Coalitional Manipulations and Generalized Proportional Rules*

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Abstract

In a unified framework of allocation problems with at least three entities (or agents), we show that "generalized proportional rules" are the only rules that are robust to coalitional manipulations. We characterize proportional rules imposing in addition *efficiency*, *dummy*, and *non-negativity*. Coalitional manipulations are considered both in the environment without any restriction on coalition formation and in the restricted environment where coalitions of only pairs are possible. In the former case, non-manipulability is formalized by *reallocation-proofness*, saying that no coalition can benefit by a reallocation of characteristic vectors (or claims, in the context of bankruptcy) of its members. In the latter case, we consider *pairwise reallocation-proofness*. Several existing and new results in specialized models are obtained as corollaries. For example, axiomatizations of the proportional rule in the context of bankruptcy or surplus sharing; "utilitarian rules" in the context of social choice with transferable utilities; the Bayesian updating rule in the context of probability updating; "linear opinion pools" in the context of probability updating.

Keywords. Reallocation-proofness; pairwise reallocation-proofness; generalized proportional rule; allocation problems

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1 Introduction

Allocation problems often have the following abstract form. There is a set of entities N and a set of issues K. Each entity $i \in N$ has a characteristic vector $c_i \in \mathbb{R}^K_+$. A profile of characteristic vectors $c \equiv (c_i)_N$ and an amount of resource $E \in \mathbb{R}_+$ determines the amount to be allocated among entities in N. Examples are:

1. "Bankruptcy problem" studied by O'Neill (1982). There is a bankrupt firm with liquidation value E. This amount has to be allocated to creditors in N. Each creditor i has a claim $c_i \in \mathbb{R}_+$. The total claim is greater than the liquidation value, $\sum_{i \in N} c_i \geq E$. We refer readers to Thomson (2002a and 2002b) for an extensive treatment of social choice rules for bankruptcy problems.

2. "Surplus sharing problem" studied by Moulin (1987). A set N of contributors is involved in a joint venture. Each contributor i contributes an amount $c_i \in \mathbb{R}_+$. And they create a total benefit E that is greater than the total contribution $\sum_{i \in N} c_i$. The benefit E has to be allocated to contributors.

3. "Social choice with transferable utilities" studied by Moulin (1985a). There is a set N of agents and a set K of public projects. Each agent i has vector $c_i \in \mathbb{R}^K_+$ consisting of utilities from projects in K. Suppose that there is also a numeraire good, or money, and all agents have quasi-linear preferences with respect to money. A project with the highest sum of individual utilities is executed. Then the maximum total utility, $\max_{k \in K} \sum_{i \in N} c_{ik}$, needs to be allocated among agents through monetary transfer.

4. "Probability updating problem", similar to the belief updating formulated by Stalnaker (1968) and Lewis (1973). Let N be an event, a subset of the set of all states of the world. Suppose that N is realized and the restriction of a prior probability distribution on N is given by $(c_i)_{i \in N} \in \mathbb{R}^N_+$. This prior distribution has to be updated, after the realization of N. Here the amount to divide is always equal to 1, the probability of the realized event N.

5. "Probability aggregation problem" studied by McConway (1981) and Rubinstein and Fishburn (1986). Let N be a set of states. Let K be a set of experts. Each expert $k \in K$ has a belief over N, a subjective probability

distribution, denoted by $(c_{ik})_{i\in N}$.[1] A probability distribution over N has to be decided, aggregating their beliefs. Here the amount to divide is always equal to 1, the probability of the entire state space.

An *allocation problem* is defined by a list $((c_i)_{i\in N}, E) \in \mathbb{R}_+^{N\times K} \times \mathbb{R}_+$ of characteristic vectors and an amount of resource. An allocation rule, or briefly, a *rule* associates with each problem a vector, in $\mathbb{R}^N$, of individual shares.

We study rules that are robust to coalitional manipulation by reallocations of characteristic vectors. This condition is formalized as *reallocation-proofness* saying that no group of entities can increase its aggregate share by reallocating members' characteristic vectors. We introduce and study a weaker axiom, called *pairwise reallocation-proofness*, pertaining to only two person coalitions, namely, pairs. In the framework of variable population, we study *merging-splitting-proofness*, similar to *reallocation-proofness*, and its pairwise version. These axioms have natural interpretations in various contexts and have been studied by a number of earlier authors.

In the context of bankruptcy problems, *reallocation-proofness* prevents a coalition of creditors from gaining through a reallocation of their claims. Thus, it is also called "no advantageous reallocation" or "strategy-proofness" (see O'Neill 1982, Moulin 1985a, 1985b, and 1987, Chun 1988, and de Frutos 1999). The same interpretation applies for surplus sharing and quasi-linear bargaining problems. When there are restrictions on coalition formations, one may well think that *reallocation-proofness* is not entirely appropriate. However, *pairwise reallocation-proofness* is appealing as long as the formation of minimal coalitions, namely, pairs, is possible.

In the context of probability updating, *reallocation-proofness* requires that the updated probability of each event should not change, when the prior changes *only* over this event (and so the prior probability of the event itself does not change). In the context of probability aggregation, it requires that the aggregated probability of an event should not change, when individual beliefs change *only* over this event (McConway 1981 studies a stronger axiom called "strong setwise function property").

Summary of Main Results

When there is only one issue as in the contexts of bankruptcy, surplus sharing, and probability updating, a best known rule is the "proportional rule" (also

[1] McConway (1981) deals with a richer environment in which a certain variety of σ-algebras are admissible. In our model, we only consider the fixed state space N and the fixed σ-algebra 2^N. See also Wilson (1975) for an abstract model of algebraic aggregation.

called the "Bayesian updating rule" in the context of probability updating). It allocates the total amount in proportion to c_i's. When there are more than one issue, we can generalize this rule in the following two steps. First, for each issue $k \in K$, the division according to the k-th issue is made in proportion to c_{ik}'s. Second, we take a weighted average of these issuewise proportional divisions. Assume that the weight we use in the second step is given by a function $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \Delta^{K-1}$, relying on the sum of characteristic vectors $\bar{c} \equiv \sum_{i\in N} c_i$ and the amount of resource E. This rule is called the *proportional rule associated with weight function* W.

We identify a greater family of rules, including all proportional rules. These rules are described by two functions $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^K$ and $a\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^N$, relying on the sum of characteristic vectors $\bar{c} \equiv \sum_{i\in N} c_i$ and the amount of resource E. Function a is composed of component functions $a_1, \cdots, a_N$, where for each $i \in N$, a_i describes how much is allocated to entity i when i has the zero characteristic vector. For each problem (c, E), each entity i receives $a_i(\bar{c}, E)$ and, in addition, $(W_k(\bar{c}, E))_{k\in K}$-linear combination of issuewise proportional divisions. This rule is called *generalized proportional rule associated with* (W, a). Note that for proportional rules, W is a weight function (its range is Δ^{K-1}) and a assigns constantly the zero vector.

We first show that when there are at least three entities, generalized proportional rules are the only rules satisfying *reallocation-proofness*. We pin down smaller families of rules adding some combinations of the following standard axioms. *Non-negativity* requires that a non-negative amount should be allocated to each entity. *Efficiency* requires that the sum of individual shares should be equal to the amount to divide. *Dummy* requires that no entity with the zero characteristic vector should receive a positive amount. *Anonymity* requires that the names, or labels, of entities should not matter. Finally, *No transfer paradox* (Moulin 1985a) requires that no entity can increase its share by transferring some values of its characteristic vector to others'. In particular, we show that proportional rules are the only rules satisfying *reallocation-proofness*, *efficiency*, *dummy*, and *non-negativity* (or *no transfer paradox*).

We next establish alternative characterizations based on *pairwise reallocation-proofness*. *Reallocation-proofness* clearly implies the pairwise version. But the converse does not hold. The gap between the two axioms is filled with *pairwise non-bossiness*, saying that no pair can change others' shares without changing its total share. This axiom is similar, in spirit, to "non-bossiness" introduced by Satterthwaite and Sonnenschein (1981). We provide an alternative axiomatization of generalized proportional rules imposing *pairwise reallocation-proofness* and *pair-*

wise non-bossiness. However, for the axiomatization of proportional rules, we do not need *pairwise non-bossiness*. We show that proportional rules are the only rules satisfying *pairwise reallocation-proofness*, *efficiency*, *dummy*, and *non-negativity* (or *no transfer paradox*).

In the variable population framework, we characterize rules satisfying *merging-splitting-proofness*. Although this axiom is similar to *reallocation-proofness*, we reach somewhat different conclusions. We show that only those generalized proportional rules with $a(\cdot) = 0$ are *merging-splitting-proof*. We also show that proportional rules are the only rules satisfying *pairwise merging-splitting-proofness*, *efficiency*, and *non-negativity* (or *no transfer paradox*).

Several existing and new results in specialized contexts are obtained as corollaries: in particular, the axiomatizations of the proportional rule by O'Neill (1982), Chun (1988), de Frutos (1999), and Chambers and Thomson (2002), the axiomatizations of "ESCD rules" and "utilitarian rules" by Moulin (1985a), the axiomatization of "linear opinion pools" by McConway (1981) etc.

The rest of the paper is organized as follows. We discuss related literature in Section 2. We introduce the model and basic concepts in Section 3. The main results and corollaries are in Section 4. In Section 5, we consider the variable population model. We discuss applications of our results in Section 6.

2 Related Literature

Both *reallocation-proofness* and *merging-splitting-proofness* have been studied by various authors in the context of bankruptcy problem. The first formal study is O'Neill (1982). While bankruptcy problems are extremely simple to describe, they are surprisingly rich and provide an ideal framework to discuss the issue of fairness. A number of interesting numerical examples of problems and allocation rules have appeared in the ancient literature (e.g., the Talmud) and inspired the recent formal literature. For a comprehensive review of the large literature, see Thomson (2002a, 2002b).

In the context of bankruptcy problem, Moulin (1985b) and Chun (1988) independently characterize the proportional rule imposing *reallocation-proofness* (called "no advantageous reallocation" by them) and, in addition, *anonymity* and *continuity* (Theorem 2 in Chun 1988 and Theorem 5 in Moulin 1985b). This result is obtained as a corollary of our results and, moreover, we show that both *anonymity* and *continuity* can be dropped.

Merging-splitting-proofness has been studied by Chun (1988) and de Frutos (1999). In the context of claims problem, including both bankruptcy and surplus sharing problems, Chun (1988) characterizes the proportional rule imposing *merging-splitting-proofness*, *anonymity*, and *continuity*. A corollary of our main result shows that *anonymity* in his result can be dropped. de Frutos (1999) shows that the proportional rule is the only rule that satisfies *merging-splitting-proofness* and *non-negativity*. This result is also obtained as a corollary.

In the context of surplus sharing, Moulin (1987) shows that the "equal sharing rule" and the proportional rule are the only rules that satisfy *reallocation-proofness*, *efficiency*, *anonymity*, and any two of "separability," "path independence," and "additivity." He also show that convex combinations of the two rules are the only rules satisfying *reallocation-proofness*, *efficiency*, *anonymity*, *non-negativity*, "claims monotonicity", and "homogeneity". This result is obtained as a corollary to our results.

Moulin (1985a) considers social choice problems with transferable utilities, also called, "quasi-linear bargaining problems". In this context, a characteristic vector is composed of utilities from public projects. Thus, it is natural to require that any translation of a characteristic vector should not make any essential difference. This requirement is called *translation invariance*. Moulin (1985a) characterizes an interesting subfamily of generalized proportional rules, called, "equal sharing above a convex decision", imposing *reallocation-proofness*, *efficiency*, *translation invariance*, *no transfer paradox*, and *anonymity*. Adding *dummy*, he characterize a subfamily of proportional rules, called, "utilitarian rules." We obtain these two results as corollaries. Moreover, we show that *anonymity* can be dropped and *reallocation-proofness* in the second result can be replaced with the pairwise version.

Probability updating problem is formulated by Stalnaker (1968) and Lewis (1973). In Economics literature, a similar problem is recently studied by Rubinstein and Zhou (1999).

Probability aggregation problems are studied by McConway (1981) and Rubinstein and Fishburn (1986). It is an example of the general model of algebraic aggregation considered by Wilson (1975). In this context, each proportional rule is associated with a vector of weights $(w_k)_{k\in K}$ and it aggregates beliefs by taking the weighted sum according to $(w_k)_{k\in K}$. Thus these rules are called "linear opinion pools": see McConway (1981). In a certain "richer environment," McConway characterizes linear opinion pools imposing *strong setwise function property*, the requirement that there exists a function associating with any list of probability assessments of an event by experts an aggregated probability of the same event.[2] This axiom implies *reallocation-proofness*. Thus his result is also obtained as a corollary.

3 Definitions

3.1 Allocation Problems

There are $N \geq 2$ *entities*, interpreted often as "agents", and K *dimensions* of the characteristics of each entity. Let us also denote the set of entities by $N \equiv \{1, \cdots, N\}$ and the set of dimensions by $K \equiv \{1, \cdots, K\}$. Each entity $i \in N$ is identified by its *characteristic vector* $c_i \in \mathbb{R}_+^K$. Let $c \in \mathbb{R}_+^{N \times K}$ be the *characteristic matrix* consisting of N rows $c_1, \cdots, c_N$. An *allocation problem* is a list $(c, E) \in \mathbb{R}_+^{N \times K} \times \mathbb{R}_+$ of a characteristic matrix c and an amount E of resource. The amount to divide depends on the sum of characteristic vectors, $\bar{c} \equiv \sum_{i \in N} c_i$, and resource E. Let $e\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}_+$ be a *feasibility function* associating with each pair $(\bar{c}, E) \in \mathbb{R}_+^K \times \mathbb{R}_+$ the amount $e(\bar{c}, E)$ to divide among N entities. We rule out uninteresting problems by assuming that for each problem (c, E), $\sum_{i \in N} c_i \neq 0$ or $e(\bar{c}, E) = 0$.[3] Throughout the paper, we fix $e(\cdot)$.

