Collaboration Between and Within Groups*

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August 20, 2023

Abstract

We study the ability of multi-group teams to undertake binary projects in a decentralized environment. The equilibrium outcomes of our model display familiar features in collaborative settings, including inefficient gradualism, inaction, and contribution cycles, wherein groups alternate taking responsibility for moving the project forward. Expected delay grows more than proportionally with project size, and some welfare-enhancing projects are not completed, even as agents become arbitrarily patient. A team composed of two equally large groups can complete larger projects than a fully homogenous team, even as the difference in preferences for completion among the two groups is arbitrarily small. Moreover, if the project is sufficiently large, the two-group team always completes the project strictly faster.

*Iaryczower gratefully acknowledges financial support from NSF grant SBE 1757191. Oliveros gratefully acknowledges the support of the Economic and Social Research Council (UK), ES/S01053X/1.

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1 Introduction

In the textbook notion of a team, agents work together to attain a common goal. In many real life scenarios, however, teams are composed of distinct groups. In these contexts, agents collaborate with both peers and "outsiders," who can have a different valuation for completion of the project, or a different effectiveness in moving the project forward. In a research lab, for instance, senior scholars collaborate with junior researchers, for whom a publication can have a larger career impact. In corporations, new product launches usually involve both Product and Operations personnel, who jointly develop new infrastructure to make the product available to clients. In open source software development, both amateur and professional developers write the code that drives projects like Linux and Mozilla Firefox. Collaboration between groups is also paramount in applications within political economy. In lobbying, beneficiaries of a piece of legislation who seek to influence legislators typically belong to different industries, each of which has a different stake in the bill's success.

In this paper, we study the ability of such multi-group teams to "get things done." In particular, we are interested in understanding how heterogeneity in costs or preferences across groups shapes the expected time to complete projects, as well as the size of projects that the team can complete. On the behavioral (positive) side, we analyze how heterogeneity across groups affects the pattern of contributions by members of the two groups. Does the group with a higher valuation for completion of the project contribute more? Do agents belonging to this group contribute in earlier or later phases of the project? We study these questions in a decentralized environment, without commitment. By this we mean that the identity of the contributor and the amount of the contribution at each point in time are not determined by a central authority or contract, but instead come from random opportunities to add value (e.g. ideas), and individual incentives to cooperate. This is particularly relevant in environments in which effort is elicited through a horizontal structure, as in open source contributions or cross-functional teams in firms. In line with our examples, we treat projects as binary; that is, as *threshold* public goods (i.e., payoffs accrue only when the paper is completed, when the product is launched, when the bill passes, etc).

In the model, two groups of internally identical agents – say junior and senior researchers– collaborate on a binary project. To capture the decentralized nature of contributions, we assume that in each period an agent is selected at random to make a contribution to the project. We allow juniors and seniors to have a different probability of having an opportunity to contribute, but assume that this is equal among agents of the same group. In line with the binary project assumption, we assume that agents of group m get a payoff $\theta^m > 0$ when the project is completed, and zero otherwise. To distinguish strategic considerations from technological incentives for gradualism, we assume that the cost of contributing is linear, and assume there are no deadlines, or asymmetric information among agents. To maintain within-group homogeneity, we focus on group-symmetric subgame perfect equilibria (SPE).

We show that the game has a generically unique equilibrium in pure strategies. The equilibrium has the feature that agents' strategies are history-independent (i.e., Markov), conditioning only on the remaining effort s required to complete the project. Exploiting the Markovian structure of equilibrium, we solve the game recursively from a subgame starting in the state $s = \phi_1 > 0$, the largest at which members of one group, say juniors, finish the project outright. We call this subgame a *small project*.

Our first main result characterizes the final small project, which juniors complete without delay. We show that in equilibrium, seniors are only willing to make partial contributions towards completion, moving the project forward in a series of steps, which decrease in size as the project gets farther from completion. Remarkably, the size of these contributions converges to zero at some point $\overline{s}_1 < \phi_1$. This means that there is a set of states $(\overline{s}_1, \phi_1]$ in which seniors do nothing, fully free-riding on juniors for the completion of the project in one shot. This *inaction* by seniors implies that at ϕ_1, the project is stalled with positive probability, and only moves forward (and in fact is completed outright) when a junior member has an opportunity to play.

Seniors' slugishness in contributing from states $s < \overline{s}_1$ is due to in-group free riding. Their inaction at ϕ_1, on the other hand, is due to out-group incentives. As the number of steps and the associated delay to completion introduced by senior agents increases, juniors' willingness to contribute *increases* as well. For them, the more sluggish seniors become, the more attractive completion becomes, to avoid costly delay. Formally, for any additional step senior members take to complete the project, the maximal state ϕ_1 at which juniors are willing to complete the project outright increases by a larger amount than the size of the seniors' last step. This assures the existence of a set of states in which only juniors contribute to advance the project.

Our second main result builds on the recursivity of equilibrium from ϕ_1 on. We show that the overall game is strategically equivalent to a sequence of small projects $\{\phi_\tau\}$. As in the case of the final small project, each small project is completed exclusively and at once by members of one active group, while members of the other group stay put. As a result, the equilibrium has two different sources of delay (on path): *inaction* and *gradualism*. The project is partitioned into multiple steps (small projects) precisely due to in-group free-riding (i.e., because each group is composed of multiple members). In fact, if "groups" are composed of single agents, the project is completed in two large steps.[1] With multi-member groups, instead, incentives to contribute are never extinguished in one-shot, but rather conserved by the existence of a dependable peer on whom an agent of the active group can free ride. The size of each small project is thus reduced by in-group free riding as the project is farther from completion (in the initial stages of the project).

In our third main result, we turn to how effort is divided among groups throughout the project. We show that in sufficiently large projects, the combination of inaction and gradualism leads to endogenous *contribution cycles*, wherein juniors and seniors alternate taking responsibility for moving the project forward, while members of the other group do nothing (Figure 1). This alternation of effort on the path of play emerges since "owning" a small project is costly. Thus, the continuation value of the group that anticipates being active contributors in the future decreases faster, as the project moves farther away from completion, than that of the passive group.

Different from the one-to-one alternation typically assumed in the literature, these contribution cycles are generally asymmetric, with one group taking on stints of multiple consecutive projects at a time, while members of the other group only undertake a single small project at a time. Crucially, relative preference intensity or cost differ-

[1]With only two identical members (two single-member "groups" without heterogeneity), and linear costs, Admati and Perry (1991) show that there exist two equilibria, with and without gradualism. Compte and Jehiel (2003) show that if the two individuals have different valuation for completion of the project, only the non-gradual equilibrium survives. Other explanations for gradualism include history-dependent outside options (Compte and Jehiel (2004)), history-dependent strategies (Marx and Matthews (2000)) or moral hazard (Bonatti and Hörner (2011), Giorgiadis (2014)).

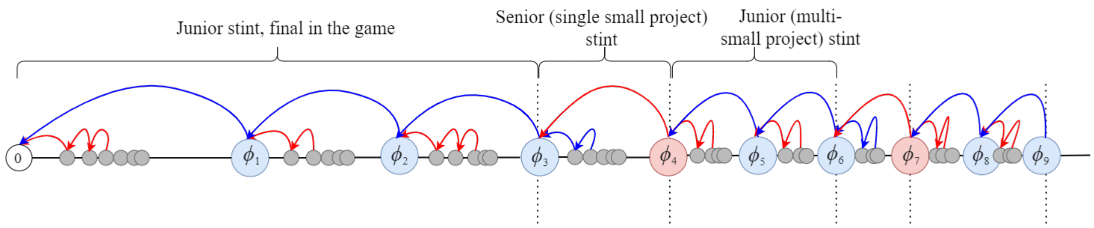

Figure 1: Junior (blue) and senior (red) agents alternate contributing in stints of small projects towards completion of the project.

entials among groups only affect the identity of the group which finishes the project, but do not affect contribution cycles in earlier phases of the game. This is because in equilibrium, the last contribution stint by juniors is precisely long enough to eliminate all differences in willingness to pay across groups due to preference intensity. As a result, contribution cycles are determined solely by free-riding incentives. In particular, we show that the group of agents who are active contributors in stints of multiple consecutive small projects is the group for which in-group free-riding incentives are *larger* (i.e., the largest group), and out-group free-riding incentives are *smaller* (i.e., the group which is more likely to have opportunities to contribute).

As is clear by now, equilibrium outcomes are inefficient due to the delay caused by gradualism *and inaction* along the path of play. Indeed, expected delay grows more than proportionally with project size, and goes to infinity as the project size approaches the maximum level that is completed in equilibrium. But delay is not the only manifestation of inefficiency. In fact, we show that some welfare-enhancing projects are not completed. What's more, due to in-group free riding, this is true even as agents become arbitrarily patient.

A natural question is how this inefficiency compares with the case of no heterogeneity across groups. We show that across-team heterogeneity increases the size of the largest implementable project by an amount bounded away from zero, for any fixed discount factor and group size. As the size of the group grows large, this benefit decreases and becomes zero in the limit. In terms of delay, we show that for large group size and discount factor, projects of relatively small size are completed faster by homogenous teams. As the total project size increases, however, eventually the two-group team always dominates in performance.

Our results speak to an influential observation by Mancur Olson in his landmark work *The Logic of Collective Action* (1965, p. 35):

> This suboptimality or inefficiency [in provision of a collective good] will be somewhat less serious in groups composed of members of greatly different size or interest in the public good. In such unequal groups, on the other hand, there is a tendency toward an arbitrary sharing of the burden of providing the collective good. The largest member, ... who would on his own provide the largest amount of the good, bears a disproportionate share of the burden of providing the collective good.

Olson's suggestion that inefficiency in public good provision would attenuate with greater within-team heterogeneity, is borne out by our analysis. We show that a team composed of two equally large groups can complete larger projects than a fully homogenous team, even as the difference in preferences for completion among the two groups is arbitrarily small. In addition, if the project is sufficiently large, the two-group team always completes the project strictly faster in expectation. Olson's intuition that *asymmetries* across the agents will lead to work being shared unequally is also reflected in our model locally, since only a subset of the agents bear the burden of each small project. However, we show that in the context of multi-group teams, the overall share of work within the project borne by a given group m is not determined only (or primarily) by preference intensity, as Olson suggests, but by in-group and out-group free-riding incentives.

2 Related Literature

Our paper is related to the literature on dynamic contributions to public goods. This literature studies, in particular, how dynamic considerations affect free-riding incentives.

Two early papers are Fershtman and Nitzan (1991) and Admati and Perry (1991). Fershtman and Nitzan (1991) consider a problem in which the public good delivers continuous benefits, and study the efficiency of public good provision. Admati and Perry (1991), instead, consider a binary project, in which benefits are attained only when the project is completed, as in our paper. The distinction between continuous and binary public goods is fundamental for dynamic free-riding incentives. As pointed out by Kessing (2007), in continuous public goods, present contributions lead to a reduction of effort in the future. In binary public goods, instead, a higher contribution by one agent induces larger contributions in the future. The latter is a basic feature of our model, as well as of Admati and Perry (1991), Compte and Jehiel (2003, 2004)), Kessing (2007), Giorgiadis (2014), and Bowen, Georgiadis, and Lambert (2016). On the other hand, Battaglini, Nunnari, and Palfrey (2014) and Harstad (2016), consider models in which the public good gives continuous benefits, as in Fershtman and Nitzan (1991), while Marx and Matthews (2000) consider both cases.

Our paper is most closely related to Admati and Perry (1991) and Compte and Jehiel (2003). Admati and Perry (1991) consider a model in which two identical agents take turns contributing to the public good. When the cost of contributing is strictly convex, the game has an essentially unique SPE path, in which agents make gradual contributions to the project. If the contribution cost is linear, the inefficient equilibrium with gradual contributions still exists, although there is also a no-delay, "one-step completion" equilibrium. Compte and Jehiel (2003) consider a variant of Admati and Perry's model in which agents differ in their valuation of the good and focus on the case of linear costs. In contrast to Admati and Perry's model, in equilibrium, each player makes only one large contribution. Moreover, whenever the project is socially desirable, it is completed as players grow arbitrarily patient. Thus, the gradual equilibrium of Admati and Perry appears not to be robust to asymmetries in agents' valuations.

Our paper considers an environment similar to that in Compte and Jehiel (2003), but with multiple agents of each type, and no pre-specified order of play. We show that cycles emerge endogenously in equilibrium if the project is large enough. In this equilibrium, agents make partial contributions to the public good, recovering the gradualism in Admati and Perry (1991). Moreover, we show that in the special case in which there is exactly one agent of each type, our model produces the result of Compte and Jehiel (2003) in which the project is completed in two steps. We conclude that gradualism and alternation can appear as a purely strategic phenomenon provided there are in-group free-riding incentives. The emergence of endogenous contribution cycles is understudied, and in particular, the emergence of repeated cycles of endogenous length is, to the best of our knowledge, new to the literature.

The gradualism present in the equilibrium of our model is captured in the literature through different incentive structures. Compte and Jehiel (2004) consider a bargaining variant of the model in Admati and Perry (1991) in which each party's contribution is interpreted as a voluntary "concession" to their share of surplus. Parties have the option to terminate bargaining, in which case the pay-offs obtained depend on the history of offers or concessions made in the bargaining process. They show that history-dependent outside options can lead to gradualism. Kessing (2007) generalizes the gradualism of Admati and Perry (1991) that arises from convex costs to a differential game counterpart, with $n > 1$ homogeneous agents. Differently to most papers in the literature, Marx and Matthews (2000) focus on history-dependent strategies. They show that allowing contributions to be made slowly over time enhances efficiency in some equilibria, even though individual contributions are private information. In these equilibria, players' contributions are supported by the threat of stopping cooperation. Finally, Bonatti and Hörner (2011) consider an "experimentation" model of public good provision. Completing the project requires a breakthrough, which in turn requires effort. Some projects can never be completed, and achieving a breakthrough is the only way to ascertain the project's type. Agents' choice of effort is unobserved by the other agents, which leads them to postpone effort, as in other work reviewed here. However, unlike these related papers (including our own), the effort expended decreases over time due to growing pessimism.

