Additive rules for the quasi-linear bargaining problem

Christopher P. Chambers* and Jerry R. Green†

Jan. 2004
(very preliminary, please do not cite or quote)

Abstract

We study the class of additive rules for the quasi-linear bargaining problem introduced by Green. We provide a characterization of the class of all rules that are efficient, translation invariant, additive, and continuous. We present two subfamilies of rules which are nested in each other: the one parameter family of t-Shapley rules and the larger family of weighted coalitional rules, and we discuss additional properties that solutions in these families possess.

1 Introduction

This paper is concerned with decisions that affect a group of n players. These players' preferences depend upon a decision (x) and their receipt or payment of a divisible transferable resource (t), which we can call money. In the domain of problems we study, preferences are quasi-linear in money and are completely general with respect to the decision: hence the term quasi-linear bargaining problems. Preferences can be represented by $U(x,t) = u(x) + t$.

*Assistant Professor of Economics, Division of the Humanities and Social Sciences, 228-77, California Institute of Technology, Pasadena, CA 91125. Phone: (626) 395-3559. Email: chambers@hss.caltech.edu.

†John Leverett Professor in the University, David A. Wells Professor of Political Economy, Harvard Business School. Email: jgreen@hbs.edu.

Our approach is normative and welfarist. We seek rules that choose good decisions and equitable vectors of monetary transfers among the players. These rules are allowed to depend on the way in which decisions affect players, but not on the nature of the decisions themselves. Any two problems, which give rise to the same set of feasible utility allocations, before any transfers of money are made, should lead to the same outcome. Thus we envision the role of monetary transfers as capturing the compensation that players should make among themselves – such compensation being due to the fact there can be desirable decisions for some players that are inefficient for the group as a whole.

This model is introduced by Green [10, 11] and studied further by Moulin [12, 13, 14], Chun [3, 4, 5], etc. In this literature, several axioms are standard and will be accepted throughout our work. First, the chosen decision should be efficient. Given the form of the utility functions, the sum of players' willingnesses to pay should be maximized. Second, because there is a one-parameter family of equivalent utility representations for each player, we do not want the selection of a particular numerical representation to affect the outcome. This condition is expressed as the translation invariance of the solution with respect to the set of feasible utility allocations. Third, the solution should not be excessively sensitive to errors of measurement or errors in judgment. Defining a natural topology on problems, we thus require the solution to be continuous in this topology.

There is an unmanageably large collection of solutions satisfying these three conditions. The key additional property we study is motivated by the idea that if a problem can be decomposed into two sub-problems which do not interact at all, then one should arrive at the same outcome whether the original problem is solved as given or the two sub-problems are solved independently. This property amounts to the additivity of the solution. Solutions that are not additive will be subject to complex agenda-setting manipulations and will exhibit other pathologies and inconsistencies.

Green [11] obtains a characterization of additive solutions for the two-player case under a further condition that he calls recursive invariance (to be explained below). In this paper we develop a characterization for the general n-player case. That is, we extend his results to any number of players and we drop the recursive invariance condition. Although the set of solutions we obtain is very large, it has a mathematical characterization that can enable further analysis and refinement.

We then undertake two such refinements by studying two subfamilies of these general solutions. The first are called the t-Shapley rules, where t is a real valued parameter. These rules correspond to the two agent rules in Green. However, they form only a small subfamily of rules in the general case. The second subfamily we study is called the weighted coalitional rules. These rules include the t-Shapley rules but are more general. However, in the case of more than two players, some weighted coalitional rules exhibit pathological properties. By laying bare the geometrical foundation of this family of solutions, the characterization we provide should enable the study of further requirements and desirable properties in the general case.

2 A general representation for n-agents

Let N be a finite set of **agents**. Say that a subset $\mathbf{B} \subset \mathbb{R}^{\mathbf{N}}$ **is bounded above** if there exists some $x \in \mathbb{R}^N$ such that $B \subset \{y : y \leq x\}$.[1] A **problem** is a nonempty subset of $\mathbb{R}^N$ which is closed, convex, comprehensive, and bounded above. By $\mathcal{B}$, we mean the set of all problems.

