On the Robustness of Majority Rule*

by

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Abstract

In this paper we demonstrate that, in a large population, if, for some domain of individual preference profiles, a voting rule satisfying anonymity, neutrality, and the Pareto rule is transitive, then so is the (simple) majority rule transitive. We also demonstrate that, unless a voting rule, F, is itself the majority rule, there exists some domain of individual preference profiles on which majority rule is transitive, but F is not. The two results, when combined, capture the sense in which majority rule is robust. A characterization of the rule, one based on the idea of "maximal" robustness, is then provided. It complements the result in May (1952). We illustrate the results by identifying the restricted domains of individual preference profiles on which majority rule and a number of other well-known voting rules satisfy anonymity, neutrality, the Pareto rule, and transitivity. We then study the "tightness" of the assumptions underlying our main robustness result by relaxing, in turn, transitivity, anonymity, and neutrality. We show that if neutrality is replaced by independence of irrelevant alternatives (and a certain technical condition), then the unanimity rule with a given order of precedence is robust.

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Contents

1. Introduction and Motivation

The rules that transform individual preferences into collective choice differ widely across institutions. In his pioneering work, Arrow (1951) offered a particular axiomatization of democratic rules and showed that they are not always coherent: unless restrictions are placed on the domain of individual preferences, the rules generate cycles for some configurations of preferences.

Although Arrow's formulation of democratic rules encompassed far more than majoritarianism, advocates of democracy have frequently adopted the narrower view that it is majority rule that characterizes an essential part of the democratic process. As Dahl (1989: 135) observes: "... virtually everyone assumes that democracy requires majority rule in the weak sense that support by a majority ought to be necessary to passing a law. But ordinarily supporters of majority rule mean it in a much stronger sense. In this stronger sense, majority rule means that majority support ought to be not only necessary but also sufficient for enacting laws." (Emphasis in the original.) While Dahl (1989, Ch. 10) observes that the term "majority rule" is not unambiguous, that it refers to a family of decision rules, it is the simple majority rule which receives by far the greatest attention in his exposition. Since, in the political science literature the qualifier is often absent when reference is made to the "simple majority rule", we will, for brevity, do the same and refer to the simple majority rule simply as the majority rule.

In view of the prominence that continues to be given in theories of democracy to majority rule, it is ironic that the most famous illustration of Arrow's Impossibility Theorem continues to be the Condorcet-cycle. To illustrate the Condorcet-cycle, consider three voters, who rank three alternatives (labelled x, y, z) as, respectively, "x over y over z", "y over z over x", and "z over x over y". Majority rule is intransitive under this configuration of preferences. To confirm this, note that, since two of the voters prefer x to y, majority rule requires that x be ranked over y; likewise, since two of the voters prefer y to z, the rule requires that y be ranked over z. By transitivity, x should be ranked over z. But since two of the voters prefer z to x, the rule requires that z be ranked over x, which is a contradiction.

Majority rule is, nevertheless, intuitively appealing. This is because it possesses several compelling properties, especially when applied to choices over political candidates. First, it satisfies the Pareto rule: if all voters prefer alternative x to alternative y, the rule ranks x over y. Secondly, it is anonymous: the rule treats all voters symmetrically, in the sense that the ranking is independent of voters' labels. Anonymity captures one of Dahl's (five) criteria for democratic decision-making (Dahl, 1989, Ch. 9): voting equality among citizens. And thirdly, majority rule satisfies neutrality: its ranking over any pair of alternatives depends only on the pattern of voters' preferences over that pair, not on the alternatives' labels.

Neutrality is symmetry with respect to alternatives. In the context of representative democracy, neutrality is a natural requirement of a voting rule because it prohibits procedural discrimination against candidates. Rules that violate neutrality have built into them preconceived rankings, for example, favouring the status-quo. If preconceived social rankings are to be avoided, neutrality is the condition that can ensure its avoidance.

But majority rule is not the only voting rule to satisfy anonymity, neutrality, and the Pareto rule. There is a vast array of others; for example, the 2/3-majority rule, under which two alternatives are considered to be socially indifferent unless at least two-thirds of the voters prefer one to the other; the Pareto-extension rule, wherein two alternatives are considered to be socially indifferent unless all voters prefer one to the other; and, more generally, the δ-majority rule ($\delta > \frac{1}{2}$), under which two alternatives are considered to be socially indifferent unless a proportion of voters at least as large as δ prefer one to the other. Note though that if a voting rule is anonymous, it is non-dictatorial. It can be shown that if a voting rule is neutral, it satisfies Arrow's condition of the "independence of irrelevant alternatives". From Arrow's Impossibility Theorem we may thus conclude that no voting rule that satisfies anonymity, neutrality, and the Pareto rule can be transitive if the domain of individual preference profiles is unrestricted.

As is well known, Arrow's theorem applies not only to direct democracy, but to many other species of democracy as well, including representative democracy (Arrow, 1963; Bergson, 1976). So it pays to work within Arrow's original formulation, which we recount in this section.

Let X denote the set of alternatives. X is assumed to be a finite set, containing at least three elements. We suppose that there are N voters (i = 1,..., N). It is assumed that voter i has a complete preference ordering over X. Let **R**(i) denote this. So **R**(i) is a reflexive, transitive, and complete binary relation. In some applications, **R**(i) should be thought of as representing i's ethical preferences; in others, i's personal preferences; and so on. We may write the (N-tuple) profile of individual preference orderings, (**R**(1), ..., **R**(i), ..., **R**(N)), as **R**. Let F be a functional relation that maps profiles of individual preference orderings from a given domain of profiles to a reflexive binary relation R on X. We write this as $R = F(\mathbf{R}(1),...,\mathbf{R}(N)) \equiv F(\mathbf{R})$. We will say that F is a <u>voting rule</u> if it is a complete, reflexive binary relation.

Voting rules need not be transitive. We say that a voting rule is a <u>social welfare function</u> if it <u>is</u> transitive. In other words, a social welfare function is a voting rule, F, the range of which belongs to the set of complete orderings over X. Arrow's Impossibility Theorem addresses the question concerning the existence of a social welfare function

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We have gone into some of the wider aspects of Dahl's theory of democracy in Dasgupta and Maskin (1999).

satisfying a particular set of democratic criteria. The Condorcet-cycle, which illustrates that majority rule is not a social welfare function on an unrestricted domain of individual preferences, is merely the most well-known illustration of Arrow's theorem.

In this paper we will offer a new defense of majority rule: robustness. We will show that among all voting rules satisfying anonymity, neutrality and the Pareto rule, majority rule has the greatest reach, in that it is transitive on the widest class of domains of individual preferences, and is the unique such voting rule. Formally, we will demonstrate that if, for some domain of individual preference profiles, a voting rule satisfying anonymity, neutrality, and the Pareto rule is transitive (i.e. it is a social welfare function on this domain), then so is majority rule transitive on this domain (i.e. it too is a social welfare function on this domain). We will also demonstrate that, unless a voting rule, F, is itself the majority rule, there exists some domain of individual preference profiles on which majority rule is transitive, but F is not. The two results, when combined, capture the sense in which majority rule is robust.

Maskin (1995) stated and proved this result for the case where N is odd. But the oddness restriction is discomfitting. It was necessary to invoke it in order to avoid pathologies that arise when, for example, exactly half the population prefers x to y and the other half prefers y to x. In this paper we will be interested in large organizations. To formalize the idea that in such environments knife-edge cases are knife-edge cases, we will assume that the number of voters is a continuum (the notation will be introduced in Section 2) and prove a similar result in Section 3 (Theorem 1).

This restriction, that the number of voters is odd, was also invoked by Black (1948a,c) to show that if the domain of individual preferences are "single-peaked", then majority rule is a transitive voting rule. To remind ourselves of what can happen if the number of voters is even, suppose N = 2 and that, among three alternatives, x, y, and z, the first voter ranks x over y over z, while the second voter ranks y over z over x. These preferences are single-peaked but, as can readily be checked, majority rule is intransitive.

Kirman and Sondermann (1972) have shown that Arrow's Impossibility Theorem holds if there is a continuum of voters, in that if a social welfare function satisfies the Arrow axioms, there is an

In a remarkable early paper, May (1952) proved a characterization theorem for majority rule (reproduced here as Theorem 2). Theorem 1 suggests an alternative characterization, which explicitly invokes the idea of what we will call "maximal" robustness. We formulate it and prove it in Section 4 (Theorem 3).

In proving Theorem 1 we will need to determine which orderings must be omitted from a domain of individual preference orderings if majority rule is to be transitive on the domain. Theorem 1 tells us that if any other voting rule is to satisfy anonymity, neutrality, the Pareto rule, and transitivity, at least as many orderings must be omitted. Therefore, for comparison, we will examine specifically which orderings need be omitted from a domain if certain prominent voting rules are to satisfy these conditions on it. In Section 5 we examine the rank-order rule (or the Borda-count), the Pareto-extension rule, and the 2/3-majority rule (Theorems 4-6); in Section 6 random dictatorship and the plurality rule are examined (Theorems 7-8).

Sections 7 and 8 are about the extent to which the assumptions in Theorem 1 can be weakened. Transitivity and anonymity are relaxed in Section 7; neutrality in Section 8.

Perhaps the most familiar weakening of transitivity to have been explored in the social choice literature is quasi-transitivity (Sen 1969). However, Gibbard (1969) showed that relaxing the notion of collective rationality in this manner does not make much purchase, because voting rules satisfying Arrow's other axioms on an unrestricted domain of preference profiles, while not dictatorial, are nevertheless "oligarchic". In Section 7.1 we extend the idea of non-oligarchic voting rules to a continuum of voters

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"invisible" dictator.

δ-majority rules were discussed by Black (1948b). Blair (1979) has explored their axiomatic basis. Under Blair's axioms, δ is a function of the profile of individual preferences. So he refers to the voting rules implied by his axioms the "variable majority rule". Neither May (1952) nor Blair (1979), however, was concerned with the robustness of voting rules.

and prove (Theorem 9) that majority rule is robust among rules that satisfy anonymity, neutrality, the Pareto rule, and quasi-transitivity. In Section 7.2 anonymity is replaced by a weaker condition, individual responsiveness and it is proved (Theorems 10-11) that, while majority rule is robust, it is not maximally robust: weighted majority rules are also robust.

In Section 8 neutrality is replaced by the weaker (and much-studied) condition of "independence of irrelevant alternatives". We add a technical condition on voting rules, which enables certain kinds of ties to be broken. Analogous to Theorem 1, we prove (Theorem 12) that among rules satisfying these conditions the unanimity rule (with a given order of precedence) is robust.

Majority rule and the unanimity rule offer genuinely different ethical viewpoints, differences that have been much discussed by political scientists and social philosophers. The arguments that advocates of these two voting rules have advanced often differ sharply. However, social choice theory has on a number of occasions revealed that seemingly minor changes in the conditions imposed on voting rules can be significant changes after all. Theorems 1 and 12 can be read in this light.

Section 9 contains a summary of the results.

2. Voting Rules and Majoritarianism in a Large Population

Let X be a (finite) set of social alternatives containing at least 3 elements. It would simplify the analysis greatly if we were to assume that individuals have strict preference orderings on X (that is, if we were to rule out indifference on the part of voters). So let $\wp_X$ be the set of all strict orderings of X. There is a large population of voters represented by the unit interval [0,1]. For any $\wp \subseteq \wp_X$ we say that preferences lie in domain $\wp$ if they can be expressed by a profile $\mathbf{P}$:[0,1] $\rightarrow \wp$, where, for i $\in$ [0,1], $\mathbf{P}$(i) is

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Spitz (1984) is a fine advocacy of majority rule. The classic on the unanimity rule is Buchanan and Tullock (1962) voter i's strict preference ordering. We will confine attention to profiles **P** that are well-behaved in the sense that their inverse images are Lebesgue-measurable, that is, **P**-1(P) is Lebesgue-measurable for each $P \in \wp_X$.