3 The Model

We consider a model in which n agents collaborate to produce a public good, which requires q units of investment. There is an infinite horizon, $t = 1, 2, \ldots$. In each period t, an agent $i(t)$ selected at random has the opportunity to invest $e \geq 0$ towards the public good. Contributing e units has a cost $C(e) = ce$, for $c > 0$.[2] We let $e^i(t)$ denote the contribution made by agent i in period t ($e^i(t) = 0$ for all $i \neq i(t)$) and

[2]With a convex cost, gradualism may be optimal, inducing cycles for technological reasons and not for strategic reasons. We restrict attention to linear costs to isolate strategic incentives.

$e(t)$ denote the contribution made in period t by the agent selected in that period. The amount of effort outstanding for completion of the good at the end of period t is the *state* $s_t = \max\{s_{t-1} - e(t), 0\}$, with $s_0 = q$.

There are two groups of agents, $m = 1, 2$, and $n^m > 1$ agents of type $m = 1, 2$ (we cover single member groups as special cases). For convenience, we refer to group 1 agents as *juniors*, and to group 2 agents as *seniors*. The probability that any particular individual of group m is selected is $\pi^m = \tilde{\pi}^m / n^m$, where $\tilde{\pi}^1 + \tilde{\pi}^2 = 1$. The payoff of a group-$m$ agent i is:

$$U^i = (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} \left[u^m(s_t) - ce^i(t) \right],$$

where $u^m(s) = 0$ for $s > 0$ and $u^m(0) = \theta^m > 0$.

A time-t *history* $h(t)$ is the tuple $(\{i(\tau)\}_{\tau=1}^{t-1}, \{e(\tau)\}_{\tau=1}^{t-1})$, which encodes previous game play, along with the state $s = q - \sum_{\tau=1}^{t-1} e(\tau)$. We say $s^{h(t)}$ is the *corresponding state* for $h(t)$. Let H^t be the set of all time-t histories, and $H := \cup_{t \in \mathbb{N}} H^t$ be the set of all histories in the game. An *equilibrium* is a (group) symmetric subgame perfect equilibrium in pure strategies. A pure strategy for a group-m agent i is a function $\sigma^m(\cdot) : H \to \mathbb{R}_+$, where $\sigma^m(h)$ denotes the contribution of an agent of group-m from history h when she has the opportunity to contribute.

Consider any history h and let h_e be the history induced by group-m agent i contributing $e \in [0, s^h]$ at h. The payoff to i from contributing e is

$$W^m(e, h) = \delta V^m(h_e) + (1-\delta) \left[u^m(s^h - e) - ce \right], \tag{3.1}$$

where the value of a player i of type m at history h, $V^m(h)$, satisfies *(i)* $V^m(h) = \theta^m$ for any $h \in H$ for which $s^h = 0$, and *(ii)* for any h with $s^h > 0$, letting $\sigma^m(h) = \arg\max_{e \geq 0} W^m(e, h)$,

$$V^m(h) = \sum_{j=1}^{2} \tilde{\pi}^j \left[\delta V^m(h_{\sigma^j(h)}) + (1-\delta) u^m(s^h - \sigma^j(h)) \right] - \pi^m c (1-\delta) \sigma^m(h). \tag{3.2}$$

We formally define our notion of equilibrium discussed above, as follows.

Definition 3.1. *A (group) symmetric subgame perfect equilibrium in pure strategies is a pair* $(\sigma^1(\cdot), \sigma^2(\cdot))$ *such that* $\sigma^m(h) = \arg\max_{e\geq 0} W^m(e,h)$ *for all* $m = 1, 2$ *and all histories* h *with* $s^h \leq q$.

4 Results

In this section, we present our results. We begin by analyzing "small" projects in Section 4.1. We define a project as small if it starts at a history h such that there exists an equilibrium in which the project is completed immediately with positive probability, and denote the size of the largest small project by ϕ_1. We show that in equilibrium, all relevant information about the history is captured by the state, which uniquely pins down agents' strategies. After characterizing equilibria of small projects, we turn to our main analysis in Section 4.2. We show that any project of size $q > \phi_1$ can be seen as a sequence of small projects. Thus, the equilibrium can be characterized recursively. Using these tools, we characterize equilibrium outcomes in large projects, in particular contribution cycles and efficiency.

4.1 Small Projects

We first establish that in equilibrium, there is a unique threshold $w(1) > 0$ such that the project is completed without delay if and only if $s^h \leq w(1)$. We also show that equilibrium strategies from any history with corresponding state $s \leq w(1)$ are *history-independent*, and uniquely determined by s. As will soon become clear, the state is the relevant information for players at *any* history of the game.

Lemma 4.1. *In equilibrium,* $\sigma^m(h) = s^h$ *for all* $m \in \{1, 2\}$ *if and only if* $s^h \leq w(1)$, *where*

$$w^m(1) \equiv \left(\frac{1}{1-\delta\pi^m}\right)\frac{\theta^m}{c} \quad \textit{and} \quad w(1) \equiv \min\{w^1(1), w^2(1)\}. \tag{4.1}$$

The logic for the result is straightforward. First, given any history h, when the corresponding state s^h is sufficiently small, it is a dominant strategy for any agent to finish the project outright.[3] There must therefore be a *maximal* state $w^m(1)$ such

[3]This is intuitive: if the agent does not finish outright, she can at best expect another agent to

that any agent of group m is willing to complete the project outright from any history h with $s^h \leq w^m(1)$, given that all other agents do the same. Therefore, outright completion by any agent occurs if and only if $s^h \leq w(1) \equiv \min\{w^1(1), w^2(1)\}$.

In all that follows, we consider the generic case (true for all but a measure zero of the parameter-values) where $w^1(1) \neq w^2(1)$. For presentation purposes we assume that juniors have a higher willingness to finish the project than seniors; i.e., $w^2(1) < w^1(1)$. Since we will show that the equilibrium is Markovian, in the main text we write "from state s" or "at s" to mean "from any history with corresponding state s." We also slightly abuse notation to write $\sigma^m(s)$ as the strategy played by, and $V^m(s)$ as the value to, any group-m agent from any h with $s^h = s$.

As we have seen, seniors do not immediately finish the project when they have an opportunity to play from any $s > w(1)$. Yet, because contributions are strategic substitutes, there is $s' > w^1(1)$ such that any junior still completes the project immediately whenever she is recognized at $s \leq s'$. Let ϕ_1 denote the largest such s'; that is, the largest state from which juniors finish the project outright in equilibrium. We show that for all s below some threshold $\overline{s}_1 \leq \phi_1$, seniors use a *cutpoint strategy* in equilibrium, gradually moving the project towards completion when they have an opportunity to play.[4] Remarkably, though, we show that this process *must come to a halt at a point $\overline{s}_1$ strictly below ϕ_1*. Thus, there is a region $[\overline{s}_1, \phi_1]$ within which seniors do not contribute at all, while juniors finish the project outright (see Figure 2). This is an extreme form of free riding from seniors, who in this region of the state space fully rely on juniors finishing the project right away.

Proposition 4.2. *In equilibrium,*

1. *Juniors finish the project outright for all $s \leq \phi_1$; i.e., $\sigma^1(s) = s$ for all $s \leq \phi_1$, where*

$$\phi_1 = \left(\frac{1}{1 - \delta + \delta\tilde{\pi}^1 \left(\frac{n^1-1}{n^1}\right)} \right) \frac{\theta^1}{c}. \tag{4.2}$$

finish the project tomorrow for sure. Thus, the payoff for waiting is at most $\delta\theta^m$. On the other hand, finishing outright gives a payoff $\theta^m - c(1-\delta)s^h$ which is higher than $\delta\theta^m$ whenever $cs^h < \min\{\theta^1, \theta^2\}$.

[4] We say that σ^m is a cutpoint strategy in $[a, b] \subseteq S$ if there is a strictly increasing sequence $\{s(k)\}_{k=0}^K$, with K possibly infinite, such that (i) $s(0) \equiv a$, and $\lim_{k \to K} s(k) = b$, and (ii) $\sigma^m(s) = s - s(k-1)$, $\forall s \in (s(k-1), s(k)]$ and $\forall k : 1 \leq k \leq K$.

2. *Seniors follow a cutpoint strategy in* $[0, \overline{s}_1] \subset [0, \phi_1]$ *with cutpoints given by*

$$s_1(k) = \frac{1 - \left(\delta\tilde{\pi}^2 \left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1-\delta\tilde{\pi}^2} \frac{\theta^2}{c} < \overline{s}_1 \equiv \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} \quad \forall k \geq 1, \tag{4.3}$$

3. $\overline{s}_1 < \phi_1$*; seniors do not contribute (*$\sigma^2(s) = 0$*) for all* $s \in [\overline{s}_1, \phi_1]$.

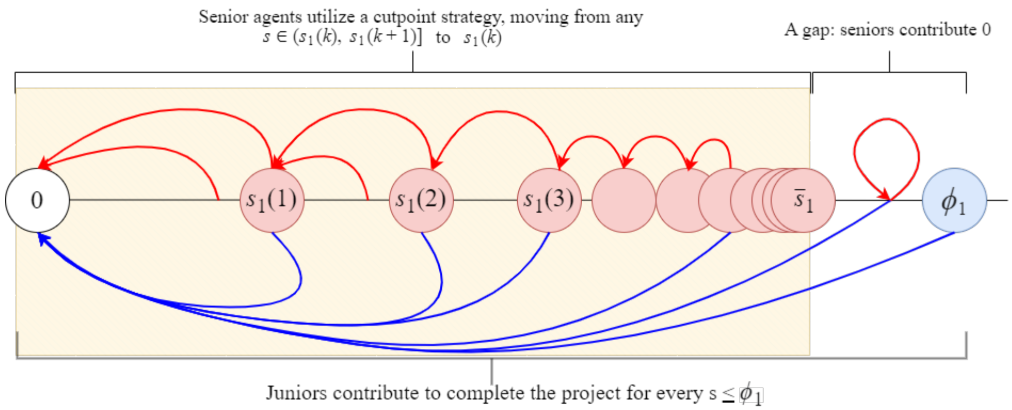

Figure 2: Equilibrium in Small Projects, in the generic case $w^1(1) > w^2(1)$.

To understand the determination of the cutpoints in part (2), consider an arbitrary cutpoint $s_1(k)$ in the sequence. Note that at $s = s_1(k) + \epsilon$, seniors must prefer to contribute $\epsilon > 0$ than overshooting to any other state $s' < s_1(k)$. Similarly, at $s = s_1(k) - \epsilon$, seniors must prefer to contribute in order to reach the cutoff $s_1(k-1)$ instead of staying in state $s_1(k) - \epsilon$. Taking the limit as $\epsilon \to 0$ we get

$$\delta V^2(s_1(k)) = \theta^2 - c(1-\delta)s_1(k). \tag{4.4}$$

On the other hand, for all k in the sequence, and all $s \in (s_1(k-1), s_1(k)]$, the value (3.2) for a group-m agent is

$$V^m(s) = \tilde{\pi}^1\theta^m + \tilde{\pi}^2\delta V^m(s_1(k-1)) - \pi^m c(1-\delta)\sigma^m(s) \tag{3.2b}$$

Combining (4.4) with (3.2b) for $m = 2$ gives

$$s_1(k) = \frac{1}{1-\delta\pi^2}\frac{\theta^2}{c} + \frac{\delta\pi^2}{1-\delta\pi^2}(n^2-1)s_1(k-1). \tag{4.5}$$

Solving this difference equation allows us to characterize the equilibrium values $V^m(s)$ for both types, for all $s \leq \min\{\phi_1, \lim_{k\to K} s(k)\}$, using equation (3.2b). Moreover, an immediate implication of the contribution "gap" in proposition 4.2 is that the values at the "beginning" of the small project are independent of the details of seniors' cutpoint strategy, and given by

$$V^1(\phi_1) = \left(\frac{(1-1/n^1)\tilde{\pi}^1}{1-\delta+(1-1/n_1)\delta\tilde{\pi}^1}\right)\theta^1 \quad \text{and} \quad V^2(\phi_1) = \left(\frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\theta^2. \tag{4.6}$$

To understand the inaction gap ($\overline{s}_1 < \phi_1$), let $\hat{k} \in \{1, 2, \ldots, \infty\}$ be the number of senior cut-points below ϕ_1. As $\hat{k}$ increases, the incentives for juniors to complete the project are compounded, as seniors take progressively smaller steps. A junior thus becomes *more* willing to contribute to avoid even more periods of delay, in expectation, were she not to. That is, ϕ_1 is an increasing function of $\hat{k}$. We show that for each finite $\hat{k}$, $s_1(\hat{k}+1) < \phi_1(\hat{k})$, which rules out any equilibrium with finitely many cutpoints. The intuition for this result lies in the asymmetry in contribution strategies between the two groups in equilibrium: while seniors remove only a small amount of delay by contributing to their next cutpoint $s_1(\hat{k})$, a contribution from juniors completes the project outright. Because the marginal value of a contribution relative to shirking is far higher for juniors than seniors, their willingness to increase their contribution as $\hat{k}$ increases is also larger than the contribution seniors are willing to make. This implies that the movement from $s_1(\hat{k})$ to $s_1(\hat{k}+1)$ is always smaller than the movement from $\phi_1(\hat{k}-1)$ to $\phi_1(\hat{k})$.

In-group and Out-group Effects. At the core of these results is the tradeoff between delay and free-riding on both peers and outsiders. The incentive to free-ride on peers (parametrized by n^m) is the *in-group* effect, while the incentive to free-ride on out-group members ($\tilde{\pi}^m$) is the *out-group* effect. Consider first in-group effects. Intuitively, reducing the number of juniors n^1 increases the maximum size of the project that juniors are willing to complete outright, ϕ_1, reducing $V^1(\phi_1)$ (eqs. (4.2) and (4.6)).[5] Similarly, reducing the number of seniors in the team increases the size of each increment $s_1(k) - s_1(k-1)$ along the sequence of seniors' strategy cutpoints (eq. (4.5)). Because of the contribution gap, though, this has no effect on $V^2(\phi_1)$ or $V^1(\phi_1)$. Increasing $\tilde{\pi}_1$, on the other hand, *shrinks* the size of the largest project that juniors are willing to complete outright, ϕ_1. To understand why this is the case, note that since only juniors contribute at ϕ_1, the beneficial effect of free-riding to *out-group* agents is quashed. Indeed, whenever a senior agent is given an opportunity to contribute, progress on the project remains stalled. Thus, as $\tilde{\pi}^1$ increases, equilibrium expected delay *decreases*, lowering the cost of shirking. To restore indifference at the cutpoint, ϕ_1 must decrease, increasing $V^1(\phi_1)$ *and* $V^2(\phi_1)$.

