On Representative Democracy

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Abstract

We study axioms which define "representative democracy" in an environment in which agents vote over a finite set of alternatives. We focus on a property that states that whether votes are aggregated directly or indirectly makes no difference. We call this property *representative consistency*. *Representative consistency* can also be understood as a "gerrymandering-proofness" requirement for voting rules. We characterize the class of rules satisfying *unanimity*, *anonymity*, and *representative consistency*. We call these rules "partial priority rules." A partial priority rule can be interpreted as a rule in which each agent can "veto" certain alternatives. We investigate the implications of imposing other axioms to the list specified above. We also study the partial priority rules in the context of specific economic models.

1 Introduction

Our objective here is to discuss an interpretation of "representative democracy" in a voting model. In so doing, we provide an axiomatic definition of the meaning of representative.

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In the context of binary social choice, the only voting rule which is *anonymous*, *neutral*, and *positively responsive* is majority rule (May's Theorem, [12]). Majority rule is not compatible with "representative democracy," in an important sense. The alternatives selected by this rule are generally not the same as the alternatives selected when agents are first divided into "districts," where each "district" of agents selects a "representative vote" for its members according to the rule.[1] Thus, in a representative democracy with single-member districts, the way in which districts are drawn can change the outcome of a vote. In common language, we say that "gerrymandering" is possible. Thus, majority rule leaves itself open to manipulation by whoever determines which agents vote in which district.

If one believes in representative democracy with single-member districts, a natural question is whether or not there *exist* voting rules which can never be manipulated in the sense described above. Our aim is to formally study this question, providing a workable and sufficiently general notion of a "representative alternative" for a group of agents by assuming that the "districts" into which agents may be grouped are not prespecified.

Society's collective goal is to select one of a *finite* set of social alternatives. Each agent in society submits a vote for one of the alternatives. An alternative is selected by aggregating agents' votes according to some rule. Our interest is in studying such rules. This model is a reduced form model, as preferences of agents are not specified.

When we say a voting rule is democratic, we mean that it i) selects any alternative which receives unanimous approval and ii) treats all agents identically. These requirements are captured by the properties *unanimity* and *anonymity*, respectively.

Our objective is to determine which democratic voting rules can be understood as "representative democracies" under a single-member district system. To make this objective precise, we specify notions of "direct democracy" and "indirect democracy." These two notions are discussed with respect to a given voting rule. In a "direct democracy," each member of society votes for an alternative, and an alternative is selected by directly aggregating these votes according to the rule. In an "indirect democracy," agents are partitioned into districts, and each agent votes for an alternative. Each *district* then selects an alternative according to the rule based on the votes in the district, resulting in a "representative vote." Each agent in society is treated as hav-

[1] For a study of these types of rules, see Fishburn [10].

ing voted for his district's "representative vote." The rule is then applied to the "representative votes," leading to a social alternative. A voting rule is a "representative democracy" if "direct democracy" and "indirect democracy" lead to the same chosen alternative, independently of the partitioning of agents. This equivalence of direct and indirect vote aggregation is logically independent from the notion of "democracy," so we give it an independent name: *representative consistency*.

Thus, *representative consistency* can be understood as a condition of "gerrymandering-proofness." If a voting rule is *representative consistent*, the selected social alternative is independent of how voting districts are drawn. Intuition suggests that *representative consistency* is a difficult condition to satisfy. Take the case of majority rule. Suppose some agent is in charge of deciding who votes in which district. There are two ways in practice that he may gerrymander. One is to draw districts so that the agents who will vote for his least preferred alternative are concentrated in as few districts as possible. The other way is to "spread" the agents who will vote for his least preferred alternative in as many districts as possible so they do not gain a majority in any district. In fact, with almost *any* voting rule one can think of, gerrymandering is sometimes possible. In a sense, there will almost always be situations in which one can draw districts so that some agents' votes are "diluted." However, there are exceptions to this rule.

The conditions *unanimity*, *anonymity*, and *representative consistency* are compatible, and characterize a class of rules that we believe has not been studied before. These rules are easiest to understand when there are only two alternatives. In this case, one of the alternatives is "special," and the rule chooses the "special" alternative unless all agents vote for the other alternative. Thus, each agent has a "veto" and can force society to choose the "special" alternative just by voting for it. Hence, no agents are "powerless" against gerrymandering.

These rules can be described in a different way. Think of the two alternatives as being *ordered*, where the "special" alternative comes first in the order. Among the alternatives receiving votes, the rule selects the one that comes first in the order. Our main contribution is to show that, for any finite number of alternatives, each "representative democracy" can be described by a *partial* order over the set of alternatives. If two alternatives are ranked according to this partial order, then we say that the two alternatives are prioritized. Of these two alternatives, the alternative which comes first according to the partial order has a higher priority. Such a rule selects the lowest priority alternative which has a priority (weakly) higher than all of the alternatives which receive votes. This alternative may or may not have been voted for by some of the agents. We call such a rule a "partial priority rule."

A typical example of a partial priority rule is found in jury trials. A jury must decide between the guilt or the innocence of a suspect. A suspect is considered innocent unless the jurors unanimously agree that she is guilty. The partial order corresponding to this partial priority rule ranks "guilty" after "innocent." Partial priority rules with three or more alternatives can be described by introducing the concept of a "hung jury." Here, if all agents agree on "guilty," the suspect is guilty. If all agents agree on "innocent," the suspect is innocent. If there is not unanimous agreement for either alternative; however, we say that there is a "hung jury." This corresponds to a partial priority rule which ranks both "guilty" and "innocent" after "hung jury," but leaves "guilty" and "innocent" incomparable.

The characterization of "representative democracy" confirms the intuition that *representative consistency* is very difficult to satisfy, and thus our result may be viewed in a negative light. Nevertheless, the class of partial priority rules is nonempty, and there is room to impose other interesting properties on rules. We investigate the compatibility of additional properties with our main axioms. For example, the *positive vote-share* condition requires that any selected alternative must receive at least one vote. When coupled with our other axioms, this condition is equivalent to the condition that the partial order corresponding to a partial priority rule is a linear order.

One advantage of our model is its generality. In fact, we assume no structure on the set of alternatives except that it is finite. However, in specific examples, restrictions on the set of alternatives are always warranted. To establish that our model is interesting in concrete economic environments, we study the partial priority rules in the context of spatial competition. Thus, we assume that the alternatives are exogenously ordered according to some linear order. For example, policies may be ordered from "left" to "right."

We discuss conditions relating to this exogenous order. *Betweenness* is the requirement that the selected alternative lie between the maximal and minimal votes according to the linear order. If it is implicitly understood that agents have single-peaked preferences over the alternatives and votes are taken to be the agents' peaks, *betweenness* is equivalent to *Pareto-efficiency*. *Vote monotonicity* states that if the votes of all agents move in a certain direction, then so should the selected alternative. *Vote monotonicity* can be understood as a simple *positive responsiveness* condition. Lastly, we study *strategy-proofness*, assuming that agents have preferences over the alternatives which are single-peaked. *Strategy-proofness* states that, in a strategic situation in which agents are required to announce an alternative, it is a dominant strategy for each agent to announce her peak. It turns out that when coupled with our other axioms, these axioms lead to three progressively restrictive classes of rules, which we characterize.

It is not necessarily the case that society will desire a single-member district system. In fact, often, democracies opt for *proportional representation*. We capture this idea by introducing lotteries into the model. We define an entire class of new rules in this model, which we call the *quasi-proportional rules*. The quasi-proportional rules are not partial priority rules, but they satisfy the principles which we take to define representative democracy.

Section 2 discusses the model and the main axioms. Section 3 proves our main theorem and several simple corollaries. Section 4 discusses our theorem in a spatial model. Section 5 is devoted to a discussion of proportional representation. Section 6 studies a natural concept of "duals" to partial priority rules. Section 7 concludes.

2 The model

2.1 Preliminaries

Let $\mathbb{N}$ be a set of "potential agents." Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. Let X be a finite set. Elements of X are "alternatives," to be interpreted as policies, candidates, etc. Each agent submits a "vote," or an element of X. A **rule** is a function $f : \bigcup_{N\in\mathcal{N}} X^N \to X$. Thus, a rule specifies a "representative" alternative for any population of agents and any list of votes these agents may submit. This "representative" alternative should be interpreted as the social alternative selected for this particular population of agents and votes.

2.2 Democracy

The first property we discuss in this section states that a rule should respect the "will of the people" when this "will" is unambiguous. It is an extremely weak axiom.

For all $N \in \mathcal{N}$, $x \in X$, let x^N be a vector of x's in X^N. For all $N \in \mathcal{N}$, all $x \in X^N$, and all $M \subset N$, let x_M be the projection of x onto X^M.

Unanimity: For all $N \in \mathcal{N}$ and all $x \in X$, $f\left(x^N\right) = x$.

The next axiom states that a rule should be ignorant of the names of agents. This is a mild principle of equality, capturing the "one-man, one-vote" principle.[2]

Anonymity: For all $N, N' \in \mathcal{N}$ such that $|N| = |N'|$, for all bijections $\sigma : N \to N'$, and all $x \in X^N$, $f(x) = f(\sigma \circ x)$.

One would expect most democratic voting rules to satisfy these two properties.