Let $\overline{x} : \mathcal{B} \to \mathbb{R}$ be defined as $\overline{x}(B) \equiv \max_{x \in B} \sum_N x_i$. We say x is **feasible** for a problem B if $\sum_N x_i \leq \overline{x}(B)$. Our interest is in providing a method for solving problems. To this end, a **rule** is a function $f : \mathcal{B} \to \mathbb{R}^N$ such that for all $B \in B$, $f(B)$ is feasible for B. A rule associates with any given problem a unique feasible solution. In particular, it allows us to make recommendations across problems.

Let H be a function defined on the set of problems which maps to the set of hyperplanes of $\mathbb{R}^N$. Specifically, let $H(B)$ be defined as $H(B) \equiv \left\{x \in \mathbb{R}^N : \sum_N x_i = \overline{x}(B)\right\}$. Thus, $H(B)$ is the set of efficient points that the agents can achieve by making transfers.

For all sets A, $\mathcal{K}(A)$ is the convex and comprehensive hull of A.

For two problems B, B', define the sum $B + B' \equiv \{x + x' : x \in B, x' \in B'\}$.[2]

We posit the following axioms.

Our first axiom states that for all problems, all solutions should be efficient.

Efficiency: For all $B \in \mathcal{B}$, $f(B) \in H(B)$.

[1] For $x, y \in \mathbb{R}^N$, $y \leq x$ means for all $i \in N$, $y_i \leq x_i$.

[2] The operator '+' is sometimes referred to as the **Minkowski sum**.

Our next axiom specifies a robustness of the rule to the underlying utility specification. Formally, any two problems $B, B' \in \mathcal{B}$ such that $B' = B + x$ for some $x \in \mathbb{R}^2$ can be viewed as arising from the same underlying preferences. Hence, a rule should recommend the same social alternative and transfers in the new problem as in the old problem. But the utility value induced by this solution for the new problem is simply the old utility value, translated by x.

Translation invariance: For all $B \in \mathcal{B}$ and all $x \in \mathbb{R}^N$, $f(B+x) = f(B) + x$.

Additivity: For all $B_1, B_2 \in \mathcal{B}$, $f(B_1 + B_2) = f(B_1) + f(B_2)$.

The next property states that if two problems are "close," then their solutions should be "close." In order to define this, we first define the **Hausdorff extended metric** on the space $\mathcal{C}$ of closed subsets of $\mathbb{R}^N$.[3] Let $d : \mathbb{R}^N \times \mathbb{R}^N$ be the Euclidean metric. Define the **distance** $d^* : \mathbb{R}^N \times \mathcal{C} \rightarrow \mathbb{R}_+$ as

$$d^*(x, B) \equiv \inf_{y \in B} d(x, y).$$

Finally, the Hausdorff extended metric, $d_{\text{Haus}} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}_+ \cup \{\infty\}$, is defined as

$$d_{\text{Haus}}(B, B') \equiv \max \left\{ \sup_{x \in B'} d^*(x, B), \sup_{x \in B} d^*(x, B') \right\}.$$

It can be verified that, restricted to the class of problems, d_{Haus} is actually a metric.

Continuity: There exists $M > 0$ such that for all $B_1, B_2 \in \mathcal{B}$, $d(f(B_1), f(B_2)) \leq M d_{\text{Haus}}(B_1, B_2)$.

Continuity tells us that the Euclidean distance between the solutions of two problems is uniformly bounded by some scale of the distance of the two problems.

[3] For d to be an **extended metric**, the following must be true:
i) For all $B, B' \in \mathcal{K}$, $d(B, B') \geq 0$ with equality if and only if $B = B'$
ii) For all $B, B' \in \mathcal{K}$, $d(B, B') = d(B', B)$
iii) For all $A, B, C \in \mathcal{K}$, $d(A, C) \leq d(A, B) + d(B, C)$.
The function d is a **metric** if it only takes real values.

The main theorem is a result characterizing all rules satisfying efficiency, translation invariance, additivity, and continuity. The basic idea is to identify problems with their support functions, and then provide an integral representation of rules as additive functions on the set of such support functions.

Here, S_+^N refers to the intersection of the positive orthant with the unit sphere in $\mathbb{R}^N$.

The main representation theorem follows.

