Credible Group-Stability in General Multi-Partner Matching Problems*

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Abstract

Pairwise-stable matching mechanisms are practically used and perform very well in the real world two-sided matching markets (e.g., the US hospital-intern market, and the British hospital-intern markets: see Roth 1984 and 1991, respectively). It is known, however, that in a two-sided many-to-many matching market (the British market), pairwise-stability is not logically related with the (weak) core unlike in a one-to-many matching market (the US market) [see Blair (1988) and Roth and Sotomayor (1990)]. In this paper, we use a graph representation of matching problems, and we define strong group-stability, a matching-variation of a solution concept in the network literature (Jackson and van den Nouweland, 2002). Although strong group-stability is equivalent to pairwise-stability in a one-to-many matching market, unfortunately, the solution concept is too strong for many-to-many matching markets. Therefore, we proceed to define executable coalitional deviations to discuss the credibility of coalitional deviations, and define credibly group-stable matchings. We prove that credible group-stability is equivalent to pairwise-stability in a wide class of matching problems. We also prove the equivalence between the set of pairwise-stable matchings and the set of matchings resulting from coalition-proof Nash equilibria in an appropriately defined strategic form game.

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1 Introduction

Two-sided matching problems have been extensively studied theoretically, and they also have practical applications in the real world.

In the United States, medical interns compete for roughly 20,000 positions offered by hospitals every year. Interns seek for exactly one position each, but hospitals may offer multiple positions. From 1945 to 1951, the market was characterized by chaotic last-minute recontracting with interns seeking more preferable positions. This chaos lasted until a central matching procedure was introduced in 1952 to replace a decentralized market procedure (the National Resident Matching Program, NRMP). In this program, interns and hospitals submit preference orderings over hospitals and interns, respectively, and a matching algorithm matches them up. Under *responsive* preferences,[1] Roth (1984a) demonstrates that the NRMP matching algorithm is actually a *deferred-acceptance algorithm* introduced in Gale and Shapley (1962),[2] and produces a *pairwise-stable* matching. A matching is pairwise-stable if and only if (i) no intern is matched with an unacceptable hospital, (ii) no hospital is matched with an unacceptable intern, and (iii) no intern and hospital would rather be matched with one and other than some agents in their current match (by getting rid of some current matches or using an unfilled position).[3] Pairwise-stability requires that a matching is immune to any one or two person deviations, and is a necessary condition for a matching to be stable in the following sense: These small size deviations are practically very important since they do not require much information, and as a result, they can be organized and carried out easily. However, Roth (1984a) shows that pairwise-stability is also sufficient. If a bigger size coalition can deviate from a matching, then either a size one or two coalition can also deviate from the matching. Thus, a pairwise-stable matching is also *group-stable* in the sense that it is in the *weak*

[1] An agent's preferences are *responsive* if and only if she prefers a partner to another partner (including an empty set) irrespective of the composition of her other partners. A partner is *acceptable* to an agent if and only if she is preferred to empty set.

[2] Actually this observation is true in the simple model. In the more complex and realistic model, two doctors can declare themselves as a *couple* and search for jobs together in the same geographic region. In the paper, we will not permit the existence of complementarities, like couples, in our theoretical model.

[3] In 1998, the hospital-proposing matching algorithm was replaced by a newly designed applicant-proposing algorithm (Roth and Peranson, 1998). Both algorithms produce pairwise-stable matchings. About the deferred-acceptance algorithm, see Gale and Shapley (1962), and Roth and Sotomayor (1990).

core (see also Roth and Sotomayor, 1990). Roth argues that it is reasonable to say that the introduction of the centralized matching procedure in the US medical intern market was extremely successful in calming down the market.

In the United Kingdom, medical intern markets are slightly different, but in important ways. First, matches are regional rather than national. There are many different matching mechanisms, which are (were) used in different regions. Second, each intern needs to experience two (medical and surgical) positions in twelve months to be eligible for full registration. These two positions are arranged separately by different agents (Roth, 1991). As a result, this is a two-to-many matching problem (with a special structure). It may seem natural that we can apply the deferred-acceptance algorithms for these two categories separately to such a matching problem. Although such a matching mechanism still finds a pairwise-stable matching, it may be no longer group-stable in a two-to-many matching problem unlike in a one-to-many matching problem (Roth, 1984b, 1985, Blair, 1988, and Roth and Sotomayor, 1990). The same statement applies even if (i) there are two categories, (ii) interns' preferences are *separable* between these categories, and (iii) hospitals' preferences are both responsive within each category and separable across categories.[4] The fact that a pairwise-stable matching may not be group-stable makes the celebrated deferred-acceptance algorithms (and pairwise-stable matchings) less attractive in many-to-many matching problems. Nevertheless, Roth (1991) reports an intriguing observation in the history of the U.K. medical intern markets. In different regions, central matching programs adopted different algorithms to match up hospitals and interns. Interestingly, central matching programs have been abandoned in many regions that adopted pairwise-unstable algorithms, while they survived in the regions that adopted deferred-acceptance algorithms. This observation says that pairwise-stability itself may have some importance for robustness of the matching programs.

In this paper, we provide a general class of matching problems that includes many-to-many two-sided matching problems. Our approach utilizes graph representation of matchings (borrowed from the literature on network formation games). Graph representation has two advantages. One is that it enables to describe a many-to-many matching problem more easily. The other is more significant. In the matching literature, group-stability is usually defined by utilizing characteristic function form games: that is, members in a

[4] See an example in the appendix. Preferences over two categories are *separable* if and only if preference ordering over one category of positions are preserved whatever position in the other category is given.

coalition cannot keep matches (links) with outsiders in order to participate in a deviation. However, this is the very reason why there is no logical relationship between pairwise-stability and weak core (in characteristic function form game) in many-to-many matching problems. One of our examples even shows the weak core can be empty, although there always exists a pairwise-stable matching in the domain.

By using the graph representation, we introduce an appropriate definition of coalitional deviations. Then, we first define a solution concept, *strong group-stability* that has a close relationship with a *strongly stable network* in the network literature (Jackson and van den Nouweland, 2001). In one-to-many matching problems, the set of strongly group-stable matchings, the set of pairwise-stable matchings and the weak core all coincide with each other. However, in many-to-many matching problems, the situation is very different. Although strong group-stability implies pairwise-stability (thus, there is a logical relationship between these two unlike between pairwise-stability and the weak core), there is no equivalence between strong group-stability and pairwise-stability: a strongly group-stable matching may not even exist in a many-to-many matching problem.

However, a close look at coalitional deviations from pairwise-stable matchings reveals that these deviations are not self-enforcing in certain ways. Even if a group of agents agree on deviating by reorganizing their (matches) links, some members may not have incentives to follow the suggested reorganization of links. Thus, unless a coalition can form a binding agreement, these coalitions cannot be carried out, and we say these deviations are not "executable". We call a coalitional deviation is *executable* if and only if no member of a coalition has an incentive to betray her new partners by recovering her link with any of her former partners in the original matching instead of some of new partners (or simply cutting some of links with her new partners). We say that a matching is *credibly group-stable* if and only if it is immune to any executable coalitional deviations.

Under responsive and separable preferences, we prove that the set of credibly group-stable matchings is equivalent to the set of pairwise-stable matchings in a large domain of *multi-partner matching problems* that contain many matching problems as special cases, such as one-sided and two-sided many-to-many matching problems, hierarchical organizations, and the British hospital-intern matching problems.

Although credible group-stability might seem arbitrary (since it considers only a deviation and a counter-deviation like in *bargaining sets*, see Aumann and Maschler, 1974), it has two nice features. One is that it is simple and informationally not demanding. Checking executability requires only limited amount of information about agents' preferences and their thoughts about other players' behavior unlike sophisticated solution concepts. Thus, the proposed members of a coalition can decide whether they join the coalition or not after checking if it is going to be executed. The other nice feature is that credible group-stability actually has a deep game theoretic foundation. Consider a strategic form game of a multi-partner matching problem in which each agent simultaneously announces a set of agents that she wants to be matched with, and the outcome of the game is a matching which matches each pair of agents who announced one and other. We show that the set of matchings generated as outcomes of *coalition-proof Nash equilibria* (Bernheim, Peleg, and Whinston, 1987) of this game coincides with the set of credibly group-stable matchings (thus with the set of pairwise-stable matchings).[5] This implies that although our credibly group-stable matching only considers deviations and counter-deviations, it is actually immune to any credible coalitional deviations defined recursively as in Bernheim, Peleg, and Whinston (1987).