2.3 Representative consistency

We here introduce *representative consistency*. Informally, the axiom states that for any population of agents and any collection of votes, we may without loss of generality partition the set of agents into "districts," find the choice for each district, and then treat each district as if each agent in the district had voted for the outcome selected for the district.[3]

Representative consistency: For all $N \in \mathcal{N}$, all $M \subset N$, and all $x \in X^N$, $f(x) = f\left(f(x_M)^M, x_{N \setminus M}\right)$.

Under the *unanimity* principle, *representative consistency* is equivalent to the statement that for all $N \in \mathcal{N}$, all partitions $\{N_1, ..., N_K\}$ of N, and all $x \in X^N$, $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_K})^{N_K}\right)$.

[2] This principle is violated in elections for the United States Senate, for example. In this case, each *state* is treated equally, regardless of population. *Anonymity* is closer to the principle underlying elections for the United States House of Representatives.

[3] *Representative consistency* bears resemblance to the *path-independence* condition for choice functions, first formalized by Plott [15]. See also Asah and Sanver [3].

3 Results

3.1 Preliminaries

Any *anonymous* rule can be specified without reference to the specific names of agents. In the proofs of results in which *anonymity* plays a role, we often exploit this fact without mention, disregarding the variable N.

The following axiom can be interpreted as meaning that only the *proportions* of votes received for each alternative are used in determining the social alternative.

Let m be an integer, let $N \in \mathcal{N}$, and let $x \in X^N$. Let $N' \in \mathcal{N}$ be such that $|N'| = m\,|N|$. A vector $x' \in X^{N'}$ is an **m-replica of x** if there exists a partition of N' into m sets of size $|N|$, say $\{N_1, ..., N_m\}$ such that for all N_i, there exists a bijection $\sigma_i : N \rightarrow N_i$ for which $x'_{N_i} = \sigma_i \circ x$. For all $N \in \mathcal{N}$, $x \in X^N$, $m \in \mathbb{N}$, x^m denotes an m-replica of x.[4]

Replication invariance: Let m be an integer. Let $N \in \mathcal{N}$ and let $x \in X^N$. Let x' be an m-replica of x. Then $f(x') = f(x)$.

The following trivial observation is useful:

Lemma 1: If a rule satisfies *unanimity*, *anonymity*, and *representative consistency*, then it satisfies *replication invariance.*

Proof: Let $N \in \mathcal{N}$ and let $x \in X^N$. Let x' be an m-replica of x. Then by definition of x', $f(x') = f\left(\underbrace{x, ..., x}_{m}\right)$. By *representative consistency*, $f(x') = f\left(\underbrace{f(x), ..., f(x)}_{m|N|}\right)$. By *unanimity*, $f\left(\underbrace{f(x), ..., f(x)}_{m|N|}\right) = f(x)$. Conclude that $f(x') = f(x)$.■

Our main result is a characterization of the class of rules satisfying *unanimity*, *anonymity*, and *representative consistency.* The following example illustrates this class in the two-alternative case. The description of the general class follows.

[4] An m-replica of a given vector is not uniquely defined.

Example 1: Let $X \equiv \{y, z\}$. Then there are only two rules satisfying *unanimity*, *anonymity*, and *representative consistency*. One such rule selects $f(x) = y$ unless all agents vote for z. The other such rule always selects z, unless all agents vote for y.

We now generalize the preceding example to the case of an arbitrary (finite) number of alternatives. A **partial order** is a binary relation which is *i*) reflexive, *ii*) transitive, and *iii*) anti-symmetric. A partial order need not be complete.[5] For a partial order $\preceq$, $\prec$ denotes the asymmetric part.[6] A pair $(Y, \preceq)$ is a **partially ordered set** if Y is a set and $\preceq$ is a partial order on Y. For all $x, y \in Y$, where $(Y, \preceq)$ is a partially ordered set, $x \wedge y \in Y$ is the **meet of x and y** if it is the unique greatest lower bound for x and y according to $\preceq$. Generally, two elements $x, y \in Y$ need not possess a meet. A **meet-semilattice** is a partially ordered set such that any pair of elements possesses a meet. For more on these definitions, see Birkhoff [4].

Figure 1 displays a typical meet-semilattice. Each meet-semilattice can be pictured as a directed graph with a unique root. Here, a is the root of the tree. An alternative precedes another alternative in terms of the partial order if one can construct a path emanating from the first alternative down the graph to the other alternative. Thus, in Figure 1, $a \preceq c$ as there exists a path going down the graph starting at a and ending at c. Moreover, $a \preceq e$. However, the alternatives b and c are unrelated according to $\preceq$, as there exists no directed path between them. The meet of two alternatives is easily found as the lowest common predecessor of the two alternatives. Thus, $b \wedge e = a$ in the example, whereas $c \wedge e = c$.

A rule f is a **partial priority rule** if there exists a partial order $\preceq$ over X so that $(X, \preceq)$ is a meet-semilattice, and for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) = \bigwedge_{i \in N} x_i$.[7,8]

[5] Reflexive: For all $x \in X$, $x \succeq x$.
Transitive: For all $x, y, z \in X$, if $x \succeq y$ and $y \succeq z$. then $x \succeq z$.
Anti-symmetric: For all $x, y \in X$, if $x \succeq y$ and $y \succeq x$, then $x = y$.
Complete: For all $x, y \in X$, either $x \succeq y$ or $y \succeq x$.

[6] That is, $x \succ y$ if and only if $x \succeq y$ is true and $y \succeq x$ is false.

[7] The notation $\bigwedge_{i \in N} x_i$ refers to the meet of the set of elements $\{x_i\}_{i \in N}$, which exists for any meet-semilattice.

[8] There is no significance to the fact that partial priority rules select the *meet* of the alternatives receiving a positive number of votes. Every partial priority rule can equivalently be described through the use of a partially ordered set in which every pair of elements has a **join** (unique least upper bound), where the rule selects the join of the alternatives

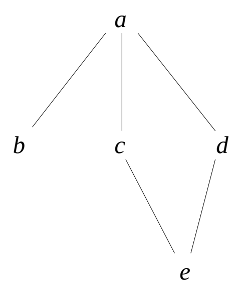

Figure 1: A meet-semilattice

Thus, with any partial priority rule, the *proportion* of votes received for each alternative does not matter. To select an alternative, one only needs to determine which alternatives receive a positive number of votes.

3.2 Interpreting partial priority rules

Let f be a partial priority rule with partial order $\preceq$. Let $x, y \in X$ be ordered by $\preceq$, so that $x \preceq y$. We may interpret this statement as reading "x has a higher priority than y." In other words, x is "more important" than y. This interpretation is possible because y can be selected in place of x only if all agents vote for y in lieu of x. Thus, each agent can "veto" y by voting for x.

However, not all alternatives need be related by $\preceq$. Suppose in fact that x and y are unrelated according to $\preceq$. Then we may interpret this as meaning that neither x nor y can be deemed a more important alternative. In this case, some "compromise" must be struck between the two. The compromise between x and y is decided using the partial priority. It is the lowest priority alternative which has priority higher than both x and y.

3.3 The main theorem

The main theorem states that a voting rule is a "representative democracy" if and only if it is a partial priority rule. The proof of this theorem is constructive and is divided into five steps. Given is a rule satisfying the

receiving a positive number of votes.

axioms describing "representative democracy." In the first step, an order on the alternatives is defined from the rule, and it is verified that this order is a partial order. This construction is always valid–even if X is not finite. The most significant obstacle is in showing that this partial order forms a meet-semilattice when coupled with X. To prove this, it is necessary that X be finite. We establish this fact in Steps 2, 3, and 4. Step 2 shows that the rule is "monotonic" in the partial order. Step 3 shows that for two-agent environments, the selected social alternative lies below the two alternatives voted for in terms of the partial order. Step 4 uses Steps 2 and 3 to show that the partial order is actually a meet-semilattice and that for two-agent environments, the rule selects the meet of the alternatives which are voted for. Step 5 establishes that the rule always selects the meet of the alternatives which receive votes.

Theorem 1: A rule satisfies *unanimity*, *anonymity*, and *representative consistency* if and only if it is a partial priority rule.

Proof: If f is a partial priority rule, then it clearly satisfies *unanimity*, *anonymity*, and *representative consistency*. For the other direction, let f be a rule which satisfies *unanimity*, *anonymity*, and *representative consistency*.

Step 1: Construction of the order and verification of its properties

For all x,$y \in X$, define $x \preceq y$ if $f(x, y) = x$. Let $x \in X$. By *unanimity*, $f(x, x) = x$, so that $x \preceq x$. Thus, $\preceq$ is reflexive. Let $x, y \in X$. Suppose $x \preceq y$ and $y \preceq x$. Then by definition, $f(x, y) = x$ and $f(x, y) = y$. Thus, $x = y$ and $\preceq$ is anti-symmetric.

We claim that $\preceq$ is transitive. Let $x, y, z \in X$. Suppose $x \preceq y$ and $y \preceq z$. By *replication invariance*,

$$f(x, z) = f(x, x, z, z).$$

As $x \preceq y$, by definition,

$$f(x, x, z, z) = f(f(x, y), f(x, y), z, z).$$

By *representative consistency*,

$$f(f(x, y), f(x, y), z, z) = f(x, y, z, z).$$

By *representative consistency*,

$$f(x,y,z,z) = f(x, f(y,z), f(y,z), z).$$

As $y \preceq z$, by definition,

$$f(x, f(y,z), f(y,z), z) = f(x,y,y,z).$$

By *representative consistency*,

$$f(x,y,y,z) = f(x, y, f(y,z), f(y,z)).$$

As $y \preceq z$, by definition,

$$f(x, y, f(y,z), f(y,z)) = f(x,y,y,y).$$

By *representative consistency*,

$$f(x,y,y,y) = f(f(x,y), f(x,y), y, y).$$

As $x \preceq y$, by definition,

$$f(f(x,y), f(x,y), y, y) = f(x,x,y,y).$$

By *replication invariance*,

$$f(x,x,y,y) = f(x,y).$$

As $x \preceq y$, by definition,

$$f(x,y) = x.$$

Thus, $f(x,z) = x$, so that by definition, $x \preceq z$. Therefore, $\preceq$ is transitive.

