Political Competition and Strategic Voting in Multi-Candidate Elections

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Abstract

We develop a model of strategic voting in a spatial model with multiple candidates when voters have both expressive and instrumental concerns. The model endogenizes the strategic coordination of voters, yet is flexible enough to allow the analysis of political platform competition by policy-motivated candidates. We fully characterize all strategic voting equilibria in a three-candidate setting. The result upends the standard calculus both for models with purely sincere voters and those where voters have only instrumental concerns, i.e., where voters solely care about pivotality. Highlighting the utility of our approach, we analyze a setting with the two mainstream and a spoiler candidate, showing that the spoiler can be made better off from entering, even though she has no chance of winning the election and reduces the winning probability of her preferred mainstream candidate.

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1 Introduction

There is manifest evidence that voters have both expressive and instrumental voting considerations, that is, they both receive a direct payoff from voting for a particular candidate, and they care about who wins the election both in real world election data (e.g., Spenkuch (2018); Fujiwara (2011); Pons and Tricaud (2018)) and in laboratory experiments (e.g., Bouton et al. (2015); Duffy and Tavits (2008); Esponda and Vespa (2014); Forsythe et al. (1996); Van der Straeten et al. (2010) and the survey in Palfrey (2009)). In a two-candidate election there is no conflict between these two objectives: voting *for* a preferred candidate is equivalent to voting *against* a disfavored candidate, so there is no reason to vote strategically. This calculus changes when there are more than two candidates because expressive and instrumental concerns can now easily be mis-aligned, giving rise to strategic voting.

The contribution of this paper is to develop a tractable spatial model with voters who have both expressive and instrumental concerns that allows for voters to coordinate strategically when there are more than just two candidates. In order to integrate political competition between policy-motivated candidates, the model must allow for a continuum of citizens. The formulation must also be rich enough to characterize how the primitives of the political environment affect the possible natures of strategic coalition formation. In particular, one wants to describe how changes in candidate policies and characteristics affect the incidence of strategic voting and electoral outcomes.

The standard approach in the strategic voting literature (e.g., Myatt (2007); Bouton (2013); Bouton and Gratton (2015); Xefteris (2019)) is to focus on a finite number of strategic voter types who individually only care about how their voting decision affects a candidate's probability of winning. Such pivotal voting models have a long tradition (e.g., Palfrey and Rosenthal (1985); Myerson and Weber (1993); Kawai and Watanabe (2013)), but they do not work in large elections when the strategic voters have any expressive considerations at all, as such considerations swamp the tiny pivot probabilities. Further, with our required continuum of voters, pivotal probabilities are always zero. To skirt this issue, some authors assume that voters' perceived pivot probabilities are exogenous and large (e.g., Chapter 1, Aldrich et al. (2018)). While this approach can accommodate both instrumental and expressive voter concerns, it cannot provide causal links between policy platforms and pivot probabilities.

We take a different direction that circumvents all of these issues. To do this, we borrow concepts from the ethical voting literature (see Feddersen and Sandroni (2006)), which formalizes the idea that some voters internalize the welfare of similarly-situated citizens and hence do not want to free-ride on the voting efforts of other "group" members (Goodin and Roberts, 1975). We must modify this approach to account for two fundamental differences. First, ethical voting models are designed to explain voter participation in large elections when participation is costly and pivot probabilities are small, so they suppose that group members only differ in their voting costs. Thus, citizens only need to decide whether or not to vote for their most preferred candidate. Ethical voting considerations mean that citizens incur psychic costs of free riding on the efforts of group members who incur the cost of voting.[1] In contrast, in our model, support for a particular candidate is necessarily determined endogenously, so we have to model *how* voting coalitions form, i.e., who joins a group of citizens to vote strategically for a less-preferred candidate in order to raise the probability of defeating an even less-preferred candidate. Our model shares the feature of ethical voting models that there is a benefit of voting for a certain candidate if others in the group also do so, and that a psychic cost of letting the group down prevents free riding.

We embed our model of strategic voting in a two-stage game and show how it affects candidate platform competition. First, policy-motivated candidates choose their platforms in a spatial setting, and then citizens decide which strategic voting coalitions (if any) to form. There are two types of voters, partisans who always vote on party lines, and non-partisans whose votes hinge on candidate policies and what they think other voters will do. A nonpartisan's utility is given by the weighted sum of the expressive and instrumental payoffs, i.e., of the expressive payoff from voting for a given candidate and the instrumental payoff derived from the winning candidate's policy. When candidates choose policies they are uncertain about the level of each candidate's partisan support. Partisan support becomes public information before nonpartisans choose whether and how to vote strategically. We impose the following structure on strategic coalition formation. First, individual rationality mandates that each coalition member must be better off in expectation if the coalition forms than if all members voted expressively. Second, unaffiliated citizens should not be able to make themselves better off by joining an existing coalition. Finally, equilibrium demands that no additional coalition can form that would make all members of the new coalition better off.

We characterize all equilibria to this voting subgame in a setting with three candidates who have arbitrary policy platforms. We first establish ways in which the centrist has an electoral advantage. If the centrist is preferred by a plurality of partisan and expressive voters, then there is a unique pure strategy equilibrium, and in this equilibrium, all citizens vote expressively and the centrist wins. Intuitively, expressive centrist supporters get their most preferred candidate and hence will not vote strategically, and strategic voting by expressive non-centrist supporters only makes sense if it can influence electoral outcomes for the better.

This is not true for a non-centrist with an initial plurality advantage. This non-centrist requires a sufficient plurality advantage to win in a pure strategy equilibrium. For example, when the centrist's base support disadvantage is small enough, the centrist can always win in a pure strategy equilibrium by attracting strategic support from voters on the other side of the political spectrum. Such voters would always be willing to

[1] Herrera and Martinelli (2006a) also borrow this notion to endogenize the costly choice to become a vote leader to coordinate like-minded voters with two candidates. See also Ali and Lin (2013); Levine and Mattozzi (2020). Bouton and Ogden (2021) use this notion in a setting with three candidates and three types of voters, where random turnout is high enough that each candidate can win, and type *A* and *B* voters share a common dispreference for candidate *C*, creating a role for strategic voting.

coordinate strategically to overcome the centrist's initial base support shortfall, voting for the centrist against their expressive interests. So, too, if the rival non-centrist's base support disadvantage is sufficiently small that enough expressive centrist supporters would be willing to coordinate on the rival to overcome the initial shortfall, then the initially-advantaged non-centrist cannot win in a pure strategy equilibrium. Strategic voting only makes sense if it can influence the electoral outcome for the better, so the initially-advantaged non-centrist can never attract strategic support in a pure strategy equilibrium. It follows that the non-centrist with an initial plurality advantage can only win in a pure strategy equilibrium when the potential maximum strategic support for each of the other two candidates is less than that advantage. If the non-centrist has such a large advantage, then all citizens vote for their expressively-preferred candidates, and the winning margin is necessarily non-trivial.

When the non-centrist's initial plurality advantage over the rival non-centrist candidate is smaller, one of two possible strategic voting equilibria will form in which the rival non-centrist either sometimes or always wins. First, it may be that the centrist's and non-centrist's policies are sufficiently similar relative to the initially-advantaged non-centrist's distant policy. In this case, the rival non-centrist may be a sufficiently close substitute for the centrist that expressive centrist supporters would vote strategically for the rival non-centrist in such numbers that the that the rival receives more votes than the initially-advantaged non-centrist could possibly attract. In the resulting pure strategy equilibrium, due to strategic voting, the rival non-centrist necessarily wins by a non-trivial margin that exceeds the maximum possible strategic support that the initially-advantaged non-centrist could attract from expressive centrist supporters. Thus, the model identifies (possibly rare) conditions under which strategic voting is necessarily associated with large winning margins.

When these demanding conditions do not hold, equilibrium is in mixed strategies where expressive centrist supporters on each side coordinate stochastically to vote for the non-centrist candidates. Stochastic coordination on both sides is required because a citizen will only vote against her expressive preferences if her coalition has a positive probability of influencing the electoral outcome. The game and the equilibrium share similarities with all-pay auctions. For example, we show that with strictly positive probability there is no strategic voting on each side, and both non-centrists have to be able to win with strictly positive probability. This means that even if there is no exogenous uncertainty about expressive support, strategic voting may have to generate endogenous uncertainty about who wins and the size of the winning margin. We also show that these mixed strategy equilibria are well-behaved. When one non-centrist has an initial plurality advantage, the rival "underdog" non-centrist's maximum potential strategic support is always greater. Further, a larger initial plurality implies a reduction of strategic voting on both sides. The equilibrium also behaves continuously as we approach the boundary cases of either purely instrumental or purely expressive voting.

We show how strategic voting can sharply alter the logic and comparative statics of voting. Consider an election with two main candidates, symmetrically situated around the median, and a spoiler candidate on the far right. With purely expressive voting, the center-right candidate would reduce her vote share by moving toward the spoiler. With strategic voting this need not be true. In particular, as the center-right candidate moves toward the spoiler she (i) better differentiates herself from the center-left candidate, raising the benefits of strategic coordination by voters; and (ii) is politically closer to the spoiler, lowering the costs of strategic coordination. By facilitating strategic voting in this way, the center-right candidate can actually increase her vote share.

To illustrate the political campaigning consequences of this observation, we endogenize the positions of the center-left and center-right candidates as a function of the spoiler's policy platform. When voters place little weight on expressive concerns, the spoiler's entry has no impact on equilibrium outcomes, as even extreme expressive supporters vote strategically. However, when voters' weights on expressive benefits are increased beyond this point, both of the main candidates move their policy positions towards the spoiler, and we show that the spoiler is strictly better off from entering. In equilibrium, the spoiler's preferred mainstream candidate is less likely to win due to the spoiler's entry (consistent with Pons and Tricaud (2018)), but this is outweighed from the spoiler's perspective by the benefits of the policy shifts in the platforms of both mainstream candidates. Further increases of the weight on expressive benefits eventually harm the spoiler, because the spoiler takes too many votes away from the spoiler's preferred mainstream candidate.

Our model can reconcile the entry of a spoiler candidate such as Ralph Nader who has no chance of winning, even though the spoiler does not receive ego rents. Our model shows that with strategic voting spoilers can gain by having mainstream candidate adopt some of their policies. For example, in an interview in 2019 in the Washingtonian[2], Ralph Nader indicate that he had hoped to "push the Democrats toward a more progressive agenda," understanding that his campaign could end up costing Al Gore the presidency. Similarly, the entry of populist in Europe has encouraged mainstream parties to move to the right, with center-right parties moving further.

1.1 Related Literature

There is a large literature that tries to formalize the idea of ethical voting, An early formalization is Harsanyi (1977), who postulates that some voters are "rule utilitarians," i.e., they receive utility for choosing a strategy that maximizes social welfare, assuming that a socially inferior candidate receives a fixed fraction of votes. Feddersen and Sandroni (2006) remove this assumption, endogenizing support for both candidates by introducing preference diversity into the Harsanyi framework while preserving Harsanyi's Kantian calculus of duty. Coate and Conlin (2004) consider rules that maximize the group's welfare, if adopted by all ethical

[2]see `https://www.washingtonian.com/2019/11/03/ralph-nader-is-opening-up-about-his-regrets/`, Ralph Nader Is Opening Up About His Regrets, Rob Brunner, November 3, 2019

voters in the group. Levine and Mattozzi (2020) model group voting that complements ethical voting by stipulating that social norms of voting participation must be enforced through costly peer monitoring and punishment. In contrast to our model, the groups are ex-ante clearly defined and within-group differences are only non-ideolgoical (e.g., voting costs).

Our framework abstracts from the details of how exactly groups come to form. One foundation for group formation can stem from leaders who mobilize groups such as unions, churches, or environmental organizations, as proposed by Uhlaner (1989), and formalized by Morton (1987),Morton (1991). Herrera and Martinelli (2006b) introduce a model of group mobilization in which both leaders and groups are endogenous.

Beginning with Duverger (1959), researchers have noted that the winner-take-all system generates strong forces for only two parties to be competitive, as voters who care about electoral outcomes do not want to "waste" their votes on candidates who are sure to lose. Cox (1997) observes that the "reduction of parties" in single member districts reflects the coordination of voters on parties. Applied to the United Kingdom, supporters of the LDP and Labour may have incentives to coordinate to defeat the Conservatives (c.f., Aldrich et al. (2018), Figure 2.2). Similar coordination can be found in Canada (Merolla and Stephenson, 2007), and elsewhere (Blais et al., 2019). The extent of strategic voting can be substantial: Abramson et al. (2018) finds that the incidence of strategic voting—voting for a second-choice candidate—was almost 40% in some constituencies for the 2010 British general election, while Daoust et al. (2018) finds that 22.6% of voters selected their second-choice in the 2015 Canadian general election. Daoust et al. (2018) also illustrates what Cox describes as the challenge of voter strategic coordination, and "the rapidity with which vote intentions change when coordination takes off", with the NDP's polling share falling from 37% to 20% in the last two months of the campaign, with two-thirds of the drop occurring in the last month and the Liberals winning a majority due to this shift. Spenkuch (2018) provides evidence that "voters cannot be neatly categorized into sincere and strategic "types", and that voters, instead, weigh both expressive and instrumental voting considerations in their choices of whether or not to vote strategically.[3] Consistent with this, Abramson et al. (2018) shows that the extent of strategic voting varies with the perceived ability to sway the outcome: while only 1.4% of voters with minimal strategic incentives reported an intention to vote for their second-choice party, it was 27.1% of those with the strongest incentives.[4]

These empirical findings suggest the following key features of voting in multi-candidate elections:

1. Voters sometimes coordinate strategically to try to defeat a less preferred candidate.

2. Voters trade off expressive and instrumental voting considerations, and are more likely to vote strate-

[3]The observation that voters have both expressive and instrumental voting considerations dates back to at least Riker and Ordeshook (1968).

[4]Palfrey (2009) summarizes extensive evidence of strategic voting in experimental laboratory settings.

gically if the chance of changing the electoral outcome is higher.

3. Coordination can be difficult and can quickly generate large shifts in candidate support, suggesting that there can be multiple ways to coordinate, and that small changes in candidate strengths may shift coordination from one candidate onto another.

4. Duverger's law says that only two candidates are competitive, but does not imply that only two candidates receive meaningful vote shares.

We develop a formal theory of strategic voting that is rich enough to generate these salient features, and yet is sufficiently tractable to permit the analysis of political competition by policy-motivated candidates who understand how platform choices affect strategic voting.

2 Model

There are $i = 1, \dots, n$ parties/candidates and a continuum of citizens. To illustrate that our model of strategic voting can be applied to analyze political competition in platforms, we consider a two-stage game. In the first stage a subset of candidates choose policy positions $x_i \in \mathbb{R}$, while the positions of the remaining candidates may be fixed. The main focus of this paper is on the second stage, where citizens choose which candidate to support given the candidates' policy positions. The candidate with the plurality of votes wins. We assume, solely for simplicity, that candidates are purely policy motivated: candidate i with ideology θ_i cares only about the policy adopted by the winning candidate, x_w, obtaining payoff $v_{\theta_i}(x_w) = -(\theta_i - x_w)^2$.

We formulate strategic voting in a setting with n candidate choices to illustrate its general nature. We then specialize to three candidates to provide an exhaustive characterization of strategic voting equilibria. The voting game features two types of citizen voters, partisans and non-partisans. A candidate i partisan always votes for i regardless of the policy positions taken. Thus, the partisan support for a candidate i is summarized by the number $\rho_i \geq 0$ of i's partisan supporters. Non-partisan voters have both expressive and instrumental voting considerations. The utility of a non-partisan voter with ideology $\theta \in \mathbb{R}$ who votes for a candidate with policy x when the candidate with policy x_w wins is given by

$$u_\theta(x, x_w) = \beta v_\theta(x) + (1 - \beta) v_\theta(x_w),$$

where $0 \leq \beta < 1$ measures the weight placed by non-partisan voters on expressive relative to instrumental considerations. The ideal policies θ of non-partisans are distributed according to $\Phi(\cdot)$. We assume that the distribution Φ has a density ϕ that is symmetric and single-peaked at zero. In the second stage, before citizens vote, the levels ρ_i of partisan support for each party i are realized and observed by all citizens. That is, to highlight how strategic voting can generate endogenous uncertainty about who wins, we assume away all extrinsic sources of uncertainty at the voting stage.