Let F be a function which, for each (well-behaved) profile **P** in $\wp_X$, assigns a "social preference" ranking, R = F(**P**), where R is a reflexive binary relation. We will say that F is a voting rule if it is a complete, reflexive binary relation. In what follows, we shall denote the asymmetric binary relation induced by R (also called the "asymmetric factor" of R) by P and the symmetric binary relation induced by R (also called the "symmetric factor" of R) by I. P and I will be interpreted as "strict social preference" and "social indifference", respectively.

F satisfies anonymity on $\wp$ if, for any (well-behaved) permutation $\pi:[0,1] \to [0,1]$, all profiles **P** in $\wp$, all $Y \subseteq X$, and all $x,y \in Y$, $xF(\mathbf{P}^{\pi})y$ if and only if $xF(\mathbf{P})y$, where, for all i, $\mathbf{P}^{\pi}(i) = \mathbf{P}(\pi(i))$. That is, a voting rule satisfies anonymity if its ranking over pairs of alternatives remains unchanged when the labels of the voters are permuted.

F satisfies neutrality if, for all profiles **P** and **P′** in $\wp$, all permutations $\chi:X \to X$, and all $Y \subseteq X$, for all $x,y \in Y$, and all i, [$x\mathbf{P}(i)y$ if and only if $\chi(x)\mathbf{P}'(i)\chi(y)$] implies [$xF(\mathbf{P})y$ if and only if $\chi(x)F(\mathbf{P}')\chi(y)$]. In words, a voting rule satisfies neutrality if its ranking over any pair of alternatives depends only on the pattern of voters' preferences over that pair, not on the alternatives' labels. This is the standard definition of neutrality (Sen, 1970). It is stronger than Arrow's "independence of irrelevant alternatives" condition. To see this, recall that a voting rule F satisfies independence of irrelevant alternatives on $\wp$ if, for all profiles **P** and **P′** in $\wp$, all $Y \subseteq X$, and all $x, y \in Y$, $\mathbf{P}|_Y = \mathbf{P}'|_Y$ implies that $xF(\mathbf{P})y \Leftrightarrow xF(\mathbf{P}')y$, where $\mathbf{P}|_Y$ and $\mathbf{P}'|_Y$ are, respectively, the restrictions of **P** and **P′** to Y. Notice that our version of "neutrality" incorporates independence of irrelevant alternatives, in that, if a voting rule is neutral on $\wp$, it is independent of irrelevant alternatives on $\wp$, but the reverse implication does not hold.

F satisfies the Pareto rule on $\wp$ if, for all profiles **P** in $\wp$, all Y $\subseteq$ X, and all x,y $\in$ Y, x**P**(i)y for almost all i implies ~{yF(**P**)x}. In words, if everyone prefers x to y, then x is socially preferred to y.

F is transitive on $\wp$ if, for all profiles **P** in $\wp$, F(**P**) is transitive. In that case R = F(**P**) can be thought of as representing the "social ordering" and F is referred to as a social welfare function.

The voting rule, FM, corresponds to majority rule if, for all profiles **P**, all Y $\subseteq$ X, and all x, y $\in$ Y,

$$xF^M(\mathbf{P})y \text{ if and only if } \mu\{i \mid x\mathbf{P}(i)y\} \geq \mu\{i \mid y\mathbf{P}(i)x\},$$

where μ is the Lebesgue measure. That is, x is judged to be socially at least as good as y if the proportion of voters who prefer x to y is at least as large as the proportion who prefer y to x.

It is easy to confirm that FM is anonymous, neutral, and satisfies the Pareto rule on the domain $\wp = \wp X$. But FM is not transitive on $\wp X$. For example, if about a third each of the population has the preferences [x,y,z], [y,z,x], and [z,x,y], respectively (where [x,y,z] is the ordering in which x is preferred to y and y is preferred to z), then social preferences are intransitive (this is the Condorcet paradox). Nevertheless, on certain smaller domains $\wp$, FM avoids such intransitivities. For example, suppose that the domain of preferences on {x,y,z} is single-peaked; that is, for any ordering in the domain, if x is preferred to y then y is preferred to z, and if z is preferred to y then y is preferred to x, so that if [x,y,z] and [z,y,x] belong to the domain, then at most [y,z,x] and

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There is a weaker version of neutrality, which separates it from the independence of irrelevant alternatives condition. In this version a voting rule F satisfies neutrality if for all profiles **P** in $\wp$ and for all permutations χ:X $\rightarrow$ X, F(χ(**P**)) = χ(F(**P**)), where χ(**P**) is the permutation of **P** over X. Our main result, Theorem 1 below, holds if the version of neutrality in the text were to be replaced by this weaker version and the independence of irrelevant alternatives condition.

[y,x,z] can also belong to the domain. It is simple to confirm that, except for profiles that are non-generic (in the sense that, for some x,y $\in$ X, $\mu\{i \mid x\mathbf{P}(i)y\} = \mu\{i \mid y\mathbf{P}(i)x\}$), FM is transitive.

3. Robustness of Majority Rule

We will show that FM is unique among voting rules in satisfying anonymity, neutrality, the Pareto rule, and transitivity on the biggest possible collection of domains of preferences. To make this precise, consider a profile **P** and two alternatives x, y $\in$ X. The <u>measure</u> of (x,y) for **P**, call it $m_{\mathbf{P}}(x,y)$, is the proportion of the population who prefer x to y; that is, $m_{\mathbf{P}}(x,y) \equiv \mu\{i \mid x\mathbf{P}(i)y\}$. We will say that a set S $\subseteq$ (0,1) is <u>exceptional</u> for a voting rule F and domain of preferences $\wp \subseteq \wp_X$, if for all profiles **P** in $\wp$ such that, for all x, y $\in$ X, $m_{\mathbf{P}}(x,y) \notin S$ (such a profile is called <u>regular</u> for S), F(**P**) is transitive. In other words, S is exceptional if it corresponds to population proportions that are problematic for transitivity. For example, the set S = {½} is exceptional for majority rule and a single-peaked domain of preferences: it is only when exactly half the population prefer one alternative to another that majority rule can fail to be transitive on a single-peaked domain.

We say that a voting rule F is transitive <u>generically</u> on $\wp$ if there exists a finite exceptional set, S, for F and $\wp$. In other words, F is transitive generically on $\wp$ if it is transitive for those profiles on $\wp$ that are regular for S. Call a voting rule <u>reasonable</u> on $\wp$ if it satisfies anonymity, neutrality, the Pareto rule, and is transitive generically on $\wp$. We now have:

Theorem 1: If F is reasonable on $\wp \subseteq \wp_X$, then F^M is reasonable on $\wp$. Moreover, if S is exceptional for F and $\wp$ and there exists a profile **P** on $\wp$ that is

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The definition motivates the assumption that exceptional sets do not contain the point 1 (resp. 0). If 1 (resp. 0) were exceptional, w would be ruling out the profile in which everyone prefers x to y (resp. y to x). In other words, we would be ruling out the potency of the Pareto rule.

regular for S, such that $F(\mathbf{P}) \neq F^M(\mathbf{P})$, then there exists $\wp' \subseteq \wp_X$ on which F^M is reasonable but F is not.

To prove this, we note the following

Lemma 1: F^M is generically transitive on domain $\wp$ if, and only if, for all triples of alternatives {x,y,z}, one of the orderings from the Condorcet cycle {[x,y,z], [y,z,x], [z,x,y]} and one of the orderings from the Condorcet cycle {[x,z,y], [z,y,x], [y,x,z]} are excluded from $\wp$.

Proof: The "only if" part is the Condorcet example. If for some triple {x,y,z}, there are preference orderings in $\wp$ corresponding to each of [x,y,z], [y,z,x], and [z,x,y], then for all profiles **P** in which approximately a third have the ranking [x,y,z], approximately a third have the ranking [y,z,x], and the remainder the ranking [z,x,y], the social ranking is intransitive. That is, all points in some interval $[1/3 - \varepsilon, 1/3 + \varepsilon]$, for some $\varepsilon > 0$, are in the exceptional set (the exceptional set is not finite). Hence F^M is not transitive generically.

To prove the "if" part, consider a domain $\wp$ on which F^M is not transitive generically. Then, in particular, the set S = {½} is not exceptional for F^M and $\wp$. Thi means that there exists a profile **P**, regular for {½}, for which the corresponding majority ranking, R^M, is intransitive; say, xR^My, yR^Mz, but zP^Mx for some triple {x,y,z}, where P^M is the strict relation corresponding to R^M. Because **P** is regular for {½}, xR^My implies that over half the population must prefer x to y. Similarly over half the population must prefer y to z. Therefore, the ordering [x,y,z] must belong to $\wp$. Analogously, so must [y,z,x] and [z,x,y]. _

Proof of Theorem 1:

See Sen and Pattanaik (1969) for a proof for the case where the number of voters is finite and odd. They called preference domains obtained from such deletions <u>value restricted</u>. In the Lemma we have stated the condition in a form somewhat different from the way value restriction was originally defined.

Begin with the first part of the theorem. Consider a domain $\wp$ on which the voting rule F is reasonable. Let S be the corresponding exceptional set. Suppose, contrary to the theorem, that F^M is not reasonable on $\wp$. Because F^M satisfies anonymity, neutrality, and the Pareto rule, it is not generically transitive on $\wp$. From Lemma 1 we know there exists a triple {x,y,z}, such that the Condorcet cycle {[x,y,z], [y,z,x], [z,x,y]} belongs to $\wp$. Because F is generically transitive, the exceptional set S is finite. Hence there exists an integer q (> 2) such that, if we divide [0, 1] into q equal subsets and so construct **P** that, for each subset [(k-1) / q, k / q), where k = 1,..., q, **P** assigns everyone in the subset the same ordering in $\wp$, then **P** is regular automatically. So F satisfies anonymity, neutrality, the Pareto rule, and transitivity on the domain of profiles **P** in $\wp$ where, foe each k = 1,...,q, everyone with index in the interval [(k-1) / q, k / q) has the same preference ordering.

Consider a profile $\mathbf{P}_1$ in $\wp$ in which voters in [0, 1 / q] prefer x to y and everyone else prefers y to x. Let $R_1 \equiv F(\mathbf{P}_1)$. Then either xR_1y or yR_1x. Suppose for the moment that yR_1x.