Let $\mathcal{D}^N$ be a non-empty collection of admissible problems. We only require the following condition of *richness*: for each $(c, E) \in \mathcal{D}^N$ and each $c' \in \mathbb{R}_+^{N \times K}$, if $\sum_{i \in N} c'_i = \sum_{i \in N} c_i$, then $(c', E) \in \mathcal{D}^N$. That is, any reallocation of each component in $\sum_{i \in N} c_i$ among N generates another admissible problem. Throughout the paper, we will assume richness of $\mathcal{D}^N$. For each $(c, E) \in \mathcal{D}^N$, let $\mathcal{D}^N(\bar{c}, E) \equiv \{(c', E) : c' \in \mathbb{R}_+^{N \times K} \text{ and } \bar{c}' = \bar{c}\}$. Then richness says that for each $(c, E) \in \mathcal{D}^N$, $\mathcal{D}^N(\bar{c}, E) \subseteq \mathcal{D}^N$.

An "allocation rule", or briefly, a *rule* is a function f that associates with each problem $(c, E) \in \mathcal{D}^N$ a vector $f(c, E) \in \mathbb{R}^N$ and that satisfies the following "bound condition": for each $(\bar{c}, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$, there exists $i \in N$ such that f_i is bounded above or below over $\mathcal{D}^N(\bar{c}, E)$, where $f_i(c, E)$ denotes the amount allocated to entity $i \in N$. We call $\mathcal{D}^N$ the *domain* of f. The bound condition excludes only those extreme functions of which image of the compact set $\mathcal{D}^N(\bar{c}, E)$ is unbounded in every component.[4] This condition is implied by some standard

[2] McConways (1981) assumes that every σ-algebra over a state space is admissible. This assumption playes an important role in his result.

[3] If there is a problem (c, E) with $\bar{c} = 0$ and $E > 0$, then the two basic axioms, "efficiency" and "dummy", to be defined later, are not compatible.

[4] Note that since $\mathcal{D}^N(\bar{c}, E)$ is compact, any continuous function satisfies the bound condition.

axioms such as “non-negativity” or “no transfer paradox”, to be introduced below. Thus, it can be dropped in all of our results where these axioms are assumed.

The following requirements for rules, or axioms, are standard. The first axiom says that each entity should receive non-negative amount.

Non-Negativity. For each $(c, E) \in \mathcal{D}^N$ and each $i \in N$, $f_i(c, E) \geq 0$.

The next axiom says that the total amount allocated should be equal to the amount to divide, that is, no surplus or deficit.

Efficiency. For each $(c, E) \in \mathcal{D}^N$, $\sum_{i \in N} f_i(c, E) = e(\bar{c}, E)$.

The next axiom says that nothing should be allocated to an entity with the zero characteristic vector.

Dummy. For each $i \in N$ and each $(c, E) \in \mathcal{D}^N$, if $c_i = 0$, $f_i(c, E) = 0$.

The next axiom says that changing labels of entities should not matter. Let $\tau \colon N \to N$ be a permutation. For each $c \in \mathbb{R}_+^N$, let c^τ be such that for each $i \in N$, $c_i^\tau \equiv c_{\tau(i)}$.

Anonymity. For each permutation τ on N, each $(c, E) \in \mathcal{D}^N$, and each $i \in N$, $f_i(c^\tau, E) = f_{\tau(i)}(c, E)$.

The last axiom says that no entity can increase its share by transferring some values of its characteristic vector to others'.

No Transfer Paradox. (Moulin 1985) For each $E \in \mathbb{R}_+$, each $c, c' \in \mathbb{R}_+^{N \times K}$, and each $i \in N$ with $(c, E), (c', E) \in \mathcal{D}^N$,

$$\text{if } c_i \leq c'_i,\ \bar{c} = \bar{c}', \text{ and for each } j \neq i,\ c_j \geq c'_j, \text{ then } f_i(c, E) \leq f_i(c', E).$$

Although these axioms are standard, we do not require them in all of our results. Also standard is *feasibility*, saying that for each $(c, E) \in \mathcal{D}^N$, $\sum_{i \in N} f_i(c, E) \leq e(\bar{c}, E)$. In our results, we do not need to impose *feasibility*. However, adding this extra condition will refine our results in a straightforward manner. Clearly, *efficiency* implies *feasibility*.

If f satisfies *no transfer paradox*, then for each (c, E) and each $i \in N$, f_i over $\mathcal{D}^N(\bar{c}, E)$ is bounded *above* by $f_i(c', E)$, where $c'_i = \bar{c}$ and $c'_j = 0$ for each $j \neq i$ (note that c' is this *only* profile that has the maximal vector for entity i over $\mathcal{D}^N(\bar{c}, E)$). So, this axiom implies the bound condition on rules. Also

The bound condition is also satisfied in the bankruptcy problems if we assume either “non-negativity” or “claims boundedness”.

non-negativity implies the bound condition. Hence the bound condition can be dropped in all of our results with *non-negativity* or *no transfer paradox*.

We give a few examples of interesting classes of allocation problems.

Example 1 (Bankruptcy). The first example is bankruptcy problems (O'Neill 1982). The problem is to divide the liquidation value E of a bankrupt firm among the set of creditors N. In this problem, $K = 1$, and $c_i \in \mathbb{R}_+$ is the amount of claim that creditor i has against the bankrupt firm. And the amount to divide is the liquidation value E, so $e(\bar{c}, E) = E$. It is assumed that the liquidation value is not sufficient to satisfy everyone's claim. Thus we set $\mathcal{D}^N \equiv \{(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+ : \sum_{i \in N} c_i \geq E\}$.

In practice, a firm issues a variety of financial assets and bankruptcy laws distinguish the types of financial assets that the creditors hold. This motivates the following generalization of the bankruptcy problems:

Example 2 (Multi-Dimensional Bankruptcy). As in Example 1, E is the liquidation value of a bankrupt firm and N is the set of creditors. Let K denote the set of types of financial assets and c_{ik} denote the amount of claim that creditor i holds in the form of asset k. Let $e(c, E) \equiv E$ and $\mathcal{D}^N \equiv \{(c, E) \in \mathbb{R}_+^{N \times K} \times \mathbb{R}_+ : \sum_{k \in K} \sum_{i \in N} c_{ik} \geq E\}$.

Example 3 (Surplus Sharing). The problem is to divide the profits from a project among the contributors. Here N is the set of contributors, $K = 1$, E is the total amount of profits that the project generates if cooperation succeeds, and $c_i \in \mathbb{R}_+$ is the amount that agent i earns if cooperation fails. Thus $E - \sum_{i \in N} c_i$ is the total gains from cooperation and assumed to be non-negative. Let $e(\bar{c}, E) \equiv E$ and $\mathcal{D}^N \equiv \{(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+ : 0 < \sum_{i \in N} c_i \leq E\}$.

Example 4 (Claims Problems). The class of general claims problems is simply the union of the classes of (single-dimensional) bankruptcy and surplus-sharing problems. Thus, the problem is to divide an amount $E > 0$ among a set of agents N, taking into account that each agent $i \in N$ has a claim $c_i \in \mathbb{R}_+$, and no relation between $\sum_{i \in N} c_i$ and E is imposed.

Example 5 (Social Choice with Transferable Utilities). In this class of problems, N is the set of agents and K is the set of alternatives. Assume that E is fixed. So we skip this notation below. Preferences are quasi-linear and c_{ik} is agent i's valuation for alternative k. Assume that monetary transfers are feasible. Then the highest total valuations among K alternatives determines the amount to divide. Thus, for each $c \in \mathbb{R}_+^{N \times K}$, $e(\bar{c}) \equiv \max_{k \in K} \bar{c}_k$. Let $\mathcal{D}^N \equiv \mathbb{R}_+^{N \times K}$. This class of problems differs from the previous ones since the amount to divide $e(\bar{c})$ depends on $\bar{c}$.

Example 6 (Probability Updating). Let N^* be the set of all states of the world and an agent initially has a belief, i.e., probability distribution over the states. Now, the agent is informed that the true state is actually in a subset $N \subseteq N^*$ and he has to update his belief. For each state $i \in N$, let $c_i \in \mathbb{R}_+$ be the probability that the agent initially assigns to state i (thus $K = 1$). Since $N \subseteq N^*$, we have $\sum_{i \in N} c_i \leq 1$. We assume $\sum_{i \in N} c_i > 0$, i.e., the agent initially assigns a positive probability to the event N that has occurred. Let E be fixed at 1. So we omit E below. We then have $\mathcal{D}^N = \{c \in \mathbb{R}_+^N : 0 < \sum_{i \in N} c_i \leq 1\}$ and $e(\bar{c}) = 1$ for each $c \in \mathcal{D}^N$. The updating rule used commonly is the "Bayesian updating rule". The problem of probability updating is studied by Stalnaker (1968) and Lewis (1973).

In the context of probability updating, *efficiency* means that the updated belief assigns probability 1 to the event N that has occurred. On the other hand, *dummy* means that if the agent initially assigns probability 0 to a state $i \in N$, then he does so under the updated belief as well: informally speaking, if the agent initially believed that a state $i \in N$ never occurs, then he continues to believe so after realizing the event N.

Example 7 (Probability Aggregation). In this class of problems, N is the set of states of the world. Initially, there are K probability distributions over the states. That is, for each $k \in K$, $(c_{ik})_{i \in N} \in \Delta^{N-1}$ is a probability distribution over N. For example, K may be the set of experts, who have different beliefs over the states. The problem is to aggregate these probability distributions and produce a single distribution. Let E be fixed at 1. So we omit E below. We then have $\mathcal{D}^N = \{c \in \mathbb{R}_+^{N \times K} : \text{for each } k \in K, \sum_{i \in N} c_{ik} = 1\}$ and $e(\bar{c}) = 1$ for each $c \in \mathcal{D}^N$. This problem of probability aggregation is studied by McConway (1981) and Rubinstein and Fishburn (1986).

In the context of probability aggregation, *efficiency* has an obvious meaning. *Dummy* means that if state i has the zero probability under all of the individual probability distributions, then state i should also have the zero probability after the aggregation.

3.2 Proportional Rules

For the single-dimensional case, namely, when $K = 1$, a simplest and best-known rule is the proportional rule, which simply divides the total amount proportional to c_i's. Formally:

Definition. Assume $K = 1$. The *proportional rule* is the rule f defined by

$$f_i(c, E) \equiv \frac{c_i}{\bar{c}} \times e(\bar{c}, E), \tag{1}$$

for each $i \in N$ and each $(c, E) \in \mathcal{D}^N$ with $\bar{c} \neq 0$; if $\bar{c} = 0$ (and so $e(\bar{c}, E) = 0$), $f_i(c, E) = 0$ for each $i \in N$.

In the context of probability updating, the proportional rule is nothing but the Bayesian updating rule.

For the case when $K \geq 2$, (1) is not well-defined. We generalize the proportional rule in the single-dimensional case, by using the following notion. A *weight function* is defined as a function $W\colon \mathbb{R}^K_+ \times \mathbb{R}_+ \to \Delta^{K-1}$ such that for each $(c, E) \in \mathcal{D}$ with $c \neq 0$ and each $k \in K$, if $\bar{c}_k = 0$, $W_k(\bar{c}, E) = 0$. Thus $\sum\limits_{k \in K\colon \bar{c}_k > 0} W_k(\bar{c}, E) = 1$.

Definition. A rule f is a *proportional rule* if there exists a weight function W such that for each problem $(c, E) \in \mathcal{D}^N$,

$$f_i(c, E) = \left(\sum_{k \in K\colon \bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) \right) \times e(\bar{c}, E).$$

We denote by P^W the proportional rule associated with weight function W.

This rule P^W first applies the proportional rule to each single dimensional sub-problem (c^k, E), where $k \in K$ and $c^k \equiv (c_{ik})_{i \in N}$, and then takes the weighted average of the solutions to the sub-problems using the vector of weights $W(\bar{c}, E)$. Note that the weights depend on the problem being considered, but depend only on $(\bar{c}, E)$.

The proportional rules satisfy *efficiency* because of $\sum_{k \in K\colon \bar{c}_k > 0} W_k(\bar{c}, E) = 1$. These rules also satisfy *dummy*. It is evident that when $K = 1$, all proportional rules coincide.

In the context of probability aggregation (Example 7), all problems have $E = 1$ and $\bar{c}_k = 1$ for each $k \in K$. This means that only the weight vector $w \equiv W((1, \cdots, 1), 1)$ of weight function W is crucial for the definition of P^W. The proportional rule P^W then simply takes a weighted average of the probability distributions using the fixed vector of weights, w. Therefore, proportional rules are called *linear opinion pools* (McConway 1981).

3.3 Reallocation-Proofness

Our main objective is to study rules that are robust to coalitional manipulations through reallocations of characteristic vectors. Such a robustness is formalized by the requirement that the total amount allocated to each group $S \subseteq N$ should not be affected by any reallocation of c_i's within S. Formally:

Definition. An allocation rule f is *reallocation-proof* if for each $(c, E) \in \mathcal{D}^N$, each $S \subseteq N$, and each $c' \in \mathbb{R}_+^{S \times K}$, if $\sum_{i \in S} c'_i = \sum_{i \in S} c_i$,

$$\sum_{i \in S} f_i(c', c_{N \setminus S}, E) = \sum_{i \in S} f_i(c, E). \tag{2}$$

In the context of claims problems and their variants, the axiom means that no group of agents can change their aggregate share by reallocating claims within the group. If the left-hand side of (2) is larger than the right-hand side, then group S with claim profile $(c_i)_{i \in S}$ can gain by reallocating their claims to c'_S (and making appropriate side-payments). If the reverse inequality holds, then group S with claims $(c'_i)_{i \in S}$ can gain.