3 Strategic Non-partisan Voting

3.1 Motivation

Non-partisan voters with an expressive preference for a candidate i can coordinate their voting behavior and commit to vote for some candidate $j \neq i$ if it is beneficial for them to do so. We say that such non-partisan voters are "strategic voters". There are three conceptual issues with describing strategic voting in a large electorate. First, each individual voter has a negligible impact on the electoral outcome, so absent other considerations, voters would always support their expressively-preferred candidates. Thus, when describing strategic voting in such an environment one must consider groups or coalitions of strategic voters who coordinate in some way to increase a candidate's chance of being elected. Second, given that coalitions rather than individuals matter in such an environment, voters may have incentives to convince others to vote strategically, but then free ride and vote for their expressively preferred candidates. Third, we must describe which voter coalitions can reasonably form and would be robust to both defections and solicitation of additional members.

The idea that voters incur costs for not doing their civic duty was first formalized in Riker and Ordeshook (1968), who propose that some voters effectively incur an exogenous fixed cost from failing to do their civic duty of voting. Feddersen and Sandroni (2006) endogenize this cost, recognizing that some individuals have higher voting costs than others, and hence should not incur a utility penalty from not voting, while citizens with low voting costs should participate and feel an ethical cost from free-riding on the voting efforts of others.[5] As far as strategic voting is concerned, it does not make sense to appeal to a "civic duty" of voting strategically. Instead, individual voters whose group benefits from strategic voting should feel guilty if they free ride by voting expressively rather than in the interest of their (endogenously determined) fellow coalition members. This is exactly the feature of Feddersen and Sandroni (2006) that we adopt for our model: Instead of incurring exogenous costs of participating in elections, our strategic voters incur endogenously-determined "costs" from voting against their expressive preferences.

The remaining issue is to characterize the strategic coalitions that can form in equilibrium. In practice this can be done by candidates' get-out-the-vote efforts, via social media, or leaders coordinating particular voter blocs. An example of the latter is the ongoing effort of the Strategic Voting Project developed in 2008 by Hisham Abdel-Rhaman, a software engineer who sought to coordinate progressive voters in Canada in each electoral riding on either the NDP or the Liberal candidate with the best chance of defeating the Conservative candidate (see http://www.strategicvoting.ca). This coordination can also happen from voters evaluating candidates after debates, or their earlier primary performances for US presidential primaries. In

[5] Their notion of ethical voting formalizes earlier ideas proposed in the political science literature (c.f., Goodin and Roberts (1975)) that voters care about the welfare of others who have similar views to theirs.

the following we describe the possible equilibrium outcomes of such coordination processes.

Before providing the formal definitions, we make two observations. First, one should not expect a concept of coordination to always a unique prediction. This feature is well known, as it arises in standard coordination games such as the battle of the sexes. In some coordination games, equilibrium refinements can be used to reduce the number of equilibria. However, in our analysis we choose to remain agnostic as to which equilibrium may arise. Importantly, most equilibria yield the same candidate winning probabilities, in which case candidate location does not vary with equilibrium in the voting subgame. Second, we describe an equilibrium rather than the process that leads to the equilibrium. Again, we choose to remain agnostic as to the nature of how voters reach that equilibrium.

3.2 Coalition Formation and Equilibrium

Non-partisan voters are distinguished ex ante by their expressive preferences. Let E_i be the set of voters with an expressive preference for candidate i, i.e.,

$$E_i = \left\{\theta \;\middle|\; v_\theta(x_i) > v_\theta(x_j), \text{ for all } j \neq i\right\}. \tag{1}$$

For simplicity of exposition we assume that all policies differ, i.e., $x_i \neq x_j$ for $i \neq j$, in which case $E_i \neq \emptyset$. We discuss the special case where $x_i = x_j$ for some $i \neq j$ below.

Unlike in Fedderson and Sandroni, the preference intensities of voters in E_i differ. For example, if E_i is an interval then voters on the opposite ends of the interval have different incentives of voting against their expressive interests. Candidate i may receive votes from strategic voters, who expressively prefer some other candidate j. Let S_i be the set of all voters who vote for candidate i. We say that S_i is a strategic coalition if and only if $S_i \setminus E_i \neq \emptyset$, i.e., if and only if it includes some citizens who strategically vote against their expressive interests. Let I be the index set of all strategic voting coalitions, i.e., $I = \{i \in \{1, \dots, n\} | S_i \backslash E_i \neq \emptyset\}$.

Once a strategic coalition is formed, all non-partisan voters in S_i vote according to their common interests, similar to the ethical voters in Fedderson and Sandroni. However, in their setting the supporters of a candidate are pre-determined, and their choice is whether or not to vote. In our case, instrumental voters come together to form coalitions, and we must allow for randomized coordination of coalition members. This means that, in equilibrium, a particular coalition S_i may have only a probabilistic understanding of rival coalition formations. We now describe this possibly stochastic coalition formation.

Recall that $I \subset \{1, \dots, n\}$ are the indices of all strategic coalitions. Let $\bar{S}_i \subset \Theta$ be the support of a strategic voting coalition for candidate i: $\bar{S}_i$ consists of instrumental voters who either always vote expressively for candidate i or who sometimes vote strategically for i, but expressively prefer another candidate. Let $\mathfrak{S}_i$ be a collection of subsets of $\bar{S}_i$. Randomized coalition formation is described by a probability distribution λ_i over

$\mathfrak{S}_i$. The probability distributions λ_i, $i \in I$ are independent, reflecting that coordination can only occur within groups and not across groups. Let λ be the joint probability distribution over coalitions, S. Because the collection of realized sets must be pairwise disjoint, independence implies the supports $\bar{S}_i$ do not overlap.

Consider a realized collection S of strategic voter coalitions $S_i, i \in I$. Then the total number of votes for a candidate i equals

$$V_i(S) = \begin{cases} \rho_i + \Phi(S_i) & \text{if } i \in I; \\ \rho_i + \Phi\left(E_i \setminus \bigcup_{j \in I} S_j\right) & \text{if } i \notin I, \end{cases} \tag{2}$$

where, abusing notation, we use Φ to describe the measure of non-partisan voter support for each candidate. The candidate with the plurality of votes wins. In case of a tie we assume that each candidate wins with strictly positive probability as the exact split does not affect our results. Let $W_i(S)$ be an indicator function that assumes the value 1 if and only if party i wins. Then candidate i's winning probability is given by

$$P_i(\lambda) = \int W_i(S)\, d\lambda(S). \tag{3}$$

Let $\lambda_i = \delta_{S_i}$ be the probability distribution that places probability one on coalition S_i. Note that if all λ_i take this form then there is no randomization. The expected instrumental payoff conditional on the realization of a particular voting coalition $S_i \in \mathcal{S}_i$ is given by

$$U_\theta(\lambda_{-i}, S_i) = \sum_{k=1}^{n} P_k(\lambda_{-i}, \delta_{S_i}) v_\theta(x_k). \tag{4}$$

We next introduce our notion of a strategic voter equilibrium in which groups of citizens coordinate their votes to make their group better off. Individual members of a strategic coalition do not cheat on other members by secretly changing their votes without telling other coalition members due to the large negative psychic payoffs incurred from being "unethical" in this way. However, it would not be unethical for a member to tell other coalition members that they are not willing to follow the coalition's recommendation. In such an event, the coalition breaks up. A minimal requirement for a strategic coalition is that all member should benefit in expectation from their membership—nobody can be forced to be in a realized coalition that makes them worse off than if the coalition were to break up and all coalition members were to vote expressively.

Definition 1 (Individual Rationality) *Let S_i be a realized coalition of strategic voters. Then breaking up S_i by having all former members instead vote expressively cannot make some previous members strictly better off. That is, there does not exist $\theta \in S_i \cap E_j$, $j \neq i$ such that $\beta v_\theta(x_j) + (1-\beta)U_\theta(\lambda_{-i}, \emptyset) > \beta v_\theta(x_i) + (1-\beta)U_\theta(\lambda_{-i}, S_i)$.*

Equilibrium also demands that anyone can join a coalition S_i, i.e., also vote strategically for candidate i, if it is their interest to do so. In other words, existing coalition members cannot prevent strategic voting of other citizens. Given that we have a continuum of agents, and choose not to make topological assumptions (e.g, that all sets S_i are closed), we assume that all new coalition members must be uniformly better off. It should be noted that with three candidates all existing members are also strictly better off, because the winning probability of candidate i is increased.

Definition 2 (Inclusivity) *For any realized voter coalition S_i for candidate i, there does not exist an $\epsilon > 0$ and a set T of supporters of other candidates who currently vote expressively but would all gain at least $\epsilon > 0$ by joining S_i, That is, there does not exist a set $T \subset \bigcup_{k \neq i} E_j \setminus \bigcup_{k \neq j} \bar{S}_k$ and an $\epsilon > 0$ such that*

$$\beta v_\theta(x_i) + (1-\beta)U_\theta(\lambda_{-i}, S_i \cup T) \geq \beta v_\theta(x_j) + (1-\beta)U_\theta(\lambda_{-i}, S_i) + \epsilon, \text{ for all } \theta \in T,\ j \neq i. \tag{5}$$

Coalitions must satisfy these two conditions. In addition, in equilibrium it should not be optimal for a new coalition to enter that satisfies individual rationality and inclusivity. A new coalition cannot be formed with voters who are already in an existing strategic coalition with strictly positive probability. The set of voters who are not affiliated with an existing strategic coalition is given by $\Theta \setminus \bigcup_{i \in I} \bar{S}_i$.

Definition 3 (Strategic Voting Equilibrium) *A collection of probability distributions $\lambda = (\lambda_i)_{i \in I}$ over coalitions is a strategic voting equilibrium if and only if*

1. *All realized strategic coalitions S_i satisfy individual rationality and inclusivity.*
2. *There does not exist a new strategic voter coalition $S_j \subset \Theta \setminus \bigcup_{i \in I} \bar{S}_i$ such that all members of S_j are at least as well off and some strictly better off if the coalition is formed.*

In a strategic voting equilibrium, a deviating coalition takes the distributions over the other coalitions, λ_{-i} as given, as in a Nash equilibrium. Note that if there is no randomization, we can replace λ_{-i} by the collection of known coalitions other than S_i.

Finally, consider the case where a set of candidates C has the same policy position x_C. Define $E_C = \{\theta \mid v_\theta(x_C) > v_\theta(x_j),\ j \notin C\}$. Because $\beta < 1$ in any strategic voting equilibrium, all non-partisan voters in E_C who vote for someone in C will coordinate on the same candidate. This means that we can drop all but one candidate in C and proceed as above.

4 Strategic Voting Equilibrium

We now analyze equilibria of subgames after candidates have chosen their positions. We focus on a setting with three candidates.

Proposition 1 *Given any set of candidate policy choices x_1, x_2, and x_3 from the first stage and all possible levels of partisan supports ρ_1, ρ_2, and ρ_3, at least one equilibrium exists in the second-stage voting game.*

We prove this result in a series of propositions below that exhaustively characterize all voting equilibria that emerge given the different possible parameterizations. This result is also necessary for an equilibrium to the entire game to exist, as one needs existence of a voting equilibrium given any (possibly off-equilibrium) set of candidate policy choices. We use a running example to illustrate each equilibrium.

We begin by characterizing the trivial equilibrium outcomes that emerge if multiple candidates adopt the same policy position. If two candidates have the same policy position x, it immediately follows that the setting reduces to the standard case with two candidates. As indicated above, if two candidates adopt policy x, voters who expressively prefer x to the policy proposed by the third candidate would coordinate on only one of the candidates who proposes x. In this case there is only a pure strategy equilibrium in which all citizens vote according to their expressive preferences. If all three candidates adopt the same policy, then any voting behavior is an equilibrium, because the policy outcome is always the same. Thus, without loss of generality, we can assume that $x_1 < x_2 < x_3$.

Let $\theta_{ij} = (x_i + x_j)/2$. Then the set of expressive supporters for the three candidates are $E_1 = (-\infty, \theta_{12})$, $E_2 = (\theta_{12}, \theta_{23})$, and $E_3 = (\theta_{23}, \infty)$. Let $N_i = \Phi(E_i) + \rho_i$ be the number of votes for party i if all citizens vote according to their expressive preferences. We call N_i the base support for candidate i; that is, N_i corresponds to candidate i's vote total if all citizens vote according to their expressive preferences (i.e., as if $\beta = 1$). Without loss of generality, we can assume that $N_1 \geq N_3$.

4.1 The Centrist's Strategic Advantage

We first show in Proposition 2 and Example 1 that if the centrist candidate 2's base support is at least as large as that for each of the two non-centrist candidates, i.e., if $N_2 > N_1$, then, no matter how small or large that base advantage is, a pure strategy equilibrium always exists in which all citizens vote for their expressively-preferred candidate and the centrist wins. This result reflects that (i) no voter views the centrist as their least favorite candidate, (ii) no voter will ever vote for their least favored candidate, and (iii) expressive centrist supporters get their most-preferred candidate if everyone votes for their expressively-preferred candidate. While this result might seem to suggest that the candidate with the greatest ex-ante support always wins, this is not true in general. Indeed, Proposition 3 and Example 2 show that the vote cushion of a leading non-centrist candidate must be sufficiently large to win in a pure strategy equilibrium; and even when $N_2 > N_1$, a mixed strategy equilibrium may exist in which all three candidates have positive probabilities of winning.

Example 1 Suppose that the candidate locations are $x_1 = -1$, $x_2 = 0$, $x_3 = 0.4$, and that Φ is a uniform distribution on $[-1, 1]$. Then the cutoffs between the parties are $\theta_{12} = -0.5$, $\theta_{13} = -0.3$, and $\theta_{23} = 0.2$.

Further, suppose that $\rho_1 = 0.2$, $\rho_3 = 0$, and $\rho_2 > \rho_1 + 0.1$. The number of voters (absent strategic voting) that would support each party are $N_1 = 0.25 + \rho_1 = 0.45$, $N_2 = 0.35 + \rho_2 > 0.45$, and $N_3 = 0.4 + \rho_3 = 0.4$. In this case in the unique pure strategy equilibrium all non-partisan voters vote for their expressively-preferred candidates. In particular, for candidate 1 to win, some voters types in $E_2 = (\theta_{12}, \theta_{23})$ would have to switch to candidate 1, which would make them strictly worse off, so a strategic voter coalition of this type would violate individual rationality. For the same reason no strategic voter coalition in favor of candidate 3 would form. ■

This result is general in nature. If the centrist candidate has a plurality when all citizens vote according to their expressive preferences, then expressive voting is always an equilibrium.

Proposition 2 *If $N_2 \geq N_1 \geq N_3$, then a pure strategy equilibrium exists in which citizens vote according to their expressive preferences and the centrist candidate 2 always wins. If $N_2 > N_1$, there is no other pure strategy equilibrium.*

The result reflects that in this equilibrium, expressive supporters of the centrist candidate 2 always get their most preferred outcome—they vote for candidate 2 and 2 wins. Thus, they have no incentive to vote strategically. Further, it does not make sense for expressive supporters of non-centrist candidates 1 and 3 to vote strategically for the centrist because the centrist already wins. So, too, supporters of say candidate 1 would not vote strategically for candidate 3, because they are better off if the centrist wins.

Proposition 2 does not preclude the possibility that a second equilibrium may exist that features coordination failure by expressive supporters of candidate 2 in which some support candidate 1, while others support candidate 3. In such an equilibrium, some left-of-center candidate 2 expressive supporters are concerned that if they do not vote for candidate 1 then candidate 3 may win due to strategic coordination by enough right-of-center candidate 2 expressive supporters on candidate 3; while right-of-center candidate 2 supporters have the opposing concern. Proposition 2 establishes that pure strategy equilibria cannot take this form, as only a tie could incentivize voters on both sides of center to vote against their expressive interests, but with a tie, a slight increase in strategic voting for one of the extreme candidates would lead to a discontinuous increase in the probability that the candidate wins.

However, a mixed strategy equilibrium may exist with precisely those features. In particular, if N_1, N_2 and N_3 are sufficiently close, there may be a mixed strategy equilibrium in which some expressive candidate 2 supporters to the left sometimes vote for candidate 1, while some on the right sometimes vote for candidate 3. Both candidates 1 and 3 win with positive probability in equilibrium, and candidate 2 wins when all non-partisan voters vote according to their expressive preferences. We omit this characterization as we analyze a similar mixed strategy equilibrium in Proposition 6, where $N_2 < N_1$, which differs largely in that candidate 1 wins rather than candidate 2 when all non-partisan voters vote according to their expressive preferences. Of note, expressive candidate 2 supporters are worse off in the mixed strategy equilibrium than in the pure strategy equilibrium in which candidate 2 always wins, and, indeed, it may be that all voters are worse off due to this form of coordination failure by candidate 2 expressive supporters.

