text{int}(X)}$, establishing the first genericity result. The second result follows along the lines above. □

Proof of Theorem 18: The proof builds off the proof of Theorem 17. We extend notation for matrices as follows: given groups $G_1, \ldots, G_k \subseteq N \cup \{0\}$, natural numbers $r_1, \ldots, r_k$, vector $x \in \Re^d$, and a set $L \subseteq K$, let $\mathcal{M}^L_{r_1,\ldots,r_k}[x|G_1; \ldots, ; G_k]$ denote the set of matrices such that: for all $j = 1, \ldots, k$, the rows corresponding to members of G_j together with the gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $r_j + |L|$.

We consider the case of n even, inclusive majority rule in some detail and give further remarks to indicate the lines of argument for other voting rules. Given L with $|L| < d - d^* = d - 2$ and $x \in F(L)$, we define the class $\mathcal{M}^L_{1,1,1}[x|0;0i;0j]$ of matrices such that: row zero together with the gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; rows zero and i together with the gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; and rows zero and j together with the gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$. Define the set

$$W^L_{i,j} \;\; = \;\; \{(x, y, M) \mid x \in F(L), y \in \Re^{n+1}, M \in \mathcal{M}^L_{1,1,1}[x|0;0i;0j]\},$$

a manifold of dimension

$$\begin{aligned} \dim(W^L_{i,j}) \;\; &= \;\; (d - |L|) + (n+1) + (d + 1 + |L|) + (1 + |L|) + (n + 1 - 3)d \\ &= \;\; d + n + 1 + (n+1)d - 2d + |L| + 2 \end{aligned}$$

and codimension

$$\operatorname{codim}(W^L_{i,j}) \;\; = \;\; d + n + 1 + (n+1)d - \dim(W^L) \;\; = \;\; 2d - |L| - 2.$$

Given any $f \in C^2(\Re^d, \Re^{n+1})$, recall that the 1-*jet* of f is the mapping $j^1 f \colon \Re^d \to \Re^{d+(n+1)+(n+1)d}$ defined by

$$j^1 f(x) \;\; = \;\; (x, f(x), d^1 f(x)).$$

By the jet transversality theorem, the set $\mathcal{F}^L_{i,j}$ of mappings such that $j^1 f \pitchfork W^L_{i,j}$ is a residual subset of $C^2(\Re^d, \Re^n)$ with the Whitney topology, as is the intersection $\mathcal{F}^L$ of these sets over pairs of voters. Let $\hat{\mathcal{G}}^L = \{f|_X \mid f \in \mathcal{F}^L\} \cap \hat{\mathcal{U}}_X$ be the set of differentiably concave restrictions to X of functions in $\mathcal{F}^L$, a set that is dense in $\hat{\mathcal{U}}_X$ with the relative topology of C^2-uniform convergence.

Let $Y = \bigcup_{L' : L \subsetneq L'} F(L')$ be the union of lower-dimensional faces adjacent to $F(L)$. For each natural number m, let $K_m = X \setminus B_{\frac{1}{m}}(Y)$ be the alternatives that are at least a distance $\frac{1}{m}$ from Y, so that K_m is compact. Define the set

$$\mathcal{V}_m \;\; = \;\; \{u \in \hat{\mathcal{U}}_X \mid TC_X(u) \cap F(L) \cap K_m = \emptyset\}$$

of vector utility functions such that if there is a tangent core point belonging to $F(L)$, then such alternatives are within a distance of $\frac{1}{m}$ of the "corners" of $F(L)$. Note that $\hat{\mathcal{U}}^L_X = \bigcap_{m=1}^\infty \mathcal{V}_m$. Furthermore, each $\mathcal{V}_m$ is open in the relative topology of C^2-uniform convergence. For denseness, we show that $\hat{G}^L \subseteq \hat{\mathcal{U}}^L_X$. Indeed, consider any $u \in \hat{\mathcal{G}}$, so there exists $f \in \mathcal{F}^L$ with $u = f|X$. From $f \in \mathcal{F}^L$, it follows that for all distinct voters $i, j \in N$, the set

$$\{x \in \Re^d \mid j^1 f(x) \in W^L_{i,j}\}$$

is a manifold with codimension $2d - |L| - 2$. Since $|L| < d - 2$, we have $2d - |L| - 2 > d$, so the above set is empty. If there were an alternative $x \in TC_X(u) \cap F(L) \cap$