The generalized proportional rules P^W are *reallocation-proof*, since the total amount that group S receives is given by

$$\sum_{i \in S} P_i^W(c, E) = \sum_{k \in K:\, \bar{c}_k > 0} W_k(\bar{c}, E) \frac{\sum_{i \in S} c_{ik}}{\bar{c}_k} e\left(\bar{c}, E\right)$$

and the right-hand side is invariant with respect to reallocations of c_i's within S.

This axiom has been introduced by Moulin (1985a) and Chun (1988) in the contexts of social choice with transferable utilities and claims problems, respectively. In these contexts, it is also called "no advantageous reallocation".

In the context of probability updating (Example 6), *reallocation-proofness* deals with two beliefs μ and μ' that differ only on an event $S \subseteq N$ but assign the same total probability to S. The axiom then says that the updated versions of μ and μ' should also assign the same total probability to S.

In the context of probability aggregation (Example 7), this axiom deals with two profiles of beliefs $(\mu_k)_{k \in K}$ and $(\mu'_k)_{k \in K}$ such that for each $k \in K$, μ_k and μ'_k differ only on an event $S \subseteq N$ but assign the same total probability to S. The axiom then says that the aggregated (social) beliefs under the two profiles should assign the same total probability to S.

3.4 Pairwise Reallocation-Proofness and Non-Bossiness

Reallocation-proofness pertains to all subsets of N, or coalitions. When there are restrictions on coalition formation, this axiom may not be entirely appropriate.

We next introduce a weaker axiom that is associated only with the minimal coalitions, namely, pairs. It says that no pair can increase the aggregate share by reallocating characteristic vectors among its members.

Definition. An allocation rule f is *pairwise reallocation-proof* if for each $(c, E) \in \mathcal{D}^N$, each $i, j \in N$ with $i \neq j$, and each $c' \in \mathbb{R}_+^{S\times K}$, if $c'_i + c'_j = c_i + c_j$ and $c'_{N\setminus\{i,j\}} = c_{N\setminus\{i,j\}}$,

$$f_i(c', E) + f_j(c', E) = f_i(c, E) + f_j(c, E). \tag{3}$$

The pairwise version is particularly relevant for claims problems since it is reasonable to believe that strategic reallocations of claims are easier to implement for smaller groups of agents ("transaction costs").

Clearly, *reallocation-proofness* implies *pairwise reallocation-proofness*. However, the converse does not hold, as shown by Example 8. We will show that the gap between the two axioms are filled with the next axiom. It says that any pair cannot change, through a reallocation of characteristic vectors, the shares of others, without affecting its own aggregate share. This axiom is similar, in spirit, to "non-bossiness" in economic environments introduced by Satterthwaite and Sonnenschein (1981).

Definition. An allocation rule f is *pairwise non-bossy* if for each $(c, E) \in \mathcal{D}^N$, each $i, j \in N$ with $i \neq j$, and each $c' \in \mathbb{R}_+^{S\times K}$, if $c'_i + c'_j = c_i + c_j$, $c'_{N\setminus\{i,j\}} = c_{N\setminus\{i,j\}}$, and $f_i(c', E) + f_j(c', E) = f_i(c, E) + f_j(c, E)$,

$$f_{N\setminus\{i,j\}}(c', E) = f_{N\setminus\{i,j\}}(c, E). \tag{4}$$

Although a pair $\{i, j\}$ may not affect their share by a reallocation of their characteristic vectors, they may change others' shares (this is shown by Example 8). Thus *pairwise reallocation-proofness* does not imply *pairwise non-bossiness*.

If (4) does not hold, say, $f_h(c', E) > f_h(c, E)$, for some $h \in N\setminus\{i, j\}$, then the coalition $S \equiv \{h, i, j\}$ gains by a reallocation of characteristic vectors which changes (c, E) to (c', E). This is a violation of *reallocation-proofness*. Thus, *reallocation-proofness* implies *pairwise non-bossy*. We will show later that the combination of *pairwise reallocation-proofness* and *pairwise non-bossiness* is equivalent to *reallocation-proofness*.

4 Main Results

Our main results are composed of two parts. First, we characterize *reallocation-proof* rules. We pin down smaller families of these rules, adding *dummy*, *efficiency*, *anonymity*, *non-negativity*, and *no transfer paradox*. Second, we establish alternative characterizations considering *pairwise reallocation-proofness* and *pairwise non-bossiness*.

Reallocation-Proofness

We first characterize *reallocation-proof* rules. These rules are described by two functions $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^K$ and $a\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^N$. Given $\bar{c} \in \mathbb{R}_+^K$ and $E \in \mathbb{R}_+$, $a(\bar{c}) \equiv (a_i(\bar{c}))_{i\in N}$ describes the list of shares allocated to each entity whenever it has the zero characteristic vector. Let us refer to the difference between entity i's share at (c, E) and $a_i(\bar{c})$ as entity i's surplus. Entity i's surplus is determined by the "$W(\bar{c}, E)$-linear" combination of the proportional allocations according to $(c_{ik})_{i\in N}$ for each $k \in K$. Proportional rules are special examples where the value of a is constantly the zero vector and the range of W is the simplex Δ^{K-1}. Formally,

Definition. A rule f is a *generalized proportional rule* if there exist two functions $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^K$ and $a\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^N$ such that for each $(c, E) \in \mathcal{D}^N$ and each $i \in N$,

$$f_i(c,E) = a_i(\bar{c},E) + \sum_{k\in K\colon \bar{c}_k>0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c},E) e(\bar{c},E). \tag{5}$$

Note that given $(\bar{c}, E)$, for each $i \in N$, $\{c_i : c_i$ is i's characteristic vector for some $(c, E) \in \mathcal{D}^N(\bar{c}, E)\}$ is bounded above by $\bar{c}$ and below by 0. Then $a_i(\bar{c}, E)+\sum_{k\in K\colon \bar{c}_k>0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E)e(\bar{c}, E)$ is bounded both above and below. Hence generalized proportional rules satisfy the bound condition.

It should be noted that W is not required to be a weight function. Also note that in the definition, the value of $W_k(\bar{c}, E)$ when $\bar{c}_k = 0$ does not play any role and so it could be set arbitrarily. Therefore, for simplicity, we will set $W_k(\bar{c}, E) = 0$ whenever $\bar{c}_k = 0$.

We show that generalized proportional rules are the only rules that are *reallocation-proof.*

Theorem 1. *Assume $N \geq 3$. Then, a rule over $\mathcal{D}^N$ is* reallocation-proof *if and only if it is a generalized proportional rule.*

Theorem 1 can easily generate characterizations of rules that also satisfy combinations of *non-negativity*, *dummy*, *efficiency*, *anonymity*, and *no transfer paradox*.

We provide necessary and sufficient conditions on (W, a) for each of these axioms.

Fact 1 (Non-Negativity). *The generalized proportional rule associated with* (W, a) *satisfies* non-negativity *if and only if for each* $(c, E) \in \mathcal{D}^N$,

$$a_i(\bar{c}, E) \geq 0, \text{ for each } i \in N; \tag{6}$$

$$\min_{j \in N} a_j(\bar{c}, E) + \sum_{k \in K:\, \bar{c}_k > 0} \min\{0, W_k(\bar{c}, E)\} e(\bar{c}, E) \geq 0. \tag{7}$$

Fact 2 (Efficiency). *The generalized proportional rule associated with* (W, a) *satisfies* efficiency *if and only if for each* $(c, E) \in \mathcal{D}^N$,

$$\sum_{k \in K:\, \bar{c}_k > 0} W_k(\bar{c}, E) e(\bar{c}, E) = \left[e(\bar{c}, E) - \sum_{i \in N} a_i(\bar{c}, E) \right]. \tag{8}$$

When $K = 1$, (8) implies that for each $i \in N$,

$$f_i(c, E) = a_i(\bar{c}, E) + \frac{c_i}{\bar{c}} \left[e(\bar{c}, E) - \sum_{i \in N} a_i(\bar{c}, E) \right]. \tag{9}$$

This rule first allocates $a_i(\bar{c}, E)$ to each i, and then divides the remaining amount among i's proportional to c_i's. It satisfies *non-negativity* if and only if for each $(c, E) \in \mathcal{D}^N$ and each $j \in N$,

$$\sum_{i \in N \setminus \{j\}} a_i(\bar{c}, E) \leq e(\bar{c}, E). \tag{10}$$

Fact 3 (Dummy). *The generalized proportional rule associated with* (W, a) *satisfies* dummy *if and only if for each* $(c, E) \in \mathcal{D}^N$ *and each* $i \in N$,

$$a_i(\bar{c}, E) = 0. \tag{11}$$

Interestingly, this implies that all allocation rules that satisfy *dummy* and *reallocation-proofness* also satisfy *anonymity*. Note that when $K = 1$, by (10), *efficiency* and *dummy* imply *non-negativity*. Note also that, under (11), (7) is equivalent to the condition that for each $k \in K$, $W_k(\bar{c}, E) \geq 0$.

Fact 4 (Anonymity). *The generalized proportional rule associated with* (W, a) *satisfies* anonymity *if and only if for each* $(c, E) \in \mathcal{D}^N$,

$$a_1(\bar{c}, E) = \cdots = a_N(\bar{c}, E).$$

Fact 5 (No Transfer Paradox). *The generalized proportional rule associated with* (W, a) *satisfies* no transfer paradox *if and only if* W *is non-negative valued, that is, for each* $(c, E) \in \mathcal{D}^N$ *and each* $k \in K$,

$$W_k(\bar{c}, E) \geq 0.$$

Proof. Let f be the generalized proportional rule associated with (W, a). Assume that W is non-negative valued. Then it is clear from (5) that when $\bar{c}$ and E are fixed, for each $i \in N$, f_i is non-decreasing in c_i. So f satisfies *no transfer paradox.*

To show the converse, suppose to the contrary that for some $(\bar{c}, E) \in \mathbb{R}_+^K \times \mathbb{R}_+$ and $k \in K$, $\bar{c}_k > 0$ and $W_k(\bar{c}, E) < 0$. Let c be such that $c_{1k} > 0$ and $c_{1l} = 0$ for each $l \neq k$. Then $f_1(c, E) = a_1(\bar{c}, E) + \frac{c_{1k}}{\bar{c}_k} W_k(\bar{c}, E)\, e(\bar{c}, E)$. Then if some amount in the k-th component of c_1 is transferred to vectors of other entities (no change in other components), the share for entity 1 increases. Thus f violates *no transfer paradox.* ∎

If the generalized proportional rule associated with (W, a) satisfies *dummy*, *efficiency*, and *non-negativity* (or *no transfer paradox*), then by (8) and (11), we may let W be the weight function. Thus, we obtain an axiomatization of proportional rules. Formally:

Corollary 1. *Assume* $N \geq 3$. *Then, a rule over* $\mathcal{D}^N$ *satisfies* reallocation-proofness*,* dummy*,* efficiency*, and* non-negativity *(or* no transfer paradox*) if and only if it is a proportional rule.*

Remark 1. When $K \geq 2$, the independence of the four axioms can be established easily. In particular, without *non-negativity* (or *no transfer paradox*), we axiomatize the family of rules f that is represented by a function $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}$ such that for each $(\bar{c}, E) \in \mathbb{R}_+^K \times \mathbb{R}_+$, $\sum_{k \in K} W_k(\bar{c}, E) = 1$, in the following form: for each $(c, E) \in \mathcal{D}^N$ and each $i \in N$,

$$f_i(c, E) = \left(\sum_{k \in K\colon \bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) \right) \times e(\bar{c}, E).$$

Note that W is not necessarily a weight function, since W may have negative values. For such W, f violates *non-negativity.* Note also that since each of *non-negativity* and *no transfer paradox* implies the bound condition in the definition of rules, this condition can be dropped. This remark also applies to Theorems 3 and 6 and Corollaries 3 and 4.

Remark 2. When $K = 1$ as in the contexts of bankruptcy and surplus sharing, *non-negativity* (or *no transfer paradox*) can be dropped. However in this case, the bound condition in the definition of rules is crucial. This remark also applies to Theorems 3 and 6 and Corollaries 3 and 4.

A *capacity function* is a function $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^K$ such that for each $(\bar{c}, E) \in \mathbb{R}_+^K \times \mathbb{R}_+$, $W(\bar{c}, E) \geqq 0$ and $\sum_{k \in K} W_k(\bar{c}, E) \leq 1$. We next axiomatize generalized proportional rules represented by capacity functions.