We now show that the result of Proposition 2 does not extend to a non-centrist who has the most base support, i.e., when $N_1 > N_2$. Such candidates can only be assured of victory if they have a sufficiently large vote cushion. If the cushion is too small we get an "underdog effect." Specifically, in pure strategy equilibria, only a candidate who is behind receives strategic voting, and it must sway the election, resulting in a loss for the candidate with the greatest base support.

We next illustrate equilibrium outcomes when candidate 1's base support advantage is sufficiently large that 1 wins in a pure strategy equilibrium. We then state the formal result, which quantifies "sufficiently large".

Example 2 Consider the setup from Example 1, but assume that $\beta = 6/7$, and $\rho_2 = \rho_3 = 0$. If $\rho_1 > 0.15$ candidate 1 would win if everyone voted expressively, because $N_1 = 0.25 + \rho_1$, $N_2 = 0.35$, and $N_3 = 0.4$. However, this can only be an equilibrium if either of the forms of strategic voting below is optimal:

1. Enough candidate 3 supporters vote strategically for the centrist candidate 2 that candidate 2 wins;
2. Enough candidate 2 supporters vote strategically for candidate 3 that candidate 3 wins.

Case 1: For candidate 2 to win, a coalition $[\theta_{23}, y_2]$ of expressive candidate 3 supporters has to form such that candidate 2 gets enough votes to win, i.e., $0.5(y_2 - \theta_{12}) > N_1 = 0.25 + \rho_1$, where $\theta_{12} = -0.5$. This implies $y_2 > 2\rho_1$. Such a strategic coalition would form if and only if it is in type y_2's interest to strategically vote for candidate 2, i.e., if and only if $-\beta(y_2 - x_3)^2 - (1-\beta)(y_2 - x_1)^2 \le -(y_2 - x_2)^2$. With candidate positions $x_1 = -1$, $x_2 = 0$, and $x_3 = 0.4$, this implies $y_2 = 0.7$. Thus, a strategic voting equilibrium in which candidate 2 wins exists if (and only if) $\rho_1 < 0.35$.

Case 2: For candidate 3 to win, a strategic voter coalition $[y_3, \theta_{23}]$ must form such that $0.5(1 - y_3) > N_1 = 0.25 + \rho_1$, which implies that $y_3 < 0.5 - 2\rho_1$. For this coalition to form, i.e., for individual rationality to hold, it must be that $-(y_3 - 0.4)^2 \ge -\beta y_3^2 - (1-\beta)(y_3 + 1)^2$, which implies $y_3 \ge -17/40$. Both conditions hold if and only if $\rho_1 < 23/95 \sim 0.2421$.

Thus, the pure strategy equilibrium exists in which candidate 1 wins if and only if $\rho_1 \ge 0.35$ instead of $\rho > 0.15$ which would be required under purely expressive voting. As a non-centrist, it is not enough for candidate 1 to have a base support advantage in order to win; candidate 1's ex-ante cushion must be large enough to discourage strategic voting for candidates 2 and 3. In a pure strategy equilibrium, citizens do not vote strategically for the party that is initially ahead — there is no point in doing that. In contrast, strategic voting makes sense to boost an underdog to victory. In the example, it is very attractive for expressive candidate 3 supporters to vote for candidate 2 because their positions are close together but far from candidate 1's. This underlies the large vote cushion that candidate 1 needs to win, as $N_1 - N_2$ must exceed 0.25. ■

We now provide a general characterization for the case when candidate 1 can win and characterize the required vote gap. Strategic voting is irrelevant if candidate 1's base support comprise a majority, i.e., if $N_1 \geq N_2 + N_3$. Proposition 3 characterizes when candidate 1 always wins given lesser pluralities.

Proposition 3 *Suppose that* $\max\{N_2, N_3\} < N_1 < N_2 + N_3$. *Then:*

1. *If* $\beta \leq (x_3 - x_2)/(x_3 - x_1)$ *then there does not exists a pure strategy equilibrium in which candidate 1 wins.*
2. *If* $\beta > (x_3 - x_2)/(x_3 - x_1)$ *then a pure strategy equilibrium in which candidate 1 wins and all citizens vote according to their expressive preference exists if and only if the vote gap* $N_1 - \max\{N_2, N_3\}$ *is sufficiently large. When this equilibrium exists it is unique.*
3. *The required base support vote gap is decreasing in* β *and converges to zero as* $\beta \to 1$.

As in Example 2, for candidate 1 to win, two types of strategic voting have to be ruled out. Most importantly, enough candidate 3 supporters must not have an incentive to support the centrist candidate that they would overcome candidate 1's initial advantage. Indeed, when $\beta \leq \bar{\beta} = (x_3 - x_2)/(x_3 - x_1)$, all candidate 3 supporters are better off supporting the centrist, and hence candidate 1 can only win if he has an expressive majority. Second, more rightist supporters of candidate 2 cannot have an incentive to vote strategically for candidate 3. These conditions requires that citizens have sufficiently weak instrumental preferences and that candidate 1's plurality advantage be sufficiently large, where the required gap is lower when instrumental preferences are weaker.

It follows that if $N_1 > N_2$ but the required base support vote gap is not large enough (given β), then any equilibrium must involve strategic voting in which some citizens vote with positive probability against their expressive preferences. We next characterize when such strategic voting equilibria arise and their properties.

4.2 Pure strategy strategic voting equilibria

We first show that it is "easy" for the centrist candidate 2 to draw strategic support from candidate 3 supporters when non-centrist candidate 1's base advantage over 2 is small enough. In particular, this pure strategy strategic voting equilibrium only requires that at least $N_1 - N_2$ moderate expressive supporters of candidate 3 be willing to coordinate on candidate 2 in order to defeat candidate 1, whom they least prefer. Importantly, when this is so, candidate 1 cannot win any strategic support—since expressive supporters of candidate 2 get their preferred winner in this equilibrium, they do not want to deviate from voting expressively.

To illustrate this, consider the continuation of Example 2.

Example 3 Consider again the setup in Example 2. We have seen that for $\beta \leq 5/7$, all candidate 3 supporters would be willing to vote strategically for candidate 2 if it were necessary to sway the election away from candidate 1 to candidate 2. When $\beta > 5/7$, the marginal potentially strategic voter is given by $y_2 = (21\beta - 25)/(70\beta - 50)$. For $\beta < 45/49$, the coalition is sufficiently large that candidate 2 gets more votes than candidate 1. This is an equilibrium because expressive candidate 2 supporters cannot improve by voting strategically for candidate 1 or 3. ■

Proposition 4 *Suppose that* $\max\{N_2, N_3\} < N_1 < N_2 + N_3$. *Then:*

1. *If $\beta \leq (x_3 - x_2)/(x_3 - x_1)$ then there exists an equilibrium in which enough expressive candidate 3 supporters vote strategically for candidate 2 and candidate 2 wins.*

2. *If $\beta > (x_3 - x_2)/(x_3 - x_1)$ then a pure strategy equilibrium in which candidate 2 wins due to strategic voting by at least $N_1 - N_3$ expressive candidate 3 supporters exists if and only if $N_1 - N_2$ is not too large. The larger is β, the smaller $N_1 - N_2$ must be for this strategic voting equilibrium to exist, requiring $N_1 - N_2 \rightarrow 0$ as $\beta \rightarrow 1$.*

3. *The equilibrium is unique if $N_1 - N_3$ is not too small. The vote gap $N_1 - N_3$ for uniqueness decreases in β, reaching a maximum of $\Phi([0.5(x_1 + x_3), 0.5(x_2 + x_3)])$ at $\beta = 0$, and going to zero as $\beta \rightarrow 1$.*

While all voters who expressively prefer candidate 2 will vote for 2, the set of voters who expressively prefer candidate 3, but strategically coordinate on candidate 2 can be given by any set of expressive candidate 3 voters with measure at least $N_1 - N_2$ that unanimously prefer to vote as a group for candidate 2 against their expressive interests. This is the requirement of individual rationality. However, the size of the coalition is indeterminate and not pinned down by inclusivity. The key feature is that in all such equilibria, the strategic support for candidate 2 is enough to ensure 2's victory—all equilibria take the same qualitative form, so there is no need to further refine the set.

When the third statement of the proposition does not hold, there is either a pure strategy equilibrium in which candidate 3 wins due to strategic voting by expressive supporters of the centrist, or there is a mixed strategy equilibrium in which leftist supporters of the centrist candidate vote strategically for candidate 1, and rightist supporters for candidate 3. The (pure or mixed) form of such equilibria depends on the relative numbers of expressive centrist supporters who are prepared to vote strategically for candidate 3 vs. 1.

We first show that while a pure strategy equilibrium can exist in which right-of-center candidate 2 expressive supporters coordinate on candidate 3 in sufficient numbers to defeat candidate 1, the conditions that must hold are far more demanding than those required to deliver the centrist's victory. In particular, enough more expressive centrist supporters must be willing to vote for candidate 3 than 1. It is not enough that candidate 3 draw strategic support from $N_1 - N_3$ right-of-center expressive candidate 2 supporters because some left-of-center expressive candidate 2 supporters would be willing to throw strategic support to candidate 1 to defeat 3. A pure strategy strategic voting equilibrium requires that candidate 3's winning margin be so large that it exceeds the maximal amount of left-of-center voters who would be in principle be willing to vote for candidate 1 (even though in equilibrium candidate 1 does not receive any strategic support). For example, to establish this support differential, candidate 1 would have to be sufficiently far away from the centrist relative to the center-right position of candidate 3 as in Example 4 below. The example shows not only that large winning margins may be associated with strategic behavior, but also that such strategic equilibria may *require* such large winning margins.

Example 4 Consider the same setup as in the previous examples, where $x_1 = -1$, $x_2 = 0$ and $x_3 = 0.4$ so that candidate 3 is much closer to the political center than candidate 1. We have shown in Example 2 that if $\beta < 14/15$ then a coalition $[y_3, \theta_{23}]$ of candidate 2 supporters would be willing to vote strategically for candidate 3, in order to prevent candidate 1 from being elected, where $y_3 < 0.1$. However, for this to be an equilibrium, a coalition $[\theta_{12}, y_1]$ of left-leaning candidate 2 supporters should not be able to coordinate strategically to swing the election to candidate 1.

Coalition $[\theta_{12}, y_1]$ would satisfy individual rationality if $-(y_1 - x_1)^2 \geq -\beta(y_1 - x_2)^2 - (1-\beta)(y_1 - x_3)^2$. This implies $y_1 \leq -(21 + 4\beta)/(70 - 20\beta)$. This potential coalition of strategic 1 supporters should not be able to change the election outcome. Thus, an equilibrium in which candidate 2 supporters vote strategically for candidate 3 to deliver 3's victory exists if $N_1 + 0.5(y_1 - \theta_{12}) < N_3 + 0.5(\theta_{23} - y_3)$. Substituting the above values for y_1 and y_3 yields that the instrumental considerations of voters must be strong enough that $\beta < 0.7(\sqrt{69} - 7) \approx 0.914637$. In this pure strategy equilibrium, candidate 3 must win the votes of at least $N_1 + 0.5(y_1 - \theta_{12})$ voters. Anticipating this margin, candidate 1 will only receive N_1 votes, as left-leaning expressive supporters of candidate 2 will not vote against their expressive preferences to support a losing cause. Thus, in this equilibrium, the winning vote margin will be at least $0.5(y_1 - \theta_{12})$. ■

A casual observer of this equilibrium outcome might conclude that there is "excessive strategic coordination" by right-of-center voters on candidate 3 because candidate 3 receives at least $\Phi([\theta_{12}, y_1])$ more votes than candidate 1. However, this conclusion is misplaced because if fewer right-of-center voters coordinated on candidate 3 (and instead voted expressively for candidate 2), then left-of-center voters would have an incentive to coordinate on candidate 1 in sufficient numbers to defeat candidate 3, making those right-of-center voters worse off, breaking the equilibrium.

Proposition 5 *Suppose $N_1 > \max\{N_2, N_3\}$, and let y_1 solve $-(x_1 - y_1)^2 = -\beta(x_2 - y_1)^2 - (1-\beta)(x_3 - y_1)^2$. Then if*

$$-(x_3 - y_3)^2 > -\beta(x_2 - y_3)^2 - (1-\beta)(x_1 - y_3)^2 \tag{6}$$

where $y_3 < \theta_{23}$ solves $N_3 + \Phi([y_3, \theta_{23}]) = N_1 + \Phi([\theta_{12}, y_1])$, there exists a pure strategic voting equilibrium in which all $\theta \in [y_3, \theta_{23}]$ vote for candidate 3 against their expressive preference for candidate 2 in order for candidate 3 to defeat candidate 1. In any pure strategy equilibrium, candidate 3 wins with a vote total of at least $N_3 + \Phi([y_3, \theta_{23}])$.

If $x_2 \geq 0$, then the underdog candidate 3 can only win in a pure strategy equilibrium when $x_3 < |x_1|$, and hence can attract more potential strategic supporters.

The condition described in (6) simply says that not enough left-oriented expressive candidate 2 supporters are willing to support candidate 1 to defeat candidate 3 if candidate 3 draws enough strategic support from right-oriented expressive candidate 2 supporters. The set of voters who expressively prefer candidate 2, but strategically coordinate on candidate 3 is again not uniquely pinned down. What is pinned down is that (i) they must all prefer to vote for candidate 3 in order to deliver candidate 3's victory rather than vote expressively and have candidate 1 win, and (ii) together with the expressive supporters of candidate 3, they must comprise a measure of at least $N_3 + \Phi([y_3, \theta_{23}])$.

4.3 Mixed Strategy Equilibria

We now show what happens when enough expressive supporters of candidate 2 on both sides are willing to vote strategically for an extreme candidate in order to defeat the extreme candidate whom they like least. The resulting equilibrium must be in mixed strategies, as each extreme candidate must have a chance of winning in order to draw strategic support from expressive candidate 2 supporters.

Example 5 Continuing the example, Example 2 showed that candidate 1 wins if expressive preferences are significant enough that $\beta \geq 14/15 \approx 0.9333$. Example 3 showed that if $\beta < 45/49 \approx 0.9184$ then there exists an equilibrium in which candidate 2 wins, but not if $\beta > 45/49$. This equilibrium always coexists with an equilibrium in which candidate 3 either sometimes or always wins. Example 4 showed that if $\beta < 0.7(\sqrt{69} - 7) \approx 0.9146$ then a pure strategy equilibrium exists in which candidate 3 wins due to strategic voting by enough expressive candidate 2 supporters, but not if $\beta > 0.7(\sqrt{69} - 7)$. Putting these together, it follows that a pure strategy equilibrium does not exist for $45/49 \leq \beta < 14/15$. However, when $0.7(\sqrt{69} - 7) \leq \beta < 14/15$ there exists a mixed strategy equilibrium in which leftist candidate 2 supporters sometimes vote for candidate 1, and rightist candidate 2 supporters sometimes vote for candidate 3. For $0.7(\sqrt{69} - 7) \leq \beta < 45/49$ this mixed strategy equilibrium co-exists with the pure strategy equilibrium in which candidate 2 wins due to strategic support by expressive candidate 3 supporters.

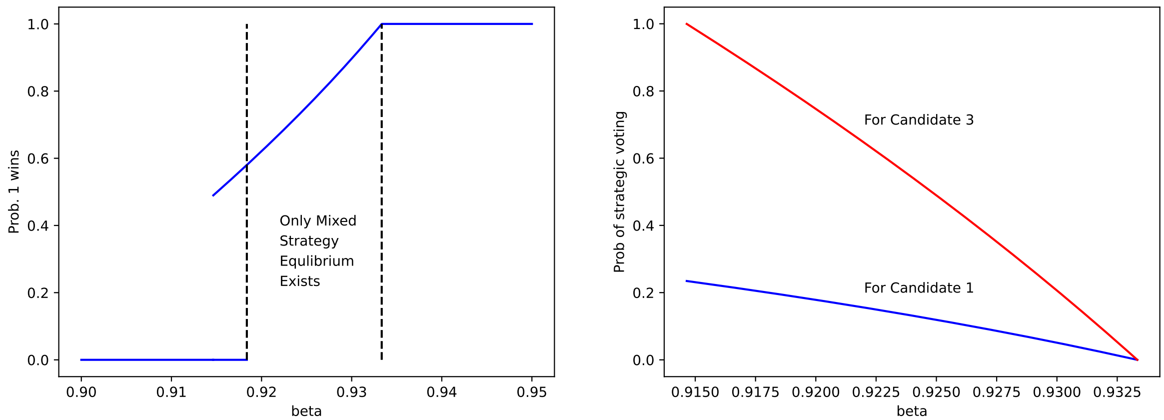

Figure 1: Left: Candidate 1's winning probabilities in all equilibria as a function of β. Right: Probability that expressive candidate 2 supporters strategically coordinate on extreme candidates 1 and 3 as a function of β in the mixed strategy equilibrium.