Now, let us go back to the observations made in Roth (1991) on the British hospital-intern markets. Although different regions used different matching programs, only the matching programs that generate pairwise-stable matchings have survived historically. We know that a pairwise-stable matching may not be necessarily group-stable. However, survival of pairwise-stable programs may make sense, since pairwise-stability has both theoretical and practical grounds: A pairwise-stable matching is indeed a coalition-proof Nash equilibrium of a strategic form game, and agents can check relatively easily if a coalitional deviation is executable.[6]

The rest of the paper is organized as follows. In Section 2, we discuss the two-sided many-to-many matching problem, discuss the solution concepts in the literature, and introduce new solutions. We provide examples that illustrate the differences among them. In Section 3, we introduce general multi-partner matching problems and define our solution concepts formally. Then, we proceed to prove the equivalence between pairwise-stability and credible group-stability (Theorem 1). In Section 4, we consider a strategic form game

[5] Note that the set of matchings generated as outcomes of *strong Nash equilibria* of this game (Aumann, 1959) coincides with the set of strongly group-stable matchings.

[6] Unlike executability, tremendous amount of information is needed to check coalition-proofness due to its recursive nature of the definition.

of our general multi-partner matching problems, and show the equivalence between the set of the resulting matchings from coalition-proof Nash equilibria and the credibly group-stable matchings. This result together with Theorem 1 shows the equivalence between the set of matchings resulting from coalition-proof Nash equilibria and the set of pairwise-stable matchings (Corollary 1). Section 5 concludes.

2 The Two-Sided Matching Problem and Stability Concepts

2.1 The Model

The model is based on Roth (1984, 85) and Blair (1988). We have finite (disjoint) sets of firms F and workers W. The set of agents is $P = F \cup W$. A firm hires a subset of W, i.e., an element of 2^W. A subset may be feasible for some firms and not for others. For each firm, there is a linear (strict) ordering over feasible subsets, which indicates the firm's preferences (strict preferences). In the same way, each worker works for a subset of F, i.e. an element of 2^F, and there is a preference ordering associated with the feasible subsets of firms for each worker. Agent i's preference ordering is denoted by $\succsim_i$: she is indifferent between two elements if and only if they are the same elements.[7] Her strict preference ordering is denoted by $\succ_i$. Let $\succsim = (\succsim_i)_{i \in P}$ denote the preference profile of agents. For each agent $i \in P$, there is a positive integer quota $q_i \in \mathbb{Z}_{++}$. Let $q = (q_i)_{i \in P}$. List $(F, W, q, \succsim)$ is called a **many-to-many two-sided matching problem**. Let a many-to-many two-sided matching problem $(F, W, q, \succsim)$ be fixed in the rest of the section.

For agent $i \in F$ ($i \in W$), preference ordering $\succsim_i$ is **responsive** if and only if for any $T \subset W$ ($T \subset F$) and any $j, j' \in W - T$ ($j, j' \in F - T$) we have (i) $T \cup \{j\} \succ_i T \cup \{j'\} \Leftrightarrow j \succ_i j'$, (ii) $T \cup \{j\} \succ_i T \Leftrightarrow j \succ_i \emptyset$, and (iii) $T \succ_i T \cup \{j'\}$ $\Leftrightarrow \emptyset \succ_i j'$.[8] We say that an agent j is **acceptable** for i if and only if $j \succsim_i \emptyset$. It is sometimes convenient to represent matchings in our problem by **graphs**. Let g^P be the collection of all elements of $F \times W$. This graph is called the complete graph. An element $(i, j) \in F \times W$ describes a match (or a link in graph-theoretic terminology). We will make use of the graph representation

[7] Without confusion, we abuse notations: $j \succsim_i j'$, $\emptyset \succsim_i j$ and $j \succsim_i \emptyset$ denote $\{j\} \succsim_i \{j'\}$, $\{j\} \succsim_i \emptyset$ and $\emptyset \succsim_i \{j\}$, respectively, for any j, j'.

[8] For any two sets S and T, $S - T$ denotes a set substraction ($S \backslash T$).

of a matching in the rest of the analysis. A **matching** is a subset of g^P. Let $\pi(i,g) = \{j \in P : (i,j) \in g \text{ or } (j,i) \in g\}$ be the set of partners of i under matching $g \subseteq g^P$.[9] A **feasible matching** is a matching $g \subseteq g^P$ such that for any $i \in P$ we have $|\pi(i,g)| \le q_i$. Let $\mathcal{G}$ be the collection of feasible matchings.

2.2 Pairwise-Stability, Core, and Weak Core

The central solution concept in the (two-sided) matching literature is pairwise-stability. In order to introduce the notion, we first define blocking pairs. We say that a pair $(i,j) \notin g$ **blocks** a feasible matching g if and only if (i) j is acceptable for i, and either $|\pi(i,g)| < q_i$ or j is more preferable than one of agents in $\pi(i,g)$, and (ii) i is acceptable for j, and either $|\pi(j,g)| < q_j$ or i is more preferable than one of agents in $\pi(j,g)$. We say that a matching g is **pairwise-stable** if and only if (i) it is feasible, (ii) for all $i \in P$ we have every $j \in \pi(i,g)$ is acceptable for i, and (iii) there is no pair $(i,j) \notin g$ that blocks g. Condition (i) says that each agent is matched with a number of partners no more than her quota. Condition (ii) says that nobody has an incentive to cut an existing match. Condition (iii) says that no pair of a worker and a firm who are not matched with each other under g has any incentive to form a new match by either replacing an existing match or simply establishing a new match.

We introduce two notions of group-stability (Roth and Sotomayor, 1990). We say that a feasible matching g' **dominates** a feasible matching g **via coalition** $T \subseteq P$ if and only if (i) for all $i \in T$, $j \in \pi(i,g')$ implies $j \in T$, and (ii) $\pi(i,g') \succ_i \pi(i,g)$ holds for every $i \in T$. The **core** of the problem is the set of feasible matchings that are not dominated by any other feasible matching. We say that a feasible matching g' **weakly dominates** a feasible matching g **via coalition** $T \subseteq P$ if and only if for any $i \in T$, $j \in \pi(i,g')$ implies $j \in T$, and $\pi(i,g') \succsim_i \pi(i,g)$, and $\pi(i,g') \succ_i \pi(i,g)$ holds for at least for some $i \in T$. The **weak core** of the problem is the set of feasible matchings that are not weakly dominated by any other feasible matching.[10] Under strict preferences,

[9] In the standard matching literature, a matching is defined as a function $\mu : P \to 2^P$ such that (i) If $i \in \mu(j)$ then $j \in \mu(i)$, (ii) If $i \in F$, then $\mu(i) \in 2^W$, and (iii) If $i \in W$, then $\mu(i) \in 2^F$. Obviously, $\pi(i,g) = \mu(i)$ if g and μ represent the same matching. However, in many-to-many matching problems or in even more general problems, graph representation seems simpler and more useful.

[10] The weak core is sometimes called the *strict core*. Allowing weakly dominating coalitional deviations increases the number of possible deviations. Indeed, the weak core is contained in the core.

it is known that in one-to-one matching problems, the above three concepts coincide, and that in one-to-many matching problems, the set of pairwise-stable matchings and the weak core coincide (the core is bigger). The following simple example (a simplified version of Example 2.6 in Blair, 1988) illustrates the difference between the set of pairwise-stable matchings and the weak core in many-to-many matching problems.

Example 1 *There are two firms (F_1 and F_2) and two workers (w_1 and w_2) with the following preference orderings. Quota for the number of matches for each agent is two.*

F_1	F_2	w_1	w_2
w_1	w_2	F_2	F_1
w_1, w_2	w_1, w_2	F_1, F_2	F_1, F_2
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
w_2	w_1	F_1	F_2

In this game there is a unique pairwise-stable matching, an empty matching $\emptyset$, and a unique weak core matching, a complete matching g^P with $\pi(i, g^P) = F$ if $i \in W$ and $\pi(i, g^P) = W$ if $i \in F$. It is easy to see that empty graph $\emptyset$ is the unique pairwise-stable matching, since for any pair of a firm and a worker, if one wants to be matched with the other, then the other does not want to be matched. The complete matching is in the weak core, since g^P is strictly individually rational, and no pair can improve upon g^P. It is also easy to see that g^P is the only weak core matching. □

A weak core matching is sometimes called a *group-stable* matching. However, in many-to-many matching problems, group-stability does not make too much sense. It can be seen from the fact that in the above example g^P is not even pairwise-stable. What is wrong? It is basically because a coalitional deviation T (including a single agent deviation) has to act within T, and the members have to cut all the matches with members of $P - T$. For example, consider F_1. Under g^P, F_1 is matched with w_1 and w_2, but she does not want to be matched with w_2 yet wants to keep a match with w_1. In the definition of weak core, if F_1 alone wants to deviate, F_1 needs to cut all matches. Thus, it is not allowed to cut a match with w_2 only. However, why does w_1 need to cut the match with F_1 when F_1 decides to cut the match with w_2? It is not clear especially because w_1 does not care about what happens to a match between F_1 and w_2: there is not such spillover or externality in this game. Actually, this is precisely the reason why the weak core and the core are not the same in one-to-many matching problems under strict preference orderings. Without including unaffected agents in a coalition, a pair of agents cannot form a new link. However, it is still possible to argue that pairwise-stability is a relevant game-theoretic concept, since strengthening the solution concept from core to weak core does keep the equivalence between the set of pairwise-stable matchings and the weak core in the one-to-many matching problems.