Corollary 2. *Assume $N \geq 3$. Then, a rule over $\mathcal{D}^N$ satisfies* reallocation-proofness, anonymity, *and* efficiency *if and only if it is a generalized proportional rule represented by a function $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}$ as follows: for each $(c, E) \in \mathcal{D}^N$ and each $i \in N$,*

$$f_i(c, E) = \frac{1}{n}\left(1 - \sum_{k:\bar{c}_k > 0} W_k(\bar{c}, E)\right) e(\bar{c}, E) + \sum_{k \in K:\bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E)\, e(\bar{c}, E)\,. \quad (12)$$

Adding non-negativity *and* no transfer paradox*, we axiomatize generalized proportional rules represented by a capacity function.*

Proof. Let f be the generalized proportional rule associated with (W, a). Assume that f satisfies the four axioms. Then by *anonymity* and *efficiency*, for each $(c, E) \in \mathcal{D}^N$ and each $i \in N$,

$$a_i(\bar{c}, E) = \frac{1}{n}\left(1 - \sum_{k:\bar{c}_k > 0} W_k(\bar{c}, E)\right) e(\bar{c}, E)\,.$$

By *no transfer paradox*, W is non-negative valued. Adding *non-negativity*, $a_i(\bar{c}, E) \geq 0$ and so $\sum_{k:\bar{c}_k > 0} W_k(\bar{c}, E) = \sum_{k \in K} W_k(\bar{c}, E) \leq 1$. Then W is a capacity function. ∎

As in Corollary 1, the bound condition in the definition of rules can be dropped for this result.

We now prove Theorem 1.

Proof of Theorem 1. Suppose $N \geq 3$. Since the "if" part has been already discussed, we only prove the "only if" part. Let f be a *reallocation-proof* rule over $\mathcal{D}^N$. We fix $E > 0$ and $d \in \mathbb{R}_+^K \setminus \{0\}$ throughout the proof and focus $(c, E) \in \mathcal{D}^N$ such that $\bar{c} = d$. Let $\mathcal{C} \equiv \{c \in \mathbb{R}_+^{K \times N} : (c, E) \in \mathcal{D}^N \text{ and } \bar{c} = d\}$. If $\mathcal{C} = \emptyset$, there is nothing to prove. Thus we assume $\mathcal{C} \neq \emptyset$ in what follows.

Note that by *reallocation-proofness* applied to N, the total share of all entities is constant over $\mathcal{C}$. Using this and *reallocation-proofness*, we can show that for each $c \in \mathcal{C}$ and each $S \subseteq N$, $\sum_{i\in S} f_i(c, E)$ depends only on S and $\sum_{i\in S} c_i$ (in addition to E and $\sum_{i\in N} c_i = d$, which are fixed). This enables us to define $g(S, x)$, which denotes the total amount received by group S at $f(c, E)$ for each $c \in \mathcal{C}$ with $\sum_{i\in S} c_i = x$. The domain of the function is the set of all $(S, x) \in 2^N \times \mathbb{R}^K_+$ such that $\emptyset \neq S \subsetneq N$ and $0 \leq x \leq d$. The value $g(S, x)$ is well-defined for all these (S, x) because of the richness assumption.

Step 1. Let $x \in \mathbb{R}^K$ satisfy $0 \leq x \leq d$. Let $\emptyset \neq S \subsetneq N$ and $i \in S$. By richness, there exists $c \in \mathcal{C}$ such that $c_i = 0$ and $\sum_{j\in S\setminus i} c_j = x$ (for simplicity, we sometimes write i instead of writing $\{i\}$). Then, at $f(c, E)$, S receives $g(S, x)$, $S \setminus i$ receives $g(S \setminus i, x)$, and i receives $g(i, 0)$. Hence

$$g(S, x) = g(S \setminus i, x) + g(i, 0). \tag{13}$$

Applying this equation to each pair $\{i, j\} \subseteq N$ yields

$$\begin{aligned} g(\{i, j\}, x) &= g(j, x) + g(i, 0) \\ &= g(i, x) + g(j, 0), \end{aligned}$$

which implies

$$g(i, x) - g(i, 0) = g(j, x) - g(j, 0).$$

Since this holds for any $i, j \in N$, it follows that $g(i, x) - g(i, 0)$ is independent of i. Thus we can denote the common value by $\xi(x)$. Then for each $i \in N$,

$$g(i, x) = \xi(x) + g(i, 0).$$

Note that by the bound condition, ξ defined over $\{x \in \mathbb{R}^K_+ : x \leq d\}$ is bounded above or below. Without loss of generality, assume that it is bounded below (the same argument can be applied for the bounded above case). Let $B \in \mathbb{R}$ be a lower bound (when f satisfies the non-negativity constraint, we can use $-\min_{i\in N} g(i, 0)$ as a lower bound). By definition, $\xi(0) = 0$. A repeated application of (13) yields that for each non-empty $S \subsetneq N$ and each $j \in S$,

$$\begin{aligned} g(S, x) &= g(j, x) + \sum_{i\in S\setminus\{j\}} g(i, 0) \qquad (14) \\ &= \xi(x) + \sum_{i\in S} g(i, 0). \qquad (15) \end{aligned}$$

Step 2. We show that for each $x, y \in \mathbb{R}^K_+$ such that $x + y \leq d$,

$$\xi(x) + \xi(y) = \xi(x + y).$$

To show this, partition N into three non-empty subsets S_1, S_2, and S_3, which is possible since $N \geq 3$. By richness, there exists $c \in \mathcal{C}$ such that $\sum_{i \in S} c_i$ is equal to x for $S = S_1$, y for $S = S_2$, and $d - x - y$ for $S = S_3$. Then by (15), S_1 receives $\xi(x) + \sum_{i \in S_1} g(i, 0)$, S_2 receives $\xi(y) + \sum_{i \in S_2} g(i, 0)$, and $S_1 \cup S_2$ receives $\xi(x + y) + \sum_{i \in S_1 \cup S_2} g(i, 0)$. Therefore, $\xi(x) + \xi(y) = \xi(x + y)$.

Step 3. We show that for each $x \in \mathbb{R}^K_+$ such that $x \leq d$ and for each $r \in (0, 1)$,

$$\xi(rx) = r\xi(x). \tag{16}$$

If r is a rational number, (16) follows from Step 2 (this can be shown by using a standard argument). To prove (16) for each real numbers $r \in (0, 1)$, we exploit the lower bound B. Let $r \in (0, 1)$. Then for any rational number $s \in (0, 1)$ with $s \geq r$,

$$\xi(rx) = \xi(s(rx/s)) = s\xi(rx/s) \geq sB,$$

where the inequality follows because B is a lower bound of ξ. Since this holds for any rational number $s \in (0, 1)$ with $s \geq r$ and the last term is continuous in s, we obtain: for each $r \in (0, 1)$,

$$\xi(rx) \geq rB. \tag{17}$$

On the other hand, by Step 2 and (17), for any rational number $s \in (0, 1)$ with $s \geq r$,

$$\begin{aligned} s\xi(x) - \xi(rx) &= \xi(sx) - \xi(rx) \\ &= \xi((s - r)x) \\ &\geq (s - r)B. \end{aligned}$$

Since this holds for any rational number $s \in (0, 1)$ with $s \geq r$ and the first and the last terms are continuous in s, we obtain

$$r\xi(x) - \xi(rx) \geq 0.$$

The symmetric argument that uses rational numbers $s \leq r$ yields the reverse inequality. Hence we obtain $r\xi(x) = \xi(rx)$.

Step 4. Let $e^k \in \mathbb{R}^K_+$ be the unit vector that has 1 in the k-th component and 0 in other components. Then for each $x \in \mathbb{R}^K_+$ with $x \leq d$,

$$\begin{aligned} \xi(x) &= \sum_{k \in K:\, d_k > 0} \xi(x_k e^k) \\ &= \sum_{k \in K:\, d_k > 0} \frac{x_k}{d_k} \xi(d_k e^k). \end{aligned}$$

Recall that E and d are fixed at the beginning of the proof. This implies that functions g_i's and ξ depend on E and d. Thus, we can denote $\xi(d_k e^k)/e\,(d,E)$ by $W_k(d,E)$ (if $d_k = 0$, let $W_k\,(d,E) \equiv 0$) and $g(i,0)$ by $a_i(d,E)$. Then, for each $c \in \mathcal{C}$ (thus $\bar{c} = d$) and each $i \in N$,

$$\begin{aligned} f_i(c,E) &= g(i,c_i) \\ &= g(i,0) + \xi(c_i) \\ &= a_i(\bar{c},E) + \sum_{k \in K\,:\,\bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c},E) e\,(\bar{c},E)\,, \end{aligned}$$

which proves (5). ∎

Remark 3. It is known that when an additive real valued function over $\mathbb{R}$ is bounded above or below on a set of positive Lebesgue measure, it is linear (Theorem 2.1.8 in Aczél and Dhombres 1989). Step 3 can be simplified when this fact is used.[5]

Pairwise Reallocation-Proofness

When *reallocation-proofness* in Theorem 1 is weakened to *pairwise reallocation-proofness*, the characterization no longer holds. Example 8 below shows this. However, adding *pairwise non-bossiness*, we establish an alternative characterization. Formally:

Theorem 2. *Assume $N \geq 3$. Then, a rule over $\mathcal{D}^N$ is* pairwise reallocation-proof *and* pairwise non-bossy *if and only if it is a generalized proportional rule.*

This result follows directly from Theorem 1 and the following lemma.

Lemma 1. *The combination of* pairwise reallocation-proofness *and* pairwise non-bossiness *is equivalent to* reallocation-proofness.

Proof. We will prove that *pairwise reallocation-proofness* and *pairwise non-bossiness* imply *reallocation-proofness*. The converse has already been discussed.

We first define some technical terms. For all profiles $c, c' \in \mathbb{R}_+^{N \times K}$ admissible in domain $\mathcal{D}^N$, c' is a *pairwise variant of* c if there exists $S \subseteq N$ such that $|S| = 2$, $\sum_{i \in S} c'_i = \sum_{i \in S} c_i$, and $c'_{N \setminus S} = c_{N \setminus S}$. Profile c' is an *iterative pairwise variant of* c if there exist a finite number of profiles $c^1, \cdots, c^k \in \mathbb{R}_+^{N \times M}$ admissible in $\mathcal{D}^N$ such that $c^1 = c$, $c^k = c'$, and for each $l = 1, \cdots, k$, c^l is a pairwise variant of c^{l-1}.

[5] This was pointed out by Hervé Moulin.

Step 1. We show that any two admissible profiles of characteristic vectors in $\mathbb{R}_+^{N\times K}$ with identical sum of components are iterative pairwise variant of each other.

Case 1. $K = 1$. Let $c, c' \in \mathbb{R}_+^{N\times K}$ be admissible in $\mathcal{D}^N$ and satisfy $c \neq c'$, $\bar{c} = \bar{c}'$, and $c_{N\setminus S} = c'_{N\setminus S}$ for some $S \subseteq N$. The proof is trivial when S has only two entities. Suppose by induction that the claim holds when S has at most k entities. Now consider the case when S has $k+1$ entities. We may assume $S \equiv \{1, \cdots, k+1\}$. Then since $K = 1$, there exists $i \in S$ such that $c'_i < c_i$. Therefore, since $c_{i+1}+c_i-c'_i \geq 0$, then by richness, $\hat{c} \equiv (c_1, \cdots, c_{i-1}, c'_i, c_{i+1} + c_i - c'_i, c_{i+1}, \cdots, c_N)$ is also admissible.[6] Clearly, c is a pairwise variant of $\hat{c}$. On the other hand, note that $\hat{c}$ and c' differ only on $S\setminus i$. Thus, applying the induction hypothesis, $\hat{c}$ is an iterative pairwise variant of c'. Therefore, c is an iterative pairwise variant of c'.

Case 2. $K \geq 2$. In this case, we apply the result in Case 1 componentwise as follows. Let $c, c' \in \mathbb{R}_+^{N\times K}$ be admissible in $\mathcal{D}^N$, $c \neq c'$, $\bar{c} = \bar{c}'$, and $c_{N\setminus S} = c'_{N\setminus S}$, for some $S \subseteq N$. For each $i \in S$ and each $k \in K$, let $c_i^k \equiv (c'_{i1}, \cdots, c'_{ik}, c_{ik+1}, \cdots, c_{iK})$. Let $c^k \equiv \left(\left(c_i^k\right)_{i\in S}, c_{N\setminus S}\right)$. Let $c_i^0 \equiv c_i$ and $c^0 = c$. For each $k = 1, \cdots, K$, applying Case 1 for the k-th component, we show that c^k is an iterative pairwise variant of c^{k-1}.

Step 2. To complete the proof, let f be a rule satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness*. Let $(c, E) \in \mathcal{D}^N$. Consider $S \subseteq N$. Let $c' \in \mathbb{R}_+^{N\times K}$ be such that $\sum_{i\in S} c'_i = \sum_{i\in S} c_i$ and $c'_{N\setminus S} = c_{N\setminus S}$. Then by Step 1, there exist $k \geq 2$ and $c^1, \cdots, c^k \in \mathbb{R}_+^{N\times K}$ such that $c^1 = c$, $c^k = c'$, and for each $l = 2, \cdots, k$, c^l is a pairwise variant of c^{l-1}. Let $x^l \equiv f\left(c^l, E\right)$, for each $l = 1, \cdots, k$. By *pairwise reallocation-proofness* and *pairwise non-bossiness*, $\sum_{i\in S} x_i^l = \sum_{i\in S} x_i^{l-1}$ for each $l = 1, \cdots, k$. Through the successive application of the two axioms, we show $\sum_{i\in S} f_i(c, E) = \sum_{i\in S} f_i(c', E)$. ∎

We show, by examples, the independence of the two axioms in Theorem 2.

