Random matching under dichotomous preferences

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Abstract

We consider bilateral matching problems where each person views those on the other side of the market as either acceptable or unacceptable: an acceptable mate is preferred to remaining single, and the latter to an unacceptable mate; all acceptable mates are welfare-wise identical.

Using randomization, many efficient and fair matching methods define strategyproof revelation mechanisms. Randomly selecting a priority ordering of the participants is a simple example. Equalizing as much as possible the probability of getting an acceptable mate accross all participants stands out for its normative and incentives properties: the profile of probabilities is Lorenz dominant, and the revelation mechanism is groupstrategyproof for each side of the market.

Our results apply to the random assignment problem as well.

Keywords: matching, assignment, strategyproofness, dichotomous preferences

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1 Introduction

The bilateral matching problem occupies a special place in the mechanism design literature, combining strong empirical relevance and an interesting mathematical structure (see Roth and Sotomayor [1990]). The celebrated Gale-Shapley algorithm selects an efficient matching with strong incentive properties: it is stable in the sense of the core, and strategyproof with respect to the side of the market actively proposing to the other side (though not with respect to the passive side).

We consider the important special case of the bilateral matching problem where each man (resp. woman) evaluates each woman (resp. man) as acceptable or unacceptable: being matched with an acceptable (resp. unacceptable) mate is better (resp. worse) than remaining single, and acceptable mates yield the same welfare. Abusing language slightly, we speak of dichotomous preferences, to capture the idea that the preferences of an individual are entirely described by the subset of his or her acceptable mates[1].

For convenience we use the marriage terminology, but we have in mind other kinds of matching than sharing a life—for which a single binary criterion is an utterly insufficient model! Examples relevant to our model include matching managers to support staff, when a staff person is acceptable to a manager if and only if he has certain skills, and a manager is acceptable to a staff person if and only if she is not requesting "hard" tasks; or matching professors to research assistants, pilots to copilots, nurses to doctors, and so on. Time sharing is the simplest way to deal fairly with the indivisibilities of matching markets: think of a set of workers sharing their time among a set of employers (Baïou and Balinski (2000), Alkan and Gale (2001)). In our model randomization over matchings is formally equivalent to time-sharing, and to fix ideas we use the former terminology.

When one side of the market finds all agents on the other side acceptable, we can think of these agents as passive *objects* and speak of an *assignment* problem, an interesting special case of our general model.

Dichotomous preferences are equally natural in the assignment problems. Think of housemates distributing single rooms, when a "good" room may be one with a private bath for some agent, one with a private phone for another agent, and so on; another example is the assignment of softwares to workers when a given software can be compatible or not with a worker's own machine; or the scheduling of a list of jobs by a single server among a given set of time-

[1] Yet individual's preferences have three indifferences classes corresponding to being matched with an acceptable mate, or an unacceptable one, or remaining single. Note that preferences among unacceptable mates do not matter, as we only consider voluntary (individually rational) matchings.

slots: each customer requesting one job finds only certain time-slots acceptable (e.g., the job is useless after next Tuesday, or can only be done on week-ends, and so on).

The dichotomous domain is much smaller in size than—yet not contained in—the strict preferences domain. The first simplification is that a (deterministic or random) matching is core stable if and only if it is efficient and voluntary (no one is matched to an unacceptable partner). Among voluntary matchings, the efficient ones are the (inclusion) maximal ones, and in all maximal matchings the number of matched men (or women) is the same. We call this number e the *efficiency size* of the matching problem[2].

For a randomized matching we use the probability of an acceptable match as the canonical utility representation. Then a voluntary random matching is efficient if and only if its total utility is e, namely with probability one it matches e women to e men. In particular, the notions of ex ante and ex post efficiency coincide.

The Gallai-Edmonds decomposition of bipartite graphs, a well-known result in graph theory, gives further details on the structure of core-stable deterministic matchings. There is a set of M^d of *disposable* men competing for the set W^o of overdemanded women: all *overdemanded* women are matched to a strict subset of disposable men. There is similarly a set W^d of disposable women competing for the set M^o of overdemanded men. And the remaining men (in $M \diagdown M^o \cup M^d$) can be perfectly matched to the remaining women (in $W \diagdown W^o \cup W^d$). This bipartition of M and W is unique for each profile of dichotomous preferences: see Section 3.

The particularly simple structure of the core in our model makes it easy to achieve at the same time the three enduring goals of mechanism design, i.e., efficiency, incentive-compatibility (interpreted here as strategyproofness) and fairness.

The simplest example uses the familiar idea of selecting randomly, and with uniform probability, a priority ordering of the participants. We call it the *random priority* solution. For any fixed ordering of the set of men and women, the corresponding lexicographic maximization of utilities over voluntary matchings yields an efficient deterministic matching (unique utility-wise): this matching method is clearly strategyproof. When we randomize with fixed probabilities over all orderings of the set of agents, strategyproofness is preserved, and so is efficiency. The former statement does not depend upon the assumption of dichotomous preferences, the latter one does.

Recall that in the classical domain of strict preferences, strategyproofness for both men and women is incompatible with core stability (Roth [1982]). Our main results (Theorems 1 and 2) show that in the dichotomous domain, we can achieve much more than strategyproofness on both sides of the market. To this end, we introduce a natural randomized solution of our matching problem, that stands out for its normative and incentive properties.

[2]Interestingly a similar result holds in the strict preference domain: the set of men and women matched is the same in any core stable matching (Theorem 2.22 in Roth and Sotomayor [1990]).

The *egalitarian* solution equalizes as much as possible the individual probilities of being matched. We show that the corresponding profile of utilities first-order stochastically dominates any other feasible profile of utilities arranged increasingly, a property known as *Lorenz-dominance.* Thus the egalitarian solution uniquely maximizes any additively separable strictly concave collective utility function, for instance, the Nash collective utility function. It also admits a competitive interpretation and is easily computed: Theorem 1.

On the incentive front, the egalitarian solution is no less remarkable, as it is robust against potential manipulations by any all-male or all-female coalition: Theorem 2. We call this property *groupstrategyproofness* with respect to one side of the market.

We do not provide an axiomatic characterization of the egalitarian solution. We state instead two results showing that this solution reaches the possibility frontier. First, no deterministic core-stable solution is groupstrategyproof with respect to every all-male coalition, or to every all-female coalition: Lemma 7. This result stands in contrast to the situation in the classical domain of strict preferences, where the Gale-Shapley algorithm delivers a deterministic solution that is groupstrategyproof with respect to one side of the market. In the dichotomous domain, randomization is essential to achieve groupstrategyproofness and core stability; on the other hand we can guarantee groupstrategyproofness with respect to both sides of the market.

When we allow for manipulations by coalitions mixing some men and some women, strong impossibility results obtain. No core-stable solution treating equals equally is even weakly groupstrategyproof; the same is true of any solution selecting a utility profile without specifying the probabilistic matching to implement it: Theorem 3.

The paper is organized as follows. After a review of the small related literature in Section 2, the model is defined in Section 3. Efficient voluntary (i.e., core-stable) matchings, deterministic as well as random, are characterized in Section 4. The egalitarian solution is defined and characterized in Section 5. Strategyproofness and groupstrategyproofness are the subject of Section 6. Section 7 gathers some concluding comments. All proofs are in the Appendix.

2 Relation to the literature

The economic theory of bilateral matching under strict preferences—surveyed in the classic book by Roth and Sotomayor [1990]—does not address fairness by randomization, but examines in great details strategyproofness and core stability.

The small literature on random assignment under strict preferences is very relevant to our work. Hylland and Zeckhauser [1979] defined a fair and efficient solution, adapting to the random assignment problem the familiar competitive equilibrium with equal incomes. Yet this competitive solution is not incentive compatible[3]. Our results show that, on the contrary, in the dichotomous do-

[3] Zhou [1990] establishes the general impossibility of achieving ex ante efficiency (when

main the randomization/time-sharing approach succesfully achieves efficiency, incentive compatibility, and fairness. And the egalitarian solution that we recommend is precisely equal income competitive (Theorem 1).