Single-Agent Teams. So far, we have assumed that $n^m > 1$ for all $m \in M$. The basic structure of equilibrium remains essentially unchanged if there is only one agent in any group. However, equilibrium outcomes have some noteworthy features, resulting from the elimination of in-group free riding. The case $n^1 = 1 < n^2$ is unremarkable, with the exception that having a single junior member maximizes the set of projects that are completed immediately with positive probability. Consider next $n^2 = 1 < n^1$. Note that by (4.3), $s_1(k) = w(1) = \overline{s}_1$ for all $k \in \mathbb{N}$. In this case, the single senior agent takes one step whenever $s \leq \overline{s}_1$, and remains inactive in $[\overline{s}_1, \phi_1]$. Since $\overline{s}_1$ and ϕ_1 are not a function of n^2, making $n^2 = 1$ has an unambiguous effect of reducing delay in small projects. Not surprisingly, when $n^2 = 1 = n^1$ we have a combination of both cases: more projects are completed immediately with positive probability, and there is less delay in states close to completion.

Efficiency. From Proposition 4.2, for any small project with $q \in (w(1), \phi_1]$, there is inefficient delay with positive probability. Given our equilibrium characterization,

[5] With fewer juniors, the chance that any single junior member would have to contribute in the next period, were she to shirk in the current period, increases. Since at ϕ_1 a junior must be indifferent between shirking and finishing outright, the amount of effort required to finish the project must increase.

we can compute the expected time for completion of any project $q < \overline{s}_1$ as,

$$\mathcal{E}(q) = (1 - (\tilde{\pi}^2)^{\psi(q)})/\tilde{\pi}^1, \quad \text{where} \quad \psi(q) \equiv \left\lceil \frac{\log\left(1 - \frac{cq}{\theta^2}(1 - \delta\tilde{\pi}^2)\right)}{\log\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)} \right\rceil, \tag{4.7}$$

where we have used (4.3), and $\mathcal{E}(q) = 1/\tilde{\pi}^1$ for $q \in [\overline{s}_1, \phi_1]$.

Note that $\psi(q)$, and therefore $\mathcal{E}(q)$, are weakly increasing in the total cost of the project cq, and weakly decreasing in the value that senior agents put on completion of the project, θ^2. Moreover, as we discussed before, increasing the number of senior agents reduces the size of each increment $s_1(k) - s_1(k-1)$ along the sequence of seniors' strategy cutpoints, and does not affect $\overline{s}_1$ or ϕ_1. As a result, the expected delay for $q < \overline{s}_1$ is also weakly increasing in the number of senior agents in the group. Interestingly, expression (4.7) shows that there is an upper bound on expected delay for any $q \leq \phi_1$, given by $1/\tilde{\pi}^1$. Thus, even when a change of parameters leads to a large increase in the k such that $q \in (s_1(k-1), s_1(k)]$, this does not lead to a proportional increase in the expected time for completion.

In addition to immediate completion, efficiency also requires the project to be completed if and only if total benefits exceed total costs, i.e.,

$$\sum_m n^m\theta^m \geq c(1-\delta)q \Leftrightarrow q \leq \frac{n^1\theta^1 + n^2\theta^2}{c(1-\delta)},$$

Since $w(1) = \left(\frac{1}{1-\delta\pi^2}\right)\frac{\theta^2}{c}$, it follows immediately that there is a nonempty set of projects for which the equilibrium is inefficient also in this sense.

4.2 Large Projects

We now study the equilibrium of the contribution game for any possible project length q. Our first result justifies the attention we paid to understanding small projects. We show that any project of size $q > \phi_1$ can be seen as a sequence of small projects. Thus, the equilibrium can be characterized recursively, following the analysis of Section 4.1.

Theorem 4.3. *There exists an almost everywhere unique equilibrium in pure strategies. The equilibrium is characterized by an increasing sequence* $\{\phi_\tau\}_{\tau=0}^{\overline{\tau}}$*, with* $\phi_0 = 0$

and $\overline{\tau}$ possibly infinite, such that:

(i) For any $\tau \geq 1$, there is a unique $m_\tau \in \{1,2\}$ such that

  (a) $\sigma^{m_\tau}(s) = s - \phi_{\tau-1}$ for all $s \in (\phi_{\tau-1}, \phi_\tau]$, and

  (b) σ^{-m_τ} is a cutpoint strategy in $[\phi_{\tau-1}, \overline{s}_\tau]$ for $\overline{s}_\tau < \phi_\tau$, and $\sigma^{-m_\tau} = 0$ in $(\overline{s}_\tau, \phi_\tau]$.

(ii) The sequence $\{\phi_\tau\}$ converges to a limit $\tilde{\phi} > 0$, and for $\tau > 1$,

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1 - \delta + \delta\pi^{m_\tau}(n^{m_\tau} - 1)}\right)\frac{\delta V^{m_\tau}(\phi_{\tau-1})}{c}. \tag{4.8}$$

(iii) If $q < \tilde{\phi}$, the project is finished in finite time, and otherwise, the project is never started; i.e., $\sigma^m(s) = 0\ \forall s \in [\tilde{\phi}, q]$ and $m \in \{1,2\}$.

The basic intuition for this result relies on two key facts. First, note that at any $s > \phi_1$, agents of both groups are not willing to contribute more than the needed amount to reach ϕ_1, i.e. $\sigma^m(s) \leq s - \phi_1$. It follows that no agent moves the project beyond ϕ_1 at any $s > \phi_1$ (see Lemma A.1 in the Appendix). This implies that for the project to be completed from $s > \phi_1$, the project at some point reaches ϕ_1 and has to transit through ϕ_1 on the equilibrium path. It follows then that the game can be truncated at

$$\phi_1 = \left(1 + \frac{\delta(1-\tilde{\pi}^1)}{1 - \delta\pi^1 - \delta(1-\tilde{\pi}^1)}\right) w^1(1)$$

by considering ϕ_1 as the terminal state, which pays out $\delta V^m(\phi_1)$ when ϕ_1 is attained.

Second, recall that the equilibrium characterization in Proposition 4.2 relied solely on the labeling assumption that $w^2(1) < w^1(1)$. In a similar fashion, when considering the truncated game at ϕ_1, we define

$$w^m(2) \equiv \frac{1}{1-\delta\pi^m}\frac{\delta V^m(\phi_1)}{c} = w^m(1)\frac{\delta V^m(\phi_1)}{\theta^m}$$

Proposition 4.2 gives that $m_2 = \{m = 1,2 : w^m(2) > w^{-m}(2)\}$ is the active contributor in this new small project. Equilibrium behavior in this small project is described by replacing juniors (group 1) for group m_2, $w^m(1)$ for $w^m(2)$, and θ^m for $\delta V^m(\phi_1)$.

Note that generically, $w^1(\tau) \neq w^2(\tau)$ for any $\tau \geq 1$, so we can continue to characterize the equilibrium for the full game, by recursively truncating the game at each state ϕ_τ that determines the upper bound of step τ, for a given $\phi_{\tau-1}$. Continuing this process, we obtain the same structure recursively. Proceeding in this fashion, parts *(i)* and *(ii)* of the theorem follow immediately from our analysis in Section 4.1. For *(iii)*, note that if $\phi_\tau \to \tilde{\phi} < q$, then for each $m \in \{1, 2\}$,

$$w^m(\tau) := \frac{\delta V^m(\phi_{\tau-1})}{c(1 - \delta\pi^m)} \to 0,$$

for else, $\phi_\tau \geq \phi_{\tau-1} + \min\{w^1(\tau), w^2(\tau)\}$, which would contradict convergence of $\{\phi_\tau\}$. Hence, $V^m(\phi_\tau) \to 0$, so that $V^m(\tilde{\phi}) = 0$. It follows that at any $s > \tilde{\phi}$, contributing nothing is strictly preferred by both types to contributing any positive amount.

4.2.1 Endogenous Contribution Cycles

In this section, we explain the emergence of contribution cycles. From Theorem 4.3, there is generically a unique contributor at each cutpoint ϕ_τ. We call $m(\tau)$ the *active contributor* in step τ, and let $-m(\tau) = \{m \in M : m \neq m(\tau)\}$. From (4.6), and using $\delta V^m(\phi_1)$ as the terminal payoffs in the truncated game, for all $\tau > 1$ we have:

$$V^{m_\tau}(\phi_\tau) = \alpha^{m_\tau} V^{m_\tau}(\phi_{\tau-1}) \quad \text{and} \quad V^{-m_\tau}(\phi_\tau) = \beta^{-m_\tau} V^{-m_\tau}(\phi_{\tau-1}) \tag{4.9}$$

where

$$\alpha^m \equiv \frac{\delta\pi^m(n^m - 1)}{\delta\pi^m(n^m - 1) + (1 - \delta)} \quad \text{and} \quad \beta^m \equiv \frac{\delta(1 - \tilde{\pi}^m)}{1 - \delta\tilde{\pi}^m}.$$

Clearly, both $\alpha^m < 1$ and $\beta^m < 1$, so agents' values decrease as we move farther away from completion. Moreover, from (4.9),

$$\frac{V^{m_\tau}(\phi_\tau)}{V^{-m_\tau}(\phi_\tau)} = \Delta^{m_\tau} \frac{V^{m_\tau}(\phi_{\tau-1})}{V^{-m_\tau}(\phi_{\tau-1})} \quad \text{where} \quad \Delta^m \equiv \frac{\alpha^m}{\beta^{-m}} < 1, \tag{4.10}$$

reflecting the fact that the value of the active contributor decreases faster than that of the non-active contributor. Note that the equilibrium payoff of non-active contributors only decreases with τ due to impatience and delay, which occurs when one of them is selected to contribute. Thus their value decreases with τ faster the smaller is the probability that a contributor is selected to contribute, $\tilde{\pi}^{m_\tau}$, and the lower is the discount factor δ. On the other hand, the contributor's value decreases with τ due to impatience, delay, *and* due to the cost of moving the project forward. This opens the possibility that seniors can be active contributors in earlier phases of the game.

Note that for any $\tau \geq 1$, an agent of group m is the active contributor for small project τ if and only if

$$\Omega^m(\phi_\tau) \equiv w^m(\phi_\tau)/w^{-m}(\phi_\tau) > 1.$$

Since $w^m(\tau) = \frac{\delta V^m(\phi_{\tau-1})}{c(1-\delta\pi^m)}$, the growth rate of $\Omega^{m_\tau}(\phi_\tau)$ is equal to that of $V^{m_\tau}(\phi_{\tau-1})/V^{-m_\tau}(\phi_{\tau-1})$, and then by (4.10), we have:

$$\Omega^{m_\tau}(\phi_{\tau+1}) = \Delta^{m_\tau}\Omega^{m_\tau}(\phi_\tau). \tag{4.11}$$

Expression (4.11) allows us to determine the identity of the active contributor recursively. Note that by construction we must have $\Omega^{m_\tau}(\phi_\tau) > 1$. Given this, $m_{\tau+1} = m_\tau$ if and only if $\Omega^{m_\tau}(\phi_{\tau+1}) > 1$, or equivalently $\Delta^{m_\tau}\Omega^{m_\tau}(\phi_\tau) > 1$. We use this result to study the emergence of contribution cycles.

We begin by providing a necessary and sufficient condition for agents of different types to alternate in the role of active contributors. Recall that by assumption, $m_1 = 1$ (i.e., junior agents finish the last small project). Suppose juniors remain in the role of active contributors for a stint of $j(1) \geq 1$ consecutive steps. Noting that $\Omega^1(\phi_\tau) = (\Delta^1)^{\tau-1}\Omega^1(\phi_1)$ for all $\tau \leq j(1)+1$, there is a switch in the identity of the active contributor at $j(1)+1$ if and only if $\Omega^1(\phi_{j(1)+1}) < 1 < \Omega^1(\phi_{j(1)})$, or

$$(\Delta^1)^{j(1)} < \Omega^2(\phi_1) < (\Delta^1)^{j(1)-1}.$$

Since $\Delta^1 < 1$, and by assumption $\Omega^2(\phi_1) < 1$, there exists a unique $j(1) \in \mathbb{Z}_+$ that satisfies this inequality, which is given by

$$j(1) = \left\lfloor 1 + \frac{\log w^2(1) - \log w^1(1)}{\log(\Delta^1)} \right\rfloor. \tag{4.12}$$

Together with (4.8), which pins down the size of each small project $\tau \leq j(1)$, (4.12) pins down the maximal project size for which there is no alternation in active contributors. We thus have the following result:

Remark 4.4. *The equilibrium has no contribution cycles if and only if*

$$q < \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\tilde{\pi}^1/n^1}\right)\frac{1-(\alpha^1)^{j(1)}}{(1-\alpha^1)}\frac{\theta^1}{c} = \phi_{j(1)}.$$

Provided $q > \phi_{j(1)}$, the final stint of $j(1)$ small projects in which juniors are active contributors is preceded by a stint of steps in which seniors are active contributors. How does this process evolve for q large? Suppose seniors remain in the role of active contributors for a stint of $j(2) \geq 1$ consecutive steps. Note that for all τ such that $j(1)+1 \leq \tau \leq j(1)+j(2)+1$,

$$\Omega^1(\phi_\tau) = \frac{1}{(\Delta^2)^{\tau-j(1)-1}}\Omega^1(\phi_{j(1)+1}) = \frac{(\Delta^1)^{j(1)}}{(\Delta^2)^{\tau-j(1)-1}}\Omega^1(\phi_1).$$

Thus, there is a switch in the identity of the active contributor at $h(1)+h(2)+1$ if and only if $\Omega^1(\phi_{j(1)+j(2)}) < 1 < \Omega^1(\phi_{j(1)+j(2)+1})$, or

$$\frac{(\Delta^1)^{j(1)}}{(\Delta^2)^{j(2)-1}} < \Omega^2(\phi_1) < \frac{(\Delta^1)^{j(1)}}{(\Delta^2)^{j(2)}}.$$

Since $\Delta^1 < \Omega^2(\phi_1)/(\Delta^1)^{j(1)-1} < 1$ by definition of $j(1)$, there is an $j(2) \geq 1$ that satisfies this inequality. This alternation between juniors and seniors in the role of active contributors continues until the associated sequence of contributions $\phi_\tau - \phi_{\tau-1}$ in each step τ converges to zero, or the cumulative contributions reach q (whichever happens earlier). This gives one of our main results.