Step 2: The rule is monotonic in the partial order

We claim that for all $x,y,z \in X$, if $x \preceq y$, then $f(x,z) \preceq f(y,z)$. Thus, let $x,y,z \in X$ and suppose $x \preceq y$. By *replication invariance* and *representative consistency*,

$$f(f(x,z), f(y,z)) = f(x,y,z,z).$$

By *representative consistency* and *replication invariance*,

$$f(x,y,z,z) = f(f(x,y), f(x,y), z, z) = f(f(x,y), z).$$

As $x \preceq y$, by definition,

$$f\left(f\left(x,y\right),z\right) = f\left(x,z\right).$$

Therefore,

$$f\left(f\left(x,z\right),f\left(y,z\right)\right) = f\left(x,z\right),$$

so that, by definition, $f\left(x,z\right) \preceq f\left(y,z\right)$.

Step 3: The rule maps every two elements into an element which precedes them in the partial order

We claim that for all $x, y \in X$, $f\left(x,y\right) \preceq x$. Let $x, y \in X$. We define an auxiliary function F derived from f. Let $P \equiv \mathbb{Q} \cap [0,1]$ (this will be the domain of F).

For all $p \in P$, p can be written as $p \equiv \frac{m(p)}{n(p)}$, where $m\left(p\right), n\left(p\right) \in \mathbb{N}$.[9] For all $p \in P$, define $F\left(p\right) = f\left(x^{m(p)}, y^{n(p)-m(p)}\right)$. By *replication invariance*, which follows from Lemma 1, F is well-defined.

We claim that for all $p, p' \in P$ and $z \in X$ such that $p < p'$, if $F\left(p\right) = z$ and $F\left(p'\right) = z$, then for all $p'' \in \left(p, p'\right) \cap \mathbb{Q}$, $F\left(p''\right) = z$.

Thus, let $p, p' \in P$ such that $F\left(p\right) = F\left(p'\right) = z$. Let $p'' \in \left(p, p'\right) \cap \mathbb{Q}$. Then there exists $\alpha \in \mathbb{Q}$ such that $p'' = \alpha p + \left(1-\alpha\right)p'$. Since $\alpha \in \mathbb{Q}$, $\alpha = \frac{m(\alpha)}{n(\alpha)}$ for $m\left(\alpha\right), n\left(\alpha\right) \in \mathbb{N}$. Thus $p'' = \frac{m(\alpha)}{n(\alpha)}\frac{m(p)}{n(p)} + \frac{(n(\alpha)-m(\alpha))}{n(\alpha)}\frac{m(p')}{n(p')}$. We can write $p'' = \frac{m(\alpha)m(p)n(p')+(n(\alpha)-m(\alpha))m(p')n(p)}{n(\alpha)n(p)n(p')}$. Thus, by appropriately arranging terms, $F\left(p''\right)$ is equal to:

$$f\left(\left(x^{m(p)}, y^{n(p)-m(p)}\right)^{n(p')m(\alpha)}, \left(x^{m(p')}, y^{n(p')-m(p')}\right)^{n(p)(n(\alpha)-m(\alpha))}\right) \tag{1}$$

By assumption, $f\left(x^{m(p)}, y^{n(p)-m(p)}\right) = f\left(x^{m(p')}, y^{n(p')-m(p')}\right) = z$, so that by *representative consistency*, (1) is equal to $f\left(z^{n(p)n(p')m(\alpha)}, z^{n(p')n(p)(n(\alpha)-m(\alpha))}\right)$. By *unanimity*, $f\left(z^{n(p)n(p')m(\alpha)}, z^{n(p')n(p)(n(\alpha)-m(\alpha))}\right) = z$. Therefore, $F\left(p''\right) = z$. Hence, F is "convex."[10]

We next show that F is constant on $(0,1)$. Since X is finite and $(0,1) \cap P$ is countable, there exists $z \in X$ and a sequence $\{p^n\}_{n=1}^{\infty} \subset (0,1) \cap P$ such that $p^n \to 1$, for which $F\left(p^n\right) = z$ for all n. Conclude by the previous

[9] This representation of p is obviously not unique.

[10] In the proof of an important theorem, Young [18] uses a similar technique of defining an auxiliary function based on proportions, and showing it is "convex."

paragraph (using the "convexity" of F), that there exists $p^* < 1$ such that for all $p \in (p^*, 1) \cap \mathbb{Q}$, $F(p) = z$.

We claim that for all $p, q \in P$, $\alpha \in \mathbb{Q} \cap (0,1)$, if $F(p) = F(q)$, then $F(\alpha p) = F(\alpha q)$. Thus, let p, q, α satisfy these hypotheses and suppose $F(p) = F(q) = z$. Write $p = \frac{m(p)}{n(p)}$, $q = \frac{m(q)}{n(q)}$, and $\alpha = \frac{m(\alpha)}{n(\alpha)}$, where $m(p), n(p), m(q), n(q), m(\alpha)$, and $n(\alpha) \in \mathbb{N}$. By definition, $F(p) = f\left(x^{m(p)}, y^{n(p)-m(p)}\right)$ and $F(q) = f\left(x^{m(q)}, y^{n(q)-m(q)}\right)$. By definition,

$$F(\alpha p) = f\left(\left(x^{m(p)}, y^{n(p)-m(p)}\right)^{m(\alpha)}, y^{n(p)(n(\alpha)-m(\alpha))}\right) \tag{2}$$

and

$$F(\alpha q) = f\left(\left(x^{m(q)}, y^{n(q)-m(q)}\right)^{m(\alpha)}, y^{n(q)(n(\alpha)-m(\alpha))}\right). \tag{3}$$

By *replication invariance*, we obtain (by replicating the inside of (2) $n(q)$ times)

$$F(\alpha p) = f\left(\left(x^{m(p)}, y^{n(p)-m(p)}\right)^{n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right)$$

and (by replicating the inside of (3) $n(p)$ times)

$$F(\alpha q) = f\left(\left(x^{m(q)}, y^{n(q)-m(q)}\right)^{n(p)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right).$$

By *representative consistency*,

$$F(\alpha p) = f\left(f\left(x^{m(p)}, y^{n(p)-n(p)}\right)^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right)$$

and

$$F(\alpha q) = f\left(f\left(x^{m(q)}, y^{n(q)-m(q)}\right)^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right).$$

As $f\left(x^{m(p)}, y^{n(p)-m(p)}\right) = f\left(x^{m(q)}, y^{n(q)-m(q)}\right) = z$,

$$F(\alpha p) = f\left(z^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right)$$

and

$$F(\alpha q) = f\left(z^{n(p)n(q)m(\alpha)}, y^{n(p)n(q)(n(\alpha)-m(\alpha))}\right).$$

Thus $F(\alpha p) = F(\alpha q)$.

Let $q^* \in P$ satisfy $q^* \in (p^*, 1)$. Then $F(q^*) = z$ and $F(p^*) = z$. Moreover, there exists $\alpha^* \in (0,1) \cap \mathbb{Q}$ such that $p^* = \alpha^* q^*$. For all $k = 1, ..., \infty$, let $q_k \equiv (\alpha^*)^k q^*$. As $\alpha^* < 1$, $q_k \to 0$, and for all k, $q_k > 0$.

We claim that for all $k = 1, ..., \infty$, $F(q_k) = z$. By definition, $F(q_0) = F(q_1) = z$. Proceed by induction. Suppose that for all $k < K$, $F(q_k) = z$. We show that $F(q_K) = z$. Thus, $F(q_K) = F(\alpha^* q_{K-1})$ and $F(q_{K-1}) = F(q_{K-2}) = z$, so that by the result in the previous paragraph, $F(\alpha^* q_{K-1}) = F(\alpha^* q_{K-2})$. But by definition, $F(\alpha^* q_{K-2}) = F(q_{K-1}) = z$, so that $F(q_K) = F(\alpha^* q_{K-1}) = z$. Thus, for all k, $F(q_k) = z$. By the "convexity" of F, we conclude that F is constant and equal to z on $(0,1)$.

Therefore, $f(f(x,y), x) = f(x,x,x,y) = F(3/4) = F(1/2) = f(x,y)$, so that by definition, $f(x,y) \preceq x$.

Step 4: For two agents, the rule chooses the meet of the alternatives receiving votes

Let $x, y \in X$. We claim that $f(x,y)$ is the unique greatest lower bound for x and y according to $\preceq$. By Step 3, $f(x,y)$ is a lower bound for x and y. Suppose there exists another lower bound for x and y, say, z. Then, by definition, $z \preceq x$ and $z \preceq y$. By Step 2, $f(x,z) \preceq f(x,y)$. As $z \preceq x$, by definition, $f(x,z) = z$, so that $z \preceq f(x,y)$. Thus, by the anti-symmetry of $\preceq$, $f(x,y)$ is the unique greatest lower bound for x and y. Therefore, $f(x,y) = x \wedge y$. Thus, $(X, \preceq)$ is a meet-semilattice. Therefore, for all $N \in \mathcal{N}$ and all $x \in X^N$, $\bigwedge_{i \in N} x_i$ is well-defined (as $\wedge$ is associative–see Birkhoff [4], p. 8-10).

