Aggregation of Rankings in Figure Skating*

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Abstract

We scrutinize and compare, from the perspective of modern theory of social choice, two rules that have been used to rank competitors in Figure Skating for the past decades. The first rule has been in use at least from 1982 until 1998, when it was replaced by a new one. We also compare these two rules with the Borda and the Kemeny rules. The four rules are illustrated with examples and with the data of 30 Olympic competitions. The comparisons show that the choice of a rule can have a real impact on the rankings. In these data, we found as many as 19 cycles of the majority relation, involving as many as nine skaters. In this context, the Kemeny rule appears as a natural extension of the Condorcet rule. As a side result, we show that the Copeland rule can be used to partition the skaters in such a way that it suffice to find Kemeny rankings within subsets of the partition that are not singletons and then, to juxtapose these rankings to get a complete Kemeny ranking. We also propose the concept of the mean Kemeny ranking, which when it exists, may obviate the multiplicity of Kemeny rankings. Finally, the fours rules are examined in terms of their manipulability. It appears that the new rule used in Figure Skating may be more difficult to manipulate than the previous one but less so than the Kemeny rule.

Key words: figure skating, ranking rules, vote aggregation, cycles, maximum likelihood, Kemeny, Copeland, Borda, manipulation.

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Résumé

Nous analysons et comparons, à la lumière de la théorie moderne du choix social, deux règles qui ont été utilisées au cours des dernières décennies pour classer les compétiteurs en patinage artistique. La première a été en usage de 1982 jusqu'à 1998, année où elle a été remplacée pas une nouvelle règle. Nous comparons également ces deux règles à celles de Borda et de Kemeny. Les quatre règles sont illustrées à l'aide d'exemples et avec les données de 30 compétitions olympiques. Les comparaisons montrent que différentes règles peuvent donner des résultats forts différents. Dans ces données, nous avons trouvé 19 cycles de la relation majoritaire, impliquant jusqu'à neuf compétiteurs. Dans ce contexte, la règle de Kemeny apparaît comme une extension naturelle de celle de Condorcet. Comme autre résultat, nous montrons que la règle de Copeland peut être utilisée pour partitionner les compétiteurs de telle sorte qu'il suffit de trouver un ordre de Kemeny sur chaque sous-ensemble de la partition et de juxtaposer ensuite ces derniers pour obtenir un ordre de Kemeny complet. Nous proposons également le concept d'ordre de Kemeny moyen qui, lorsqu'il existe, peut palier la multiplicité des ordres de Kemeny. Finalement, les quatre règles sont examinées du point de vue des possibilités de manipulation. Il ressort que la nouvelle règle utilisée en patinage artistique a des chances d'être plus difficile à manipuler que l'ancienne mais moins que la règle de Kemeny.

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1 Introduction

An extensive literature has been devoted to the design of social welfare functions, i.e. rules for aggregating individual preferences or rankings on a set of alternatives into a collective preference or final ranking. Yet, there are not very many instances in real life where the preoccupation is to arrive at a collective preference or ranking, as opposed to merely choosing an alternative or a subset of alternatives. A notable exception is professional sport, where a wide variety of methods are used to aggregate individual rankings into a final ranking. These methods are especially prominent in judged sports such as diving, synchronized swimming, gymnastics, and figure skating, but we also find examples of their use in some professional sports where ranking could be made from objective data. See for instances Jech (1983), Benoit (1992), and Levin and Nalebuff (1995).

Among judged sports, figure skating has probably become the most popular. Almost everybody knows that skaters are ranked from the marks that they receive from a panel of judges. However, very few people know that these marks merely serve to establish a ranking of the skaters for each judge, displayed as *ordinals* under the marks. These ordinal rankings (not the marks) are then aggregated into a final ranking according to a complex rule that has changed through time. Given the complexity of the procedure used in some periods, this ignorance is quite understandable.

In this paper, we scrutinize and compare two rules that have been prescribed by the International Skating Union (ISU) for the past decades, from the perspective of modern theory of social choice. The first one is described, inter alia, in the Regulations of the ISU for 1982 and for 1994. We call it the ISU-94 rule. It has been replaced in 1998 by a simpler rule that we call the ISU-98 rule. We also compare them to other rules, including the Kemeny rule which seems well suited to the context of figure skating.

The ISU-94 rule uses four different criteria to arrive at a final ranking. The first is the median rank of the skaters, i.e. the best rank that a skater obtains from a majority of judges. If two skaters obtain the same median rank, one tries to break the tie by reverting to the numbers of judges contributing to the median rank of each skater. If this is not sufficient to break all ties, one then tries to do so by using the sum of ranks that are smaller or equal to the median rank. If there still remains some ties after applying this third criterion, one uses the sum of ranks given by all judges, which is equivalent to the Borda count. Finally, if all ties are not resolved after these four principles have been applied, competitors who tie for a rank obtain the same rank. We show that the ISU-94 rule can also be defined as the successive application of approval rules, where the number of acceptable competitors is increased by one at each step.

Bassett and Persky (1994), hereafter BP, analyze the ISU-94 rule. They concentrate their analysis on the first criterion that they call the Median Rank Principle (MRP). They show that this principle responds positively to increased marks by each judge and respects the view of a majority of judges when this majority agrees on a skater's rank. They also demonstrate that (MRP) is the only one to possess these two properties. Moreover, they claim that this principle provides strong safeguards against manipulation by a minority of judges.

Based on the findings of Saari (1990)[1], we challenge this view of BP. We also stress the fact, also mentioned by BP, that this principle may conflict with another prominent majority principle advocated by Condorcet (1785), which prescribes that if a competitor is ranked ahead of another competitor by an absolute majority of judges, he should be ahead in the final ranking. It is well known that this principle may fail to give a consistent ranking because of a cycle in the majority relation, a possibility that Condorcet was well aware of. For example, A may be judged better than B by a strict majority of judges, who may be judged better than C, who may be judged better than D, and D may be judged better than A.

These cycles are not just a theoretical possibility. We found as many as 19 cycles in the data of 30 Olympic competitions. They involve as many as nine skaters. These are often middle ranked skaters, as if the disagreements between the judges occur mainly for competitors who are not medal contenders. However, in one case, these are famous skaters. In another case, one of the skaters in the cycle actually obtained the third place in the ISU-94 ranking. These cycles never prevented the occurrence of a Condorcet winner, except in one instance where two skaters tied for this title under the ISU-94 rule.

Despite the possibility of cycles, there is still something to be drawn from Condorcet's prescription. A more general criterion, called the Extended Condorcet Criterion (XCC), is proposed in this paper. Loosely speaking, it says that if a competitor is ranked consistently

[1]Saari shows that among all scoring voting procedures (including approval rules), the Borda rule is the method that either minimizes, or comes close to minimizing, the likelihood of a successful manipulation by a small group of individuals. Thus, as a combination of scoring rules, (MRP) cannot do better than the Borda rule, which is well known for its susceptibility to manipulation.

ahead of another competitor by an absolute majority of judges, he or she should be ahead in the final ranking. Consistency here refers to the absence of cycle involving these two skaters. No need to say, the ISU-94 rule violates this more general principle.

It is hard to know whether the ISU was conscious of this fact but, with the new rule put forward in 1998, the ISU abandoned the first three criteria of the ISU-94 rule, including (MRP), and replaced them with a criterion that is a small modification of a rule known in the literature as the Copeland rule. It consists in ranking alternatives with respect to the number of other alternatives that each alternative beats by a majority of votes or with which they tie.[2] This rule respects the Extended Condorcet Criterion. The Borda count is still used as a second criterion, to break ties left by the Copeland rule. The combination of these two criteria is all the more curious that the Copeland and the Borda rules may rank candidates in reverse orders.

The different tables that appear at the end of the paper show that the ISU-94 and ISU-98 rules may give very different rankings. Had the ISU-94 rule still been in use at the Olympic Games of 2002, it could have given rankings quite different from the new rule then in use.[3] For example, Table 11 shows that in the short program of the women competition, skater labeled[4] I would have obtained rank 9 instead of 11 while J would have obtained rank 10 instead of 8. In short, there are six inversions between the ISU-94 and the ISU-98 rankings in the short program, one in the free program, and four in the final ranking.[5]

Condorcet's objective was to find a most probable ranking on the set of alternatives. He showed that the majority relation, whenever consistent, is this most probable ranking under the hypothesis that every judge or voter can pick the best of two alternatives with a probability greater than one half and that this judgment is independent between pairs of alternatives and judges. If the majority relation produces cycles, some way must be found to break them. The Copeland rule, which is the first component of the ISU-98 rule, is one way to do so while respecting (XCC). However, it does not necessarily provide a most probable ranking on these subsets. Young (1988) shows that the Kemeny rule is the rule to use in

[2] Counting ties as wins is what makes the rule used by the ISU different from the original Copeland rule.

[3] As an illustration of the ignorance concerning these rules, the CBS Sportline posted on its website for the Olympic Games of 2002 the description of the 1994-rule.

[4] Names are replaced by labels.

[5] The final ranking is now given by a weighted sum of the ranks obtained in the short program and the free program, the latter receiving twice as much weight as the former.

order to get a most probable ranking in the presence of cycles. We shall present this rule and contrast it with the two ISU and the Borda rules.

Finding a Kemeny order by considering all possible permutations of the skaters, as the definition suggests, may prove prohibitive when the number of competitors is large as in figure skating. We show that the Copeland rule itself can be used to partition the skaters in such a way that it suffice to find Kemeny rankings within subsets of the partition that are not singletons. Then, a complete Kemeny order is obtained by concatenating the Kemeny rankings on the subsets of the partition.

It is also well known that there may exist many Kemeny orders. We propose the concept of the mean Kemeny ranking, which when it exists, may obviate partially this problem.

The two ISU, the Borda, and the Kemeny rules are illustrated with examples and with the data of 30 Olympic competitions: men (M), women (W), and pairs (P), short and free programs, for 1976, 1988, 1992, 1994, and 2002. A summary of these comparisons appears in Tables 12 and 13. The measure of the disagreement between two rankings is the number of pairs of competitors for which the relative ranks are inverted. For example, there are two inversions between the orders *cab* and *abc* namely one for the pair $\{a, c\}$ and another one for the pair $\{b, c\}$. If two competitors obtain the same rank in one ranking and different ranks in another ranking, this is counted as half a difference. More details on the comparisons are given in the conclusion.

The last issue treated in this paper is manipulation, or the misrepresentation of one's true (preferred) ranking by a judge in order to change the final ranking for one that he or she prefers. For example, a judge may prefer rankings that favour a particular competitor. There is a famous impossibility theorem in this context, due to Gibbard (1973) and Satthertwaite (1975), which says that all social choice functions, i.e. functions that select a winner, are manipulable. More generally, a judge may prefer a ranking to another, not just because of the winner, but because of the whole ranking. Bossert and Storcken (1992) extend the Gibbard-Satthertwaite theorem to this context, i.e. to social welfare functions or ranking rules.

The ISU rules do not escape these theorems. This paper shows, by means of examples, how the different principles involved in these rules, including the Median Rank Principle, are prone to manipulation. Some ranking rules may be more so than others. The question is then whether or not the ISU rules do well in this respect. Some considerations on this matter are offered, drawing on recent work by Saari (1990). It is argued that the Copeland and especially the Kemeny rules require more sophistication in order to be manipulated. However, the present state of research does not permit a clear-cut answer on this subject.

The paper is organized as follows. Section 2 introduces the notation including the formal definitions of most rules considered in this paper. Section 3 examines these rules from the perspective of properties that are often encountered in the theory of social choice. Section 4 describes the cycles of the majority relation found in the data of the 30 Olympic competitions. The Kemeny rule is the object of section 5. Manipulability is taken up is section 6. A summary of the paper and concluding remarks on the choice of a ranking rule are offered in section 7.

2 Definitions

Let $X = \{a, b, c, ...\}$ or $X = \{A, B, C, ...\}$ be the set of *competitors, skaters* or *alternatives,* with cardinality $|X| = m$. The terms competitor, skater and alternative will be used interchangeably, depending on the context. The first two are more appropriate to our context while the term alternative is more usual in the theory of social choice. This term will often be used when referring to this theory.

We denote by $\mathcal{B}$ the set of *binary relations* on X, by $\mathcal{R}$ the subset of *complete weak orders* or *rankings* (reflexive and transitive binary relations) on X, and by $\mathcal{L}$ the subset of *linear orders* (complete, transitive, and asymmetric binary relations) on X. A complete weak order on X can be represented in three different ways. The first is given by a vector $r = (r_a, r_b, r_c, \ldots)$, where r_a is the rank of a, r_b the rank of b, and so on. Equivalently, a weak order can be represented by an $(m \times m)$ binary matrix $N = [\nu_{st}]_{s,t \in X}$ where:

$$\nu_{st} = \begin{cases} 1 & \text{if } s \neq t \text{ and } r_s \leq r_t \\ 0 & \text{otherwise} \end{cases}$$

Conversely, given a binary representation N of a weak order, we get the representation r by setting $r_s = m - \sum_{t=1}^{m} \nu_{st}$. Note that $\nu_{st} = \nu_{ts} = 1$ if and only if $r_s = r_t$. The third representation is a sequence $s_1 s_2 \ldots$, where s_1 and s_2 are the competitors with ranks 1 and 2 respectively, etc. Parentheses are used to identify competitors with the same rank, as in $ab(cde)fg$. We shall use all three representations in this paper.

To complete the description of the problem, there is a set $J = \{1, 2, \ldots, n\}$ of *voters* or *judges.* For each judge j, we have a ranking $r^j \in \mathcal{R}$, also called a *vote*, on the set X. A *profile of votes* is a vector $R = (r^1, \ldots, r^n) \in \mathcal{R}^n$. A profile may also be given in the form of binary matrices $(N^1, \ldots, N^n)$. Once the voters or judges have cast their votes, the problem is to aggregate these votes into a final ranking. We formalize this idea in the following definition.

Definition 1 An *aggregation* or *ranking rule* is a function $\Gamma : \mathcal{R}^n \to \mathcal{R}$ that assigns to each profile R, a final ranking $\Gamma(R)$ of the alternatives. $\Gamma_s(R)$ represents the rank of alternative s in the final ranking $\Gamma(R)$.

In the language of the theory of social choice, Γ is a *social welfare function.* Note that some of the rules discussed in this paper are defined in terms of the *aggregate binary matrix:*

$$N(R) \equiv [\nu_{st}]_{s,t \in X} = \sum_{j=1}^{n} N^j = \sum_{j=1}^{n} \left[\nu_{st}^j\right]_{s,t \in X}$$

In Figure Skating, the marks given by the judges to the competitors merely serve to establish a ranking r^j (ordinals) for each judge j. Then, an aggregation or ranking rule, defined by the International Skating Union (ISU), is applied to the profile thus obtained to arrive at a final ranking. This rule has changed through time. We shall describe the rule in use from at least 1982 until 1998 and the one by which it was replaced in 1998. The Borda rule is a component of these two rules. A variant of the Copeland rule, which is also often discussed in the literature on aggregation of votes, is a component of the new rule. As is well known, the Borda rule is a scoring rule. Interestingly, the first ISU rule can be seen as an iterative application of successive positional or scoring rules with tie-breaking criteria. Hence, we first recall the definitions of the Borda and the Copeland rules and the definition of scoring rules, before giving a definition of the two ISU rules.