Theorem 4.5 (Contribution Cycles)**.** *For any integer $k \geq 1$, define recursively the total number of steps up to and including stint k as $J(k) \equiv J(k-1)+j(k)$, letting $J(0)=0$. Let $\iota(k)=1$ if k is odd, $\iota(k)=2$ if k is even, and $\Omega^1(\phi_0) \equiv \frac{1-\delta\pi^2}{1-\delta\pi^1}\frac{\theta^1}{\theta^2}$. Then* *for all* $k \geq 1$*:*

$$m_\tau = \begin{cases} 1 & \text{if } J(2k-2) < \tau \leq J(2k-1) \\ 2 & \text{if } J(2k-1) < \tau \leq J(2k), \end{cases}$$

where the number of steps in stint k*,* $j(k)$*, is given recursively by:*

$$\Omega^{\iota(k)}(\phi_{J(k)}) = [\Delta^{\iota(k)}]^{j(k)}\Omega^{\iota(k)}(\phi_{J(k-1)}) \qquad \text{and} \qquad \Delta^{\iota(k)} < \Omega^{\iota(k)}(\phi_{J(k)}) < 1. \tag{4.13}$$

Theorem 4.5 shows that in the generically unique equilibrium, agents belonging to distinct sub-groups within the team endogenously alternate actively contributing to the public good, until either the sequence $\{\phi_\tau\}$ converges or q is reached. This defining feature of equilibrium endogenizes the alternation in player contributions that was heretofore exogenously imposed in the related literature, following Admati and Perry (1991).

In-group and Out-group Effects. In general, each contribution stint k can consist of multiple small projects. However, in the special case in which $\Delta_1 = \Delta_2 = \Delta$ – this happens, for instance, if both groups have equal in-group and out-group free-riding incentives – the ratio of values changes at the same rate when junior and senior agents are active contributors. As a result, with the exception of the final stint, juniors and seniors take turns completing a *single* small project. When instead $\Delta_1 \neq \Delta_2$, the rate of change of $\Omega^m(\phi_\tau)$ varies with the identity of the active contributor. In order to restore equilibrium, then, the length of contribution stints by each group must be asymmetric. In our next result, we characterize the asymmetry in contribution cycles.

Proposition 4.6. *If* $\Delta^{m'} < \Delta^m$*, contribution cycles alternate between single-step stints by type* m' *agents and stints of either* x^m *or* $x^m + 1$ *small projects by group-m agents, where*

$$x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$$

Two points are worth emphasizing. First, note that the length of each stint $k > 1$ *does not depend on relative preference intensity.* This is because in equilibrium, the last contribution stint by juniors eliminates all differences in willingness to pay across members of both groups due to differences in preference intensity. As a result, in earlier stages of the game, the determination of the identity of the active contributor solely depends on free-riding incentives, as captured by n^m and $\tilde{\pi}^m$. Second, note that if both groups have equal size, then $\Delta^1 > \Delta^2$ if and only if juniors are more likely to be selected to contribute ($\tilde{\pi}^1 > 1/2$). Similarly, if both groups are equally likely to contribute, then $\Delta^1 > \Delta^2$ if and only if juniors outnumber seniors.

The first case balances out in-group free riding incentives. Proposition 4.6 says that the group of agents who are active contributors in "long" stints of small projects is the group for which out-group free-riding incentives are *smaller*. While this result might appear counterintuitive at first glance, the logic is familiar to us from the analysis of Section 4.1. What determines the length of a contribution stint is how fast the ratio of values $V^{m_\tau}(\phi_{\tau-1})/V^{-m_\tau}(\phi_{\tau-1})$ decreases towards balance when group m is the active contributor. As we showed in Section 4.1, the reason for the counterintuitive sign of the out-group effect on group m agents is that at the cutpoint $\phi(\tau)$, group $-m$ agents don't make positive contributions. Thus, shifting recognition probability to the opposite group effectively does not increase free-riding opportunities for type m agents, and increases delay. As a result, increasing $\tilde{\pi}^{m_\tau}$ leads to a lower depreciation of the value of the active contributors. The second case balances out-group effects across groups. Proposition 4.6 says that the group of agents who are active contributors in "long" stints of contribution steps is precisely the group for which in-group free-riding incentives are larger. This is because a larger group size means a smaller expected dent in the value of the active contributors in each step, since other agents of the same group can also foot the bill. Thus, it takes a longer stints of small projects to achieve balance in the value ratio.

Single-Agent Teams. Suppose $n^1 = n^2 = 1$. From our analysis in Section 4.1, the equilibrium for small projects in this case is characterized by cutpoints

$$s_1(1) = \frac{\theta_2/c}{1 - \delta\tilde{\pi}_2} \quad \text{and} \quad \phi_1 = \frac{\theta_1/c}{1-\delta}$$

such that $\sigma^1(s) = s \ \forall s \leq \phi_1$, $\sigma^2(s) = s \ \forall s \leq s_1(1)$, and $\sigma^2(s) = 0 \ \forall s \in (s_1(1), \phi_1]$. Since at the cutpoint ϕ_1 the single junior agent is indifferent between completing the project and not contributing, we must have $V^1(\phi_1) = 0$. This implies that for all $s > \phi_1$, only the senior member contributes. It is easy to see then that $\sigma^2(s) = s - \phi_1$

for all $s \in (\phi_1, \phi_2]$ with

$$\phi_2 = \phi_1 + \frac{\delta \tilde{\pi}_1}{1 - \delta \tilde{\pi}_2} \frac{\theta_2/c}{1 - \delta}$$

and $V^2(\phi_2) = 0$. It follows that the project is completed if and only if

$$q \leq \frac{1}{1 - \delta} \left[\frac{\theta_1}{c} + \frac{\delta(1 - \tilde{\pi}_2)\theta_2}{c(1 - \delta \tilde{\pi}_2)} \right], \tag{4.14}$$

and if completed, is completed in at most two steps.

This result is analogous to the main theorem of Compte and Jehiel (2003). We can think of the above result as extending their finding to the case where recognition among two heterogeneous agents is stochastic instead of sequentially determined. Theorem 4.7 shows that this result depends crucially on there being a single agent of each group, and thus on the non-existence of in-group free-riding incentives.

Next, suppose $n^2 > 1$ and $n^1 = 1$. As in the previous case, here $\phi_1 = (\theta_1/c)/(1-\delta)$ and $V^1(\phi_1) = 0$, and the single type 1 agent is unwilling to contribute for any $s > \phi_1$. The game truncated at final node ϕ_1 now only involves senior agents, who are fully homogeneous, as in Admati and Perry (1991). The equilibrium of the truncated game is characterized by the sequence

$$\phi_\tau = \phi_1 + \frac{\delta \tilde{\pi}^1}{1 - \delta \tilde{\pi}^1} \frac{\Delta^2}{(n^2 - 1)} \left[\frac{1 - (\Delta^2)^\tau}{1 - \Delta^2} - 1 \right] \bar{s}_1 \qquad \text{for } \tau > 1$$

with associated values

$$V^2(\phi_\tau) = (\Delta^2)^{\tau - 1} \frac{\tilde{\pi}_1}{1 - \delta \tilde{\pi}_2} \theta_2,$$

recovering the structure of the equilibrium in Admati and Perry (1991).

4.2.2 Efficiency Considerations

We consider two distinct questions related to efficiency: (i) whether all welfare-improving projects are completed, and (ii) what is the delay incurred in finishing efficient projects. Finally, we also compare both the completion of welfare-improving projects and delay to those within a team of *homogenous* agents.

Our normalization of utility by factor $1-\delta$ affects both the project's infinite-lasting flow of payoffs and the one-stage cost of each paid contribution. This normalization simplifies the expressions of the value function as θ^m represents both a flow payoff and the discounted utility of the project. For any $\delta < 1$ this double role of θ^m does not affect the arguments above and, in particular, does not affect the efficiency considerations. However, as $\delta \to 1$, any finite contribution in a single period vanishes relative to the discounted utility of the infinite-lasting project. Hence, when discounting disappears, efficiency considerations are more subtle. To simplify the expressions in this section, we consider a project that, once completed, lasts for T finite periods and delivers θ^m in each period. The utility of such a finished project to agent i of group $m = 1, 2$ is

$$U^i = (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\theta^m = (1-\delta)\hat{\theta}^m \quad \text{where} \quad \hat{\theta}^m \equiv \left(\frac{1-\delta^T}{1-\delta}\right)\theta^m.$$

Implementable Projects. By expression (4.8) in Theorem 4.3, for any $\tau > 1$, if group m is the unique contributor at τ,

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1-\delta\tilde{\pi}^{-m}-\delta\pi^m}\right)\frac{\delta V^m(\phi_{\tau-1})}{c}.$$

where

$$V^m(\phi_{\tau-1}) = \begin{cases} \alpha^m V^m(\phi_{\tau-2}) & \text{if } m(\tau-1) = m \\ \beta^m V^m(\phi_{\tau-2}) & \text{if } m(\tau-1) \neq m \end{cases}$$

Since $\alpha_m < \beta_m < 1$, the value $V_m(\phi_\tau)$ is decreasing in τ, and goes to zero as $\tau \to \infty$. It follows that the incremental contribution $\phi_\tau - \phi_{\tau-1}$ is decreasing in τ and goes to zero as $\tau \to \infty$. This means that the sequence $\{\phi_\tau\}$ converges to a point $\tilde{\phi}$. The threshold $\tilde{\phi}$ gives the size of the smallest project that will not be completed in equilibrium. Using (4.8) and the derived values for each agent at the end of any given contribution stint, we can calculate $\phi_{J(k)}$ for any $k \geq 1$. In particular, note that the values at the end of any given stint k only depend on the number of times each agent contributes up to the k^{th} stint. For instance, for $k \geq 1$ odd, we have

$$V^1(\phi_{J(k)}) = [\alpha^1]^{j(k)}[\beta^1]^{j(k-1)}V^1(\phi_{J(k-2)}), \quad V^2(\phi_{J(k)}) = [\beta^2]^{j(k)}[\alpha^2]^{j(k-1)}V^2(\phi_{J(k-2)})$$

Proposition 4.6 then allows us to obtain an upper bound for the limit $\tilde{\phi}$ (we present the exact characterization of this bound in the proof of Theorem 4.7). Using this bound, we address the question of whether all projects which would be efficient to pursue – i.e., projects such that $cq < (n^1\hat{\theta}^1 + n^2\hat{\theta}^2)$ – are in fact completed. We call these projects *efficient* for short. We show that not all efficient projects are completed, even in the limit as $\delta \to 1$.

Theorem 4.7. *For any $\delta < 1$, $c\tilde{\phi} < \hat{\theta}^1 + \hat{\theta}^2$. Hence, (i) for any fixed $\delta < 1$, some efficient projects are not completed. Moreover, (ii) since $n^m > 1$ for any $m \in \{1,2\}$, then even in the limit as $\delta \to 1$, some efficient projects are not completed.*

The qualification that $n^m > 1$ for any $m \in \{1,2\}$ is important. As we discussed in the previous section, when $n^1 = n^2 = 1$ the project is completed if and only if (4.14) holds, and if completed, it is completed in at most two steps. It follows that

$$c\tilde{\phi}_{n^1=n^2=1} = \hat{\theta}^1 + \frac{1-\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\delta\hat{\theta}^2$$

which implies that in the limit as $\delta \to 1$, all efficient projects are completed. We thus see that as agents become arbitrarily patient, *in-group* free riding is uniquely responsible for inefficiency in project completion.

Delay. As we have shown before, the incremental contribution $\phi_\tau - \phi_{\tau-1}$ in (4.8), is decreasing in τ and goes to zero as the sequence $\{\phi_\tau\}$ approaches the limit $\tilde{\phi}$. It follows that (i) the number of steps required to finish a project of size q, $J(\hat{k}(q))$, where $\hat{k}(q) \equiv \min\{k : \phi_{J(k)} \geq q\}$, is an increasing and convex function of q, and (ii) $J(\hat{k}(q)) \to \infty$ as $q \to \tilde{\phi}$. This means that the number of contribution stints required to complete a project of size q, $\hat{k}(q)$, is also an increasing and convex function of q, such that $\hat{k}(q) \to \infty$ as $q \to \tilde{\phi}$. In our next result, we use the structure of equilibrium strategies characterized in the previous section to compute the expected delay $\mathcal{E}\,(q)$ associated with completion of a project of size q. We first show that the expected completion time of a project that requires $\hat{k}(q)$ contribution stints is given by

$$\tilde{\mathcal{E}}\,(q) \equiv \mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \sum_{\ell=1}^{\hat{k}(q)} \left(j(\ell) + \tilde{\pi}_{m_{-\ell}}(\tilde{\pi}_{m_\ell})^{j(\ell)-2}\right),$$

where $\iota(k) = 1$ if k is odd and $\iota(k) = 2$ if k is even.

By Proposition 4.6, moreover, we know that if $\Delta^{m'} < \Delta^m$, contribution cycles alternate between stints of either x^m or $x^m + 1$ steps by group-m agents, and single-step stints by group-m' agents, while if $\Delta^{m'} = \Delta^m$, both groups alternate in single-step stints. We can then split the terms of the above expression in stints taken by each group to compute tight bounds on expected delay for a project of size q.