Consider a profile **P′** in which the voters in [0, 1 / q] have ordering [x,y,z], the voters in [1 / q, 2 / q] have ordering [y,z,x], and everybody else has the ordering [z,x,y]. Let R′ ≡ F(**P′**). Notice that everyone except the voters [0, 1 / q] prefers z to x. Hence, because F satisfies neutrality, yR_1x implies zR′x. Observe also that everybody except the voters [1 / q, 2 / q] prefers x to y. Hence from neutrality and anonymity, yR_1x implies xR′y. Transitivity implies that zR′y. Notice also that, under **P′**, the voters in [0, 2 / q] prefer y to z, and everyobody else prefers z to y. Consider now a profile $\mathbf{P}_2$ in $\wp$ which has the property that voters in [0, 2 / q] prefer x to y and everyone else prefers y to x. Since F is neutral, we may conclude that yR_2x, where $R_2 \equiv F(\mathbf{P}_2)$. Continuing iteratively, we can show for each k = 1,...,(q-1) that, for a profile $\mathbf{P}_k$, in which voters in [0, k / q] prefer x to y and everyone else prefers y to x, yR_kx, where $R_k \equiv F(\mathbf{P}_k)$. But consider $\mathbf{P}_{q-1}$, in which everyone in [0, (q-1) / q] prefers x to y. By the argument just completed, $yR_{q-1}x$, where $R_{q-1} \equiv F(\mathbf{P}_{q-1})$. From anonymity and neutrality and the fact that yR_1x, however, we conclude that $xR_{q-1}y$. Hence $yI_{q-1}x$. Consider finally a profile $\mathbf{P}''$ in $\wp$ such that all voters in [0, (q-1) / q] have the ordering [x,y,z], whereas the remaining voters have the ordering [y,z,x]. Because $yI_{q-1}x$, anonymity and neutrality imply that $xI''y$ and $xI''z$, where $R'' \equiv F(\mathbf{P}'')$. But from the Pareto rule, we have $yP''z$. Hence R'' is intransitive, a contradiction. We conclude that F^M is, generically, transitive after all.

We next prove the second assertion in the theorem. For ease of exposition, we will first offer a sketch of the proof. Details will then be filled in.

Assume there exists $\mathbf{P}$ in $\wp$ such that $F(\mathbf{P}) \neq F^M(\mathbf{P})$, where $\mathbf{P}$ is non-pathologic. This means there exist x and y such that

(1) $xF(\mathbf{P})y$ and $\sim[xF^M(\mathbf{P})y]$.

Choose P_*', $P_*'' \in \wp$ such that $xP_*'y$ and $yP_*''x$. To illustrate the argument we will deploy, suppose there is an alternative, z, such that $zP_*'xP_*'y$ and $zP_*''yP_*''x$, where z, x, and y are contiguous in P_*', and z, y, and x are contiguous in P_*''.

Let $\mathbf{P}_*$ be a profile such that

(2) $\mathbf{P}^*(i) \in \{P_*', P_*''\}$ for all i, and

(3) $m_{\mathbf{P}*}(x,y) = m_{\mathbf{P}}(x,y)$.

Note that $m_{\mathbf{P}}(x,y) < ½$. Notice also that, because $\mathbf{P}$ is non-pathologic, (2) and (3) imply $\mathbf{P}_*$ is non-pathologic as well. Since $\{P_*', P_*''\} \subseteq \wp$ and F is, generically, anonymous on $\wp$, we may infer from (1) that

(4) $xF(\mathbf{P}_*)y$.

Since $\{P_*', P_*''\}$ consists of just two orderings, F^M is transitive on this domain.

In saying that z, x, and y are contiguous in P_*' we mean that there are no alternatives lying between z and x and between x and y in the preference ordering P_*'. As we are at this point merely sketching the proof of the theorem, we assume that z exists. When we come to the complete proof, it will be shown that z does exist.

Let us now choose $\bar{P} \in \wp X$ such that

(5) $y\bar{P}z\bar{P}x$, and

(6) $\bar{P}\,|_{X-\{z\}} = P_*''\,|_{X-\{z\}}$.

It can be shown (see below) that there do not exist three alternatives for which $\{P_*', P_*'', \bar{P}\}$ constitutes a Condorcet triple. This means that F^M is reasonable on $\{P_*', P_*'', \bar{P}\}$.

We now demonstrate that F is not reasonable on $\{P_*', P_*'', \bar{P}\}$. Assume then that it is reasonable.

Consider a profile $\bar{\mathbf{P}}''$ such that

(7) $\mu\{i\,|\,\bar{\mathbf{P}}''(i)=P_*''\} = m_{\mathbf{P}}(y,x) - m_{\mathbf{P}}(x,y) > 0$,

(8) $\mu\{i\,|\,\bar{\mathbf{P}}''(i)=\bar{P}\} = m_{\mathbf{P}}(x,y)$,

(9) $\mu\{i\,|\,\bar{\mathbf{P}}''(i)=P_*'\} = m_{\mathbf{P}}(x,y)$.

From (7)-(9), we have $m_{\bar{\mathbf{P}}''}(x,y) = m_{\mathbf{P}}(x,y)$. But F is anonymous and $xF(\mathbf{P})y$. Therefore,

(10) $xF(\bar{\mathbf{P}}'')y$.

Moreover, from (7)-(9) we conclude that $m_{\bar{\mathbf{P}}''}(y,z) = m_{\mathbf{P}}(x,y)$. But F is anonymous and neutral, and $xF(\mathbf{P})y$. Therefore,

(11) $yF(\bar{\mathbf{P}}'')z$.

On combining (10) and (11) and using the fact that F is transitive, we have

(12) $xF(\bar{\mathbf{P}}'')z$.

But (12) violates the Pareto rule and the fact that $z\bar{\mathbf{P}}''(i)x$ for all i. Hence, we can take $\wp' = \{P_*', P_*'', \bar{P}\}$ to confirm the theorem.

It remains to fill in the details.

Choose $P', P'' \in \wp$, such that $xP'y$ and $yP''x$. Let $\mathbf{P}'$ be a profile with the property that

(13) $\mathbf{P}'(i) \in \{P', P''\}$ for all i, and

By $P\,|_{X-\{w\}} = P'\,|_{X-\{w\}}$ we mean that P and P′ are identical on the set of alternatives X-{w}.

(14) $m_{\mathbf{P}'}(x,y) = m_{\mathbf{P}}(x,y)$.

Because **P** is non-pathologic, (13) and (14) imply **P′** is also non-pathologic. Since $\{P', P''\} \subseteq \wp$ and F is, generically, anonymous on $\wp$, we may infer from (14) that

(15) $xF(\mathbf{P}')y$.

Since $\{P', P''\}$ consists of only two orderings, F^M is transitive on this domain.

Suppose, for the moment, that there exists $w \in X$ such that

(16) $yP''wP''x$.

If there exists more than one w satisfying (16), let w* be the one such that,

(17) $wP''w^*P''x$ for all $w \neq w^*$ satisfying (16).

Choose $P_o'' \in \wp$ such that

(18) $P_o''\big|_{X-\{w^*\}} = P''\big|_{X-\{w^*\}}$, and

(19) $xP_o''w^*P_o''v$ for all $v \neq w^*$ such that $xP''v$.

We claim that F^M is transitive on $\{P', P'', P_o''\}$. To see this, note that, for an intransitivity, there would have to exist three alternatives with respect to which $\{P', P'', P_o''\}$ constitutes a Condorcet triple. But from (17)-(19) the only way in which P'' and P_o'' differ is in how they rank x and w^o. Hence, a Condorcet triple is impossible. If F is not reasonable on $\{P', P'', P_o''\}$, we are done. Assume, therefore, that F _is_ reasonable on this domain. (In particular, this implies that if **P*** is a profile on this domain such that $m_{\mathbf{P}^*}(x,y) = m_{\mathbf{P}}(x,y)$, then $xF(\mathbf{P}^*)y$.) It follows _a fortiori_ that F is reasonable on $\{P', P_o''\}$. Notice that one can think of P_o'' as being a converted version of P'' in which (16) does not hold when $w = w^o$. Continuing iteratively, we can "convert" P'' into an ordering P_*'' such that, for _all_ $w \in X$, (16) does not hold, i.e.,

(20) $\forall w \notin \{x,y\}$, either $wP_*''y$ or $xP_*''w$,

so that F is reasonable on $\{P', P_*''\}$. Moreover, if **P*** is a profile on this domain with $m_{\mathbf{P}^*}(x,y) = m_{\mathbf{P}}(x,y)$, then $xF(\mathbf{P}^*)y$.

Similarly, we can convert P' into an ordering P_*' for which

(21) $\forall w \notin \{x,y\}$, either $wP_*'x$ or $yP_*'w$,

so that F is reasonable on $\{P_*', P_*''\}$. Moreover, if $\mathbf{P}^*$ is a profile on this domain such that $m_{\mathbf{P}^*}(x,y) = m_{\mathbf{P}}(x,y)$, then $xF(\mathbf{P}^*)y$.

Suppose there exists $z \in X$ such that

(22) $zP_*'x$ and $zP_*''y$.

If there are multiple such z, choose the one that is <u>lowest</u> in the ordering P_*''. Now choose $\bar{P} \in \wp X$ such that

(23) $y\bar{P}z\bar{P}x$, and

(24) $\bar{P}\,|_{X-\{z\}} = P_*''\,|_{X-\{z\}}$.

We claim that there do not exist three alternatives for which $\{P_*', P_*'', \bar{P}\}$ constitutes a Condorcet triple. If, contrary to the claim, there were such alternatives, (24) would imply that one would have to be z. Denote the other two by u and v, and assume, without loss of generality, that $u\bar{P}v$. From (24), we have

(25) $P_*''\,|_{\{u,v\}} = \bar{P}\,|_{\{u,v\}}$.

If say, $z\bar{P}v$, then from (23) and (24) we have

(26) $P_*''\,|_{\{z,v\}} = \bar{P}\,|_{\{z,v\}}$.

But (25) and (26) contradict the Condorcet-triple hypothesis. Hence, we may assume that

(27) $u\bar{P}v\bar{P}z$.

Since $uP_*''v$, (27) implies that for a Condorcet triple we must have

(28) $zP_*''uP_*''v$, and

(29) $vP_*'zP_*'u$.

Now (22) and (29) imply

(30) $vP_*'x$.

Therefore, $v \neq y$. Furthermore, (24), (27), and (28) imply

(31) $zP_*''vP_*''y$.

But (30) and (31) together contradict the assumption that z is the alternative satisfying (22) that is lowest in ordering P_*''. Hence there exist no Condorcet triples in $\{P_*', P_*'', \bar{P}\}$. This proves that F^M is reasonable on $\{P_*', P_*'', \bar{P}\}$. We have already shown that F is not reasonable on $\{P_*', P_*'', \bar{P}\}$.

We have been assuming that there exists $z \in X$ such that (22) holds. A similar argument applies if there exists $z \in X$ such that

(32) $yP_*'z$ and $xP_*''z$.

It remains to consider the case where, for all $z' \in X$,

(33) $z'P_*'x \Leftrightarrow xP_*''z'$.

Let z be the alternative immediately below x in P_*'', i.e.,

(34) $xP_*''zP_*''w$

for all $w \neq z$ such that $xP_*''w$. From (33) we have

(35) $zP_*'x$.

Choose Pz'' such that

(36) $Pz''|_{X\text{-}\{z\}} = P_*''|_{X\text{-}\{z\}}$

and, for all w such that $wP_*''y$,

(37) $wPz''zPz''y$.

Now, if there is a Condorcet triple in $\{P_*', P_*'', Pz''\}$ for some triple of alternatives, (36) implies that z must be one of the alternatives. Let {u,v} be the other two alternatives. From (36) we have

(38) $Pz''|_{\{u,v\}} = P_*''|_{\{u,v\}}$.

Hence, for a Condorcet triple, (34), (37), and (38) imply that {u,v} = {x,y}. But because $yP_*''xP_*''z$ and $zPz''yPz''x$, we must have $xP_*'zP_*'y$, which contradicts (35).