A recent flurry of papers on the deterministic assignment of indivisible goods bears some relation to our work. The central question of that literature is to characterize the set of efficient and incentive compatible (strategyproof) assignment mechanisms: Papai [2000], Ehlers and Klaus [2000], Ehlers, Klaus and Papai [2000]. Most papers work in the strict preference domain, but three exceptions are Svensson [1994], [1999], and Bogomolnaia, Ehlers and Deb [2000]. These authors consider the deterministic assignment in the full domain of complete and transitive preference relations, allowing for strict and for dichotomous preferences. Fixed priority mechanisms are strategyproof and non bossy. Conversely, these two properties force the mechanism to resemble closely a fixed priority one; the additional property of neutrality captures precisely the fixed priority mechanisms.

Finally our egalitarian solution is closely related to the solution with the same name for supermodular cooperative games , due to Dutta and Ray [1989]. We can interpret our solution as the egalitarian solution of a certain cooperative game derived from the matching problem (Section 5).

3 The model

A matching problem consists of a finite set M of "men", a finite set W of "women", and two $M \times W$ zero-one matrices RM and RW, representing dichotomous preferences of men over women and of women over men respectively. An entry $RM_{mw} = 1$ if woman w is *acceptable* for man m, and $RM_{mw} = 0$ if she is not acceptable for him (and similarly for RW). Thus, each row RM_m of RM represents the preferences of a man m, and each column RW^w of RW represents the preferences of a woman w.

Each person prefers to be matched to an acceptable person of the opposite gender to being unmatched, but would rather be alone than matched to an unacceptable person.

We assume throughout the paper that matching is voluntary, namely two individuals can be matched only if they like each other. We refer to this important assumption as the *individual rationality* restriction. It implies that all the information about feasible matchings and relevant preferences is conveyed by a single $M \times W$ zero-one matrix R, equal to the entry by entry product of RM and RW: $R_{mw} = 1$ if and only if $RM_{mw} = 1$ and $RW_{mw} = 1$, i.e., if and only if man m and woman w are mutually acceptable (we then say that they are "compatible"). We call the triple (M, W, R) the individually rational (ir) reduced problem of the problem (M, W, RM, RW).

Thus we work most of the time with the ir-reduced model (M, W, R). The only exception is Section 6 devoted to strategic behavior: there the matching

preferences over lotteries are described by von Neumann-Morgenstern utilities), fairness (in the minimal sense of equal treatment of equals), and strategyproofness.

mechanism computes the ir-reduced problem from the reported preferences RM and RW.

As this will cause no confusion, we use the notation R_m (resp. R^w) both for the row (resp. the column) of R and for the subset of women (resp. men) who are compatible with man m (resp. woman w). For any subset S of men (resp. subset B of women)—also called a *coalition*—we write $R_S = \cup_S R_m$ (resp. $R^B = \cup_B R^w$) for the set of people of the opposite gender, compatible with at least one person in S (resp. B).

When all agents on one side of the market finds all agents on the other side acceptable, we speak of an *assignment* problem, and of the passive agents as objects. For instance $RM \equiv 1$ implies $R = RW$ and the objects in M are allocated to agents in W.

A deterministic matching μ of the ir-reduced problem (M, W, R) is a subset of $M \times W$, such that $(m, w) \in \mu$ only if $R_{mw} = 1$, and any person appears there at most once: for any $m \in M$ (resp. $w \in W$) there exist at most one $\widetilde{w} \in W$ (resp. $\widetilde{m} \in M$) such that $(m, \widetilde{w}) \in \mu$ (resp. $(\widetilde{m}, w) \in \mu$). Persons who appear as a component of a pair from μ, and only those, are *matched* by μ. We write $\mathcal{A}(M, W, R)$ for the set of voluntary deterministic matchings.

A *random matching* is a lottery π on $\mathcal{A}(M, W, R)$. For all our results, the only relevant information about a random matching π is the *random allocation matrix* Z, giving for all m and w the probability z_{mw} that man m and woman w are matched, i.e., the probability that π selects a deterministic matching μ such that $(m, w) \in \mu$. Thus the $M \times W$ matrix Z is substochastic, that is to say it is non-negative and the sum of each row and each column is at most 1; moreover z_{mw} is positive only if man m and woman w are compatible.

The mapping from a random matching π to its allocation matrix Z is clearly not one-to-one. A variant of the Von Neumann-Birkhof theorem on bistochastic matrices (see Lemma 2.1 in Bogomolnaia and Moulin [2002]), implies that it is *onto* the following set of $M \times W$ matrices:

$$\mathcal{Z}(M, W, R) = \{Z \mid Z \text{ is substochastic and } R_{mw} = 0 \Rightarrow z_{mw} = 0\} \qquad (1)$$

For agent m with dochotomous preferences RM_m, stochastic dominance is a complete relation among the feasible m-rows Z_m, therefore the sum of the entries in this row is the canonical utility representation of his preferences over random matchings. It is simply the probability that he gets an acceptable match. Denoting similarly Z^w for the w-column of Z, our utility functions are:

$$u_m(Z_m) = \sum_W z_{mw} = \sum_{R_m} z_{mw}, \qquad v_w(Z^w) = \sum_M z_{mw} = \sum_{R^w} z_{mw} \qquad (2)$$

We denote by $\mathcal{UV}(M, W, R)$ the set of feasible utility vectors (of length $|M| + |W|$), namely the image of $\mathcal{Z}(M, W, R)$ under the utility functions (2).

A *solution* is a mapping $(M, W, R) \rightarrow Z$ associating a random allocation matrix to any problem. A *welfarist solution* only keeps track of the utility profile. It does not specify which particular allocation matrix Z implements the utility profile. Formally it is a mapping $(M, W, R) \to (u, v) \in \mathcal{UV}(M, W, R)$.

4 Efficiency

A person who is not compatible with anybody ($R_m = \emptyset$ or $R^w = \emptyset$) has no bearing on efficiency, and can simply be ignored. We assume $R_m \neq \emptyset$ and $R^w \neq \emptyset$ for all m, w in this section and the following one. For future reference, note that such a "null" person could play a strategic role in a group manipulation of the kind we discuss in Section 6 (by sending a non-null report).

We discuss first the structure of efficient matchings in the ir-reduced problem (M, W, R). Any such matching μ is core-stable because a pair (m, w), has a blocking objection only if m and w are mutually acceptable and neither is matched at μ: this would contradict the efficiency of μ. The converse statement is just as obvious, therefore for deterministic matchings core-stability and efficiency + individual rationality are identical properties.

In the following statement, $\{S_1, S_2, S_3\}$ is called a *partition* of S if S_1, S_2, S_3 are disjoint subsets of S, their union is S, and at least one of S_1, S_2, S_3 is non-empty. Also, we say that an ir-reduced problem (M, W, R) is a *perfect match* if there is $\mu \in \mathcal{A}(M, W, R)$ at which all men and all women are matched.

Lemma 1 *(Gallai-Edmonds decomposition)*

Given a matching problem (M, W, R) with $R_m \neq \emptyset$, $R^w \neq \emptyset$ for all m, w, there is a unique pair of partitions $\{M^o, M^p, M^d\}$ of M and $\{W^o, W^p, W^d\}$ of W such that:

*i) W^d is only compatible with M^o, and M^o is **overdemanded** by W^d: $R^{W^d} = M^o$ and*

$$\left|R_S \cap W^d\right| > |S| \quad \textit{for all } S \subseteq M^o \tag{3}$$

ii) the restricted problem (M^p, W^p, R) is a perfect match

*iii) M^d is only compatible with W^o, and W^o is **overdemanded** by M^d: $R_{M^d} = W^o$ and*

$$\left|R^B \cap M^d\right| > |B| \quad \textit{for all } B \subseteq W^o \tag{4}$$

*We call M^o, W^o the sets of **overdemanded** persons, M^p, W^p the sets of **perfect** persons, and M^d, W^d the sets of **disposable** persons.*

Note in particular, that $|M^o| < |W^d|$, $|M^p| = |W^p|$ and $|M^d| > |W^o|$. The problem is trivial when the only non-empty sets are M^p and W^p.

The women in W^d are compatible only with men in M^o and are enough to make everyone in M^o happy: we can assign M^o to W^d. In fact (3) implies $\left|R_S \cap W^d \diagdown \{w\}\right| \geq |S|$ for any w in W^d, hence by Hall's theorem applied to the assignment of overdemanded men, M^o can be assigned to $W^d \setminus \{w\}$. This justifies our "disposable" terminology for the women in W^d.