Proposition 4.8. *Suppose $q = \phi_\ell$ for $\ell > j(1)$, and let $\mathcal{E}_1 \equiv j(1) + \tilde{\pi}_2(\tilde{\pi}_1)^{j(1)-2}$. If $\Delta^m \geq \Delta^{m'}$, the expected delay associated with completion of a project of size q is:*

$$\tilde{\mathcal{E}}\,(q) = \mathcal{E}_1 + \left(\frac{\hat{k}(q) - 1}{2}\right)\left\{(1 + x) + \frac{\tilde{\pi}_m}{\tilde{\pi}_{m'}} + \tilde{\pi}_{m'}(\tilde{\pi}_m)^{x-2}\right\}.$$

for some $x \in [x^m, x^m + 1]$, where $x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$.[6] *In particular, $\tilde{\mathcal{E}}\,(\cdot)$ is an increasing and convex function, and $\lim_{q \to \tilde{\phi}} \tilde{\mathcal{E}}\,(q) = \infty$.*

Together with the characterization for delay in small projects given by (4.7), Proposition 4.8 gives a full characterization of delay for arbitrary project length q. The result shows that the expected delay increases non-linearly with the size of the project, and goes to infinity as the size of the project approaches the smallest project not completed in equilibrium, $\tilde{\phi}$. In this way, expected delay is a smooth measure of the two components of inefficiency discussed in this paper: delay and the non-completion of some efficient projects.

Comparison with Homogenous Teams. We compare the performance of a team in which $2n$ agents have an identical value of θ for completing the project, to one in which there are two payoff-distinct groups— n group 1 members obtain payoff $\theta+\varepsilon$ and n group 2 members $\theta - \varepsilon$, for some $\varepsilon \in [0, \theta]$, upon completion. We assume all agents are equally likely of being recognized to contribute, and have per-unit contribution

[6] We state the result for an odd number of contribution stints. In the proof of Proposition 4.8, we provide the result for the case of an even number of contribution stints. Note also that the expression of expected delay in the proposition assumes that the project starts exactly at a cutpoint, i.e., $q = \phi_\ell$ for some $\ell > j(1)$. It is straightforward to compute expected delay when the project starts at a point interior to a small project, using the expression (4.7) for expected delay in small projects.

cost c, making the set of welfare-enhancing projects the same across teams.[7]

Implementable Projects. We begin by noting that equilibrium behavior within the homogenous set of agents is akin to that of a two-group team in which the recognition probability of the outside group vanishes. We denote the cut-points used by the agents $\{t(k)\}_{k=1}^{\infty}$. Using expression (4.3), we obtain

$$t(k) = \frac{1}{1-\delta}\left[1 - \left(\delta \cdot \frac{2n-1}{2n-\delta}\right)^k\right]\frac{\theta}{c} < \bar{t} = \frac{\theta}{c(1-\delta)}, \tag{4.15}$$

where $\bar{t} \equiv \lim_{k\to\infty} t(k)$ represents the largest project implementable by the homogeneous team. On the other hand, our previous analysis directly allows us to obtain the maximum project size $\tilde{\phi}$ undertaken by the two-group team. In particular, our assumptions guarantee that $\Delta^1 = \Delta^2$, so that $j(k) = 1$ for $k > 1$. Note that group 1 agents— who have a higher value for the project— will be active near completion. Thus, from Theorem 4.3, the values to members of each group at the start of the final small project are given by $V^1(\phi_1) = \alpha/\delta \cdot (\theta + \varepsilon)$ and $V^2(\phi_1) = \beta/\delta \cdot (\theta - \varepsilon)$, where

$$\alpha = \frac{\delta(n-1)}{2n - \delta(n+1)}, \qquad \beta = \frac{\delta}{2-\delta}.$$

Clearly, $m_2 = 2$ if and only if $V^2(\phi_1) > V^1(\phi_1)$. We thus obtain a global one-to-one alternation in group responsibility over small projects, whenever $\varepsilon < \left(\frac{1-\delta}{(2-\delta)n-1}\right)\theta$. Suppose this inequality holds. Using (4.8) and (4.9) recursively, we directly obtain

$$\phi_k = \frac{2n}{c(2n - \delta\,(n+1))}\left[(\theta+\varepsilon)\sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor}(\alpha\beta)^{\ell} + \beta\cdot(\theta-\varepsilon)\sum_{\ell=0}^{\lfloor\frac{k}{2}\rfloor-1}(\alpha\beta)^{\ell}\right],$$

with

$$\lim_{k\to\infty}\phi_k \equiv \tilde{\phi} = \frac{2n}{c(2n-\delta\,(n+1))}\left(\frac{(1+\beta)\theta + (1-\beta)\varepsilon}{1-\alpha\beta}\right).$$

[7] As in the rest of the paper, here we restrict to symmetric SPE. A possible question is whether we could support the equilibrium behavior of the two-group case within a homogenous team, by arbitrarily dividing members into two equal size groups. This is in fact a valid equilibrium construction; *it is however not robust.* In particular, when costs deviate even an arbitrarily small degree from linearity to become convex, an equilibrium of this form no longer exists. On the other hand, the equilibrium we consider for homogenous agents is robust to this perturbation. The same is true for the equilibrium described for the two-group case throughout the paper, where groups are distinguished by tangible heterogeneity in valuations or recognition probabilities.

We see from above that, within the prescribed range for ε, the size of the largest implementable project is *increasing* in inter-group heterogeneity ε.

Consider next the *marginal introduction* of heterogeneity within the team by letting $\varepsilon \to 0$ above. The largest implementable project in the limit is

$$\tilde{\phi} = R_\delta(n) \cdot \frac{\theta}{c}, \qquad \text{where} \quad R_\delta(n) \equiv \frac{2n}{2n - 2\delta(n+1) + \delta^2},$$

By comparison with $\bar{t}$ in (4.15), the gain in the size of the largest implementable project in the two-group setting over the homogenous team is given by

$$\Gamma_\delta(n) \cdot \frac{\theta}{c}, \qquad \text{with } \Gamma_\delta(n) \equiv R_\delta(n) - 1/(1-\delta).$$

It follows that the two-group team is able to complete a larger set of welfare-enhacing projects whenever $\Gamma_\delta(n) > 0$. But, by direct inspection, $\Gamma_\delta(n)$ is a strictly decreasing function of n, with $\lim_{n\to\infty} \Gamma_\delta(n) = 0$. Hence, this is true for *any* δ, n. We conclude that the introduction of within-team heterogeneity improves the size of the largest implementable project by an amount bounded away from 0, for any fixed δ, n. As n grows large, this benefit decreases and becomes zero in the limit.

Delay. We now compare project completion times across the two team structures, focusing again on the case where $\varepsilon \searrow 0$. The analysis we have done so far is nearly sufficient to undertake this comparison. Suppose $q = \phi_k$, as in the characterization of delay given by Proposition 4.8. In the two-group team, the expected completion time for a project of size q is exactly $2k$ periods, since each small project takes *two* periods to complete in expectation. On the other hand, the homogenous team moves precisely one cutpoint in each period. Hence, the two-group team achieves project completion faster than the homogenous team if and only if $\phi_k > t(2k)$. Moreover, because $\lim_{k\to\infty} \phi_k = \tilde{\phi} > \bar{t} = \lim_{k\to\infty} t(2k)$, it follows that for any δ, n, there exists $k^*(\delta, n) \in \mathbb{N}$ such that $\phi_k > t(2k)$ whenever $k \geq k^*(\delta, n)$. The threshold $k^*(\delta, n)$ is non-trivial; i.e., for large n and δ, projects of relatively small size are completed more efficiently with homogenous teams. As the total project size increases, however, eventually the two-group team always dominates in performance.

5 Conclusions

In this paper, we study inter-group collaboration within a team of agents through a model of sequential contributions to a joint project, in a decentralized environment. We assume the joint project is a *binary public good*, delivering benefits inexcludably to the team of agents only once completed, and that there is no deadline or asymmetric information among agents. Crucially, we focus on a team naturally divided into two sub-groups, each of which has a distinct value from completing the project.

We show that in large projects, the equilibrium of the model displays endogenous contribution cycles, in which agents from different groups within the team alternate making gradual contributions towards project completion. Importantly, *delay*, *inaction*, and *alternation* in work between the sub-groups — familiar features of the teamwork setting— all emerge as features of equilibrium. Neither collaboration between homogenous, nor fully heterogenous agents captures these stylized facts. This points to the salience of the *inter-group* case as a descriptive framework for collaborative settings whenever contributions are transparent and decentralized.

In the paper, we focused on binary public goods/joint projects. In other cases, agents may *also* obtain a flow payoff even if the project is incomplete; i.e., $u^m(s(t)) = r^m(q - s(t))$ for $s(t) > 0$ and $u^m(0) = qr^m + \theta^m > 0$. All of our analysis extends to this case, with suitable modifications, when the flow payoff r^m is "small." Thus, our conclusions can be extended to applications in which the terminal payoff dominates. When the flow payoff r^m is large relative to terminal payoffs θ^m, however (e.g., collaboration across countries to reduce carbon emissions), the strategic analysis changes qualitatively. We believe that extending the analysis in this paper to the continuous public good case would be a promising avenue for future research.

In addition to the theoretical contributions we have emphasized throughout, our paper also provides guidance for applied research. In the equilibrium of our model, agents with the higher valuation for completion ex ante will assume responsibility for finishing the project in the final stages, but do not necessarily make larger contributions than agents with lower ex ante valuation at all points in time. Moreover, the prevalence of inaction across groups, and the length of contribution stints by agents in both groups are solely determined by in- and out-group free riding incentives, and not ex ante valuations for completion. This suggests that strategic considerations are a key element to be considered in linking parameters to observed data.

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A Proofs

Proof of Lemma 4.1. (a) First we show that $\exists b > 0$ such that $\sigma^m(h) = s^h$ for all m and all histories h such that $s^h \leq b$. Note that for $s^h \leq \min\{\theta^1, \theta^2\}/c$,

$$W^m(s^h, h) = \theta^m - (1-\delta)c \cdot s^h \geq \delta\theta^m, \ \forall m,$$

where the right-hand side is an upper bound on $W^m(e, h)$ for any $e < s^h$, since it is the value from zero contribution and the project being completed by another agent in the subsequent period. Letting $b = \min\{\theta^1, \theta^2\}/c$ completes the step.

(b) We now determine $w(1) \equiv \sup\{b : \sigma^m(s) = s \ \forall s \leq b, \forall m \in M\}$. Note that from any h, that for $e < s^h$: $W^m(e, h) = \delta V^m(h_e) - (1-\delta)ce$. Take any h with $s^h \leq w(1)$. From (3.2), and since $\sigma^m(h') = s(h')$ for all m for histories h' with $s(h') < w(1)$, for any $e < s^h$, $V^m(h_e) = \theta^m - \pi^m c(1-\delta)(s^h - e)$. Thus for $e < s^h$:

$$W^m(e, h) = \delta\theta^m - c(1-\delta)\left(e + \delta\pi^m(s^h - e)\right).$$

Note that for $e \in (0, s^h)$, $s^h < w(1)$, $W^m(0, h) - W^m(e, h) > 0$. Thus, when all agents finish the project on their turn, an agent is better off contributing zero than contributing $e \in (0, s^h)$. Moreover, $W^m(s^h, h) \geq W^m(0, h)$ if and only if

$$s^h \leq \frac{\theta^m}{c\,(1-\delta\pi^m)} \equiv w^m(1),$$

with equality uniquely at $s = w^m(1)$. Thus $\sigma^m(h) = s^h$ for $m = 1, 2$ if $s < w(1) \equiv \min_m w^m(1)$ and only if $s \leq w(1)$. In particular, from any history h with $s^h = w(1)$, we see it is always an equilibrium that $\sigma^m(h) = w(1)$ for $m = 1, 2$.

(c) We show that equilibrium behavior at any history with corresponding state $w(1)$ is unique and entails completion of the project. Note that if $w^m(1) > w(1)$, group m agents have a strict incentive to contribute at such an h, so we need only consider m for which $w^m(1) = w(1)$. From previous work, we know that $\sigma^m(h) \in \{0, w(1)\}$. Consider therefore an equilibrium in which $\sigma^m(h) = 0$. In such an equilibrium,

$V^m(h) = \frac{\tilde{\pi}^{m'}\theta^m}{1-\delta\tilde{\pi}^m}$. A deviation to contributing $w(1)$ at h is profitable, since

$$W^m(w(1), h) = \theta^m - c(1-\delta)w(1) = \delta\theta^m \frac{1-\pi^m}{1-\delta\pi^m} = \delta V^m(h)\frac{\frac{1-\pi^m}{1-\delta\pi^m}}{\frac{1-\tilde{\pi}^m}{1-\delta\tilde{\pi}^m}} > \delta V^m(w(1)) \tag{A.1}$$

(d) It remains to show that if $w^m(1) > w^{m'}(1)$, then $\sigma^m(h) = s^h$ for all h with $s^h \in (w^{m'}(1), w^m(1)]$. Consider first $s^h \in (w^{m'}(1), w^m(1))$. As before, $W^m(s^h, h) = \theta^m - c(1-\delta)s^h$. Since group m' agents don't finish outright, for any $e \in [0, s^h)$,

$$W^m(e, h) < \delta\theta^m - c(1-\delta)\left[e + \delta\pi^m(s^h - e)\right] \leq \delta\left(\theta^m - c(1-\delta)\pi^m s^h\right)$$

Thus $W^m(s^h, h) > W^m(e, h)$ for all $e \in [0, s^h)$ if and only if $s^h < w^m(1)$. Now consider a history h for which $s^h = w^m(1)$. Using similar arguments we have that $W^m(w^m(1), h) > W^m(e, h)$ for all $e \in (0, w^m(1))$, so we only need to consider equilibria with $\sigma^m(h) \in \{0, w^m(1)\}$. The definition of $w^m(1)$ gives that $\sigma^m(h) = w^m(1)$ is indeed an equilibrium. For uniqueness, assume that $\sigma^m(h) = 0$ from some history h with corresponding state $w^m(1)$. It follows then that

$$V^m(h) = \frac{\tilde{\pi}^{m'}\delta}{1-\tilde{\pi}^m\delta}V^m\left(h_{\sigma^{m'}(h)}\right)$$

Since no type finishes the project outright, we must have that

$$V^m\left(h_{\sigma^{m'}(h)}\right) \leq \theta^m - \pi^m c(1-\delta)\left(w^m(1) - \sigma^{m'}(h)\right)$$

Therefore, there is a profitable deviation for type m if $\delta V^m(h) < \theta^m - c(1-\delta)w^m(1)$, which holds, since $\frac{1-\tilde{\pi}^m}{n^m-\tilde{\pi}^m}\sigma^{m'}(h) < w^m(1)$. □

Proof of Proposition 4.2. We prove this in a series of steps.