We conclude that F^M is transitive on $\{P_*', P_*'', Pz''\}$ and hence on $\{P_*', Pz''\}$. Moreover $F \neq F^M$ on the latter domain. The rest of the argument is thus the same as that for the case in which (22) holds. _

Remark: Caplin and Nalebuff (1988) have studied δ-majority rules when the set of alternatives is compact and convex in the n-dimensional Euclidean space ($n \geq 1$). In contrast to our undertaking here, they were not seeking to locate conditions on the domain of individual preference orderings for which voting rules are transitive. They instead sought conditions, both on individual preference orderings and on the domain of profiles, for which a given δ-majority rules generates a "social choice function" (SCF). It will be recalled that an SCF, generated by a voting rule, is a mapping that, for each profile of individual preference orderings and each subset, Y, of the set of social alternatives, selects a non-empty subset of Y, each of whose elements is at least as good as any other element of Y. It will also be recalled that a voting rule can generate an SCF and yet violate transitivity (see Section 6, below). Caplin and Nalebuff uncovered a set of conditions on individual preference orderings and their profiles, such that $\delta = 0.64$ (more precisely, $\delta = (1 - e^{-1})$) is the minimum value of δ for which the δ-majority rule generates an SCF, regardless of the size of n. But as they did not insist that the δ-majority rule be transitive, their finding is consistent with Theorem 1, even though the two results may appear contradictory.

4. Characterization of Majority Rule in Terms of Maximal Robustness

We shall say that a voting rule F is <u>positively responsive</u> on $\wp$ if, for all {x,y}, all profiles **P**, **P′** belonging to $\wp$, and all $V \subseteq [0,1]$, such that, $\mathbf{P}|_{X-\{x,y\}} = \mathbf{P'}|_{X-\{x,y\}}$, **P**(i) = **P′**(i) for all $i \notin V$, and $\mu(V) > 0$, if for all $i \in V$, {y**P**(i)x and x**P′**(i)y} implies {xF(**P**)y implies xP(F(**P′**)y}, where P(F(**P′**)) is the asymmetric factor of F(**P′**).

May (1952) established the following characterization of majority rule:

Theorem 2: A voting rule F is anonymous, neutral, and positively responsive on $\wp X$ if and only if $F = F^M$.

Notice that positive responsiveness and neutrality together imply the Pareto rule.

Theorem 1 enables us to obtain an alternative characterization of majority rule. Suppose that F is reasonable on the collection of domains $\underline{\wp} = \{\wp 1, \ldots, \wp_k\}$. We shall say that F maximally robust if there is no voting rule F′ that is reasonable on a bigger collection $\underline{\wp}' \supset \underline{\wp}$. We then have

Theorem 3: A voting rule F is anonymous, neutral, satisfies the Pareto rule on $\wp$X, and is maximally robust if and only if $F = F^M$.

Proof: If $F = F^M$, then we know that F is anonymous, neutral and satisfies the Pareto rule. Moreover, the first part of Theorem 1 establishes that F^M is reasonable. Hence F^M is maximally robust. Suppose then that F ≠ FM. It follows from the second part of Theorem 1 that there exists a domain $\wp'$ on which FM is reasonable but F is not. We conclude that no other voting rule satisfying anonymity, neutrality and the Pareto rule on $\wp$X is maximally robust. _

Remark: If we do not impose the requirement that F satisfies anonymity, neutrality, and the Pareto rule on $\wp$X, there exist maximally robust voting rules other than FM. For example, consider the voting rule F that coincides with FM for any profile **P** in a domain on which FM is reasonable, but for all other **P**, and all Y ⊆ X and all x, y ∈ Y, xI(**P**)y, where I(**P**) is the symmetric factor of F(**P**). It will be observed that such an F is reasonable on the same class of domains on which FM is reasonable. It differs from FM only on other domains.

From Lemma 1, we know that, for any triple {x,y,z} (at least) two of the six possible strict orderings {[x,y,z], [y,z,x], [z,x,y], [x,z,y], [z,y,x], [y,x,z]} must be omitted from the domain $\wp$ (one from the first group of three, and one from the second group of three) if FM is to be reasonable on $\wp$. Theorem 1 tells us that at least as many strict orderings must be omitted if any other voting rule is to be reasonable. For comparison, it will prove useful to examine specifically which orderings need be omitted if certain prominent voting rules are to be reasonable. In the next two sections we will investigate this.

5. Examples of Other Voting Rules, 1: Quasi-agreement

We say that domain $\wp$ reflects <u>quasi-agreement</u> if for all triples {x,y,z} there exists a member, say, x, such that (a) for all P $\in$ $\wp$, xPy and xPz; or (b) for all P $\in$ $\wp$, yPx, and zPx; or (c) for all P $\in$ $\wp$, either yPxPz or zPxPy.

Remark: Under quasi-agreement, for any triple {x,y,z}, (at least) four of the six possible strict orderings {[x,y,z], [y,z,x], [z,x,y], [x,z,y], [z,y,x], [y,x,z]} must be omitted from the domain $\wp$ (two from the first group of three, and two from the second group of three).

5.1 Rank-order Voting (Borda-count)

Let Q be the cardinality of X. In rank-order voting, each voter assigns weight Q to his favorite alternative, Q-1 to his next favorite, and so on. For any profile **P** in $\wp$ and any alternative x let w(x,**P**(i)) be the weight (from 1 to Q) that voter i assigns to x when his preference ordering is **P**(i). Then, for all **P** in $\wp$, all Y $\subseteq$ X and all x,y $\in$ Y, the <u>rank-order voting rule</u> (or the <u>Borda-count</u>), F^B, satisfies

$$xF^B(\mathbf{P})y \text{ if and only if } \{\int w(x,\mathbf{P}(i))d\mu(i) \geq \int w(y,\mathbf{P}(i))d\mu(i)\}.$$

Clearly, F^B is a voting rule. Moreover, it is easy to confirm that F^B satisfies anonymity, the Pareto rule, and transitivity on the unrestricted domain $\wp X$. Only neutrality presents a problem. However, F^B satisfies even this property on certain restricted domains.

Theorem 4: F^B is reasonable on $\wp$ if and only if $\wp$ reflects quasi-agreement.

Proof: Suppose first that $\wp$ reflects quasi-agreement. It suffices to show that F^B satisfies neutrality on $\wp$.

Consider two profiles **P** and **P′** in $\wp$, a pair {x,y}, and a permutation χ:X $\rightarrow$ X

As we will see in Section 8, only the "independence" part of neutrality presents a problem.

such that **P** ranks x and y the same way that **P′** ranks χ(x) and χ(y). If, for all i, x**P**(i)y, then because F^B satisfies the Pareto rule, xP(F^B(**P**))y and χ(x)P(F^B(**P′**))χ(y), where P(F^B(**P**)) and P(F^B(**P′**)) are the asymmetric factors of F^B(**P**) and F^B(**P′**), respectively. Assume therefore, that there exist two non-empty subsets of voters V1 and V2 such that, if i $\in$ V1 , then x**P**(i)y, and if i $\in$ V2, then y**P**(i)x. We claim that for all i $\in$ V1 and all j $\in$ V2

(39) w(x,**P**(i)) - w(y,**P**(i)) = w(y,**P**(j)) - w(x,**P**(j)).

To see that (39) holds, note that if for i $\in$ V1 and z $\in$ X,

(40) x**P**(i)z**P**(i)y

then quasi-agreement implies that

(41) y**P**(j)z**P**(j)x

for all j $\in$ V2. Similarly, if (41) holds for some j $\in$ V2 and z $\in$ X, then (40) holds for all i $\in$ V1.

Analogous to (39), we can show that, for all i$\in$V1 and j$\in$V2,

(42) w(χ(x),**P′**(i)) - w(χ(y),**P′**(i)) = w(χ(y),**P′**(j)) - w(χ(x),**P′**(j)).

But from (39) and (42) we conclude that {x**P**(i)y if and only if χ(x)**P′**(i)χ(y)} implies {xF^B(**P**)y if and only if χ(x)F^B(**P′**)χ(y)}. But this is neutrality.

Next suppose $\wp$ is a domain that does not reflect quasi-agreement. Then, for some triple {x,y,z}, there exist P,P′ $\in$ $\wp$ such that

(43) xPyPz, and

(44) yP′zP′x.

From (43) and (44), we have

(45) w(x,P) - w(y,P) < w(y,P′) - w(x,P′), and

(46) w(x,P) - w(z,P) > w(z,P′) - w(x,P′).

Now, from (45) and (46), we can find a generic profile **P** in which voters have either P or P′ for a preference ordering, such that

(47) yP(F^B(**P**))x and xP(F^B(**P**))z.

But (47) contradicts neutrality. _

5.2 Pareto-Extension Rule

The Pareto-extension rule, F^P, is defined as follows: for all **P** in $\wp$, all $Y \subseteq X$, and all x, y ∈ Y, (i) F^P satisfies the Pareto rule, and (ii) $xI^P(\mathbf{P})y$ if ~{y**P**(i)x for almost all i} and ~{x**P**(i)y for almost all i}, where I^P is the symmetric factor of F^P.

Observe first that F^P is a voting rule. By definition, F^P satisfies the Pareto rule. It is a simple matter also to verify that F^P satisfies anonymity and neutrality.

Theorem 5: F^P is reasonable on $\wp$ if and only if $\wp$ reflects quasi-agreement.

Proof: Suppose first that $\wp$ reflects quasi-agreement. It suffices to show that F^P is transitive. Consider a profile **P** in $\wp$ such that, for some {x,y,z}, xR^Py and yR^Pz, where $R^P \equiv F^P(\mathbf{P})$. We must show that there exists a set $V \subseteq [0,1]$ with positive Lebesgue measure, such that, for all i ∈ V, x**P**(i)z.

Because xR^Py and yR^Pz, the sets V1 = {i | x**P**(i)y} and V2 = {i | y**P**(i)z} have a positive Lebesgue measure. Now, we might as well assume that, for almost all i∈V1,

(48) z**P**(i)x**P**(i)y,

otherwise, we are done. But because $\wp$ reflects quasi-agreement, (48) implies that y**P**(i)x**P**(i)z for all i ∈ V2. Hence take V = V2.

Next suppose that $\wp$ is a domain that does not reflect quasi-agreement. Then, for some triple {x,y,z}, there exist P, P′ ∈ $\wp$ such that (43) and (44) hold. Let **P** be a generic profile in which all voters have either P and P′ as a preference ordering (and there is a positive measure of each). Since $R^P \equiv F^P(\mathbf{P})$, we have $xI^P(\mathbf{P})y$, $yP^P(\mathbf{P})z$, and $zI^P(\mathbf{P})x$, where I^P and P^P are, respectively, the symmetric and asymmetric factors of R^P. Hence, R^P is intransitive. _

5.3 2/3-Majority Rule

2/3-majority rule, $F^{2/3}$ is defined as follows: for all **P** in $\wp$, all $Y \subseteq X$, and all x, y ∈ Y, xF2/3(**P**)y if and only if μ{i | y**P**(i)x} < 2/3.

Note first that F2/3 is a voting rule. It is an easy matter to confirm that F2/3 is anonymous, neutral and satisfies the Pareto rule.

Theorem 6: F2/3 is reasonable on $\wp$ if and only if $\wp$ reflects quasi-agreement.

Proof: Suppose that $\wp$ satisfies quasi-agreement. It suffices to show that F2/3 is generically transitive on $\wp$. Consider a generic profile **P** in $\wp$ such that, for some {x,y,z},

(49) $xR^{2/3}y$, and

(50) $yR^{2/3}z$,

where $R2/3 \equiv F^{2/3}(\mathbf{P})$.