2.1 The Borda and Other Scoring Rules

Borda (1784) presented his rule to the Académie des sciences de France, which used it until Napoléon got elected to this institution. Given a profile R, alternative s receives $b^*(s) = \sum_{j=1}^{n} (m - r_s^j)$ points. This number is the *Borda score* of s. If the profile is given in the form of binary matrices $(N^1, \ldots, N^n)$, then $(m - r_s^j) = \sum_{t=1}^{m} \nu_{st}^j$ so that $b^*(s) = \sum_{j=1}^{n} \sum_{t=1}^{m} \nu_{st}^j = \sum_{t=1}^{m} \nu_{st}$. With the Borda rule, alternatives are ordered inversely to their Borda scores.

Now, let us define $b(s) = \sum_{j=1}^{n} r_s^j$. Clearly, $b^*(s) = mn - b(s)$. Thus, alternatively, the Borda rule orders alternatives according to the numbers $b(s)$. Let us define $b = [b(s)]_{s=1}^{m}$ and $b^* = [b^*(s)]_{s=1}^{m}$. Formally, we have the following definition.[6]

Definition 2 The *Borda rule* is the function $B : \mathcal{R}^n \rightarrow \mathcal{R}$ defined by:

$$\forall s, t \in X : B_s(R) \leq B_t(R) \Leftrightarrow b(s) \leq b(t) \Leftrightarrow b^*(s) \geq b^*(t)$$

$B(R)$ is the *Borda ranking.*

A scoring rule is one in which a vector of weights $w = (w_1, \ldots, w_m)$, with $w_i \geq w_{i+1},\ i = 1, \ldots, m-1,\ w_1 > w_m = 0$, is used to tally the ballots. Let $\bar{n}_i(s) = |\{j \in J : r_s^j = i\}|$ and $p^w(s) = \sum_{i=1}^{m} \bar{n}_i(s)\, w_i$. The competitors are ranked according to numbers of points $p^w(s)$ that they receive. The Borda rule is the scoring rule with $w = (m-1, m-2, \ldots, 1, 0)$.

Definition 3 The *scoring rule* with vector w is the function $P^w : \mathcal{R}^n \rightarrow \mathcal{R}$ defined by:

$$\forall s, t \in X : P_s^w(R) \leq P_t^w(R) \Leftrightarrow p^w(s) \geq p^w(t)$$

2.2 The Copeland Rule

Let $\mu_{st} : \mathcal{R}^n \rightarrow \mathbb{R}$, be a function defined for every pair of alternatives (s, t) and every profile R by:

$$\mu_{st}(R) = \begin{cases} 1 & \text{if } \nu_{st} > \nu_{ts} \\ \frac{1}{2} & \text{if } \nu_{st} = \nu_{ts} \\ 0 & \text{otherwise} \end{cases}$$

Then, for each alternative s, let:

$$c(s) = \sum_{t \in X \setminus \{s\}} \mu_{st}(R)$$

This number is the *Copeland score* of s. This is the number of alternatives over which s gets a strict majority of votes plus half the number of alternatives with which it ties.[7]

[6] Formally, the numbers that have just been defined depend on R. This is not indicated in the notation for the sake of simplicity.

[7] Alternatively, the three numbers $1, \frac{1}{2}, 0$ could be replaced by $2, 1, -1$, as in Moulin (1988).

Definition 4 The *Copeland rule* is the function $C : \mathcal{R}^n \to \mathcal{R}$ defined by:

$$\forall s, t \in X : C_s(R) \leq C_t(R) \Leftrightarrow c(s) \geq c(t)$$

$C(R)$ is the *Copeland ranking.*

2.3 The ISU-94 Rule

This rule, which has been used from at least 1982 until 1998, is described in rule 371 of the International Skating Union Regulations of 1982 and 1994, reproduced in Appendix B. The ISU-94 rule, as we call it, can be formally defined as follows. Its definition involves four principles or criteria. When a principle has been applied, the next one is used only if there remain ties between some competitors.

The first criterion is the median rank $\rho(s)$ of each skater s, i.e. the smallest rank such that a majority of judges have placed skater s at one of the ranks $1, \ldots, \rho(s)$ (par. 1, 2, 6)[8]. The smaller the median rank the better. If two skaters, say s and t, obtain the same median rank, one tries to break the tie by reverting to the numbers of judges $n_\rho(s)$ and $n_\rho(t)$ who gave the median or a better rank to s and t respectively (par. 3). The more the better. If this is not sufficient to break all ties, one then tries to do so by using the sum of ranks of tied skaters according to the judges who gave the median rank or a better rank to these competitors (par. 4). This number, for skater s, is written as $n_\rho(s)$. The smaller the better. If there still remains some ties after applying this third criterion, one uses the sum of ranks given to s by all judges (par. 5), or equivalently the Borda scores $b(s)$, as a breaking criterion. Finally, if all ties are not resolved after these four principles have been applied, competitors who tie for a rank obtain the same rank (par. 10).

Formally, these criteria are defined as follows. First, let:

$J_i(s) = \{j \in J : r_s^j \leq i\}$, and $n_i(s) = |J_i(s)|$

Then, $\rho(s) = \min\limits_{i \in \{1,\ldots,m\}} i$ such that $n_i(s) > \frac{|J|}{2}$,

Next, for the sake of simplicity, we write $J_\rho(s) = J_{\rho(s)}(s)$ and $n_\rho(s) = n_{\rho(s)}(s)$

Finally, let $b_\rho(s) = \sum\limits_{j \in J_\rho(s)} r_s^j$ and recall that $b(s) = \sum\limits_{j \in J} r_s^j$.

[8] These are the paragraphs of the regulations translated in this criteria.

These variables depend on the profile R, which we drop for the sake of simplicity. Note that $\dfrac{b(s)}{n}$ is the mean rank of skater s while $\dfrac{b_\rho(s)}{n_\rho(s)}$ is the mean rank of skater s according to the judges of $J_\rho(s)$. Now, for each candidate $s \in X$, define the vector $U(s) \in \mathbb{R}^4$ by

$$U(s) = (\rho(s),\ n - n_\rho(s),\ b_\rho(s),\ b(s))$$

and let $\leq_\ell$ be the lexicographic weak order[9] on $\mathbb{R}^4$. We can now give the definition of the ISU-94 rule.

Definition 5 The *ISU-94 rule* is the function $ISU^{94} : \mathcal{R}^n \to \mathcal{R}$ defined by:

$$\forall s, t \in X : ISU^{94}_s(R) \leq ISU^{94}_t(R) \Leftrightarrow U(s) \leq_\ell U(t)$$

$ISU^{94}(R)$ is the *ISU-94 ranking.*

Note that if $U(s) = U(t)$ for some s and t, then s and t obtain the same rank. The left panel of Table 1 illustrates the computation of the four criteria of this rule. See also the tables starting on page 47 for additional illustrations. The left part of the panel in these tables gives the profile, i.e. the rankings of the judges, indexed from 1 to 5 or 7. The ranking resulting from the application of the four criteria is given in the third panel. More details on reading these tables are given at the end of the next subsection.

Remark 1 The ISU-94 rule can also been described as an iterative procedure, where a scoring rule is applied at each iteration. This will prove useful when examining the strategic aspects of this rule. The scoring rule used at iteration $i < m$ is defined by $w^i = (\overbrace{1, ..., 1}^{i \text{ times}}, \overbrace{0, ..., 0}^{m-i \text{ times}})$. This is an approval rule. Skaters $s : p^{w^i}(s) > \frac{n}{2}$ who have not been ranked in the previous iterations are given the rank $P^{w^i}_s(R)$. Ties in this ranking are then broken whenever possible, by using the scoring rule with $w = (m-1, m-2, \ldots, 1, 0)$ first on the subset of judges $J_\rho(s)$ and if necessary, over the whole set of judges. Note that $\sum_{i \in J_\rho(s)} \bar{n}_i(s) w_i = mn_\rho(s) - b_\rho(s)$. In the last iteration, the skater left obtains the last rank. There can be at most one skater left at this stage.

This iterative procedure is illustrated in the middle panel of Table 1. At iteration 1, none of the skaters is ranked since all numbers $p^{w^1}(s)$ are smaller than $\frac{5}{2}$. At iteration 2, skaters A

[9] Let us first define $<_\ell$. For $a, b \in \mathbb{R}^L$, $a <_\ell b$ if $\exists k \in \{1, \ldots, L\} : a_k < b_k$ and $a_l = b_l\ \forall l < k$. Then, $a \leq_\ell b$ if either $a = b$ or $a <_\ell b$. The reader can check that $\leq_\ell$ is transitive.

and B are ranked first and second respectively. At iteration 3, no additional skater is ranked. At iteration 4, skaters D and E tie for the third place. However, since $b_\rho(C) < b_\rho(D)$, C obtains the third rank and D the fourth.[10] In the last iteration, there remains only E, who gets the last rank.

	Profile					Criteria-94			
s	1	2	3	4	5	ρ	n_ρ	b_ρ	b
A	1	2	1	2	2	2	5	8	8
B	2	1	3	2	1	2	4	6	9
C	5	4	2	4	3	4	4	13	18
D	3	3	4	5	4	4	4	14	19
E	4	5	5	1	5	5	5	21	20

	Iteration				
s	1	2	3	4	5
A	2	**5**	5	5	5
B	2	**4**	5	5	5
C	0	1	2	**4**	5
D	0	0	2	**4**	5
E	1	1	1	2	**5**

s	Final ranking
A	1
B	2
C	3
D	4
E	5

Table 1: Illustration of the ISU-94 rule

2.4 The ISU-98 Rule

This rule represents an important simplification with respect to the previous one. It also incorporates a different majority principle as we shall see later. Now, skaters are first ranked according to the numbers of other skaters that they defeat according to a majority of judges, or with whom they tie. The Borda rule is once again used to break ties according to the first criterion.[11]

Formally, the rule is based on a slight modification of the Copeland rule obtained by replacing the number $\frac{1}{2}$ in the definition of μ_{st} by 1. Thus, a tie is given the same weight as a win. We shall denote the corresponding scores and rule by c^* and C^* respectively. Let $V(s) = (c^*(s), b^*(s))$. The formal definition of the ISU-98 rule is the following.[12]

Definition 6 The *ISU-98 rule* is the function $ISU^{98} : \mathcal{R}^n \to \mathcal{R}$ defined by:

$$\forall s, t \in X : ISU_s^{98}(R) \leq ISU_t^{98}(R) \Leftrightarrow V(s) \geq_\ell V(t)$$

$ISU^{98}(R)$ is the *ISU-98 ranking.*

[10] Note that $J_\rho(C) = \{2, 3, 4, 5\}$ and $J_\rho(C) = \{1, 2, 3, 5\}$.

[11] This combination of the Copeland and the Borda rules is all the more curious that they may rank candidates in reverse orders. See Straffin (1980) and Moulin (1988, Exercice 9.1).

[12] The new rule is described and illustrated in Communication No. 997 of the ISU, available on the web site www.isu.org.

In plain words, skaters are first ranked according to their Copeland scores $c^*(s)$. If two skaters, say s and t, obtain the same Copeland score, one tries to break the tie by comparing the Borda scores $b^*(s)$ and $b^*(t)$. Finally, if all ties are not resolved after the application of these two criteria, i.e. if $V(s) = V(t)$ for some s and t, competitors who tie for a rank obtain the same rank. Table 2 illustrates the application of this rule. Note that we use the binary matrix $N(R)$ as the starting point since we can easily compute the number of wins under the majority rule from this matrix. Accordingly, b^* instead of b is the breaking criterion. Recall that $b^*(s) + b(s) = mn$.

	Profile							
s	1	2	3	4	5	6	7	8
A	1	5	1	4	3	3	3	3
B	2	2	3	3	5	2	4	1
C	3	4	4	1	1	4	5	2
D	4	3	2	5	2	5	1	4
E	5	1	5	2	4	1	2	5

	Binary matrix					Criteria-98	
s	A	B	C	D	E	c^*	b^*
A	0	4	4	5	4	4	17
B	4	0	6	5	3	3	18
C	4	2	0	5	5	3	16
D	3	3	3	0	5	1	14
E	4	5	3	3	0	2	15

s	Final ranking
A	1
B	2
C	3
D	5
E	4

Table 2: Illustration of the ISU-98 rule

Remark 2 The example of Table 2 has been chosen on purpose. It shows that C^* may be different from C. Indeed, $c = (2.5, 2.5, 2.5, 1, 1.5)$ and $C(R) = (1, 1, 1, 5, 4)$ while $C^*(R) = (1, 2, 2, 5, 4)$. Had c been chosen instead of c^* in the definition of ISU^{98}, the final ranking would have been $(2, 1, 3, 5, 4)$ instead of $(1, 2, 3, 5, 4)$. Moreover, as will be explained in Remark 6, C^* does not have all the properties of C.

Yet, in most practical situations, C and C^* will be identical. The discrepancy between C^* and C in the example of Table 2 is due to the fact that A ties with three other skaters while B has only one tie. We have not found similar situations in the other examples of this paper and in the data of Olympic competitions. In this paper, we shall not distinguish between the original Copeland rule and its modified version, except when necessary. The context will indicate whether an affirmation applies to the original rule, its modified version, or both.

The tables starting on page 47 give other illustrations of the two ISU rules, and of the Borda and the Kemeny rule, to be defined in section 5. In particular, the modified profile of Table 7 illustrates the extent to which the two ISU rules may be different. Noteworthy, they may gives rankings that are very different from the Borda ranking despite the fact that the

Borda rule is a component of both. The winner according to the Borda rule is ranked last by the ISU-94 rule[13] and fourth by the ISU-98 rule.[14]

The fourth panel of each part of the tables gives a measure of the disagreements between rankings. This measure is the Kemeny metric to be also defined in section 5. In plain words, this metric is the number of pairs of competitors for which the relative ranks are inverted. For example, there are two inversions between the orders *cab* and *abc*, namely one for the pair $\{a, c\}$ and another one for the pair $\{b, c\}$. If two competitors obtain the same rank in one ranking and different ranks in another ranking, this is counted as half a difference.

3 Properties of Ranking Rules

In the theory of social choice, properties are often imposed on ranking rules or stated as desirable. The most famous are probably the four axioms originally defined by Arrow (1951). Their definitions are adapted to the context of figure skating in the first subsection below. The second subsection deals with two majority principles.