Figure 1 depicts the compatibility matrix R of a 10 men - 10 women example. One checks easily:

$$M^o = \{m_1, m_2, m_3\}; M^p = \{m_4, m_5, m_6\}; M^d = \{m_7, m_8, m_9, m_{10}\}$$
$$W^d = \{w_6, w_7, w_8, w_9, w_{10}\}; W^p = \{m_3, m_4, m_5\}; W^o = \{m_1, m_2\}$$

A matching $\mu \in \mathcal{A}(M, W, R)$ is efficient if and only if the subset of $M \cup W$ that it matches is inclusion maximal: by Lemma 1 all inclusion maximal matchings in $\mathcal{A}(M, W, R)$ match the same number of agents.

Lemma 2 *Notations as in Lemma 1. Define the* ***efficiency size*** *of problem* (M, W, R) *as*

$$e(M, W, R) = |M^o| + |W^o| + |M^p| = |M^o| + |W^o| + |W^p| \tag{5}$$

A deterministic matching $\mu \in \mathcal{A}(M, W, R)$ *is efficient (Pareto optimal w.r.t. the utilities (2)) if and only if exactly* $2e$ *agents (*e *men and* e *women) are matched by* μ.

In all efficient matchings, M^o *is matched to a proper subset of* W^d *,* W^o *is matched to a proper subset of* M^d*, and* M^p *and* W^p *are perfectly matched.*

From the point of view of welfare, all that matters is which pairs of coalitions (S, B) of size e, where $S \subset M$, $B \subset W$, can be matched and which cannot. We call the former pairs *efficient* and denote their set by $E(M, W, R)$. A person is disposable if and only if there is an efficient coalition (S, B) to which he or she does not belong. Note that, by Lemma 2, there is also an efficient coalition to which he or she belongs (recall our assumption $R_w, R_m \neq \emptyset$).

The only aspect of the efficiency frontier not determined by the partition of Lemma 1 is this: which coalitions from M^d can be (simultaneously) matched? and which coalitions from W^d can be matched? In the example of Figure 1 the answer is, respectively, every pair of $M^d = \{m_7, m_8, m_9, m_{10}\}$ and seven triples from $W^d = \{w_6, w_7, w_8, w_9, w_{10}\}$ (the three triples containing w_7 and w_8 are excluded).

We turn our attention to efficient and individually rational random matchings. Let π be a lottery over $\mathcal{A}(M, W, R)$, the set of deterministic matchings. A necessary condition for the Pareto optimality (efficiency) of π with respect to the utilities (2) is that its support contains only efficient matchings (this is the familiar ex-post efficiency property). This condition is sufficient as well: by Lemma 2 every efficient deterministic matching μ maximizes the joint utility $\sum_M u_m + \sum_W v_w = 2e(M, W, R)$, and so does every lottery over these matchings.

Lemma 3 *Notations as in Lemmas* 1, 2.

i) A random matching is efficient in the ir-reduced problem if and only if, with probability one, it matches $2e(M, W, R)$ *persons.*

ii) A random allocation matrix Z is efficient if and only if the sum of its entries is $e(M, W, R)$.

iii) A random allocation matrix Z for matching problem is efficient if and only if $z_{mw} > 0$ only for $(m, w) \in (M^o, W^d) \cup (M^p, W^p) \cup (M^d, W^o)$, and its restriction to (M^p, W^p) is bistochastic, to (M^o, W^d) is row-stochastic, and to (M^d, W^o) is column-stochastic.

A consequence of Lemma 3 is that a random matching is core-stable if and only if its allocation matrix is efficient. Indeed the only agents whose utility may be smaller than 1 are those in M^d and W^d, but there is no mutually acceptable match between these agents (statement i in Lemma 1).

The efficiency size $e(M, W, R)$ has some sub- and supermodularity properties that play a crucial role in the sequel. In what follows we consider the efficiency size $e(S, B, R)$ of the restriction of our original problem (M, W, R) to subsets $S \subseteq M$ and $B \subseteq W$. We abuse notation slightly by writing the restriction of R to $S \times B$ simply as R. The set of disposable men, or women, has a simple characterization by means of the efficiency size function:

$$m \in M^o \cup M^p \Leftrightarrow e(M \diagdown m, W, R) = e(M, W, R) - 1 \tag{6}$$

$$m \in M^d \Leftrightarrow e(M \diagdown m, W, R) = e(M, W, R) \tag{7}$$

The efficiency size $e(M, W, R)$ is submodular in M and in W, and supermodular in $M \times W$. In the following inequalities, we write $e(S, B)$ instead of $e(S, B, R)$:

$$\begin{aligned}
e(S, W) + e(T, W) &\geq e(S \cup T, W) + e(S \cap T, W), \text{ for all } S, T \subseteq M \\
e(M, B) + e(M, C) &\geq e(M, B \cup C) + e(M, B \cap C), \text{ for all } B, C \subseteq W \\
e(S, B) + e(S', B') &\geq e(S, B') + e(S', B), \text{ for all } S \subseteq S' \text{ and all } B \subseteq B'
\end{aligned}$$

Finally, the efficiency size function allows a compact description of the efficient utility profiles in the ir-reduced problem. These are the utility profiles of the core-stable random matchings.

Lemma 4 *Given an ir-reduced problem (M, W, R), a utility vector $(u, v) \in [0, 1]^{M+W}$ is feasible and efficient if and only if it is a solution of the following system:*

$$\begin{aligned}
\sum_M u_m &= \sum_W v_w = e(M, W, R); \sum_S u_m \leq e(S, W, R) \textit{ for all } S \subseteq M, \\
\sum_B v_w &\leq e(M, B, R) \textit{ for all } B \subseteq W
\end{aligned} \tag{8}$$

We denote by $\mathcal{UV}^e(M, W, R)$ the set of efficient feasible utility vectors.

5 The egalitarian solution

The egalitarian solution picks an efficient matching equalizing as much as possible the individual utilities, i.e., the probabilities of an acceptable match. When full equality is not compatible with efficiency, the solution maximizes the familiar leximin ordering.

In the example of Figure 1, women w_7 and w_8 are only compatible with m_2, therefore $\min\{v_7, v_8\} \leq 0.5$ for any feasible utility profile (u, v). Men m_7, m_8, m_9 and m_{10} have to share w_1 and w_2, so $\min\{u_7, u_8, u_9, u_{10}\} \leq 0.5$ as well. If we reserve m_2, w_1 and w_2 for these six disposable persons, they each end up with a probability $\frac{1}{2}$ to be matched. The remaining disposable women w_6, w_9 and w_{10} can share men m_1 and m_3, so as to get utility $\frac{2}{3}$ each (give $\frac{2}{3}$ of m_3 to w_{10}, $\frac{1}{3}$ of m_3 and $\frac{1}{3}$ of m_1 to w_9, and $\frac{2}{3}$ of m_1 to w_6). All other persons get utility 1. This is the egalitarian solution.

We define first the solution by means of a simple algorithm, before deriving its equalizing properties in Theorem 1 below. We use the notations $r(T) = |R_T|$ and $r(T, B) = |R_T \cap B|$ for $T \subseteq M$, $B \subseteq W$. For any real valued function h, the expression $\arg\min_S h(T)$ stands for the subset T of S minimizing h over all subsets of S; if several subsets of S reach the minimum, T is the largest one in the sense of inclusion: this is well-defined in our case, because, for our choice of the function h, if two subsets minimize h on 2^S, so will their union.

The last piece of notation concerns the sequence T_k, $k = 1, 2, ...,$ of *disjoint* subsets of M constructed in the Definition. We write $T_{1,...,k}$ for the union of $T_1, T_2, ..., T_k$.

Given an ir-reduced problem (M, W, R), and an efficient (core-stable) utility profile, the only agents whose utility may be less than 1 are the disposable men M^d and women W^d. In view of Lemma 1, we can leave aside the perfectly matched agents in M^d and W^d and solve two separate matching subproblems (M^d, W^o, R) and (M^o, W^d, R). Since overdemanded agents always get a match we can think of W^o and M^o as *objects* and of these subproblems as, respectively the assignment of W^o to the agents in M^d, and of M^o to the agents in W^d. The algorithm defining the egalitarian solution is only given for the subproblem (M^d, W^o, R); for the subproblem (M^o, W^d, R) we simply exchange the roles of men and women.