Example 8. For each $(c, E) \in \mathcal{D}^N$, if $c_1 = 0$, then for each $i \neq 1$, $f_i(c, E) = E/N$; if $c_1 \neq 0$, $f_1(c, E) \equiv 2E/N$ and for each $i \neq 1$, $f_i(c, E) = 0$. Then given $E \geq 0$, for each $i \in N\setminus 1$, the share for the pair $\{1, i\}$ is constantly $2E/N$. Hence such a pair cannot increase its share through a reallocation of characteristic vectors. Now consider a pair $\{i, j\}$, $i \neq j$, not containing agent 1. For each $(c, E) \in \mathcal{D}^N$, if $c_1 = 0$ (or $c_1 \neq 0$, respectively), then $f_i(c, E) + f_j(c, E) = 2E/N$ (or 0, respectively) and the sum does not changes by any reallocation of

[6] Note that $\hat{c} \geq 0$, since $c_i - c'_i > 0$.

characteristic vectors of i and j. Thus, f is *pairwise reallocation-proof*. However, f is not *reallocation-proof*. This is because, when $c_1 \neq 0$, any triple $S \equiv \{1, i, j\}$ containing entity 1 can increase its aggregate share from $2E/N$ to $3E/N$ by decreasing entity 1's characteristic vector to the zero vector. Note that f violates *pairwise non-bossiness* because any pair containing entity 1 can change others' shares without affecting its share. Also note that f violates both *dummy* and *efficiency*.

Example 9. Fix two agents i and j. Let f^{ij} be the rule defined as follows. For each $(c, E) \in \mathcal{D}^N$, if $\sum_{k \in K} c_{ik} \geq \sum_{k \in K} c_{jk}$, $f_i^{ij}(c, E) \equiv E$ and $f_h^{ij}(c, E) \equiv 0$ for each $h \neq i$; if $\sum_{k \in K} c_{ik} < \sum_{k \in K} c_{jk}$, $f_j^{ij}(c, E) \equiv E$ and $f_h^{ij}(c, E) \equiv 0$ for each $h \neq j$. Then it is easy to show that f^{ij} is *pairwise non-bossy* but is not *pairwise reallocation-proof*.

It follows from Theorem 2 and Corollary 1 that proportional rules are the only rules satisfying *pairwise reallocation-proofness*, *pairwise non-bossiness*, *efficiency*, *dummy*, and *non-negativity*. In fact, *pairwise non-bossiness* can be dropped, as stated in the next result.

Theorem 3. *Assume $N \geq 3$. Then, a rule over $\mathcal{D}^N$ satisfies* pairwise reallocation-proofness, efficiency*,* dummy*, and* non-negativity *(or* no transfer paradox*) if and only if it is a proportional rule.*

Proof. Suppose $N \geq 3$. Let f be a rule over $\mathcal{D}^N$, satisfying the four axioms. For each $S \subseteq N$, let $\mathcal{D}_S^N \equiv \{(c, E) \in \mathcal{D}^N : c_{N \setminus S} = 0\}$. Then by *dummy*, we can treat problems in $\mathcal{D}_S^N$ as problems with only agents in S.

Note that when $|S| = 3$, *pairwise reallocation-proofness* and *efficiency* imply *reallocation-proofness* over $\mathcal{D}_S^N$. Thus, by Corollary 1, f coincides with a proportional rule over $\mathcal{D}_S^N$. Denote the weight function by W^S. Note that for each $S, T \subseteq N$ with $|S| = |T| = 3$, if $|S \cap T| \geq 2$, $\mathcal{D}_S^N \cap \mathcal{D}_S^N \neq \varnothing$. Then we can show that $W^S = W^T$. Hence weight functions for all triples are identical and we may write them simply by W. And we conclude that f coincides with the proportional rule (associated with W) over $\cup_{|S| \leq 3} \mathcal{D}_S^N$.

Let $k \geq 3$. Suppose by induction that f coincides with the proportional rule (associated with W) over $\cup_{|S| \leq k} \mathcal{D}_S^N$. Let $S \subseteq N$ contain $k+1$ agents. Let $(c, E) \in \mathcal{D}_S^N$ and $x \equiv f(c, E)$. Consider a pair $\{i, j\} \subseteq S$. Let $c' \in \mathbb{R}^{S \times M}$ be such that $c'_i \equiv c_i + c_j$, $c'_j \equiv 0$, and for each $h \neq i, j$, $c'_h \equiv c_h$. Then by *pairwise reallocation-proofness*, $x_i + x_j = f_i(c', E) + f_j(c', E)$. Since $(c', E) \in \mathcal{D}_{S \setminus j}^N$, then by the induction hypothesis, $x_i + x_j$ is the sum of allocations for i and j made by the proportional rule associated with W. Since this holds for each $i, j \in S$, f

coincides with the proportional rule also over $\mathcal{D}^N_S$, for each S with $k+1$ agents. ∎

As mentioned in Remark 1, when $K \geq 2$, the axioms in this characterization are independent. Since each of *non-negativity* and *no transfer paradox* implies the bound condition in the definition of rules, this condition can be dropped. As mentioned in Remark 2, when $K = 1$ as in the contexts of bankruptcy and surplus sharing, *non-negativity* (or *no transfer paradox*) can be dropped. However in this case, the bound condition is crucial for the result.

5 Variable Population

This section discusses an axiom, called *merging-splitting-proofness*, which is closely related to *reallocation-proofness*. In the context of claims problems, *merging-splitting-proofness* says that no group $S \subsetneq N$ can increase its share by merging its members' claims, and no agent $i \in N$ can increase his share by splitting his claim among dummy agents and himself.

To state the axiom formally, we have to consider a model where the set of entities N is variable. Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathcal{N}$ be the set of all non-empty finite subsets of $\mathbb{N}$.

For each $N \in \mathcal{N}$, let $\mathcal{A}^N$ denote the class of all allocation problems associated with N. For each $N \in \mathcal{N}$, let $\mathcal{D}^N \subseteq \mathcal{A}^N$ and $\mathcal{D} = \cup_{N \in \mathcal{N}} \mathcal{D}^N$. In what follows, we assume that $\mathcal{D}$ satisfies the following condition of *richness**: for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, each non-empty $N' \subseteq N$, and each $c' \in \mathbb{R}^{N'}_+$, if $\sum_{i \in N'} c'_i = \sum_{i \in N} c_i$, then $(c', E) \in \mathcal{D}^{N'}$. Clearly, if $\mathcal{D}$ satisfies richness*, then for each $N \in \mathcal{N}$, the fixed-population domain $\mathcal{D}^N$ satisfies richness. A *rule* is now a function that associates with each $N \in \mathcal{N}$ and each allocation problem $(c, E) \in \mathcal{D}^N$ a vector $f(c, E) \in \mathbb{R}^N$ and that satisfies the bound condition, that is, for each $N \in \mathcal{N}$ and each $(\bar{c}, E) \in \mathbb{R}^K_+ \times \mathbb{R}_+$, there exists $i \in N$ such that f_i is bounded above or below over $\mathcal{D}^N(\bar{c}, E)$.

Definition. A rule f is *merging-splitting-proof* (O'Neill 1982) if for each $N \in \mathcal{N}$, each non-empty $S \subseteq N$, each $i \in S$, each $(c, E) \in \mathcal{D}^N$, and each $c'_i \in \mathbb{R}^K_+$, if $c'_i = \sum_{j \in S} c_j$, then

$$f_i(c'_i, c_{N \setminus S}, E) = \sum_{j \in S} f_j(c, E). \tag{18}$$

This axiom is introduced by O'Neill (1982) in the context of bankruptcy problems.[7] In this context, if the left-hand side of (18) is larger than the right-hand

[7] O'Neill calls this axiom "strategy-proofness".

side, then in problem (c, E), group S gains by merging the members' claims and having member i represent the total claim of the group. On the other hand, if the right-hand side is larger, then in problem $(c'_i, c_{N\setminus S}, E)$, agent i gains by creating dummy agents $S \setminus k$ and splitting his claim among S.

Note that the left-hand side of (18) is well-defined since $\mathcal{D}$ satisfies richness*.

The following pairwise version of *merging-splitting-proofness* deals only with mergers of two entities' characteristic vectors and splits of a single entity's vector into two.

Definition. A rule f is *pairwise merging-splitting-proof* if for each $N \in \mathcal{N}$, each $i, j \in N$, each $(c, E) \in \mathcal{D}^N$, and each $c'_j \in \mathbb{R}_+$, if $c'_j = c_i + c_j$, then

$$f_j(c'_j, c_{N\setminus\{i,j\}}, E) = f_i(c, E) + f_j(c, E). \tag{19}$$

The axioms in the previous sections can be extended to the variable-population domain in an obvious way. We begin our analysis by proving some logical relations among the axioms.

Lemma 2. Merging-splitting-proofness *implies* reallocation-proofness. *Also* pairwise merging-splitting-proofness *implies* pairwise reallocation-proofness.

Proof. Let f be a rule that is *merging-splitting-proof*. Let $N \in \mathcal{N}$, $S \subsetneq N$, $i \in S$, $(c, E) \in \mathcal{D}^N$, and $c'_i \in \mathbb{R}_+$ be such that $c'_i = \sum_{j\in S} c_j$. Then by *merging-splitting-proofness*,

$$f_i(c'_i, c_{N\setminus S}, E) = \sum_{j\in S} f_j(c, E).$$

This obviously implies that $\sum_{j\in S} f_j(c, E)$ is not affected by any reallocation of claims among agents in S. Thus, f is *reallocation-proof*. The proof for the pairwise part is obtained when S is a pair. ∎

Note that the converse of this lemma does not hold because *reallocation-proofness* does not relate problems with different populations, while *merging-splitting-proofness* does. To explain this by example, let $N \in \mathcal{N}$ and $N' \subsetneq N$. Let W and W' be two distinct weight functions. Let f coincide with the proportional rule associated with W over $\mathcal{D}^N$ and with the proportional rule associated with W' over $\mathcal{D}^{N'}$. Then f is not *merging-splitting-proof* but is *reallocation-proof*.

We now introduce an axiom that is closely related to *dummy* but meaningful only in the variable population framework.

Definition. A rule f is *null-consistent* if for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, and each $i \in N$, if $c_i = 0$, then $f_j(c_{N\setminus i}, E) = f_j(c, E)$ for each $j \in N \setminus i$.

When $c_i = 0$, it is reasonable to view (c, E) and $(c_{N\backslash i}, E)$ as the same allocation problem. *Null-consistency* simply says that each $j \neq i$ receives the same amount at these two problems. *Dummy* implies only that $f_i(c, E) = 0$, and allows that $j \neq i$ receives different amounts at (c, E) and $(c_{N\backslash i}, E)$. *Null-consistency* is introduced by Chun (1988) in the context of claims problem.

The first result on the variable population domain $\mathcal{D}$ is an extension of Theorem 1.

Theorem 4. *A rule over $\mathcal{D}$ is* reallocation-proof *and* null-consistent *if and only if it is a generalized proportional rule f, that is, there exist two functions $W\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^K$ and $a\colon \mathbb{R}_+^K \times \mathbb{R}_+ \to \mathbb{R}^{\mathbb{N}}$ such that for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, and each $i \in N$,*

$$f_i(c, E) = a_i(\bar{c}, E) + \sum\nolimits_{k \in K\colon \bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) e\left(\bar{c}, E\right).$$

Proof. Let f be a rule satisfying *reallocation-proofness* and *null-consistency*. Let $N \in \mathcal{N}$ contain at least three agents. Let $N' \subseteq N$. By Theorem 1, f coincides with a generalized proportional rule over $\mathcal{D}^N$. By *null-consistency*, any problem where agents in $N \backslash N'$ are nulls should be solved as problems in $\mathcal{D}^{N'}$. Thus, f coincides with this generalized proportional rule also over $\mathcal{D}^{N'}$. ∎

Remark 4. Note that the range of function a is now $\mathbb{R}^{\mathbb{N}}$, an infinite dimensional space. It is easy to show that the two axioms are independent. In particular, without *null-consistency*, any rule that coincides with a generalized proportional rule on each fixed population domain $\mathcal{D}^N$ is *reallocation-proof.* Note that such a rule does not necessarily have "consistency" across different populations and so is not necessarily a generalized proportional rule over $\mathcal{D}$. Moreover, *reallocation-proofness* does not impose any restriction on the fixed population domains $\mathcal{D}^N$ with $|N| \leq 2$.

We now study rules satisfying *merging-splitting-proofness.* Although *merging-splitting-proofness* is, in spirit, similar to *reallocation-proofness*, our results to be explained below are somewhat different from the previous results. This is mainly because *merging-splitting-proofness* implies both *dummy* and *null-consistency*, as shown in the following lemma.

Lemma 3. Merging-splitting-proofness *implies* dummy *and* null-consistency*.*

Proof. Let f be a rule over $\mathcal{D}$ satisfying *merging-splitting-proofness.* Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{D}^N$ be such that $|N| \geq 3$ and $c_h = 0$ for some $h \in N$. Let $x \equiv f\left(c, E\right)$ and $y \equiv f\left(c_{N\backslash h}, E\right)$. Let $j \in N \backslash h$ and α be the amount j receives in the single entity problem in which j has the characteristic vector $\bar{c}$ and the resource amount is E. Then applying *merging-splitting-proofness* to both (c, E) and $\left(c_{N\backslash h}, E\right)$, we have $\sum_{i \in N} x_i = \alpha = \sum_{i \in N\backslash h} y_i$. On the other hand, for each $i \in N\backslash h$, by *merging-splitting-proofness* for pair $\{i, h\}$, $x_i + x_h = y_i$. Hence $\sum_{i \in N} x_i + (|N| - 2)\, x_h = \sum_{i \in N\backslash h} y_i$. Since $\sum_{i \in N} x_i = \sum_{i \in N\backslash h} y_i$ and $|N| \geq 3$, $x_h = 0$ and for each $i \in N\backslash h$, $x_i = y_i$.