3.1 Arrow's Conditions

Binary Independence (BI) A ranking rule Γ satisfies binary independence[15] if, given two profiles R and $\tilde{R}$ and two skaters s and t such that

$$\forall j \in J : r_s^j \leq r_t^j \Leftrightarrow \tilde{r}_s^j \leq \tilde{r}_t^j \text{ and } r_s^j \geq r_t^j \Leftrightarrow \tilde{r}_s^j \geq \tilde{r}_t^j,$$

on has:

$$\begin{aligned} \Gamma_s(R) &\leq \Gamma_t(R) \Leftrightarrow \Gamma_s\left(\tilde{R}\right) \leq \Gamma_t\left(\tilde{R}\right) \text{ and} \\ \Gamma_s(R) &\geq \Gamma_t(R) \Leftrightarrow \Gamma_s\left(\tilde{R}\right) \geq \Gamma_t\left(\tilde{R}\right). \end{aligned}$$

[13] This is hardly surprising since the ISU-94 rule is a combination or scoring rules. We know from Fishburn (1981) that given two vectors of weights w and w', there exists a profile such that the scoring rule P^w gives an order, say $s_1 s_2 \cdots s_m$, while $P^{w'}$ gives the reverse order $s_m s_{m-1} \cdots s_1$. Actually, Benoit (1992) finds the ranking of the baseball players, in the vote for the Most Valuable Player, to be quite sensitive to the particular weights. Saari (1992) has more striking theoretical results.

[14] This also comes as no surprise. As pointed out in footnote 11, the Borda and the Copeland rules can rank candidates in reverse orders. We also know from Saari and Merlin (1996) that the Borda and the Copeland rankings can be as different as desired when there are at least five candidates.

[15] This property is often called *independence of irrelevant alternatives* (IIA).

Weak Pareto Principle (WP) A ranking rule Γ satisfies this principle if, given two skaters s and t, one has:

$$r_s^j < r_t^j \ \forall j \in J \Rightarrow \Gamma_s(R) < \Gamma_t(R)$$

Non-Dictatorship (ND) A ranking rule Γ satisfies this condition if there exists no $j \in J$ such that $\Gamma(R) = r^j \ \forall R$.

Monotonicity (M) A ranking rule Γ satisfies monotonicity if, given two profiles R and $\tilde{R}$ and two skaters s and t such that

$$\begin{aligned} r_s^j &< r_v^j \Rightarrow \tilde{r}_s^j < \tilde{r}_v^j \quad \forall j \in J, \ \forall v \neq s \\ r_s^j &= r_v^j \Rightarrow \tilde{r}_s^j \leq \tilde{r}_v^j \quad \forall j \in J, \ \forall v \neq s \\ r_u^j &\leq r_v^j \Leftrightarrow \tilde{r}_u^j \leq \tilde{r}_v^j \quad \forall j \in J, \ \forall u, v \neq s \end{aligned}$$

the following holds:

$$\begin{aligned} \Gamma_s(R) &< \Gamma_t(R) \Rightarrow \Gamma_s\left(\tilde{R}\right) < \Gamma_t\left(\tilde{R}\right) \quad \text{and} \\ \Gamma_s(R) &= \Gamma_t(R) \Rightarrow \Gamma_s\left(\tilde{R}\right) \leq \Gamma_t\left(\tilde{R}\right). \end{aligned}$$

(BI) says that only the relative rankings of two skaters should matter in establishing the final relative ranking of these two skaters. (WP) says that if all judges are unanimous on the relative rankings of two skaters, the final relative ranking of these two skaters should agree with the unanimous view of the judges. (ND) prescribes that no judge be able to impose his or her ranking as the final ranking in all circumstances. (M) says that if all judges were to maintain or improve the relative ranking of skater s with respect to other skaters without changing the relative ranking of all other skaters, then the final ranking of skater s should remain at least as good as it was before the change.

The syntax of the premise of (M) is the one usually found in the literature. See, for instance, Arrow (1951) or Moulin (1988). However, the following lemma provides a different formulation that will be more convenient for the proof of Theorem 5.

Lemma 1 *The premise of (M) can be written equivalently as:*

$$\tilde{r}_s^j \leq r_s^j \quad \forall j \in J$$

$$r_u^j \leq r_v^j \Leftrightarrow \tilde{r}_u^j \leq \tilde{r}_v^j \quad \forall j \in J, \ \forall u, v \neq s$$

Proof. Given $\tilde{r}_s^j \leq r_s^j$ and $r_u^j \leq r_v^j \Leftrightarrow \tilde{r}_u^j \leq \tilde{r}_v^j$ $\forall u, v \neq s$, we must have $\tilde{r}_v^j \geq r_v^j$ $\forall v \neq s$. Hence, $r_s^j < r_v^j \Rightarrow \tilde{r}_s^j < \tilde{r}_v^j$ and $r_s^j = r_v^j \Rightarrow \tilde{r}_s^j \leq \tilde{r}_v^j$ $\forall v \neq s$. Conversely, suppose that the premise of (M) holds but that $\tilde{r}_s^j > r_s^j$ for some j. Then, $\exists v \in X : r_s^j \leq r_v^j$ and $\tilde{r}_s^j > \tilde{r}_v^j$, a contradiction. ■

Lemma 2 *Given a profile R and a judge i, the numbers $\rho(s)$, $n - n_\rho(s)$, $b_\rho(s)$, $b(s)$, defined in connection with the ISU-94 ranking rule, are non-decreasing with respect to r_s^i. The numbers $c(s)$, $c^*(s)$, $b^*(s)$, defined in connection with the Copeland and the ISU-98 ranking rules, are non-increasing with respect to r_s^i. Thus, a ranking based on any of these numbers satisfies (M).*

Proof. The first two assertions are immediate. As for the last assertion, consider two profiles R and $\tilde{R}$ and two skaters s and t such that $\tilde{r}_s^j \leq r_s^j$ and $\tilde{r}_t^j \geq r_t^j$ $\forall j \in J$. Suppose that $\rho(s) < \rho(t)$ under R. Then, clearly $\tilde{\rho}(s) < \tilde{\rho}(t)$ under $\tilde{R}$, as requested by (M). Similarly, $\rho(s) = \rho(t)$ implies $\tilde{\rho}(s) \leq \tilde{\rho}(t)$. The same holds for all other numbers, with reversed inequalities for the last three. ■

Theorem 3 *The Borda, the Copeland, the ISU-94, and the ISU-98 rules satisfy (ND), (WP), and (M).*

Proof. This assertion follows from the definitions of the rules and Lemma 2. ■

Corollary 4 *None of the Borda, the Copeland, the ISU-94, and the ISU-98 rules satisfies (BI).*

Proof. This follows from Theorem 3 and the impossibility theorem of Arrow (1951), which says that there is no ranking rule that satisfies (BI), (ND), (WP) and (M). ■

The non-binary character of the aforementioned rules is immediate from their definitions. In establishing the ranking between two alternatives, the ranks of other alternatives or the majority of these two alternatives in comparison to other alternatives play a crucial role. Actually, it is not hard to find examples of violation of (BI). Hence, in Table 8, judge 5 changes her ranking to give the modified profile. In doing so, she does not change her relative ranking of A and B. Yet, this changes permits B to ravish the first place to A according to the Borda, the ISU-94, and the ISU-98 rules, in violation of (BI). In Table 9, we have a violation of (BI) by the Copeland rule and again by the Borda and the ISU-98 rules. In this example, judge 3 changes her ranking to give the modified profile. In doing so, she does not change her relative ranking of A and B. Yet, as in the previous example, this changes permits B to ravish the first place to A according to the Copeland, the Borda and the ISU-98 rules. In particular, A defeats all other candidates under the majority rule with the original profiles but defeats only two candidates with the modified profile.

3.2 Majority Principles

Bassett and Persky (1994) point out that the first criterion of the ISU-94 rule, which they call the *Median Rank Principle* (MRP), translates a majority principle, which they call the *Rank Majority Principle* (RMP). We now give their formal definition of this principle, followed by their characterization of (MRP) in terms of (RMP) and (M).

Rank Majority Principle (RMP) Recall the definition $\bar{n}_i(s) = \left|\{j \in J : r_s^j = i\}\right|$. A ranking rule Γ satisfies this principle if, given a profile R, two skaters s and t, and two ranks i and k such that $\bar{n}_i(s) > \frac{n}{2}$, $\bar{n}_k(t) > \frac{n}{2}$, $i < k$, one has $\Gamma_s(R) < \Gamma_t(R)$. In plain words, if skaters s and t obtain a strict majority for exactly rank i and k respectively, with $i < k$, then s should obtain a final lower rank than t.

Theorem 5 (Bassett and Persky) *An aggregation rule $\Gamma : \mathcal{R}^n \to \mathcal{R}$ satisfies (M) and (RMP) if and only if*

$$\forall s, t \in X : \Gamma_s(R) \leq \Gamma_t(R) \Leftrightarrow \rho(s) \leq \rho(t) \tag{MRP}$$

Proof. The "only if" part is immediate. Conversely, suppose that (M) and (RMP) hold but that (MRP) fails, i.e. $\exists R \in \mathcal{R}^n$, $s, t \in X$, and positive integers i, k such that $i = \rho_s(R) < \rho_t(R) = k$ but $\Gamma_s(R) \geq \Gamma_t(R)$. Then, consider another profile $\hat{R}$ such that $r_s^j \leq i \Rightarrow \hat{r}_s^j = i$ (which implies $r_s^j \leq \hat{r}_s^j$) $\forall j \in J$, and $r_u^j \leq r_v^j \Leftrightarrow \hat{r}_u^j \leq \hat{r}_v^j$ $\forall j \in J$, $\forall u, v \neq s$. Then, by (M), $\Gamma_s\left(\hat{R}\right) < \Gamma_t\left(\hat{R}\right) \Rightarrow \Gamma_s(R) < \Gamma_t(R)$ or equivalently:

$$\Gamma_t(R) \leq \Gamma_s(R) \Rightarrow \Gamma_t\left(\hat{R}\right) \leq \Gamma_s\left(\hat{R}\right) \tag{1}$$

Next, consider a profile $\tilde{R}$ such that $\hat{r}_t^j \geq k \Rightarrow \tilde{r}_t^j = k$ $\forall j \in J$, and $\tilde{r}_u^j \leq \tilde{r}_v^j \Leftrightarrow \hat{r}_u^j \leq \hat{r}_v^j$ $\forall j \in J$, $\forall u, v \neq s$. Then, by (M),

$$\Gamma_t\left(\hat{R}\right) \leq \Gamma_s\left(\hat{R}\right) \Rightarrow \Gamma_t\left(\tilde{R}\right) \leq \Gamma_s\left(\tilde{R}\right) \tag{2}$$

Combining (1) and (2), we get:

$$\Gamma_t(R) \leq \Gamma_s(R) \Rightarrow \Gamma_t\left(\tilde{R}\right) \leq \Gamma_s\left(\tilde{R}\right)$$

However, with profile $\tilde{R}$, we now have $\bar{n}_i\,(s) > \frac{n}{2}$ and $\bar{n}_k(t) > \frac{n}{2}$. By (RMP), we should have $\Gamma_t\left(\tilde{R}\right) > \Gamma_s\left(\tilde{R}\right)$, a contradiction. ■

Remark 3 BP states their monotonicity condition with the premise: given two profiles R and $\tilde{R}$ and two skaters s and t such that $\tilde{r}_s^j \leq r_s^j$ and $\tilde{r}_t^j \geq r_t^j\ \forall j \in J$. They omit the part $r_u^j \leq r_v^j \Leftrightarrow \tilde{r}_u^j \leq \tilde{r}_v^j \quad \forall j \in J,\ \forall u, v \neq s$, given in Lemma 1. Their definition is thus the one of strong monotonicity (SM) given in subsection 6.1. As will be seen in subsection 6.1, (MRP) does not satisfy (SM). Actually, BP implicitly use (M) in the proof of their theorem, instead of (SM), which is fine.

Remark 4 (RMP) almost characterizes (MRP) since most interesting rules satisfy (M). However, (RMP) alone does not provide a complete ranking. Thus, (RMP) alone does not imply (MRP).

As pointed out by BP, (RMP) is by no means the only majority principle of interest in this context. Condorcet advocated another principle, which says that if in a head to head comparison, a majority of judges give a better rank to skater s than to skater t, then s should obtain a better final rank than t. Accordingly, skaters should be ranked on the basis of these head to head comparisons. This idea leads to the definition of the correspondence $M : \mathcal{R}^n \to \mathcal{B}$:

$$s\,M(R)\;t \Leftrightarrow \nu_{st} > \nu_{ts}$$

The binary relation $M(R)$ is the *majority relation* issued from profile R. As is well known, this relation is not necessarily transitive: it may contain cycles, a term to be made precise below. Thus, M is not an aggregation rule. Yet, if $M(R)$ is a linear order on X, we call it the *Condorcet ranking*. An alternative s such that $s\,M(R)\;t\ \forall t \neq s$ is a *Condorcet winner*. Note that there may exist a Condorcet winner even if there is no Condorcet ranking. To complicate the problem, in the context of figure skating where ties are allowed, $M(R)$ is not necessarily complete. This problem will be remedied in the next section by extending $M(R)$ to the weak majority relation.

Although M is not necessarily an aggregation rule, it is often considered as desirable that an aggregation rule gives a result that agrees with the Condorcet ranking whenever possible. This prompt the following definition.

Condorcet Consistency (CC) An aggregation rule $\Gamma : \mathcal{R}^n \to \mathcal{R}$ is *Condorcet consistent* if $\Gamma(R) = M(R)$ for every profile R such that $M(R)$ is a linear order.

Theorem 6 *The Copeland rule and the ISU-98 ranking rule satisfy (CC).*

Proof. Consider a sequence $s_1, s_2, \ldots, s_m$ and a profile R such that $\nu_{s_1} > \nu_{s_2} > \cdots > \nu_{s_m}$. Then, $c(s_1) > c(s_2) > \cdots > c(s_m)$ and thus, $C(R)$ is a singleton satisfying $C_{s_1}(R) < C_{s_2}(R) < \cdots < C_{s_m}(R)$ as requested by (CC). Since there is no tie, $ISU^{98}(R) = C^*(R) = C(R)$, establishing that the ISU-98 rule satisfies (CC). ■

Remark 5 The Borda rule may violate (CC). This possibility has been known since the lifetime of Borda and Condorcet, who debated passionately over the respective merits of their rules. In as much as the Borda rule is a component of the ISU-94 rule, the latter does not respect (CC) either. However, (MRP), i.e. the first criterion of the ISU-94 rule, may itself be at odds with (CC). The original profile of Table 9 provides an illustration of a conflict between (MRP) and (CC). Skater B is ranked first according to (MRP) but she should be ranked second according to (CC).

Remark 6 Henriet (1985) characterizes the Copeland rule in terms of three properties. The modified Copeland rule violates two of these properties, namely a monotonicity condition stronger than (M) and the independence of the ranking with respect to the reversal of cycles in the majority relation. See Appendix A for the details.

4 Cycles in Olympic Competitions

A cycle of the majority relation $M(R)$ is a subset of X, say $\{s_1, \ldots, s_\kappa\}$ such that:

$$s_1 M(R) s_2 M(R) \cdots M(R) s_\kappa M(R) s_1$$

With both profiles of Table 7, there is a cycle of M on the whole set X. This possibility was well known by Condorcet himself and for this reason, is often called the *Condorcet paradox*.