K_m, then Theorem 16 would yield voters $i, j \in N$ such that $d^1 f(x) = d^1 u(x) \in \mathcal{M}^L_{1,1,1}[x|0; 0i; 0j]$, which is impossible. Therefore, $TC_X(u) \cap F(L) \cap K_m = \emptyset$, i.e., $u \in \mathcal{V}_m$. We conclude that $\hat{\mathcal{U}}^L_X$, as the intersection of sets that are open and dense, is residual in $\hat{\mathcal{U}}_X$ with the relative topology of C^2-uniform convergence.

For n odd, majority rule, we focus here on the set $\mathcal{M}^L_{1,1,2}[x|0; 0k; 0ij]$ of matrices such that: row zero together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; rows zero and k together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; and rows zero, i, and j together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $2 + |L|$. The manifold

$$W^L_{i,j,k} \;=\; \{(x, y, M) \mid x \in F(L), y \in \Re^{n+1}, M \in \mathcal{M}^L_{1,1,2}[x|0; 0k; 0ij]\}$$

has dimension

$$\begin{aligned} \dim(W^L_{i,j,k}) &= (d - |L|) + (n + 1) + (d + 1 + |L|) + (2 + |L|) + (n + 1 - 3)d \\ &= d + n + 1 + (n + 1)d - 2d + |L| + 3 \end{aligned}$$

and codimension $\operatorname{codim}(W^L_{i,j}) = d + n + 1 + (n + 1)d - \dim(W^L) \;=\; 2d - |L| - 3$. This exceeds d by assumption $|L| < d - d^*$, and the above arguments can be applied.

For n even, exclusive majority rule, let $\mathcal{M}^L_{1,2,2}[x|0; 0hi; 0jk]$ be the matrices such that: row zero together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; rows zero, h, and i together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $2 + |L|$; and rows zero, j, and k together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $2 + |L|$. The manifold

$$W^L_{h,i,j,k} \;=\; \{(x, y, M) \mid x \in F(L), y \in \Re^{n+1}, M \in \mathcal{M}^L_{1,2,2}[x|0; 0hi; 0jk]\}$$

has dimension $\dim(W^{L,a}_{i,j,k}) = d + n + 1 + (n + 1)d - 2d + |L| + 3$ and codimension $\operatorname{codim}(W^L_{i,j}) = 2d - |L| - 4$. Also, let $\mathcal{M}^L_{1,1,3}[x|0; 0k; 0hij]$ be the matrices such that: row zero together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; rows zero and k together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $1 + |L|$; rows zero, h, i, and j together with gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $3 + |L|$. The manifold

$$W^{L,b}_{h,i,j,k} \;=\; \{(x, y, M) \mid x \in F(L), y \in \Re^{n+1}, M \in \mathcal{M}^L_{1,1,3}[x|0; 0k; 0hij]\}$$

also has dimension $\dim(W^L_{i,j,k}) = d + n + 1 + (n + 1)d - 2d + |L| + 3$ and codimension $\operatorname{codim}(W^L_{i,j}) = 2d - |L| - 4$. This exceeds d by assumption $|L| < d - d^*$, and the above arguments can be applied.

Finally, for a general non-collegial voting rule, let $\mathcal{M}^L[x|r]$ be the matrices M such that the rows of M together with the gradients $\nabla f^\ell(x)$, $\ell \in L$, have rank $r + |L|$. When $r = \underline{m}$, the main case of interest here, the manifold

$$W^L_{\underline{m}} \;=\; \{(x, y, M) \mid x \in F(L), y \in \Re^{n+1}, M \in \mathcal{M}^L[x|\underline{m}]\}$$

has dimension

$$\dim(W^L_{\underline{m}}) \;=\; (d-|L|)+(n+1)+\underline{m}d+(n+1-\underline{m})(\underline{m}+|L|)$$

and codimension

$$\begin{aligned}\operatorname{codim}(W^L_{i,j}) &= d+n+1+(n+1)d-\dim(W^L)\\ &= (n+1-\underline{m})(d-\underline{m}-|L|).\end{aligned}$$