Figure 1 displays the equilibrium outcomes. The figure on the left plots the probability that candidate 1 wins in all equilibria as a function of the weight β that non-partisans place on expressive preferences. The figure on the right displays the probabilities with which some coalition forms that includes candidate 2 expressive supporters who vote strategically for extreme candidates 1 and 3. Posed differently, one minus these probabilities yields the probabilities q_1 and q_3 that strategic coalitions do not form for the respective extreme candidates.

At the top end of the mixed strategy equilibrium, i.e., at $\beta = 14/15$, a measure $N_1 - N_3 = 0.05$ of expressive candidate 2 would be willing to strategically vote for candidate 3 if doing so would lead to candidate 3's victory. That is, there are just enough expressive candidate 2 supporters close to candidate 3 who would be willing to vote for candidate 3 in order to defeat candidate 1 only if their votes were surely decisive to sway the electoral outcome. However, at this point, the slightest offsetting strategic voting by expressive candidate 2 supporters close to candidate 1 would lead to 1's victory, in which case the strategic voters for candidate 3 would regret their votes. As a consequence, at $\beta = 14/15$, expressive candidate 2 supporters almost always vote their expressive preferences, i.e., $q_1 = q_3 = 1$, and hence candidate 1 wins with probability 1, smoothly meeting the pure strategy equilibrium that obtains for higher values of β.

Reducing β from 14/15 raises the weight on instrumental preferences making expressive candidate 2 supporters more willing to vote strategically. Importantly, while candidate 1 has a larger base support than candidate 3, i.e., $N_1 - N_3 = 0.05$, candidate 2 is closer to candidate 3 than candidate 1, making it easier for candidate 3 to attract more strategic voters. This means that as voters care more and more about who wins the election rather than voting expressively, candidate 3 attracts differentially more potential strategic supporters than candidate 1.

The sizes of the realized strategic coalitions that form vary, but any strategic coalition that forms consists of expressive supporters of candidate 2 who are closest to the extreme candidates. The requirements of internal and inclusivity mean that the marginal strategic supporter in each realized coalition is indifferent between strategically voting for an extreme party and instead having all coalition members vote according to their expressive preferences. As β is reduced, the population of potential strategic voters rises faster for candidate 3: the rate at which candidate 3 supporters vote strategically must rise faster to preserve indifference of the marginal strategic supporter of candidate 1. At the lower bound of $0.7(\sqrt{69} - 7)$, candidate 3 supporters always vote strategically, and the size of the largest strategic coalition that supports candidate 3 is just big enough that there are not enough expressive candidate 2 supporters to the left who would be willing to vote for candidate 1 in order to defeat candidate 3. Posed differently, the equilibrium condition that there must be a tie between candidates 1 and 3 when both sides engage in maximal strategic voting (else one side is coordinating excessively) pins down q_3; and at $0.7(\sqrt{69} - 7)$, $q_3 = 0$, implying that a further reduction in β below $0.7(\sqrt{69} - 7)$, raises the strategic support that candidate 3 can acquire above that for candidate 1, implying that the equilibrium is in pure strategies and candidate 3 always wins.

For $0.7(\sqrt{69} - 7) \leq \beta < 45/49$, there are two equilibria, one where candidate 2 always wins, and one where candidates 1 and 3 both win with positive probability. Inspection yields that on this range, even though candidate 3 sometimes wins in the mixed strategy equilibrium, expressive supporters of candidate 3 are better off in the pure strategy equilibrium where candidate 2 always wins. To see this, note that in the mixed strategy equilibrium, candidate 3 wins less than half of the time, so the winner's expected location is to the left of candidate 2. It follows that candidate 3 supporters prefer the sure thing of candidate 2's victory, both from a less risky lottery perspective, and from a higher mean perspective. In contrast, an expressive candidate 1 supporter at x_1 strictly prefers the mixed strategy equilibrium: even at the lower end of the support where candidate 3 wins half the time, we have $-0.5\left(0^2 + 1.4^2\right) = -0.98 > -1$. However, a slight shift to the right of x_2 to 0.6 is enough that the welfare of voter $\theta = x_1$ can also be lower in the mixed strategy equilibrium than in the pure strategy equilibrium in which the centrist draws strategic support to win. ■

We next establish the existence of mixed strategy equilibria. The inequalities conditions under which this equilibrium exists is a superset of the set of parameter values for which pure strategy equilibria do not exist. Hence, with this next proposition we establish existence of equilibria for all parameters, including arbitrary candidate positions and base turnouts.

Proposition 6 *Suppose that $N_1 \geq \max\{N_2, N_3\}$. Let $\bar{S}_1 = (-\infty, \bar{y}_1]$ and $\bar{S}_3 = [\bar{y}_3, \infty)$ be the largest possible strategic voting coalitions that could form, if the supported candidate wins with probability one, i.e.,*

$$-(x_1 - \bar{y}_1)^2 = -\beta(x_2 - \bar{y}_1)^2 - (1-\beta)(x_3 - \bar{y}_1)^2;$$
$$-(x_3 - \bar{y}_3)^2 = -\beta(x_2 - \bar{y}_3)^2 - (1-\beta)(x_1 - \bar{y}_3)^2.$$

If $\Phi(\bar{S}_1) \geq \Phi(\bar{S}_3) > N_1$ then there exists a mixed strategy equilibrium in which some expressive candidate 2 supporters vote for candidate 1, and some vote for candidate 3, and either candidate 1 or 3 wins depending on the realized coalitions. In equilibrium, both candidates 1 and 3 win with strictly positive probability.

The proof in the Appendix provides the explicit construction of the equilibrium mixed strategies over coalition formation, and Example 5 provides further intuition. A key intuition is that there are two possibly opposing effects at play in a mixed strategy equilibrium:

1. One extreme candidate may have a larger number N_i of base support and expressive voters. Without loss of generality we assume that this is the case for candidate 1.

2. One extreme candidate may differentially appeal to expressive supporters of the centrist candidate. This is the case in our Example 5 because candidate 3 is closer to the centrist than candidate 1.

If β is large, so that it is difficult to attract strategic voters, then the base-support advantage drives the equilibrium. That is, the advantaged candidate, in this case candidate 1, is very likely to win. In particular, candidate 1 would automatically win if a strategic coalition for candidate 3 does not form, which occurs with probability q_3. Because overcoming the vote deficit is costly and only pays off if the differential strategic support for candidate 3 exceeds the base support advantage of candidate 1, any time such coalition forms the winning probability must be large. This is possible only if q_1 and hence q_3 are close to 1.

As β is decreased it becomes easier to attract strategic voters, so candidate 3's voter deficit $N_3 - N_1$ matters less. The strategic support for candidate 1 consists of voters in some interval $[\theta_{12}, \theta_{12} + z_1]$, where $z_1 \geq 0$. The strategic support for candidate 3 consists of voter types $[\theta_{23} - Z - z_3, \theta_{23}]$, where $z_3 \geq 0$ and $[\theta_{23} - Z, \theta_{23}]$ is the minimum strategic support needed to overcome the voter deficit, i.e., Z is implicitly defined by $N_3 - N_1 = \Phi(\theta_{23}) - \Phi(\theta_{23} - Z)$.

In equilibrium, the sizes of the realized coalitions vary. When a strategic coalition say of $[\theta_{12}, \theta_{12} + z_1]$ forms, the marginal member $\theta_{12} + z_1$ of the coalition must be indifferent between expressively voting for candidate 2 and strategically voting for candidate 1. Were $\theta_{12} + z_1$ to strictly prefer strategic voting, it would violate inclusivity, because there would then be a set T of expressive candidate 2 supporters close to $\theta_{12} + z_1$, who are currently outside the strategic voter coalition, but would all receive a uniformly higher expected payoff by joining, i.e., everyone in T is made better of by at least some amount $\epsilon > 0$. Similarly, if type

$\theta_{12} + z_1$ is strictly worse off from strategic voting, it would violate individual rationality, as no coalition member can be made strictly worse if the coalition forms. In turn, this indifference condition pins down the distribution over strategic coalitions by potential strategic supporters of candidate 3.

The equilibrium is then determined by probability distributions $F_1(z_1)$ and $F_3(z_3)$, respectively. Candidate 3 would win if $\Phi(\theta_{12} + z_1) - \Phi(\theta_{12}) < \Phi(\theta_{23} - Z) - \Phi(\theta_{23} - Z - z_3)$, and candidate 1 wins if the inequality is reversed. Because of the discontinuity of payoffs at a tie, it follows that F_1 and F_3 are continuous. Further, as mentioned above, strategic coalitions do not form with probabilities q_1 and q_3, respectively. The probability mass q_1 on no strategic coalition forming to support candidate 1 is pinned down by the indifference condition at the lower end of the support for z_3, as that coalition must win with strictly positive probability to offset the fact that the expressive cost to $\theta_{23} - z_3$ of supporting candidate 3 is strictly bounded away from zero whenever $N_1 > N_3$. The condition that there must be a tie between candidates 1 and 3 when both sides engage in maximal strategic voting (else one side is coordinating excessively) pins down q_3. The distributions F_1 and F_3 are pinned down by the fact that the marginal voter in any realization coalition must be indifferent between forming and not forming a coalition.

Now decrease β. Example 5 shows that q_3 goes to 0, i.e., strategic coalitions favoring candidate 3 form with probability close to 1 despite the ex-ante base disadvantage in votes. For any β that is marginally smaller than that associated with $q_3 = 0$, there are too few possible strategic 1 supporters to defeat candidate 3.

We next consider a case in which the ex-ante vote advantage and the ease of appealing to strategic voters both favor candidate 1. Figure 2 illustrates outcomes when candidate positions are $x_1 = -0.6$, $x_2 = 0$, $x_3 = 0.64$ and, $\rho_1 = \rho + 0.01$, $\rho_2 = 0$, $\rho_3 = \rho$. Now $\theta_{12} = -0.3$, $\theta_{23} = 0.32$, and hence $N_1 = 0.36 + \rho$, $N_2 = 0.31$, $N_3 = 0.32 + \rho$. Because $x_2 = 0$ is further from x_3 than from x_1, and zero is also the position of the median voter, it follows that candidate 1 has an advantage over candidate 3 in attracting strategic voters, as the figure on the right illustrates.

When ρ is sufficiently large, e.g., $\rho \geq 0.3$, then a pure strategy equilibrium does not exist in which candidate 2 wins, because even if all expressive candidate 3 supporters voted for candidate 2, candidate 1 would still win. This means that there is a unique equilibrium. For sufficiently large β, candidate 1 always wins. For lower values there is only the mixed strategy equilibrium.

When, as in this example, the same candidate has both the ex-ante vote advantage and also appeals more strongly to potential strategic voters that candidate must always win when β is very large or very small. This delivers the $\cup$-shaped relationship between β and candidate 1's probability of winning. In the figure on the right this is reflected by the fact that candidate 3 has no strategic support both for $\beta = 0$ and for β sufficiently large. For large β there is no strategic voting, but candidate 1's base vote advantage ensures victory.

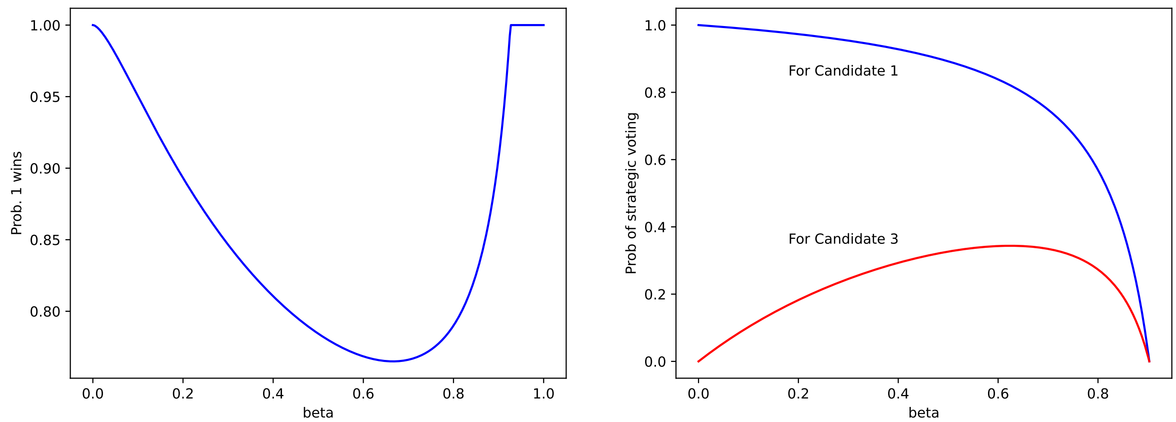

Figure 2: Left: Candidate 1's winning probabilities in the mixed strategy equilibrium as a function of β. Right: Probability that expressive candidate 2 supporters strategically coordinate on extreme candidates 1 and 3 as a function of β in the mixed strategy equilibrium.

4.4 Continuity of Mixed Strategy Equilibria and Comparative Statics

We now show that the mixed strategy equilibria are smoothly well behaved with natural comparative statics. It is immediate that as $\beta \to 1$ mixed strategy equilibria disappear, converging to the outcome with purely sincere voting. In fact, unless $N_1 = N_3$ the mixed strategy equilibrium disappears before reaching $\beta = 1$ as Figure 2 illustrates. When $\beta \to 0$ such continuous behavior is less immediate.

We can see in example 2 that candidate 1's winning probability converges to 1 as $\beta \to 0$, which corresponds to the pure strategy equilibrium at $\beta = 0$ in which sufficiently many center left voters strategically support candidate 1 so that there are not enough right-of-center candidate 2 supporters who could swing the election to candidate 3. We now show that this result is general in nature.

Proposition 7 *Let $N_1 > \max\{N_2, N_3\}$ and suppose that the mixed strategy equilibrium exists for all $\beta > 0$ in the neighborhood of $\beta = 0$. Then the winning probability of candidates 1 and 3 converge to the winning probabilities in the case with purely instrumental voting, i.e., when $\beta = 0$. That is, candidate 1 wins with probability approaching 1 if $N_1 + \Phi([\theta_{12}, \theta_{13}]) > N_3 + \Phi([\theta_{13}, \theta_{23}])$, and candidate 3 wins with probability approaching 1 if the opposite strict inequality holds.*

Proposition 7 shows that the mixed strategy equilibrium is well-behaved, converging continuously in electoral outcomes to those at $\beta = 0$, where the equilibrium is in pure strategies with the non-centrist candidate who is preferred by the majority of voters winning. We next show that the mixed strategy equilibrium also has the natural comparative static features that when a non-centrist's ex ante advantage is greater, voters are less likely to engage in strategic behavior by voting against their expressive interests.

Proposition 8 *Suppose $N_1 > \max\{N_2, N_3\}$, and that the mixed strategy equilibrium exists. In the mixed strategy equilibrium, an increase in ρ_1 to ρ_1' that increases candidate 1's base support advantage*

1. *reduces the probability of strategic voting for both candidates 1 and 3, i.e., $q_1' > q_1$ and $q_3' > q_3$.*
2. *reduces the maximal coalition supporting candidate 1 by $\rho_1' - \rho_1$ and more generally shifts the distribution over the measure of realized strategic coalition sizes for candidate 1 to the left by $\rho_1' - \rho_1$.*
3. *does not affect the size of the maximal strategic coalition that forms for candidate 3, but increases the size of the smallest (non-trivial) strategic coalition that forms for candidate 1 by $\rho_1' - \rho_1$.*

Increasing candidate 1's base support advantage immediately implies the the smallest viable (possibly winning) candidate 3 coalition increases, implying that $q_1' > q_1$. If candidate 3 does not receive strategic support, candidate 1 is sure to win regardless of the size of candidate 1's base support advantage (as long as the equilibrium continues to be in mixed strategies). It follows that the indifference condition for a marginal realized candidate 3 strategic coalition member is the same, implying that *when* that coalition forms it must have the same probability of winning, and that the size of the largest possible candidate 3 strategic coalition is unaffected. The requirement that there is a tie when both strategic coalitions are maximal then means that the size of the maximal strategic coalition that forms for candidate 1 must shrink by $\rho_1' - \rho_1$; and since the maximal marginal coalition member given ρ_1' is closer to candidate 1 (incurring a smaller expressive voting cost for voting for 1 and hence a higher payoff from voting strategically), his indifference to not voting strategically means that his payoff from not voting strategically must also rise, yielding $q_3' > q_3$.