The literature contains numerous analyses of the theoretical probability of the occurrence of cycles. Gerhlein (2002) provides an excellent survey of many of theses analyses for the case of three alternatives. In his words, “models have been developed which attempt to determine these probability estimates for situations which are contrived to make the outcome most likely to occur. In general, intuition suggests that we would be most likely to observe this paradox on the pairwise majority rule relations when voters’ preferences are closest to being balanced between all pairs of candidates.” This notion of balanced preferences is given different forms in the literature. One of them is Impartial Anonymous Culture (IAC). With (IAC), it is assumed that all preference profiles, for a fixed number of voters, are equally likely to be observed. A second assumption is Maximal Culture (MC). It differs from (IAC) in not requiring that there is a fixed number of voters in the population. Another assumption is Impartial Culture (IC), in which each voter independently selects his preference type according to a uniform probability distribution. A fourth assumption is the Dual Culture Condition (DC) under which “each possible ranking has the same probability of being observed as its reversed, or dual, ranking.”

For the case of nine voters, as in Figure Skating, these different assumptions produce probabilities of a cycle on three alternatives that range between 0.056 and 0.17. However, in his conclusion, Gerhlein makes clear that the probability of a cycle would certainly be significantly lower “for typical situations that are not contrived to make majority rule cycles most likely to occur.” He adds: “Given all of this, if we wonder whether or not majority rule cycles should be expected to be routinely observed in three-candidate elections, the answer is clearly ‘No.’

In Figure Skating, judges are supposed to make an impartial evaluation of the competitors. Thus, their rankings should be close to one another (positively correlated). They should be far from being balanced according to the different definitions given in the previous paragraphs, diminishing if not eliminating the possibility of cycles. It was thus a surprise to find out as many as 19 cycles, i.e. cycles on 19 disjoint subsets of skaters, in the data of 30 Olympic competitions.

Many of the cycles found in Olympic competitions involve some ties between skaters. Thus, in order to examine the data of Olympic competitions, we must extend the majority relation to take the possibility of ties into account.

Competition	Set	Cycle	Rel.
M-1976, short	$\{3,4,5,6,7,9\}$	$4T3M5T7T9M6T4$	W
		$4T3M7T5M9M6T4$	W
		$4T3M7M9T5M6T4$	W
M-1976, free	$\{6,7,8\}$	$7M6M8T7$	W
M-1988, short	$\{5,6,7\}$	$6M5M7T6$	W
	$\{12,13,14\}$	$12M13M14M12$	M
	$\{20,21,22\}$	$20M22M21T20$	W
M-1992, short	$\{14,15,16\}$	$14M16T15T14$	W
M-1992, free	$\{4,5,6,7,8\}$	$4M7M8M5M6T4$	W
	$\{5,6,7,8\}$	$5M6M7M8M5$	M
	$\{12,13,14\}$	$12M13M14T12$	W
W-1988, short	$\{12,\ldots,20\}$	$15T13M14M12M20T16M18M19M17M15$	W
		$13M17T14M12M20T16M18M19M15T13$	W
W-1992, short	$\{7,8,9\}$	$8M7M9T8$	W
	$\{18,19,21\}$	$19M21M18T19$	W
W-1992, free	$\{16,17,18\}$	$16M17M18T16$	W
W-1994, short	$\{8,9,10\}$	$8M10M9T8$	W
	$\{17,18,19,20\}$	$17M18M19M17$	M
W-2002, short	$\{6,7,8,9,10,11\}$	$6M7M8M11M9M10M6$	M
	$\{20,21,22\}$	$20M22M21M20$	M
P-1976, short	$\{10,11,12,13\}$	$10M12M13M11M10$	M
P-2002, short	$\{13,14,15\}$	$13M14M15M13$	M

Table 3: Cycles in 12 Olympic Competitions

We thus define the *weak majority* relation $W : \mathcal{R}^n \to \mathcal{B}$ by:

$$sW(R)\,t \Leftrightarrow \sum_{i=1}^{n} \nu_{st}^{i} \geq \sum_{i=1}^{n} \nu_{ts}^{i}$$

Accordingly, a cycle of $W(R)$ is a subset of X, say $\{s_1, \ldots, s_\kappa\}$ such that

$$s_1 W(R)\, s_2 W(R) \cdots W(R)\, s_\kappa W(R)\, s_1$$

We also define the *tie* relation $T : \mathcal{R}^n \to \mathcal{B}$ by:

$$s\,T(R)\ t \Leftrightarrow \sum_{i=1}^{n} \nu_{st}^{i} = \sum_{i=1}^{n} \nu_{ts}^{i}$$

A cycle of $T(R)$ is a subset of X, say $\{s_1, \ldots, s_\kappa\}$ such that all elements tie. Note that M is the asymmetric component of W and T its symmetric component.

As explained above, we found cycles of $W(R)$ in 12 of the 30 Olympic competitions that we examined. In some cases, there are cycles on two or three disjoint subsets, for a total of 19 subsets.[16] These cycles are described in Table 3. We sometimes list more than one cycle for a particular subset. However, we do not pretend to be exhaustive in this respect. The last column indicates which relation, M or W, is involved in the cycle. The names of the competitors are replaced by the numbers corresponding to their ISU-94 ranking in the competition under scrutiny, with ties broken in an arbitrary way.

Let us single out a few facts from Table 3. Most cycles involve middle ranked skaters, as if the disagreements between the judges occur mainly for these competitors. However, in one case these were famous skaters. In another case, one of the skaters in the cycle actually obtained the third place in the ISU-94 ranking. These cycles never prevented the occurrence of a Condorcet winner, except in the 1992 and 1994 women free programs, where two skaters tied for this title.

In the men 1976 short program, the skater ranked ninth by the ISU-94 rule ties with the skater ranked fifth according to the majority rule. The skater ranked ninth by the ISU-94 rule also defeats the skater ranked sixth. In the women 1988 short program, a majority of judges has given a better rank to skater 19 than to skater 15 while the ISU-94 rule has ranked them not only in the reverse order but four ranks apart. We found the longest cycle of W (9 skaters) in the women 1988 short program and the longest cycle of M (6 skaters) in the

[16] We do not include cycles of T in this count. The latter will show up as ties in the Kemeny rankings.

women 2002 short program. Most cycles were found in the short programs. Paradoxically, the short program is supposed to be the most objective part of a competition.

5 The Kemeny Rule

In our opinion, Condorcet Consistency is more compelling than the Rank Majority Principle, which together with Monotonicity characterizes the Median Rank Principle. Indeed, it is hard to resist to the principle requiring that the ranking between two skaters reflect the view of a majority of judges whenever possible, i.e. when this can be done in a consistent way. Actually, there is a much more profound foundation to this criterion, due to Condorcet (1785). More important, Condorcet's approach can be extended to deal with cycles in the majority relation. The result is the Kemeny rule.

5.1 Condorcet's Maximum Likelihood Approach

Condorcet's objective was to justify the majority principle. On this, he was certainly inspired by Rousseau (1762) in his *Social Contract*, for whom the opinion of the majority is legitimate because it expresses the "general will."

> When in the popular assembly a law is proposed, what the people is asked is not exactly whether it approves or rejects the proposal, but whether it is in conformity with the general will, which is their will. Each man, in giving his vote, states his opinion on that point; and the general will is found by counting votes. When therefore the opinion that is contrary to my own prevails, this proves neither more nor less that I was mistaken, and that what I thought to be the general will was not so. [Rousseau (1913), p. 93]

Condorcet's objective was to formulate this proposition rigorously, using the calculus of probability, which was new at the time of his writing. In the context of figure skating, his approach may be cast as follows. There is a best skater, in terms of the performance being assessed, a second best, etc. Put differently, there is a true or objective order of the skaters, base on their performances. Judges may have different opinions because they are imperfect observers. Some details of the performances, quick as they can be, may escape them. However, if they are right more often than they are wrong, then the opinion of the majority should yield the true order of the skaters.

In figure skating, this is a legitimate point of view since the intent of the ISU regulations is clearly to have judges furnish an evaluation of the relative merits of the competitors in terms of scores. They are even instructed on how to subtract points for different types of mistakes. The different scores attributed to the different elements of the competition are then aggregated and the ranking r^j of each judge is determined from these aggregated scores. Thus, the profile R is supposed to contains these evaluations and not the preferences of the judges. The recent scandals that have afflicted this sport and the findings of Campbell and Galbraith (1996), on which we shall come back, may cast doubt on this presumption. Yet, if we take the rankings of a profile R as independent evaluations of the true ranking of the competitors according to the established rules, a pertinent question is: which final ranking $\Gamma(R)$ is most likely to be the true ranking of the competitors?

Assuming that every judge can pick the best of two skaters with a probability p satisfying $1/2 < p \leq 1$, and that this judgment is independent between pairs of skaters and judges,[17] Condorcet showed that if the binary relation M is an order on X, then it is the most probable order on X. This was one of the first applications of maximum likelihood estimation and of statistical hypothesis testing.

Condorcet also proposed a method for breaking cycles. His prescription is to eliminate some of the propositions (sMt is a proposition), starting with the one with the weakest majority and so on until the cycle disappears. This works fine if $m = 3$ but unfortunately this method may give ambiguous results or a partial order for larger values of m. Young (1988) develops a correct application of Condorcet's maximum likelihood approach for the more general case. He shows that a most likely order is in fact a Kemeny order, i.e. an order that minimizes a "distance" between an order r an a profile R proposed by Kemeny (1959) and Kemeny and Snell (1962), hence the name given to this rule. Note that the Copeland rule, which is used as the first criterion in the ISU-98 rule, is another method to deal with cycles while respecting (CC). However, it departs from Condorcet's maximum likelihood approach.

[17] Drissi and Truchon (2004) relax the assumption that this probability is the same for all pairs of alternatives. Other authors have relaxed the assumption that it is the same for all voters.

5.2 Definition of the Rule

The definition of the rule involves the Kemeny "distance" between a weak order and a profile. First, let $\gamma_{st} : \mathcal{R}^2 \to \mathbb{R}$ be a function defined for every pair of alternatives (s, t) such that $s < t$ and every pair of weak orders $(\hat{r}, r)$ by:

$$\gamma_{st}(\hat{r}, r) = \begin{cases} 1 & \text{if } \hat{r}_s < \hat{r}_t \text{ and } r_s > r_t \\ \frac{1}{2} & \text{if } \hat{r}_s < \hat{r}_t \text{ and } r_s = r_t \\ \frac{1}{2} & \text{if } \hat{r}_s = \hat{r}_t \text{ and } r_s > r_t \\ 0 & \text{otherwise} \end{cases}$$

Then, the *Kemeny metric* on $\mathcal{R}$ is the function $c^K : \mathcal{R}^2 \to \mathbb{R}$ defined by:

$$c^K(\hat{r}, r) = \sum_{s \in X} \sum_{t \in X} \gamma_{st}(\hat{r}, r) \tag{3}$$

Finally, the Kemeny "distance" δ^K between a weak order r and a profile R is given by:

$$\delta^K(r, R) = \sum_{j=1}^{n} c^K\left(r, r^j\right)$$

Definition 7 The *Kemeny rule* is the correspondence[18] $K : \mathcal{R}^n \to \mathcal{R}$ that assigns to each profile R, the subset $K(R) = \arg\min_{r \in \mathcal{R}} \delta^K(R)$. The elements of $K(R)$ are *Kemeny rankings*.

The value of $\delta^K(R)$ may be seen as the total number of pairwise supports for r in profile R, i.e. the total number of judges who rank pairs of skaters as in r. A maximum likelihood order is thus one that has the greatest total support from the judges.

5.3 Results on Kemeny Orders

Given that there may be ties in the majority relation, the search for a Kemeny ranking should ideally be done in the space of weak orders. However, the space of weak orders being considerably larger than the space of linear orders, we shall limit the search to linear orders. Yet, as we shall see below, we do not abandon weak orders completely. With this provision in mind, the Kemeny rule may be given an equivalent definition. Let:

$$\kappa(r, R) = \sum_{s \in X} \sum_{\substack{t \in X \\ r_s < r_t}} \nu_{st}(R)$$

[18] Contrary to the rules previously defined, we now have a correspondence instead of a single value function.

We call $\kappa(r,R)$ the *Kemeny score* of order r. If the rows and the columns of $N(R)$ are ordered according to the vector r, then, $\kappa(r,R)$ is the sum of the elements above the diagonal of $N(R)$. Actually, $\kappa(R)$ is a positive transformation of the maximum likelihood function. Hence, a Kemeny order has the maximum Kemeny score. This is established in the following lemma.

Lemma 7 $\arg\min_{r\in\mathcal{L}} \delta^K(R) = \arg\max_{r\in\mathcal{L}} \kappa(R)$

Proof. Let $\tau_{st}(R) = \left|j \in J : r_s^j = r_t^j\right|$ and note that $\nu_{st}(R) = \left|j \in J : r_s^j \leq r_t^j\right|$. If r is a linear order such that $r_s < r_t$, we have:

$$\begin{aligned}\sum_{j=1}^{n} \gamma_{st}\left(r, r^j\right) &= \left|j \in J : r_s^j > r_t^j\right| + \frac{1}{2}\left|j \in J : r_s^j = r_t^j\right| \\ &= n - \nu_{st}(R) + \frac{1}{2}\tau_{st}(R)\end{aligned}$$

If r is a linear order such that $r_s > r_t$, we have $\gamma_{st}\left(r, r^j\right) = 0\ \forall j$. Thus,

$$\delta^K(r,R) = \sum_{s\in X}\sum_{t\in X}\sum_{j=1}^{n} \gamma_{st}\left(r, r^j\right) = \sum_{s\in X}\sum_{\substack{t\in X \\ r_s<r_t}} \left(n - \nu_{st}(R) + \frac{1}{2}\tau_{st}(R)\right) \tag{4}$$

Note that the term $\frac{1}{2}\sum_{s\in X}\sum_{\substack{t\in X \\ r_s<r_t}} \tau_{st}(R)$ is common to all orders r, whether $r_s < r_t$ or $r_s > r_t$. Indeed, if $r_s > r_t$, then $\tau_{st}(R)$ is replaced by $\tau_{ts}(R)$ in this term but both have the same value. Thus, minimizing $\delta^K(r,R)$ is equivalent to maximizing $\kappa(r,R)$. ■

Lemma 8 *Let* $X = \{1, 2, \ldots, m\}$ *and suppose* $r^K = (1, 2, \ldots, m)$ *is a Kemeny order for a given profile* R*. Then,* $\nu_{s,s+1}(R) \geq \nu_{s+1,s}(R)$, $s = 1, \ldots, m-1$.

Proof. If this was not the case for some s, one could obtain an order with a better Kemeny score by simply interchanging the ranks of s and $s+1$. ■

5.4 Finding Kemeny Orders

Finding a Kemeny order by considering all possible permutations of the skaters, as the definition suggests, may prove prohibitive when the number of competitors is large as in figure skating. There are many contributions in the literature that address this specific problem.[19] Whatever the method used, one way to ease the task it to break the problem into many subproblems, using an extension of Condorcet Consistency that we shall rightly call the Extended Condorcet Criterion. In essence, it says that if the set of alternatives can be partitioned in such a way that all members of a subset of this partition defeat all alternatives belonging to subsets with a higher index, then the former should obtain a better rank than the latter. For most practical applications, a Kemeny order on each subset of the partition can be found by simple enumeration. Then, a complete Kemeny order is obtained by juxtaposing the Kemeny orders on the subsets of the partition.