This exceeds d if and only if

$$d \;>\; \underline{m}+|L|+\frac{\underline{m}+|L|}{n-\underline{m}},$$

consistent with the case $L=\emptyset$ in Theorem 17, or equivalently, if and only if $|L| < d-d^*$, which holds by assumption. □

A.4 Detailed example of convergence

Assume $n=2$ and $m=1$, so that the support of either voter 1 or voter 2 is sufficient to for a proposal to pass. Assume that the contract curves for $\{0,1\}$ and $\{0,2\}$ intersect at $\tilde{x}$; assume that in an open set G containing $\tilde{x}$, this intersection is unique; and assume that both voters strictly prefer the status quo to $\tilde{x}$, consistent with Theorem 10, and that the agenda setter has the opposite preference. We will show that there is a sequence of stationary bargaining equilibria in non-degenerate proposal strategies that converges to $\tilde{x}$ as the voters become patient, and this will be demonstrated in a somewhat constructive way. Let y denote an alternative on the contract curve for $\{0,1\}$, and given y, let z denote the alternative on the contract curve for $\{0,2\}$ that makes the agenda setter indifferent, i.e., $u_0(y)=u_0(z)$. Assume without loss of generality that the agenda setter weakly prefers voter 1's ideal point, i.e., $u_0(\hat{x}^1)\geq u_0(\hat{x}^2)$, so when $y=\hat{x}^1$, the agenda setter is indifferent between y and some alternative $\hat{z}$ on the contract curve for $\{0,2\}$. For simplicity, assume that voter 2 weakly prefers $\hat{z}$ to q, which can be interpreted as saying the status quo is not too bad for the agenda setter.

We know that in an equilibrium with non-trivial mixing, the agenda setter mixes with some probability, say $\eta>0$, on an alternative y and remaining probability, $1-\eta>0$, on z such that: y is on the contract curve for $\{0,1\}$, z is on the contract curve for $\{0,2\}$, and $u_0(y)=u_0(z)$. In addition, these proposals must make the corresponding voters indifferent between acceptance and rejection. For voter 1, this means

$$u_1(y) \;=\; (1-\delta)u_1(q)+\delta[\eta u_1(y)+(1-\eta)u_1(z)],$$

and solving for η, we obtain

$$\eta \;=\; 1-\frac{(1-\delta)(u_1(q)-u_1(y))}{\delta(u_1(y)-u_1(z))}.$$

For voter 2, this means

$$u_2(z) \;=\; (1-\delta)u_2(q) + \delta[\eta u_2(y) + (1-\eta)u_2(z)],$$

so that η also satisfies

$$\eta \;=\; \frac{(1-\delta)(u_2(q)-u_2(z))}{\delta(u_2(z)-u_1(y))}.$$

This analysis implies that the indifference condition holds for both voters when y (and thus $z \neq y$) is chosen so that

$$\frac{(1-\delta)(u_1(q)-u_1(y))}{\delta(u_1(y)-u_1(z))} + \frac{(1-\delta)(u_2(q)-u_2(z))}{\delta(u_2(z)-u_1(y))} \;=\; 1,$$

or equivalently,

$$\frac{u_1(q)-u_1(y)}{u_1(y)-u_1(z)} + \frac{u_2(q)-u_2(z)}{u_2(z)-u_1(y)} \;=\; \frac{\delta}{1-\delta}. \tag{13}$$

Given $\delta \in (0,1)$, there is a choice of y (and thus $z \neq y$) that solves the above equation. Indeed, as $y \to \tilde{x}$, our assumption that the contract curves intersect uniquely at $\tilde{x}$ implies that $z \to \tilde{x}$, and thus, the left-hand side of the equation diverges to infinity; and as $y \to \hat{x}_1$, we have $z \to \hat{z}$, and thus the left-hand side converges to a negative quantity. We conclude that there is a mixed strategy equilibrium in which the agenda setter mixes between appropriately chosen y and z.