4.5 Non-Montonicities of Winning Probabilities in Candidate Positions

To illustrate the potential non-monotonicities of winning probabilities in candidate positions, we consider a setting with a uniform distribution of voter ideologies. We first show that with strategic voting, a centrist candidate can increase its probability of winning by moving further away from the stronger candidate even though this increases the stronger candidate's expressive support.

With two candidates, when candidate 1 is located to the left of the median, candidate 1's vote share grows when candidate 2 locates further to the right of x_1. Similarly, with three candidates, and expressive voting, when candidate 1 has a larger base than candidate 3 even when candidate 2 locates closely, then candidate 1's vote share and winning probability rise as candidate 2 moves further away. This is because moving further away strengthens the stronger candidate 1 while leaving candidate 2's vote share unaffected.

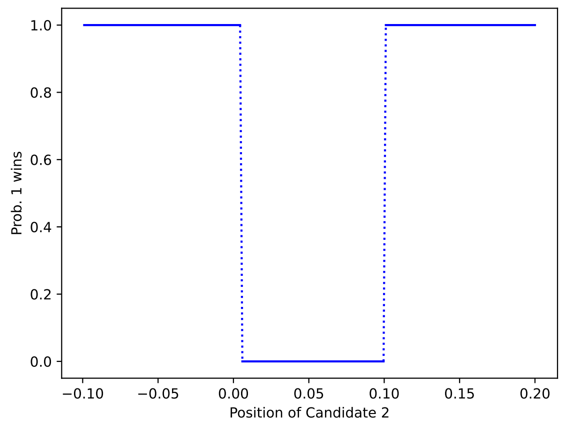

Figure 3: Candidate 1's winning probability as a function of the candidate's position, x_2.

Figure 3 illustrates how strategic voting can change this calculus. Now as x_2 is shifted to the right, N_1 rises, N_2 stays unchanged, and N_3 falls. However, with strategic voting, this does not imply that candidate 1's winning probability increases as x_2 is shifted to the right. In our example $x_1 = -0.1$ and $x_3 = 1$, $\beta = 0.2$ (so that instrumental voting considerations are high) and there are no partisan voters. With expressive voting, candidate 1 would always win whenever $x_1 < x_2 < x_3$. Now suppose that instrumental preferences matter. As candidate 2 shifts x_2 to the right, it has three effects. First it increases N_1, increasing the amount of strategic voting needed to defeat candidate 1. Second, x_2 increasingly differentiates itself from x_1 by locating closer to expressive candidate 3 supporters. This starker contrast with candidate 1 makes strategic voting more attractive *if* it can deliver a victory for candidate 2. Third, locating closer to candidate 3 also facilitates strategic voting by reducing the expressive loss to strategic voters. At $x_2 < 0$, candidate 2 fails to attract strategic voters, both due to insufficient differentiation from candidate 1 and because the expressive voting cost is too high. For somewhat larger x_2, strategic voting occurs, until $x_2 = 0.1$. For $x_2 > 0.1$, candidate 1's expressive vote share is large enough to render forming a sufficiently large strategic coalition infeasible, implying that candidate 1 always wins.

5 Political Competition with an Extreme Spoiler Party

5.1 Overview

In this section, we illustrate how one can endogenize the platform choices of two policy-motivated parties. We consider an otherwise symmetric setting in which we endogenize the entry choice by a third, extreme spoiler "citizen-candidate" party. We contrast the equilibrium platform choices with strategic and purely-expressive voting. We focus on a setting with two mainstream parties that have symmetrically opposing ideal policies $\theta_1 = -\theta_2$. The position of party 3 is fixed at its ideal policy $x_3 = \theta_3 > \theta_2$ that is sufficiently far to the right that in equilibrium either candidate 1 or candidate 2 wins. We use a symmetric setting so that existence of a local equilibrium is guaranteed, given that that no general existence results are available for models of candidate competition with policy-motivated candidates as in Wittman (1983). For similar reasons we assume that the spoiler is a citizen candidate who only chooses whether or not to enter, before the other candidates choose their platforms.

We now assume that when candidates 1 and 2 simultaneously choose policy positions x_1 and x_2 there is uncertainty about the extent of partisan support ρ_i for each party i. In a standard two-candidate model with policy motivation, candidates face a basic tradeoff between moving away from the other candidate's policy by locating closer to their own ideal point versus increasing their chance of winning by moving closer to their rival. This calculus can change when voters are strategic. In particular, consider again the example depicted in Figure 3. Candidate 2 is disadvantaged because candidate 3 siphons off voters on the far right. By moving further away from candidate 1, candidate 2 increases candidate 1's expressive support, hurting himself, as in Wittman (1983), but now this shift can induce strategic right wing voters to support him, more than offsetting the classical effect. There is a further secondary effect, because candidate 1 will also locate further to the right to reduce the likelihood that right-wing voters strategically support candidate 2.

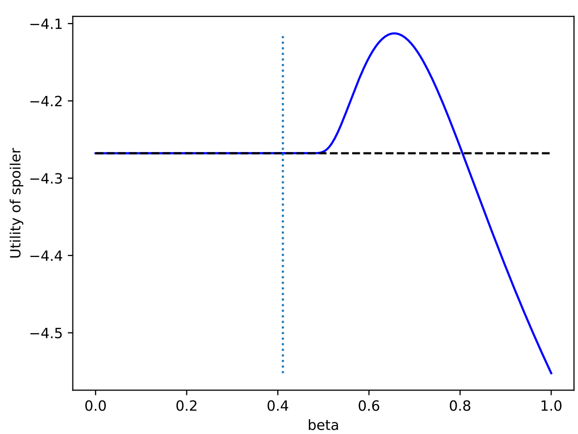

Figure 4: The spoiler's expected utility as a function of β for $-\theta_1 = \theta_2 = 1$, Φ is $N(0, 1)$ distributed, and G is a $N(0, 1)$ distribution truncated to the interval $[-1, 1]$.

To summarize, the possibility of strategic voting by candidate 3 supporters can provide both candidate 1 and 2 incentives to move toward the spoiler's position, albeit for different reasons: Candidate 1 does it to reduce strategic voting, while candidate 2 does it to increase strategic voting. In practice, this means that candidates 1 and 2 adopt some of the spoiler's platform. We will show that in equilibrium candidate 2 moves further to the right than candidate 1, implying that candidate 1 becomes more likely to win. Thus, whether the spoiler benefits from these shifted positions depends on the relative magnitudes of the rightward shift versus the change in winning probabilities. We show that the first effect dominates the second whenever β takes on intermediate values.

Figure 4 illustrates the spoiler's utility as a function of β when $-\theta_1 = \theta_2 = 1$, $\theta_3 = x_3 = 2$, Φ is a standard normal distribution and G is a standard normal distribution truncated to the interval $[-1, 1]$. The dotted vertical line indicates the critical value $\bar{\beta}$ at which the spoiler's presence begins to matter. For $\beta \leq \bar{\beta}$, voters care so much about who win that no nonpartisan votes for the spoiler—even if their ideal position is arbitrarily far to the right. Increasing β past $\bar{\beta}$, first causes candidates 1 and 2 to move their policies to the right in order to alter the incidence of strategic voting for candidate 2, as explained above. As a result, the spoiler's utility first rises as voters care more and more about their expressive component of preferences before falling sharply as β approaches 1 so that voters overwhelmingly weigh about expressive considerations. That is, candidate 3 is best off when voters' instrumental preferences are intermediate, neither too low nor too high.

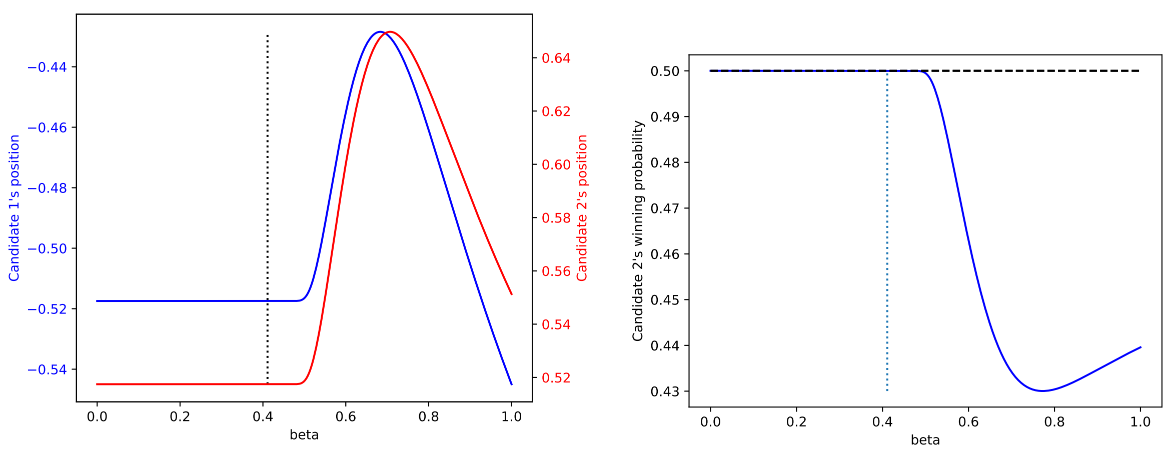

Figure 5: Candidate positions and candidate 2's winning probability as a function of β for $-\theta_1 = \theta_2 = 1$, Φ is $N(0, 1)$ distributed, and G is a $N(0, 1)$ distribution truncated to the interval $[-1, 1]$.

Figure 5 provides the intuition. Intermediate values of β induce the mainstream candidates to locate more closely to the spoiler, and this "closer location" effect more than offsets the roughly 14% reduction induced in candidate 2's winning probability, i.e., in the chances that the spoiler's preferred mainstream candidate wins.[6] In contrast, when instrumental considerations are higher, the mainstream parties ignore the

[6]Although this example is just illustrative, Pons and Tricaud (2018) use a regression discontinuity design to show that the

spoiler, and when expressive considerations matter too much to voters, the spoiler is hurt in three ways. First, once β is sufficiently high, further increases in β increasingly disadvantage candidate 2 because more right wing citizens vote for the spoiler, making it harder for candidate 2 to attract the strategic voters needed to defeat candidate 1. Second, reflecting this difficulty, candidate 2 now starts to retreat away from the spoiler to reduce candidate 1's base expressive support. Third, candidate 1 also moves to the left away from the spoiler due to the reduced risk of strategic voting by expressive candidate 3 supporters for candidate 2. This rational is also reflected in the non-monotonicity of candidate 2's winning probability. For intermediate values of β the winning probability drops, because candidate 2 loses votes to the spoiler and candidate 1 moves to the right to reduce the incidence of strategic voting, thereby becoming more attractive to the median voter. However, once β becomes sufficiently large, preventing strategic voting matters less to candidate 1 leading her to shift toward a more leftist policy by enough that candidate 2's winning probability rises.

Of note, comparing purely expressive with purely strategic voter preferences, we see that expressive preferences give rise to greater polarization—candidate 1 moves to the left, closer to his ideal policy, because the spoiler draws votes from candidate 2, who is now less likely to win (and the marginal effect on the probability is lower), while candidate 2 moves to the right, toward his ideal policy, both because candidate 1 has moved to the left, and because moving to the right wins candidate 2 some expressive supporters away from candidate 3. The consequence is that if β is sufficiently high, the spoiler is hurt by entry—relative to a two-candidate setting (equivalently relative to $\beta \leq \bar{\beta}$), entry both increases the variance in electoral outcomes, and it shifts the expected policy outcome to the left, away from the spoiler's ideal policy.

5.2 Formal Analysis

We next set out the structure for our formal analysis. Because candidate 3 is a spoiler with zero probability of winning, only the net-difference $\rho = \rho_2 - \rho_1$, between party 2 and party 1 stalwarts matters. First candidates choose policy positions x_1 and x_2, and then ρ is realized. Let G be the cdf of ρ. We make the following assumptions:

Assumption 1

1. *G is twice continuously differentiable, with a density g that is symmetric around 0.*
2. *The distribution of voter types Φ is twice continuously differentiable and symmetric around zero.*
3. *The fourth moment of Φ is finite.*

presence of a spoiler in French parliamentary and local elections reduces the chances of the ideologically-closest candidate by about one-fifth. Our model can reconcile why the spoiler may want to enter despite the impact on winning probabilities.

Let y_{23} be the most extreme voter type that would be prepared to vote for candidate 2 in order to defeat candidate 1. Then y_{23} is the largest voter type consistent with individual rationality, i.e., satisfying

$$-(x_2 - y_{23})^2 \geq -\beta(x_3 - y_{23})^2 - (1-\beta)(x_1 - y_{23})^2, \tag{7}$$

which is equivalent to

$$2y_{23}((1-\beta)x_1 + \beta x_3 - x_2) \leq (1-\beta)x_1^2 + \beta x_3^2 - x_2^2. \tag{8}$$

It is immediate that (8) holds with a strict inequality for $y_{23} = 0.5(x_2 + x_3)$ if $\beta < 1$, because a voter with bliss point $0.5(x_2 + x_3)$ is expressively indifferent between candidates 2 and 3, but is strictly better off if candidate 2 wins rather than candidate 1.

First consider $(1-\beta)x_1 + \beta x_3 \leq x_2$, which always holds if voters care enough about who wins relative to voting for their expressively preferred candidate. Then (8) does not constrain y_{23} because raising $y_{23} > 0$ lowers the left-hand side of the equation. Thus, arbitrarily large coalitions of voters $\theta \geq 0.5(x_2 + x_3)$ will form as long as these coalitions can swing the vote to candidate 2.

Now suppose that $(1-\beta)x_1 + \beta x_3 > x_2$. Then there is a maximum coalition size that can obtain, and given the unbounded support for θ the size of the strategic voter coalition that can obtain is increasing in x_2, as closer location to right-wing voters makes strategic coordination more attractive, and decreasing in x_1 as then right-wing voters mind it less when candidate 1 wins. From (8), the right-most voter who would just be willing to join the coalition is given by

$$y_{23} = \frac{1}{2}\frac{(1-\beta)x_1^2 + \beta x_3^2 - x_2^2}{(1-\beta)x_1 + \beta x_3 - x_2}, \tag{9}$$

Observe that if $\beta \leq \bar{\beta}$, where $\bar{\beta} = \frac{x_2 - x_1}{x_3 - x_1}$ solves $(1-\bar{\beta})x_1 + \bar{\beta}x_3 = x_2$, then $y_{23} = \infty$. That is, if the instrumental considerations of voters is sufficiently strong, a strategic coalition of right-wing voters will form to vote for candidate 2, whenever doing so can achieve victory.

Next note that (8) implies that raising x_1 raises the right-hand side of (8), because $y_{23} \geq 0.5(x_2 + x_3) > x_1$. This, in turn, implies that the constraint (8) becomes more binding. Hence, $\bar{y}_{23}$ must decrease. That is, by shifting x_1 to the right, candidate 1 can reduce the ex-ante probability that voters will strategically coordinate on candidate 2 to defeat 1. Conversely, increasing x_2 raises the left-hand side of (8), making the constraint less binding, and causing y_{23} to increase. That is, just as candidate 1 can reduce strategic voting for candidate 2 by making her policy more attractive to right-wing voters thereby reducing the cost to those voters of having her win, candidate 2 can increase her strategic support from right-wing voters by making her policy more attractive to them. Thus, $\partial y_{23}/\partial x_1 < 0$ and $\partial y_{23}/\partial x_2 > 0$.

The votes for candidates 1 and 2 if there is strategic voting are $V_1 = \Phi(0.5(x_1 + x_2))$, and $V_2 = \Phi(y_{23}) - \Phi(0.5(x_1 + x_2)) + \rho$, respectively. Strategic instrumental voting will occur if and only if the coalition $[0.5(x_1 +$

$x_2), y_{23}]$ suffices to deliver victory to candidate 2. Let $\bar{\rho}$ be the value of ρ at which $V_1 = V_2$, i.e.,

$$\bar{\rho} = 2\Phi(0.5(x_1 + x_2)) - \Phi(y_{23}). \tag{10}$$

Candidate 1's winning probability is $G(\bar{\rho})$.