Extended Condorcet Criterion (XCC) Let $\mathbb{X} = \{X_1, X_2, \ldots, X_p\}$ be a partition of X such that:

$$\alpha < \beta, s \in X_\alpha, t \in X_\beta \Rightarrow \nu_{st} > \nu_{ts}$$

A partition satisfying the above property is called a *Condorcet partition*. Let $\mathcal{P}(X)$ be the class of all Condorcet partitions of X.

Then, an aggregation rule[20] $\Gamma : \mathcal{R}^n \to \mathcal{R}$ satisfies the *Extended Condorcet criterion* (XCC) if for any selection $\gamma(R) \in \Gamma_s(R)$ and any partition $\mathbb{X} \in \mathcal{P}(X)$, the following holds:

$$[\alpha < \beta, s \in X_\alpha, t \in X_\beta] \Rightarrow \gamma_s(R) < \gamma_t(R)$$

(XCC) implies (CC). Indeed, if $M(R)$ is a linear order, then the finest Condorcet partition of X is a set of singletons. Thus, $\Gamma(R) = M(R)$ as requested by (CC).

If there is a cycle of W on some subset of alternatives, then these alternatives must belong to a same subset X_α of any Condorcet partition of X. In particular, this must be the case for two alternatives s and t such that sTt. In the finest partition of this class, the sets X_α are cycles of maximal length of W (possibly a set of tied skaters) or singletons. X_1 is also called the *top cycle* of W or the *Condorcet set*, a solution concept introduced by Good (1971) and Schwartz (1972) for the strict majority relation. X_2 is the top cycle on $X \backslash X_1$, etc. X_p is sometimes called the *bottom cycle*.

When $W(R)$ contains cycles, (XCC) does not say how to rank alternatives within an X_α. In other words, (XCC) yields only a partial order in these circumstances. The Kemeny rule

[19] Several of these contributions are based on mathematical programming. See for instance Remage and Thompson (1966), de Cani (1969), Marcotorchino and Michaud (1979), Barthelemy, Guénoche, and Hudry (1989).

[20] Γ is now a correspondence.

can then be used to complete a partial order yielded by (XCC). We explain how below, after proving a few more results.

The Condorcet partition is by no means new. Laslier (1997) uses the same kind of decomposition in the context of tournaments, which are defined by an asymmetric and complete binary relation on a set of alternatives. He calls the finest partition of $\mathcal{P}(X)$ a scaling decomposition. He proves that such a decomposition is unique. Laslier also gives many references where this idea can be found. Depending on the authors, the subsets of this partition are called adjacent sets, convex parts, autonomous parts, etc. One finds the same kind of decomposition in Saari and Merlin (2000), whose concern is closer to ours, namely the study of Kemeny orders. They call the subsets of a partition "layers". However, most of these authors rule out the possibility of ties in the majority relation and no one seems to have made an explicit use of such a partition to find a Kemeny order on the set of all alternatives.

Corollary 9 (to Lemma 8) *The Kemeny rule satisfies (XCC).*

Proof. Given a profile R, suppose that r^K is Kemeny order that violates (XCC), i.e. there exists a partition $\mathbb{X} \in \mathcal{P}(X)$, $X_\alpha, X_\beta \in \mathbb{X}$ with $\alpha < \beta$, and $s \in X_\alpha, t \in X_\beta$ such that $r_t^K < r_s^K$. By definition of $\mathcal{P}(X)$, we must have sMt. Thus, by Lemma 8, s cannot be the immediate successor of t in the Kemeny order. Thus, there must exist other alternatives, say $a, \ldots, k$, between t and s in this Kemeny order. Using Lemma 8 again, we must have $tMaM \ldots MkMs$. Since we also have sMt, there is a cycle of M on the set $\{t, a, \ldots, b, s\}$. Using the definition of $\mathcal{P}(X)$ again, $\{t, a, \ldots, b, s\}$ should belong to the same subset of the partition $\mathbb{X}$. We thus have a contradiction since, at the outset, s and t belonged to different X_α and X_β. ■

Lemma 10 *The Copeland rule satisfies (XCC).*

Proof. Given a profile R and a partition $\mathbb{X} \in \mathcal{P}(X)$, take any $s \in X$ and suppose it belongs to $X_\alpha \in \mathbb{X}$. For any $u \in X$ that belongs to some $X_\beta \in \mathbb{X}$ with $\alpha < \beta$, we have by definition of $\mathcal{P}(X)$, $\nu_{su} > \nu_{us}$. Thus, given two alternatives s and t such that $s \in X_\alpha, t \in X_\beta$ with $\alpha < \beta$, alternative s defeats more other alternatives than t. This implies $c^*(s) = c(s) > c(t) = c^*(t)$ and by definition, $C_s^*(R) = C_s(R) < C_t(R) = C_t^*(R)$ as requested by (XCC). ■

Remark 7 Since a Kemeny and a Copeland rankings both satisfy (XCC), they must agree on all singletons of the finest partition of $\mathcal{P}(X)$. However, they may order elements of the subsets X_α that are not singletons in quite different ways. As an illustration, with the original profile of Table 7, for which there is a cycle on the whole set X, the Copeland ranking is $C^*(R) = C(R) = (1,2,2,4,4,6)$, i.e. the weak order A(BC)(DE)F, while the unique Kemeny ranking is $r^K = (2,3,4,5,1,6)$, i.e. the order EABCDF. Note also that C^* and C may disagree on subsets of the finest partition of $\mathcal{P}(X)$ that are not singletons. According to the proof of 10, they may disagree only on these subsets.[21]

Remark 8 According to Lemma 10, given a partition $\{X_1, X_2, \ldots, X_h, \ldots, X_p\} \in \mathcal{P}(X)$, the elements of X_1 are the competitors with the first $|X_1|$ ranks in $C(R)$ or $C^*(R)$,[22] the elements of X_2 are the next $|X_2|$ competitors in this ranking and so on. In particular, if $X_h = \{h\}$, then the rank of h in a ranking satisfying (XCC) is $|X_1| + |X_2| + \cdots + |X_{h-1}|$. Now, let $\hat{C}(R)$ be the Copeland ranking with ties broken in such a way that the condition of Lemma 8 is satisfied on each subset of tied skaters.[23] Then, the finest partition of $\mathcal{P}(X)$ can be found by examining each competitor in turn in the order given by $\hat{C}(R)$. Suppose that the subsets $X_1, X_2, \ldots, X_h$ have already been identified as the first elements of the partition and let s be the next alternative according to $\hat{C}(R)$. If $\nu_{st} > \nu_{ts}\ \forall t : \hat{C}_s(R) < \hat{C}_t(R)$,[24] then $X_{h+1} = \{s\}$ and the rank of s in any Kemeny order is the same as in $\hat{C}(R)$. Otherwise, let t^* be the alternative with the smallest rank $\hat{C}_{t^*}(R)$ such that $\nu_{ut} > \nu_{tu}\ \forall t > t^*$ and $\forall u : s \leq u \leq t^*$. Then, $X_{h+1} = \left\{u \in X : \hat{C}_s(R) \leq \hat{C}_u(R) \leq \hat{C}_{t^*}(R)\right\}$. Once this partition found, there merely remains to find a Kemeny ranking x_α (in the form of a sequence) on each subset X_α of the partition that is not a singleton. Juxtaposing these orders gives a complete ranking $x_1 x_2 \cdots x_p$ on X. The next result asserts that this ranking is a complete Kemeny order on X.

Theorem 11 *Given a profile R, take any partition $\mathbb{X} = \{X_1, X_2, \ldots, X_p\} \in \mathcal{P}(X)$ and the sequence $x_1 x_2 \cdots x_p$, where x_α is a Kemeny order on X_α under profile R restricted to X_α, $\alpha = 1, \ldots, p$. Then x^*, or equivalently the corresponding $r^*(x^*)$, is a Kemeny order on X.*

[21] In the example of Table 2 where C^* and C disagree, there is a cycle of $W(R)$ on the whole set X.
[22] According to the previous remark, it does not matter which of $C(R)$ or $C^*(R)$ is used here.
[23] This can be done easily by finding a Kemeny order on the set of tied skaters.
[24] Alternatively, if $c(s) = c^*(s) = m - \hat{C}_s(R)$.

Proof. Contrary to the assertion, suppose that x^* is not a Kemeny order on X but that x is. Then, x cannot be different from x^* in respect only to alternatives that belong to the same X_α since this would violate the assumption that x_α is a Kemeny order on X_α. Thus, there exist $X_\alpha, X_\beta \in \mathbb{X}$ with $\alpha < \beta$, and $s \in X_\alpha, t \in X_\beta$ such that $r_t(x) \leq r_s(x)$ instead of $r_s(x) < r_t(x)$ as with x^*. By Corollary 9, r cannot be a Kemeny order since (XCC) is violated. ■

Theorem 11 is akin to Theorem 6 in Saari and Merlin (1997), which says that, in any Kemeny order and for any α and β, alternatives from X_α are strictly ranked above alternatives from X_β when $\alpha < \beta$, and that dropping an alternative from X_α has no effect on the ranking within X_β, $\beta \neq \alpha$. Our theorem is somewhat stronger in that it applies to any Condorcet partition and not just to the finest of these partitions. Moreover, it allows for ties in the majority relation.

This approach can be illustrated with the short program of the women 2002-Olympic competition. Using the Copeland rule and the method described in Remark 8, we find the finest partition of $\mathcal{P}(X)$: {{A}, {B}, {C}, {D}, {E}, {F, G, H, I, J, K}, {L}, {M}, {N}, {O}, {P}, {Q}, {R}, {S}, {T, U, V}, {W}, {X}, {Y}, {Z}, {ZZ}}.[25] There is a cycle of the majority relation M on the subset {F, G, H, I, J, K}, namely $\mathrm{F}\, M\, \mathrm{G}\, M\, \mathrm{H}\, M\, \mathrm{I}\, M\, \mathrm{J}\, M\, \mathrm{K}\, M\, \mathrm{F}$ and the cycle $\mathrm{T}\, M\, \mathrm{V}\, M\, \mathrm{U}\, M\, \mathrm{T}$ on {T, U, V}. There are 6!, i.e.120, possible orders on the subset {F, G, H, I, J, K}. However, using Lemma 8, one can eliminate all orders involving two successive alternatives s and t such that $\nu_{st} < \nu_{ts}$. This leaves 31 orders of which we list 10 in Table 4, together with their values $\kappa(\cdot, R_{\mathrm{FGHIJK}})$ where R_{FGHIJK} is the restriction of R to the set {F, G, H, I, J, K}. There are thus four Kemeny orders on the subset {F, G, H, I, J, K}. In the case of the subset {T, U, V}, there are two Kemeny orders, UTV and TVU. The next subsection deals with the multiplicity of Kemeny orders.

[25] The Copeland Rule gives rank 7 to G and H, 9 to I and J, and 20 to T, U, and V. However, $\nu_{\mathrm{GH}} > \nu_{\mathrm{HG}}$ and $\nu_{\mathrm{IJ}} > \nu_{\mathrm{JI}}$. Calling the Kemeny routine on these two sets returns GH and IJ as the Condorcet rankings on these two pairs. On the subset {T, U, V}, the routine just says that there is indeed a cycle and two Kemeny orders on this set. Thus, any Kemeny order can be used to break the ties left by the Copeland rule.

Order	$\kappa(\,\cdot\,, R_{\text{FGHIJK}})$
FIGHJK	80
FGHKIJ	80
FGHJKI	80
FGHIJK	80
GJFHKI	79
GHJFKI	79
FHIGJK	78
FKIGHJ	76
JFGHKI	75
FHKIGJ	75

Table 4: Ten orders satisfying Lemma 8

5.5 The Mean Kemeny Ranking

The Kemeny rule may yield more than one order. This occurrence can sometimes be resolved by reverting to the concept of the *mean Kemeny ranking.* Given a set of Kemeny orders $\left\{\hat{r}^1, \ldots, \hat{r}^k\right\}$, consider the weak order $\tilde{r}$ defined by:

$$\forall s, t \in X : \tilde{r}_s \leq \tilde{r}_t \Leftrightarrow \sum_{q=1}^{k} \hat{r}_s^q \leq \sum_{q=1}^{k} \hat{r}_t^q$$

This weak order is a ranking according to the mean ranks of alternatives over all Kemeny orders. It will be called the *mean Kemeny ranking* if and only it weakly agrees with at least one order in $\left\{\hat{r}^1, \ldots, \hat{r}^k\right\}$, i.e. if there exists an order $\hat{r}^q \in \left\{\hat{r}^1, \ldots, \hat{r}^k\right\}$ such that:

$$\forall s, t \in X : \hat{r}_s^q < \hat{r}_t^q \Rightarrow \tilde{r}_s \leq \tilde{r}_t$$

The ranking $\tilde{r}$ represents a compromise between the Kemeny orders. Thus, its choice makes sense, at least when it weakly agrees with one Kemeny order. There is a mean Kemeny order on the subset {F, G, H, I, J, K} of the women 2002 Olympic competition. This is the fourth Kemeny order FGHIJK.

There may be instances where the mean Kemeny ranking may fail to exist, i.e. where $\tilde{r}$ disagrees with all Kemeny orders. This is the case on the subset {T, U, V} of the women 2002 Olympic competition. There are two Kemeny orders on this subset, UTV and TVU.

The order corresponding to $\tilde{r}$ is TUV. It disagrees with both Kemeny orders. Thus, choosing $\tilde{r}$ in this case would be inconsistent with the Condorcet-Kemeny-Young approach since this order is less probable than any Kemeny order.

In such a case, a Kemeny order can be chosen at random or according to some other criterion. For example, the Kemeny order closest (according to the Kemeny metric) to the Copeland or to the Borda ranking could be chosen. In the case of the subset {T, U, V}, the two orders UTV and TVU have the same distance from the Borda order TUV and from the Copeland ranking. The second order has been chosen for no particular reason. The concatenation of TVU and of FGHIJK with the singletons in the finest Condorcet partition of X (or, equivalently with the rank given by the Copeland rule on these singletons) yields the complete Kemeny order of Table 11 on p. 51.

According to the definition of the mean Kemeny ranking, if all competitors of a set X_α in the partition tie for a rank, then they will be declared ex aequo according to $\tilde{r}$. Indeed, all permutations of the elements of X_α are Kemeny orders on this set. Taking the mean of the ranks of each competitor in X_α over all Kemeny orders yield the same number for each. Declaring these competitors ex aequo makes sense since all Kemeny orders on this set are equally probable.[26]

There are other instances in which some competitors could be declared ex aequo. In the original profile of Table 7, if judge 3 were to interchange her ranking of C and F, there would then be three Kemeny orders: EABFCD, EABDFC, and EABCDF. This means that CDF, DFC and FCD have the same likelihood. Not surprisingly, with this change, there is a cycle C M D M F M C on the set $\{C, D, F\}$. There is good ground here to declare these three competitors ex aequo since they have the same likelihood of being in any of the last three positions and this is what happens with $\tilde{r}$.