From (49), we have

(51) $\mu\{i \mid x\mathbf{P}(i)y\} > 1/3$

and, from (50)

(52) $\mu\{i \mid y\mathbf{P}(i)z\} > 1/3$.

We wish to show that $xR^{2/3}z$. Suppose it is not. Then we have

(53) $\mu\{i \mid z\mathbf{P}(i)x\} > 2/3$.

Now, (51)-(53) imply that

(54) $\mu\{i \mid z\mathbf{P}(i)x\mathbf{P}(i)y\} > 0$, and

(55) $\mu\{i \mid y\mathbf{P}(i)z\mathbf{P}(i)x\} > 0$.

But (54)-(55) together contradict quasi-agreement.

Next suppose that $\wp$ is a domain that does not reflect quasi-agreement. Then, for some triple, {x,y,z} there exist P, $P' \in \wp$ such that (43) and (44) hold. Let **P** be a generic profile in which about half the voters have preference ordering P and the remainder have P′. Writing by $I^{2/3}$ and $P^{2/3}$, respectively, for the symmetric and asymmetric factors of $R^{2/3}$, we have $xI^{2/3}y$, $yP^{2/3}z$, and $zI^{2/3}x$. But this means that $R^{2/3}$ is intransitive. _

6. Examples of Other Voting Rules, 2: Strong Quasi-agreement

We say that $\wp$ reflects strong quasi-agreement if for all triples {x,y,z} there exists a member, say x, such that either (a) for all P $\in \wp$, xPy and xPz; or (b) for all P $\in \wp$, yPx and zPx.

Remark: Under strong quasi-agreement, for any triple {x,y,z}, (at least) four of the six possible strict orderings {[x,y,z], [y,z,x], [z,x,y], [x,z,y], [z,y,x], [y,x,z]} must be omitted from the domain $\wp$ (two from the first group of three, and two from the second group of three). But note that strong quasi-agreement is a more demanding condition than quasi-agreement.

In order to discuss the next two examples of voting rules that we wish to contrast with majoritarianism, it will be necessary to introduce further definitions:

We will say that a voting rule, F, satisfies contraction consistency (also called "Nash's independence of irrelevant alternatives" and "property α") on $\wp$ if, for all **P** in $\wp$, all $Z \subseteq Y \subseteq X$, and all $x \in Z$, {xF(**P**)y for all $y \in Y$} implies {xF(**P**)y for all $y \in Z$}. We will say that a voting rule, F, satisfies expansion consistency (also called "property ß") on $\wp$ if, for all **P** in $\wp$, all $Z \subseteq Y \subseteq X$, and all x, $y \in Z$, {xF(**P**)z for all $z \in Z$, yF(**P**)z for all $z \in Z$, and xF(**P**)z for all $z \in Y$} implies {yF(**P**)z for all $z \in Y$}. It is a simple matter to prove that, if F is a voting rule, contraction consistency and expansion consistency are together equivalent to transitivity.

6.1 Random Dictatorship

Random dictatorship, F^{RD}, is defined as follows. For all **P** in $\wp$, all $Y \subseteq X$, and all x, $y \in Y$, xF^{RD}(**P**)y if and only if, either (a) μ{i | x**P**(i)z for all $z \in Y$, $z \neq x$} > 0, or (b) μ{i | x**P**(i)z for all $z \in Y$, $z \neq x$} = μ{i | y**P**(i)z for all $z \in Y$, $z \neq y$} = 0 and μ{i | x**P**(i)y} > 0.

Theorem 7: F^{RD} is reasonable on $\wp$ if and only if $\wp$ reflects strong quasi-agreement.

Proof: We note first that F^{RD} is a voting rule, satisfying anonymity, neutrality, and the Pareto rule on $\wp_X$. Suppose then that $\wp$ reflects strong quasi-agreement. We need to show that F^{RD} is transitive on Y.

It is a simple matter to show that F^{RD} satisfies contraction consistency. We need, therefore, to prove that F^{RD} satisfies expansion consistency as well. Consider $Z \subseteq Y \subseteq X$, **P** in $\wp$, and x, $y \in Z$, such that $xF^{RD}(\mathbf{P})z$ and $yF^{RD}(\mathbf{P})z$ for all $z \in Z$ and $xF^{RD}(\mathbf{P})z$ for all $z \in Y$. We must show that $yF^{RD}(\mathbf{P})z$ for all $z \in Y$. Suppose the contrary. Then there exists $z \in Y$ such that $zP(F^{RD}(\mathbf{P}))y$, where $P(F^{RD}(\mathbf{P}))$ is the asymmetric factor of $F^{RD}(\mathbf{P})$. This can be if either

(56) For almost all i, $z\mathbf{P}(i)y$; or

(57) For almost all i, there exists $z' \in Y$ such that $z'\mathbf{P}(i)y$ and $\mu\{i \mid z\mathbf{P}(i)z'$ for all $z' \in Y$, $z \neq z'\} > 0$.

Suppose (56) holds. Let $V_1 = \{i \mid y\mathbf{P}(i)x\}$. Because $yF^{RD}(\mathbf{P})z$ for all $z \in Z$, $\mu(V_1) > 0$. Furthermore,

(58) $z\mathbf{P}(i)y\mathbf{P}(i)x$ for almost all $i \in V_1$.

Let $V_2 = \{i \mid x\mathbf{P}(i)y\}$. Because $xF^{RD}(\mathbf{P})z$ for all $z \in Z$, $\mu(V_2) > 0$. Furthermore, in view of (58) and strong quasi-agreement, we have

(59) $z\mathbf{P}(i)x\mathbf{P}(i)y$ for almost all $i \in V_2$. But $V1 \cup V2 = [0,1]$; and so, (58), (59), and the Pareto rule imply that there exists $z \in Y$ such that $zF^{RD}(\mathbf{P})x$ and $\sim\{xF^{RD}(\mathbf{P})z\}$, which is a contradiction. We conclude that $yF^{RD}(\mathbf{P})z$ for all $z \in Y$ after all.

Assume next that (57) holds. Once again, let $V_1 = \{i \mid y\mathbf{P}(i)x\}$ and $V_2 = \{i \mid x\mathbf{P}(i)y\}$. As before, we may conclude that $\mu(V_1), \mu(V_2) > 0$. Consider $z \in Y$ such that for a positive measure of voters in V_1, $z\mathbf{P}(i)y$. (As X is finite, z exists.) For these voters, $z\mathbf{P}(i)y\mathbf{P}(i)x$. In view of strong quasi-agreement, however, $z\mathbf{P}(i)x\mathbf{P}(i)y$ for all $i \in V_2$. From this we may also conclude that $z\mathbf{P}(i)y\mathbf{P}(i)x$ for all $i \in V_1$. But $V1 \cup V2 = [0,1]$. This implies that $zF^{RD}(\mathbf{P})x$ and $\sim\{xF^{RD}(\mathbf{P})z\}$, which is a contradiction. So $yF^{RD}(\mathbf{P})z$ for all $z \in Y$ after all.

To prove the converse, assume that $\wp$ does not reflect strong quasi-agreement. Consider a non-pathologic **P** belonging to $\wp$, such that $xF^{RD}(\mathbf{P})z'$ and $yF^{RD}(\mathbf{P})z'$ for all z′

$\in$ Z, $xF^{RD}(\mathbf{P})z'$ for all $z' \in Y$, and there exists $z \in Y\text{-}Z$, with the property that $z\mathbf{P}(i)y\mathbf{P}(i)x$ for $i \in V_1$ and <u>either</u> (a) $x\mathbf{P}(i)z\mathbf{P}(i)y$ for $i \in V_2$, where, as before, $V_1 = \{i \mid y\mathbf{P}(i)x\}$ and $V_2 = \{i \mid x\mathbf{P}(i)y\}$, <u>or</u> (b) $x\mathbf{P}(i)y\mathbf{P}(i)z$. We know that $\mu(V_1)$, $\mu(V_2) > 0$ and $V1 \cup V2 = [0,1]$. By construction, **P** violates strong quasi-agreement. Now note that $zP(F^{RD}(\mathbf{P}))y$ in either case (a) or case (b). This is inconsistent with expansion consistency. _

6.2 Plurality Rule

<u>Plurality rule</u>, F^{PL}, is defined as follows. For all **P** in $\wp$, all $Y \subseteq X$, and all $x, y \in Y$, $xF^{PL}(\mathbf{P})y$ if and only if <u>either</u> (a) $\mu\{i \mid x\mathbf{P}(i)z \text{ for all } z \in Y\} \geq \mu\{i \mid y\mathbf{P}(i)z \text{ for all } z \in Y\}$, where $\mu\{i \mid x\mathbf{P}(i)z \text{ for all } z \in Y\} > 0$, <u>or</u> (b) $\mu\{i \mid x\mathbf{P}(i)z \text{ for all } z \in Y\} = \mu\{i \mid y\mathbf{P}(i)z \text{ for all } z \in Y\} = 0$ and $\mu\{i \mid x\mathbf{P}(i)y\} > 0$.

Theorem 7: F^{PL} is reasonable on $\wp$ if and only if $\wp$ satisfies strong quasi-agreement.

Proof: Similiar to that of Theorem 6. _

7. Relaxing Transitivity and Anonymity

Are the conditions on voting rules postulated in Theorem 1 tight? In this section we shall relax transitivity and anonymity. (In Section 8 we will explore one particular route to weakening neutrality.)

As is well known (see Remark following the proof of Theorem 1), a voting rule does not need to be transitive if it is to yield a collective decision over elements of Y ($\subseteq$ X). In this context, a condition that has been much studied is quasi-transitivity (Sen, 1969). In Section 7.1 we shall study the robustness of majority rule if quasi-transitivity replaces transitivity as a requirement on voting rules. In Section 7.2 we will demonstrate that anonymity can also be relaxed considerably even while retaining a part of Theorem 1.

7.1 Quasi-transitivity

We shall say that a voting rule F satisfies <u>quasi-transitivity</u> (also called <u>P-</u>

transitivity) on $\wp$ if, for all profiles **P** in $\wp$, xPy and yPz imply xPz, where P is the asymmetric factor of $R \equiv F(\mathbf{P})$. There are voting rules (e.g. the Pareto extension rule) that satisfy anonymity, neutrality, the Pareto rule, and quasi-transitivity generically on $\wp X$. However, Gibbard (1969) showed that, in the case of a finite number of voters, all such voting rules represent "oligarchic" forms of collective choice: for any such voting rule there is an identifiable and unique group of voters such that, for all $Y \subseteq X$ and all $x, y \in Y$, if any member of this group strictly prefers x to y, then xRy, and if all members of this group strictly prefer x to y, then xPy. In short, members of the oligarchy have veto power.

The idea of an oligarchy can be extended to the case of a continuum of voters. We shall say that an ε-veto set for a voting rule F on $\wp$ is a subset $C \subseteq [0,1]$ with $\mu(C) = \varepsilon$ such that, for all profiles **P** in $\wp$ and for all x,y, if x**P**(i)y for all $i \in C$, then xF(**P**)y. We shall say that F satisfies no veto-power if there exists $\varepsilon^* > 0$ such that, for all $\varepsilon < \varepsilon^*$, F does not have an ε-veto set.

Theorem 9: Suppose that F satisfies anonymity, neutrality, the Pareto rule, quasi-transitivity, and no veto-power generically on $\wp$. Then F^M does too.