In many-to-many matching problems, the problem with the weak core is severer. In the above example, in order for F_1 to cut a match with w_2 without affecting the match with w_1, F_1 needs to include w_1 in the deviating coalition. However, unlike in one-to-many matching problems, w_1 is linked with another agent F_2, and the deviating coalition needs to include F_2 in order to keep the same payoff for w_1. Unfortunately, F_2 is matched with both w_1 and w_2. Thus, in order for F_1 to cut the match with w_2 unilaterally, w_2 needs to be in the coalition, which is not possible. This is why a weak core matching needs not even be pairwise-stable. Our observation points out the limitation of describing a matching problem in a characteristic function form game.

2.3 Coalitional Deviations in Networks and Executability

The main problem is that in a characteristic function form game, ability of a coalition is limited to the set of feasible matchings within the coalition. Here, we give more power to coalitional deviations by allowing them to keep existing links if they wish. For this purpose, a graph representation of a matching is useful. We adopt (and then modify) a solution concept introduced in Jackson and van den Nouweland (2001) in network formation games.[11] In a many-to-many two-sided matching problem, we say a feasible matching g' is **obtainable from** a feasible matching g **via deviation by** T if and only if (i) $(i,j) \in g'$ and $(i,j) \notin g$ implies $\{i,j\} \subseteq T$, and (ii) $(i,j) \in g$ and $(i,j) \notin g'$ implies $\{i,j\} \cap T \neq \emptyset$. A **coalitional deviation** from a feasible matching g is a coalition and feasible matching pair (T, g') such that (i) g' is obtainable from g via T, and (ii) for any $i \in T$ we have $\pi(i,g') \succ_i \pi(i,g)$. One way to interpret this concept is the following: A coalition T is organized by telephone calls

[11] Our solution is the same as the strongly stable network in Jackson and van den Nouweland (2001). However, note that "pairwise-stability" in a network game (Jackson and Wolinsky, 1996) is a very different solution concept from the standard notion of pairwise-stability in matching problems due to existence of quotas.

among the members, and they discuss how they reorganize the links after the deviation. They can form new links and cut old links within the coalition, or they can unilaterally cut some of existing links with outsiders. We say that a matching g is **strongly group-stable** if and only if (i) it is feasible, and (ii) for any $T \subseteq P$, for any g' that is obtainable from g via deviation by T, and for any $i \in T$ such that $\pi(i, g') \succ_i \pi(i, g)$, there exists $j \in T$ with $\pi(j, g) \succ_j \pi(j, g')$. Consider one-to-many (or one-to-one as a special case) matching problems. In this case, the weak core, the set of pairwise-stable matchings, and the set of strongly stable matchings all coincide with each other (the argument is obvious). Although strong group-stability also makes sense in a many-to-many matching problem, unfortunately, the set of strongly group-stable matchings may be empty. It is indeed empty in Example 1. The two strongly group-stable matching candidates $\emptyset$ and g^P are not strongly group-stable. This is because empty graph $\emptyset$, which is pairwise-stable, is dominated by g^P via P, and g^P, the unique weak core matching, is not immune to any one agent deviation. Thus, there is no strongly group-stable matching in Example 1.

However, it is easy to see that the coalitional deviation (P, g^P) is somewhat unreasonable. In this coalitional deviation, agents are matched up with unacceptable partners. This implies that even if the coalition deviates from the empty graph $\emptyset$, the coalitional deviation itself would fall apart by members' individual deviations to cut off unacceptable partners. In this sense, the coalitional deviation itself is not credible. The readers may think that we can get around the nonexistence problem of a group-stable matching by prohibiting a coalition to have links with unacceptable partners. However, having unacceptable partners is not the only source of such phenomenon. A little more subtle example is the following.

Example 2 *Consider the following many-to-many two-sided matching problem. Quotas are all two. Their preferences are as follows:*

F_1	F_2	F_3	F_4	w_1	w_2	w_3	w_4
w_1w_3	w_2w_4	w_1	w_2	F_2F_3	F_1F_4	F_1	F_2
w_2w_3	w_1w_4	$\emptyset$	$\emptyset$	F_1F_3	F_2F_4	$\emptyset$	$\emptyset$
w_1w_2	w_1w_2			F_1F_2	F_1F_2		
w_3	w_4			F_3	F_4		
w_1	w_2			F_2	F_1		
w_2	w_1			F_1	F_2		
$\emptyset$	$\emptyset$			$\emptyset$	$\emptyset$		

This preference profile is responsive. The unique pairwise-stable matching is

$$g = \{(F_1, w_3), (F_2, w_4), (F_3, w_1), (F_4, w_2)\}.$$

Now, consider a coalitional deviation $P' = \{F_1, F_2, w_1, w_2\}$ *with* $g^{P'}$ *(fully matched up within* P'*). This is improving for* P'*, and it kills* g*. However,* $g^{P'}$ *is not stable, since say agent* F_1 *would cut a link with* w_1 *and form a link with* w_3 *(keeping a link with* w_2*). Thus, there is no strongly stable matching in this example.* □

The coalitional deviation $(P', g^{P'})$ from g does not have unacceptable partners in the links, but it may not be credible in the following sense. Notice that F_1 is originally matched with w_3, and discusses with another member of P' to cut a link with w_3 and to be matched with w_1 and w_2. Certainly, it is beneficial to F_1. However, after the deviation is agreed, F_1 wants to exclude w_2 secretly and to form a link only with w_1 by keeping the link with w_3 intact. In our interpretation of a *coalitional deviation*, even if members make phone calls and agree on forming $g^{P'}$, they may not follow the suggestion on their link formation. In this sense, a coalitional deviation P' with g is not *executable* (or not self-enforcing). To state the definition formally, we need one more notation. Let $\beta_i(V) \in V$ be such that $j \succsim_i \beta_i(V)$ for any V ($V \subseteq W$ if $i \in F$, and $V \subseteq F$ if $i \in W$) i.e., β_i selects the least preferable (or the bottom) element. We say that coalitional deviation (T, g') is **executable**, if and only if (i) for any $(i, j) \in g' - g$, agents i and j are mutually acceptable ($j \succ_i \emptyset$ and $i \succ_j \emptyset$), and (ii) for any $(i, j) \in g - g'$, either (a) j is unacceptable for i ($\emptyset \succ_i j$), or $\beta_i(\pi(i, g')) \succ_i j$ and $|\pi(i, g')| = q_i$, or (b) i is unacceptable for j ($\emptyset \succ_j i$), or $\beta_j(\pi(j, g')) \succ_j i$ and $|\pi(j, g')| = q_j$. That is, an executable coalitional deviation (T, g') from a feasible matching g means is the following: a member of a coalition T may be suggested to cut some matches in $g - g'$, and form new matches in $g' - g$. All the members of T strictly improve by this, but some may prefer to keep some of links in $g - g'$ and to discard some of ones in $g' - g$ so that they can do even better. If there is such an agent, then (T, g') will not be carried out even if it is agreed beforehand. In Examples 1 and 2, planned coalitional deviations are not immune to cheating by the members. Then, why do they agree to deviate in the first place? We may say that we can dismiss such nonexecutable deviations when we check stability of a matching. In these examples, if we do not allow nonexecutable deviations, then pairwise-stable matchings survives group deviations as well. This is not a coincidence: as we will see in the next section, *a matching is immune to executable coalitional deviations if and only if it is pairwise-stable.*

Although credible group-stability may seem to be arbitrary, it has a deep game theoretic foundation. By introducing a strategic form game of multi-partner matching problems, we can actually show that the set of matchings generated from *coalition-proof Nash equilibria* (Bernheim, Peleg, and Whinston, 1987) of this game coincides with the set of credibly group-stable matchings (see Section 4).[12] This gives a strong game-theoretical support for a simple solution concept like pairwise-stability.

2.4 An Example of Empty Core

Before closing this section, we provide an example that has an empty core. In many-to-many two-sided matching problems, there always exists a pairwise-stable matching under responsive preferences. However, as the following example shows, the core may be empty in the characteristic function form game.[13]

[12] A coalition-proof Nash equilibrium is a strategy profile that is immune to any credible coalitional changes in the members' strategies, and the credibility of coalitional deviations is defined recursively in a consistent manner (see Bernheim, Peleg, and Whinston, 1987). Our equivalence result gives us another reason that our network approach is more preferable than characteristic function approach in matching problems. The counterpart of a coalition-proof Nash equilibrium in a characteristic function form game is *the credible core* in Ray (1989) that checks credibility of coalitional deviations in a recursive way. However, as is shown in Ray (1989), the core and the credible core are equivalent in characteristic function form games. This implies that in characteristic function form game, credibility argument does little in justifying pairwise stability.

[13] Although this example does not fit with the British hospital-intern market problem as it is, the same logic goes through (see an example in the Appendix).