The second profile of Table 7 provides another example of multiple Kemeny orders. There are six Kemeny orders: EADFBC, EADBCF, EABCDF, DFEABC, DEAFBC, DEABCF. Here, $\tilde{r}$ does not agree weakly with any of the Kemeny orders. Thus, there is no mean Kemeny ranking. The third one has been chosen for the simple reason that it happens to be the unique Kemeny order with the original profile.

[26]This also makes sense in terms of the Kemeny distance. Indeed, if r is Kemeny order and if sTt, then the term $\sum_{j=1}^{n} \gamma_{st}\left(r, r^j\right)$ is diminished when r is changed into a weak order by giving s and t the same rank.

Interestingly, in this case, the Borda ranking is $(4, 2, 6, 2, 1, 4)$. Thus, B and D tie for the second rank and A and F tie for the fourth rank. Moreover, only one point in the Borda scores separates adjacent ranks, thus confirming the close competition between all skaters. The ISU-94 ranking in this case is $(1, 2, 4, 3, 6, 5)$, completely at odds with the Borda and with the six Kemeny orders.

5.6 Properties of the Kemeny Rule

It is easy to check that the Kemeny rule satisfies (ND), (WP), and (M). Thus, by Arrow's theorem, it does not satisfy (BI). In the original profile of Table 7, if A and B had not shown up for the competition or equivalently if all the judges had ranked them in the last two places without changing the marks of all other competitors, then the unique Kemeny and Condorcet order would become CDFE. Hence E would become the Condorcet loser, in violation of (BI). This is also a violation of strong monotonicity (SM) defined in the next section.

However, Young and Levenglick (1978) show that the Kemeny rule satisfies a weaker independence condition that they call *local independence of irrelevant alternatives* (LIIA). Its definition involves the concept of interval for orders. An interval for an order is any subset of alternatives that occurs in succession in that order. For example, $abcd, bcd, bc, cd, cde$ are intervals of the order $abcde$. (LIIA) requires that the ranking of alternatives within any interval be unaffected by the presence of alternatives outside this interval. This condition implies that the ranking of alternatives toward the top of the list is unaffected by the removal of those at the bottom, etc.

The Kemeny rule is also *symmetric* (S) (it puts all judges on an equal footing), *neutral* (N) (it treats all skaters in the same way). Moreover, it satisfies *reinforcement* (R) (whenever two distinct groups of judges both reach the same ranking of the skaters, this ranking is also the consensus for the two groups of judges merged together). The Borda rule and more general scoring methods also satisfy these properties but the Kemeny rule is the only one to satisfy (LIIA) as well.

With the original profile of Table 7, if B was dropped from the list, then C would move ahead of A in the Borda ranking. Given that the Borda rule is a component of both the ISU-94 and the ISU-98 rules, none of them satisfies (LIIA). Actually, the (MRP) and the Copeland rule themselves may violate (LIIA). With the original profile of Table 10, we have the vector $\rho = (2, 3, 4, 3, 5)$. Were we to remove D and E from X, we would get $\rho = (2, 1, 3)$ for the set $\{A, B, C\}$, in violation of (LIIA). As for the Copeland rule, with the removal of B from the original profile of Table 7, A and C tie for the first place while A defeats C under the original profile. A more probing example is given at the end of Appendix A.

6 On the Manipulability

Although judges are supposed to be as objective as possible in assessing the performance of skaters, it is in the human nature no to be always so. In their empirical study of the results of 24 Olympic competitions, Campbell and Galbraith (1996) find strong evidence of the presence of a small national bias, the latter being more marked for medal contenders than for less strong competitors. Actually, such biases are easily detected by a quick look at the rankings of the judges. In addition, a notorious fact at least since the 2002 Olympic Games, sophistication may go as far as involving some logrolling between two or more competitions.

Manipulation refers to strategic behavior (insincere voting) rather than to biased voting, which is usually sincere. A judge may report a ranking different from his or her preferred one in order to change the final ranking for one that he or she prefers. This is what manipulation is about. This is different from biased voting, say in favour of a fellow skater.

A judge might be interested only in the winner of a competition and could thus try to manipulate the ranking procedure with this objective in mind. We could then see the ranking procedure as a *social choice function*, i.e. a function that selects a winner. There is another famous impossibility theorem in this context, proved independently by Gibbard (1973) and Satthertwaite (1975), which says that all non-dictatorial (ND) social choice functions are manipulable. There is a strong link between these impossibility results and the one of Arrow.

More generally, a judge may prefer a ranking to another one, not just because of the winner, but because of the whole ranking. One way to formalize this preference is to use the Kemeny metric between two rankings. The smaller the Kemeny distance between a ranking and the preferred ranking of a judge, the better. Bossert and Storcken (1992) extend the Gibbard-Satthertwaite theorem to this context. Thus, it appears that all rules considered in this paper are manipulable. To get a better picture of the situation, let us turn to another monotonicity condition.

6.1 Strong Monotonicity and Manipulation

Muller and Satthertwaite (1977) establish that a social choice function is not manipulable if and only if it satisfies a strong monotonicity condition. We shall define this condition and show that it is violated by the Copeland, the Borda, the ISU-94, the ISU-98, and the Kemeny rules. Hence, these rules are all manipulable.

Strong Monotonicity (SM): A ranking rule Γ satisfies *strong monotonicity* if, given two profiles R and $\tilde{R}$ and two skaters s and t such that

$$\begin{aligned} r_s^j &< r_v^j \Rightarrow \tilde{r}_s^j < \tilde{r}_v^j \quad \forall j \in J,\ \forall v \neq s \\ r_s^j &= r_v^j \Rightarrow \tilde{r}_s^j \leq \tilde{r}_v^j \quad \forall j \in J,\ \forall v \neq s \end{aligned}$$

the following holds:

$$\begin{aligned} \Gamma_s(R) &< \Gamma_t(R) \Rightarrow \Gamma_s\left(\tilde{R}\right) < \Gamma_t\left(\tilde{R}\right) \quad \text{and} \\ \Gamma_s(R) &= \Gamma_t(R) \Rightarrow \Gamma_s\left(\tilde{R}\right) \leq \Gamma_t\left(\tilde{R}\right). \end{aligned}$$

(SM) has the same premise as (M) except that the relative ranking of skaters other than s are not required to remain the same. Both have the same conclusion. In Remark 3, it has been pointed out that BP gave the definition of (SM) instead of (M), although they used the latter in their characterization of (MRP) (Theorem 5). This may explain why they labeled their condition "incentive compatibility" and why they contended that "Median ranks provide strong safeguard against manipulation by a minority of judges."

The example of Table 10 provides a violation of (SM) by (MRP) and hence by the ISU-94 rule. In this example, judge 4 changes his ranking. Despite the fact that he gives A a better rank in the modified profile, the latter looses her first rank to the profit of B, who obviously also gets a push from the part of judge 4.[27] Note that the final ranking of A and B under the ISU-94 rule depends on (MRP) alone with both profiles. According to the Muller-Satthertwaite theorem, (MRP) is thus manipulable. In our opinion, this principle and the ISU-94 rule are actually quite prone to manipulation, more so than the other rules considered in this paper, including the ISU-98 rule.

[27] This example also illustrates the violation of (BI). The final ranking of A relative to B is inverted despite the fact that judge 4 has not changed his relative ranking of these two skaters.

This is not to say that the latter are not manipulable. In the example of Table 9, judge 3, by changing his ranking from BACEDF to BCEDAF, permits his favorite candidate B to ravish the first rank to A according to B, K, C, and ISU^{98}.[28] Another example, given in Table 8, illustrates the manipulation of B, ISU^{94}, and ISU^{98} again. Here, by changing her ranking from BACDE to BCDAE, judge 5 is able to make B the winner instead of A, according to B, ISU^{94} and ISU^{98}.[29]

6.2 On the Ease of Manipulation

There are many aspects to manipulation:

- Some ranking rules may be more prone to manipulation than others in that there exist more profiles for which manipulation can be successful.
- The impact of manipulation in terms of the final results may be more or less important.
- Some ranking rules may require more sophistication than others in order to be manipulated successfully. In some cases, manipulation by individuals or small groups may be possible. In other cases, manipulation may require large coalitions.

An analysis of manipulation requires combining these three aspects. Let us give a brief look at the rules discussed in this paper in this respect.

The Borda Rule

This rule has been considered as highly manipulable for a long time. Many authors have constructed examples showing that a coordinated action by many voters may have a dramatic impact on the final Borda ranking. Already in the 19th Century, the French mathematician Laplace (1812) pointed out that even honest voters could be tempted to give the last ranks to the strongest candidates in order to favour their own candidate. This, in his opinion, would give a great advantage to mediocre candidates. He added that, for this reason, this

[28] We have $c^* = (5, 4, 3, 2, 1, 0)$ with the first profile and $c^* = (2, 4, 4, 3, 2, 0)$ with the second profile. Thus, with the second profile, B would tie with C for the first rank under C. Under ISU-98, C is the unique winner, thanks to the Borda criterion. By reporting $\{5, 1, 3, 4, 2, 6\}$ instead of $\{5, 1, 2, 4, 3, 6\}$, judge 3 could make sure that C is the unique winner under C alone.

[29] With the original profile, A is the winner thanks to the second criterion. With the modified profile, B becomes the winner, thanks to the fourth criterion.

rule had been abandoned by institutions that had previously adopted it. Borda's answer is well known: "My method should be used only with honest people."

Saari (1990) develops a measure of susceptibility to manipulation for scoring voting procedures. He shows that, at least for $m = 3$, it is the Borda rule for which manipulation can have the largest expected strategic impact. His study is done under the following assumptions. It is equally likely for any pair of alternatives to be the target of a manipulation attempt; all profiles of rankings are equally likely; and it is equally likely that a strategic voter or a small coalition of strategic voters has any particular ranking.

However, his most surprising finding is that, among all scoring voting procedures, the Borda rule is the method that either minimizes, or comes close to minimizing, the likelihood of a successful manipulation by a small group of individuals (say less than 5). This result is essentially due to the fixed value for the successive differences between the weights w_i used in the Borda method. However, this same property makes this rule vulnerable to carefully coordinated manipulation by large groups.

The ISU-94 Rule

Depending on the profile, this rule can be manipulated via any of its four criteria. However, as seen in Remark 1, the ISU-94 rule is the combination of many scoring procedures. Among them are those defined by the weight vectors $(1, 0, ..., 0)$ and $(1, 1, ..., 1, 0)$, which define the plurality and the anti-plurality rules respectively. According to Saari (1990), these are the two scoring rules that are the most susceptible of being manipulated by a small coalition, i.e. the ones for which there exist more profiles that can be manipulated successfully.

The ISU-94 rule being a combination of many procedures that are susceptible to manipulation, is it more or less susceptible to manipulation than any of its components? There is probably no clear cut answer to this question. On the one hand, the fact that manipulation within a scoring procedure may imply the recourse to a different one may reduce the opportunities for manipulation within the first procedure. On the other hand, this same fact may open additional possibilities to the manipulators. Do these additional possibilities compensate for the lost ones? The answer probably depends on the circumstances. However, in as much as the Borda rule is rarely used in this context, it is safe to assert that the performance of the whole ISU-94 rule cannot be as good as that of the Borda rule in terms of susceptibility to manipulation.

The Copeland, the ISU-98, and the Kemeny Rules

The Copeland and the Kemeny rules satisfy (XCC). This puts a heavy burden on those who wish to manipulate these rules. Indeed, suppose that the original profile yields a linear order as the majority relation $M(R)$. If a judge or a subset of judges are unable to introduce a cycle in this relation by reporting different rankings, then they are unable to manipulate the final ranking in the sense of their preferences. Under these circumstances, the best way for these judges to make sure that the final ranking resembles their own ranking is to report the latter. This applies also to the ISU-98 rule since in this case, it would be identical to the Copeland rule.

Thus, in order for a subset of judges to be able to manipulate these rules successfully, there must be a cycle in $M(R)$ under the original profile or they must be able to produce a cycle by altering their rankings. In the first case, they would have to be aware of the presence of this cycle before hand. In the second case, their strategic behavior, to be successful, should in addition to producing a cycle, yield a ranking preferable to the final ranking that would be obtained otherwise. Moreover, manipulation would be successful, if at all, only on those subsets of the finest partition of $\mathcal{P}(X)$ that are not singletons.

The example of Table 9 shows that the kind of sophistication described above is possible. In this example, there is a Condorcet order, which is also both a unique Copeland and Kemeny order. By changing his preference, judge 3 alone is able to produce the cycle BCDEAB and force a change in both the Copeland and the Kemeny rankings so that his favorite candidate is now the winner under both rules.[30]

Whether the Kemeny rule is more or less susceptible to manipulation than the ISU-94, the ISU-98, or any scoring rule seems to remain an open question. However, the fact that it involves complex computations as compared to the other rules certainly makes it more difficult to manipulate. Moreover, the fact that it satisfies the local independence condition also limits the possibilities of its manipulation.

In an attempt to diminish cheating and manipulation, the ISU is now experimenting with a new system in which there is a larger number of judges but the results of only a fraction of them, chosen at random, are retained.[31] This new system will not prevent manipulation but

[30] Ex aequo with C under C. By reporting $\{5,1,3,4,2,6\}$ instead of $\{5,1,2,4,3,6\}$, judge 3 could make sure that C is the unique winner under both rules.

[31] With a minimum of 8 judges, 5 are drawn to form the result, with a minimum of 10 judges 7 are drawn at random, and with a minimum of 12 up to 14 judges, 9 are drawn to form the result.

it will make it more difficult and render its success less likely. Moreover, this new system will not prevent judges from reporting biases scores in favour of their fellow competitors. In as much as the score now remain anonymous, this new system may provide judges with more latitude to report extravagant scores.[32]

7 Conclusion

We may wonder whether the prescription of a new rule by the ISU may have a real impact in real competitions and whether the use of a different rule such as the Kemeny or the Borda rule could produce significantly different rankings. Merlin, Tataru, and Valognes (2000) give interesting indications on these questions. They estimated the probability that all Condorcet consistent rules, all scoring rules, and a large family of runoff rules elect the same candidate in a three-candidates election to be around 0.5, under the Impartial Culture (IC) condition defined at the beginning of section 4. This probability falls to less than 0.02 on the set of profiles yielding a cycle of the majority relation. We should also expect it to fall with more candidates. However, the requirement that a very large set of rules yield the same result is very strong.

In real life, things could be very different. Hence, Levin and Nalebuff (1995), using data from 30 British Union elections, find that many different electoral systems would not have given different top choices. The systems differed in the ranking of the lower candidates. They suggest that, when voters' preferences are sufficiently similar, a variety of voting systems lead to similar choices, and these choices have desirable properties. The difficulties in aggregating preferences would arise when there is a lack of consensus. In this case, the choice of an electoral system can make the greatest difference.

To dig further into this question, we have applied the two ISU, the Borda, and the Kemeny rules to the data of 30 Olympic competitions: men (M), women (W), and pairs (P), short and free programs, for 1976, 1988, 1992, 1994, and 2002. The results are summarized in Tables 12 and 13 by the measure of the disagreement between the rankings as given by the Kemeny metric c^K.[33]

[32] This kind of behavior has been observed the first time that the new system has been put on trial at a Skate Canada competition held in Québec City in the fall 2002.