Step 5: Extending the result to arbitrary finite numbers of agents

We establish that for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) = \bigwedge_{i \in N} x_i$. The proof proceeds by induction on the cardinality of the set of agents N. Suppose that $|N| = 1$. By *unanimity*, $f(x) = x = \bigwedge_{i \in N} x_i$. Suppose that $|N| = 2$. Then by Step 4, $f(x) = \bigwedge_{i \in N} x_i$.

Let K be an integer such that $K > 2$. Suppose that for all $N \in \mathcal{N}$ such that $|N| < K$, for all $x \in X^N$, $f(x) = \bigwedge_{i \in N} x_i$. Let $N^* \in \mathcal{N}$ such that $|N^*| = K$. Without loss of generality, write $N^* \equiv \{1, ..., K\}$. Let $x \in X^{N^*}$. By *representative consistency*,

$$f(x_1, ..., x_K) = f(f(x_1, ..., x_{K-1}), ..., f(x_1, ..., x_{K-1}), x_K).$$

By the induction hypothesis,

$$f\left(f\left(x_1,...,x_{K-1}\right),...,f\left(x_1,...,x_{K-1}\right),x_K\right)=f\left(\bigwedge_{i=1}^{K-1}x_i,...,\bigwedge_{i=1}^{K-1}x_i,x_K\right).$$

By *representative consistency* (using the fact that $K>3$),

$$f\left(\bigwedge_{i=1}^{K-1}x_i,...,\bigwedge_{i=1}^{K-1}x_i,x_K\right)=f\left(\begin{array}{c}\bigwedge_{i=1}^{K-1}x_i,f\left(\bigwedge_{i=1}^{K-1}x_i,...,\bigwedge_{i=1}^{K-1}x_i,x_K\right),...,\\ f\left(\bigwedge_{i=1}^{K-1}x_i,...,\bigwedge_{i=1}^{K-1}x_i,x_K\right)\end{array}\right).$$

By the induction hypothesis, the previous expression is equal to

$$f\left(\bigwedge_{i=1}^{K-1}x_i,\bigwedge_{i=1}^{K}x_i,...,\bigwedge_{i=1}^{K}x_i\right).$$

By *representative consistency*,

$$f\left(\bigwedge_{i=1}^{K-1}x_i,\bigwedge_{i=1}^{K}x_i,...,\bigwedge_{i=1}^{K}x_i\right)=f\left(\begin{array}{c}f\left(\bigwedge_{i=1}^{K-1}x_i,\bigwedge_{i=1}^{K}x_i\right),f\left(\bigwedge_{i=1}^{K-1}x_i,\bigwedge_{i=1}^{K}x_i\right),\\ \bigwedge_{i=1}^{K}x_i,...,\bigwedge_{i=1}^{K}x_i\end{array}\right).$$

By the induction hypothesis, the previous expression is equal to

$$f\left(\bigwedge_{i=1}^{K}x_i,...,\bigwedge_{i=1}^{K}x_i\right).$$

By *unanimity*, $f\left(\bigwedge_{i=1}^{K}x_i,...,\bigwedge_{i=1}^{K}x_i\right)=\bigwedge_{i=1}^{K}x_i$. Therefore, our induction hypothesis is true. ■

The following is a natural example of a partial priority rule.

Example 2: Sets of alternatives Let A be a finite set. We study the power set of A. Thus, 2^A is naturally ordered by set inclusion, and it is clear that $\left(2^A,\subset\right)$ forms a meet-semilattice, where for all $B,C\in 2^A$, $B\wedge C=B\cap C$. Define f so that, for all $N\in\mathcal{N}$ and all $B\in\left(2^A\right)^N$, $f\left(\left(B_i\right)_{i\in N}\right)\equiv\bigcap_{i\in N}B_i$. Thus, f selects all "alternatives" which are voted for by all agents.

The partial priority rules satisfy the three axioms no matter what the cardinality of X is. But in order for them to be the *only* rules satisfying the three axioms, it is necessary that X be finite. Here is an example of a set X which is countably infinite, and a rule f on X satisfying the three axioms which is *not* a partial priority rule.

Example 3: A countably infinite set of alternatives Let X be countably infinite. Let $r : X \to \mathbb{Q}$ be a bijection (such a bijection exists as X and $\mathbb{Q}$ have the same cardinality). Define $f : \bigcup_{N \in \mathcal{N}} X^N \to X$ as

$$f\left((x_i)_{i\in N}\right) \equiv r\left(\frac{\sum_{i\in N} r^{-1}(x_i)}{|N|}\right).$$

It is simple to verify that f satisfies the three axioms listed in the theorem. To see that f is not a partial priority rule, let N be a two-agent set and N' be a three agent set. Let $x, y \in X$ such that $x \neq y$. Then $(x, y) \in X^N$ and $(x, x, y) \in X^{N'}$. Suppose f is a partial priority rule. Then $f(x, y) = x \wedge y = (x \wedge x) \wedge y = x \wedge x \wedge y = f(x, x, y)$. By definition, $f(x, y) = r\left(\frac{r^{-1}(x)+r^{-1}(y)}{2}\right)$ and $f(x, x, y) = r\left(\frac{2r^{-1}(x)+r^{-1}(y)}{3}\right)$. As r is a bijection, conclude $\frac{r^{-1}(x)+r^{-1}(y)}{2} = \frac{2r^{-1}(x)+r^{-1}(y)}{3}$. Thus, $r^{-1}(x) = r^{-1}(y)$. As r is a bijection, $x = y$. But we supposed $x \neq y$, a contradiction. A partial order $\preceq$ can be defined from f as in the proof of Theorem 1, Step 1. However, it is immediate that if $x \neq y$, then $f(x, y) \notin \{x, y\}$, and thus $\preceq$ is the trivial partial order in which all alternatives are only comparable to themselves. Thus, $(X, \preceq)$ does not form a meet-semilattice.

We do not know what is the most general class of rules satisfying the three axioms for arbitrary sets X. However, there is much related work in the functional equations literature [1].

3.4 Positive vote-share and priority orderings

A partial priority rule cannot generally be interpreted as prioritizing all alternatives. However, there are some partial priority rules which prioritize all alternatives. The following axiom states that an alternative cannot be selected without receiving any votes. It is natural to impose when X is interpreted as a set of candidates, for example.

Positive vote-share: For all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \in \{x_i\}_{i \in N}$.

The following corollary states that partial priority rules which satisfy *positive vote-share* are exactly those partial priority rules which prioritize all alternatives.[11]

Corollary 1: A rule f satisfies *unanimity*, *anonymity*, *representative consistency*, and *positive vote-share* if and only if there exists a linear order $\preceq$ over X such that for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \equiv \min_{\preceq} \{x_i\}_{i \in N}$.

Proof: It is simple to verify that the axioms are satisfied by any such rule.

For the other direction, let $\preceq$ be the partial order constructed from f in Theorem 1. For all $x, y \in X$, *positive vote-share* implies that $f(x, y) \in \{x, y\}$. By definition of $\preceq$, either $x \preceq y$ or $y \preceq x$. Hence $\preceq$ is complete, so that it is a linear order. ■

The preceding corollary can be proved directly without referring to Theorem 1. In fact, a crucial step in the proof of Theorem 1 relies on the fact that X is finite. However, the preceding corollary can be proved even without the finiteness of X. The reasoning is simple. In the proof of the main theorem, we define the partial order over X by $x \preceq y$ if $f(x, y) = x$. However, we had to determine that $(X, \preceq)$ is a meet-semilattice (Step 2, Step 3, and Step 4). When *positive vote-share* is satisfied, $(X, \preceq)$ is easily seen to be a meet-semilattice, as $\preceq$ is complete.

3.5 Representative preference

Let A be a finite set, and let X be the set of linear orders over A, interpreted as **preference relations**. A typical preference relation is written as $R \in X$.

The goal of this section is to discuss social welfare functions which satisfy *representative consistency*. Thus, given a set of agents with preference relations, we may define a "representative preference."[12]

The added structure of this section leads us naturally to the concept of Pareto efficiency.

[11] A **linear order** is any binary relation which is *i*) complete, *ii*) transitive, and *iii*) anti-symmetric.

[12] This notion is not related to the notion of representative agent of consumer theory.

Efficiency: For all $N \in \mathcal{N}$, all $R \in X^N$, and all $a, b \in A$, if for all $i \in N$, aR_ib, then $af(R)b$.

Efficiency is standard. Moreover, *efficiency* obviously implies *unanimity*. Thus, any rule satisfying *efficiency*, *anonymity*, and *representative consistency* is a partial priority rule. It is natural to ask how much more structure is implied on partial priority rules by *efficiency*.

The following corollary is simple and is basically a reshuffling of the definitions. For all $a, b \in A$, let $X_{a,b} \equiv \{R \in X : aRb\}$.

Corollary 2: A partial priority rule with partial order $\preceq$ satisfies *efficiency* if and only if for all $a, b \in A$, the restriction of $\preceq$ to $X_{a,b}$ is a meet-semilattice.