Theorem: A rule f satisfies efficiency, translation invariance, additivity, and continuity if and only if there exists a countably additive, nonnegative measure μ on the Borel subsets of S_+^N and an integrable function $h : S_+^N \to \mathbb{R}^N$ such that $f(B) \equiv \int_{S_+^N} h(u) \left(\sup_{x \in B} x \cdot u\right) d\mu(u)$, where h and μ satisfy the following restrictions:

i) $\sum_{i \in N} h_i(u) = \left\{ \begin{array}{c} 0 \text{ if } u \neq \left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}} \right) \\ \sqrt{|N|}/\mu\left(\left\{ \left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}} \right) \right\} \right) \text{ otherwise} \end{array} \right.$, μ-almost surely

ii) for all $i \in N$, $\int_{S_+^N} h_i(u) u_i d\mu(u) = 1$

iii) for all i, j such that $i \neq j$, $\int_{S_+^N} h_i(u) u_j d\mu(u) = 0$.

Proof: Step 1: Embedding problems into the space of support functions

Define a function $\sigma : \mathcal{B} \to C\left(S_+^N\right)$ which maps each problem into its support function, defined as $\sigma(B)(x) \equiv \sup_{y \in B} x \cdot y$. The function σ is one-to-one. Hence, on $\sigma(\mathcal{B})$, we may define $T : \sigma(\mathcal{B}) \to \mathbb{R}^N$ as $T(\sigma(B)) = f(B)$. It is easy to verify that T is positively linearly homogeneous, additive, and Lipschitz continuous in the sup-norm topology (the last statement follows from the well-known fact that $d_{\text{Haus}}(B, B') = d_{\sup}(\sigma(B), \sigma(B'))$, when the support function is defined on the unit sphere).

Step 2: Defining a rule on the class of support functions

Write $T = (T_i)_{i \in N}$. Each T_i is then positively linearly homogeneous, additive, and Lipschitz continuous with Lipschitz constant M. Extend T_i to the linear hull of $\sigma(\mathcal{B})$, *i.e.* $\sigma(\mathcal{B})-\sigma(\mathcal{B})\equiv\{f-g: f\in\sigma(\mathcal{B}), g\in\sigma(\mathcal{B})\}$. Call the extension T_i^*. This extension is itself Lipschitz continuous; that is, let $g-g', h-h' \in \sigma(\mathcal{B})-\sigma(\mathcal{B})$. Then $d(T^*(g-g'), T^*(h-h')) = d(T(g)-T(g'), T(h)-T(h'))$. Moreover, $d(T(g)-T(g'), T(h)-T(h')) = d(T(g)+T(h'), T(h)+T(g'))$. But since T is additive, we conclude $d(T(g)+T(h'), T(h)+T(g')) = d(T(g+h'), T(g'+h))$. By Lipschitz continuity of T, $d(T(g+h'), T(g'+h)) \leq Md_{\sup}(g+h', g'+h)$. But the latter is equal to $Md_{\sup}(g-g', h-h')$. Hence $d(T^*(g-g'), T^*(h-h')) \leq Md_{\sup}(g-g', h-h')$, so that T^* is Lipschitz continuous. This establishes that T^* is also continuous.

By efficiency, for all $g\in\sigma(\mathcal{B})$, $\sum_{i\in N} T_i(g) = \sqrt{|N|}g\left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right)$. Fix $j\in N$. For all $i\neq j$, we may extend T_i^* to all of $C\left(S_+^N\right)$ so that the extension is continuous, using an appropriate version of the Hahn-Banach Theorem (e.g. Dunford and Schwartz [8], II.3.11). Call this extension T_i^{**}. For j, define $T_j^{**}(g) = \sqrt{|N|}g\left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right) - \sum_{i\neq j} T_i^{**}(g)$. Clearly, T_j^{**} is continuous and is an extension of T_j^*, and for all $g\in C\left(S_+^N\right)$, $\sum_{i\in N} T_i^{**}(g) = \sqrt{|N|}g\left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right)$.

Step 3: Uncovering the integral representation agent-by-agent

Each T_i^{**} has an integral representation, by the Riesz representation theorem (for example, Aliprantis and Border [1], Theorem 13.14). Thus, $T_i^{**}(g) = \int_{S_+^N} g(x)\, d\mu_i(x)$. The (possibly signed) measures μ_i are each countably additive and of bounded variation, since S_+^N is compact and Hausdorff.