Example 3 *Consider the following matching problem. Quotas are all two.*

F_1	F_2	F_3	F_4	F_5	w_1	w_2	w_3	w_4	w_5
w_2w_3	w_1w_3	w_1w_2	w_2	w_1	F_1F_4	F_2F_5	F_3	F_2	F_1
w_2w_4	w_3w_4	w_1w_4	$\mathbf{w}_1\mathbf{w}_2$	$\mathbf{w}_1\mathbf{w}_2$	F_1	F_2	$\mathbf{F}_1\mathbf{F}_3$	$\mathbf{F}_1\mathbf{F}_2$	$\mathbf{F}_1\mathbf{F}_2$
w_3w_4	w_1w_4	w_2w_4	$\emptyset$	$\emptyset$	F_1F_5	F_2F_4	$\mathbf{F}_2\mathbf{F}_3$	$\emptyset$	$\emptyset$
w_2	w_3	w_1			$\mathbf{F}_1\mathbf{F}_2$	$\mathbf{F}_2\mathbf{F}_3$	$\emptyset$		
w_2w_5	w_3w_4	w_2			F_4	F_5			
w_3	w_1	$\mathbf{w}_1\mathbf{w}_3$			$\mathbf{F}_4\mathbf{F}_5$	$\mathbf{F}_4\mathbf{F}_5$			
w_3w_5	w_1w_4	$\mathbf{w}_2\mathbf{w}_3$			$\mathbf{F}_1\mathbf{F}_3$	$\mathbf{F}_1\mathbf{F}_2$			
$\mathbf{w}_1\mathbf{w}_2$	$\mathbf{w}_2\mathbf{w}_3$	$\emptyset$			$\emptyset$	$\emptyset$			
w_4	w_5								
$\mathbf{w}_4\mathbf{w}_5$	$\mathbf{w}_4\mathbf{w}_5$								
$\mathbf{w}_1\mathbf{w}_3$	$\mathbf{w}_1\mathbf{w}_2$								
$\emptyset$	$\emptyset$								

Preferences are responsive, and we listed all individually rational choices. Choices in bold characters are the relevant choices that compose individually rational matchings. Note that F_1, F_2, and F_3 do not want to be matched with w_1, w_2, and w_3, respectively, unless one is matched with one of the other two. However, w_1, w_2, and w_3 want to be matched with F_1, F_2, and F_3: they do not mind being matched with one of the others only when they get their favorites. Note also that F_1 and F_2 (w_1 and w_2 by symmetry) do not want to be matched with w_5 and w_4 (F_5 and F_4), respectively, unless they are matched with w_4 and w_5 (F_4 and F_5), respectively. We list the graph representations of all individually rational matchings:

$$
\begin{aligned}
g_1 &= \{(F_1, w_1), (F_1, w_2), (F_2, w_1), (F_2, w_2)\},\\
g_2 &= \{(F_2, w_2), (F_2, w_3), (F_3, w_2), (F_3, w_3)\},\\
g_3 &= \{(F_1, w_1), (F_1, w_3), (F_3, w_1), (F_3, w_3)\},\\
g_4 &= \{(F_1, w_1), (F_1, w_3), (F_2, w_2), (F_2, w_1), (F_3, w_3), (F_3, w_2)\},\\
g_5 &= \{(F_1, w_1), (F_1, w_2), (F_2, w_2), (F_2, w_3), (F_3, w_3), (F_3, w_1)\},\\
g_6 &= \{(F_1, w_4), (F_1, w_5), (F_2, w_4), (F_2, w_5)\},\\
g_7 &= \{(F_4, w_1), (F_4, w_2), (F_5, w_1), (F_5, w_2)\},\\
g_8 &= \emptyset.
\end{aligned}
$$

We can see that none of the above matchings is in the core, although an empty matching g_8 is pairwise stable. There is always a coalitional deviation that yields another matching: $g_1 \rightarrow_{\{F_2,F_3,w_2,w_3\}} g_2$, $g_2 \rightarrow_{\{F_1,F_3,w_1,w_3\}} g_3$,

$g_3 \to_{\{F_1,F_2,w_1,w_2\}} g_1$, $g_4 \to_{\{F_1,F_2,w_4,w_5\}} g_6$, $g_5 \to_{\{F_4,F_5,w_1,w_2\}} g_7$, $g_6 \to_{\{F_1,F_2,F_3,w_1,w_2,w_3\}}$ g_5, $g_7 \to_{\{F_1,F_2,F_3,w_1,w_2,w_3\}} g_4$, and g_8 is dominated by all other individually rational matchings. Thus, the core (and the weak core) is empty. □

3 General Multi-Partner Matching Problems

3.1 The Setup

So far, we discussed only two-sided matching problems. Although two-sidedness is sometimes crucial in getting many interesting properties of pairwise-stable matchings (including existence), the solution concepts we introduce are applicable to one-sided or multi-sided matching problems (as is pairwise-stability).[14] A **general multi-partner matching problem** is a list $(P, ((M_i^k, q_i^k)_{k\in K}, \succsim_i)_{i\in P})$ such that (i) P is a finite set of agents, (ii) K is the set of categories of agent i's partners, (iii) $M_i^k \subseteq P - \{i\}$ is the set of feasible partners of agent i in category k, (iv) q_i^k is a quota for category k partners of agent i, (v) $\succsim_i$ is agent i's preference relation over subsets of $\Pi_{k\in K} M_i^k$. In the rest of the paper, we fix a general multi-partner matching problem $(P, ((M_i^k, q_i^k)_{k\in K}, \succsim_i)_{i\in P})$.

Let $g^P \equiv \{S \in 2^P : |S| = 2\} \times K$ be the complete graph which matches every pair of agents under each category. A **matching** is a subset of g^P. For any matching $g \subseteq g^P$, let $\pi(i,k,g) = \{j \in M_i^k : (i,j;k) \in g\}$ be the set of agents whom i is matched with in category k under g, and let $\pi(i,g) = (\pi(i,k,g))_{k\in K} \subseteq \Pi_{k\in K} M_i^k$. A **feasible matching** is a matching $g \subseteq g^P$ such that (i) for any $(i,j;k) \in g$ we have $j \in M_i^k$ and $i \in M_j^k$, and (ii) we have $|\pi(i,k,g)| \le q_i^k$ for any $i \in P$, for any $k \in K$. Let $\mathcal{G}$ be the set of all feasible matchings.

Preferences are separable among categories: we say that agent i's preference relation is **separable** across categories if and only if for any $k \in K$, any $S, T \subseteq M_i^k$, and any $(\pi(i,l,g))_{l\in K-\{k\}}$ and $(\pi(i,l,g'))_{l\in K-\{k\}}$ we have

$$\begin{aligned}\left(S, (\pi(i,l,g))_{l\in K-\{k\}}\right) &\succsim_i \left(T, (\pi(i,l,g))_{l\in K-\{k\}}\right) \Leftrightarrow \\ \left(S, (\pi(i,l,g'))_{l\in K-\{k\}}\right) &\succsim_i \left(T, (\pi(i,l,g'))_{l\in K-\{k\}}\right).\end{aligned}$$

[14] It may be said that "pairwise-stability" does not make any sense in N-sided matching problems ($N \ge 3$). This is absolutely true if a N-sided matching means a coalition formed by N different agents from every side. However, such a definition works only for one-to-one matching problems. If an agent can be matched with multiple agents from each side, a coalition does not make sense again. In this paper, we naively assume that an agent can be matched with multiple agents from each group (side). Thus, a "match" still means a link between two players in this paper.

Thus, within category k, agent i's preferences can be written as $\succsim_i^k$, where for any $S, T \subseteq M_i^k$ we have $S \succsim_i^k T$ if and only if $\left(S, (\pi(i,l,g))_{l \in K-\{k\}}\right) \succsim_i \left(T, (\pi(i,l,g))_{l \in K-\{k\}}\right)$ for any feasible matching $g \in \mathcal{G}$. We assume that $\succsim_i^k$ is **responsive** for each category $k \in K$ and agent $i \in P$. For a feasible matching g, we say that $(i,j;k) \notin g$ **blocks** g if and only if (i) j is acceptable for i in category k, and either $|\pi(i,k,g)| < q_i^k$ or j is more preferable than one of agents in $\pi(i,k,g)$ in category k, and (ii) i is acceptable for j, and either $|\pi(j,k,g)| < q_j^k$ or i is more preferable than one of agents in $\pi(j,k,g)$ in category k. We say that a matching g is **pairwise-stable** if and only if (i) it is feasible, (ii) for all $i \in P$, for all category $k \in K$, we have every $j \in \pi(i,k,g)$ is acceptable for i, and (iii) there is no pair and category $(i,j;k) \notin g$ that blocks g.[15] Note that in our class of problems, we cannot guarantee the existence of pairwise-stable matching, since it covers one-sided matching problems.[16]

This general definition covers not only many-to-many (thus, one-to-many and one-to-one) two-sided matching problems, but also many interesting problems including the British hospital-intern markets. In many-to-many two-sided matching markets, set $|K| = 1$, and partition P into two groups F and W in which for any $i \in F$, $M_i = W$ and for any $i \in W$, $M_i = F$. In British hospital-intern markets, K has two elements: m, denoting medical positions, and s, denoting surgical positions. The set of agents P is partitioned into two sets, H and I (hospitals and interns, respectively). Any hospital $h \in H$ has a quota for each category, q_h^m and q_h^s, and any intern $i \in I$ has unit quota for each category, $q_i^m = q_i^s = 1$. Actually, the framework is also good for one-sided

[15]Note that our definitions of blocking and pairwise-stability are *category-wise*. Consider the following example: Agent j (i) is not acceptable for i (j) in category k (k'), while j (i) is quite preferable candidate for i (j) in category k' (k). In such a case, forming two links between i and j in categories k and k' is *not* considered as pairwise-stable. We adopt this definition since such a "trade" is not self-enforceable, unless i and j can write a binding contract on the matches in the two categories.