[33] We recall that the final ranking is given by a weighted sum of the ranks obtained in the short program and the free program, the latter receiving twice as much weight as the former. We have not done that kind

They show that the choice of a rule can have a real impact on the results. Many disagreements have been found between the two ISU, the Borda, and the Kemeny rankings.[34] Many of the differences occur for middle places, as in the comparisons reported by Levin and Nalebuff (1995). Yet, disagreements for these places may be important for the skaters concerned since participation in future competitions may depend on being ranked in the first ten places or so. In one case, the ISU-94 rule gives a fourth place while the Kemeny rule gives a third place. In two instances, the Kemeny rule gives a tie for the first rank instead of ranks 1 and 2. In one instance, it gives a tie for the second place instead of ranks 2 and 3. Finally, in four instances, it gives a tie for the third place instead of ranks 3 and 4. The differences between the Kemeny and the ISU-98 rankings are essentially due to the presence of cycles in the majority relation, of which many occurrences have been found. The latter are thus not only a theoretical possibility.

The choice of a particular ranking rule should be based on its properties. While the ISU-94 puts forward a majority principle as the basis of its ranking rule, this principle is at odds with the usual Condorcet criterion, which says that if a competitor is ranked ahead of all other competitors by an absolute majority of judges, he should be first in the final ranking. The ISU-94 rule also conflicts with an extension of this principle to other ranks, which goes as follows: If a competitor is ranked consistently ahead of another competitor by an absolute majority of judges, he or she should be ahead in the final ranking. Consistency refers to the absence of cycle involving these two skaters. The ISU-98 rule represents an improvement in this respect since its first criterion is the Copeland rule, which satisfies this Extended Condorcet Criterion.

This criterion may give a complete ranking of all competitors in ideal circumstances. In the case of cycles, an additional criterion is however needed to break cycles while retaining Condorcet's objective, which was to find a ranking with the highest probability of being the true ranking. The Copeland rule incorporates a way of breaking these cycles. However, if we stick to Condorcet's objective, the Kemeny rule is a better choice to break these cycles.

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of weighted sum with the rules other than the ISU rules. Hence, for the final rankings (both programs), there is only one number, showing the discrepancy between the two ISU rules.

[34] In the case of multiple Kemeny orders, the mean Kemeny order was chosen when it existed. Otherwise, the Kemeny order closest to the Copeland ranking was chosen so as to minimize the disagreement with the ISU-98 ranking.

Young (1995) advocates this rule for social decision purposes. Le Breton and Truchon (1997) propose a measure of how far from the Borda rule a social choice function may be. They find that the Kemeny rule fares best, as compared to other rules satisfying the Condorcet criterion. With 20 skaters and 9 judges, it does almost twice as better as the Copeland rule.

Another alternative would be to use uniquely the Borda rule. The latter has much to command for itself. It is a scoring rule, which satisfies interesting requirements such as another reinforcement condition (different from the one defined in Section 7) and participation. The first one says that, if two distinct panels of judges select the same winner, then the joint panel should also select this winner. Participation means that if an additional judge succeeds in changing the winner, it can only be in the sense that he or she favours. There is no Condorcet consistent rule, such as the Kemeny rule, that satisfies these two requirements. The reader is referred to Young (1974) or Moulin (1988) for more details on these rules. Based on the findings of Saari (1990), the Borda rule also appears to be less susceptible to manipulation by a minority of judges than the ISU-94 rule. Manipulation by a large coalition of judges is not really an issue in rating skating. However, the ISU-98 rule and especially the Kemeny rule may be more difficult to manipulate than the Borda rule.

As stressed in this paper, the ranking procedures used in figure skating are defined on the domain of ordinal rankings provided by the judges. Is there a good theoretical basis for throwing away the raw marks of the judges and retaining only the relative rankings drawn from the marks? In a Monte Carlo experiment, BP contrast the ISU-94 rule with a method that consists in summing the raw marks of the judges. They start with true marks for all competitors that result in a complete ranking, the true ranking. The marks of each judge are then obtained by adding a random term to the true marks, truncating the result at 6.0. With true marks ranging from 4.8 to 5.8, they find that the ISU-94 rule outperforms the simple addition of the raw marks in picking the true winner (54% of the time versus 46%).

BP's simulations suggest that neglecting part of the information may play an important role in smoothing out errors in the judges' marks. We might also think of a role for this procedure in reducing the scope for manipulation. However, both contentions would need a firmer theoretical justification.

Appendices

A Characterization of the Copeland Rule

The Copeland rule has been defined in terms of the binary matrix $N(R)$. We might as well have defined the rule in terms of the weak majority relation $W(R)$ defined in section 4. Actually, we may abstract from R completely and define the rule in terms of any binary relation W on X, with M and T as the asymmetric and the symmetric components of W. With some abuse of notation, we shall define $\mu_{st}(W)$, $\Gamma(W)$, and $C(W)$.

Let $\mu_{st} : \mathcal{B} \to \mathbb{R}$, be a function defined for every pair of alternatives (s, t) and every binary relation W by:

$$\mu_{st}(W) = \begin{cases} 1 & \text{if } sMt \\ \frac{1}{2} & \text{if } sTt \\ 0 & \text{if } tMs \end{cases}$$

Then, for each alternative s, let:

$$c(s) = \sum_{t \in X \setminus \{s\}} \mu_{st}(W)$$

Then, the *Copeland rule* is the function $C : \mathcal{B} \to \mathcal{R}$ defined by:

$$\forall s, t \in X : C_s(W) \leq C_t(W) \Leftrightarrow c(s) \geq c(t)$$

$C(W)$ is the *Copeland ranking.*
Henriet (1985) has given a characterization of the Copeland rule in terms of three properties.

Neutrality (N) Let $\sigma(X)$ be a permutation of the elements of X and given a binary relation W on X, let W^σ be the binary relation on X defined by $sW^\sigma t \Leftrightarrow \sigma^{-1}(s)\, W \sigma^{-1}(t)$. Then, a ranking rule $\Gamma : \mathcal{B} \to \mathcal{R}$ is *neutral* (N) if $\forall \sigma(X),\ \forall s, t : \Gamma_s(W^\sigma) = \Gamma_{\sigma^{-1}(s)}(W)$.

In plain words, interchanging alternatives in X with the proper change in W should bring the same interchange of ranks in the final ranking.

Positive Responsiveness (PR) A ranking rule $\Gamma : \mathcal{B} \to \mathcal{R}$ satisfies Positive Responsiveness if, given two binary relations W and $\tilde{W}$ and two skaters s and t such that

$$\exists u \in X : [uMs \text{ and } s\tilde{T}u] \text{ or } [sTu \text{ and } s\tilde{M}u]$$
$$\forall u, v \neq s, t : uWv \Leftrightarrow u\tilde{W}v$$

then,

$$\Gamma_s(W) \leq \Gamma_t(W) \Rightarrow \Gamma_s\left(\tilde{W}\right) < \Gamma_t\left(\tilde{W}\right)$$

In plain words, if W and $\tilde{W}$ are identical except that alternative s gets an improved position with respect to some alternative u in $\tilde{W}$ then, s gets a better rank than t under $\tilde{W}$ if this was so under W and she gets also a better (improved) rank than t if she tied with t under W.

(PR) is stronger than (M) in two respects. Its premise is stronger than the premise of (M) in that s may get an improved position with respect to some alternative u in $W\left(\tilde{R}\right)$ without all the judges maintaining or improving the relative ranking of skater s with respect to other skaters while maintaining the relative ranking of all other skaters. Moreover, (PR) requires that s gets an improved rank with respect to t under $\tilde{W}$ if she tied with t under W.[35]

Independence from reversals of cycles (IRC) Suppose that W and $\tilde{W}$ are two binary relations that are identical except that if there is a cycle of $W(R)$ say $\{s_1, \ldots, s_\kappa\}$, then the same cycle also appears in $\tilde{W}$ but its order is reversed. More precisely, for each $s_k \in \{s_1, \ldots, s_\kappa\}$

$$\begin{aligned} s_k M s_{k+1} &\Rightarrow s_{k+1} \tilde{T} s_k \\ s_k T s_{k+1} &\Rightarrow s_{k+1} \tilde{M} s_k \end{aligned}$$

with the usual provision $s_{\kappa+1} = s_1$. Then, (IRC) requires that $\Gamma(W) = \Gamma\left(\tilde{W}\right)$.

(N) is a very mild condition that is satisfied by all rules discussed in this paper. It is easy to check that the original Copeland rule satisfies (PR) and (IRC). In the case of (IRC), there are four possible patterns concerning the relation of s_k with s_{k-1} and s_{k+1}. They are listed below. Let $\gamma(s_k)$ be the contribution of a particular pattern to $c(s_k)$. It can be checked that in each case, $\gamma(s_k)$ remains the same when going from W to $\tilde{W}$. Since W and $\tilde{W}$ are identical except for the reversal of the cycle, this means that $c(s_k)$ remains the same when going from W to $\tilde{W}$.

W	$\tilde{W}$	$\gamma(s_k)$
$s_{k-1} M s_k M s_{k+1}$	$s_{k+1} \tilde{T} s_k \tilde{T} s_{k-1}$	1
$s_{k-1} T s_k M s_{k+1}$	$s_{k+1} \tilde{M} s_k \tilde{T} s_{k-1}$	1.5
$s_{k-1} M s_k T s_{k+1}$	$s_{k+1} \tilde{T} s_k \tilde{M} s_{k-1}$	0.5
$s_{k-1} T s_k T s_{k+1}$	$s_{k+1} \tilde{M} s_k \tilde{M} s_{k-1}$	1.0

Theorem 12 (Henriet) *The Copeland rule $C : \mathcal{B} \to \mathcal{R}$ is the only ranking rule that satisfies simultaneously (N), (PR), and (IRC).*

[35] Monotonicity as defined in section 3 is often called Non-Negative Responsiveness. The Strong Monotonicity defined in section 6.1 is actually called Strong Positive Association by Muller and Satterthwaite. The strength of their condition comes from the fact that the rankings of alternatives other than s and t are not required to remain the same.

It is easy to see that the modified Copeland rule used in the ISU-98 rule does not satisfy (IRC). Indeed, with W, we have $\gamma^*(s_k) = 1, 2, 1, 2$ for the four respective patterns listed above. With $\tilde{W}$, we have $\gamma^*(s_k) = 2, 1, 2, 1$. Thus, $c^*(s_k)$ may change when a cycle implying s_k is reversed.

The modified Copeland rule does not satisfy (PR) either. To see this, consider the two binary matrices of Table 5. They define two binary relations W^1 (left panel) and W^2 (right panel). By a theorem of McGarvey (1953), there exists profiles R^1 and R^2 such that $W^1 = W(R^1)$ and $W^2 = W(R^2)$. The only change when going from W^1 to W^2 is that the tie between B and D is broken in favour of B. To meet (PR), this push for B should break the equality $c^*(\text{A}) = c^*(\text{B})$ in favour of B, which does not happen.

s	A	B	C	D	E	c^*
A	0	1	1	0	1	3
B	0	0	1	1	1	3
C	0	0	0	1	0	1
D	1	1	0	0	0	2
E	0	0	1	1	0	2

s	A	B	C	D	E	c^*
A	0	1	1	0	1	3
B	0	0	1	1	1	3
C	0	0	0	1	0	1
D	1	0	0	0	0	1
E	0	0	1	1	0	2

Table 5: Example showing the violation of (PR) by the modified Copeland rule

Since the Borda, the Kemeny and the two ISU rules use more information than simply the relation $W(R)$, they do not satisfy (PR) and (IRC). The premises of these two conditions could be satisfied with all kinds of changes in the profiles that could trigger changes in the rankings contrary to the conclusions of (PR) and (IRC).

To conclude this appendix, we give another example showing that the original and the modified Copeland rules do not satisfy (LIIA). Consider the binary matrix of Table 6, which defines a complete strict majority relation M on $X = \{\text{A, B, C, D, E, F}\}$. It yields the ranking $C = C^* = (2, 1, 4, 4, 4, 2)$. Removing E and F from X, we get $C = C^* = (1, 2, 3, 4)$, in violation of (LIIA).

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	0	0	1	1	1	1
C	0	0	0	1	1	0
D	0	0	0	0	1	1
E	1	0	0	0	0	1
F	1	0	1	0	0	1

Table 6: Example showing the violation of (LIIA) by the Copeland rule

B Rule 371 of the ISU (1982 and 1994)

Determination of results of each part of a competition

1. The competitor[36] placed first by the absolute majority of Judges in a part of the competition is first; he who is placed second or better by an absolute majority is second and so on.
2. For this purpose, the place numbers 1 and 2 count as second place; place numbers 1, 2 and 3 count as third place, and so on.
3. If two or more competitors have obtained a majority for the same place, the first among them is he who has been so placed by the greater number of Judges.
4. If such majorities are equal, then the lowest total of place numbers of those Judges forming the majority determines between them.
5. If the total of the place number is equal according to paragraph 4, the sum of the place numbers of all Judges determines the result; if this is also equal the competitors are tied.
6. If there is no absolute majority for a place, the result for such place must be ascertained by seeking the best majority for the following place; and if there is no such majority then by seeking the best majority for the next following place and so on.
7. If such majorities are equal under paragraph 6, the systems referred to in paragraphs 4 and 5 must be applied.
8. The ascertainment of each place must first be made in accordance with paragraphs 1 through 5, and thereafter according to paragraphs 6 and 7 in the above mentioned order.
9. a) If two or more competitors are temporarily tied with majorities for the same place, the place must be awarded to one of those competitors on the basis of paragraphs 3, 4 and 5. After awarding the place, the remaining temporarily tied competitor(s) must be awarded the next following place(s) on the basis of paragraphs 3, 4 and 5 without considering any additional competitors.
   b) In awarding the subsequent places thereafter, the unplaced competitors with a majority for the lowest numbered place shall be given first consideration.
10. If the foregoing rules fail to determine the award of any place, then the competitors tied for that place must be announced as tied. If two competitors so tie for first place, the next place to be awarded is third place (not second). If two skaters so tie for second place, the next place to be awarded is fourth place (not third) and so on.

[36] The rule always adds “or the team” after “competitor”. This has been omitted to simplify the text.