Now consider N containing only two entities, say, 1 and 2. Let $(c, E) \in \mathcal{D}^N$ be such that $c_2 = 0$. Let $x \equiv f(c, E)$. If $x_2 > 0$, any pair of null entities including entity 2 in a three agents problem $((c_1, c_2, 0), E)$ will be able to increase its share by merging the two zero vectors into the zero vector of entity 2 in (c, E), violating *merging-splitting-proofness*. If $x_2 < 0$, then entity 2 can increase the share by splitting $c_2 = 0$ into two zero vectors of himself and any other entity except 1 and 2. Thus, $x_2 = 0$. By *merging-splitting-proofness*, $x_1 + x_2 = x_1 = f_1(c_1, E)$. ∎

It follows directly from Theorem 4 and Lemmas 2 and 3 that only the generalized proportional rules with $a_i(\cdot) \equiv 0$ for each $i \in \mathbb{N}$ are *merging-splitting-proof.*

Theorem 5. *A rule over $\mathcal{D}$ is* merging-splitting-proof *if and only if it is a generalized proportional rule f associated with a function $W\colon \mathbb{R}^K_+ \times \mathbb{R}_+ \to \mathbb{R}^K$ such that for each $N \in \mathcal{N}$ and each $i \in N$,*

$$f_i(c, E) = \sum_{k \in K:\, \bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) e(c, E)\,.$$

Thus *merging-splitting-proofness* also implies *anonymity.* Adding *efficiency* and *non-negativity* (or *no transfer paradox*), we axiomatize proportional rules. Formally:

Corollary 3. *A rule over $\mathcal{D}$ satisfies* merging-splitting-proofness, efficiency*, and* non-negativity (*or* no transfer paradox) *if and only if it is a proportional rule.*

When $K \geq 2$, the independence of the three axioms can be established easily. Moreover, in this case, by *non-negativity* (or *no transfer paradox*), the bound condition in the definition of rules can be dropped. When $K = 1$, *non-negativity* (or *no transfer paradox*) can be dropped. However, in this case, the bound condition is crucial.

We will show next that *merging-splitting-proofness* can be weakened to *pairwise merging-splitting-proofness.* We first establish two useful lemmas.

Lemma 4. *The combination of* pairwise merging-splitting-proofness *and* efficiency *imply* dummy *and* null-consistency*.*

Proof. Let f be an allocation rule that is defined over a rich* domain and that is *pairwise merging-splitting-proof.* Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{D}^N$ be such that $c_k = 0$ for some $k \in N$.

First assume $|N| \geq 3$. By *pairwise merging-splitting-proofness*,

$$f_i(c_{N\setminus\{k\}}, E) = f_i(c, E) + f_k(c, E). \tag{$*$}$$

for each $i \in N \setminus \{k\}$. Adding up the equation for each $i \in N \setminus \{k\}$ yields

$$\begin{aligned} E = \sum_{i \in N\setminus\{k\}} f_i(c_{N\setminus\{k\}}, E) &= \sum_{i \in N} f_i(c, E) + (|N| - 2) f_k(c, E) \\ &= E + (|N| - 2) f_k(c, E), \end{aligned}$$

where the first and the last equalities hold by *efficiency.* Then, since $|N| \geq 3$, $f_k(c, E) = 0$. By $(*)$, $f_i(c_{N\setminus\{k\}}, E) = f_i(c, E)$.

To deal with the case $|N| = 2$, we can argue as in Proof of Lemma 3. ∎

Lemma 5. *The combination of* pairwise reallocation-proofness*,* efficiency*, and* null-consistency *implies* dummy *and* pairwise merging-splitting-proofness*.*

Proof. Let f be an allocation rule that is *efficient*, *null-consistent*, and *pairwise reallocation-proof.* Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{D}^N$, and $\{i, j\} \subseteq N$. Let $c' \in \mathbb{R}^N_+$ be defined by $c'_i \equiv 0$, $c'_j \equiv c_i + c_j$, and for each $h \in N \setminus \{i, j\}$, $c'_h \equiv c_h$. By *pairwise reallocation-proofness* and *null-consistency*,

$$\begin{aligned} f_i(c, E) + f_j(c, E) &= f_i(c', E) + f_j(c', E) \\ &= f_i(c', E) + f_j(c'_{N\setminus i}, E). \end{aligned} \tag{$\star$}$$

Null-consistency also implies that for each $h \in N \setminus i$, $f_h(c', E) = f_h(c'_{N\setminus i}, E)$. Then by *efficiency*,

$$\begin{aligned} E = \sum_{h \in N} f_h(c, E) = \sum_{h \in N} f_h\,(c', E) &= f_i(c', E) + \sum_{h \in N\setminus i} f_h(c'_{N\setminus i}, E) \\ &= f_i(c', E) + E. \end{aligned}$$

Thus, $f_i(c', E) = 0$, which shows that f satisfies *dummy.* This and $(\star)$ show that f is also *pairwise merging-splitting-proof.* ∎

Now we are ready to show that proportional rules are the only rules satisfying *pairwise merging-splitting-proofness*, *efficiency*, and *non-negativity* (or *no transfer paradox*). Formally:

Theorem 6. *A rule over $\mathcal{D}$ satisfies* pairwise merging-splitting-proofness, efficiency, *and* non-negativity (*or* no transfer paradox) *if and only if it is a proportional rule.*

Proof. Let f be a rule that satisfies the three axioms. By Lemmas 2 and 4, f satisfies *dummy*, *null-consistency*, and *pairwise reallocation-proofness*. By Theorem 3, f is a proportional rule over each fixed-population domain $\mathcal{D}^N$, where $N \in \mathcal{N}$ has three or more entities. Then there is a weight function W such that f is the proportional rule associated with W over $\mathcal{D}^N$. For each $N' \in \mathcal{N}$, since f is also a proportional rule over $\mathcal{D}^{N \cup N'}$. By *null-consistency*, problems in $\mathcal{D}^{N \cup N'}$ with the zero characteristic vectors for entities in N' can be considered as problems in $\mathcal{D}^N$. Therefore, the associated weight function of f over $\mathcal{D}^{N \cup N'}$ coincides with W. Again by *null-consistency*, all problems in $\mathcal{D}^{N \cup N'}$ with zero characteristic vectors for entities in N can be considered as problems in $\mathcal{D}^{N'}$. Therefore, f is also the proportional rule associated with W over $\mathcal{D}^{N'}$. ∎

Replacing *pairwise merging-splitting-proofness* with *pairwise reallocation-proofness* and *null-consistency*, we obtain:

Corollary 4. *A rule over $\mathcal{D}$ satisfies* pairwise reallocation-proofness, efficiency, null-consistency, *and* non-negativity (*or* no transfer paradox) *if and only if it is a proportional rule.*

Proof. By Lemma 5, if a rule satisfies the four axioms, it satisfies *pairwise merging-splitting-proofness*. Therefore, the result follows from Theorem 6. ∎

When $K \geq 2$, axioms considered in Theorem 6 and Corollary 4 are independent. When $K = 1$, *non-negativity* (or *no transfer paradox*) can be dropped.

6 Applications

The special models in Examples 1–7 satisfy the richness condition. Also, these examples can be extended to variable population models in a straightforward manner so that the extended models satisfy the richness* condition. Therefore our results can be applied to these examples.

6.1 Fixed Population Models

We discuss implications of our results in various fixed population models. We obtain several earlier results in the literature as corollaries.

Social Choice with Transferable Utilities

We first consider social choice problems with transferable utilities (Example 5) studied by Moulin (1985a). In this context, there are K alternatives and c_i is a vector of i's valuations for these alternatives. Therefore, it is reasonable to require that how these valuations are normalized should not matter. Indeed, the following invariance axiom is considered in Moulin (1985a). Let $\mathbf{1} \in \mathbb{R}^K$ denote the vector consisting of 1 only. An allocation rule f is *translation invariant* if for each $(c, E) \in \mathcal{D}$, each $i \in N$, and each $\lambda \in \mathbb{R}_+$, $f_i((c_i + \lambda\mathbf{1}, c_{-i}), E + \lambda) = f_i(c, E) + \lambda$ and $f_j((c_i + \lambda\mathbf{1}, c_{-i}), E + \lambda) = f_j(c, E)$ for each $j \neq i$. For each $c \in \mathbb{R}_+^{N \times K}$, let $\bar{c}_{\max} \equiv \max_{k \in K} \bar{c}_k$.

Definition. A rule is an "equal sharing above a convex decision", briefly, an *ESCD rule* if there exists a weight function $\rho \colon \mathbb{R}_+^K \to \Delta^{K-1}$ such that

$$\rho(\bar{c} + \lambda\mathbf{1}) = \rho(\bar{c}) \text{ for each } \bar{c} \in \mathbb{R}_+^K \text{ and each } \lambda \geq 0; \tag{20}$$

$$f_i(c) \equiv c_i \cdot \rho(\bar{c}) + \frac{1}{n}(\bar{c}_{\max} - \bar{c} \cdot \rho(\bar{c})), \text{ for each } c \in \mathbb{R}_+^{N \times K} \text{ and each } i \in N. \tag{21}$$

We denote the ESCD rule associated with ρ by ES^ρ.

Clearly, ES^ρ is *efficient*. Note that (20) is needed for *translation invariance*. If for all $\bar{c} \in \mathbb{R}_+^K$ and all $k \in K$ with $\bar{c}_k = 0$, $\rho_k(\bar{c}) = \rho'_k(\bar{c})$, then $ES^\rho = ES^{\rho'}$; that is, when $\bar{c}_k = 0$, the value of $\rho_k(\bar{c})$ is irrelevant in the definition of $ES^\rho(c)$. Note that

$$\begin{aligned} ES_i^\rho(c) &= \sum_{k \in K} c_{ik}\rho_k(\bar{c}) + \frac{1}{n}\left(\bar{c}_{\max} - \sum_{k \in K} \bar{c}_k \rho_k(\bar{c})\right) \\ &= \frac{1}{n}\bar{c}_{\max} + \sum_{k \in K}\left(c_{ik} - \frac{1}{n}\bar{c}_k\right)\rho_k(\bar{c}). \end{aligned}$$

For each $k \in K$, if $\bar{c}_k \neq 0$, let $W_k^\rho(\bar{c}) \equiv \frac{\bar{c}_k \rho_k(\bar{c})}{\bar{c}_{\max}}$; if $\bar{c}_k = 0$, let $W_k^\rho(\bar{c}) \equiv 0$. Then $\sum_{k \in K} c_{ik}\rho_k(\bar{c}) = \sum_{k \in K : \bar{c}_k > 0} c_{ik}\rho_k(\bar{c}) = \sum_{k \in K : \bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k^\rho(\bar{c})\bar{c}_{\max}$ (when $\bar{c}_k = 0$, $0 \leq c_{ik} \leq \bar{c}_k$ and so $c_{ik} = 0$). Thus, for each $c \in \mathbb{R}_+^{N \times K}$ and each $i \in N$,

$$ES_i^\rho(c) = \frac{1}{n}\left(\bar{c}_{\max} - \sum_{k \in K : \bar{c}_k > 0} W_k^\rho(\bar{c})\bar{c}_{\max}\right) + \sum_{k \in K : \bar{c}_k > 0} \frac{c_{ik}}{\bar{c}_k} W_k^\rho(\bar{c})\bar{c}_{\max}.$$

Therefore, if we let

$$a_i^\rho(\bar{c}) \equiv \frac{1}{n}\left(\bar{c}_{\max} - \sum_{k \in K} \bar{c}_k \rho_k(\bar{c})\right) = \frac{1}{n}\left(\bar{c}_{\max} - \sum_{k \in K : \bar{c}_k > 0} W_k^\rho(\bar{c})\bar{c}_{\max}\right),$$

for each $i \in N$ and each $c \in \mathbb{R}_+^{N \times K}$, then ES^ρ is a generalized proportional rule associated with (W^ρ, a^ρ). Note that for each $k \in K$ with $\bar{c}_k > 0$, $W_k^\rho(\bar{c}) \frac{\bar{c}_{\max}}{\bar{c}_k} \in [0,1]$ and $\sum_{k:\bar{c}_k>0} W_k^\rho(\bar{c}) \frac{\bar{c}_{\max}}{\bar{c}_k} \leq 1$.

Applying Corollary 2, we obtain the following axiomatization of ESCD rules, established by Moulin (1985a).

Corollary 5 (Moulin 1985a, Theorem 1). *Assume $N \geq 3$. In the context of social choice with transferable utilities, $\mathcal{D}^N \equiv \mathbb{R}_+^{N \times K}$, a rule satisfies* reallocation-proofness*,* efficiency*,* no transfer paradox*,* translation invariance*, and* anonymity *if and only if it is an ESCD rule.*

Proof. Let f be a rule satisfying the five axioms. Then by Corollary 2, f is a generalized proportional rule represented by a non-negative valued function $W\colon \mathbb{R}_+^K \to \mathbb{R}_+^K$. Without loss of generality, we assume that for each $\bar{c} \in \mathbb{R}_+^K$ and each k, if $\bar{c}_k = 0$, $W_k(\bar{c}) = 0$.

Let $c \in \mathbb{R}_+^{N \times K}$ be such that $\bar{c} >> 0$. For each $k \in K$, let $\rho_k(\bar{c}) \equiv W_k(\bar{c}) \frac{\bar{c}_{\max}}{\bar{c}_k}$. Let $\xi(c_i, \bar{c}) \equiv \sum_{k \in K:\bar{c}_k>0} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c})\, \bar{c}_{\max} = \sum_{k \in K} c_{ik}\rho_k(\bar{c})$.