[16]In one-to-many two-sided matching problems, there exists a pairwise stable matching, and a randomized myopic adjustment process brings a pairwise-stable matching with probability one (see Roth and Vande Vate, 1990, and in a somewhat different context, Corollary 2 in Jackson and Watts, 2002) (note that Jackson and Watts use different terminologies since the paper is on network: their core stable network corresponds to our pairwise-stable matching). In more general one-sided matching problems, Chung (2000) generalizes Roth-Vande Vate argument by introducing a "no odd rings condition", and Diamantoudi, Miyagawa, and Xue (2002) show a convergence whenever a pairwise stable matching exists by confining their attention to the strict preference domain. However, it is not clear if these discussions extend to a general multi-partner matching problem even when there exists a pairwise-stable matching.

matching problems as well. We can just let $M_i = P - \{i\}$ for any agent i, and $|K| = 1$. This framework can also incorporate hierarchical organization structure by crafting M_i's to describe the hierarchy as long as an agent cares only about her immediate bosses and immediate supporting people (by creating many categories so that we can classify bosses and supporting people of each agent in different categories).

What our framework cannot cover are N-sided matching problems with $N \geq 3$: a match is composed of an agent each from N sides (for three-sided matching problems, see Alkan, 1988). Thus, in N- sided matching problems with $N \geq 3$, a match no longer means a link between a pair: it means a coalition formed by N-sides. Our multi-partner matching problem definition is only valid for the cases where a match means a link between two agents. Similarly, Shapley and Scarf's (1974) *housing market problem* and Sönmez's (1999) *generalized matching problem* are not included in this domain by the same reason.[17] In these problems, matchings create coalitions (say, the top trading cycle in the housing market problems), and "pairwise" stability does not make much sense.

3.2 Credible Group-Stability

Now, we generalize the solution concepts introduced in the previous section. Let g be a feasible matching. A feasible matching g' is **obtainable from** g **via deviation by** T if and only if (i) $(i, j; k) \in g'$ and $(i, j; k) \notin g$ for some category k implies $\{i, j\} \subseteq T$, and (ii) $(i, j; k) \in g$ and $(i, j; k) \notin g'$ for some category k implies $\{i, j\} \cap T \neq \emptyset$. A matching g is **strongly group-stable** if and only if (i) it is feasible and (ii) for any $T \subseteq P$, for any feasible matching g' that is obtainable from g via deviation by T, and for any $i \in T$ such that $\pi(i, g') \succ_i \pi(i, g)$, there exists $j \in T$ with $\pi(j, g) \succ_j \pi(j, g')$. **A coalitional deviation** from g is a coalition and a feasible matching pair (T, g') such that (i) g' is obtainable from g via T, and (ii) for any $i \in T$ we have $\pi(i, g') \succ_i \pi(i, g)$. For any agent i, let $\beta_i^k(V) \in V$ be such that $j \succsim_i^k \beta_i^k(V)$ for any $V \subseteq M_i^k$: i.e., β_i^k selects the least preferable (or the bottom) element within category k. We say that coalitional deviation (T, g') from g is **executable**, if and

[17] The generalized matching problem in Sönmez (1999) actually covers both the housing market problem and one-to-one two-sided matching problem. The fact that it does not cover many-to-one matching problem shows the intrinsic difference between our network-based approach and his coalition-based approach (recall that we need the weak core instead of the core for an equivalence with pairwise stability in many-to-one matching problem).

only if (i) for any $(i,j;k) \in g' - g$, agents i and j are mutually acceptable in category k ($j \succ_i^k \emptyset$ and $i \succ_j^k \emptyset$), and (ii) for any $(i,j;k) \in g - g'$, either (a) j is unacceptable for i in category k ($\emptyset \succ_i^k j$), or $\beta_i(\pi(i,k,g')) \succ_i^k j$ and $|\pi(i,k,g')| = q_i^k$, or (b) i is unacceptable for j in category k ($\emptyset \succ_j^k i$), or $\beta_j(\pi(j,k,g')) \succ_j^k i$ and $|\pi(j,k,g')| = q_j^k$. Condition (i) says that any newly matched agents in T are acceptable to each other within the category they got matched, and condition (ii) says that if a match between i and j in category k in the previously existing matching is dismissed in an executable coalitional deviation, if and only if one of them, say i, is unacceptable to the other, say j, or i is acceptable to j but i is less preferable than the worst partner of j in the new matching and j's quota is binding under the new matching. If these conditions are met, for any member of T, there is no reason to keep some of the matches that she is supposed to cut, or not to form a match with an agent she is supposed to form a match. That is why we say that such a coalitional deviation is executable. We say a matching g is **credibly group-stable** if and only if it is feasible and there is no executable coalitional deviation from g. The main result of this section is as follows:

Theorem 1 *In a general multi-partner matching problem with separable and responsive preferences, the set of pairwise-stable matchings is equivalent to the set of credibly group-stable matchings.*

In order to show the equivalence, we need to prove two directions in order. Hence, we prove Theorem 1 by two propositions.

Proposition 1 *A credibly group-stable matching is pairwise-stable.*

Proof. Suppose that a credibly group-stable matching g is not pairwise-stable. Then, either (i) for some agent there is an unacceptable match in some category, or (ii) there is a blocking pair in some category but there is no unacceptable match. Suppose that case (i) is true. It means that some agent i is willing to cut a link in some categories in g. Let her cut any possible link in the most preferable way, and obtain a new matching g'. Letting $T = \{i\}$, it is obvious that i does not want to cut any more link nor to recover links. Thus, such a deviation $(\{i\}, g')$ is executable. However, this contradicts with g being credibly group-stable. Thus, suppose that case (ii) is true: there is no acceptable match in any category, and there is a pair (i,j) that blocks g in all categories in $K' \subseteq K$ ($K' \neq \emptyset$). Consider a coalitional deviation $(\{i,j\}, g')$ such that (i) $|\pi(i,k,g')| \leq q_i^k$ and $\pi(i,k,g') \succsim_i^k S$ for any $S \subseteq \pi(i,k,g') \cup \{j\}$ with

$|S| \leq q_i^k$ for all $k \in K'$, and (ii) $|\pi(j,k,g')| \leq q_j^k$ and $\pi(j,k,g') \succsim_j^k S$ for any $S \subseteq \pi(j,k,g') \cup \{i\}$ with $|S| \leq q_j^k$ for all $k \in K'$. Since there is no unacceptable match, agents i and j has no incentive to cut any link in categories in $K - K'$ under separability of their preferences. Since (i,j) blocks g in all categories K' and not in categories $K - K'$, we have that $j \in \pi(i,k,g') \succ_i^k \pi(i,k,g)$ and $i \in \pi(j,k,g') \succ_i^k \pi(j,k,g)$ for all $k \in K'$. This deviation is improving for both agents i and j by separability of their preferences, moreover it is obviously executable. These contradict with the supposition that g is credibly group-stable. $\square$

The proof of the other direction is more involved.

Proposition 2 *A pairwise-stable matching is credibly group-stable.*

Proof. *Suppose that g is pairwise-stable, and (T,g') is an executable coalitional deviation from g.* First note that for any $(i,j;k) \in g' - g$, we have $i,j \in T$, since g' is obtainable from g. Moreover, for any $(i,j;k) \in g' - g$, we have $j \succ_i^k \emptyset$ and $i \succ_j^k \emptyset$ by executability. We first prove the following claims.

Claim 1: For any $i,j \in T$ and category k with $(i,j;k) \in g' - g$, either $\beta_i^k(\pi(i,k,g)) \succ_i^k j$ or $\beta_j^k(\pi(j,k,g)) \succ_j^k i$.

Proof of Claim 1. Suppose not. Then, under g, there are agents i and j and a category k such that agents i and j can cut $\beta_i^k(\pi(i,k,g))$ and $\beta_j^k(\pi(j,k,g))$, respectively, and establish a link $(i,j;k)$. Both i and j are better off by responsiveness and separability of preferences. This implies that g is not pairwise-stable. This is a contradiction. $\diamondsuit$

Claim 2: For any $i,j \in T$ and category k with $(i,j;k) \in g' - g$, either $\beta_i^k(\pi(i,k,g)) \succ_i^k j$ and $|\pi(i,k,g)| = q_i^k$, or $\beta_j^k(\pi(j,k,g)) \succ_j^k i$ and $|\pi(j,k,g)| = q_j^k$.