Now, let $\delta \to 1$, so that for given y and z, the right-hand side of (13) becomes arbitrarily large. In equilibrium, the left-hand side becomes commensurately large, and thus we have $|y-z| \to 0$, which implies $y \to \tilde{x}$ and $z \to \tilde{x}$, demonstrating the power of the agenda setter as voters become patient. In case $\tilde{x} \neq \hat{x}^0$, this finding is consistent with the result of Theorem 9. Furthermore, it shows that in a simple environment, the property of being a constrained core point with respect to the agenda setter's gradient is sufficient for an alternative to be the limit of equilibrium proposal strategies as the voters become patient. Thus, general necessary conditions in addition to those of Theorem 9 are not immediately forthcoming.

References

[1] V. Anesi and J. Duggan (2016) "Dynamic Bargaining and Stability with Veto Players," *Games and Economic Behavior,* forthcoming.

[2] J. Banks (1995) "Singularity Theory and Core Existence in the Spatial Model," *Journal of Mathematical Economics,* 24: 523–536.

[3] J. Banks and J. Duggan (2000) "A Bargaining Model of Collective Choice," *American Political Science Review,* 94: 73–88.

[4] J. Banks and J. Duggan (2006) "A General Bargaining Model of Legislative Policy-making," *Quarterly Journal of Political Science,* 1: 49–85.

[5] D. Baron and J. Ferejohn (1989) "Bargaining in Legislatures," *American Political Science Review,* 83: 1181–1206.

[6] S.-J. Cho and J. Duggan (2003) "Uniqueness of Stationary Equilibria in a One-dimensional Model of Bargaining," *Journal of Economic Theory,* 113: 118–130.

[7] D. Diermeier and P. Fong (2011) "Legislative Bargaining with Reconsideration," *Quarterly Journal of Economics,* 126: 947–985.

[8] D. Diermeier and P. Fong (2012) "Characterization of the von Neumann-Morgenstern Stable Set in a Non-cooperative Model of Dynamic Policy-making with a Persistent Agenda Setter," *Games and Economic Behavior,* 76: 349–353.

[9] J. Duggan (2014) "Majority Voting over Lotteries: Conditions for Existence of a Decisive Voter," *Economics Bulletin,* 34: 263–270.

[10] J. Duggan (2017) "Existence of Stationary Bargaining Equilibria" (2017) *Games and Economic Behavior,* 102: 111–126.

[11] C. Gibson, K. Wirthmüller, A. de Plessis, and E. Looijenga (1976) *Topological Stability of Smooth Mappings,* Lecture Notes in Mathematics 552, Springer: New York.

[12] M. Hirsch (1976) *Differentiable Topology,* Springer: New York, NY.

[13] H. Jongen, P. Jonker, and F. Twilt (2000) *Nonlinear Optimization in Finite Dimensions,* Nonconvex Optimization and Its Applications 47, Springer Science+Business Media: Dordrecht.

[14] T. Kalandrakis (2004) "A Three-player Dynamic Majoritarian Bargaining Game," *Journal of Economic Theory,* 116: 294–322.

[15] T. Kalandrakis (2010) "Minimum Wining Coalitions and Endogenous Status Quo," *International Journal of Game Theory,* 39: 617–643.

[16] A. Mas-Colell (1985) *The Theory of General Economic Equilibrium: A Differentiable Approach,* Cambridge University Press: New York, NY.

[17] R. McKelvey (1976) "Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control," *Journal of Economic Theory,* 12: 472–482.

[18] R. McKelvey (1979) "General Conditions for Global Intransitivities in Formal Voting Models," *Econometrica,* 47: 1085–1112.

[19] C. Plott (1967) "A Notion of Equilibrium and Its Possibility under Majority Rule," *American Economic Review,* 57: 787–806.

[20] D. Primo (2002) "Rethinking Political Bargaining: Policymaking with a Single Proposer," *Journal of Law, Economics, and Organization,* 18: 411–427.

[21] T. Romer and H. Rosenthal (1978) "Political Resource Allocation, Controlled Agendas, and the Status Quo," *Public Choice,* 33: 27–43.

[22] Schofield (1978) "Instability of Simple Dynamic Games," *Review of Economic Studies,* 45: 575–594.

[23] Schofield (1983) "Generic Instability of Majority Rule," *Review of Economic Studies,* 50: 695–705.