Definition 1 *Given the subproblem (M^d, W^o, R) of the ir-reduced problem (M, W, R), we define recursively an increasing sequence of numbers $\alpha_k, k = 1, ..., K$, where $o < \alpha_k < 1$, and a partition of M^d by a sequence $T_k, k = 1, ..., K$, of non-empty coalitions:*

$$\begin{aligned} M_0 &= M^d, W_0 = W^o; M_k = M^d \setminus T_{1,...,k},\ W_k = W^o \setminus R_{T_{1,...,k}}; \\ \alpha_k &= \min_{M_{k-1}} \frac{r(T, W_{k-1})}{|T|}; \qquad (9) \\ T_k &= \arg\min_{M_{k-1}} \frac{r(T, W_{k-1})}{|T|} \quad \text{if } \alpha_k < 1 \end{aligned}$$

The sequences α_k, T_k stop at K where $T_{1,...,k} = M^d$. The egalitarian utility profile is: $u^e_m = \alpha_k$ if $m \in T_k$, for $k = 1, ..., K$

The egalitarian solution is *welfarist:* we only define its utility profile without specifying which random allocation matrix should be used to implement it. For instance, the perfectly matched agents in M^p, W^p receive utility 1, and there may be a number of ways to match them.

Yet the matrices $Z \in \mathcal{Z}(M^d, W^o, R)$ implementing the egalitarian utility profile u^e are largely determined by the algorithm (9). The matrix Z must assign all the objects in R_{T_1} to coalition T_1 in order to achieve the total utility $\sum_{T_1} u^e_m = r(T_1) = |R_{T_1}|$. All the objects in $R_{T_2} \cap W_1 = R_{T_2} \diagdown R_{T_1}$ must be assigned to coalition T_2 to secure the utility $\sum_{T_2} u^e_m = r(T_2, W_1) = |R_{T_2} \cap W_1|$, given that R_{T_1} is not available. The key fact is that the new maximum utility α_2 is greater than α_1, thus the decision to reserve R_{T_1} to T_1 is vindicated on egalitarian grounds. By repeating this argument we see that Z must assign precisely the objects $R_{T_k} \cap W_{k-1} = R_{T_k} \diagdown R_{T_{1,...,k-1}}$ to coalition T_k for $k = 1, ..., K$. In short, coalition T_k receives at stage k all the objects it likes among those not assigned in earlier stages.

Our next result states two distinct characterizations of the egalitarian solution, justifying its normative appeal as a fair outcome of our matching problem. Recall that the *Lorenz dominance* is the following partial orderings of vectors in $\mathbb{R}^{M \cup W}$: (u, v) dominates (t, w) if upon rearranging their $p =| M | + | W |$ coordinates increasingly as $x^*$and y^*, we have $\Sigma_1^k (x^*_i - y^*_i) \geq 0$ for all $k = 1, ..., p$.

Theorem 1 *i) Given an ir-reduced problem (M, W, R), the egalitarian utility profile is Lorenz dominant in the feasible utility set $\mathcal{UV}(M, W, R)$.*

ii) Given positive prices p_w for each woman and p^m for each man, we say that an efficient allocation matrix $Z \in \mathcal{Z}(M, W, R)$ is competitive at p with equal income of one if we have for all m, w

$z_{mw} > 0 \Rightarrow p_w = \min_{y \in R_m} p_y$, $p^m = \min_{x \in R^w} p^x$

and $\sum_{y \in W} p_y \, z_{my} = \sum_{x \in M} p^x z_{xw} = 1$

The allocation matrix Z is equal income competitive if and only if it implements the egalitarian profile.

We interpret statement i first. If the arbitrary convex set $\mathcal{U}$ of utility profiles contains a Lorenz dominant element u^*, this profile has a very strong claim to fairness within the efficiency frontier. Indeed, it achieves the maximum over $\mathcal{U}$ of any collective utility function averse to inequality (in the sense of the Pigou-Dalton transfer principle); it is the unique maximum if the collective utility is strictly averse to inequality. Thus u^* maximizes not only the leximin ordering but also the Nash collective utility $\sum_M \log u_m$, and any collective utility $\sum_M f(u_m)$ for any increasing and concave function f (if f is strictly concave, u^* is the unique maximum), and more.

Turning to statement ii, we note that an allocation Z implementing the egalitarian profile u^e is equal income competitive for the following prices:

$$p_w = p^m = 1 \text{ if } m \in M^d \cup M^p \text{ and } w \in W^d \cup W^p$$
$$p_w = \frac{1}{\alpha_k} \text{ if } w \in R_{T_k} \cap W_{k-1} \text{ for } k = 1, ..., K$$
$$p^m = \frac{1}{\beta_l} \text{ if } m \in R_{S_l} \cap M_{l-1} \text{ for } l = 1, ..., L$$

where α_k, T_k, W_k is the sequence constructed in (9), so that $R_{T_k} \cap W_{k-1}$ covers W^o as k varies from 1 to K. The sequence β_l, S_l, M_l is the similar sequence when the algorithm is used to assign the agents W^d to M^o(upon exchanging the roles of men and women).

Consider the competitive demand at the above prices p, for an agent m in T_k with income of 1. This agent will buy only (a fraction of) the cheapest objects in R_m. The discussion following Definition 1 shows that these are precisely the objects in the non-empty set $R_m \cap R_{T_k} \cap W_{k-1}$. They all cost $\frac{1}{\alpha_k}$ therefore with a budget of one, agent m buys a fraction (probability or time-share) α_k of his good objects. Thus every matrix Z implementing u^e is equal income competitive. Statement ii says that the converse holds true as well.

The main result in Dutta and Ray [1989], is related to—and used in the proof of—our Theorem 1. By Lemma 1 we can interpret the set $\mathcal{U}^e(M^d, W^o, R)$ of efficient utility vectors for M^d as the core of the submodular cooperative game $S \to e(S, W, R)$. Theorem 3 in Dutta and Ray [1989] implies that $\mathcal{U}^e(M^d, W^o, R)$ contains a Lorenz dominant element (the first statement of our Theorem 1) and computes it by an algorithm identical to (9), except that $r(T, B)$ is replaced everywhere by $e(T, B)$, the efficiency size of the reduced problem (T, B, R) See the proof of Theorem 1. Note that $r(T, B)$ is computed by direct inspection of the matrix R, whereas computing $e(T, B)$ is the harder task of discovering the Gallai-Edmonds decomposition of the subproblem (T, B, R).

6 Strategyproofness and groupstrategyproofness

We investigate in this section the strategic opportunities in the direct revelation mechanisms associated with solutions (or welfarist solutions) of the matching problems. Recall that a solution g maps every problem (M, W, R) into a random allocation matrix Z, whereas a welfarist solution f maps (M, W, R) into a utility profile u (or (u, v)). Thus every solution g projects onto the following welfarist solution $f(M, W, R) = u(g(M, W, R))$.

Strategyproofness is not a welfarist concept: its definition requires to specify how the allocation matrix is affected when the reported preferences change. When we speak below of a welfarist solution (such as the egalitarian and priority solutions) being strategyproof (or any group variant of this property) we always refer to the most demanding interpretation, namely "f is strategyproof" means that "every solution g projecting onto f is strategyproof".

Given a (non welfarist) solution g (Section 2), the associated direct revelation mechanism works as follows. Men and women report their preferences RM and RW respectively, and the solution g applied to the ir-reduced problem $RM \bullet RW$ implements the random allocation matrix Z. Here the notation A$\bullet$B stands for the entry-by-entry product of the matrices A, B of the same size.

Definition 2 *The solution g is called male-strategyproof if for all $m \in M$, and any three matrices RM, RM' and RW with $g(RM \bullet RW) = Z$ and $g(RM' \bullet RW) = Z'$:*

$$\{RM_{m'} = RM'_{m'} \quad \text{for all } m' \neq m\} \Rightarrow u_m(Z_m) \geq u_m(Z'_m)$$

The solution is called strategyproof if it is both male- and female-strategyproof.