Proof of Claim 2. Let $(i,j;k) \in g' - g$ for some agents $i,j \in T$ and category k. By Claim 1, without loss of generality let $\beta_i^k(\pi(i,k,g)) \succ_i^k j$. There are two cases.

1. Suppose that $|\pi(i,k,g)| = q_i^k$. Then the proof of Claim 2 is complete.

2. Suppose that $|\pi(i,k,g)| < q_i^k$. Since g is pairwise-stable, there are no blocking pairs. In particular, $(i,j;k)$ cannot block g. Since i and j are mutually acceptable by executability, $\beta_j^k(\pi(j,k,g)) \succ_j^k i$ and $|\pi(j,k,g)| = q_j^k$ must follow. $\diamondsuit$

Claim 2 allows us to order any pair $(j, i; k)$ in $g' - g$ in such a way that the second argument i satisfies $\beta_i^k(\pi(i,k,g)) \succ_i^k j$ and $|\pi(i,k,g)| = q_i^k$. That is, (i) the latter argument i is matched with a partner who is less preferable than any other partner that i was matched with under g, and (ii) i used up her quota under g. We say that i is **pointed in category** k by j if $(j, i; k) \in g' - g$. Since (T, g') is improving for such an agent i, there exists $h \in T$ such that $(i, h; k) \in g' - g$ and $l, l' \in g - g'$, and $h \succ_i^k l \succ_i^k l' \succ_i^k j$. It says that since i was pointed by a less preferable partner j in category k, she needs to be compensated by pointing a more preferable partner h. Since we know $h \succ_i^k \beta_i(\pi(i,k,g))$, h is indeed pointed in category k. However, then, h needs to be compensated by pointing a more preferable partner than i, say i''. Thus, we need $(h, i''; k) \in g' - g$, and so on and so forth. The same agent can be pointed by different agents, but whenever she is pointed by someone different, she must point someone who she has not pointed yet. Thus, to be consistent with the finiteness of agents, we need to have agent j who pointed i first to be pointed by somebody in this chain. This creates a cycle, and the number that each agent i shows up as the first argument of the elements of $g' - g$ must be equal to the one that i shows up as the second arguments of them.

However, a close look leads us to a contradiction. Agent i was pointed by j and pointed h in order to be compensated. However, since her quota was binding, she needed to cut two links with l and l'. Focus on l. Note that executability of (T, g') also implies for l that $l \in T$. Otherwise, i will come back to l by cutting j off in category k. Hence $\pi(l,k,g') - \pi(l,k,g) \neq \emptyset$. For any $(i', l; k) \in g' - g$, $i' \succ_l i$ must follow. Otherwise, i will come back to l by cutting j off and l will come back to i by cutting i' contradicting to executability of (T, g'). By Claim 2, l points i' in category k. This creates a chain of pointing agents, by finiteness of T. However, l cannot be pointed by anybody unlike the last case: more precisely, for any $h' \in \pi(l,k,g')$, we have $h' \succ_l^k i$. Otherwise, (T, g') is not executable since i and l can recover their link by cutting inferior partners. Now, l cannot be pointed in category k by anybody in a chain, but l also initiates the chain.

This process necessarily generates an infinite number of chains or an infinite chain. Since the number of players is finite, we reach a contradiction. $\square$

4 Strategic Form Games

We can rewrite our matching problem in a strategic form: a player announces a subset of players who she wants to be matched with, and a match is made if and only if a pair of players announce each other's name. Here, we show that this game is useful to clarify the relationships among the notions of stable matchings in matching problems. Consider a general multi-partner matching problem $(P, ((M_i^k, q_i^k)_{k \in K}, \succsim_i)_{i \in P})$. A **strategic form game** is a list $G(P) = (P, (S_i, u_i)_{i \in P})$, where (i) $S_i = \{(T^k)_{k \in K} \subseteq \Pi_{k \in K} M_i^k : |T^k| \leq q_i^k \text{ for all } k \in K\}$ is the strategy set for player $i \in P$ (a typical element $s_i \in S_i$ is such that $s_i = (s_i^k)_{k \in K} \subseteq \Pi_{k \in K} M_i^k$); and (ii) $u_i : \Pi_{j \in P} S_j \to \mathbb{R}$ is player i's payoff function such that $u_i(s) \geq u_i(s')$ if and only if $m_i(s) \succsim_i m_i(s')$, where $m_i(s) = (m_i^k(s^k))_{k \in K} = \Pi_{k \in K}\{j \in P : j \in s_i^k \text{ and } i \in s_j^k\}$ is the list of the sets of players who are matched with i in each category under a matching resulting from strategy profile $s \in \Pi_{j \in P} S_j$. Obviously, a resulting matching g from s satisfies $\pi(i, g) = (\pi(i, k, g))_{k \in K} = (m_i^k(s^k))_{k \in K} = m_i(s)$ for all $i \in P$. A strategy profile s^* is a **strong Nash equilibrium** of $G(P)$ if and only if for any $T \subseteq P$, any $s'_T \in \Pi_{j \in T} s'_j$, there exists $i \in T$ such that $u_i(s'_T, s^*_{-T}) \leq u_i(s^*)$ (Aumann, 1959). Note that a coalitional deviation T in this strategic form game not only rearranges matches within any deviating coalition, but also can keep some of preexisting matches between the members of T and the outsiders if the members wish. If we apply the notion of a strong Nash equilibrium, to a one-to-many (and, of course, to a one-to-one) matching game, the set of the matchings generated from strong Nash equilibria and the set of pairwise-stable matchings are equivalent without invoking the *weak* core (by the reason described before).[18] Although the strong Nash equilibrium concept makes sense in a many-to-many matching game, unfortunately, it may not exist. It does not exist indeed in Example 1, since the two candidates, empty graph $\emptyset$ is dominated by g^P via P (attainable by calling all feasible links), and g^P is not immune to any one player deviation (attainable by removing an unacceptable player's name).

Now, we define a weaker solution concept based on credibility of coalitional deviations: **coalition-proof Nash equilibrium (CPNE:** Bernheim, Peleg, and Whinston, 1987).[19] For $P' \subseteq P$, consider a **reduced game** $G(P', s_{-P'})$

[18] One of the results in Kara and Sönmez (1997) shows that in a one-to-many matching problem, the same game form implements pairwise-stable matching in strong Nash equilibrium.

[19] In a network formation problem, Dutta and Mutuswami (1998) use CPNE of a strategic

that is a strategic form game with players in P' and is created from $G(P)$ by setting each player $j \in P - P'$ to be a passive player who plays a given $s_j \in S_j$ no matter what happens. (i) For any $P' = \{i\}$ and any $s_{-P'} \in \Pi_{j \in P-P'} S_j$, $s^*_i \in S_i$ is a coalition-proof Nash equilibrium if and only if there is no $s'_i \in S_i$ with $u_i(s'_i, s_{-P'}) \geq u_i(s^*_i, s_{-P'})$. (ii) For a positive integer ℓ, for any $P' \subset P$ with $|P'| \leq \ell$, and any $s_{-P'} \in \Pi_{j \in P-P'} S_j$, all CPNE of a reduced game $G(P', s_{-P'})$ have been identified. Then, (a) for any $\tilde{P} \subset P$ with $|\tilde{P}| = \ell + 1$, $s^*_{\tilde{P}}$ is **self-enforcing** in a reduced game $G(\tilde{P}, s_{-\tilde{P}})$ if and only if $s^*_{P''}$ is a CPNE of a reduced game $G(P'', (s_{-\tilde{P}}, s^*_{\tilde{P}-P''}))$ of $G(\tilde{P}, s_{-\tilde{P}})$ with $P'' \subset \tilde{P}$. (b) for any $\tilde{P} \subset P$ with $|\tilde{P}| = \ell + 1$, $s^*_{\tilde{P}}$ is a CPNE of a reduced game $G(\tilde{P}, s_{-\tilde{P}})$ if and only if $s^*_{\tilde{P}}$ is self-enforcing in reduced game $G(\tilde{P}, s_{-\tilde{P}})$, and there is no other self-enforcing $s'_{-\tilde{P}}$ that dominates $s^*_{-\tilde{P}}$. That is, CPNE can be interpreted as *a strategy profile that is immune to any credible coalitional deviation (deviation itself is a CPNE in the reduced game)*. Let $CPNE(G(P', s_{-P'}))$ denote the set of CPNE strategy profiles on P' for the game $G(P', s_{-P'})$ for all $P' \subseteq P$ and strategy profile s.

Recall that $m_i(s) = (m^k_i(s^k))_{k \in K} = \Pi_{k \in K}\{j \in P : j \in s^k_i \text{ and } i \in s^k_j\}$ for any agent $i \in P$. An immediate observation is the following.