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Original Profile

	Profile							Criteria-94			
s	1	2	3	4	5	6	7	ρ	n_ρ	b_ρ	b
A	2	2	2	6	6	6	1	2	4	7	25
B	3	3	3	1	1	1	6	3	6	12	18
C	4	4	4	4	4	4	2	4	7	26	26
D	5	5	6	3	3	2	3	3	4	11	27
E	1	1	1	5	5	5	5	5	7	23	23
F	6	6	5	2	2	3	4	4	4	11	28

	Binary matrix						Criteria-98	
s	A	B	C	D	E	F	c^*	b^*
A	0	4	4	4	1	4	4	17
B	3	0	6	6	3	6	3	24
C	3	1	0	4	4	4	3	16
D	3	1	3	0	4	4	2	15
E	6	4	3	3	0	3	2	19
F	3	1	3	3	4	0	1	14

Skater	ISU-98	ISU-94	Borda	Kemeny
A	1	1	3	2
B	2	2	1	3
C	3	4	4	4
D	5	3	5	5
E	4	6	2	1
F	6	5	6	6

	Kemeny	Borda	ISU-94
ISU-98	3	3	3
ISU-94	6	6	
Borda	2		

Disagreements between rankings

Modified Profile

	Profile							Criteria-94			
s	1	2	3	**4**	**5**	**6**	7	ρ	n_ρ	b_ρ	b
A	2	2	2	6	6	6	1	2	4	7	25
B	3	3	3	3	3	3	6	3	6	18	24
C	4	4	4	4	4	4	2	4	7	26	26
D	5	5	6	1	2	2	3	3	4	8	24
E	1	1	1	5	5	5	5	5	7	23	23
F	6	6	5	2	1	1	4	4	4	8	25

	Binary matrix						Criteria-98	
s	A	B	C	D	E	F	c^*	b^*
A	0	4	4	4	1	4	4	17
B	3	0	6	3	3	3	1	18
C	3	1	0	4	4	4	3	16
D	3	4	3	0	4	4	3	18
E	6	4	3	3	0	3	2	19
F	3	4	3	3	4	0	2	17

Skater	ISU-98	ISU-94	Borda	Kemeny
A	1	1	4	2
B	6	2	2	3
C	3	4	6	4
D	2	3	2	5
E	4	6	1	1
F	5	5	4	6

	Kemeny	Borda	ISU-94
ISU-98	7	9	5
ISU-94	6	9	
Borda	5		

Disagreements between rankings

Table 7: Example with a cycle, illustrating the difference between four rules

Original Profile

	Profile					Criteria-94			
s	1	2	3	4	5	ρ	n_ρ	b_ρ	b
A	1	2	1	2	2	2	5	8	8
B	2	1	3	2	1	2	4	6	9
C	4	4	2	4	3	4	5	17	17
D	3	3	4	5	4	4	4	14	19
E	5	5	5	1	5	5	5	21	21

	Binary matrix					Criteria-98	
s	A	B	C	D	E	c^*	b^*
A	0	3	5	5	4	4	17
B	3	0	4	5	4	4	16
C	0	1	0	3	4	2	8
D	0	0	2	0	4	1	6
E	1	1	1	1	0	0	4

Skater	ISU-98	ISU-94	Borda	Kemeny
A	1	1	1	1
B	2	2	2	1
C	3	3	3	3
D	4	4	4	4
E	5	5	5	5

	Kemeny	Borda	ISU-94
ISU-98	0.5	0	0
ISU-94	0.5	0	
Borda	0.5		

Disagreements between rankings

Modified Profile

	Profile					Criteria-94			
s	1	2	3	4	**5**	ρ	n_ρ	b_ρ	b
A	1	2	1	2	4	2	4	6	10
B	2	1	3	2	1	2	4	6	9
C	4	4	2	4	2	4	5	16	16
D	3	3	4	5	3	3	3	9	18
E	5	5	5	1	5	5	5	21	21

	Binary matrix					Criteria-98	
s	A	B	C	D	E	c^*	b^*
A	0	3	4	4	4	4	15
B	3	0	4	5	4	4	16
C	1	1	0	3	4	2	9
D	1	0	2	0	4	1	7
E	1	1	1	1	0	0	4

Skater	ISU-98	ISU-94	Borda	Kemeny
A	2	2	2	1
B	1	1	1	1
C	3	4	3	3
D	4	3	4	4
E	5	5	5	5

	Kemeny	Borda	ISU-94
ISU-98	0.5	0	1
ISU-94	1.5	1	
Borda	0.5		

Disagreements between rankings

Table 8: Example showing the manipulability of the Borda and the ISU rules

Original Profile

	Profile					Criteria-94			
s	1	2	3	4	5	ρ	n_ρ	b_ρ	b
A	1	3	2	3	4	3	4	9	13
B	2	1	1	4	6	2	3	4	14
C	3	2	3	5	3	3	4	11	16
D	5	4	5	1	1	4	3	6	16
E	4	5	4	2	2	4	4	12	17
F	6	6	6	6	5	6	5	29	29

	Binary matrix						Criteria-98	
s	A	B	C	D	E	F	c^*	b^*
A	0	3	3	3	3	5	5	17
B	2	0	4	3	3	4	4	16
C	2	1	0	3	3	5	3	14
D	2	2	2	0	3	5	2	14
E	2	2	2	2	0	5	1	13
F	0	1	0	0	0	0	0	1

Skater	ISU-98	ISU-94	Borda	Kemeny
A	1	2	1	1
B	2	1	2	2
C	3	3	3	3
D	4	5	3	4
E	5	4	5	5
F	6	6	6	6

	Kemeny	Borda	ISU-94
ISU-98	0.0	0.5	2.0
ISU-94	2.0	2.5	
Borda	0.5		

Disagreements between rankings

Modified Profile

	Profile					Criteria-94			
s	1	2	**3**	4	5	ρ	n_ρ	b_ρ	b
A	1	3	5	3	4	3	3	7	16
B	2	1	1	4	6	2	3	4	14
C	3	2	2	5	3	3	4	10	15
D	5	4	4	1	1	4	4	10	15
E	4	5	3	2	2	3	3	7	16
F	6	6	6	6	5	6	5	29	29

	Binary matrix						Criteria-98	
s	A	B	C	D	E	F	c^*	b^*
A	0	3	2	2	2	5	2	14
B	2	0	4	3	3	4	4	16
C	3	1	0	3	3	5	4	15
D	3	2	2	0	3	5	3	15
E	3	2	2	2	0	5	2	14
F	0	1	0	0	0	0	0	1

Skater	ISU-98	ISU-94	Borda	Kemeny
A	4	3	4	5
B	1	1	1	1
C	2	2	2	2
D	3	5	2	3
E	4	3	4	4
F	6	6	6	6

	Kemeny	Borda	ISU-94
ISU-98	0.5	0.5	2.0
ISU-94	2.5	2.5	
Borda	1.0		

Disagreements between rankings

Table 9: Example showing the manipulability of the Borda, Kemeny, Copeland, and ISU-98 rules; also illustrating a conflict between (MRP) and (CC)

Original Profile

s	Profile					Criteria-94			
s	1	2	3	4	5	ρ	n_ρ	b_ρ	b
A	1	2	1	4	3	2	3	4	11
B	3	1	3	3	1	3	5	11	11
C	4	4	2	5	2	4	4	12	17
D	2	3	4	2	4	3	3	7	15
E	5	5	5	1	5	5	5	21	21

	Binary matrix					Criteria-98	
s	A	B	C	D	E	c^*	b^*
A	0	2	4	4	4	3	14
B	3	0	4	3	4	4	14
C	1	1	0	2	4	1	8
D	1	2	3	0	4	2	10
E	1	1	1	1	0	0	4

Skater	ISU-98	ISU-94	Borda	Kemeny
A	2	1	1	2
B	1	2	1	1
C	4	4	4	4
D	3	3	3	3
E	5	5	5	5

	Kemeny	Borda	ISU-94
ISU-98	0.0	0.5	1.0
ISU-94	1.0	0.5	
Borda	0.5		

Disagreements between rankings

Modified Profile

s	Profile					Criteria-94			
s	1	2	3	**4**	5	ρ	n_ρ	b_ρ	b
A	1	2	1	2	3	2	4	6	9
B	3	1	3	1	1	1	3	3	9
C	4	4	2	5	2	4	4	12	17
D	2	3	4	4	4	4	5	17	17
E	5	5	5	3	5	5	5	23	23

	Binary matrix					Criteria-98	
s	A	B	C	D	E	c^*	b^*
A	0	2	4	5	5	3	16
B	3	0	4	4	5	4	16
C	1	1	0	2	4	1	8
D	0	1	3	0	4	2	8
E	0	0	1	1	0	0	2

Skater	ISU-98	ISU-94	Borda	Kemeny
A	2	2	1	2
B	1	1	1	1
C	4	4	3	4
D	3	3	3	3
E	5	5	5	5

	Kemeny	Borda	ISU-94
ISU-98	0	1	0
ISU-94	0	1	
Borda	1		

Disagreements between rankings

Table 10: Example showing the violation of (SM) and (LIIA) by (MRP) and the ISU-94 rule

Short Program

Skater	I-98	I-94	Bor	Kem
A	1	1	1	1
B	2	2	2	2
C	3	3	3	3
D	4	4	5	4
E	5	5	4	5
F	6	6	6	6
G	7	7	7	7
H	8	9	8	8
I	9	11	9	9
J	10	8	10	10
K	11	10	11	11
L	12	12	12	12
M	13	13	13	13
N	14	14	14	14
O	15	15	15	15
P	16	16	16	16
Q	17	17	17	17
R	18	19	18	18
S	19	18	19	19
T	20	21	20	20
U	21	20	21	22
V	22	23	22	21
W	23	22	23	23
X	24	24	24	24
Y	25	25	25	25
Z	26	26	26	26
ZZ	27	27	27	27

	Kem	Bor	ISU-94
ISU-98	1	1	6
ISU-94	7	7	
Borda	2		

Free Skating

Skater	I-98	I-94	Bor	Kem
A	3	3	3	3
B	2	2	1	2
C	4	4	4	4
D	1	1	1	1
E	6	6	6	6
F	8	8	8	8
G	5	5	5	5
H	7	7	7	7
I	16	15	15	16
J	17	17	17	17
K	12	12	13	12
L	10	10	10	10
M	24	24	24	24
N	15	16	16	15
O	11	11	11	11
P	13	13	12	13
Q	14	14	14	14
R	9	9	9	9
S	22	22	22	22
T	21	21	21	21
U	20	20	20	20
V	18	18	18	18
W	19	19	19	19
X	23	23	23	23
Y	25	25	25	25
Z	26	26	26	26
ZZ	27	27	27	27

	Kem	Bor	ISU-94
ISU-98	0.0	2.5	1.0
ISU-94	1.0	1.5	
Borda	2.5		

Final Ranking

Skater	I-98	I-94
A	3	3
B	2	2
C	4	4
D	1	1
E	6	6
F	8	7
G	5	5
H	7	8
I	13	13
J	16	15
K	10	10
L	9	9
M	24	24
N	15	17
O	12	12
P	14	14
Q	17	16
R	11	11
S	22	21
T	21	22
U	20	20
V	18	18
W	19	19
X	23	23
Y	25	25
Z	26	26
ZZ	27	27

	ISU-94
ISU-98	4

Table 11: Results of the 2002 women Olympic competition

Short Program

Competition		Kem	Bor	ISU-94
M-1976	ISU-98	1.0	2.0	3.5
	ISU-94	4.5	2.5	
	Borda	3.0		
W-1976	ISU-98	1.5	0.0	1.0
	ISU-94	2.5	1.0	
	Borda	1.5		
P-1976	ISU-98	1.0	2.5	1.0
	ISU-94	2.0	1.5	
	Borda	3.5		
M-1988	ISU-98	3.0	1.5	1.5
	ISU-94	4.5	3.0	
	Borda	4.5		
W-1988	ISU-98	5.0	2.0	5.0
	ISU-94	8.0	5.0	
	Borda	6.0		
P-1988	ISU-98	0.0	1.0	0.0
	ISU-94	0.0	1.0	
	Borda	1.0		
M-1992	ISU-98	0.5	1.0	0.0
	ISU-94	0.5	1.0	
	Borda	1.5		
W-1992	ISU-98	2.5	3.0	8.0
	ISU-94	7.5	7.0	
	Borda	5.5		

Free Skating

Competition		Kem	Bor	ISU-94
M-1976	ISU-98	1.0	1.0	1.0
	ISU-94	2.0	0.0	
	Borda	2.0		
W-1976	ISU-98	1.0	2.5	1.0
	ISU-94	1.0	3.5	
	Borda	3.5		
P-1976	ISU-98	0.0	1.0	1.0
	ISU-94	1.0	0.0	
	Borda	1.0		
M-1988	ISU-98	1.0	2.0	1.0
	ISU-94	2.0	3.0	
	Borda	3.0		
W-1988	ISU-98	0.0	3.5	1.5
	ISU-94	1.5	2.0	
	Borda	3.5		
P-1988	ISU-98	0.5	0.0	0.0
	ISU-94	0.5	0.0	
	Borda	0.5		
M-1992	ISU-98	2.0	1.5	2.0
	ISU-94	0.0	1.5	
	Borda	1.5		
W-1992	ISU-98	2.0	2.5	1.5
	ISU-94	1.5	4.0	
	Borda	4.5		

Both Programs

Competition		ISU-94
M-1976	ISU-98	2
W-1976	ISU-98	0
P-1976	ISU-98	2
M-1988	ISU-98	2
W-1988	ISU-98	3
P-1988	ISU-98	0
M-1992	ISU-98	5
W-1992	ISU-98	6

Table 12: Comparison of the results of 16 Olympic competitions

Short Program | **Free Skating** | **Both Programs**

P-1992

	Kem	Bor	ISU-94
ISU-98	0.0	1.0	0.0
ISU-94	0.0	1.0	
Borda	1.0		

	Kem	Bor	ISU-94
ISU-98	0.5	0.0	1.0
ISU-94	1.5	1.0	
Borda	0.5		

	ISU-94
ISU-98	1

M-1994

	Kem	Bor	ISU-94
ISU-98	0.0	1.5	2.0
ISU-94	2.0	1.5	
Borda	1.5		

	Kem	Bor	ISU-94
ISU-98	0.5	2.5	1.0
ISU-94	1.5	3.5	
Borda	3.0		

	ISU-94
ISU-98	2

W-1994

	Kem	Bor	ISU-94
ISU-98	2.0	2.5	2.5
ISU-94	4.5	3.0	
Borda	4.5		

	Kem	Bor	ISU-94
ISU-98	1.0	1.0	1.0
ISU-94	2.0	2.0	
Borda	2.0		

	ISU-94
ISU-98	2

P-1994

	Kem	Bor	ISU-94
ISU-98	0.0	1.0	1.0
ISU-94	1.0	2.0	
Borda	1.0		

	Kem	Bor	ISU-94
ISU-98	0.5	0.0	1.0
ISU-94	0.5	1.0	
Borda	0.5		

	ISU-94
ISU-98	1

M-2002

	Kem	Bor	ISU-94
ISU-98	0.0	3.0	2.0
ISU-94	2.0	3.0	
Borda	3.0		

	Kem	Bor	ISU-94
ISU-98	0.5	2.0	2.0
ISU-94	1.5	2.0	
Borda	2.5		

	ISU-94
ISU-98	0

W-2002

	Kem	Bor	ISU-94
ISU-98	1	1	6
ISU-94	7	7	
Borda	2		

	Kem	Bor	ISU-94
ISU-98	0.0	2.5	1.0
ISU-94	1.0	1.5	
Borda	2.5		

	ISU-94
ISU-98	4

P-2002

	Kem	Bor	ISU-94
ISU-98	2.0	1.0	0.0
ISU-94	2.0	1.0	
Borda	3.0		

	Kem	Bor	ISU-94
ISU-98	0.0	0.5	0.0
ISU-94	0.0	0.5	
Borda	0.5		

	ISU-94
ISU-98	0

Table 13: Comparison of the results of 14 additional Olympic competitions