The solution is called male-groupstrategyproof if for all $S \subseteq M$, and any three matrices $RM, RM'$and RW, with $g(RM \bullet RW) = Z$ and $g(RM' \bullet RW) = Z'$:

$$\begin{aligned} \{RM_{m'} &= RM'_{m'} \text{ for all } m' \notin S \text{ and } u_m(Z_m) \geq u_m(Z'_m) \text{ for all } m \in S\} \\ &\Rightarrow \{u_m(Z_m) = u_m(Z'_m) \text{ for all } m \in S\} \end{aligned}$$

The simplest example of a strategyproof and core-stable solution is a *priority solution*. Given a priority ordering $\succ$ of $M \cup W$, we define the $\succ$-priority utility profile as the $\succ$-lexicographic maximum over the utility set $\mathcal{UV}$. Thus the highest priority person i_1 gets utility 1 because he or she is compatible with at least one person; the next person in the priority line, i_2, gets utility 1 if there is a way to match him or her without affecting the utility of person i_1, otherwise this person's utility is 0, and so on.

It is easy to check that a priority solution is core-stable, as well as strategyproof. As both properties are preserved by convex combinations with fixed weights (Lemma 3), a natural idea is to randomly select a priority solution, with uniform probability over all orderings of $M \cup W$. The resulting *random priority solution* is fair, core-stable and strategyproof for both sides of the market.

Thus in the dichotomous domain, core stability and strategyproofness are satisfied by a large class of solutions (including all convex combinations of priority solutions, the egalitarian solution, and more). This stands in sharp contrast with the situation in the classic domain of strict preferences where core stability is only compatible with strategyproofness on one side of the market (Roth [1982]). For instance, a priority solution in the classic domain is strategyproof on both sides, but it does not always pick a matching in the core.

Back to the dichotomous domain, it is a much taller order to discover a solution in the core that is also strategyproof with respect to one-gender coalitions. For instance, a priority solution is not male-groupstrategyproof. Consider the problem with three men and two women where each woman finds all men acceptable, m_1 finds both women acceptable whereas w_1 is the only acceptable woman for m_2 and w_2 the only acceptable woman for m_3. Given the priority ordering $m_1 \succ m_2 \succ m_3$, the truthful outcome matches $m_1$and m_2 However m_1 can help m_3 at no cost to himself by announcing that w_1 is his only acceptable mate.

Our next result says that every deterministic and core-stable solution is similarly vulnerable.

Lemma 5 *With three or more agents and two or more objects, no deterministic core-stable solution is either male- or female-groupstrategyproof.*

On the other hand, if randomization is feasible, this incompatibility disappears.

Theorem 2 *The egalitarian solution is both male-groupstrategyproof and female-groupstrategyproof.*

We are not aware of another mechanism design problem where randomization is *necessary* to combine efficiency and strong incentive compatibility properties.

We note that the random priority solution is not even weakly-groupstrategyproof with respect to all male or all female coalitions. That is to say, there are profiles where a coalition of men, say, can strictly improve upon every member's utility by jointly misreporting their preferences. An example is given in Bogomolnaia and Moulin [2001].

Finally, we turn to joint misreports by coalitions mixing some men and some women. A simple example shows that no core-stable solution is groupstrategyproof.

Assume $M = \{m\}$, $W = \{w_1, w_2\}$, and both women are compatible with the unique man. Here m can enforce matching (m, w) by reporting that he only likes w, so if a matching mechanism chooses (m, w_i) with positive probability, the coalition m, w_j, $j \neq i$, can manipulate to the benefit of woman w_j.

Thus the strong form of groupstrategyproofness is out of reach in our model, given our primary concern with core stability. However, there is some hope to find a reasonable solution meeting the weaker version of this property where we only rule out joint misreports from which all members of the group strictly benefit. For instance, a *priority* solution is thus weakly groupstrategyproof (Definition 3). Indeed, consider a member of the deviating coalition with the highest priority. He or she can improve only if somebody with higher priority, who was matched initially, does not get a match after manipulation. The only way this can happen is when another member of the deviating coalition pretends to like such a person, is matched to this person by some implementation of the priority solution, but refuses this match ex post. But this means that under the manipulation the utility of this last member of the deviating coalition is zero, so this person does not improve.

Definition 3 *The solution g is called weakly groupstrategyproof if for all $S \subseteq M$, all $B \subseteq W$, and any four matrices RM, RW, RM', RW', with $Z = g(RM \bullet RW)$ and $Z' = g(RM' \bullet RW') \bullet RM \bullet RW$:*

$$\begin{aligned}\{RM_{m'} &= RM'_{m'} \text{ for all } m' \notin S \text{ and } RW_{w'} = RW'_{w'} \text{ for all } w' \notin B\} \Rightarrow \\ \{u_m(Z_m) &\geq u_m(Z_{m'}) \text{ for some } m \in B \text{ or } v_w(Z^w) \geq v_m(Z'^w) \text{ for some } w \in B\}\end{aligned}$$

Definition 3 uses the full force of the individual rationality assumption. Following the misreport RM', RW' by coalitions S and B, the solution g implements the allocation matrix $\widetilde{Z} = g(RM' \bullet RW')$. The pair (m, w) is incompatible if $RM_{mw} \cdot RW_{mw} = 0$; if $\widetilde{z}_{mw} > 0$ for such a pair, the match (m, w) will be implemented by g with positive probability, given the reported preferences, yet it will not happen ex post because it is not voluntary for at least one of the two persons. Thus the allocation matrix actually implemented under individual rationality is $\widetilde{Z} \bullet RM \bullet RW$, as shown in Definition 3.

Unfortunately, weak groupstrategyproofness is not compatible with even the elementary test of horizontal equity known as equal treatment of equals.

Consider the following example with four men and four women. Let m_1 and w_1 like persons of the opposite gender numbered 2, 3 and 4, m_2 and w_2 like persons 1, 3 and 4, while m_3, m_4 and w_3, w_4 only like person 1. We have:

$$RM = \begin{array}{ccccc} & w_1 & w_2 & w_3 & w_4 \\ m_1 & 0 & 1 & 1 & 1 \\ m_2 & 1 & 0 & 1 & 1 \\ m_3 & 1 & 0 & 0 & 0 \\ m_4 & 1 & 0 & 0 & 0 \end{array}, \; RW = \begin{array}{ccccc} & w_1 & w_2 & w_3 & w_4 \\ m_1 & 0 & 1 & 1 & 1 \\ m_2 & 1 & 0 & 0 & 0 \\ m_3 & 1 & 1 & 0 & 0 \\ m_4 & 1 & 1 & 0 & 0 \end{array},$$

$$\text{so } R = \begin{array}{ccccc} & w_1 & w_2 & w_3 & w_4 \\ m_1 & 0 & 1 & 1 & 1 \\ m_2 & 1 & 0 & 0 & 0 \\ m_3 & 1 & 0 & 0 & 0 \\ m_4 & 1 & 0 & 0 & 0 \end{array}$$

Suppose now all men in $S = \{m_3, m_4, w_3, w_4\}$, pretend to also like person 2. We get:

$$RM' = \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}, \; RW' = \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}, \text{ so } R' = \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

Note that the preference matrix R' allows a perfect match.

Consider an efficient solution g that also *treats equals equally*, namely for all problem (M, W, R) with $g(M, W, R) = Z$ and all m, m', w, w' :

$$R_m = R_{m'} \Rightarrow Z_m = Z_{m'}; \; R^w = R^{w'} \Rightarrow Z^w = Z^{w'}$$

The above property plus efficiency determine g at R and R' :

$$g(R) = \begin{array}{cccc} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{array}, \; g(R') = \begin{array}{cccc} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array}$$

Because persons 3, 4 reject, ex post, a match with person 2, the allocation matrix actually implemented is:

$$Z' = \begin{matrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{matrix},$$

thus the misreport strictly benefits everyone in S. We conclude that g violates weak groupstrategyproofness.

A welfarist solution is *WGSP* if all solutions projecting onto it are *WGSP*. In the Appendix, we use the same 4×4 example to prove the second statement in our next result.