Step 1. *Suppose $s_1(2) > w(1)$ is such that for all histories h with $s^h \leq s_1(2)$, $\sigma^1(h) = s^h$. Then:*

1. For any h with corresponding state $s^h \in (s_1(1), s_1(2)]$, $\sigma^2(h) = s^h - w(1)$.

2. For any h with $s^h > s_1(2)$, $\sigma^2(h) \leq s^h - s_1(2)$.

Proof. Suppose h is such that $s^h \in (s_1(1), s_1(2)]$, and consider the problem of a senior (i.e., group 2) agent, i. We have shown that for any history h' with $s^{h'} > w(1)$, $\sigma^2(h') < s^{h'}$. Contributing $e \in (s^h - w(1), s^h)$ from h is clearly dominated, since by moving the state to $w(1)$, the project will be finished at the same time, with lower expected costs. Thus, $\sigma^2(h) \in [0, s^h - w(1)]$. Suppose first $\sigma^2(h) = s^h - w(1)$ for all histories h with $s^h \in (s_1(1), s_1(2)]$. Note that since all agents finish the project outright for any history h' with $s^{h'} \leq w(1)$, if agent i contributes $e = s^h - w(1)$, from the history h, she gets a payoff

$$\begin{aligned} W^2(s^h - w(1), h) &= \delta V^2(h_{s^h - w(1)}) - (1-\delta)c(s^h - w(1)) \\ &= \delta\left[\theta^2 - \pi^2 c(1-\delta)w(1)\right] - (1-\delta)c(s^h - w(1)). \end{aligned} \tag{A.2}$$

Now suppose i deviates and contributes $e \in [0, s^h - w(1))$, say $e = s^h - w(1) - \varepsilon$, with $0 < \varepsilon \leq s^h - w(1)$. This gives her a payoff

$$\begin{aligned} W^2(e,h) &= \delta V^2(h_e) - (1-\delta)c(s^h - w(1) - \varepsilon) \\ &= \delta\theta^2(\tilde{\pi}^1 + \delta\tilde{\pi}^2) - (\delta)^2\tilde{\pi}^2(1-\delta)\pi^2 c w(1) - (1-\delta)c(s - w(1)) + \varepsilon(1-\delta)c(1-\pi^2\delta), \end{aligned} \tag{A.3}$$

Note that this expression is increasing in ε. Thus, the binding deviation is $e = 0$, or $\varepsilon = s^h - w(1)$. This is not a profitable deviation iff

$$s^h \leq \left(\frac{\delta\tilde{\pi}^2}{1-\delta\pi^2}\right)\frac{\theta^2}{c} + \left\{\frac{1-\delta\pi^2(2-\delta\tilde{\pi}^2)}{1-\delta\pi^2}\right\}w(1) = w(1) + w(1)\left[\frac{\delta\pi^2(n^2-1)}{1-\delta\pi^2}\right] = s_1(2).$$

Suppose instead that in equilibrium $\sigma^2(h) < s^h - w(1)$ for some history h with $s^h \in (s_1(1), s_1(2)]$. Note that the equilibrium payoff for a senior agent at h is bounded above by the RHS of A.3. A deviation to $e = s^h - w(1)$ at h gives a payoff A.2. Since $s^h \in (s_1(1), s_1(2)]$, it follows that this is a profitable deviation, and thus in equilibrium we must have $\sigma^2(h) = s^h - w(1)$ for all histories h with $s^h \in (s_1(1), s_1(2)]$.

For part 2, note that our previous argument shows that $\sigma^2(h) < s^h - w(1)$ for all h with $s^h > s_1(2)$. Now, for any such h, moving the project to any history with corresponding state $s \in (s_1(2), w_1)$ is clearly dominated by moving the project to a history with corresponding state $s_1(2)$, since the project will be finished at the same time, with lower expected costs. Thus $\sigma^2(h) \leq s^h - s_1(2)$ for all $s^h > s_1(2)$.

By definition 4, when senior agents play a cutpoint strategy in $[0, b]$, their strategy is *history-independent* from all histories h for which $s^h \leq b$. We thus use the convention of writing $\sigma^m(s)$ in place of $\sigma^m(h)$, and $V^m(s)$ in place of $V^m(h)$ for any history h such that $s^h = s$, whenever group-m agents play a cutpoint strategy.

Step 2. *Suppose there is $b \in \mathbb{R}_+$ such that senior (group 2) agents follow a cutpoint strategy in $[0, b]$ and $\sigma^1(h) = s^h$ at all h for which $s^h \leq b$. Then (1) the sequence of cutpoints is described by (4.3), and (2) the values for junior (group 1) agents are given by*

$$V^1(s_1(1)) = \theta^1 - \pi_1 n^1 c(1-\delta)s_1(1), \tag{A.4}$$

and for $k > 1$,

$$\begin{aligned} V^1(s_1(k)) = &\left(\frac{1-(\tilde{\pi}^2\delta)^k}{1-\tilde{\pi}^2\delta}\right)\left[\tilde{\pi}^1\theta^1\right] + (\tilde{\pi}^2\delta)^k\theta^1 \\ &- \pi^1\left(\frac{1-\delta}{1-\delta\tilde{\pi}^2}\right)\left[\left(\frac{1-(\delta\tilde{\pi}^2)^k}{1-\delta\tilde{\pi}^2}\right) - \left(\delta\tilde{\pi}^2\right)^k\left(\frac{n^2-\delta\tilde{\pi}^2}{1-\delta\tilde{\pi}^2}\right)\left[1-\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right]\right]\theta^2 \end{aligned} \tag{A.5}$$

Proof. Using (4.4) for $s_1(k)$ and $s_1(k-1)$ in (3.2b) gives

$$s_1(k) = w_2(1) + \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)s_1(k-1) \tag{A.6}$$

Solving the difference equation, we have:

$$s_1(k) = \left[\sum_{\ell=0}^{k-1}\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^\ell\right]\left[\frac{1}{c}\left(\frac{n^2}{n^2-\delta\tilde{\pi}^2}\right)\theta^2\right]$$

and using that $\sum_{\ell=0}^{k-1} x^\ell = \frac{1-x^k}{1-x}$ we get (4.3). The values of junior agents for $k = 1$ follows trivially, as agents in both groups contribute to complete the project at $s_1(1)$. Consider $k > 1$. The value for a junior at $s_1(k)$ is

$$V^1(s_1(k)) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(s_1(k-1)) - \pi^1 c(1-\delta)s_1(k), \quad \text{and} \tag{A.7}$$

$$V^1(s_1(k-1)) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(s_1(k-2)) - \pi^1 c(1-\delta)s_1(k-1)$$

So substituting, recursively,

$$V^1(s_1(k)) = \left(\frac{1-(\tilde{\pi}^2\delta)^{k-1}}{1-\tilde{\pi}^2\delta}\right)\tilde{\pi}^1\theta^1 + (\tilde{\pi}^2\delta)^{k-1}\theta^1 - \pi^1 c(1-\delta)\left[\sum_{\ell=0}^{k-1}(\tilde{\pi}^2\delta)^\ell s_1(k-\ell)\right].$$

Substituting $s_1(k-\ell)$, and simplifying, step 2 follows.

Step 3. *In any symmetric SPE, σ^2 is a cutpoint strategy in $[0,\overline{s}_1]$ for $\overline{s}_1 \equiv \sup_{k\geq 1} s_1(k)$ if $\sigma^1(h) = s^h$ for all histories h with $s^h \leq \overline{s}_1$.*

Proof. We proceed by induction. First, by lemma 4.1 and step 1 above, the statement is true up to $k = 2$, with $s_1(0) \equiv 0$, $s_1(1) \equiv w(1)$, and $s_1(2)$ as defined in step 1. Next, suppose the statement is true up to k, and consider any history h with $s^h \in (s_1(k), s_1(k+1)]$. Note that if for any such h, $\sigma^2(h) \leq s^h - s_1(k)$, an argument analogous to that of step 1 establishes the induction step. To prove this, we show that for any history h with $s^h \in (s_1(k), s_1(k+1)]$, if $\sigma^2(h) > s^h - s_1(k)$, a senior (group 2) agent would gain by deviating to contributing zero. Since moving to a non-cutpoint is dominated by moving to a cutpoint, it suffices to consider $\sigma^2(h) = s^h - s_1(k-\ell)$ $\forall$ $\ell : 1 \leq \ell < k$. In the proposed equilibrium, a senior that gets to contribute has payoff

$$W^2(s^h - s_1(k-\ell), h) = \delta V^2(s_1(k-\ell)) - (1-\delta)c(s - s_1(k-\ell)) = \theta^2 + r^2 q - (1-\delta)cs$$

A deviation to zero gives $W^2(0,h) = \delta V^2(h)$. Since by assumption $\sigma^2(h) = s^h - s_1(k-\ell)$ and we know $\sigma^1(h') = s^{h'}$ for all histories with $s^{h'} \leq \overline{s}_1$, at s we have

$$V^2(h) = \tilde{\pi}^2\delta V^2(s_1(k-\ell)) + (1-\tilde{\pi}^2)\theta^2 - \pi^2 c(1-\delta)[s^h - s_1(k-\ell)].$$
$$\Longrightarrow W^2(0,h) = \delta\theta^2 - \delta\tilde{\pi}^2(1-\delta)cs_1(k-\ell) - \delta\pi^2 c(1-\delta)[s^h - s_1(k-\ell)]$$

Thus, contributing zero is a profitable deviation iff

$$\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)s^h - \delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2}\right)s_1(k-\ell) > \frac{\theta^2}{c}$$

Note that if this is satisfied for $\ell = 1$, it is satisfied for all $\ell : 1 \leq \ell < k$. Moreover, recall that $s^h > s_1(k)$, so writing $s^h = s_1(k) + \varepsilon$, for $\varepsilon > 0$, this is profitable iff

$$\left(\frac{n^2 - \delta\tilde{\pi}^2}{n^2}\right)(s_1(k) + \varepsilon) - \delta\tilde{\pi}^2\left(\frac{n^2 - 1}{n^2}\right)s_1(k-1) > \frac{\theta^2}{c}$$

and using (A.6), the inequality becomes $\left(\frac{n^2-\delta\tilde{\pi}^2}{n^2}\right)\varepsilon > 0$, which always holds.

Having shown that for any history h with $s^h \in (s_1(k), s_1(k+1)]$, $\sigma^2(h) \leq s^h - s_1(k)$, an argument analogous to that of step 1 shows that in equilibrium $\sigma^2(h) = s^h - s_1(k)$ for any h with $s^h \in (s_1(k), s_1(k+1)]$. This concludes the proof of step 3.

Step 4. *Suppose $\overline{s}_1 < \phi_1$ and $\sigma^1(h) = s^h$ from any h with $s^h \leq \phi_1$. Then $\sigma^2(h) = 0$ from all histories h with $s^h \in [\overline{s}_1, \phi_1]$.*

Proof. (i) First, note that for all h with $s^h \in [\overline{s}_1, \phi_1]$, $\sigma^2(h) < s^h - s_1(k)$ for all k in the sequence. The formal argument is the same the proof of step 3, and therefore omitted here. (ii) Now suppose for $s^h \in [\overline{s}_1, \phi_1]$, $\sigma^2(h) \in [0, s^h - \overline{s}_1]$, say $\sigma^2(h) = s^h - \overline{s}_1 - \varepsilon$, with $0 \leq \varepsilon \leq s^h - \overline{s}_1$. This gives a group-2 agent payoff

$$W^2(e, h) = \delta V^2(h_e) - (1 - \delta)c(s^h - \overline{s}_1 - \varepsilon)$$

Now, by part (i), any contribution $e \in [0, s^h - \overline{s}_1]$ will never result in a move to a state in which a senior agent advances the project to any $s_1(k)$ in the sequence. Thus $V^2(h_e) \leq (1 - \tilde{\pi}^2)\theta^2$. Hence, for any $e = s - \overline{s}_1 - \varepsilon$, with $0 \leq \varepsilon \leq s - \overline{s}_1$, $W^2(e, h) \leq \delta(1 - \tilde{\pi}^2)\theta^2 - (1 - \delta)c(s^h - \overline{s}_1 - \varepsilon)$. It follows that if $\sigma^2(h) > 0$ for any such h, a senior (group 2) agent would gain by deviating to contributing zero. To complete the proof, we show that if $\sigma^2(h) = 0$ for all h with $s^h \in [\overline{s}_1, s_1(1)]$, seniors don't have a profitable deviation in this interval. For any such h:

$$V^2(h) = \left(\frac{1 - \tilde{\pi}^2}{1 - \delta\tilde{\pi}^2}\right)\theta^2.$$

A deviation to any cutpoint $s_1(k)$ gives

$$\begin{aligned} W^2(s^h - s_1(k), h) &= \delta V^2(s_1(k)) - c(1 - \delta)(s - s_1(k)) \\ &= \theta^2 - (1 - \delta)cs_1(k) - c(1 - \delta)(s - s_1(k)) = \theta^2 - (1 - \delta)cs \end{aligned}$$

where we used (4.4). This is a profitable deviation iff $W^2(s^h - s_1(k), s) > \delta V^2(h)$ or

$$s^h < \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} = \bar{s}_1 \Rightarrow\Leftarrow .$$

Step 5. *In any equilibrium, there are infinitely many cupoints $\{s_1(k)\}$ below ϕ_1.*

Proof. Assume, towards a contradiction, that $\exists\ \hat{k} < \infty$ such that $s_1(\hat{k}) < \phi_1(\hat{k}) \leq s_1(\hat{k}+1)$. Recall that, by definition, $\phi_1(\hat{k})$ coincides with ϕ_1 when there are exactly $\hat{k}$ cutpoints of juniors below ϕ_1. The assumed condition is thus necessary to sustain an equilibrium with exactly $\hat{k}$ senior cutpoints. The above steps show that seniors play a history-independent strategy from any h with $s^h < \phi_1$; moreover, juniors complete outright from any h with $s^h \leq \phi_1$. So, we use the notation $\sigma^m(s)$ and $V^m(s)$, again, to mean the strategy and value of group-m agents from any h with $s^h = s$.