Proof: The argument follows that of Theorem 1. As before, let q (> 2) be an integer. We consider profiles in which all voters with index in [k / q, (k+1) / q] have the same preference ordering. The aim is to show that if F satisfies the conditions of the theorem generically on $\wp$, then $\wp$ cannot contain a Condorcet triple. The result will then follow from Lemma 1.

Consider a profile **P** in which, $\forall i \in [0, 1/q]$, x**P** (i)y and $\forall i \notin [0, 1/q]$, y**P**(i)x. Write by P the asymmetric factor of F(**P**). Then we can rule out xPy and yPx the same way as in the proof of Theorem 1. Suppose, therefore, that xIy, where I is the symmetric

See also Guha (1972), Mas-Colell and Sonnenschein (1972), and Blau and Brown (1989) for related results.

factor of F(**P**). We can choose q big enough so that $(1/q) < \varepsilon^*$, in which case xIy violates the condition that there is no veto-power. _

7.2 Individual Responsiveness

We can relax anonymity quite a bit even while retaining part of Theorem 1. We shall call a voting rule F <u>individually responsive</u> on $\wp$ if, for any integer q, any pair {x,y}, and any subset of voters $V \subseteq [0,1]$ such that $\mu(V) = 1/q$, there exist preference orderings, $P, P' \in \wp$ with xPy and yP′x and a profile $\mathbf{P}_{-V}$ for voters not in V, such that ~{yF(**P**)x} and yF(**P′**)x, where **P**(i) = P and **P′**(i) = P′ for all $i \in V$, and in either case $\mathbf{P}_{-V}$ is the profile for all $i \notin V$.

We may now prove

Lemma 2: Any voting rule that satisfies anonymity and the Pareto rule on $\wp$ is also individually responsive, provided that, for all {x,y}, there exist orderings $Px, Py \in \wp$ for which xPxy and yPyx.

Proof: Consider a profile $\mathbf{P}^o$ in which everyone has ordering Px. By the Pareto rule, we have ~{yF($\mathbf{P}^o$)x}. Choose integer q > 2 and consider successively $\mathbf{P}^1, \mathbf{P}^2, \ldots, \mathbf{P}^q$ where, for each k = 1, 2,..., q, **P**k is the profile in which voters [0,k / q] have ordering P_y and voters [k / q, 1] have ordering P_x. By the Pareto rule, we have yF($\mathbf{P}^q$)x. But this means that there exists $k^* \in \{1,\ldots, q\}$ such that ~{yF($\mathbf{P}^{k^*-1}$)x} and {yF($\mathbf{P}^{k^*}$)x}.

Consider the subset $V(k^*,q) \equiv [(k^*-1)/q,\ k^*/q]$. Let $P = P_x$, $P' = P_y$. Define the profile $\mathbf{P}_{-V(k^*,q)}$ as: $\mathbf{P}_{-V(k^*,q)}(i) = P_y$ for $i \in [0, k^*/q]$ and $\mathbf{P}_{-V(k^*,q)}(i) = P_x$ for $i \in [k^*/q, 1]$. Notice that the conditions for individual responsiveness are satisfied. Finally observe that, since the voting rule is anonymous, the argument is not restricted to V(k*,q): we can construct the same argument for any other subset of measure 1 / q. _

Theorem 10: Suppose that a voting rule F is transitive, neutral, individually responsive, and satisfies the Pareto rule generically on $\wp$. Then F^M does so too.

Proof: In view of Lemma 1, it suffices to show that, for all {x,y,z}, at least one of [x,y,z], [y,z,x], and [z,x,y] does not belong to $\wp$.

Suppose, to the contrary, that all three do. Choose integer $q > 2$ with the property that, on the domain of profiles in $\wp$ in which, for each $k = 1,..., q$, everyone with index in $V(k,q) = [(k-1)/q, k/q]$ has the same preference ordering, F satisfies all four properties. We claim that, for all k, if $\mathbf{P}_k$ is a profile such that all voters in V(k,q) prefer x to y and all remaining voters prefer y to x, then $xF(\mathbf{P}k)y$. If this claim holds, then we can readily establish a contradiction to the hypothesis that [x,y,z], [y,z,x], and [z,x,y] belong to $\wp$. Specifically using the method of the proof of Theorem 1, we can show (if all three orderings belong to $\wp$) that the existence of a profile **P,** for which all voters in [0, 1/q] prefer x to y, those in [1/q, 1] prefer y to x, and $xF(\mathbf{P})y$ implies that, for all k, there exists a profile **P′** for which all voters in [0, 1/q] prefer x to y, those in [1/q, 1] prefer y to x, and $xF(\mathbf{P}')y$. But, for $k = q$, this conflicts with neutrality and the definition of $\mathbf{P}^q$, unless we also have $yF(\mathbf{P})x$. Furthermore, in the latter case, if we consider the profile in which everyone in [0, 1/q] has the ordering [x,y,z], and everybody in [1/q, 1] has the ordering [y,z,x], we conclude that $x\hat{I}y\hat{P}z\hat{I}x$, where $\hat{R}$ is the social preference ranking corresponding to this profile; and $\hat{I}$ and $\hat{P}$ are its symmetric and asymmetric factors, respectively. But this violates transitivity.

So then suppose, contrary to the initial claim, that there exists $\mathbf{P}_k^*$ such that $x\mathbf{P}_k^*(i)y$ for all $i \in V(k,q)$, $y\mathbf{P}_k^*(i)x$ for all $i \notin V(k,q)$, and yet

(60) $\sim\{xF(\mathbf{P}_k^*)y\}$.

From the fact that F is individually responsive, there exists a profile $\mathbf{P}_{-V(k,q)}$ for the voters not in V(k,q) and orderings P_x and P_y, such that

(61) $\sim\{yF(\mathbf{P}_x)x\}$, and

(62) $yF(\mathbf{P}_y)x$,

where $\mathbf{P}_x$ and $\mathbf{P}_y$ are the profiles in which voters in V(k,q) have orderings P_x and P_y, respectively, and voters not in V(k,m) have the profile $\mathbf{P}_{-V(k,q)}$.

Let $V_x = \{i \mid i \notin V(k,q)$ and $x\mathbf{P}_{-V(k,q)}(i)y\}$ and $V_y = \{i \mid i \in V(k,q)$ and $y\mathbf{P}_{-V(k,q)}(i)x\}$. Now, if V_x is empty, then (61) and the hypothesis that F is neutral contradict (60); whereas, if V_y is empty, then (61) and the hypothesis that F is neutral contradict (60). Hence, we may assume that V_x and V_y are non-empty.

Consider the profile $\mathbf{P}'$ in which: for all $i \in V(k,q)$, $x\mathbf{P}'(i)y\mathbf{P}'(i)z$; for all $i \in V_y$, $y\mathbf{P}'(i)z\mathbf{P}'(i)x$; and, for all $i \in V_x$, $z\mathbf{P}'(i)x\mathbf{P}'(i)y$. Let R' be the social preference ranking corresponding to $\mathbf{P}'$. From (61) and (62), $xR'yR'z$. Hence, from transitivity:

(63) $xR'z$.

But because, for all $i \in V(k,q)$, $x\mathbf{P}'(i)z$, and for all $i \notin V(k,q)$, $z\mathbf{P}'(i)x$, (63) contradicts (60). We conclude that [x,y,z], [y,z,x] and [z,x,y] cannot all belong to $\wp$. _

Remark: Note that the counterpart of the second assertion of Theorem 1 is not contained in Theorem 10. F^M is <u>not</u> uniquely maximally robust among those voting rules that satisfy neutrality, the Pareto rule, and individual responsiveness.

To see this, consider a measurable function $w:[0,1] \to R+$. We shall call F^{WM} a <u>weighted majority rule with weight w</u> if, for all $\mathbf{P}$, all $Y \subseteq X$, and all $x,y \in Y$, $xF^{WM}(\mathbf{P})y$ if and only if:

$$\int_{i\in V(\mathbf{P},x,y)} w(i)d\mu(i) \geq \int_{i\in V(\mathbf{P},y,x)} w(i)d\mu(i),$$

where $V(\mathbf{P},x,y) \equiv \{i \mid x\mathbf{P}(i)y\}$.

We now note:

Theorem 11: For any measurable function w(.), F^{WM} is transitive generically on $\wp$ if and only if F^M is transitive generically on $\wp$.

Proof: Obvious. _

Theorem 11 enables us to conclude that Theorem 10 holds when F^M is replaced by F^{WM}. In other words, weighted majority voting rules are also maximally robust.

8. Non-Neutrality and the Unanimity Rule

A voting rule F satisfies <u>independence of irrelevant alternatives</u> on $\wp$ if, for all profiles **P** and **P′** in $\wp$, all $Y \subseteq X$, and all $x, y \in Y$, $\mathbf{P}|_Y = \mathbf{P}'|_Y$ implies that $xF(\mathbf{P})y \Leftrightarrow xF(\mathbf{P}')y$, where $\mathbf{P}|_Y$ and $\mathbf{P}'|_Y$ are, respectively, the restrictions of **P** and **P′** to Y.

It is well known that neutrality is sharper than independence of irrelevant alternatives, in that, if a voting rule is neutral on $\wp$, it is independent of irrelevant alternatives on $\wp$; but the reverse implication does not hold.

In what follows, we will relax neutrality in the following way:

Consider a profile **P** such that, for all x and y,

(64) $\mu\{i \mid x\mathbf{P}(i)y\} = \mu\{i \mid y\mathbf{P}(i)x\} = ½$.

If F were to satisfy anonymity and neutrality, we would have $xI(F(\mathbf{P}))y$ for all x and y, where I is the symmetric factor of F. We wish to weaken neutrality so that ties like this might be broken. However, we will require them to be broken in a consistent way. Specifically, we will say that F satisfies <u>tie-breaking consistency</u> on $\wp$ if there exists an ordering R_F^* such that, for all x and y, and some profile **P** on $\wp$ for which (64) holds,

(65) $xR_F^*y \Leftrightarrow xF(\mathbf{P})y$.

Note that if F satisfies anonymity and neutrality, the "tie-breaker" R_F^* is the ordering in which all alternatives are deemed indifferent. Observe too that tie-breaking consistency is much weaker than neutrality: not only does it allow ties to be broken, but it applies at most to one profile satisfying (64). Indeed, tie-breaking consistency is restrictive only in that it requires R_F^* to be transitive.

Let P_o be a strict ordering on X. We define <u>unanimity rule with order of precedence</u> P_o to be the voting rule $F^u_{P_o}$ such that, for all **P**, all all $Y \subseteq X$, and all $x,y \in Y$, $xF^u_{P_o}(P)y$ _ $xP_o y$ unless $y\mathbf{P}(i)x$ for almost all i, in which case $yF^u_{P_o}(P)x$..

Remark: The ordering Po can be interpreted as prescribing an order of precedence. That is, if xPoy, then x is considered better than y unless almost everyone prefers y to x. Thus, the top-ranked alternative according to Po can be thought of as the <u>status quo</u>.

We now state a result analogous to Theorem 1.

Theorem 12: Let F be a voting rule that satisfies anonymity, the Pareto rule, tie-breaking consistency, independence of irrelevant alternatives, and transitivity on $\wp$. Then there exists a strict ordering Po such that $F^{u}_{P_o}$ satisfies these conditions too. Moreover, if there exists a profile **P** in $\wp$ such that $F(P) \neq F^{u}_{P_o}(P)$, then there exists a domain $\wp'$ on which $F^{u}_{P_o}$ satisfies these conditions but F does not.