Theorem 3 *Assume at least 4 men and 4 women. No core-stable solution treating equals equally is weakly groupstrategyproof. No core-stable welfarist solution is weakly groupstrategyproof.*

7 Two concluding comments

1. Beyond horizontal equity (Equal Treatment of Equals), we can evaluate the normative appeal of a solution by several tests familiar to the fair division literature. The *No Envy* test requires $u_m(Z_m) \geq u_m(Z_{m'})$ for all m, m', and a similar statement among women. Another popular test is *Population Monotonicity* stating that when a new woman is added to W, the utility of no man should decrease and that of no woman should increase, and a symmetrical statement by exchanging the roles of men and women. By its competitive interpretation, the egalitarian solution meets No Envy, and it is easy to deduce from the properties of the efficiency function in Section 4 that Population Monotonicity is true as well. In fact these two tests prove relatively easy to meet in our model. Examples include the random priority solution (Section 6) and a number of other solutions. See Bogomolnaia and Moulin [2001] for a detailed discussion.

2. The *roommate* problem is the natural generalization of bilateral matching where the (gender neutral) agents must form pairs (to share a hypothetical room). Under the assumption of dichotomous preferences and the restriction to voluntary (individually rational) matching, an ir-reduced problem is described by a general matching problem, namely a pair (N, G) where N is the set of agents, G is an undirected graph on N, and an edge between two agents means that they are mutually compatible. The Gallai-Edmonds decomposition generalizes to the matching problem (N, G): see Theorem 3.2.1 in Lovasz and Plummer [1986]. In particular all inclusion maximal matchings compatible with G have the same cardinality, so we can still speak of the efficiency size e of an arbitrary problem. Hence for random matchings, it is still true that ex post and ex ante efficiency coincide: a random matching is efficient if and only if, with probability one, it matches exactly $2e$ agents.

It is straightforward to extend the definition of fixed priority and random priority solutions to the roommate problem. One checks just as easily that these solutions are strategyproof. Therefore, the latter two solutions are efficient, strategyproof and treat equals equally. The egalitarian solution is well defined as the unique maximand in the utility set of the leximin ordering. However it is no longer Lorenz dominant in that set because the efficiency function is not submodular anymore. It is not groupstrategyproof either, because bilateral matching is a special case of the roommate problem.

As in bilateral matching, weak groupstrategyproofness is out of reach in the roommate problem, if we also insist on efficiency and fairness. Unlike in bilateral matching, this fact does not depend upon the ex post rejection of matches. Consider the three person problem where each one of agents 1, 2 and 3 like the other two, so that G is the complete graph. An efficient solution treating equals equally must match each pair $\{i, j\}$ with probability $\frac{1}{3}$, resulting in the utility $\frac{2}{3}$ for each. Now if 1 and 2 both report that they only like each other, they are matched with probability 1 (by efficiency).

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Appendix: Proofs

1. Proofs for Section 4

A bipartite graph between a set of men and a set of women connects some men to some women by edges. A matching is a subset of edges of the graph such that each point (man or woman) is in at most one edge. Matching theory explores the properties of inclusion maximal matchings, simply called *maximal matchings*. If we interpret the presence (resp. absence) of an edge in the graph joining m to w as "man m and woman w are mutually acceptable", then a maximal matching is precisely an efficient and voluntary matching. The decomposition of efficient matchings in two disjoint efficient assignments is then a simple reinterpretation of the well known Gallai-Edmonds decomposition of bipartite matching graphs.

Lemmas 1–3 can thus be easily retrieved from matching theory upon translating our economic terminology into the graph-theoretical terminology of matching. An excellent survey of matching theory is Lovasz and Plummer [1986], in particular, Chapter 3. More direct proofs can be found in Ore [1962] (Theorem 7.6.1).

Sub- and supermodularity properties stated in the section are the submodularity of the rank function of a matroid (see, for example, Korte and Vygen [1991], Theorem 13.10)

Lemma 4:

It is enough to check the statement for the *assignment* of W^o to M^d. For any coalition S and any deterministic assignment μ, $\mu \in \mathcal{A}(M^o, W^d, R)$, with associated utility profile u, the sum $\sum_S u_m$ is the number of agents in S assigned by μ. Therefore if μ is efficient its utility profile meets system (8). By linearity, so does the utility profile of any efficient random assignment.

To prove the converse statement, let $u \in [0,1]^{M^d}$ be a solution of system (8). By a classical result about submodular (concave) cooperative games (Shapley [1971]), u is a convex combination of *marginal contribution* profiles $u^{\succ}$. For a strict ordering $\succ$ of M^d, this vector is defined by:

$$u^{\succ}_m = e(T \cup \{m\}, W^o, R) - e(T, W^o, R) \text{ for all } m$$

where T is the set of agents preceding m in $\succ$.

It is easy to see that $u^{\succ}$ is the utility profile of the (efficient) $\succ$-priority assignment, as defined in Section 6. Thus u is the utility profile of an efficient random assignment.

2. Proofs for Section 5: Theorem 1

It is enough to check Statement i for the *assignment* subproblem (M^d, W^o, R). In what follows we write M and W instead of M^d and W^o.

The straightforward proof of Statement ii is omitted for brevity.

Step 1 Preliminary result.

Let f be a non negative submodular function defined over the non-empty subsets of M. Fix a coalition S, $S \subseteq M$, and consider the program of minimizing $\frac{f(T)}{|T|}$ over all non-empty subsets of S. It is easy to check that the set of its solutions is stable by union, therefore the largest solution is well-defined; this set is denoted $\arg\min_S \frac{f(T)}{|T|}$.

For any subset B of W, the function $T \to r(T, B) = |R_T \cap B|$ is obviously submodular, therefore the sequence T_k, $k = 1, 2...$ is well-defined by (9).

Step 2. Another algorithm

We apply the above result to the submodular function $T \to e(T, W, R)$ denoted simply $e(T)$. We construct inductively two sequences $\beta_k, S_k, k = 1, 2...$ by an algorithm similar to (9), introduced by Dutta and Ray [1989]:

$$
\begin{aligned}
N_0 &= M, \;\; e_0 = e; \text{ next for } k = 1, 2... \\
\beta_k &= \min_{N_{k-1}} \frac{e_{k-1}(T)}{|T|}; \;\; S_k = \arg\min_{N_{k-1}} \frac{e_{k-1}(T)}{|T|}; \\
N_k &= M \diagdown S_{1,...,k}; \;\; e_k(T) = e(T \cup S_{1,...,k}) - e(S_{1,...,k}) \qquad (10)
\end{aligned}
$$

By construction, the sets S_k are disjoint and there is a step L such that $S_1, ...S_L$ partition M, at which point the sequence stops.

Dutta and Ray's egalitarian solution for the submodular (concave) cooperative game $T \to e(T)$ is the utility profile x, $x_m = \beta_k$ if $m \in S_k, k = 1, ..., L$. They show that x is in the core of the game (M, e), and Lorenz dominates every other utility profile in the core. Translated with our terminology, the core property is system (8), thus the result says that x is an efficient profile, $x \in \mathcal{U}^e(M, W, R)$, and it Lorenz dominates every other feasible utility profile (even inefficient ones).

For the sake of completeness, we prove below that x is in $\mathcal{U}^e(M, W, R)$, and that is maximizes the leximin ordering over this set. We also show that the sequence β_k is strictly increasing. Then we proceed to check that (β_k, S_k) and (α_k, T_k) are two sequences of the same length, $K = L$, and that they coincide with the possible exception of the last term.

The equality $\sum_M x_m = e(M)$ is clear from (10). Feasibility of x requires $\sum_S x_m \leq e(S)$ for all S. Suppose $S \subseteq S_1$, then $\sum_S x_m = |S| \cdot \beta_1 \leq e(S)$ by definition of β_1. Suppose $S \subseteq S_{1,2}$, and set $S^i = S \cap S_i, i = 1, 2$. By definition of β_2 and submodularity of e:

$$\beta_2 \leq \frac{e(S^2 \cup S_1) - e(S_1)}{|S^2|} \leq \frac{e(S) - e(S^1)}{|S^2|}$$

Combining this with $\beta_1 \leq \frac{e(S^1)}{|S^1|}$ gives $\beta_1 \cdot \left|S^1\right| + \beta_2 \cdot \left|S^2\right| \leq e(S)$, as desired. And so on inductively.