Proposition 3 *Suppose that $s^* \in CPNE(G(P))$. Then, a matching g generated from s^* via function $(m_i)_{i \in P}$ is pairwise-stable, thus, is credibly group-stable.*

Proof. Suppose that matching g generated from $s^* \in CPNE(G(P))$ is not pairwise-stable. Then, there is a category k in which either (i) one of players is matched with an unacceptable agent, and there is no blocking pair in any category, or (ii) there is a pair who blocks g. Suppose that case (i) is true. It means that player i is willing to cut some of links in g. She can do that in $G(P)$ by simply not announcing such partners. Considering $G(\{i\}, s^*_{-\{i\}})$, we can easily see that s^*_i is not a CPNE of the reduced game. This is a contradiction. Thus, suppose that case (ii) is true, and there is a pair and a category $(i, j; k)$ which blocks g. Suppose that both i and j improve by forming a link in category k ($j \succ^k_i \emptyset$ and $i \succ^k_j \emptyset$) if and only if $k \in K'$ for some $K' \subseteq K$. Consider a coalitional deviation by (i, j) with (s'_i, s'_j), where $s'_i = (s^{k\prime}_i)_{k \in K}$ and $s'_j = (s^{k\prime}_j)_{k \in K}$ such that (i) $s^{k\prime}_i = \{S \subseteq M^k_i : |S| \leq q^k_i$ and $S \succsim^k_i T$ for any $T \subseteq s^{k*}_i \cup \{j\}$ with $|T| \leq q^k_i\}$ for $k \in K'$ and $s^{k\prime}_i = \{S \subseteq M^k_i : |S| \leq q^k_i$ and $S \succsim^k_i T$ for any $T \subseteq s^{k*}_i$ with $|T| \leq q^k_i\}$, for $k \in K - K'$; and (ii)

form game to analyze the resulting networks.

$s_j^{k\prime} = \{S \subseteq M_j^k : |S| \leq q_j^k$ and $S \succsim_j^k T$ for any $T \subseteq s_j^{k*} \cup \{i\}$ with $|T| \leq q_j^k\}$ for $k \in K'$ and $s_j^{k\prime} = \{S \subseteq M_j^k : |S| \leq q_j^k$ and $S \succsim_j^k T$ for any $T \subseteq s_j^{k*}$ with $|T| \leq q_j^k\}$, for $k \in K - K'$. This deviation is obviously improving for both agents i and j. Moreover, under separable and responsive preferences, neither i and j cannot improve upon the matching generated from $(s_i', s_j', s_{-i,j}^*)$ by changing their strategies alone or together. Thus, it is credible as well. □

We also prove the other direction, which requires a more tedious analysis. This is the main result of this section.

Proposition 4 *A credibly group-stable matching is supported by a CPNE of game $G(P)$.*

Proof. First recall that a CPNE is immune to credible coalitional deviations and a credibly group-stable matching is immune to executable coalitional deviations. Hence, if a matching generated from a credible coalitional deviation is executable, then a credibly group-stable matching is supported by a CPNE. Thus, we will show that a matching generated from a credible deviation in the strategic form game is obtainable by an executable coalitional deviation. Suppose that a coalitional deviation (T, s_T') from s is a *credible deviation*: i.e., $s_T' \in CPNE(G(T, s_{-T}))$. Let $s' = (s_T', s_{-T})$. Let $g = \{(i,j;k) \in g^P : j \in s_i$ and $i \in s_j\}$, and $g' = \{(i,j;k) \in g^P : j \in s_i'$ and $i \in s_j'\}$. Note that $s_j' = s_j$ for any $P - T$. We want to show that (T, g') is an executable deviation from g. We will prove that (i) for any $(i,j;k) \in g' - g$, $j \succ_i^k \emptyset$ and $i \succ_j^k \emptyset$, and (ii) for any $(i,j;k) \in g - g'$, either (a) $\emptyset \succ_i^k j$, or $\beta_i^k(\pi(i,k,g')) \succ_i^k j$ and $|\pi(i,k,g')| = q_i^k$, or (b) $\emptyset \succ_j^k i$, or $\beta_j^k(\pi(j,k,g')) \succ_j i$ and $|\pi(j,k,g')| = q_j^k$.

It is easy to see that Condition (i) holds: otherwise, s_T' is not immune to a single player i's deviation to exclude j (in the case of $\emptyset \succ_i^k j$), this contradicts to $s_T' \in CPNE(G(T, s_{-T}))$.

Now, we check Condition (ii). Suppose, to the contrary, that for some $(i,j;k) \in g - g'$, ($\neg$a) $j \succ_i^k \emptyset$, and either $j \succ_i^k \beta_i^k(\pi(i,k,g'))$ or $|\pi(i,k,g')| < q_i^k$, and ($\neg$b) $i \succ_j^k \emptyset$, and either $i \succ_j^k \beta_j^k(\pi(j,k,g'))$ or $|\pi(j,k,g')| < q_j^k$. By the construction of g', $i \in T$ or $j \in T$ must follow. Suppose that $i \in T$. Then, from ($\neg$a), player i would be better off by having j as a partner either by replacing $\beta_i^k(\pi(i,k,g'))$ by i, if $|\pi(i,k,g')| = q_i^k$, or by simply including j, if $|\pi(i,k,g')| < q_i^k$. Two cases are possible:

1. $j \notin T$. Then by optimally modifying s_i' in category k to include j, agent i will be better off, since $i \in s_j^k$. Since it is a unilateral optimal deviation, it is credible. This is a contradiction with $s_T' \in CPNE(G(T, s_{-T}))$.

2. $j \in T$. We have $j \notin s_i^{k\prime}$ or $i \notin s_j^{k\prime}$, otherwise $(i,j) \notin g - g'$. The question is if i or j can credibly deviate from s'_T by forming a link between them in category k. The issue is if there is a further deviation after $\{i,j\}$ deviates once. Since s'_T is a CPNE, it is also a Nash equilibrium. Thus, $s'_i \in \arg\max_{\tilde{s}_i \in S_i} u_i(\tilde{s}_i, s'_{T-\{i\}}, s_{-T})$ and $s'_j \in \arg\max_{\tilde{s}_j \in S_j} u_j(\tilde{s}_j, s'_{T-\{j\}}, s_{-T})$. Now, suppose that $\{i,j\}$ deviate from this by letting $s''_i \in \arg\max_{\tilde{s}_i \in S_i} u_i(\tilde{s}_i, \bar{s}_j, s'_{T-\{i,j\}}, s_{-T})$ and $s''_j \in \arg\max_{\tilde{s}_j \in S_j} u_j(\tilde{s}_j, \bar{s}_i, s'_{T-\{j,i\}}, s_{-T})$, where $i \in \bar{s}_j$ and $j \in \bar{s}_i$. Note that u_i is dependent on strategy of j only if strategy of j contains i. That is, agent i only cares about whether strategy of j contains i. By the construction, and since only i and j are changing strategies together or unilaterally (since it may be true that $i \in s_i^{k\prime}$ or $j \in s_j^{k\prime}$ but not both), we have $s''_i \in \arg\max_{\tilde{s}_i \in S_i} u_i(\tilde{s}_i, s''_j, s'_{T-\{i,j\}}, s_{-T})$ and $s''_j \in \arg\max_{\tilde{s}_j \in S_j} u_j(\tilde{s}_j, s''_i, s'_{T-\{j,i\}}, s_{-T})$ (recall ($\neg$a) and ($\neg$b)). This is a credible deviation, since there is no further deviation from this. This is a contradiction with $s'_T \in CPNE(G(T, s_{-T}))$. $\square$

Since the resulting matching of a CPNE is pairwise-stable, and pairwise-stability is equivalent to credible group-stability (Theorem 1), Propositions 3 and 4 conclude that the set of resulting matchings of CPNEs and the set of pairwise-stable matchings are equivalent.

Corollary 1 *Consider a multi-partner general matching problem $(P, ((M_i^k, q_i^k)_{k\in K}, \succsim_i)_{i\in P})$ with separable and responsive preferences. Then, the set of pairwise-stable matchings is equivalent to the set of matchings generated from $CPNE(G(P))$.*

5 Conclusion

In this paper, we provide theoretical foundations for pairwise-stability in general multi-partner matching problems when agents' preferences are responsive and separable across categories. Our framework covers not only many-to-many two-sided matching problems but also the British hospital-intern markets, hierarchical organizations, one-sided matching problems, and multi-unit matching with constant marginal benefits.

The domain of the problems can be further generalized allowing the case in which one agent does not distinguish two categories yet others place quotas on different categories. For example, a small college wants to hire an econometrician but it does not care if she is a time-series or a cross-section specialist, and another small college wants to hire a person who knows microeconometrics but it does not distinguish whether she is a labor economist or a cross-section econometrician. However, a big university may have a quota for each of time-series econometrics, cross-sectional econometrics, and labor economics. The logic of the proofs work perfectly in such a case, as well. We chose a simpler setting for the sake of easy explanations and readability of the proofs.