We will prove this via two Lemmas. We note first though that, starting from the voting rule F, we can create a <u>two-person</u> voting rule, $F_{1/2}$, such that, for all P1, P2 $\in \wp X$ and all x,y $\in$ X,

$xF_{1/2}(\{P1,P2\})y \Leftrightarrow xF(\mathbf{P}_{12})y$,

where $\mathbf{P}_{12}$ is the profile such that

(66)

If F satisfies tie-breaking consistency and independence of irrelevant alternatives on $\wp$, then, for all x,y and all P1, P2 $\in \wp$ such that xP1y and yP2x,

(67) $xR_F{*}y \Leftrightarrow xF_{1/2}(\{P1,P2\})y$.

We now have:

Lemma 3: Suppose that F satisfies anonymity, the Pareto rule, tie-breaking consistency and independence of irrelevant alternatives on $\wp$. $F_{1/2}$ is transitive on $\wp$, if and only if, for all x,y,z:

(i) $(xR_F{*}y, yR_F{*}z, xP_F{*}z) \Rightarrow$ either [y,z,x] or [z,x,y] $\notin \wp$,

(ii) $(xI_F{*}y, yI_F{*}z) \Rightarrow$ 2 strict orderings from each Condorcet-cycle are absent from $\wp$.

Proof: Suppose that F satisfies the hypotheses. Assume first that $F_{1/2}$ is transitive on $\wp$. Assume too that $xR_F{*}y$, $yR_F{*}z$, and $xP_F{*}z$, but $\{[y,z,x], [z,x,y]\} \subseteq \wp$. Let P1 = [y,z,x], and P2 = [z,x,y]. Then it must be that $xF_{1/2}(\{P1,P2\})y$, $yF_{1/2}(\{P1,P2\})z$, and

$zP(F_{1/2}(\{P1,P2\}))x$, where $P(F_{1/2}(.))$ is the asymmetric factor of $F_{1/2}(.)$. But this is inconsistent with transitivity of $F_{1/2}$. Hence, either $[y,z,x] \notin \wp$ or $[z,x,y] \notin \wp$. This confirms (i).

Suppose instead that $xI_F^*yI_F^*z$. If there exist two orderings in $\wp$ from the same Condorcet triple, say, [x,y,z] and [y,z,x], then if we take P1 = [x,y,z] and P2 = [y,z,x], we obtain $xF_{1/2}(\{P1,P2\})yP(F_{1/2}(\{P1,P2\}))zF_{1/2}(\{P1,P2\})x$, which again contradicts transitivity. This confirms (ii).

To establish the reverse implication in the Lemma, assume that $F_{1/2}$ is intransitive on $\wp$. Then, there exist $P1,P2 \in \wp$, such that, either

(68) $xF_{1/2}(\{P1,P2\})yF_{1/2}(\{P1,P2\})zF_{1/2}(\{P1,P2\})x$ with at least one strict preference, or

(69) $xF_{1/2}(\{P1,P2\})zF_{1/2}(\{P1,P2\})yF_{1/2}(\{P1,P2\})x$, with at least one strict preference.

Let us first assume that

(70) $xR_F^*yR_F^*z$ (with at least one strict preference) and xP_F^*z.

If (68) holds, then from (70), we may infer:

(71) zP1x and zP2x.

Since $yF_{1/2}(\{P1,P2\})z$, one of the two voters must prefer y to z. Assume then that

(72) yP1zP1x.

Now (71), (72) and $xF_{1/2}(\{P1,P2\})y$ imply that

(73) zP2xP2y.

Note now that (70), (72) and (73) together constitute a violation of the implication of (i), namely, "if (64) holds, then either $[y,z,x] \notin \wp$, or $[z,x,y] \notin \wp$."

Now suppose that (69) holds. Then, if

(74) yP_F^*z, we must have

(75) zP_1y and zP_2y.

However, $yF_{1/2}(\{P1,P2\})x$ implies that at least one voter prefers y to x. Hence, we may assume

(76) yP1x.

But (75) and (76) imply,

(77) zP1yP1x.

Therefore, because $xF_{1/2}(\{P1,P2\})z$, we must have

(78) xP2zP2y.

From (70), (78) and $yF_{1/2}(P1,P2)x$, we have

(79) $yI_F{*}x$.

Moreover, (70), (74) and (79) imply

(80) $yR_F{*}xR_F{*}z$ and $yP_F{*}z$.

However, (70), (77) and (78) constitute a violation of the implication of (i), namely, "if (80), then either $[z,y,x] \notin \wp$ or $[x,z,y] \notin \wp$."

Consider next the case where, instead of (74), we have

(81) $yI_F{*}z$.

Then it follows that

(82) $xP_F{*}y$.

Hence, because $yF_{1/2}(\{P1,P2\})x$, we may infer that

(83) yP_1x and yP_2x.

Furthermore, $xF_{1/2}(\{P1,P2\})z$ implies that at least one voter must prefer x to z. So we may assume

(84) yP1xP1z.

Since $zF_{1/2}(\{P1,P2\})y$, we have

(85) zP2yP2x.

But (70), (81) and (82) imply

(86) $xR_F{*}zR_F{*}y$ and $xP_F{*}y$.

Thus, as required, (84) and (85) constitute a violation of the implication of (i), namely, "if (86), then either $[y,x,z] \notin \wp$ or $[z,y,x] \notin \wp$."

Finally, let us assume the antecedent of (ii):

(87) $xI_F{}^*yI_F{}^*zI_F{}^*x$.

We must show that, for one of the two Condorcet triples, fewer than two orderings are omitted from $\wp$. Suppose, to the contrary, that two orderings are omitted from each Condorcet cycle. Without loss of generality assume that P1 = [x,y,z] $\in$ $\wp$.

If P2 = [x,z,y] $\in$ $\wp$, then

$xP(F_{1/2}(\{P1,P2\}))yI(F_{1/2}(\{P1,P2\}))z$ and $xP(F_{1/2}(\{P1,P2\}))z$.

If P2 = [z,y,x] $\in$ $\wp$, then

$xI(F_{1/2}(\{P1,P2\}))yI(F_{1/2}(\{P1,P2\})zI(F_{1/2}(\{P1,P2\}))x$.

If P2 = [y,x,z] $\in$ $\wp$, then

$xI(F_{1/2}(\{P1,P2\}))yP(F_{1/2}(\{P1,P2\}))z$ and $xP(F_{1/2}(\{P1,P2\}))z$.

Hence, in all cases, the social ranking is transitive. We conclude that, for an intransitivity, at most one ordering may be omitted from some Condorcet triple. _

Let $F^o_{1/2}$ be the two-person voting rule such that, for all P1,P2 $\in$ $\wp$ and all x,y $\in$ X,

$$xF^0_{1/2}(\{P_1, P_2\})\,y \_ xF^u_{P_o}(P_{12})y,$$

where $\mathbf{P}_{12}$ is the profile satisfying (66).

Note that if xP1y and yP2x then xP0y $\Leftrightarrow$ $xF^o_{1/2}(\{P_1,\ P_2\})\,y$. We then have

Lemma 4: $F^o_{1/2}$ is transitive on $\wp$ if and only if $F^u_{P_o}$ is transitive on $\wp$.

Proof: If $F^u_{P_o}$ is transitive on $\wp$, then it is immediate that $F^o_{1/2}$ is too. Assume, therefore, that $F^o_{1/2}$ is transitive on $\wp$. We must show that $F^u_{P_o}$ is transitive. Consider profile **P** on $\wp$ and three alternatives x,y, and z. Assume that

(88) xPoyPoz.

If $F^u_{P_o}(P)$ is intransitive, then either

(89) $xF^u_{P_o}(P)yF^u_{P_o}(P)zF^u_{P_o}(P)x$, or

(90) $xF^u_{P_o}(P)zF^u_{P_o}(P)yF^u_{P_o}(P)x$.

Hence, from (88), we must have either

(91) $z\mathbf{P}(i)x$ for almost all i, (from (89)), or

(92) $z\mathbf{P}(i)y$ for almost all i, (from (90)).

If (89) holds, then, there must exist i such that $x\mathbf{P}(i)y$ and j such that $y\mathbf{P}(j)z$, which in view of (91) imply that

(93) $\mathbf{P}(i) = [z,x,y]$ and $\mathbf{P}(j) = [y,z,x,]$.

But because $F^{o}_{1/2}$ is transitive on $\wp$, (88) and Lemma 3 (with P_o replacing RF*) imply that (93) is impossible.

If, on the other hand, (90) holds, then there must exist i such that $x\mathbf{P}(i)z$, which in view of (92) implies that

(94) $\mathbf{P}(i) = [x,z,y]$.

But because $x\mathbf{P}(i)y$, the relation $yF^{u}_{P_o}(P)x$ contradicts xP_oy, and so (94) cannot hold. We conclude that $F^{u}_{P_o}(P)$ is transitive. _

Proof of Theorem 12: Suppose that F satisfies anonymity, tie-breaking consistency, independence of irrelevant alternatives, the Pareto rule, and transitivity on $\wp$. Because F satisfies tie-breaking consistency, there exists an ordering RF* satisfying (64) and (65). Moreover, because F satisfies anonymity and independence of irrelevant alternatives, R_F* satisfies (65) for all profiles satisfying (64). Choose a strict ordering Po which is consistent with R_F*; that is, for all $x,y \in X$, $xP_F{*}y \Rightarrow xP_oy$, where P_F* is the asymmetric factor of R_F*. Let I_F* denote the symmetric factor of R_F*. If there do not exist $x,y \in X$ such that $xI_F{*}y$, then we may conclude P_o = RF*.

Suppose that, for some triple $x,y,z \in X$,

(95) xP_oyP_oz.

Since F is transitive on $\wp$, so is $F_{1/2}$. Hence from (95), the fact that P_o is consistent with R_F*, and Lemma 3, together imply that

(96) either $[y,z,x] \notin \wp$ or $[z,x,y] \notin \wp$.

But Lemma 3, (95), and (96) then imply that $F^{o}_{1/2}$ is transitive on $\wp$. From Lemma 4, we may conclude that $F^{u}_{P_o}$ does too.

Next suppose that for some **P** on $\wp$, $F(P)_{-\{x,y\}} \neq F^{u}_{P_o}(P)_{-\{x,y\}}$. Then there exist x,y $\in$ X such that

(97) $F(P)_{-\{x,y\}} \neq F^{u}_{P_o}(P)_{-\{x,y\}}$.

Since P_o is strict, we can assume

(98) xP_oy.

Hence, from (97) and (98), we have

(99) $x F^{u}_{P_o}(\mathbf{P})y$, and

(100) $yF(\mathbf{P})x$.

Let P_{oo} be the "reverse" of P_o; that is, $aP_{oo}b \Leftrightarrow bP_oa$, for all a,b $\in$ X. We claim that $F^{u}_{P_o}$ is transitive on $\wp \cup\{P_o,P_{oo}\}$. To see this, note that, from Lemma 3, $F^{o}_{1/2}$ is transitive on a domain $\underline{\hat{\ }}$, provided that, for all a,b,c with aP_obP_oc,

(101) either $[b,c,a] \notin \underline{\hat{\ }}$ or $[c,a,b] \notin \underline{\hat{\ }}$.