Next we check that β_k is strictly increasing. Note first that $e(T) \leq |T|$ for all T, implying $\beta_k \leq 1$ for all k. The inequality $\beta_k < \beta_{k+1}$ means $\beta_k < \frac{e_k(T)}{|T|}$ for all $T \subseteq N_k$. Assume the latter fails for some T, namely:

$$\beta_k = \frac{e_{k-1}(S_k)}{|S_k|} \geq \frac{e_k(T)}{|T|}$$

Both ratios above are at most 1 therefore:

$$\beta_k \geq \frac{e_{k-1}(S_k) + e_k(T)}{|S_k| + |T|} \geq \frac{e_{k-1}(T \cup S_k)}{|T \cup S_k|}$$

But S_k is the *largest* solution of min $\frac{e_{k-1}(T)}{|T|}$, contradiction.

It is now easy to check that x maximizes the leximin ordering over $\mathcal{U}^e$. Suppose y, another profile in $\mathcal{U}^e$, is leximin preferred to x. Then for all $m \in S_1, y_m \geq x_1^* = x_m$, and moreover:

$$e(S_1) \geq \sum_{S_1} y_m \geq \sum_{S_1} x_m = e(S_1)$$

therefore y and x coincide on S_1. Next:

$$\text{for all } m \in S_2 \qquad y_m \geq y^*_{|S_1|+1} \geq x^*_{|S_1|+1} = x_m$$

and moreover:

$$e(S_1 \cup S_2) \geq \sum_{S_1 \cup S_2} y_m \geq \sum_{S_1 \cup S_2} x_m = e(S_1 \cup S_2)$$

so that y and x coincide on $S_1 \cup S_2$. And so on.

For the sake of brevity, we refer the reader to Theorem 3 in Dutta and Ray [1989] for a proof of the fact that x is Lorenz dominant in $\mathcal{U}^e$.

Step 3. Equality of the two sequences

We check by induction that, as long as β_k remains strictly below 1, the two algorithms (9) and (10) coincide. More precisely we prove the following property $\mathcal{P}(k)$ by induction on k:

$$\beta_k < 1 \Rightarrow \{\text{for } t = 1, ..., k : \alpha_t = \beta_t, T_t = S_t\};\ e(T_{1,\ldots,k}) = r(T_{1,\ldots,k})$$

Step 3.a. Proof of $\mathcal{P}(1)$

Assume we know $e(S_1) = r(S_1)$. This implies $\beta_1 = \frac{r(S_1)}{|(S_1)} \geq \alpha_1$. On the other hand $e(T) \leq r(T)$ for all T implies $\beta_1 \leq \alpha_1$, so that $\beta_1 = \alpha_1$ and $S_1 \subseteq T_1$. Next we have $\beta_1 = \alpha_1 = \frac{r(T_1)}{|(T_1)} \geq \frac{e(T_1)}{|(T_1)|}$, implying $T_1 \subseteq S_1$. Thus the equality $e(S_1) = r(S_1)$ is enough to conclude $\alpha_1 = \beta_1$, $S_1 = T_1$.

We assume $e(S_1) < r(S_1)$ and show a contradiction of the assumption $\beta_1 < 1$. In the Gallai-Edmonds decomposition of (S_1, W, R), $S_1^a = S_1 \backslash S_1^d$ is non-empty (otherwise $e(S_1) = r(S_1)$, see Lemmas 1, 2). If S_1^d is empty, $e(S_1) = |S_1| \Rightarrow \beta_1 = 1$. Thus both S_1^a and S_1^d are non-empty and we have, with the notations of Lemmas 1, 2:

$$\frac{e(S_1^d)}{|S_1^d|} = \frac{|W|}{|S_1^d|} < \frac{|W| + |S_1^a|}{|S_1^d| + |S_1^a|} = \frac{e(S_1)}{|S_1|}$$

where the inequality follows from $|W^o| < |S_1^d|$, namely the overdemanded character of W^o. We have reached a contradiction of the definition of β_1.

Step 3.b. Induction step

Assume $\mathcal{P}(k-1)$ and $\beta_k < 1$. Consider an arbitrary coalition $T \subseteq M_{k-1}$. By $\mathcal{P}(k-1)$, $e(T_{1,\ldots,k-1}) = r(T_{1,\ldots,k-1})$, which means that all objects in $R_{T_{1,\ldots,k-1}}$ can be assigned to $T_{1,\ldots,k-1}$. Clearly this implies:

$$\begin{aligned} e(T \cup T_{1,\ldots,k-1}) &= e(T_{1,\ldots,k-1}) + e(T, W \diagdown R_{T_{1,\ldots,k-1}}) \\ \text{i.e.,} \quad e_{k-1}(T) &= e(T, W_{k-1}) \end{aligned} \tag{11}$$

We see now that the k-th step in our two algorithms are to minimize, respectively, $\frac{e(T,W_{k-1})}{|T|}$ and $\frac{r(T,W_{k-1})}{|T|}$. Upon changing W to W_{k-1}, the same argument as in Step 3.a shows now: $\alpha_k = \beta_k$, $T_k = S_k$, as well as $e(T_k, W_{k-1}) = r(T_k, W_{k-1})$. Combining this with (12) and $\mathcal{P}(k-1)$, we get:

$$e(T_{1,\ldots,k}) = e(T_{1,\ldots,k-1}) + e(T_k, W_{k-1}) = r(T_{1,\ldots,k-1}) + r(T_k, W_{k-1}) = r(T_{1,\ldots,k-1})$$

Step 4. Proof of Theorem 1 (Statement i)

The sequence β_k increases, and $\beta_k \leq 1$.

We show first that $\beta_K < 1$. Indeed, if $\beta_{K-1} < 1 = \beta_K$, then (11) is valid and $\beta_K = 1$ reads $e(T, W_{K-1}) = |T|$ for all $T \subseteq N_{K-1}$. Thus $S_K = M_{K-1}$ and the algorithm (10) stops there. Moreover $N_{K-1} = M_{K-1}$ can be assigned to W_{K-1}, and on the other hand only the agents in M_{K-1} like W_{K-1}:

$$R_{M \diagdown M_{K-1}} = R_{T_{1,\ldots,K-1}} = W \diagdown W_{K-1}$$

But this means that M_{K-1} cannot contain disposable agents. Thus, $M_{K-1} = \emptyset = W_{K-1}$, $T_{1,\ldots,K-1} = M^d$, $S_K = \emptyset$, thus the algorithm stopped before the K-th step.

Now, if $T_{1,\ldots,K} = M$ and $\beta_K < 1$, $\mathcal{P}(K)$ establishes the full equality of the two algorithms, and we are done.

4. Proofs for section 6

Lemma 5:

It is enough to consider $|M| = 3, |W| = 2$ and restrict our attention to profiles where each woman likes all three men Let a deterministic mechanism among $M = 1, 2, 3$ and $W = a, b$ be male-groupstrategyproof and efficient. We derive a contradiction by considering the following eight different preference matrices $RM = R$:

	a	b
R_1	1	1
R_2	1	1
R_3	1	1

[1];

1	0
1	1
1	1

[2];

1	0
1	0
0	1

[3];

1	0
1	0
1	1

[4]

1	1
1	0
1	1

[5];

0	1
1	0
1	1

[6];

0	1
1	0
0	1

[7];

1	1
1	0
0	1

[8]

The efficiency size of all eight problems is 2. We write an assignment (i, j) to indicate that object a goes to agent i and b to j. We write a two-person coalition simply as ij.

Suppose, without loss of generality, that our mechanism chooses $(1, 2)$ at [1]. Then it must choose $(1, 2)$ at [2] : if $u_1 = 0$ or $u_2 = 0$ at [2], coalition 12 can manipulate at [2] by $R' =$ [1]. Consider [3] next: the only two efficient assignments are $(1, 3)$ and $(2, 3)$; if $(2, 3)$ is selected, 23 manipulates at [2] by $R' =$ [3], hence $(1, 3)$ is chosen at [3]. Then $(1, 3)$ is chosen at [4] as well: the other efficient choice $(2, 3)$ allows 13 to manipulate at [4] by reporting [3] (and 23 to manipulate at [3] by [4]!). Now at [5] the efficient coalition 13 must be chosen (by way of $(1, 3)$ or $(3, 1)$) or 13 could manipulate at [5] by $R' =$ [4].