The candidates' optimization problems are therefore given by

$$\max_{x_1} -G(\bar{\rho})(x_1 - \theta_1)^2 - (1 - G(\bar{\rho}))(x_2 - \theta_1)^2, \tag{11}$$

and

$$\max_{x_2} -G(\bar{\rho})(x_1 - \theta_2)^2 - (1 - G(\bar{\rho}))(x_2 - \theta_2)^2. \tag{12}$$

The first-order conditions are

$$g(\bar{\rho})\left(\phi\left(\frac{x_1 + x_2}{2}\right) - \phi(y_{23})\frac{\partial y_{23}}{\partial x_1}\right)\left(\frac{x_1 + x_2}{2} - \theta_1\right)(x_2 - x_1) - (x_1 - \theta_1)G(\bar{\rho}) = 0 \tag{13}$$

and

$$-g(\bar{\rho})\left(\phi\left(\frac{x_1 + x_2}{2}\right) - \phi(y_{23})\frac{\partial y_{23}}{\partial x_2}\right)\left(\theta_2 - \frac{x_1 + x_2}{2}\right)(x_2 - x_1) + (\theta_2 - x_2)(1 - G(\bar{\rho})) = 0. \tag{14}$$

We have the following result:

Proposition 9 *Suppose that $\theta_2 = -\theta_1$, $x_3 \geq \theta_2$, and Assumption 1 holds. Then there exists a $\bar{\beta}$ such that in equilibrium:*

- *For $\beta \leq \bar{\beta}$, instrumental considerations of voters dominate. The outcome is same as when candidate 3 is not present. The equilibrium platforms of candidates 1 and 2 are*

$$x_2 = -x_1 = \frac{\theta_2}{1 + 4\theta_2 g(0)\phi(0)}. \tag{15}$$

- *A sufficiently small increase in β above $\bar{\beta}$ causes both candidates 1 and 2 to shift x_1 and x_2 to the right, with x_2 shifting by more than x_1, reducing candidate 2's chance of winning but raising candidate 3's expected utility.*

When β is small enough—where small enough depends on the spoiler's location—nonpartisans care so much about who wins that they all vote for either candidate 1 or 2. As a result, political competition reduces to the classical two-candidate Wittman (1983) setting, with associated symmetric locations. A slight increase in β above $\bar{\beta}$ now means that the spoiler can steal votes from extreme right-wing voters away from candidate 2. In the proof, we use the implicit function theorem at $\bar{\beta}$ to show that this induces both mainstream candidates to shift their policies to the right, with candidate 2 moving further because rightward shifts move toward candidate 2's ideal policy and away from candidate 1's. It follows that candidate 1's probability of winning rises. However, a second application of the implicit function theorem reveals that the spoiler gains more from the rightward policy shifts of the two mainstream candidates than the spoiler loses from the increased probability that the spoiler's least preferred candidate wins. As a result, even though the spoiler steals votes away from her preferred mainstream candidate, she still gains from entry.

6 Conclusion

There is extensive evidence that voters care both about which candidate they vote for, and which candidate wins. In a two candidate setting, this distinction is irrelevant because expressive and instrumental concerns coincide. However, as recent polling data over potential Republican presidential primary candidates illustrates, this distinction matters with more than two candidates—51 percent preferred a candidate with the best chance of winning versus 44 percent who wanted to agree with the candidate on everything even if the candidate would have a tougher time winning in November.[7]

We develop a model of strategic voting in a spatial model with multiple candidates when voters have both expressive and instrumental concerns. The model endogenizes the strategic coordination of citizens on a less-preferred candidate in order to raise the chances of defeating an even less-preferred candidate. We fully characterize all strategic voting equilibria in a three-candidate setting. We provide several important insights: First, even though elections may be close, one candidate may be systematically more likely to win, indicating that close elections may not be a good natural experiment.[8] Second, strategic voting does not only have to occur in close elections. Third, strategic voting can generate endogenous uncertainty about who wins. To highlight this, we assume away all extrinsic sources of uncertainty at the voting stage. The presence of endogenous uncertainty, in turn, may add to the difficulty of forecasting electoral outcomes even with accurate polling data, as voters efforts to coordinate strategically may necessarily be unpredictable.

Finally, a virtue of our formulation of strategic voting is that it is simple enough to incorporate into a standard model of political competition with policy motivated candidates. To illustrate this, we endogenize candidate policy choices with the two mainstream candidate and a spoiler who understand that voters may coordinate strategically, We show that the spoiler can be made better off from entering, even though she has no chance of winning the election and reduces the winning probability of her preferred mainstream candidate. This occurs because both mainstream candidates partially incorporate the spoiler's platform by moving toward the spoiler.

[7] See, FiveThirtyEight, "Which Republican Candidate Should Biden Be Most Afraid Of?" `https://fivethirtyeight.com/features/which-republican-candidate-should-biden-be-most-afraid-of/`

[8] See Levine and Martinelli (2022) who also make this point in a setting with campaign spending.

7 Appendix

Proof of Proposition 2. An equilibrium with only expressive voting exists if and only if it is not optimal for any group of citizens to vote strategically and make themselves better off in the process. If $N_2 \geq N_1$, then if candidate 2 supporters vote expressively so would all expressive supporters of other parties—individual rationality would be violated if expressive supporters of the other parties voted strategically for another party, as either candidate 2 still wins or they vote in sufficient numbers for their least favored party, that it wins.

Now suppose the inequality is strict and another pure strategy equilibrium exists. If candidate 2 wins with probability 1 in that equilibrium, then individual rationality mandates that supporters of candidates 1 and 3 vote according to their expressive preferences. If, instead, there is a tie in which more than one candidate receives strategic support, then either inclusivity is violated (an arbitrarily small increase in strategic voting for one of the candidates would discontinuously raise the probability that candidate wins to one, making those voters strictly better off), or if this is not possible then there is so much strategic voting for one of the candidates that those voters would all be better off voting according to their expressive preferences, violating individual rationality. ■

Proof of Proposition 3. Let

$$\bar{y}_2 = \sup y_2 \text{ s.t. } -(x_2 - y_2)^2 \leq -\beta(x_3 - y_2)^2 - (1-\beta)(x_1 - y_2)^2 \tag{16}$$

and

$$\bar{y}_3 = \inf y_2 - (x_3 - y_3)^2 \leq -\beta(x_2 - y_3)^2 - (1-\beta)(x_1 - y_3)^2. \tag{17}$$

Then $S_2 = [\theta_{12}, \bar{y}_2]$ is the largest strategic coalition that could form, i.e., that satisfy individual rationality, if it results in a victory for candidate 2. Similarly, $S_3 = [\bar{y}_3, \infty)$ is the largest strategic coalition that could form in support of candidate 3. Observe that the sizes of these coalitions are decreasing in β, and that $\bar{y}_2 = \infty$ for $\beta \leq \bar{\beta}$.

For candidate 1 to win in a pure strategy equilibrium these coalitions cannot be large enough to affect the electoral outcome. Thus, $\Phi(S_i) \leq N_1$ for $i = 2, 3$.

Conversely, if $\Phi(S_i) > N_1$ for at least one $i \in \{2, 3\}$ then a strategic coalition would form to defeat candidate 1. By construction, individual rationality holds. Further, inclusivity holds because if the existing coalition wins, then there is no benefit for additional citizens to also vote strategically for the winning candidate. ■

Proof of Proposition 4. Suppose $N_1 > \max\{N_2, N_3\}$, but

$$-(x_2 - y_2)^2 > -\beta(x_3 - y_2)^2 - (1 - \beta)(x_1 - y_2)^2, \tag{18}$$

where $y_2 > \theta_{23}$ solves $N_2 + \Phi([\theta_{23}, y_2]) = N_1$. This always holds for $\beta < (x_3 - x_2)/(x_3 - x_1)$, and more generally holds for all sufficiently small β. Then a pure strategic voting equilibrium exists in which all strategic voters with $\theta \in [\theta_{23}, y_2]$ vote for candidate 2 against their expressive preference for candidate 3 in order for candidate 2 to defeat candidate 1. Further, candidate 2 wins in every equilibrium if, in addition,

$$-(x_3 - y_3)^2 < -\beta(x_2 - y_3)^2 - (1 - \beta)(x_1 - y_3)^2, \tag{19}$$

where y_3 solves $N_1 = N_3 + \Phi([y_3, \theta_{23}])$.

Choose y_2' marginally larger than y_2 so that (18) still holds for y_2'. Then candidate 2 wins if strategic citizens in $[\theta_{23}, y_2']$ vote for candidate 2, and all expressive candidate 2 supporters vote for candidate 2. This is an equilibrium because no expressive candidate 2 supporter can earn a higher payoff by voting strategically for a different candidate, so candidates 1 and 3 cannot win. While the actual size of the coalition is not uniquely determined, the coalition is large enough that candidate 2 wins in any equilibrium. It is immediate that an equilibrium in which candidate 1 wins cannot exist, and if (19) holds, then the argument in the proof of Proposition 3 yields that there is no equilibrium in which candidate 3 wins. That is, when (19) holds not enough right-of-center expressive candidate 2 supporters are willing to vote strategically for candidate 3 in order to defeat candidate 1. ■

Proof of Proposition 5. Choose y_3' marginally smaller than y_3 so that (6) still holds for y_3'. Then candidate 3 wins if strategic citizens with $\theta \geq y_3'$ vote for candidate 3. This is an equilibrium because the vote share is large enough that the potential strategic support of expressive candidate 2 supporters $\theta \in [\theta_{12}, y_1]$ is not large enough to defeat candidate 3. Further, by (6) all strategic voters with $\theta \in [y_3', \theta_{23}]$ would prefer to vote for candidate 3 in order to defeat candidate 1. Candidate 3 requires a vote share of at least $N_3 + N([y_3, \theta_{23}])$ in the pure strategy equilibrium, else strategic voters in $(\theta_{12}, y_1]$ would want to strategically support candidate 1 resulting in 1's victory, making them all better off than if they voted expressively. ■

Proof of Proposition 6. We first establish necessary conditions that must hold in any mixed strategy equilibrium.

Claim 1: All strategic coalitions in a mixed strategy equilibrium are intervals of the from $[\theta_{12}, \theta_{12} + z_1]$ and $[\theta_{23} - z_3, \theta_{23}]$ up to a set of measure zero.

Proof: Suppose there exists a set S_i that is not an interval. We focus on the case where S_i is a coalition that votes for candidate 1, as the argument for strategic voters for candidate 3 is analogous. Then there exists a set T with positive measure such that $\theta_{12} < T < \sup S_i$ and $S_i \cap T = \emptyset$. Next, note that a voter type at $\sup S_i$ must be indifferent between being in the coalition and having everyone in S_i voting expressively. In particular, if that voter is strictly worse off, then individual rationality is violated in a neighborhood of $\sup S_i$, and if that voter is strictly better off, we can add a set $T = [\sup S_i, \sup S_i + \varepsilon]$ for some ε, that makes all existing and new coalition members strictly better off, a violation of inclusivity.

Because indifference holds at $\sup S_i$, all types strictly between θ_{12} and $\sup S_i$ are strictly better off if they join the coalition. In particular, this would be true for all members of coalition T, violating inclusivity. Hence, the set must be an interval up to a set of measure zero. $\square$

Because sets of measure zero are irrelevant for determining the winner, we can restrict attention to strategic voting in which all those who vote for a given candidate against their expressive interests comprise an interval. The intervals can be characterized by their endpoints, z_1 and z_3, respectively, i.e., voters in $[\theta_{12}, z_1]$ vote strategically for candidate 1 and those in $\theta_{23} - z_3, \theta_{23}]$ vote strategically for candidate 3.

Claim 2: In any mixed strategy equilibrium, the support of the distribution over the endpoints z_i is an interval.

Proof: Suppose by way of contradiction that one of the distributions, say the distribution F_1 over z_1 does not have an interval support. Let $V = (z_{1,L}, z_{1,H})$ be an open interval of z_1 values that occur with probability zero and $F(z_1) < 1$ for $z_1 \in V$. Let $z_{3,L}$ and $z_{3,H}$ be defined such that the vote ends in a tie when the endpoints of the intervals are $z_{1,i}$, $z_{3,i}$ respectively, for $i = L.H$. Note that there cannot be mass points at the boundaries of V, else a marginal increase of the opposing coalitions would make all coalition members strictly better off as the winning probability would be strictly increased. The voter at $z_{1,H}$ must be indifferent between being in the coalition or having no coalition by claim 1. However, this means that any member of a coalition $\tilde{S}_i = [\theta_{12}, \theta_{12} + \tilde{z}_1]$, where $z_{1,L} < \tilde{z}_1 < \tilde{z}_{1,H}$ is strictly better off being in a coalition. Thus, starting with a coalition $[\theta_{12}, \theta_{12} + z_{1,L}]$ we can add $T = [\theta_{12} + z_{1,L}, \theta_{12} + \tilde{z}_1]$, thereby making everyone strictly better off, and hence violating inclusivity. $\square$

Claim 3: The vote shares must be equal if the largest coalitions are chosen.

Proof: By claim 2 we can conclude that the supports are given by intervals $[0, \bar{z}_1]$ and $[0, \bar{z}_3]$. Suppose by way of contradiction that if coalitions $[\theta_{12}, \theta_{12} + \bar{z}_1]$ and $[\theta_{23} - \bar{z}_3, \theta_{23}]$ form then candidate 1 wins with a strict majority of votes. Recall from claim 1 that indifference must hold at z_1 for any realized coalition $[\theta_{12}, \theta_{12} + z_1]$ where $0 \leq z_1 \leq \bar{z}_1$. However, in a neighborhood of $\bar{z}_1$ the winning probability of candidate 1 remains 1. This, however, implies that if a marginally smaller coalition $[\theta_{12}, \theta_{12} + z_1]$ formed, then the marginal coalition member must be strictly better off, as the winning probability has not changed and that voter is closer to θ_{12}. This contradicts the indifference condition established in claim 1. $\square$

Claim 4: There is no point mass at the upper end of the distributions.

Proof: Using the notation of claim 3, suppose without loss of generality that there is a point mass at $\bar{z}_1$. Claim 3 established that there must be a tie when the coalitions are maximal. However, then coalition $[\theta_{23} - \bar{z}_3, \theta_{23}]$ could be marginally increased. This would result in a discrete increase in candidate 3's winning probability, because in case of tie each candidate wins with strictly positive probability. Hence, inclusivity would be violated. □

Recall that $\theta_{ij} = 0.5(x_i + x_j)$. Let Z be the minimum amount of strategic voting for candidate 3 to have a chance of winning, i.e., Z solves

$$N_1 = N_3 + \Phi(\theta_{23}) - \Phi(\theta_{23} - Z). \tag{20}$$

Let y_1 and y_3 be the cutoffs for strategic voters: $y_1 = \theta_{12} + z_1$ and $y_3 = \theta_{23} - z_3 - Z$, with $z_1, z_3 \geq 0$. where it must be that $\theta_{23} - \theta_{13} > Z$, as the voter at $\theta_{23} - Z$ must prefer candidate 3 to candidate 1. When this does not hold then one of the pure strategy equilibria in which either candidate 1 or candidate 2 wins exists. For any $z_1, z_3 \geq 0$, define the total (expressive plus strategic) vote shares for candidates 1 and 3 by

$$H_1(z_1) \equiv N_1 + \Phi(\theta_{12} + z_1) - \Phi(\theta_{12}) \text{ and } H_3(z_3) \equiv N_3 + \Phi(\theta_{23}) - \Phi(\theta_{23} - Z - z_3). \tag{21}$$

Then candidate 1 wins if $H_1(z_1) > H_3(z_3)$. Because Φ is strictly increasing, this is equivalent to $z_1 > H_1^{-1}(H_3(z_3))$. Candidate 3 wins if the inequality is reversed.

Let $F_i(z_i)$ be the mixed strategy cdf that describes the position of the most extreme strategic voters, i.e., $y_1 = \theta_{12} + z_1$ and $y_3 = \theta_{23} - Z - z_3$. Let q_1 be the mixed strategy probability of choosing $y_1 = \theta_{12}$, and let q_3 be the probability of choosing $y_3 = \theta_{23}$. That is, q_1 and q_3 are the probabilities with which voters do not coordinate on strategic voting for the extreme candidates 1 and 3, i.e., the probabilities with which the candidates only receive votes from expressive supporters.

The indifference condition for each realized marginal type y_1 is

$$\begin{aligned} &-\beta(y_1 - x_2)^2 - (1-\beta)\left(q_3(y_1 - x_1)^2 + (1-q_3)(y_1 - x_3)^2\right) \\ &\quad = -\beta(y_1 - x_1)^2 - (1-\beta)\Big(\left(q_3 + (1-q_3)F_3(H_3^{-1}(H_1(z_1)))\right)(y_1 - x_1)^2 \\ &\qquad + \left(1 - q_3 - (1-q_3)F_3(H_3^{-1}(H_1(z_1)))\right)(y_1 - x_3)^2\Big). \end{aligned} \tag{22}$$

The left-hand side is the expected payoff if all members of the realized coalition $S_1 = \{\theta : \theta \leq y_1\}$ who expressively prefer candidate 2 vote for 2, which leads to 1 winning if and only if $y_3 = \theta_{23}$, which happens with probability q_3, because $V_1^E > V_3^E$. The right-hand side is the expected payoff if all members of the realized coalition S_1 vote for candidate 1. Now, candidate 1 wins either when $y_3 \leq \theta_{23} - Z$, which happens with probability $1 - q_3$; or when the positive measure of the realized coalition S_3 is less than S_1, which happens when $z_3 \leq H_3^{-1}(H_1(z_1))$. Thus, candidate 1 wins with probability $F_3(H_3^{-1}(H_1(z_1)))$.