Condition (101) holds for $\underline{\hat{\ }} = \wp$ because $F^{u}_{P_o}$ is transitive on $\wp$. But clearly (101) continues to hold if we add P_o and P_{oo} to $\wp$ (since neither [b,c,a] nor [c,a,b] is consistent with P_o or P_{oo}). Hence, from Lemma 4, $F^{u}_{P_0}$ is transitive on $\wp_o \equiv \wp \cup\{P_o,P_{oo}\}$. If F fails to satisfy anonymity, the Pareto rule, tie-breaking consistency, independence of irrelevant alternatives, and transitivity on $\wp_o$, then we can take $\wp' = \wp_o$, and we are done. Therefore assume that F <u>does</u> satisfy these properties on $\wp_o$. In particular, this implies that the same tie-breaker R_F* applies to $\{P_o,P_{oo}\}$ as to $\wp$.

Suppose, for the moment, that y is not the lowest alternative in the ranking P_o. Let z be the alternative just below y. Let $\dot{P}_o$ be the same as P_o except that y and z are

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If instead $y F^{u}_{P_o}(\mathbf{P})x$, then from (98), we would have $y\mathbf{P}(i)x$ for almost all i. But then (97) would imply that F violates the Pareto rule.

interchanged. Let P_o* be the same as P_o except that z has moved to just above x. Finally, let P_o** be the same as P_o except that x and y are interchanged. Let $\wp^* = \{P_o, \hat{P}_o, P_{oo}, P_o^*, P_o^{**}\}$. Notice that, for all a,b,c $\in$ X such that aP_obP_oc, $[b,c,a] \notin \wp^*$. Hence $F^u_{P_o}$ is transitive on $\wp^*$. If F does not satisfy anonymity, the Pareto rule, independence of irrelevant alternatives, tie-breaking consistency, and transitivity on $\wp^*$, we are done. Therefore, assume that F <u>does</u> satisfy these properties. This means, in particular, that the same tie-breaker R_F^* applies to $\wp^*$ as to $\{P_o, P_{oo}\}$ in $\wp$. Suppose

(102) xI_F^*y.

Then, because xP_oyP_oz and R_F^* is consistent with P_o, we have

(103) $yI_F^*xR_F^*z$.

But from (103) and Lemma 3, $F_{1/2}$ is intransitive on $\wp^*$ if

(104) $[x,z,y], [z,y,x] \in \hat{\underline{\ }}$,

which holds because $x\hat{P}_oz\hat{P}_oy$ and $zP_{oo}yP_{oo}x$. We conclude that

(105) xP_F^*y.

Now suppose that $m_{\mathbf{P}}(x,y) = m_{\mathbf{P}}(y,x) = ½$. Since P_o is consistent with R_F^*, (97) and (98) imply that xI_F^*y, a contradiction of (105). It follows that either

(106) $m_{\mathbf{P}}(x,y) > m_{\mathbf{P}}(y,x)$, or

(107) $m_{\mathbf{P}}(x,y) < m_{\mathbf{P}}(y,x)$.

If (106) holds, consider the following profile $\mathbf{P}^*$ on $\wp^*$:

$\mathbf{P}^*(i)$

Now $m_{\mathbf{P}^*}(x,y) = m_{\mathbf{P}}(x,y)$. Hence from (100), we have

(108) $yF(\mathbf{P}^*)x$.

Also, $m_{P^*}(x,z) = ½$. Therefore, since xPoz, we have

(109) $xF(\mathbf{P}^*)z$.

Finally, from the Pareto rule, we conclude that

(110) $zP(F(\mathbf{P}^*))y$.

But (108)-(110) imply that F is intransitive on $\wp^*$. Hence we can take $\wp' = \wp^*$.

If (107) holds, there are two cases to consider:

Case 1 There exists a profile $\mathbf{P}'$ on $\wp^*$ such that

(111) $m_{P'}(y,z) < ½$ and $zF(\mathbf{P}')y$.

Consider the profile $\mathbf{P}^{**}$ such that

$$\mathbf{P}^{**}(i) = \begin{cases} \bar{P}_o, & i \in [0, m_P(x,y)] \\ P_o^{**}, & i \in [m_P(x,y), m_P(x,y) + m_{P'}(y,z)] \\ P_o^*, & i \in [m_P(x,y) + m_{P'}(y,z), 1] \end{cases}$$

Since $m_{P^{**}}(x,y) = m_P(x,y)$, we conclude from (100) that

(112) $yF(\mathbf{P}^{**})x$.

Moreover, because $m_{P^{**}}(y,z) = m_{P'}(y,z)$, (111) implies that

(113) $zF(\mathbf{P}^{**})y$.

Now $m_{P^{**}}(x,z) > m_{P^{**}}(z,x)$. Therefore, if $zF(\mathbf{P}^{**})x$, we can apply the same argument as in the case where (106) holds to conclude that F is intransitive on $\wp^*$. Hence assume that

(114) $xP(F(\mathbf{P}^{**}))z$.

But (112)-(114) also contradict the transitivity of F on $\wp^{**}$.

Case 2 Suppose $\mathbf{P}'$ is a profile on $\wp^*$ such that

(115) $m_{P'}(y,z) < ½$ implies

(116) $yF(\mathbf{P}')z$.

Suppose first that, for some $\mathbf{P}''$ such that

(117) $m_{P''}(x,z) < m_{P''}(z,x)$,

we have

(118) $zP(F(\mathbf{P}''))x$.

Choose $\bar{\mathbf{P}}$ such that

$$\bar{\mathbf{P}}(i) = \begin{cases} P_o^*, & i \in [0, m_{\mathbf{P}''}(x,z)] \\ P_o^{**}, & i \in [m_{\mathbf{P}''}(x,z), 1] \end{cases}$$

Notice that $m_{\bar{\mathbf{P}}}(x,y) = m_{\mathbf{P}''}(z,x) > ½$. Therefore, if $yF(\bar{\mathbf{P}})x$, we can apply the same argument as in the case where (106) holds to conclude that F is intransitive on $\wp^*$. Hence assume that

(119) $xP(F(\bar{\mathbf{P}}))y$.

Since $m_{\bar{\mathbf{P}}}(y,z) < ½$, (115) and (116) imply that

(120) $yF(\mathbf{P}')z$.

Next, observe that, because $m_{\bar{\mathbf{P}}}(x,z) < ½$, (117) and (118) imply that

(121) $zP(F(\bar{\mathbf{P}}))x$.

But (119)-(121) imply that F is intransitive on $\wp$.

Therefore, assume that, for all $\mathbf{P}''$ satisfying (117), we have

(122) $xF(\mathbf{P}'')z$.

Choose $\tilde{\mathbf{P}}$ such that

$$\tilde{\mathbf{P}}(i) = \begin{cases} P_o, & i \in [0, m_{\mathbf{P}}(x,y)] \\ P_{oo}, & i \in [m_{\mathbf{P}}(x,y), 1] \end{cases}$$

Since $m_{\tilde{\mathbf{P}}}(x,z) = m_{\mathbf{P}''}(x,z)$, (122) implies that

(123) $xF(\tilde{\mathbf{P}})z$.

But $m_{\tilde{\mathbf{P}}}(x,y) = m_{\mathbf{P}}(x,y)$. Therefore, (100) implies that

(124) $yF(\tilde{\mathbf{P}})x$.

Finally, we know that $z\tilde{\mathbf{P}}(i)y$ for all $i \in [0, 1]$. Hence

(125) $zP(F(\tilde{\mathbf{P}}))y$.

But (123)-(125) imply that F is intransitive on $\wp^*$, which completes the proof. _

9. Summary and Commentary

In the nearly-fifty years since Arrow's famous Impossibility Theorem was published, the theory of social choice has been subjected to scrutiny from a wide variety of directions. Relatively little analytical attention has been paid, however, to the robustness of various voting rules. This paper has been about robustness in a world with a continuum of voters. We have been especially interested in identifying domains of individual preference profiles for which various well known voting rules satisfy the Pareto rule and are (generically) anonymous, neutral, and transitive. Voting rules satisfying these four conditions were termed reasonable. It was shown (Theorem 1) that, if for some domain of individual preference profiles a voting rule is reasonable, then so is majority rule reasonable. It was also shown that, unless a voting rule, F, is itself the majority rule, there exists some domain of individual preference profiles on which majority rule is reasonable, but F is not. The two results, when combined, capture the sense in which majority rule is the most robust among voting rules that are required to satisfy the Pareto rule and to be (generically) anonymous, neutral, and transitive. From these observations we were able to offer a new characterization of majority rule, one based on the idea of "maximal" robustness (Theorem 3). This characterization complements the one offered by May (1952) and reproduced here as Theorem 2.

Majority rule satisfies anonymity, neutrality, and the Pareto rule on an unrestricted domain of individual preference profiles. Lemma 1 identified a necessary and sufficient condition on the domain of individual preference profiles for the rule to be generically transitive. (A different form of this condition has been called value restriction in the literature.) Theorem 1 tells us that a domain must satisfy at least as strong a condition as value restriction if any other voting rule is to be reasonable. In Section 5 three well-known voting rules, namely, the Borda-count, the Pareto-extension rule, and the 2/3-majority rule were studied. It was shown that they share the same necessary and sufficient condition for reasonableness (Theorems 4-6). A characterization of this condition, which we called quasi-agreement, was provided. As expected, it was found to be stronger than value restriction. The moral is that, if we were to judge voting rules solely in terms of their reach in being reasonable, these three widely differing aggregation procedures are the same.

In Section 6 two further voting rules - random dictatorship and the plurality rule - were studied. It was shown that they share the same necessary and sufficient condition for reasonableness (Theorems 7-8). A characterization of this condition was provided. As the condition is even stronger than quasi-agreement, we named it strong quasi-agreement. The moral is that, if we were to judge voting rules solely in terms of their reach in being reasonable, random dictatorship and the plurality rule are the same.

These results allow us to group the foregoing six voting rules in a hierarchy. In order of decreasing robustness, the hierarchy consists of three groups: {majority rule}, {Borda-count, the Pareto extension rule, 2/3 majority rule}, and {random dictatorship, plurality rule}.

In Sections 7 and 8 the "tightness" of the assumptions (in turn, transitivity, anonymity, and neutrality) underlying Theorem 1 were studied. While majority rule was found to retain something of its robustness when transitivity was weakened to "quasi-transitivity" (Theorem 9) and anonymity was replaced by "individual responsiveness" (Theorems 10-11), dropping neutrality as an ethical criterion was found to have a marked effect. It was shown that if neutrality is replaced by "independence of irrelevant alternatives" and a certain technical condition designed to break ties (which we called "tie-breaking consistency"), then it is the unanimity rule, with a given order of precedence, that is robust (Theorem 12). Put informally, if a voting rule, F, satisfies anonymity, tie-breaking consistency, independence of irrelevant alternatives, the Pareto rule, and transitivity on a domain of individual preference profiles, then so does the unanimity rule with a given order of precedence satisfy them. Moreover, unless a voting rule, F, is itself the unanimity rule with a given order of precedence, there exists some domain of individual preference profiles on which the unanimity rule satisfies these conditions, but F does not. Theorems 1 and 12, therefore, tell us that replacing neutrality by independence of irrelevant alternatives in the notion of reasonableness of voting rules has a marked effect on its robustness. Perhaps we should not have expected it to be otherwise.

A question we have not addressed relates to what is known in the literature as "strategy-proofness" of voting rules. As the number of voters has been assumed to be a continuum, sincere voting is, trivially, compatible with individual incentives. What we have not investigated here are the domains of individual preference orderings on which the various voting rules are coalition-proof against manipulation. That remains a subject for future work.

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