Step 4: Synthesizing the agents' measures to obtain one measure

Define the measure $\mu = \sum_{i\in N} |\mu_i|$.[4] Each μ_i is then absolutely continuous with respect to μ, and so the Radon-Nikodym theorem guarantees the existence of measurable functions $h_i : S_+^N \rightarrow \mathbb{R}$ so that

[4]Here, $|\mu_i|$ denotes another measure called the **absolute value of** μ_i. When μ_i is countably additive and of bounded variation (as we know it is), then $|\mu_i|$ is also countably additive, and in particular, $\mu_i(A)\neq 0$ implies $|\mu_i|(A)>0$. See Alprantis and Border [1], Corollary 9.35 and Theorem 9.55.

for all measurable g, $T_i^{**}(g) = \int_{S_+^N} h_i(u)\, g(u)\, d\mu(u)$. Thus, we may write $T^{**}(g) = \int_{S_+^N} h(u)\, g(u)\, d\mu(u)$, where $h : S_+^N \to \mathbb{R}^N$. Further, $\sum_{i\in N} T_i^{**}(g) = \int_{S_+^N} \sum_{i\in N} h_i(u)\, g(u)\, d\mu(u)$, which we know is equal to $\sqrt{|N|} g\left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right)$. This establishes that $\sum_{i\in N} h_i(u) = 0$ μ-almost everywhere, except at $x = \left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right)$, in which case $\sum_{i\in N} h_i\left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right) = \frac{\sqrt{|N|}}{\mu\left(\left\{\left(\frac{1}{\sqrt{|N|}},...,\frac{1}{\sqrt{|N|}}\right)\right\}\right)}$.

Step 5: Translating the representation back to the space of problems

Translating back into the original framework, this tells us that $f(B) = \int_{S_+^N} h(x)\left(\sup_{y\in B} x \cdot y\right) d\mu(x)$, where μ is a positive, countably additive, measure, and h is a measurable function from S_+^N into $\mathbb{R}^N$, which satisfies

$$\sum_{i\in N} h_i(x) = \begin{cases} 0 \text{ if } x \neq \left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right) \\ \frac{\sqrt{|N|}}{\mu\left(\left\{\frac{1}{\sqrt{|N|}},...,\frac{1}{\sqrt{|N|}}\right\}\right)} \text{ for } x = \left(\frac{1}{\sqrt{|N|}}, ..., \frac{1}{\sqrt{|N|}}\right) \end{cases}$$

μ-almost surely.

Step 6: Uncovering the implications of translation invariance

Under additivity, translation invariance is equivalent to the statement that for each unit vector e_i, $f(\mathcal{K}(\{e_i\})) = e_i$. The support function of $\mathcal{K}(\{e_i\})$ is given by $\sigma(\mathcal{K}(\{e_i\}))(u) = u_i$. Thus, one obtains $\int_{S_+^N} h(u)\, u_i d\mu(u) = e_i$, equivalent to the statement that for all i, $\int_{S_+^N} h_i(u)\, u_i d\mu(u) = 1$ and for all $i \neq j$, $\int_{S_+^N} h_i(u)\, u_j d\mu(u) = 0$. ■

Theorem 1 tells us that any solution satisfying the four axioms can be represented by a function h and a measure μ. It is worth discussing these objects. First, the function h maps from the nonnegative part of the unit sphere in Euclidean N-space. Elements of the unit sphere can be interpreted as a list of "weights," one for each agent. For any problem, these weights are used to compute the maximal "weighted utility" that can be achieved within the problem before making transfers. This suggests a "weighted utilitarian" notion.

The function h specifies another vector in Euclidean N-space for each such list of weights. This vector can be interpreted as a fixed list of relative utility values. It is scaled by the maximal weighted utility achieved from the list of weights. A higher weighted utility scales this vector. Thus, this can be viewed as a "payoff vector," where the payoff is scaled by the maximal weighted utility. The payoff vectors are then aggregated over, according to a measure μ.

There are many degrees of freedom in this definition. In particular, we have many degrees of freedom in choosing h and μ. We are allowed to renormalize h as long as the renormalization is accompanied by a counter-balancing renormalization of μ. Thus, there is no sense in which these parameters are "unique."