The analogous indifference condition to (22) for each realized marginal type y_3 is

$$\begin{aligned}
&-\beta(y_3-x_2)^2-(1-\beta)(y_3-x_1)^2\\
&\quad=-\beta(y_3-x_3)^2-(1-\beta)\Big(\big(q_1+(1-q_1)F_1(H_1^{-1}(H_3(z_3)))\big)(y_3-x_3)^2\\
&\qquad\qquad+\big(1-q_1-(1-q_1)F_1(H_1^{-1}(H_3(z_3)))\big)(y_3-x_1)^2\Big).
\end{aligned} \tag{23}$$

The citizen at $\theta_{13}=0.5(x_1+x_3)$ is indifferent between either extreme candidate winning. Thus, if strategic voting occurs θ_{23} must be strictly outside the voter coalition as long as voters place any weight $\beta>0$ on expressive preferences.

Note that

$$x_2-x_1=2(\theta_{23}-\theta_{13}),\ x_3-x_1=2(\theta_{23}-\theta_{12}),\text{ and } x_3-x_2=2(\theta_{13}-\theta_{12}). \tag{24}$$

Solving equation (22) for F_3 using $y_1=\theta_{12}+z_1$ and (24) yields

$$F_3(H_3^{-1}(H_1(z_1)))=\frac{\beta}{1-\beta}\frac{1}{1-q_3}\frac{(\theta_{23}-\theta_{13})\,z_1}{(\theta_{23}-\theta_{12})\,(\theta_{13}-\theta_{12}-z_1)}. \tag{25}$$

Similarly, solving (23) for F_1, using $y_3=\theta_{23}-Z-z_3$ and (24) yields

$$F_1(H_1^{-1}(H_3(z_3)))=\frac{\beta}{1-\beta}\frac{1}{1-q_1}\frac{(\theta_{13}-\theta_{12})\,(Z+z_3)}{(\theta_{23}-\theta_{12})\,(\theta_{23}-\theta_{13}-(Z+z_3))}-\frac{q_1}{1-q_1}. \tag{26}$$

Thus,

$$F_1(z)=\frac{\beta}{1-\beta}\frac{1}{1-q_1}\frac{(\theta_{13}-\theta_{12})\left(Z+H_3^{-1}(H_1(z))\right)}{(\theta_{23}-\theta_{12})\left(\theta_{23}-\theta_{13}-(Z+H_3^{-1}(H_1(z)))\right)}-\frac{q_1}{1-q_1}; \tag{27}$$

$$F_3(z)=\frac{\beta}{1-\beta}\frac{1}{1-q_3}\frac{(\theta_{23}-\theta_{13})\,H_1^{-1}(H_3(z))}{(\theta_{23}-\theta_{12})\left(\theta_{13}-\theta_{12}-H_1^{-1}(H_3(z))\right)}. \tag{28}$$

Note that $H_3^{-1}(H_1(z))$ and $H_1^{-1}(H_3(z))$ are strictly monotone in z and therefore F_1 and F_3 are strictly increasing on their supports. Further, (20) implies $H_3^{-1}(H_1(0))=0$ and $H_1^{-1}(H_1(0))=0$. Hence, $F_3(0)=0$.

Further, given our definition of q_1 we have $F_1(0)=0$. Substituting $F_1(0)=0$ and $H_3^{-1}(H_1(0))=0$ in (27) we solve for:

$$q_1=\frac{\beta}{1-\beta}\frac{(\theta_{13}-\theta_{12})Z}{(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}-Z)}. \tag{29}$$

Note that $q_1\geq 0$ because the interval of strategic voting $[\theta_{23}-Z-z_3,\theta_{23}]$ of candidate 2 supporters who vote for candidate 3 must be strictly to the right of the voter θ_{13} who is indifferent between candidates 1 and 3. If the solution has $q_1\geq 1$ then there is no mixed strategy equilibrium. Either candidate 1's expressive vote support advantage is sufficiently large to win (Proposition 3), or we get the pure strategy equilibrium in which enough expressive candidate 3 supporters vote strategically for candidate 2 that 2 wins (Proposition 5).

Next, let $[0, \bar{z}_i]$ be the support of F_i. Then claim 3 implies $H_1(\bar{z}_1) = H_3(\bar{z}_3)$. Further, claim 4 implies that there cannot be a mass point at the upper end of either distribution. Thus, in this equilibrium we set $H_1(\bar{z}_1) = H_3(\bar{z}_3)$, which pins down q_3.

Setting the right-hand sides of (27) to 1 and solving for $\bar{z}_i$, $i = 1, 3$ yields

$$H_3^{-1}(H_1(\bar{z}_1)) = \frac{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}) - (\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13}))Z}{\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13})}. \tag{30}$$

Similarly we get

$$H_1^{-1}(H_3(\bar{z}_3)) = \frac{(1-\beta)(1-q_3)(\theta_{13}-\theta_{12})(\theta_{23}-\theta_{12})}{(1-\beta)(1-q_3)(\theta_{23}-\theta_{12}) + \beta(\theta_{23}-\theta_{13})}. \tag{31}$$

Assuming that $H_1(\bar{z}_1) = H_3(\bar{z}_3)$ we get

$$\begin{aligned} &H_3\left(\frac{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}) - (\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13}))Z}{\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13})}\right) \\ &\qquad = H_1\left(\frac{(1-\beta)(1-q_3)(\theta_{13}-\theta_{12})(\theta_{23}-\theta_{12})}{(1-\beta)(1-q_3)(\theta_{23}-\theta_{12}) + \beta(\theta_{23}-\theta_{13})}\right). \end{aligned}$$

Thus, we can solve:

$$q_3 = \frac{(\theta_{13}-\theta_{12})((1-\beta)(\theta_{23}-\theta_{12}) + \beta C) - (\theta_{23}-\theta_{12})C}{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{13}-\theta_{12}-C)}, \tag{32}$$

where

$$C = H_1^{-1}\left(H_3\left(\frac{(1-\beta)(\theta_{23}-\theta_{12})(\theta_{23}-\theta_{13}) - (\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13}))Z}{\theta_{23}-\theta_{12}-\beta(\theta_{23}-\theta_{13})}\right)\right). \tag{33}$$

It remains to prove that $0 \le q_3 \le 1$. Let $\bar{y}_1$ and $\bar{y}_3$ solve

$$-(x_1-\bar{y}_1)^2 = -\beta(x_2-\bar{y}_1)^3 - (1-\beta)(x_1-\bar{y}_1)^2 \tag{34}$$

and

$$-(x_3-\bar{y}_3)^2 = -\beta(x_2-\bar{y}_3)^2 - (1-\beta)(x_1-\bar{y}_3)^2. \tag{35}$$

Further, let $N(\bar{y}_1) \ge 1 - N(\bar{y}_3)$ (recall that a pure strategy equilibrium exists if the inequality is reversed). Let $\hat{y}_1$ be the upper end of the support of F_1. If this largest coalition forms, then candidate 1 must win with probability 1 by claims 3 and 4. Substituting $F_3(\cdot) = 1$ into (22) implies

$$-\beta(\hat{y}_1-x_2)^2 - (1-\beta)\left(q_3(\hat{y}_1-x_1)^2 + (1-q_3)(\hat{y}_1-x_3)^2\right) = -(\hat{y}_1-x_1)^2 \tag{36}$$

It follows immediately that $\hat{y}_1 = \bar{y}_1$ if $q_3 = 1$ and that $\hat{y}_1$ is decreasing in q_3.

Similarly, let $\hat{y}_3$ be the upper end of the support of F_3. We can again conclude that that maximal coalition would win with probability 1, and hence (23) reduces to (35). Thus, $\hat{y}_3 = \bar{y}_3$.

We have established in claim 3 that $N(\hat{y}_1) = 1 - N(\hat{y}_3) = 1 - N(\bar{y}_3)$. Further, we have established that $N(\bar{y}_1) \ge 1 - N(\bar{y}_3)$. Finally, recall that $\hat{y}_1 = \bar{y}_1$ for $q_3 = 0$. Similarly, it is easy to see that $\bar{y}_1 = \theta_{12}$ if $q_3 = 1$.

Thus, continuity implies that there exists $0 \leq q_3 < 1$ such that $N(\hat{y}_1) = 1 - N(\hat{y}_3)$. This is the value given by (33).

Finally, $0 \leq q_1 < 1$. We have already shown that $q_1 \geq 0$. Thus, it remains to prove that $q_1 < 1$.

Let $\tilde{F}_1$ be given by (27) if we set $q_1 = 0$. Note that $\tilde{F}_1(0)$ is then equal to q_1 as defined in (29). Recall that $\theta_{23} - Z > \theta_{13}$, else strategic voting of expressive 2 supporters for candidate 3 would not generate enough votes for candidate 3 to win. Thus, $\tilde{F}_1(z_1)$ is strictly increasing in z_1. Because $\tilde{F}_1(\bar{z}_1) = 1$ at the upper end of the support of $\tilde{F}_1$ and $\bar{z}_1 > 0$, monotonicity implies that $q_1 = \tilde{F}_1(0) < \tilde{F}_1(\bar{z}_1) = 1$. □ ■

Proof of Proposition 7. Suppose without loss of generality that $N_1 + \Phi([\theta_{12}, \theta_{13}]) > N_3 + \Phi([\theta_{13}, \theta_{23}])$ holds. Suppose that the probability of victory for candidate 1 is bounded from above by $q < 1$ for arbitrarily small β. However, for such small β a type that is marginally to the left of θ_{13} would be willing to vote strategically for candidate 1 to increase the probability that 1 wins from q to 1. Further all individuals to the left of this marginal type would similarly be strictly better off. By assumption, if these voters join an existing voting coalition they would all be uniformly strictly better off joining the coalition. This violates inclusivity, a contradiction. ■

Proof of Proposition 8. In the mixed strategy equilibrium, the marginal realized coalition member is indifferent (by individual rationality and inclusivity) between having all members vote expressively or all voting strategically. Further, regardless of the level of candidate 1's base support advantage, if members of a realized coalition for candidate 3 vote expressively, candidate 1 is sure to win. It follows that the indifference condition for each possible marginal realized coalition member $\theta < \theta_{23}$ who votes for candidate 3 but expressively prefers candidate 2 given ρ_1' takes the form:

$$-\beta(\theta - x_2)^2 - (1 - \beta)(\theta - x_1)^2 = -\beta(\theta - x_3)^2 - (1 - \beta)(\Pr(1 \text{ wins})(\theta - x_1)^2 + \Pr(3 \text{ wins})(\theta - x_3)^2)$$

The left-hand side does not depend on the level of candidate 1's ex ante base support advantage. It follows that the probability that candidate 3 wins when all voters to the right of θ vote for candidate 3 given ρ_1' is the same as that given ρ_1. Since this holds for each such θ, it follows that the distribution over the sizes of strategic coalitions supporting candidate 1 is shifted to the left by $\rho_1' - \rho_1$. Further, setting Pr(3 wins) to one, it follows that the size of the maximal possible coalition supporting candidate 3 is unchanged.

Next, observe that the smallest non-trivial coalition for candidate 3 increases due to candidate 1's increased base support advantage. Because the marginal member of this coalition has the same probability of winning given ρ_1' as ρ_1, it follows that $q_1' > q_1$.

Further, the requirement that there be a tie in vote share when both coalitions are maximal together with the size of the maximal coalition supporting candidate 3 being unchanged, it follows that the size of the maximal coalition supporting candidate 1 is reduced by $\rho_1' - \rho_1$. The indifference condition for the marginal maximal coalition member takes the form:

$$-\beta(\theta - x_2)^2 - (1-\beta)(q_3'(\theta - x_1)^2 + (1-q_3')(\theta - x_3)^2 = -\beta(\theta - x_1)^2 - (1-\beta)(\theta - x_1)^2,$$

since candidate 1 always wins when the maximal coalition forms. Because the increase to ρ_1' shifts θ to the left, the right-hand side of the indifference condition increases, implying that the left-hand side must increase, i.e., $q_3' > q_3$. For smaller coalitions, the marginal coalition member's indifference condition takes the form

$$-\beta(\theta-x_2)^2-(1-\beta)(q_3'(\theta-x_1)^2+(1-q_3')(\theta-x_3)^2 = -\beta(\theta-x_1)^2-(1-\beta)(\Pr(1\text{ wins})(\theta-x_1)^2+\Pr(3\text{ wins})(\theta-x_3)^2).$$

Because $q_3' > q_3$, the left-hand side is larger given ρ_1' than ρ_1, implying that the right-hand side must also be larger, i.e., it must be that candidate 1 is more likely to win. ■

Proof of Proposition 9. If $\beta \le \bar{\beta}$, then $y_{23} = \infty$, so $\bar{\rho} = 2\Phi(0.5(x_1 + x_2)) - 1$. Hence, the terms $\phi(y_{23})\frac{\partial y_{23}}{\partial x_1}$ in the first-order conditions (13) and (14) disappear, which yields

$$g(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)\right)\left(\frac{x_1+x_2}{2} - \theta_1\right)(x_2 - x_1) - (x_1 - \theta_1)G(\bar{\rho}) = 0$$

and

$$-g(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right)\right)\left(\theta_2 - \frac{x_1+x_2}{2}\right)(x_2 - x_1) + (\theta_2 - x_2)(1 - G(\bar{\rho})) = 0.$$

Substituting $\theta_1 = -\theta_2$, and imposing symmetry, $x_1 = -x_2$, we solve these first order conditions for the equilibrium locations:

$$x_2 = -x_1 = \frac{\theta_2}{1 + 4\theta_2 g(0)\phi(0)}, \tag{37}$$

which also implies $\bar{\rho} = 0$.

We now show that we have enough structure to apply the implicit function theorem to characterize the equilibrium candidate location responses to slight increases in β above $\bar{\beta}$. Define

$$f(\beta) = \phi(y_{23})\frac{\partial y_{23}}{\partial x_1}.$$

Clearly, f is continuously differentiable for $\beta \neq \bar{\beta}$. We next show that f is also continuously differentiable at $\bar{\beta}$. Because the left-derivative of f at $\bar{\beta}$ is trivially zero, it is sufficient to show that $\lim_{\beta\downarrow\bar{\beta}} f'(\beta) = 0$.

For $\beta > \bar{\beta}$

$$\frac{\partial y_{23}}{\partial \beta} = -\frac{1}{2}\frac{(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)}{((1-\beta)x_1 + \beta x_3 - x_2)^2}. \tag{38}$$

and

$$\frac{\partial^2 y_{23}}{\partial\beta\partial x_1} = \frac{1}{2}\frac{(x_3 - x_2)^2\,(x_2 + \beta x_3 - (1+\beta)x_1)}{((1-\beta)x_1 + \beta x_3 - x_2)^3}. \tag{39}$$

Because $\frac{\partial y_{23}}{\partial \beta}$ and $\frac{\partial y_{23}}{\partial x_1}$ both go to infinity at rate $\frac{1}{(\beta-\bar{\beta})^2}$ as $\beta \downarrow \bar{\beta}$, for $\beta > \bar{\beta}$ there exists $K_1, K_2 > 0$ such that

$$\left|f'(\beta)\right| = \left|\phi'(y_{23})\frac{\partial y_{23}}{\partial \beta}\frac{\partial y_{23}}{\partial x_1} + \phi(y_{23})\frac{\partial^2 y_{23}}{\partial \beta \partial x_1}\right| \leq \left|\phi'(y_{23})\right| \frac{K_1}{(\beta-\bar{\beta})^4} + \phi(y_{23})\frac{K_2}{(\beta-\bar{\beta})^3}. \tag{40}$$

Further, y_{23} goes to infinity at the rate $1/(\beta-\bar{\beta})$ as $\beta \downarrow \bar{\beta}$. Because the fourth moment of Φ is finite, it follows that $\lim_{x\to\infty} x^4\phi(x) = 0$. Integration by parts yields that $\int_0^\infty x^4\phi'(x)\,dx = x^4\phi(x)|_0^\infty - 4\int_0^\infty x^3\phi(x)\,dx$. Hence, $\int_0^\infty x^4\phi'(x)\,dx$ is finite, which implies that $\lim_{x\to\infty} x^4\phi'(x) = 0$. This and (40) yield $\lim_{\beta\downarrow\bar{\beta}} f'(\beta) = 0$.