Note that $\frac{1}{n}\left(1 - \sum_{k \in K} W_k(\bar{c})\right)\bar{c}_{\max} = \frac{1}{n}\bar{c}_{\max} - \frac{1}{n}\xi(\bar{c}, \bar{c})$ and so for each $i \in N$, $f_i(c) = \frac{1}{n}\bar{c}_{\max} - \frac{1}{n}\xi(\bar{c},\bar{c}) + \xi(c_i, \bar{c})$. Hence for each $c \in \mathbb{R}_+^K$ and each i with $c_i - \frac{1}{n}\bar{c} \in \mathbb{R}_+^K$ and $\bar{c} >> 0$,

$$f_i(c) = \xi\left(c_i - \frac{1}{n}\bar{c}, \bar{c}\right) + \frac{1}{n}\bar{c}_{\max}. \tag{$*$}$$

Let $\lambda \geq 0$. Let $c' \equiv c + \lambda(\mathbf{1}, \cdots, \mathbf{1})$. Then by $(*)$ and *translation invariance*, for each $i \in N$, $\xi\left(c'_i - \frac{1}{n}\bar{c}', \bar{c} + n\lambda\mathbf{1}\right) + \frac{1}{n}\bar{c}'_{\max} = \xi\left(c_i - \frac{1}{n}\bar{c}, \bar{c}\right) + \frac{1}{n}\bar{c}_{\max} + \lambda$. Since $c'_i - \frac{1}{n}\bar{c}' = c_i - \frac{1}{n}\bar{c}$ and $\bar{c}'_{\max} = \bar{c}_{\max} + n\lambda$, $\xi\left(c_i - \frac{1}{n}\bar{c}, \bar{c} + n\lambda\mathbf{1}\right) = \xi\left(c_i - \frac{1}{n}\bar{c}, \bar{c}\right)$. Hence for each $i \in N$, each $c \in \mathbb{R}_+^K$ with $\bar{c} >> 0$ and $c_i \geq \frac{1}{n}\bar{c}$, and each $\lambda \geq 0$, $\sum_{k \in K}\left(c_{ik} - \frac{1}{n}\bar{c}_k\right)\rho_k(\bar{c} + n\lambda\mathbf{1}) = \sum_{k \in K}\left(c_{ik} - \frac{1}{n}\bar{c}_k\right)\rho_k(\bar{c})$. Therefore, $\rho(\bar{c} + n\lambda\mathbf{1}) = \rho(\bar{c})$, for each $\bar{c} >> 0$ and each $\lambda \geq 0$ (for example, let $k \in K$, $c_{ik} \equiv \bar{c}_k$, and $c_{il} \equiv \frac{1}{n}\bar{c}_l$ for each $l \neq k$; then the equality is reduced to $\frac{n-1}{n}\bar{c}_k\rho_k(\bar{c} + n\lambda\mathbf{1}) = \frac{n-1}{n}\bar{c}_k\rho_k(\bar{c})$; so we obtain $\rho_k(\bar{c} + n\lambda\mathbf{1}) = \rho_k(\bar{c})$). Thus for each $\bar{c} \in \mathbb{R}_{++}^K$ and each $\alpha \geq 0$, $\rho(\bar{c} + \alpha\mathbf{1}) = \rho(\bar{c})$. Hence for each $y >> 0$, each $0 \leq x \leq y$, and each $\alpha \geq 0$,

$$\xi(x, y + \alpha\mathbf{1}) = \xi(x, y). \tag{$**$}$$

Let $c \in \mathcal{D}^N$, $\lambda > 0$, and $i \in N$ be such that $\bar{c} >> 0$, $c_i - \frac{1}{n}\bar{c} \geq 0$, and $c_i - \frac{1}{n}\bar{c} + \frac{n-1}{n}\lambda\mathbf{1} \leq \bar{c}$. Let $c' \in \mathbb{R}_+^{N \times K}$ be such that $c'_i = c_i + \lambda\mathbf{1}$ and for each $j \neq i$, $c'_j = c_j$. Then by *translation invariance*, $f_i(c') = f_i(c) + \lambda$.

Hence by $(*)$, $\xi\left(c_i+\lambda\mathbf{1}-\frac{1}{n}\left(\bar{c}+\lambda\mathbf{1}\right),\bar{c}+\lambda\mathbf{1}\right)+\frac{1}{n}\left(\bar{c}_{\max}+\lambda\right)=\xi\left(c_i-\frac{1}{n}\bar{c},\bar{c}\right)+\frac{1}{n}\bar{c}_{\max}+\lambda$. Since $c_i-\frac{1}{n}\bar{c}+\frac{n-1}{n}\lambda\mathbf{1}\leq\bar{c}$, then by $(**)$, $\xi\left(c_i-\frac{1}{n}\bar{c}+\frac{n-1}{n}\lambda\mathbf{1},\bar{c}\right)=\xi\left(c_i-\frac{1}{n}\bar{c},\bar{c}\right)+\frac{n-1}{n}\lambda$, that is, $\sum_{k\in K}\left(c_{ik}-\frac{1}{n}\bar{c}_k+\frac{n-1}{n}\lambda\right)\rho_k\left(\bar{c}\right)=\sum_{k\in K}\left(c_{ik}-\frac{1}{n}\bar{c}_k\right)\rho_k\left(\bar{c}\right)+\frac{n-1}{n}\lambda$. Then $\sum_{k\in K}\rho_k\left(\bar{c}\right)=1$. Since this holds for each $\bar{c}>>0$ and W is non-negative valued, ρ is a weight function over $\mathbb{R}^K_{++}$. Relying on *translation invariance*, we can extend the domain of this weight function ρ to $\mathbb{R}^K_+$ and show that for each $c\in\mathcal{D}^N$ and each $i\in N$, $\xi\left(c_i,\bar{c}\right)=\sum_{k\in K}c_{ik}\rho_k\left(\bar{c}\right)$. ∎

Moulin (1985a) also establishes an axiomatization of the following interesting subfamily of ESCD rules.

Definition. An ESCD rule is *utilitarian* if it is represented by a weight function $W\colon\mathbb{R}^K_+\to\Delta^{K-1}$ such that for each $c\in\mathbb{R}^{N\times K}_+$,

$$W_k\left(\bar{c}\right)=0,\text{ for each }k\in K\text{ with }\bar{c}_k<\bar{c}_{\max},\tag{22}$$

$$W\left(\bar{c}+\lambda\mathbf{1}\right)=W\left(\bar{c}\right)\text{ for each }\lambda\in\mathbb{R}_+.\tag{23}$$

We denote the utilitarian rule associated with weight function W by U^W.

Note that by (22), for each $c\in\mathbb{R}^{N\times K}$,

$$\begin{aligned}U^W\left(c\right)&\equiv\left(c_i\cdot W\left(\bar{c}\right)\right)_{i\in N}\\&=\left(\sum_{k\colon\,\bar{c}_k=\bar{c}_{\max}}\frac{c_{ik}}{\bar{c}_k}W_k\left(\bar{c}\right)\bar{c}_{\max}\right).\end{aligned}$$

Clearly, utilitarian rules are proportional. They assign the zero weight on each inefficient alternative. Thus, the share of each agent is a weighted average of his valuations of efficient alternatives. When there is unique efficient alternative, each agent receives his valuation of this alternative. When agents have expected utility preferences, utilitarian rules can be considered as probabilistic selections from efficient alternatives without side payment.

Note that among ESCD rules, only utilitarian rules satisfy *dummy*. Thus, adding *dummy*, we obtain an axiomatization of utilitarian rules, established originally by Moulin (1985a, Theorem 3). Moreover, since *reallocation-proofness* and *dummy* imply *anonymity* (see Facts 3-4), *anonymity* can be dropped. Also, we can weaken *reallocation-proofness* to *pairwise reallocation-proofness*. And *non-negativity* can be replaced with *no transfer paradox*.

Corollary 6 (Moulin 1985a). *Assume $N\geq 3$. In the context of social choice with transferable utilities, $\mathcal{D}^N\equiv\mathbb{R}^{N\times K}_+$, an allocation rule satisfies* pairwise reallocation-proofness*,* efficiency*,* dummy*,* non-negativity *(or* no transfer paradox*), and* translation invariance *if and only if it is utilitarian.*

Proof. Let f be an allocation rule satisfying the five axioms. By Theorem 3, f is a proportional rule associated with a weight function W. Note that (23) holds because of *translation invariance*. We would like to show that W satisfies (22). Let $d \in \mathbb{R}^K_+$. We distinguish two cases.

Case 1: $d \gg 0$. Let $c \in \mathbb{R}^{N\times K}_+$ be such that $\bar{c} = d$ and $c_1 = (\lambda, \ldots, \lambda)$ for some $\lambda \in \mathbb{R}$ such that $0 < \lambda < \bar{c}_{\max}$. By *dummy*, $f_1(0, c_{-1}) = 0$. By *translation invariance*, $f_1(c) = \lambda$. On the other hand, since f is the proportional rule associated with W, we have

$$\begin{aligned}\lambda = f_1(c) &= \sum_{k\in K} W_k(\bar{c})\frac{\lambda}{\bar{c}_k}\bar{c}_{\max}\\ &= \lambda \sum_{k\in K} W_k(\bar{c})\frac{\bar{c}_{\max}}{\bar{c}_k}.\end{aligned}$$

This implies that if $\bar{c}_{\max} > \bar{c}_k$, then $W_k(\bar{c}) = 0$.

Case 2: $d \not\gg 0$. Let $K^* = \{k \in K : d_k > 0\}$. Note that K^* is non-empty by assumption. Let $c \in \mathbb{R}^{N\times K}_+$ be such that $\bar{c} = d$ and

$$c_{1k} = \begin{cases} \lambda & \text{if } k \in K^*,\\ 0 & \text{otherwise}\end{cases}$$

for some $\lambda \in \mathbb{R}_{++}$. To use the result of Case 1, let $\epsilon \in \mathbb{R}_{++}$ and $c_1' = c_1 + (\epsilon, \ldots, \epsilon)$. Then by Case 1, $f_1(c_1', c_{-1}) = \lambda + \epsilon$. By translation invariance, $f_1(c) = \lambda$. Since we know that f is a generalized proportional rule,

$$\begin{aligned}\lambda = f_1(c) &= \sum_{k\in K:\bar{c}_k>0} W_k(\bar{c})\frac{\lambda}{\bar{c}_k}\bar{c}_{\max}\\ &= \lambda \sum_{k\in K:\bar{c}_k>0} W_k(\bar{c})\frac{\bar{c}_{\max}}{\bar{c}_k}.\end{aligned}$$

This implies that if $\bar{c}_{\max} > \bar{c}_k$, then $W_k(\bar{c}) = 0$. ∎

Remark 5. By *translation invariance*, Corollaries 5-6 on $\mathbb{R}^{N\times K}_+$ can be easily extended to $\mathbb{R}^{N\times K}$, which is exactly the domain considered by Moulin (1985a).

Remark 6. In both Corollaries 5 and 6, since *non-negativity* (or *no transfer paradox*) implies the bound condition in the definition of rules, this condition can be dropped. In Corollary 6, when $K = 1$, *non-negativity* (or *no transfer paradox*) can be dropped.

Bankruptcy and Surplus Sharing

In the context of claims problems, Moulin (1987) studies rules satisfying *reallocation-proofness*, *efficiency*, *anonymity*, *non-negativity*, and the following two additional properties.

A rule f satisfies *claims monotonicity*, if f_i is non-decreasing in c_i for each $i \in N$. It satisfies *homogeneity* if for each $(c, E) \in \mathcal{D}^N$ and each $\lambda \geq 0$, $f(\lambda c, \lambda E) = \lambda f(c, E)$.

Moulin (1987) establishes several characterizations based on the following result, which is obtained as a corollary of our results.

Corollary 7 (Moulin 1987, Lemma 2). *Assume $N \geq 3$. In the context of claims problems, if a rule satisfies* reallocation-proofness, efficiency, anonymity, non-negativity, claims monotonicity, *and* homogeneity, *then there exists a real valued function h such that*

$$0 \leq h(x) \leq x \text{ for each } x \geq 0; \tag{24}$$

$$f_i(c, E) = \frac{1}{N}E + \left(c_i - \frac{1}{N}\bar{c}\right) h\left(\frac{E}{\bar{c}}\right) \text{ for each } c \in \mathbb{R}^N_{++} \text{ and each } E \in \mathbb{R}_+. \tag{25}$$

Proof. Let f be a rule satisfying the six axioms. By Theorem 1, f is a generalized proportional rule. By *efficiency*, (9) holds. Then by *anonymity*, there exists a function $a_0 \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that for each $(c, E) \in \mathbb{R}_+ \times \mathbb{R}_+$ and each $i \in N$,

$$f_i(c, E) = a_0(\bar{c}, E) + \frac{c_i}{\bar{c}}\left(E - Na_0(\bar{c}, E)\right).$$

By *homogeneity*, $a_0(\lambda\bar{c}, \lambda E) = \lambda a_0(\bar{c}, E)$ for each $(\bar{c}, E) \in \mathbb{R}_+ \times \mathbb{R}_+$ and each $\lambda \geq 0$. Then for each $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ and each $i \in N$,

$$f_i(c, E) = \frac{E}{N} + \left(c_i - \frac{\bar{c}}{N}\right)\left(\frac{E}{\bar{c}} - Na_0\left(1, \frac{E}{\bar{c}}\right)\right).$$

Therefore if we let $h\left(\frac{E}{\bar{c}}\right) \equiv \frac{E}{\bar{c}} - Na_0\left(1, \frac{E}{\bar{c}}\right)$, (25) holds. By *non-negativity*, a_0 is non-negative valued. Thus $h\left(\frac{E}{\bar{c}}\right) \leq \frac{E}{\bar{c}}$.