Proof: Let f be a partial priority rule satisfying *efficiency*. Let $a, b \in A$ and let $R, R' \in X_{a,b}$. Thus, aRb and $aR'b$. By *efficiency*, $a(R \wedge R')b$. Thus, $R \wedge R' \in X_{a,b}$ and $\left(X_{a,b}, \preceq|_{X_{a,b}}\right)$ is a meet-semilattice.

Next, suppose that f is a partial priority rule such that for all $a, b \in A$, $\left(X_{a,b}, \preceq|_{X_{a,b}}\right)$ is a meet-semilattice. Let $a, b \in A$, $N \in \mathcal{N}$ and $R \in X^N$ such that for all $i \in N$, aR_ib. Then for all $i \in N$, $R_i \in X_{a,b}$. By definition, $f(R) = \bigwedge_{i \in N} R_i$, and by assumption, $\bigwedge_{i \in N} R_i \in X_{a,b}$. Therefore, $a\left(\bigwedge_{i \in N} R_i\right)b$, so that $af(R)b$. Thus, f satisfies *efficiency*. ■

4 The spatial model

As noted, the partial order corresponding to a partial priority rule is interpreted as a *priority* of alternatives and not as a way of relating the similarity of alternatives. However, there is often a natural, exogenous way of ordering the set of alternatives. For example, candidates in elections are often viewed as being ordered from left to right. Tax policies can be ordered by monetary amounts. In this section, we focus on sets of alternatives possessing such an exogenous order and conditions of rules relating to such orders.

In the remainder of this section, we suppose X is exogenously endowed with a linear order $\leq^*$. We interpret $\leq^*$ as ordering the alternatives by some attribute in a fashion agreed upon by all potential agents. Thus, for all $x, y, z \in X$, $x \leq^* y \leq^* z$ is to be read as "y is more similar to x than z is" in terms of the attribute.

The analysis of this section leads to three progressively restrictive classes of rules.

4.1 Formal notions of compromise

The first condition states that the selected social alternative should be a "compromise" amongst the agents' votes, in the weak sense that it lies between the maximal and minimal votes according to $\leq^*$. As mentioned in the Introduction, if agents possess single-peaked preferences over X (to be defined formally below), *betweenness* is equivalent to requiring that a rule select *Pareto-efficient* alternatives.

Betweenness: For all $N \in \mathcal{N}$ and all $x \in X^N$, $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* f(x) \leq^* \max_{\leq^*} \{x_i\}_{i \in N}$.

The next condition is a weak monotonicity condition. If all agents' votes move in a certain direction, so should the selected social alternative. This is a simple condition reflecting the fact that the selected social alternative is "representative" of the agents' votes.

Vote monotonicity: For all $N \in \mathcal{N}$ and all $x, y \in X^N$, if for all $i \in N$, $x_i \leq^* y_i$, then $f(x) \leq^* f(y)$.

The following lemmas will be useful.

Lemma 2: If a rule satisfies *betweenness*, then it satisfies *unanimity*.

Proof: Let f be a rule that satisfies *betweenness*. Let $N \in \mathcal{N}$ and let $x \in X$. Then $\min_{\leq^*} \left\{x_i^N\right\}_{i \in N} = x$ and $\max_{\leq^*} \left\{x_i^N\right\}_{i \in N} = x$. By *betweenness*, $x \leq^* f\left(x^N\right) \leq^* x$. As $\leq^*$ is anti-symmetric, $f\left(x^N\right) = x$. Therefore, f satisfies *unanimity*. ■

Lemma 3: If a rule satisfies *unanimity* and *vote monotonicity*, then it satisfies *betweenness*.

Proof: Let f be a rule that satisfies *unanimity* and *vote monotonicity*. Let $N \in \mathcal{N}$ and let $x \in X^N$. Then for all $i \in N$, $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* x_i \leq^* \max_{\leq^*} \{x_i\}_{i \in N}$. By *unanimity*, $f\left(\left(\min_{\leq^*} \{x_i\}_{i \in N}\right)^N\right) = \min_{\leq^*} \{x_i\}_{i \in N}$ and $f\left(\left(\max_{\leq^*} \{x_i\}_{i \in N}\right)^N\right) = \max_{\leq^*} \{x_i\}_{i \in N}$. Thus, by *vote monotonicity*, $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* f(x)$ and $f(x) \leq^* \max_{\leq^*} \{x_i\}_{i \in N}$. Thus f satisfies *betweenness*. ■

4.2 Results on compromise

We begin this section by defining two subclasses of the partial priority rules, which depend on the linear order $\leq^*$.

Say that a rule f is an **interval partial priority rule** if it is a partial priority rule and for all $x, y, z \in X$, if $y, z \geq^* x$, then $y \wedge z \geq^* x$, and if $y, z \leq^* x$, then $y \wedge z \leq^* x$. Thus, a partial priority rule is an interval partial priority rule if the weak lower and upper contour sets of any alternative according to $\leq^*$ are themselves meet-semilattices under the partial order induced by $\preceq$. The term "interval" refers to the fact that restricted to any interval according to $\leq^*$, the rule is still a partial priority rule. Say that a rule f is a **separating partial priority rule** if it is an interval partial priority rule such that for all $x, y, z \in X$, if $x \leq^* y \leq^* z$ and $x, z \succeq y$, then $x \wedge z = y$. Thus, an interval partial priority rule is a separating partial priority rule if the weak lower and upper contour sets of any alternative according to $\leq^*$ never "intersect" at any element greater than the alternative itself. The term "separating" refers to the fact that these contour sets never intersect.

Figure 2 displays the meet-semilattice corresponding to a typical interval partial priority rule under the presumption that $a \leq^* b \leq^* c \leq^* d \leq^* e$. The general structure of such rules can be grasped from this diagram. The main characteristic of such a rule is that there are at most two branches emanating from any alternative. If there are two branches, one must branch into the upper contour set of the alternative, and the other must branch into the lower contour set (as in the diagram, the two branches emanating from b branch into the two different contour sets). These two branches are allowed to rejoin. It is clear that the meet-semilattice in Figure 2 does not correspond to a separating partial priority rule, as $a \leq^* b \leq^* e$ and $a,e \succeq b$, yet $a \wedge e = a$.

Figure 3 is an example of a typical separating partial priority rule. The general structure of separating partial priority rules can be understood from this diagram. Note that the alternatives a, b, c, d, and e are ordered from "left" to "right." Moreover, at most two branches emanate from any given alternative. The last important characteristic is that for any alternative, one can draw an imaginary vertical line branching down from the alternative which never crosses the graph. Thus, the graph corresponding to a separating partial priority rule is a system of interlocking upside-down "y"-shapes. Such a graph looks like a game tree in which at any given point, there are only two possible moves.

Theorem 2: A rule satisfies *betweenness*, *anonymity*, and *representative*

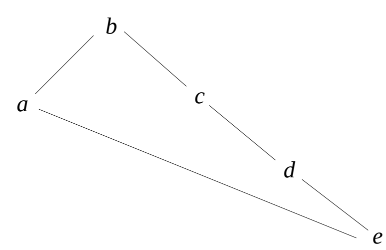

Figure 2: An interval partial priority rule

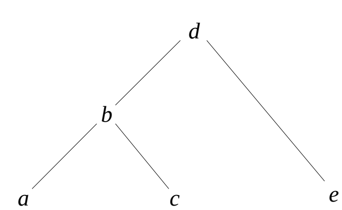

Figure 3: A separating partial priority rule

consistency if and only if it is an interval partial priority rule.

Proof: Suppose f is an interval partial priority rule. We already know that it satisfies *anonymity* and *representative consistency*, so we will show that it satisfies *betweenness*. Thus, let $N \in \mathcal{N}$ and $x \in X^N$. As f is an interval partial priority rule, and since for all $i \in N$, $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* x_i$, we obtain $\min_{\leq^*} \{x_i\}_{i \in N} \leq^* \bigwedge_{i \in N} x_i$. A similar expression holds for $\max_{\leq^*} \{x_i\}_{i \in N}$, so that f satisfies *betweenness*.

Now suppose that f is a rule satisfying the axioms. By Lemma 2, we know that it is a partial priority rule. We will verify that f is an interval partial priority rule. Let $x, y, z \in X$, and suppose that $y, z \geq^* x$. Without loss of generality, assume that $y \geq^* z \geq^* x$. By *betweenness*, $f(y, z) \geq^* z$. By the transitivity of $\leq^*$, $f(y, z) \geq^* x$. By definition, $f(y, z) = y \wedge z$. Thus, $y \wedge z \geq^* x$. A similar statement holds for the lower contour set of x.■

Theorem 3 demonstrates that a partial priority rule is *vote monotonic* if and only if it is a separating partial priority rule.

Theorem 3: A rule satisfies *unanimity*, *anonymity*, *representative consistency*, and *vote monotonicity* if and only if it is a separating partial priority rule.

Proof: By Lemma 3 and Theorem 2, if f satisfies the axioms, it must be an interval partial priority rule. Let $x, y, z \in X$ satisfy $x \leq^* y \leq^* z$ and $x, z \succeq y$. By *vote monotonicity*, $x \wedge z = f(x, z) \leq^* f(y, z) = y$ and $x \wedge z = f(x, z) \geq^* f(x, y) = y$. By the anti-symmetry of $\leq^*$, $x \wedge z = y$. Thus, f is a separating partial priority rule.