(a) Note that if $s_1(\hat{k}) < \phi_1(\hat{k}) \leq s_1(\hat{k}+1)$, it must be that $\sigma^2(s) = s - s_1(\hat{k})$ and $\sigma^1(s) = s$ for all $s \in (s_1(k), \phi_1(\hat{k})]$, so any junior agent's value at $\phi_1(\hat{k})$ is:

$$V^1\phi_1(\hat{k})) = \tilde{\pi}^1\theta^1 - \pi^1 c(1-\delta)\phi_1(\hat{k}) + \tilde{\pi}^2\delta V^1(s_1(\hat{k})) \tag{A.8}$$

where $V^1(\phi_1(\hat{k}))$ must verify $\delta V^1(\phi_1(\hat{k})) = \theta^1 - c(1-\delta)\tilde{s}(1,\hat{k})$. Replacing in (A.8),

$$\phi_1(\hat{k}) = w^1(1)\left[1 + \frac{\delta\tilde{\pi}^2}{1-\delta}\frac{\theta^1 - \delta V^1(s_1(\hat{k}))}{\theta^1}\right] \tag{A.9}$$

(b) Note that if $s_1(\hat{k}+1) < \phi_1(\hat{k})$ there is no equilibrium with $\hat{k}$ cut-points for seniors below $\phi_1(\hat{k})$. This is because while junior agents contribute to complete the project, the maximal amount any senior agent is willing to contribute to obtain the state $s_1(\hat{k})$ is $s_1(\hat{k}+1) - s_1(\hat{k}) < \phi_1(\hat{k}) - s_1(\hat{k})$. Thus, at $\phi_1(\hat{k})$, only a junior agent would be willing to contribute, contradicting the hypothesized equilibrium structure.

We now show that $\phi_1(k) > s_1(k+1)\ \forall k \geq 1$. Consider first $k \geq 2$. From part *(*a),

$$\begin{aligned}\phi_1(k) &= \left[\frac{n^1}{n^1 - \delta\tilde{\pi}^1}\right]\frac{1}{c}\left\{\left[\frac{1-\delta\tilde{\pi}^1}{1-\delta}\right]\theta^1 - \left(\frac{\delta^2}{1-\delta}\right)\tilde{\pi}^2 V^1(s_1(k))\right\} \\ &= \frac{n^1}{n^1 - \delta\tilde{\pi}^1}\cdot\frac{1}{c}\left\{\theta^1\cdot\left[\frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\delta\tilde{\pi}^2}\right] + \theta^2 T(n^2)\right\}\end{aligned}$$

where

$$T(n^2) \equiv \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)^k\left(n^2-(n^2-1)\left(1-\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right)\right)\right)}{n^1(1-\delta\tilde{\pi}^2)^2}$$

results from substituting $V^1(s_1(k))$ from (A.5) and simplifying. Noting that

$$\frac{\partial}{\partial n^2}\left(n^2-(n^2-1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k\right) = 1+k\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^{k-1} - (k+1)\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)^k,$$

and that for any k, the function $f(x) = k \cdot x^{k-1} - (k+1)x^k$ has its maximum at $x = \frac{k-1}{k+1}$ with $f(\frac{k-1}{k+1}) = \left(\frac{k-1}{k+1}\right)^{k-1}$, we see that the LHS of this equation is positive. Moreover, since the limit as $n^2 \to \infty$ of the argument in the LHS is $(1-\delta\tilde{\pi}^2)k$,

$$T(n^2) > \lim_{n^2\to\infty} T(n^2) = \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2\left(1-(\delta\tilde{\pi}^2)^k\left(1+(1-\delta\tilde{\pi}^2)\cdot k\right)\right)}{n^1(1-\delta\tilde{\pi}^2)^2} > 0 \quad \forall k \geq 2$$

Therefore, for any $k \geq 2$,

$$\frac{\phi_1(k)}{s_1(k+1)} > \frac{n^1}{n^1-\delta\tilde{\pi}^1} \cdot \frac{\theta^1}{\theta^2} \cdot \frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^{k+1}}.$$

Further using the fact that $w^1(1) > w^2(1) \implies \frac{n^1}{n^1-\delta\tilde{\pi}^1} \cdot \frac{\theta^1}{\theta^2} > \frac{n^2}{n^2-\delta\tilde{\pi}^1}$, we obtain

$$\begin{aligned}\frac{\phi_1(k)}{s_1(k+1)} &> \frac{n^2}{n^2-\delta\tilde{\pi}^2} \cdot \frac{1-(\delta\tilde{\pi}^2)^{k+1}}{1-\left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^{k+1}} \\ &= \left(1-(\delta\tilde{\pi}^2)^{k+1}\right) \cdot \left[\frac{n^2(n^2-\delta\tilde{\pi}^2)^k}{(n^2-\delta\tilde{\pi}^2)^{k+1}-(\delta\tilde{\pi}^2)^{k+1}(n^2-1)^{k+1}}\right] := R(n^2,k).\end{aligned}$$

For any fixed, arbitrary k, $\frac{\partial}{\partial n^2}R(n^2,k) < 0$ so that from above,

$$\frac{\phi_1(k)}{s_1(k+1)} > R(n^2,k) > \lim_{n^2\to\infty} R(n^2,k) = 1,$$

which yields the desired result $\phi_1(k) > \psi(k+1)$, for any $k \geq 2$.

Consider now the case where $k = 1$. From part *(a)* and step 2, we have that

$$\phi_1(1) = \frac{n^1(1 + \delta\tilde{\pi}^2)}{n^1 - \delta\tilde{\pi}_1}\frac{\theta^1}{c} + \frac{\delta^2\tilde{\pi}^1\tilde{\pi}^2 \cdot n^2}{(n^1 - \delta\tilde{\pi}^1)(n^2 - \delta\tilde{\pi}^2)}\frac{\theta^2}{c}$$

By definition, $s_1(2) = \frac{1-(\delta\tilde{\pi}^2 \cdot \frac{n^2-1}{n^2-\delta\tilde{\pi}^2})^2}{1-\delta\tilde{\pi}^2}$, so that (using $w_1(1) > w_2(1)$),

$$\frac{\phi_1(1)}{s_1(2)} > \frac{n^2(n^2 - \delta\tilde{\pi}^2)^2}{(n^2 - \delta\tilde{\pi}^2)(n^2 - 1)^2} > 1.$$

Again, we obtain the desired result in this standalone, residual case. Thus, there can be no eqbm. with finitely many senior cutpoints $\{s_1(k)\}_{k=1}^{\hat{k}}$ below ϕ_1, for any $\hat{k}$.

Step 6 (Conclusion). Step 5 rules out equilibria with finitely many senior cutpoints. Thus, any equilibrium must have infinitely many cut-points with

$$\overline{s}_1 = \lim_{k\to\infty} s_1(k) = \lim_{k\to\infty} \frac{1 - \left(\delta\tilde{\pi}^2\left(\frac{n^2-1}{n^2-\delta\tilde{\pi}^2}\right)\right)^k}{1 - \delta\tilde{\pi}^2}\frac{\theta^2}{c} = \left(\frac{1}{1 - \delta\tilde{\pi}^2}\right)\frac{\theta^2}{c} < \phi_1.$$

Here, the second equality results from substituting the expression for $s_1(k)$ from (4.3), established in Step 2, and the final inequality is a direct consequence of Step 5. We now verify that the remaining candidate is, in fact, an equilibrium, and derive an explicit expression for ϕ_1. This completes the proof of the main proposition.

Step 3 uniquely identifies senior agent behavior as a cut-point strategy in equilibrium from any history h with $s^h < \overline{s}_1 < \phi_1$. Given the full completion ($\sigma^1(s) = s$) by junior agents from any $s \leq \phi_1$, it must be the case in any equilibrium— from Step 4— that $\sigma^2(s) = 0$ for all $s \in [\overline{s}_1, \phi_1]$. That is, the prescribed behavior of seniors is a best response to juniors' putative strategy $\sigma^1(s) = s$ for all $s \leq \phi_1$. It thus only remains to show there is no profitable deviation for juniors, given seniors' behavior.

Given that $\overline{s}_1 < \phi_1$, and $\sigma^2(s) = 0$ for all $s \in (\overline{s}_1, \phi_1]$, we have $V^1(\phi_1) = \tilde{\pi}^1\theta^1 + \tilde{\pi}^2\delta V^1(\phi_1) - \pi^1 c(1-\delta)\phi_1$ and $V^2(\phi_1) = \tilde{\pi}^1\theta^2 + \tilde{\pi}^2\delta V^2(\phi_1)$, and thus we have

$$V^1(\phi_1) = \left(\frac{1}{1 - \delta\tilde{\pi}^2}\right)\left\{\tilde{\pi}^1\theta^1 - \pi^1 c(1-\delta)\phi_1\right\}, \quad \text{and} \tag{A.10}$$

$$V^2(\phi_1) = \left(\frac{1 - \tilde{\pi}^2}{1 - \delta\tilde{\pi}^2}\right)\theta^2 \tag{A.11}$$

Since in equilibrium juniors' values must verify $\delta V^1(\phi_1) = \theta^1 - c(1-\delta)\phi_1$, (A.10) directly gives (4.2). Substituting in (A.10) and (A.11) gives (4.6).

Next we show that junior agents don't have a profitable deviation from any h with s^h in $[0, \phi_1]$. At a history h' with $s^{h'} = \phi_1$, by definition junior agents are indifferent between completing the project or staying put. They clearly cannot gain by moving to a non-cutpoint point, so it is enough to consider deviations to senior agents' cutpoints. In equilibrium, $W^1(\phi_1, h') = \theta^1 - (1-\delta)c\phi_1$. A deviation to some $s_1(k)$, contributing $e' = \phi_1 - s_1(k)$, gives a payoff $W^1(e', h') = \delta V^1(s_1(k)) - (1-\delta)c(\phi_1 - s_1(k))$. This is not a profitable deviation iff

$$0 \geq \delta V^1(s_1(k)) + (1-\delta)cs_1(k) - \theta^1 \equiv \mathbb{L}^1_k \tag{A.12}$$

Using now (A.7) we have that

$$\frac{\mathbb{L}^1_k}{1-\delta} = -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}^1_{k-1} - \delta\tilde{\pi}^2 cs_1(k-1) + (1-\delta\pi^1)cs_1(k)$$

and replacing $s_1(k-1)$ with the expression from (A.6) we get

$$\begin{aligned}\frac{\mathbb{L}^1_k}{1-\delta} &= -\theta^1 + \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}^1_{k-1} + cw^2(1)\frac{n^2-\delta\tilde{\pi}^2}{n^2-1} - cs_1(k)\left(\frac{1-\delta\tilde{\pi}^2}{n^2-1} + \delta\pi^1\right) \\ &\leq \frac{\delta}{1-\delta}\tilde{\pi}^2\mathbb{L}^1_{k-1} + c\left(1-\delta\pi^1\right)\left(w^2(1) - w^1(1)\right)\end{aligned}$$

and the inequality is strict if $w^2(1) < s_1(k)$. Thus, juniors do not deviate to any $s_1(k)$ from arbitrary history h' with $s^{h'} = \phi_1$ if and only if $\mathbb{L}^1_1 \leq 0$, which is true since

$$\begin{aligned}\mathbb{L}^1_1 &= \delta\left(\theta^1 - \pi^1 c(1-\delta)w^2(1)\right) + (1-\delta)cw^2(1) - \theta^1 \\ &= (1-\delta)c(1-\delta\pi^1)(w^2(1) - w^1(1)) < 0.\end{aligned}$$

□

Lemma A.1. *For all histories h with $s^h > \phi_1$ and $m \in \{1, 2\}$, $\sigma^m(h) \leq s^h - \phi_1$.*

Proof of Lemma A.1. Consider first junior (group 1) agents. By construction, ϕ_1 is the largest s such that from any h with $s^h = s$, juniors finish the project outright, given $\sigma^2(h) = 0$. Since $\sigma^2(h) = 0$ minimizes free riding incentives, ϕ_1 is the largest state corresponding to any history from which juniors would complete the project outright, for any σ^2. Thus, at any history h with $s^h > \phi_1$, $\sigma^1(h) < s^h$. By Lemma 4.1, from any h with $s^h \in (0, \phi_1)$, a junior agent strictly prefers to finish the project outright over playing $e \in [0, s^h)$. By the linearity of costs, it follows that for h with $s^h > \phi_1$ a junior agent would weakly prefer to finish the project outright over moving the project to $s' \in (0, \phi_1)$. Hence, at any history h with $s^h > \phi_1$, $\sigma^1(h) \leq s^h - \phi_1$. Next consider senior agents. Since σ^2 is a cutpoint strategy in $[0, \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c})$, we know that for any k in the sequence, and any $s > s_1(k)$, $\sigma^2(s) \leq s - s_1(k)$. Thus, for all $s > \phi_1$, $\sigma^2(s) \leq s - \left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}$. In equilibrium $\sigma^2(s) = 0$ for all $s \in [\left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}), \phi_1]$. Thus, for any history h with $s^h > \phi_1$, moving the project to $s' \in [\left(\frac{1}{1-\delta\tilde{\pi}^2}\right)\frac{\theta^2}{c}), \phi_1)$ is dominated by moving the project to ϕ_1; this doesn't increase delay and reduces expected costs. Thus, for any h with $s^h > \phi_1$, $\sigma^2(h) \leq s^h - \phi_1$. □

Proof of Remark 4.4. For $\tau = 1$, $V^1(\phi_1) = (\alpha^1/\delta)\theta^1$, and for $2 \leq \tau \leq j(1)$: $V^1(\phi_\tau) = \alpha^1 V^1(\phi_{\tau-1}) \Rightarrow V^1(\phi_\tau) = (\alpha^1)^\tau \frac{1}{\delta}\theta^1$. On the other hand, for $\tau : 2 \leq \tau \leq j(1)^*$:

$$\phi_\tau - \phi_{\tau-1} = \left(\frac{1}{1 - \delta\tilde{\pi}^2 - \delta\tilde{\pi}^1/n^1}\right)\frac{\delta V^1(\phi_{\tau-1})}{c} = \phi_1(\alpha^1)^{\tau-1}$$

where we used (4.2) for $\tau = 1$. Summing up over all $2 \leq \tau \leq j(1)$ we get

$$\phi_{j(1)} = \phi_1\left[\sum_{\tau=0}^{j(1)-1}(\alpha^1)^\tau\right] = \phi_1\frac{1-(\alpha^1)^{j(1)}}{(1-\alpha^1)} \tag{A.13}$$