However, there is also another way that we can imagine renormalizing solutions. Thus, the fact that lists of weights lie in S_+^N is useful for the proof, but has no economic content, and moreover is not necessary. Thus, one is also free to scale any $u \in S_+^N$ by some $\alpha > 0$ as long as h is then equivalently scaled by $1/\alpha$. Thus, the maximal weighted utility is scaled up by α while the vector the weights map to is scaled by α, having no aggregate effect. Such "renormalizations" will sometimes make the nature of the problem more transparent. In such environments, integration would no longer be performed over S_+^N, but over whatever lists of weights were deemed relevant. Clearly, when considering "renormalized" lists of weights, we never need to consider situations where two lists of weights are simply scale translations of each other.

3 A class of rules: the $t-$Shapley rules

The following property was introduced by Chun [4] (he calls it *trivial independence*). We use the terminology introduced by Green [11]. The property states that if a solution is recommended for a particular problem, and this solution is added to the original utility possibility set, then reapplying the rule to this new problem results in the solution for the original problem.

Recursive invariance: For all $B \in \mathcal{B}$, $f(\mathcal{K}(B \cup \{f(B)\})) = f(B)$.

This axiom was used by Green [11] in the class of two-agent problems. Upon adding recursive invariance and a mild symmetry axiom to our four main axioms, he established a characterization of a one-parameter family of rules. In this section, we discuss a natural extension of this class to the many-agent case. These rules identify any problem with a transferable utility game, and then recommend the Shapley value [18] of this associated game for the problem.

Fix a parameter $t > 0$. For a given problem $B \in \mathcal{B}$, define the TU-game associated with a bargaining problem as follows:

For all $S \subset N$, $v_B(S) = \max_{x \in B} \sum_{i \in S} x_i - t\left(\overline{x}(B) - \sum_{i \in N} x_i\right)$. The quantity $\nu_B(S)$ is the maximal amount that coalition S can obtain, when being forced to pay some "tax" at rate t on the degree of inefficiency of the selected alternative. Given that we have defined a game in transferrable utility, we can compute the Shapley value of the game. This corresponding value will be called the t-Shapley solution for the problem B.

The t-Shapley rules enjoy many properties. In particular, they are anonymous and recursively invariant. Here, we simply show how to express the t-Shapley rules in the representation derived above. As in the previous section, we are allowed to "renormalize" the lists of weights in S_+^N. We shall do this in order to keep the analysis simple and clean.

Writing out explictly the definition of the Shapley value, we may, for all $i \in N$, and all $B \in \mathcal{B}$, compute:

$$f_i^t(B) = \sum_{\{S \subset N : i \notin S\}} \left[v_B(S \cup \{i\}) - v_B(S)\right] \frac{|S|!\,(|N \backslash S| - 1)!}{|N|!}.$$

Rewriting $\nu_B(S) = \max_{x \in B}(1+t)\sum_{i \in S} x_i + t\sum_{i \in N \backslash S} x_i - t\overline{x}(B)$, we conclude

$$f_i^t(B) = \sum_{\{S \subset N : i \notin S\}} \left[\begin{array}{c} \max_{x \in B}\left[(1+t)\sum_{j \in S \cup \{i\}} x_j + t\sum_{i \in N \backslash (S \cup \{i\})} x_j\right] \\ -\max_{x \in B}\left[(1+t)\sum_{j \in S} x_j + t\sum_{i \in N \backslash S} x_j\right] \end{array}\right] \frac{|S|!\,(|N \backslash S| - 1)!}{|N|!}.$$

Thus, for all $S \subset N$, define $u_i^S = 1 + t$ if $i \in S$, and $u_i^s = t$ if $i \notin S$.[5] Rearranging the preceding obtains:

$$f_i^t(B) = \sum_{\{S \subset N : i \in S\}} \left(\sup_{x \in B} u^S \cdot x\right) \frac{(|S| - 1)!\,|N \backslash S|!}{|N|!}$$

[5] Hence, the element in S_+^N to which this vector corresponds is the unit vector $\frac{u}{\|u\|}$.

$$-\sum_{\{S\subset N: i\notin S\}}\left(\sup_{x\in B} u^S\cdot x\right)\frac{|S|!\left(|N\backslash S|-1\right)!}{|N|!}$$