To show differentiability at $\bar{\beta}$, it is sufficient to prove that

$$\lim_{\beta\downarrow\bar{\beta}} \frac{f(\beta) - f(\bar{\beta})}{\beta - \bar{\beta}} = 0. \tag{41}$$

The argument is similar to above. Note that $f(\beta) < \hat{K}/(\beta-\bar{\beta})^2$ for β marginally larger than $\bar{\beta}$ and some $\hat{K} > 0$. Further, we have shown that $\phi(y_{23}) < \varepsilon(\beta-\bar{\beta})^4$, for β near $\bar{\beta}$ because $\lim_{x\to\infty} x^4\phi(x) = 0$. Thus, the limit in (41) exists and is zero. Hence, $f(\beta)$ is continuously differentiable, and $f'(\bar{\beta}) = 0$.

An analogous argument shows that $\phi(y_{23})\frac{\partial y_{23}}{\partial x_2}$ is continuously differentiable, and that the first derivative with respect to β at $\bar{\beta}$ is zero.

Next differentiate candidate 1's first-order condition (13) with respect to β to obtain:

$$\begin{aligned}\frac{\partial \mathrm{FOC}_1}{\partial \beta} =& \phi(y_{23})\frac{\partial y_{23}}{\partial \beta}\left(-g'(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right) - \phi(y_{23})\frac{\partial y_{23}}{\partial x_1}\right)\left(\frac{x_1+x_2}{2} - \theta_1\right)(x_2 - x_1)\right)\\ &- \phi(y_{23})\frac{\partial y_{23}}{\partial \beta}g(\bar{\rho})\left(\left(\frac{\phi'(y_{23})}{\phi(y_{23})}\frac{\partial y_{23}}{\partial x_1} + \frac{\frac{\partial^2 y_{23}}{\partial x_1 \partial \beta}}{\frac{\partial y_{23}}{\partial \beta}}\right)\left(\frac{x_1+x_2}{2} - \theta_1\right)(x_2 - x_1) + (x_1 - \theta_1)\right).\end{aligned} \tag{42}$$

Similarly, differentiating candidate 2's first-order condition (14) with respect to β yields

$$\begin{aligned}\frac{\partial \mathrm{FOC}_2}{\partial \beta} =& \phi(y_{23})\frac{\partial y_{23}}{\partial \beta}\left(g'(\bar{\rho})\left(\phi\left(\frac{x_1+x_2}{2}\right) - \phi(y_{23})\frac{\partial y_{23}}{\partial x_2}\right)\left(\theta_2 - \frac{x_1+x_2}{2}\right)(x_2 - x_1)\right)\\ &+ \phi(y_{23})\frac{\partial y_{23}}{\partial \beta}g(\bar{\rho})\left(\left(\frac{\phi'(y_{23})}{\phi(y_{23})}\frac{\partial y_{23}}{\partial x_2} + \frac{\frac{\partial^2 y_{23}}{\partial x_2 \partial \beta}}{\frac{\partial y_{23}}{\partial \beta}}\right)\left(\theta_2 - \frac{x_1+x_2}{2}\right)(x_2 - x_1) + (\theta_2 - x_2)\right).\end{aligned} \tag{43}$$

Note that the term in the large parentheses on the first lines of (42) and (43) are zero, respectively at $\beta = \bar{\beta}$. In contrast, the terms inside the large parentheses on the second lines of (42) and (43) go to infinity. Thus,

$$\lim_{\beta\downarrow\bar{\beta}} \frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = -\lim_{\beta\downarrow\bar{\beta}} \frac{\frac{\phi'(y_{23})}{\phi(y_{23})}\frac{\partial y_{23}}{\partial x_1} + \frac{\frac{\partial^2 y_{23}}{\partial x_1 \partial \beta}}{\frac{\partial y_{23}}{\partial \beta}}}{\frac{\phi'(y_{23})}{\phi(y_{23})}\frac{\partial y_{23}}{\partial x_2} + \frac{\frac{\partial^2 y_{23}}{\partial x_2 \partial \beta}}{\frac{\partial y_{23}}{\partial \beta}}} = \frac{x_3 - x_2}{x_3 - x_1}. \tag{44}$$

Next, differentiate the first-order conditions with respect to x_1 and x_2 at $\beta = \bar{\beta}$. Note that $\bar{\rho} = 0$ at $\bar{\beta}$, and hence $g'(\bar{\rho}) = 0$. Similarly, because $x_1 = -x_2$ in equilibrium at $\bar{\beta}$, $\phi'(0.5(x_1 + x_2)) = 0$. Further,

$\phi(y_{23})\frac{\partial y_{23}}{\partial x_1} = 0$ and $\frac{\partial G(\bar{\rho})}{\partial x_i} = g(0)\phi(0)$. Therefore,

$$\left.\frac{\partial \mathrm{FOC}_1}{\partial x_1}\right|_{\beta=\bar{\beta}} = -2g(0)\phi(0)(x_1-\theta_1) - \frac{1}{2} < 0, \quad \left.\frac{\partial \mathrm{FOC}_1}{\partial x_2}\right|_{\beta=\bar{\beta}} = g(0)\phi(0)(x_2-x_1) > 0, \tag{45}$$

and

$$\left.\frac{\partial \mathrm{FOC}_2}{\partial x_1}\right|_{\beta=\bar{\beta}} = g(0)\phi(0)(x_2-x_1) > 0, \quad \left.\frac{\partial \mathrm{FOC}_2}{\partial x_2}\right|_{\beta=\bar{\beta}} = -2g(0)\phi(0)(\theta_2-x_2) - \frac{1}{2} < 0. \tag{46}$$

Let $x_1(\beta)$ and $x_2(\beta)$ be the optimal policy of candidates 1 and 2 given β. Further, let $\tilde{x}_1(\beta, x_2)$ be the solution to the first-order conditions (13) with respect to x_1. Similarly, let $\tilde{x}_2(\beta x_1)$ be the solution to (14) with respect to x_2. Then $x_1(\beta) = \tilde{x}_1(\beta, x_2(\beta))$ and $x_2(\beta) = \tilde{x}_2(\beta, x_1(\beta))$. Differentiating with respect to β yields

$$x_1'(\beta) = \frac{\partial \tilde{x}_1(\beta, x_2)}{\partial \beta} + \frac{\partial \tilde{x}_1(\beta, x_2)}{\partial x_2} x_2'(\beta); \tag{47}$$

$$x_2'(\beta) = \frac{\partial \tilde{x}_2(\beta, x_1)}{\partial \beta} + \frac{\partial \tilde{x}_2(\beta, x_1)}{\partial x_1} x_1'(\beta). \tag{48}$$

Solving these equations for $x_1'(\beta)$ and $x_2'(\beta)$ yields

$$x_1'(\beta) = \frac{\frac{\partial \tilde{x}_1(\beta,x_2)}{\partial x_2}\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial \beta} + \frac{\partial \tilde{x}_1(\beta,x_2)}{\partial \beta}}{1 - \frac{\partial \tilde{x}_1(\beta,x_2)}{\partial x_2}\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial x_1}}; \tag{49}$$

$$x_2'(\beta) = \frac{\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial x_1}\frac{\partial \tilde{x}_1(\beta,x_2)}{\partial \beta} + \frac{\partial \tilde{x}_2(\beta,x_1)}{\partial \beta}}{1 - \frac{\partial \tilde{x}_1(\beta,x_2)}{\partial x_2}\frac{\partial \tilde{x}_2(\beta,x_1)}{\partial x_1}}. \tag{50}$$

The implicit function theorem implies that for $i \neq j$,

$$\frac{\partial \tilde{x}_i(\beta, x_j)}{\partial \beta} = -\frac{\frac{\partial \mathrm{FOC}_i}{\partial \beta}}{\frac{\partial \mathrm{FOC}_i}{\partial x_i}}, \text{ and } \frac{\partial \tilde{x}_i(\beta, x_j)}{\partial x_j} = -\frac{\frac{\partial \mathrm{FOC}_i}{\partial x_j}}{\frac{\partial \mathrm{FOC}_i}{\partial x_i}} \tag{51}$$

Let D be the denominator term in equations (49) and (50). Substituting (45), (46), and (51) into D yields.

$$\lim_{\beta\to\bar{\beta}} D = 1 - \frac{\frac{\partial \mathrm{FOC}_1}{\partial x_2}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\frac{\frac{\partial \mathrm{FOC}_2}{\partial x_1}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} = 1 - \frac{4g(0)^2\phi(0)^2(x_2-x_1)^2}{(1+4g(0)\phi(0)(\theta_2-x_2))\,(1+4g(0)\phi(0)(x_1-\theta_1))}. \tag{52}$$

Using symmetry, i.e., $\theta_2 = -\theta_1$ and x_1 and x_2 from the symmetric solution (37) implies

$$\lim_{\beta\to\bar{\beta}} D = \frac{(1+4g(0)\phi(0)\theta_2)^2\left(1+16g(0)^2\phi(0)^2\theta_2^2\right)}{\left(1+4g(0)\phi(0)\theta_2+16g(0)^2\phi(0)^2\theta_2^2\right)^2} > 0. \tag{53}$$

Equations (49) and (51) imply

$$D\frac{\partial x_1(\beta)}{\partial \beta} = \frac{\frac{\partial \mathrm{FOC}_1}{\partial x_2}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} - \frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}} = \frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\left(\frac{\frac{\partial \mathrm{FOC}_1}{\partial x_2}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} - \frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}\right) \tag{54}$$

$$D\frac{\partial x_2(\beta)}{\partial \beta} = \frac{\frac{\partial \mathrm{FOC}_2}{\partial x_1}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}}\frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}} - \frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}} = \frac{\frac{\partial \mathrm{FOC}_2}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial x_2}}\left(\frac{\frac{\partial \mathrm{FOC}_2}{\partial x_1}}{\frac{\partial \mathrm{FOC}_1}{\partial x_1}}\frac{\frac{\partial \mathrm{FOC}_1}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} - 1\right) \tag{55}$$

Substituting the above derivatives we get

$$\lim_{\beta \downarrow \bar{\beta}} D \frac{\frac{\partial x_1(\beta)}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = \frac{2(x_3 - x_2) + 8g(0)\phi(0)(\theta_2(x_3 - x_2) + 2x_2^2)}{(x_2 + x_3)\,(1 + 4g(0)\phi(0)(\theta_2 - x_2))^2}. \tag{56}$$

Thus, $\frac{\partial x_1(\beta)}{\partial \beta} > 0$ for β that are marginally larger than $\bar{\beta}$, because $D > 0$. Similarly,

$$\lim_{\beta \downarrow \bar{\beta}} D \frac{\frac{\partial x_2(\beta)}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = \frac{2(x_2 + x_3) + 8g(0)\phi(0)(\theta_2(x_2 + x_3) - 2x_2^2)}{(x_2 + x_3)\,(1 + 4g(0)\phi(0)(\theta_2 - x_2))^2}, \tag{57}$$

which implies $\frac{\partial x_2(\beta)}{\partial \beta} > 0$ for β marginally larger than $\bar{\beta}$, because $\theta_2, x_3 > x_2$.

Next, we prove that candidate 2 moves her policy by more to the right than candidate 1 moves her policy:

$$\lim_{\beta \downarrow \bar{\beta}} D \frac{\frac{\partial x_2(\beta)}{\partial \beta} - \frac{\partial x_1(\beta)}{\partial \beta}}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = \frac{4x_2\,(1 + 4g(0)\phi(0)\theta_2 - 8g(0)\phi(0)x_2)}{(x_2 + x_3)\,(1 + 4g(0)\phi(0)(\theta_2 - x_2))^2}, \tag{58}$$

which is strictly positive because (15) implies

$$1 + 4g(0)\phi(0)\theta_2 - 8g(0)\phi(0)x_2 = \frac{1 + 16g(0)^2\phi(0)^2\theta_2^2}{1 + 4g(0)\phi(0)\theta_2} > 0. \tag{59}$$

Finally, we prove that the spoiler is better off for β marginally larger than $\bar{\beta}$. The spoiler's utility is

$$U_3(\beta) = -G(\bar{\rho})(x_1(\beta) - x_3)^2 - (1 - G(\bar{\rho}))(x_2(\beta) - x_3)^2. \tag{60}$$

Differentiating with respect to β yields

$$\begin{aligned} U_3'(\beta) = &- g(\bar{\rho})\left(\phi\left(\frac{x_1(\beta) + x_2(\beta)}{2}\right)\frac{x_1'(\beta) + x_2'(\beta)}{2} - \phi(y_{23})\left(\frac{\partial y_{23}}{\partial \beta} + \frac{\partial y_{23}}{\partial x_1}x_1'(\beta) + \frac{\partial y_{23}}{\partial x_2}x_2'(\beta)\right)\right) \\ &\cdot \left((x_1(\beta) - x_3)^2 - (x_2(\beta) - x_3)^2\right) \\ &+ 2G(\bar{\rho})(x_3 - x_1(\beta))x_1'(\beta) + 2(1 - G(\bar{\rho}))(x_3 - x_2(\beta))x_2'(\beta). \end{aligned} \tag{61}$$

Note that

$$\lim_{\beta \downarrow \bar{\beta}} \left| \frac{\frac{\partial y_{23}}{\partial \beta}\phi(y_{23})}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} \right| = \lim_{\beta \downarrow \bar{\beta}} \left| \frac{\phi(y_{23})}{\phi'(y_{23})g(\bar{\rho})\frac{\partial y_{23}}{\partial \beta}\theta_2(x_2 - x_1)} \right| \le \lim_{\beta \downarrow \bar{\beta}} C(\beta - \bar{\beta})^{2-b} = 0, \tag{62}$$

where $C > 0$ is some constant. The last inequality follows because $|\phi'(x)/\phi(x)| > \varepsilon/x^b$ for $0 < b < 1$ and $\frac{\partial y_{23}}{\partial \beta}$ goes to infinity at the rate $1/(\beta - \bar{\beta})^2$. Further,

$$\lim_{\beta \downarrow \bar{\beta}} \frac{\frac{\partial y_{23}}{\partial x_i}x_i'(\beta)}{\frac{\partial \mathrm{FOC}_2}{\partial \beta}} = 0. \tag{63}$$

because $\frac{\partial y_{23}}{\partial x_i}$ goes to zero as $\beta \downarrow \bar{\beta}$. Thus,

$$\lim_{\beta\downarrow\bar{\beta}} \frac{U_3'(\beta)}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = \lim_{\beta\downarrow\bar{\beta}} -2g(0)\phi(0)\left(\frac{\frac{\partial x_1(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} + \frac{\frac{\partial x_2(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}\right) x_2(\bar{\beta})x_3 + (x_3 + x_2(\bar{\beta}))\frac{\frac{\partial x_1(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} + (x_3 - x_2(\bar{\beta}))\frac{\frac{\partial x_2(\beta)}{\partial\beta}}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}}. \qquad (64)$$

Substituting (56) and (57) and writing x_2 for $x_2(\bar{\beta})$ yields

$$\lim_{\beta\downarrow\bar{\beta}} D\frac{U_3'(\beta)}{\frac{\partial \mathrm{FOC}_2}{\partial\beta}} = \frac{4\,(1 + 4g(0)\phi(0)\theta_2)\,(x_3^2 - x_2^2) + 8g(0)\phi(0)x_2\left(4x_2^2 - \theta_2 x_3\right)}{(x_2 + x_3)(1 + 4g(0)\phi(0)(\theta_2 - x_2))^2}. \qquad (65)$$

The denominator of (65) is strictly positive. To verify that the numerator is strictly positive for $x_3 \geq \theta_2$ substitute the solutions for x_1 and x_2 from (37) into the numerator and evaluate at $x_3 = \theta_2$ to get

$$\frac{8g(0)\phi(0)\theta_2^3\left(7 + 28g(0)\phi(0)\theta_2 + 80g(0)^2\phi(0)^2\theta_2^2 + 64g(0)^3\phi(0)^3\theta_2^3\right)}{(1 + 4g(0)\phi(0)\theta_2)^3} > 0. \qquad (66)$$

Differentiating the numerator of (65) with respect to x_3 and again using (37) yields

$$8(1 + 4g(0)\phi(0)\theta_2)(1 - 2g(0)\phi(0)x_2)x_3 = 2(1 + 2g(0)\phi(0)\theta_2)x_3 > 0. \qquad (67)$$

Thus, (65) is strictly positive for $x_3 \geq \theta_3$. Hence, candidate 3's utility from entry is increasing in β for β slightly larger than $\bar{\beta}$. ■

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