For the other direction, let f be a separating partial priority rule. We will show that it is *vote monotonic*.

Let $\preceq$ be the partial order associated with f. Say a partial priority rule f satisfies condition $(*)$ if for all $x, y, z \in X$, if $x \leq^* y$, then $f(x, z) \leq^* f(y, z)$. We claim that if a partial priority rule satisfies condition $(*)$, then it is *vote monotonic*. To see why, suppose $(*)$ is true. Let $N \in \mathcal{N}$, and let $x, y \in X^N$. Suppose that for all $i \in N$, $x_i \leq^* y_i$. Suppose, without loss of generality, that N can be ordered as $N \equiv \{1, ..., n\}$. For all $m = 1, ..., n$, let $y_m x \equiv (y_1, ..., y_m, x_{m+1}, ..., x_n)$. We show by induction that for all $m = 1, ..., n$, $f(x) \leq^* f(y_m x)$. Let $m = 1$. Then $x_1 \leq^* y_1$. By condition $(*)$, $x_1 \wedge (\bigwedge_{i=2}^n x_i) \leq^* y_1 \wedge (\bigwedge_{i=2}^n x_i)$. Thus, $f(x) \leq^* f(y_1 x)$. Let $M \leq n$ be some integer. Now, suppose that for all $m < M$, $f(x) \leq^* f(y_m x)$. Then $x_M \leq^* y_M$. By condition $(*)$,

$$\begin{aligned} & x_M \wedge \left(\left(\bigwedge_{i=1}^{M-1} y_i \right) \wedge \left(\bigwedge_{i=M+1}^{n} x_i \right) \right) \\ \leq & \ ^* y_M \wedge \left(\left(\bigwedge_{i=1}^{M-1} y_i \right) \wedge \left(\bigwedge_{i=M+1}^{n} x_i \right) \right). \end{aligned}$$

The left-hand side of this expression is $f(y_{M-1}x)$, and the right-hand side is $f(y_M x)$. Thus, $f(y_{M-1}x) \leq^* f(y_M x)$. By transitivity of $\leq^*$, $f(x) \leq^* f(y_{M-1}x) \leq^* f(y_M x)$ implies $f(x) \leq^* f(y_M x)$. Thus, for all $m = 1, ..., n$, $f(x) \leq^* f(y_m x)$. In particular, if $m = n$, then $y_n x = y$ and $f(x) \leq^* f(y)$.

We now show that if a rule is a separating partial priority rule, then it satisfies condition $(*)$. Let $x, y, z \in X$, and suppose that $x \leq^* y$. There are three possible cases.

a) $y \leq^* z$: In this case, as f is a separating partial priority rule, it is an interval partial priority rule. Thus, $y \leq^* y \wedge z \leq^* z$. If $x \wedge z \leq^* y \wedge z$, we are done. Otherwise, $y \wedge z \leq^* x \wedge z$. As f is an interval partial priority rule, $y \wedge z \leq^* (y \wedge z) \wedge (x \wedge z) \leq^* x \wedge z$. As $(y \wedge z) \wedge (x \wedge z) = x \wedge y \wedge z$, $y \leq^* x \wedge y \wedge z \leq^* z$. Moreover, $y, z \succeq x \wedge y \wedge z$. As f is a separating partial priority rule, conclude $y \wedge z = x \wedge (y \wedge z)$. Thus, by definition, $y \wedge z \preceq x$. Thus, $x, z \succeq y \wedge z$ and $x \leq^* y \wedge z \leq^* z$. As f is a separating partial priority rule, conclude $x \wedge z = y \wedge z$. Thus, $x \wedge z \leq^* y \wedge z$.

b) $x \leq^* z \leq^* y$: In this case, as f is an interval partial priority rule, $x \leq^* x \wedge z \leq^* z \leq^* y \wedge z \leq^* y$, so that $x \wedge z \leq^* y \wedge z$.

c) $z \leq^* x$: In this case, as f is a separating partial priority rule, it is an interval partial priority rule. Thus, $z \leq^* x \wedge z \leq^* x$. If $x \wedge z \leq^* y \wedge z$, we are done. Otherwise, $y \wedge z \leq^* x \wedge z$. As f is an interval partial priority rule, $y \wedge z \leq^* (y \wedge z) \wedge (x \wedge z) \leq^* x \wedge z$. As $(y \wedge z) \wedge (x \wedge z) = x \wedge y \wedge z$, $z \leq^* x \wedge y \wedge z \leq^* x$. Moreover, $x, z \succeq x \wedge y \wedge z$. As f is a separating partial priority rule, conclude $x \wedge z = y \wedge (x \wedge z)$. Thus, by definition, $x \wedge z \preceq y$. Thus, $y, z \succeq x \wedge z$ and $z \leq^* x \wedge z \leq^* y$. As f is a separating partial priority rule, conclude $y \wedge z = x \wedge z$. Thus, $x \wedge z \leq^* y \wedge z$.■

The preceding results are surprising as one would expect that the partial priority rules satisfying our compromise conditions would be those whose partial order $\preceq$ is exactly the exogenous linear order $\leq^*$.

4.3 Strategic considerations in the spatial model

We now study a domain of *preferences* over X. Say a binary relation R on X is **single-peaked** if there exists $x(R) \in X$ such that for all $x \in X$ such that $x \neq x(R)$, $x(R)\,Px$, and for all $x, y \in X$, if $x(R) \leq^* x \leq^* y$ or if $x(R) \geq^* x \geq^* y$, xRy.[13] Let $SP(X, \leq^*)$ be the set of all single-peaked binary relations for the linearly ordered set $(X, \leq^*)$.[14]

We ask which partial priority rules satisfy the following condition, which states that if votes are taken to be agents' peaks, then no agent can ever benefit by lying about his peak. This condition was first studied in this model by Moulin [13].

Strategy-proofness: For all $N \in \mathcal{N}$, all $R \in SP(X, \leq^*)^N$, all $j \in N$, and all $R'_j \in SP(X, \leq^*)$, $f\left((x(R_i))_{i\in N}\right) R_j f\left((x(R_i))_{i \in N\setminus\{j\}}, x\left(R'_j\right)\right)$.

Say a partial priority rule f is a **target rule** if there exists $x^* \in X$ such that for all $y, z \in X$, *i)* $z \leq^* y \leq^* x^*$ implies $z \succeq y$, *ii)* $x^* \leq^* y \leq^* z$ implies $z \succeq y$, and *iii)* $y \leq^* x^* \leq^* z$ implies $y \wedge z = x^*$. Thus, a target rule is a partial priority rule with two branches, rooted at x^*, each of which agree with the order $\leq^*$. Target rules were first discussed by Thomson and Thomson and Ching [16, 17] using a different set of axioms.[15] The terminology belongs to them. The word "target" refers to the element $x^* \in X$ discussed in the definition. A target rule selects the alternative in between the agents votes which lies closest (in terms of $\leq^*$) to the target.

In Figure 4, we give a typical example of a target rule. Here, the alternative x^* corresponds to b. The partial orders corresponding to target rules all have the upside-down "v"-shape as depicted here.

Theorem 4: A rule satisfies *unanimity*, *anonymity*, *representative consistency*, and *strategy-proofness* if and only if it is a target rule.[16]

[13] As usual, for a binary relation R, P denotes the asymmetric part of R. The alternative $x(R)$ is called the **peak** of R.

[14] In this environment, a rule selects only Pareto efficient alternatives if and only if it satisfies *betweenness* with respect to the peaks.

[15] Formally, their definition was stated for rules which map from profiles of single-peaked preferences into alternatives. Moreover, their definition is stated for a fixed population of agents. However, in that context, target rules only use information about the agents' peaks, where the peaks are aggregated in the same way we describe here.

[16] For a fixed population $N \in \mathcal{N}$, it follows that the target rules must be generalized median voter rules, as defined in Moulin [13].

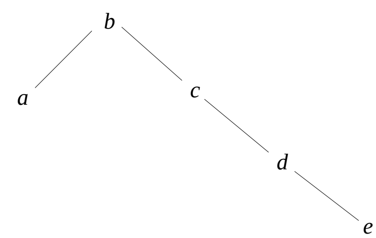

Figure 4: A target rule

Proof: Clearly, the target rules are separating partial priority rules, and thus satisfy *unanimity*, *anonymity*, and *representative consistency*. It is well-known that they also satisfy *strategy-proofness*. Thus, we will show that any rule satisfying the axioms must be a target rule.

First, we show that for all $x, y, z \in X$, if $x \preceq y \preceq z$, then either $x \leq^* y \leq^* z$ or $z \leq^* y \leq^* x$. Call this property the **chain property**. Suppose that this statement is false. We will only show that one case leads to an impossibility. The remaining three cases can be similarly proved. Thus, suppose $y \leq^* z \leq^* x$. Suppose, without loss of generality, that $y <^* z$ (this is without loss of generality, as otherwise, $z \leq^* y \leq^* x$ holds). Let $R \in SP(X, \leq^*)$ such that $x(R) = z$ and xPy. Then $f(x(R), y) = y$. However, $f(x, y) = xPy = f(x(R), y)$, in contradiction to *strategy-proofness*.

Next, we show that f is an interval partial priority rule. Thus, let $x, y, z \in X$ such that $x, y \geq^* z$. If $x = y$, then clearly $x \wedge x = x$. So suppose that $x \neq y$, and suppose $x \wedge y <^* z$. Without loss of generality, suppose that $y >^* x$. Let $R \in SP(X, \leq^*)$ satisfy $x(R) = y$. Then $f(x, x) = xP(x \wedge y) = f(x(R), x)$, a contradiction to *strategy-proofness*. The case in which $x, y \leq^* z$ is proved similarly.