Thus, there is no alternation of effort on the equilibrium path iff $q < \phi_{j(1)}$. □

Proof of Proposition 4.6. In the text we prove that if $\Delta^m < \Delta^{m'}$, then all stints of group m are one-step, except for potentially the first one, which is the case only if $m' = 1$. We prove the result for $\Delta^2 < \Delta^1$. The case of $\Delta^2 > \Delta^1$ is analogous. For $k-1$ even, (4.13) gives $\Omega^2(\phi_{J(k-1)}) = \Delta^2\Omega^2(\phi_{J(k-2)})$ and $\Delta^2 < \Omega^2(\phi_{J(k-1)}) < 1$, so

$$\Delta^2 < \Omega^1(\phi_{H(k-2)}) < 1. \tag{A.14}$$

And, for k odd, (4.13) gives $\Omega^1(\phi_{J(k)}) = [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)})$ and $\Delta^1 < \Omega^1(\phi_{J(k)}) < 1$

$$\implies \Delta^2\Delta^1 < [\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-2)}) < \Delta^2. \tag{A.15}$$

Note that the left hand side of (A.15) and $\Omega^1(\phi_{J(k-2)}) < 1$ from (A.14) imply that $\Delta^2 < [\Delta^1]^{j(k)-1}$. Now consider $k-2$ which is odd and gives $\Delta^1 < \Omega^1(\phi_{J(k-2)}) < 1$. Thus, $\Delta^1 < \Delta^2\Omega^1(\phi_{J(k-1)})$ (since $\Omega^2(\phi_{J(k-1)}) = \Delta^2\Omega^2(\phi_{J(k-2)})$) and then $[\Delta^1]^{j(k)+1} < \Delta^2$ (since $[\Delta^1]^{j(k)}\Omega^1(\phi_{J(k-1)}) < 1$). Note that it must be that for all k odd

$$[\Delta^1]^{j(k)+1} < \Delta^2 < [\Delta^1]^{j(k)-1} \Leftrightarrow \frac{\log[\Delta^2]}{\log[\Delta^1]} - 1 < j(k) < \frac{\log[\Delta^2]}{\log[\Delta^1]} + 1$$

Assume first that $\log[\Delta^2] = \ell \log[\Delta^1]$ for some $\ell \in \mathbb{Z}^+$. Then we have that $\ell - 1 < j(k) < \ell + 1$ which implies that $j(k) = \ell$. Let now $\lfloor y \rfloor \equiv \sup\{x \in \mathbb{Z}_+ : x \leq y\}$, and write $\frac{\log[\Delta^2]}{\log[\Delta^1]} = \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + \epsilon$ for some $\epsilon \in (0,1)$. It then follows that

$$\left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor \leq j(k) \leq \left\lfloor \frac{\log[\Delta^2]}{\log[\Delta^1]} \right\rfloor + 1.$$

□

Proof of Theorem 4.7. We prove the theorem for the case of public goods with an infinite-stream of payoffs θ^m in each period and then analogize to the case of a finite stream of payoffs of length T, which allows us to make the efficiency comparison as $\delta \to 1$. In particular, we prove the following result:

1. *Suppose $(\Delta^1)^y < \Delta^2 < \Delta^1$ for some $y \in \mathbb{N}$. Let y^* denote the smallest integer y^* such that this inequality holds. Then,*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi_1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^1(1-(\alpha^1)^{y^*})}{(1-\alpha^1)(1-\beta^1(\alpha^1)^{y^*})}\right] + \frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi_2)(1-\alpha^2(\beta^2)^{y^*})}.$$

2. *Suppose $(\Delta^2)^y < \Delta^1 < \Delta^2$ for some $y \in \mathbb{N}$. Let y^* denote the smallest integer y^* such that this inequality holds. Then,*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi_1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}(\beta^1)^{y^*}}{1-\alpha^1(\beta^1)^{y^*}}\right] + \frac{(\beta^2)^{j(1)}(1-(\alpha^2)^{y^*})\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi_2)(1-\alpha^2)(1-\beta^2(\alpha^2)^{y^*})}.$$

3. *Suppose $\Delta_1 = \Delta_2 = \Delta$. Then*

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi_1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1} + \frac{(\alpha^1)^{j(1)}\beta^1}{(1-\beta^1\alpha^1)}\right] + \frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi_2)(1-\alpha^2\beta^2)}.$$

Proof. We prove (1), as (2) is analogous. Part (3) follows directly from (1) or (2), making $y^* = 1$. From (4.2), we have

$$\begin{aligned}\phi_1 &= \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi^1}\right)\frac{\theta^1}{c}\\ &\vdots\\ \phi_{j(1)} &= \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi^1}\right)\frac{\theta^1}{c}\sum_{\ell=1}^{h_1}\alpha^{\ell-1} = \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi_1}\right)\left(\frac{1-(\alpha^1)^{h_1}}{1-\alpha^1}\right)\frac{\theta^1}{c}.\end{aligned}$$

At $\phi_{j(1)+1}$, the active contributor switches from group 1 to group 2. Hence,

$$\phi_{j(1)+1} = \phi_{j(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\pi^2}\right)\frac{\delta V^2(\phi_{j(1)})}{c} = \phi_{j(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\pi^2}\right)\frac{(\beta^2)^{j(1)}\theta^2}{c}.$$

In the case (1) being analyzed, $J(2) = j(1)+1$ since each stint of senior agents is of length one. Continuing forward,

$$\phi_{J(3)} = \phi_{J(2)} + \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi^1}\right)\frac{(\alpha^1)^{j(1)}\beta^1\theta^1}{c}\left(\frac{1-(\alpha^1)^{j(3)}}{1-\alpha^1}\right),$$

analogous to the formulation of $\phi_{j(1)}$. Moreover, note that by case (1), $j(3) \le y^*$, and substituting gives us

$$\phi_{J(3)} \le \phi_{J(1)} + \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\pi^2}\right)\frac{(\beta^2)^{j(1)}\theta^2}{c} + \left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi^1}\right)\frac{(\alpha^1)^{j(1)}\beta^1\theta^1}{c}\left(\frac{1-(\alpha^1)^{y^*}}{1-\alpha^1}\right).$$

In general, note that the Lth stint for $L > 1$ odd will consist of at most y^* steps actively contributed by juniors, and the $(L+1)-$th one step stint by seniors. With this pattern, for any $L \ge 1$, the following formula holds for odd numbered stints:

$$\begin{aligned}\phi_{J(2L+1)} \le \phi_{J(1)} &+ \left(\frac{1}{1-\delta\tilde{\pi}^1-\delta\pi^2}\right)\frac{(\beta^2)^{j(1)}\theta^2}{c}\sum_{k=0}^{L-1}\left(\alpha^2(\beta^2)^{y^*}\right)^k + \\ &\left(\frac{1}{1-\delta\tilde{\pi}^2-\delta\pi^1}\right)\frac{(\alpha^1)^{j(1)}\beta^1\theta^1}{c}\left(\frac{1-(\alpha^1)^{y^*}}{1-\alpha^1}\right)\sum_{k=0}^{L-1}\left(\beta^1(\alpha^1)^{y^*}\right)^k. \quad \text{(A.16)}\end{aligned}$$

Substituting for $\phi_{J(1)}$ from above and taking the limit as $L \to \infty$ in (A.16),

$$\tilde{\phi} \leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\pi^1)}\left[\frac{1-(\alpha^1)^{j(1)}}{1-\alpha^1}+\frac{(\alpha^1)^{j(1)}\beta^1(1-(\alpha^1)^{y^*})}{(1-\alpha^1)(1-\beta^1(\alpha^1)^{y^*})}\right]+\frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2(\beta^2)^{y^*})} \tag{A.17}$$

There are two possible cases. For brevity, we consider the case in which there exists minimal positive integer y^* such that $(\Delta^1)^{y^*} < \Delta^2 < \Delta^1$ (the case in which $(\Delta^2)^{y^*} < \Delta^1 < \Delta^2$ for some integer y^* is analogous.). From (A.17),

$$\begin{aligned}\tilde{\phi} &\leq \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\delta\pi^1)}\left[\frac{1}{1-\alpha^1}-\frac{(\alpha^1)^{j(1)}(1-\beta^1)}{(1-\alpha^1)(1-\beta^1(\alpha^1)^{y^*})}\right]+\frac{(\beta^2)^{j(1)}\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2(\beta^2)^{y^*})}\\ &< \frac{\theta^1}{c(1-\delta\tilde{\pi}^2-\delta\pi^1)(1-\alpha^1)}+\frac{\theta^2}{c(1-\delta\tilde{\pi}^1-\delta\pi^2)(1-\alpha^2)},\end{aligned}$$

where the second inequality follows because $\beta^2 < 1$. Now, for each group $m \in \{1,2\}$,

$$1-\alpha^m = 1-\frac{(1-1/n^m)\tilde{\pi}^m\delta}{1-\delta\tilde{\pi}^{-m}-\delta\pi^m} = \frac{1-\delta}{1-\delta\tilde{\pi}^m-\delta\pi^m}$$

Substituting into the above inequality, and re-arranging we obtain $c\tilde{\phi} < \frac{\theta^1+\theta^2}{1-\delta}$ for any fixed $\delta < 1$, in the case of infinte-lasting flow payoffs. For a project whose flow payoffs last exactly T periods, the analogous expression is

$$c\tilde{\phi} < \frac{(1-\delta^T)(\theta^1+\theta^2)}{1-\delta} = \hat{\theta}^1+\hat{\theta}^2.$$

Redacting the flow payoffs from the public good to a horizon of length T gives the analogous expression $c\tilde{\phi} \leq \hat{\theta}^1+\hat{\theta}^2$. Since for any $\delta < 1$ $c\tilde{\phi} < \hat{\theta}^1+\hat{\theta}^2$, then for $n^m > 1$ for some $m \in \{1,2\}$, $\lim_{\delta\to 1} c\tilde{\phi} \leq \hat{\theta}^1+\hat{\theta}^2 < n^1\hat{\theta}^1+n^2\hat{\theta}^2$. The second result follows since $\tilde{\phi}$ is the project size completed in equilibrium. □

Proof of Proposition 4.8. We show that the expected completion time of a project that requires L contribution stints, $\{h_1,\ldots,h_L\}$, is given by

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^L\right) = \sum_{\ell=1}^L j(\ell)+\sum_{\ell=1}^L \tilde{\pi}_{m_{-\ell}}(\tilde{\pi}_{m_\ell})^{j(\ell)-2}, \tag{A.18}$$

where $m_\ell = 1$ if ℓ is odd and 2 if ℓ is even, and $m_{-\ell} = m \in M \neq m_\ell$. The second part of the proposition follows by the argument in the text. Consider the sequence $\{j(\ell)\}_{\ell=1}^{L}$ and define the random variable $x\left(\{j(\ell)\}_{\ell=1}^{L}\right)$, as the number of periods it takes to finish the project. Note that for $k = 0, 1, 2, \ldots$

$$x\left(\{j(\ell)\}_{\ell=1}^{L}\right) = j(L) + k + x\left(\{j(\ell)\}_{\ell=1}^{L-1}\right) \qquad \text{w.p.} \qquad (\tilde{\pi}^1)^{j(L)}(\tilde{\pi}^2)^k$$

It follows that $\mathbb{E}\left(x\left(\{j(\ell)\}_{\ell=1}^{L}\right)\right) = \sum_{\ell=1}^{L} \mathbb{E}\left(x\left(j(\ell)\right)\right)$. Letting $m_\ell = 1$ if ℓ is odd and 2 if ℓ is even, and $m_{-\ell} = m \in M \neq m_\ell$, we have:

$$\mathbb{E}\left(x\left(j(\ell)\right)\right) = j(\ell) + (\tilde{\pi}_{m_\ell})^{j(\ell)}\left[\sum_{k=1}^{\infty}(\tilde{\pi}_{m_{-\ell}})^k k\right] = j(\ell) + (\tilde{\pi}_{m_\ell})^{j(\ell)-2}\left[\tilde{\pi}_{m_{-\ell}}\right],$$

$$\Rightarrow \mathbb{E}\left(x\left(\{j(\ell)\}_{\ell=1}^{L}\right)\right) = \sum_{\ell=1}^{L} j(\ell) + \sum_{\ell=1}^{L}(\tilde{\pi}_{m_\ell})^{j(\ell)-1}\left[\frac{\tilde{\pi}_{m_{-\ell}}}{\tilde{\pi}_{m_\ell}}\right]$$

Now, suppose $q = \phi_\ell$ for $\ell > j(1)$ and $\Delta^m > \Delta^{m'}$, and let $\mathcal{E}_1 \equiv j(1) + \tilde{\pi}_2(\tilde{\pi}_1)^{j(1)-2}$, as in the statement of the proposition. Since $\Delta^m > \Delta^{m'}$, before the final stint of $j(1)$ small projects by juniors, juniors and seniors alternate in a contribution cycle in which group m' completes single small projects, while group m completes either $x^m \equiv \left\lfloor \frac{\log[\Delta^{m'}]}{\log[\Delta^m]} \right\rfloor$ or $x^m + 1$ small projects before turning it over to members of the other group. We can then split the terms of (A.18) in stints taken by each group, and incorporate this information to compute expected delay for a project of size q.

We show the result for $\hat{k}(q)$ odd, noting that the even case is analogous. In addition to the last stint by group 1 agents there are $(\hat{k}(q) - 1)/2$ stints by each group, so for some $x \in [x^m, x^m + 1]$,

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \left(\frac{\hat{k}(q) - 1}{2}\right)\left\{(1 + x) + \frac{\tilde{\pi}^m}{\tilde{\pi}^{m'}} + \tilde{\pi}^{m'}(\tilde{\pi}^m)^{x-2}\right\}.$$

In particular, if $\Delta^m = \Delta^{m'}$, groups alternate in single step stints, so

$$\mathcal{E}\left(\{j(\ell)\}_{\ell=1}^{\hat{k}(q)}\right) = \mathcal{E}_1 + \left(\hat{k}(q) - 1\right)\left\{1 + \frac{(\tilde{\pi}^m)^2 + (\tilde{\pi}^{m'})^2}{2\tilde{\pi}^{m'}\tilde{\pi}^m}\right\}.$$

□