To show (24), we use the same argument that is in the last step of Moulin's (1987) proof. Fix $x > 0$ and consider two profiles (c, E) and (c', E) such that $E = \frac{2N}{2N-1}x$; $c_1 = \frac{1}{2N-1}c'_1 = \frac{1}{N-1}$; $c_i = c'_i = 1$ for each $i \neq 1$. Then by (25), $f_1(c, E) = \frac{1}{N}E - \frac{1}{2N-1}h(x)$ and $f_1(c', E) = \frac{1}{N}E$. Therefore by *claims monotonicity*, $h(x) \geq 0$. ∎

Moulin (1987) characterizes rules satisfying the above six axioms and "resource additivity" defined as follows. A rule f satisfies *resource additivity* if for each $c \in \mathbb{R}_+^N$ and each $E, E' \in \mathbb{R}_+$,

$$f(c, E + E') = f(c, E) + f(c, E'). \tag{26}$$

Corollary 8 (Moulin 1987, Theorem 3). *Assume $N \geq 3$. In the context of claims problems, a rule f satisfies* reallocation-proofness, efficiency*,* anonymity*,* non-negativity*,* claims monotonicity*,* homogeneity*, and* resource additivity *if and only if it is a convex combination of the proportional rule and the equal division rule, that is, there exists $\lambda \in [0,1]$ such that for each $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ and each $i \in N$,*

$$f_i(c, E) = \lambda \frac{c_i}{\bar{c}} E + (1 - \lambda) \frac{E}{n}.$$

Proof. By Corollary 7, there is a real valued function h such that $f_i(c, E) = \frac{1}{N} E + \left(c_i - \frac{1}{N}\bar{c}\right) h\left(\frac{E}{\bar{c}}\right)$ for each $c \in \mathbb{R}_+^N$ and each $E \in \mathbb{R}_+$. By *resource additivity*, h is *additive*. Since, by (24), h is also bounded, we can show that for some λ, $h(x) = \lambda x$ for each $x \in \mathbb{R}_+$ (See Theorem 2.1.8 in Aczél and Dhombres 1989). By (24), $\lambda \in [0,1]$. ∎

Note that by Corollary 1, when we add *dummy* and drop all of the axioms other than *reallocation-proofness*, *efficiency*, and *non-negativity*, then we are left with only the proportional rule. Moreover, by Theorem 3, we can weaken *reallocation-proofness* to *pairwise reallocation-proofness*. This characterization of the proportional rule in the fixed population model is new in the literature.

In the context of claims problems, Chun (1988, Theorem 1) characterizes rules satisfying *reallocation-proofness*, *efficiency*, *anonymity*, and *continuity*. This result is obtained as a corollary of Theorem 1. Since *continuity* implies the bound condition (note that *continuous* image of each compact set is compact and so bounded), this condition can be dropped for this result.

Here, since we impose the bound condition, we can drop *continuity* and obtain:

Corollary 9 (Chun 1988). *Assume $N \geq 3$. In the context of claims problems, a rule satisfies* reallocation-proofness, efficiency*, and* anonymity *if and only if it is a generalized proportional rule represented by a function $W\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as follows: for each $(c, E) \in \mathcal{D}^N$ and each $i \in N$,*

$$f_i(c, E) = \frac{1}{n}(1 - W(\bar{c}, E)) E + \frac{c_i}{\bar{c}} W(\bar{c}, E) E. \tag{27}$$

Proof. Let f satisfy the three axioms. By Theorem 1, f is a generalized proportional rule associated with (W, a). By *anonymity*, there is $a_0\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ such that for all $i \in N$, $a_i = a_0$. By *efficiency*, (8) holds and so $a_0(\bar{c}, E) = \frac{1}{n}(1 - W(\bar{c}, E)) E$ for all $(\bar{c}, E) \in \mathbb{R}_+ \times \mathbb{R}_+$. Therefore, (27) follows from (9). ∎

In the context of probability updating, *non-negativity* and *efficiency* are natural. Let us call rules satisfying the two axioms *updating rules.* The most commonly used updating rule is the Bayesian updating rule. Theorem 3 and Corollary 1 provide axiomatizations of this rule.

Corollary 10. *Assume that there are at least three states, that is, $N \geq 3$. In the context of probability updating, the Bayesian updating rule is the only updating rule over $\mathcal{D}^N$ that satisfies* (pairwise) reallocation-proofness *and* dummy*.*

In the context of probability aggregation, let us call rules satisfying *non-negativity* and *efficiency* (probability) *aggregation rules.* McConway (1981) considers the following axiom. An allocation rule f satisfies *strong setwise function property* if there is a function $h\colon [0,1]^K \to [0,1]$ such that for each $(c, E) \in \mathcal{D}$ and each $S \subseteq N$,

$$\sum_{i \in S} f_i(c, E) = h(\sum_{i \in S} c_i).$$

It is easily seen that this axiom implies *reallocation-proofness.* Note that h does not depend on $\sum_{i \in N} c_i$ nor E, since $\sum_{i \in N} c_i = (1, \ldots, 1)$ and $E = 1$ in any problem of probability aggregation. McConway's axiom is stronger than *reallocation-proofness* since he requires that h should be defined independently of S. Hence we obtain the following result of McConway as a corollary:

Corollary 11 (McConway 1981, Theorem 3.3). *Assume that there are at least three states, that is, $N \geq 3$. In the context of probability aggregation, the linear opinion pools are the only aggregation rules that satisfy* strong setwise function property *and* dummy*.*

6.2 Variable Population Models

We now discuss implications of our results with (*pairwise*) *merging-splitting-proofness* in variable population models.

Social Choice with Transferable Utilities

In the context of social choice with transferable utilities, we have shown that proportional rules are the only rules satisfying *pairwise merging-splitting-proofness*, *efficiency*, and *non-negativity* (or *no transfer paradox*). Adding *translation invariance* only, we characterize utilitarian rules.

Corollary 12. *In the context of social choice with transferable utilities, utilitarian rules are the only rules satisfying* pairwise merging-splitting-proofness*,* efficiency*,* non-negativity *(or* no transfer paradox*), and* translation invariance*.*

The proof of this corollary is similar to the proof of Corollary 6. Note that unlike Corollary 6, we do not need *anonymity*. This is because *anonymity* is implied by *pairwise merging-splitting-proofness* and *efficiency*.

Bankruptcy and Surplus Sharing

For the class of claims problems, Moulin (1985b, Theorem 5) and Chun (1988, Theorem 2) independently axiomatize the proportional rule based on *reallocation-proofness*, *anonymity*, and *continuity*. They do not impose the bound condition in the definition of rules. However, *continuity* implies the bound condition. Therefore, their results are obtained as corollaries of our results, in particular, Corollary 4 and Remark 2. Moreover, we show that *anonymity* in their results can be dropped[8].

Corollary 13 (Moulin 1985b; Chun 1988). *In the context of claims problems (or bankruptcy or surplus sharing), the proportional rule is the only rule satisfying* pairwise reallocation-proofness*,* efficiency*,* null-consistency*, and* continuity*.*

Chun (1988) also axiomatizes the proportion rule by using *merging-splitting-proofness*, *efficiency*, *anonymity*, and *continuity* (Theorem 3 in Chun 1988). This result is obtained as a corollary. Moreover, *anonymity* can be dropped and *merging-splitting-proofness* can be weakened to the pairwise version.

In the context of bankruptcy problems (Example 1), de Frutos (1999, Theorem 1) shows that the proportional rule is the only rule satisfying *merging-splitting-proofness*, *efficiency*, and *non-negativity*. This result is a corollary of Theorem 6. Moreover, our result shows that *merging-splitting-proofness* can be weakened to the pairwise version.

Corollary 14 (de Frutos 1999). *In the context of claims problems (or bankruptcy or surplus sharing), the proportional rule is the only rule satisfying* pairwise merging-splitting-proofness*,* efficiency*, and* non-negativity*.*

Chambers and Thomson (2002) establish an axiomatization of the proportional rule on the basis of the following axioms. A rule f satisfies *consistency* if for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, and each $N' \subseteq N$, if $x \equiv f(c, E)$, then

[8] Moulin (1985b) and Chun (1988) also use *efficiency* and *null-consistency*. Chun (1988) calls *null-consistency* "dummy".

$x_{N'} = f\left(c_{N'}, \sum_{i \in N'} x_i\right)$. It satisfies *equal treatment of equal groups* if for each $N \in \mathcal{N}$, each subsets $N', N'' \subseteq N$, and each $(c, E) \in \mathcal{D}^N$, if $\sum_{i \in N'} c_i = \sum_{i \in N''} c_i$, then $\sum_{i \in N'} f_i(c, E) = \sum_{i \in N''} f_i(c, E)$.

As pointed out by Chambers and Thomson (2002), *consistency*, *equal treatment of equal groups*, and *efficiency* imply *reallocation-proofness*. Thus, we obtain:

Corollary 15 (Chambers and Thomson 2002). *In the context of bankruptcy, the proportional rule is the only rule satisfying* equal treatment of equal groups*,* consistency*,* efficiency*, and* non-negativity *(or* no transfer paradox*).*

Proof. The following proof is due to Chambers and Thomson (2002). Let $\mathcal{D}$ be the variable population domain of bankruptcy problems defined in Example 1. Let f be a rule satisfying the four axioms.

We first show that f satisfies *anonymity*. Consider any two populations $N, N' \in \mathcal{N}$ of the same size n. Let $\pi \colon N' \to N$ be a bijection. Let $(c, E) \in \mathcal{D}^N$. Let $\bar{N}$ be disjoint from both N and N' and have the same size n as N and N'. Let $\bar{\pi} \colon \bar{N} \to N$ be a bijection. Let $(\bar{c}, 2E) \in \mathcal{D}^{N \cup \bar{N}}$ be such that $\bar{c}_N = c$ and $\bar{c}_{\bar{N}} = c^{\bar{\pi}}$, where for each $i \in \bar{N}$, $c_i^{\bar{\pi}} \equiv c_{\bar{\pi}(i)}$. Let $\bar{x} \equiv f(\bar{c}, 2E)$. Then applying *equal treatment of equal groups* to each pair of singleton groups $\{i\}$ and $\{\bar{\pi}(i)\}$ for each $i \in \bar{N}$, $\bar{x}_{\bar{N}} = \bar{x}_N^{\bar{\pi}}$, where for each $i \in \bar{N}$, $\bar{x}_{N,i}^{\bar{\pi}} \equiv \bar{x}_{\bar{\pi}(i)}$. By *efficiency*, $\sum_{i \in N} \bar{x}_i = \sum_{i \in \bar{N}} \bar{x}_i = E$. By *consistency*, $\bar{x}_N = f(\bar{c}_N, E) = f(c, E)$ and $\bar{x}_{\bar{N}} = f(\bar{c}_{\bar{N}}, E)$. Hence for each $i \in \bar{N}$, $f_{\bar{\pi}(i)}(c, E) = f_i(c^{\bar{\pi}}, E)$. Let $c' \equiv c^{\pi}$. Let $(\bar{c}', 2E) \in \mathcal{D}^{N' \cup \bar{N}}$ be such that $\bar{c}'_{N'} \equiv c'$ and $\bar{c}'_{\bar{N}} \equiv c^{\bar{\pi}}$. Let $\bar{x}' \equiv f(\bar{c}', 2E)$. By *equal treatment of equal groups*, $\bar{x}'_{\bar{N}} = \bar{x}'^{\left(\pi^{-1} \circ \bar{\pi}\right)}_{N'}$. By *consistency*, for each $i \in \bar{N}$, $\bar{x}'_i = f_i(c^{\bar{\pi}}, E) = f_{\pi^{-1} \circ \bar{\pi}(i)}(c', E) = \bar{x}'_{\pi^{-1} \circ \bar{\pi}(i)}$. Therefore, for each $i \in \bar{N}$, $f_{\bar{\pi}(i)}(c, E) = f_{\pi^{-1} \circ \bar{\pi}(i)}(c', E)$, that is, for each $i \in N'$, $f_{\pi(i)}(c, E) = f_i(c^{\pi}, E)$.

By Corollary 4, we only have to show that f satisfies *reallocation-proofness* (note that *consistency* implies *null-consistency*). Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{D}^N$, $S \subsetneq N$, and $c' \in \mathbb{R}_+^N$ be such that $c'_{N \setminus S} = c_{N \setminus S}$ and $\sum_{i \in S} c'_i = \sum_{i \in S} c_i$. Without loss of generality, we assume $N = \{1, \cdots, n\}$ and $S = \{1, \cdots, s\}$ where $s < n$. Let $N' \equiv \{n+1, \cdots, n+n\}$ and $S' \equiv \{n+1, \cdots, n+s\}$. Let $(\bar{c}, 2E; N \cup N') \in \mathcal{D}^{N \cup N'}$ be such that $\bar{c}_N = c$, $\bar{c}_{N'} = c'$, $\bar{c}_{S'} = c'_S$, and $\bar{c}_{N' \setminus S'} = c'_{N \setminus S}$. Let $x \equiv f(\bar{c}, 2E; N \cup N')$. Then by *equal treatment of equal groups*, $\sum_{i \in N} x_i = \sum_{i \in N'} x_i$ and $\sum_{i \in S} x_i = \sum_{i \in S'} x_i$. By *consistency*, $f(c, E) = x_N$ and $f(c', E) = x_{N'}$. By *anonymity*, for each $i \in N$, $f_i(c', E) = x_{n+i}$. Hence $\sum_{i=1}^{s} f(c', E) = \sum_{i=n+1}^{n+s} x_i = \sum_{i \in S} x_i = \sum_{i=1}^{s} f(c, E)$. ∎

In both Corollaries 14-15, *non-negativity* can be dropped, as long as we impose the bound condition in the definition of rules: see Remark 2.

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