Let $x^* \equiv \bigwedge_{x \in X} x$. By definition of x^*, for all $x \in X$, $x^* \preceq x$. We prove that $i)$ in the definition of target rule is true; $ii)$ follows from a symmetric argument.

Let $y, z \in X$ such that $z \leq^* y \leq^* x^*$. We show that $z \succeq y$. As f is an interval partial priority rule, $z \leq^* y \wedge z \leq^* y$. Moreover, by definition, $y \succeq y \wedge z \succeq x^*$, so that by the chain property and the fact that $x^* \geq^* y$, $x^* \geq^* y \wedge z \geq^* y$. Thus, $y \wedge z \leq^* y$ and $y \wedge z \geq^* y$, so that by the anti-symmetry of $\leq^*$, $y \wedge z = y$. Thus, by definition, $z \succeq y$.

To verify *iii*) in the definition of target rule, let $y, z \in X$ satisfy $y \leq^* x^* \leq^* z$. As $y \succeq y \wedge z \succeq x^*$, by the chain property, $x^* \geq^* y \wedge z \geq^* y$. As $z \succeq y \wedge z \succeq x^*$, by the chain property, $x^* \leq^* y \wedge z \leq^* z$. Therefore, $x^* \geq^* y \wedge z$ and $x^* \leq^* y \wedge z$. By the anti-symmetry of $\leq^*$, $y \wedge z = x^*$. ■

5 Proportional representation

5.1 Single-member districts vs. proportional representation

Theorem 1 relies on the fact that *representative consistency* models single-member district representation. This means that each district chooses a *unique* representative alternative. Such a system ensures that each district has its own representative. However, there are other natural systems of representation; the most common alternative being *proportional representation*. Proportional representation is not a winner-take-all system as is single-member district representation. Instead, a set of representatives are chosen for each district *proportionally* to the number of votes cast for each alternative. Thus, if there are one hundred votes for x and one hundred votes for y, the outcome of the vote is a fifty-fifty mixture between x and y. Usually majority rule is applied at a later stage.

We can model this type of system by introducing lotteries. Specifically, if X is the set of alternatives, let $\Delta(X)$ be the set of lotteries over X. Agents submit votes for lotteries over X; the interpretation is left open. A lottery may represent the composition of some governing body–for example, a fifty-fifty lottery between x and y means that half of Parliament should support policy x and the other half should support policy y. There are no statements made here about how such a Parliament will decide which alternative is chosen for society. Another interpretation is that a lottery represents an actual randomization–a fifty-fifty lottery between x and y is the statement that in the future, a fair coin will be tossed; if heads, the alternative selected is x, if tails, it is y.

5.2 Quasi-proportional representation

We define a class of voting rules, which we term the **quasi-proportional rules**. These rules are called quasi-arithmetic means in the literature on functional equations [1, 2].[17] Specifically, let $r : \Delta(X) \to \Delta(X)$ be a homeomorphism such that for all $x \in X$, $r(\delta_x) = \delta_x$ (here, δ_x refers to the degenerate lottery with probability one on x). Define a voting rule as follows. For all $N \in \mathcal{N}$, for all $p \in \Delta(X)^N$, $f(p) \equiv r\left(\frac{\sum_N r^{-1}(p_i)}{|N|}\right)$. Any such voting rule satisfies the conditions listed in our theorem, as well as *continuity*. The function r serves the purpose of "transforming" the space of lotteries so that the outcome of a vote is simply the average of all of the agents' votes. The **proportional rule** results when r is the identity mapping. The quasi-proportional rules satisfy the three principles we took to define representative democracy; in particular, they are "gerrymandering-proof."

By varying r, we can give more or less "importance" to different alternatives. In systems of proportional representation, to eliminate "fringe" alternatives, a quota of votes is often required for an alternative to be represented. For example, in many representative democracies, an alternative needs to receive five percent of the popular vote to be represented. While this extreme case violates the principle of *strong monotonicity* (which is satisfied by all quasi-proportional rules), by using an appropriate function r, we can define similar rules, so that fringe alternatives are given less importance than their popular vote would suggest.

A natural question is whether or not the partial priority rules and the quasi-proportional rules exhaust the class of rules satisfying *unanimity*, *anonymity*, and *representative consistency* on the space of lotteries. The answer is clearly no; we may simply choose an r in the definition of quasi-proportional rule which is not continuous, yet is still a bijection. Still, there are even more complex rules satisfying the three properties, as the following examples demonstrate. The examples work by composing the partial priority rules with the quasi-proportional rules.

Example 4: Let $X \equiv \{x, y\}$. We can identify any $p \in \Delta(X)$ with the probability that it places on alternative x. Therefore, we identify $\Delta(X)$

[17] The quasi-arithmetic means were first characterized by Kolmogorov [11] and Nagumo [14]. They were specifically concerned with the case in which $|X| = 2$. The quasi-arithmetic means are characterized by the axioms *unanimity*, *anonymity*, *representative consistency*, together with a *continuity* and *strong monotonicity* axioms.

with $[0,1]$. Define f as follows. For all $N \in \mathcal{N}$ and all $x \in [0,1]^N$, if for all $i \in N$, $x_i > \frac{1}{2}$, $f(x) \equiv \frac{\sum_{i \in N} x_i}{|N|}$, and if there exists $x_i \leq \frac{1}{2}$, $f(x) \equiv \frac{\sum_{i \in N} \min\left\{x_i, \frac{1}{2}\right\}}{|N|}$.

Example 5: Let $X \equiv \{x, y\}$. We can identify any $p \in \Delta(X)$ with the probability that it places on alternative x. Therefore, we identify $\Delta(X)$ with $[0,1]$. Define an equivalence relation $\simeq$ on $[0,1]$ as follows: for all $r, s \in [0,1]$, $r \simeq s$ if and only if $r - s \in \mathbb{Q}$. Label the set of equivalence classes of $\simeq$ by $\mathcal{P}$, with generic element P. For all P, for all $\{r_1, ..., r_n\} \subset P$, it is easily verified that $\left(\sum_{i=1}^n r_i\right)/n \in P$. By the Axiom of Choice, for all P, let $x_P \in P$. For all $P \in \mathcal{P}$, let $r_P : P \to P$ be a strictly increasing bijection. Let $\preceq$ be a linear order over P. Define the rule f as follows. Let $N \in \mathcal{N}$ and let $x \in [0,1]^N$. Let $P(x) \equiv \min_{\preceq} \{P : \text{there exists } i \in N \text{ such that } x_i \in P\}$. Let $N' \subset N$ be defined by $N' \equiv \{i \in N : x_i \in P(x)\}$. Then $f(x) \equiv r_{P(x)}\left(\frac{\sum_{i \in N'} r_{P(x)}^{-1}(x_i) + \sum_{i \in N \setminus N'} r_{P(x)}^{-1}\left(x_{P(x)}\right)}{|N|}\right)$. Note that $f(x) \in P(x)$.

6 Duals of partial priority rules

For some partial priority rules, a natural concept of a "dual" rule exists. Thus, for a partial order $\preceq$ on a set X, define the **dual of** $\preceq$, written $\preceq'$, so that for all $x, y \in X$, $x \preceq' y$ if and only if $y \preceq x$. When it exists, the meet corresponding to $\preceq'$ is the join corresponding to $\preceq$.[18] For a partial priority rule f with corresponding partial order $\preceq$, if $\preceq'$ exists, we define the **dual of f** , written f', as the partial priority rule with corresponding partial order $\preceq'$.

A **lattice** is a partially ordered set such that any pair of elements has a meet and a join. Thus, a dual of f exists if and only if $(X, \preceq)$ is a lattice. When X is finite, a simple necessary and sufficient condition for a meet-semilattice $(X, \preceq)$ to be a lattice is that it have a unique maximal element (Birkhoff, [4], p. 23). Thus, for a partial priority rule f with partial order $\preceq$, a unique maximal element x exists if and only if for all $y \in X$, $f(x, y) = y$. Thus, we have the following proposition.

[18] To see this, let $x, y \in X$, and let $x \wedge' y$ be the meet of x and y according to $\preceq'$. By definition, for all $z \preceq' x$,y, $z \preceq' x \wedge' y$. By definition of $\preceq'$, then, for all $z \succeq x, y$, $z \succeq x \wedge' y$. But then $x \wedge' y$ is exactly $x \vee y$.

Proposition 1: A partial priority rule f with partial order $\preceq$ has a dual if and only if there exists $x \in X$ such that for all $y \in X$, $f(x, y) = y$.

Thus, a partial priority rule has a dual if there exists some alternative which can always be "vetoed" by *any* other alternative. The following example provides the dual of the rule from Example 2.

Example 6: The dual of Example 2 It is simple to verify that the rule f of Example 2 has a dual. To see why, note that $A \in 2^A$ and that for all $B \in 2^A$, $f(A, B) = A \cap B = B$. Thus, by Proposition 2, f has a dual. In fact, for all $N \in \mathcal{N}$, for all $B \in \left(2^A\right)^N$, $f'\left((B_i)_{i \in N}\right) = \bigcup_{i \in N} B_i$. Thus, an "alternative" is selected if and only if at least one agent votes for it. This rule is intuitively "dual" to the rule of Example 2.