In this paper, we focused on the case of responsive and separable preferences. However, in two-sided matching problems, weaker preference restriction introduced in Kelso and Crawford (1982), *substitutable preferences*, still preserves many of the results obtained in the responsive preference domain (see Roth, 1984b, 1985, Blair, 1988, Roth and Sotomayor, 1990, Roth, 1991, and Milgrom, 2003). It is not immediately clear if our results extend in this domain, since we are concerned about group deviations, yet all other papers on this domain are only analyzing pairwise-stability. This is an open question.

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Appendix:

Here, we adjust our Example 2 in order to make it fit better with the British hospital intern matching problem. The logic to obtain empty core is identical to Example 2. One special feature of the British market is that interns are required to get two positions: medical and surgical ones. Thus, an intern can be matched twice with the same hospital twice as long as categories are different. Moreover, potentially, an intern's preference ordering over hospitals can depend on the category (some hospital is good at surgical category, but not in medical one, and some intern is good at medical one but not at surgical one). These imply that preference ordering is not defined over subsets of agents, but over ordered subsets of agents. In the following example, we simply assume that agents's preference orderings are common in each category. Thus, the ordering of elements of subsets does not matter in each agent's preferences. We assume preferences are separable.

Example 4 *Consider the British hospital-intern market with two categories, i.e.* $K = \{m, s\}$*. Assume that there are five hospitals and five interns, i.e.* $H = \{H_1, H_2, ..., H_5\}$ *and* $I = \{i_1, i_2, ..., i_5\}$*. Each hospital has a position for each category, and each intern needs to have a position in each category. Each agent has a common preference ordering irrespective of categories (only brand name matters). These intrinsic preference orderings (irrespective of categories)* *are listed as follows.*

H_1	H_2	H_3	H_4	H_5	i_1	i_2	i_3	i_4	i_5
i_2	i_3	i_1	i_2	i_1	H_1	H_2	H_3	H_2	H_1
i_3	i_1	i_2	$\emptyset$	$\emptyset$	H_4	H_5	$\emptyset$	$\emptyset$	$\emptyset$
i_4	i_5	$\emptyset$	i_1	i_2	$\emptyset$	$\emptyset$	H_1	H_1	H_2
$\emptyset$	$\emptyset$	i_3	i_3	i_3	H_5	H_4	H_2	H_3	H_3
i_5	i_4	i_4	i_4	i_4	H_2	H_3	H_4	H_4	H_4
i_1	i_2	i_5	i_5	i_5	H_3	H_1	H_5	H_5	H_5

The resulting separable and responsive preferences over pairs of agents satisfy the following: We assume that for any H_j (i_j), $(i_l, i_{l'}) \sim_{H_j} (i_{l'}, i_l)$ ($(H_l, H_{l'}) \sim_{i_j} (H_{l'}, H_l)$), i.e., H_j (i_j) is indifferent between having i_l (H_l) and $i_{l'}$ ($H_{l'}$) as medical and surgical interns (hospitals that offer medical and surgical positions), respectively, and having $i_{l'}$ ($H_{l'}$) and i_l (H_l) as medical and surgical interns (hospitals that offer medical and surgical positions), respectively.[20] *Thus, we just list unordered pair of agents in the table below.*

H_1	H_2	H_3	H_4	H_5	i_1	i_2	i_3	i_4	i_5
i_2i_2	i_3i_3	i_1i_1	i_2i_2	i_1i_1	H_1H_1	H_2H_2	H_3H_3	H_2H_2	H_1H_1
i_2i_3	i_1i_3	i_1i_2	$i_2\emptyset$	$i_1\emptyset$	H_1H_4	H_2H_5	$H_3\emptyset$	$H_2\emptyset$	$H_1\emptyset$
i_3i_3	i_1i_1	i_2i_2	$\mathbf{i}_1\mathbf{i}_2$	$\mathbf{i}_1\mathbf{i}_2$	$H_1\emptyset$	$H_2\emptyset$	$\mathbf{H}_1\mathbf{H}_3$	$\mathbf{H}_1\mathbf{H}_2$	$\mathbf{H}_1\mathbf{H}_2$
i_2i_4	i_3i_4	i_1i_4	$\emptyset\emptyset$	$\emptyset\emptyset$	H_1H_5	H_2H_4	$\mathbf{H}_2\mathbf{H}_3$	$\emptyset$	$\emptyset$
i_3i_4	i_1i_4	i_2i_4			$\mathbf{H}_1\mathbf{H}_2$	$\mathbf{H}_2\mathbf{H}_3$	$\emptyset\emptyset$		
$i_2\emptyset$	$i_3\emptyset$	$i_1\emptyset$			H_4H_4	H_5H_5			
i_2i_5	i_3i_4	$i_2\emptyset$			$H_4\emptyset$	$H_5\emptyset$			
$i_3\emptyset$	$i_1\emptyset$	$\mathbf{i}_1\mathbf{i}_3$			$\mathbf{H}_4\mathbf{H}_5$	$\mathbf{H}_4\mathbf{H}_5$			
i_3i_5	i_1i_4	$\mathbf{i}_2 i_3$			$\mathbf{H}_1\mathbf{H}_3$	$\mathbf{H}_1\mathbf{H}_2$			
$\mathbf{i}_1\mathbf{i}_2$	$\mathbf{i}_2\mathbf{i}_3$	$i_4\emptyset$			$\emptyset\emptyset$	$\emptyset\emptyset$			
i_4i_4	i_5i_5	$\emptyset\emptyset$							
$i_4\emptyset$	$i_5\emptyset$								
$\mathbf{i}_4\mathbf{i}_5$	$\mathbf{i}_4\mathbf{i}_5$								
$\mathbf{i}_1\mathbf{i}_3$	$\mathbf{i}_1\mathbf{i}_2$								
$\emptyset\emptyset$	$\emptyset\emptyset$								

Preferences are separable, and we listed all individually rational choices. Choices in bold characters are the relevant choices that compose individually rational

[20] Here, we assume indifferences in players' preferences for the sake of simplifying the analysis. It must be clear that even if we introduce small differences between, say, (i_l, i_k) and (i_k, i_l), the result does not change.

matchings. Note that H_1, H_2, and H_3 do not want to be matched with i_1, i_2, and i_3, respectively, unless it is matched with one of the other two in the other position. However, i_1, i_2, and i_3 want to be matched with H_1, H_2, and H_3: they do not mind being matched with one of the others only when they get their favorites. Note also that H_1 and H_2 (i_1 and i_2 by symmetry) do not want to be matched with i_5, and i_4 (H_5 and H_4), respectively, unless it is matched with i_4 and i_5 (H_4 and H_5) in the other position, respectively.

In the above matching problem, the core is empty. We list the graph representations of all individually rational matchings. Superscripts of parentheses represent links in categories k and k' ($k, k' \in \{m, s\}$ with $k \neq k'$):

$$\begin{aligned}
g_1 &= \{(H_1, i_1)^k, (H_1, i_2)^{k'}, (H_2, i_1)^{k'}, (H_2, i_2)^k\},\\
g_2 &= \{(H_2, i_2)^k, (H_2, i_3)^{k'}, (H_3, i_2)^{k'}, (H_3, i_3)^k\},\\
g_3 &= \{(H_1, i_1)^k, (H_1, i_3)^{k'}, (H_3, i_1)^{k'}, (H_3, i_3)^k\},\\
g_4 &= \{(H_1, i_1)^k, (H_1, i_3)^{k'}, (H_2, i_2)^k, (H_2, i_1)^{k'}, (H_3, i_3)^k, (H_3, i_2)^{k'}\},\\
g_5 &= \{(H_1, i_1)^k, (H_1, i_2)^{k'}, (H_2, i_2)^k, (H_2, i_3)^{k'}, (H_3, i_3)^k, (H_3, i_1)^{k'}\},\\
g_6 &= \{(H_1, i_4)^k, (H_1, i_5)^{k'}, (H_2, i_4)^{k'}, (H_2, i_5)^k\},\\
g_7 &= \{(H_4, i_1)^k, (H_4, i_2)^{k'}, (H_5, i_1)^{k'}, (H_5, i_2)^k\},\\
g_8 &= \emptyset.
\end{aligned}$$

We can see that none of the above matchings is group-stable. There is always a coalitional deviation that brings another matching: $g_1 \rightarrow_{\{H_2,H_3,i_2,i_3\}} g_2$, $g_2 \rightarrow_{\{H_1,H_3,i_1,i_3\}} g_3$, $g_3 \rightarrow_{\{H_1,H_2,i_1,i_2\}} g_1$, $g_4 \rightarrow_{\{H_1,H_2,i_4,i_5\}} g_6$, $g_5 \rightarrow_{\{H_4,H_5,i_1,i_2\}} g_7$, $g_6 \rightarrow_{\{H_1,H_2,H_3,i_1,i_2,i_3\}} g_5$, $g_7 \rightarrow_{\{H_1,H_2,H_3,i_1,i_2,i_3\}} g_4$, and g_8 is dominated by all other individually rational matchings. Thus, the core (thus, the weak core) is empty. □