Consider [6]: if $(2, 1)$ is selected, 12 manipulates at [5] by [6]; if $(2, 3)$ is selected, 13 manipulates at [6] by $R' =$ [4]. Thus $(3, 1)$ is selected. Now if $(2, 3)$ is selected at [7], 23 manipulates at [6] by [7]; therefore, by efficiency, $(2, 1)$ is chosen at [7]. Consider [8]: if $(1, 3)$ or $(2, 3)$ is selected, 12 manipulates at [8] by [7], thus $(2, 1)$ is chosen at [8] as well. Recall that $(1, 3)$ is chosen at [3]: therefore 13 manipulates at [8] by [3], and we have reached a contradiction.

Theorem 2:

We assume that $RW_{mw} = 1$ for all m, w, and thus $R = RM$. Clearly, under profiles where some women dislike some men, men' possibilities to manipulate can only diminish.

We fix two profiles R, R', respectively the true and the false profile, and two allocation matrices $Z \in \varphi(M, W, R)$, $Z' \in \varphi(M, W, R')$. We define the sets of losers, winners and indifferent agents in the manipulation at R by R' :

$$\begin{aligned} m &\in \mathcal{W} \text{ iff } u_m(Z'_m) = \sum_{R_m} z'_{mw} > u_m(Z_m); \\ m &\in \mathcal{I} \text{ iff } u_m(Z'_m) = u_m(Z_m); \\ m &\in \mathcal{L} \text{ iff } u_m(Z'_m) < u_m(Z_m) \end{aligned}$$

We assume $R_m = R'_m$ for all $m \in \mathcal{L}$ (the losers cannot be part of the deviating coalition) and prove by induction $\mathcal{W} = \emptyset$. This establishes groupstrategyproofness because R, R' are arbitrary.

We write α_k, T_k, W_k, and α'_l, T'_l, W'_l, for the sequences corresponding to R and R' respectively. We prove by induction the property $\mathcal{P}(k)$:

$$T_{1,\dots,k} \subseteq \mathcal{I} \cup \mathcal{L};\ z'_{mw} = 0 \text{ for } w \in R_{T_{1,\dots,k}} \text{ and } m \in (\mathcal{W} \cup \mathcal{I}) \cap T_{k+1,\dots,K}$$

We assume $\mathcal{P}(k-1)$ and prove $\mathcal{P}(k)$. In the case $k = 1$, $\mathcal{P}(0)$ is void. We assume $k \leq K - 1$, so that $u_m(Z_m) = \alpha_k$ for all $m \in T_k$. We set $T_k^- = T_k \cap \mathcal{L}$, $T_k^+ = T_k \cap (\mathcal{W} \cup \mathcal{I})$, either of which can be empty. For any $m' \in T_k^-$, $m \in T_k^+$, we have:

$$u'_{m'}(Z'_{m'}) = u_{m'}(Z'_{m'}) < u_{m'}(Z_{m'}) = \alpha_k = u_m(Z_m) \leq u_m(Z'_m) \leq u'_m(Z'_m) \tag{12}$$

where the first equality comes from $R_{m'} = R'_{m'}$ and the last inequality holds because the support of Z'_m is contained in R'_m. Therefore m' belongs to an "earlier" member of the partition T'_l—a set with a smaller index l—than m, implying that no object that m' likes (at $R_{m'} = R'_{m'}$) is assigned to m at $R' : z'_{mw} = 0$ for $w \in R_{m'}$. By the assumption $\mathcal{P}(k-1)$, Z' does not give to m any share of an object from $R_{T_{1,\dots,k-1}}$ either. Therefore:

$$\{w \in R_m \text{ and } z'_{mw} > 0\} \Rightarrow \{w \in R_{T_k} \cap W_{k-1} \setminus R_{T_k^-}\} \text{ for all } m \in T_k^+ \tag{13}$$

Now we define an allocation matrix Z^* restricted to $T_k, R_{T_k} \cap W_{k-1}$, where $Z \rightarrow p(Z)$ simply deletes the coordinates outside $R_{T_k} \cap W_{k-1}$:

$$\begin{aligned} Z^*_{m'} &= p(Z_{m'}) \text{ for } m' \in T_k^- \\ Z^*_m &= p(Z'_m) \text{ for } m \in T_k^+ \end{aligned}$$

By the definitions of T_k and W_{k-1}, $u_{m'}(Z^*_{m'}) = \alpha_k$ for all $m' \in T_k^-$, and by (17) $u_m(Z^*_m) = u_m(Z'_m) \geq \alpha_k$ for all $m \in T_k^+$. Next the support of any $Z^*_{m'}$ and that of any Z^*_m are disjoint: see (17). Therefore Z^* is feasible. By definition of T_k, all inequalities $u_m(Z^*_m) \geq \alpha_k$ must be equalities, which proves $T_k^+ \subseteq \mathcal{I}$, and the first statement in $\mathcal{P}(k)$.

Moreover Z^* must exhaust all objects of $R_{T_k} \cap W_{k-1}$. Among these, those in $R_{T_k} \cap W_{k-1} \setminus R_{T_k^-}$ are assigned in full by Z' to the agents of T_k^+: no fraction of those objects goes to anyone in $T_{k+1,\dots,K}$. To complete the proof of $\mathcal{P}(k)$ we must check that the objects in $R_{T_k^-}$ are not assigned at all to any agent n in $(\mathcal{W} \cup \mathcal{I}) \cap T_{k+1,\dots,K}$ by Z'. For such an agent n we have:

$$u'_n(Z'_n) \geq u_n(Z'_n) \geq u_n(Z_n) \geq \alpha_{k+1}$$

therefore (16) implies $u'_n(Z'_n) > u'_{m'}(Z'_{m'})$ for all $m' \in T_k^-$.

Thus m' appears in the sequence T'_l earlier than n, implying that all objects he likes (at $R_{m'} = R'_{m'}$) are allocated before n is served, i.e., n gets none of $R_{T_k^-}$.

The proof of $\mathcal{P}(K-1)$ is now complete. If $\alpha_K \leq 1$, then $u_m(Z_m) = \alpha_K$ for $m \in T_K$ and the above argument shows $\mathcal{P}(K)$. If $\alpha_K \geq 1$, then $u_m(Z_m) = 1$ implies $T_K \cap \mathcal{W} = \emptyset$. Thus $\mathcal{W}$ is empty in both cases.

Theorem 3:

The first statement is proven by means of the 4×4 example just before the theorem. To prove the second statement, fix an efficient welfarist solution f, and consider the same 4×4 example. We write $u = f(R)$ for the utility profile chosen by f. Without loss of generality we can assume $u_{m_2} \geq u_{m_3} \geq u_{m_4}$ and $v_{w_2} \geq v_{w_3} \geq v_{w_4}$; as m_1 and w_1 are overdemanded, $u_{m_1} = v_{w_1} = 1$. This implies $u_{m_3}, v_{w_3} \leq \frac{1}{2}$ and $u_{m_4}, v_{w_4} \leq \frac{1}{3}$.

Consider the same manipulation RM', RW' by the coalition $S = \{m_3, m_4, w_3, w_4\}$. By efficiency, $f(R') = (1,1,1,1)$. Let g be a solution projecting on f and such that $g(R')$ is the allocation matrix:

$$g(R') = \begin{matrix} 0 & 0 & \frac{7}{12} & \frac{5}{12} \\ 0 & 0 & \frac{5}{12} & \frac{7}{12} \\ \frac{7}{12} & \frac{5}{12} & 0 & 0 \\ \frac{5}{12} & \frac{7}{12} & 0 & 0 \end{matrix}$$

Then $Z' = g(R') \bullet R$ is:

$$Z' = \begin{matrix} 0 & 0 & \frac{7}{12} & \frac{5}{12} \\ 0 & 0 & 0 & 0 \\ \frac{7}{12} & 0 & 0 & 0 \\ \frac{5}{12} & 0 & 0 & 0 \end{matrix}$$

and the (true) utility profile of S after the misreport is $u'_{m_3} = v'_{w_3} = \frac{7}{12} > \frac{1}{2}$, $u'_{m_4} = v'_{w_4} = \frac{5}{12} > \frac{1}{3}$. Therefore g is not weakly groupstrategyproof, and neither is f.