7 Discussion and conclusion

7.1 Previous literature related to representative consistency

It is important to note that *representative consistency* is not a new condition; however, we believe that the interpretation here is. Blackorby and Donaldson [5] study a condition called the "population substitution principle," which is the same as *representative consistency*. Their interest in the condition is in comparing the welfares of varying groups of agents. The literature on functional equations has long studied this (dating back to 1930–see [11, 14], as well as [6]) and related notions. These works use the notion to characterize the quasi-arithmetic means [1, 2]. For a good introduction, see Diewert [7]. By establishing that *representative consistency* is an "averaging" condition, these works add further normative justification to *representative consistency*. The main theoretical distinction between our work and these works is that the cited works require the domain of a rule to be a continuum, usually the unit interval. To our knowledge, no other work has allowed the domain to be a finite set.

7.2 Other principles of democracy

An upsetting aspect of the partial priority rules is that they violate the principle of *neutrality*. Informally speaking, *neutrality* requires that the "names" of alternatives should not matter for a voting rule. It is often taken as a basic principle of democracy. Unfortunately, in a pure voting model, the properties *anonymity* and *neutrality* are incompatible when rules must be single-valued and environments with even populations of agents are permissible. Thus, a natural extension of our model would include rules which are allowed to be multi-valued. Such an extension would require redefining *representative consistency* for this case. Interestingly, *neutrality* can be recaptured when lotteries are permitted. Simply let f be a quasi-proportional rule such that the corresponding function r as described above is permutation-invariant.

7.3 General comments

Another negative aspect of the partial priority rules is the amount of power they give to each individual. *Any* individual can "veto" an alternative with low priority simply by voting for an alternative with a higher priority. Similar voting rules have been studied in strategic models where information about the social alternatives is incomplete [8, 9]. It is generally found that these voting rules fare worse than other standard *anonymous* rules in terms of their ability to correctly aggregate information.

In representative democracies with single-member districts, the usual practice is to select an actual agent to represent a district of voters. This is the case in the United States House of Representative, for example. Our paper implicitly identifies this representative agent with the alternative she supports. In a scenario in which representative agents actually represent districts, it is the case that these representatives are treated anonymously in some "second stage." In order to preserve *anonymity* of a voting rule, then, it is the case that generally, districts of agents must be equipopulous. A natural question to ask is what happens to Theorem 1 when such a requirement is made. It turns out that weakening *representative consistency* to accommodate this the equal population requirement does not admit any new rules. This issue is discussed further in Appendix B.

Another possible objection to the model is that the same voting rule is applied at *all* levels of government. A natural extension of the model would study *three* different voting rules, say f, g, and h. The rule f would be applied at the district level. The rule g would be used for aggregating *representative* votes at the national level (*i.e.* after applying f; as a Congress might do). The rule h would be a rule which could be used to *directly* aggregate votes at the national level. In this scenario, *representative consistency* would say that the function h should always be equal to the composition of g with f, independently of the partitioning of agents into districts. It turns out that if *unanimity* and *anonymity* are required for all three voting rules, then the three voting rules are equal and Theorem 1 still applies.

A natural question is whether Theorem 1 holds when the set of potential agents is finite. While we have no formal proof, we conjecture that if the cardinality of the set of potential agents is at least three, then a result akin to Theorem 1 holds. A potential approach to solving this problem is presented in Aczel [1] in his characterization of the quasi-arithmetic means based on the bisymmetry equation.

8 Appendix A–Independence of the axioms

The following are examples establishing the independence of our axioms. Each axiom is followed by a rule which violates it and satisfies the other two axioms of Theorem 1.

Unanimity: Let f be a constant rule–*i.e.* there exists $z \in X$ such that for all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \equiv z$.

Anonymity: Let R be a linear order on the set of agents $\mathbb{N}$. For all $N \in \mathcal{N}$ and all $x \in X^N$, $f(x) \equiv x_{\arg\max_R N}$.

Representative consistency: Let $\leq^*$ be a linear order on X. For all $N \in \mathcal{N}$ and all $x \in X^N$,

$$f(x) \equiv \min_{\leq^*} \left\{ x \in \{x_i\}_{i \in N} : |i : x_i >^* x| \geq \frac{|N|}{2} \right\}.$$

9 Appendix B–Equipopulous districts

This appendix discusses a restricted version of *representative consistency.*

Weak representative consistency: For all $N \in \mathcal{N}$, all partitions $\{N_1, ..., N_m\}$ of N such that for all i, j, $|N_i| = |N_j|$, and all $x \in X^N$, $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_m})^{N_m}\right)$.

As the name suggests, *weak representative consistency* is a weaker condition than *representative consistency* under the *unanimity* principle. Generally, however, it need not be weaker.

Weak representative consistency is a condition which stipulates that the populations of the "districts" that agents are partitioned into must have equal cardinalities. For example, in the United States House of Representatives, congressional districts are required to be of approximately equal size. If each Congressman is treated *anonymously*, the *anonymity* principle will generally require that districts be equipopulous.

Under the *unanimity* principle, *weak representative consistency* has no bite if the set of agents cannot be partitioned into nontrivial districts of equal cardinalities (for example, if $|N| = 5$). This is the case if the set of agents has a cardinality which is a prime number.

We establish that no new rules become available when requiring that districts be equipopulous. Thus, even if all districts are required to contain the same number of agents, any rule which is not a partial priority rule presents some opportunities for gerrymandering.

Theorem 5 generalizes Theorem 1 using the *weak representative consistency* condition.

Theorem 5: A rule satisfies *unanimity*, *anonymity*, and *weak representative consistency* if and only if it is a partial priority rule.

The theorem relies on the following lemma, whose proof is identical to Lemma 1. Thus, we state it without proof.

Lemma 2: If a rule satisfies *unanimity*, *anonymity*, and *weak representative consistency*, then it satisfies *replication invariance*.

The next lemma establishes that if a rule satisfies *unanimity*, *weak representative consistency*, and *replication invariance*, then it satisfies *representative consistency*. Taken together, Lemmas 2 and 3 will directly imply Theorem 5.

Lemma 3: If a rule satisfies *unanimity*, *weak representative consistency*, and *replication invariance*, then it satisfies *representative consistency*.

Proof: Let f be a rule that satisfies *unanimity*, *weak representative consistency*, and *replication invariance*. Let $N \in \mathcal{N}$ and let $x \in$

X^N. Let $\{N_1, ..., N_m\}$ be a partition of N. We claim that $f(x) = f\left(f(x_{N_1})^{N_1}, ..., f(x_{N_m})^{N_m}\right)$.

By *replication invariance*,

$$f(x) = f\left(x^{\prod_{i=1}^{m}|N_i|}\right).$$

Rewrite

$$f\left(x^{\prod_{i=1}^{m}|N_i|}\right) = f\left(\left((x_{N_k})^{\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right).$$

Again, rewrite

$$f\left(\left((x_{N_k})^{\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right) = f\left(\left(\left(x_{N_k}^{\prod_{i\neq k}|N_i|}\right)^{|N_k|}\right)_{k=1}^{m}\right).$$

For all $k = 1, ..., m$, $x_{N_k}^{\prod_{i\neq k}|N_i|}$ has cardinality $\prod_{i=1}^{m}|N_i|$. By *weak representative consistency*,

$$f\left(\left(\left(x_{N_k}^{\prod_{i\neq k}|N_i|}\right)^{|N_k|}\right)_{k=1}^{m}\right) = f\left(\left(f\left(x_{N_k}^{\prod_{i\neq k}|N_i|}\right)^{|N_k|\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right).$$

By *replication invariance*, for all $k = 1, ..., m$,

$$f\left(x_{N_k}^{\prod_{i\neq k}|N_i|}\right) = f(x_{N_k}).$$

Thus,

$$f\left(\left(f\left(x_{N_k}^{\prod_{i\neq k}|N_i|}\right)^{|N_k|}\right)_{k=1}^{m}\right)=f\left(\left(f\left(x_{N_k}\right)^{|N_k|\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right).$$

By *replication invariance*,

$$f\left(\left(f\left(x_{N_k}\right)^{|N_k|\prod_{i=1}^{m}|N_i|}\right)_{k=1}^{m}\right)=f\left(\left(f\left(x_{N_k}\right)^{|N_k|}\right)_{k=1}^{m}\right).$$

By *anonymity*, which is implied by *replication invariance*,

$$f\left(\left(f\left(x_{N_k}\right)^{|N_k|}\right)_{k=1}^{m}\right)=f\left(f\left(x_{N_1}\right)^{N_1},...,f\left(x_{N_m}\right)^{N_m}\right).$$

Thus, $f(x)=f\left(f\left(x_{N_1}\right)^{N_1},...,f\left(x_{N_m}\right)^{N_m}\right)$. As stated in the main body of the text, this condition is equivalent to *representative consistency* when *unanimity* is satisfied.■

The proof of Theorem 5 is now simple.

Proof: Let f be a partial priority rule. Then it clearly satisfies *unanimity*, *anonymity*, and *weak representative consistency.*

Let f be a rule that satisfies *unanimity*, *anonymity*, and *weak representative consistency.* By Lemma 2, f satisfies *replication invariance*. By Lemma 3, f satisfies *representative consistency.* Thus, f satisfies *unanimity*, *anonymity*, and *representative consistency.* By Theorem 1, f is a partial priority rule. ■

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