Now, write $h_i\left(u^S\right)=\frac{1}{|S|}$ if $i\in S$ and $-\frac{1}{|N\backslash S|}$ if $i\notin S$. For all S, define $\mu\left(\left\{u^S\right\}\right)=\frac{|S|!|N\backslash S|!}{|N|!}$. Then we conclude

$$f_i^t(B)=\sum_{S\subset N} h_i\left(u^S\right)\left(\sup_{x\in B} u^S\cdot x\right)\mu\left(\left\{u^S\right\}\right).$$

This gives us exactly the type of representation obtained in Theorem 1. One can easily verify that all of the conditions are satisfied.

The t-Shapley rules are intuitively appealing and enjoy many normative properties. In particular, they satisfy all of the axioms that we used in our theorem. Moreover, they are also anonymous and recursively invariant. Let us verify that they are recursively invariant.

One might wonder if, in fact, the t-Shapley rules are the unique family which satisfies all of the axioms, together with anonymity and recursive invariance. Unfortunately, the answer is no, and a much broader class of rules can be defined. In the next section, we discuss another (more general) class of rules that includes the t-Shapley rules, but also includes many pathological and ill-behaved rules. However, this framework will make it simpler to discuss the geometric properties of certain rules.

4 The weighted coalitional rules

We here introduce another family of rules. They are motivated by the following observation: Suppose a rule satisfies our primary axioms. Note that the measure μ associated with such a rule must have a support consisting of at least $|N|$ vectors. Otherwise, translation invariance cannot be satisfied.

To this end, suppose that the measure μ associated with this rule has a support of exactly $|N|$ vectors. Write the support of μ as $\{P_1,...,P_N\}$. Let P be the $|N|\times|N|$ matrix whose rows are the P_i's. We index rows by subscript and columns by superscript. We claim that for all $B\in\mathcal{B}$, $f(B)=P^{-1}\left[\sup_{x\in B} P_i\cdot x\right]_{i\in N}$.

Proposition: Suppose f satisfies the axioms listed in Theorem 1, and let μ be the measure associated with f. Suppose that the support of μ

is $\{P_1, ..., P_N\}$. Then for all $B \in \mathcal{B}$, $f(B) = P^{-1}[\sup_{x\in B} P_i \cdot x]_{i\in N}$, where P is the matrix whose rows are P_i's.

Proof: Let f satisfy the hypothesis of the proposition. The rule f can then be written so that for all $B \in \mathcal{B}$, $f(B) = \sum_{i\in N} h(P_i)\,\mu(\{P_i\})(\sup_{x\in B} P_i \cdot x)$. In particular, for all $j \in N$ and all $B \in \mathcal{B}$, $f_j(B) = \sum_{i\in N} h_j(P_i)\,\mu(\{P_i\})(\sup_{x\in B} P_i \cdot x)$.

First, we claim that $\{P_1, ..., P_N\}$ is linearly independent. To see why, let $x \in \mathbb{R}^N$ be arbitrary. By translation invariance, we establish that $x = f(\mathcal{K}(\{x\})) = \sum_{i\in N} h(P_i)(P_i \cdot x)\,\mu(\{P_i\})$. Define the $N \times N$ matrix Q as $Q_j^i = h_j(P_i)\,\mu(\{P_i\})$. The preceding expressions then read $f_j(B) = \sum_{i\in N} Q_j^i [\sup_{x\in B} P_i \cdot x]_{i\in N}$, or $f(B) = Q[\sup_{x\in B} P_i \cdot x]_{i\in N}$. We claim that $Q = P^{-1}$. By *ii*) of Theorem 1, for all $j \in N$, $\sum_{i\in N} Q_j^i P_i^j = 1$. Thus, $Q_j \cdot P^j = 1$. By *iii*) of Theorem 1, if $j \neq k$, $\sum_{i\in N} Q_j^i P_i^k = 0$. Thus, $Q_j \cdot P^k = 0$. These two statements imply that $QP = I$. Since P and Q are each $|N| \times |N|$ matrices, we conclude that $Q = P^{-1}$. Hence $f(B) = P^{-1}[\sup_{x\in B} P_i \cdot x]_{i\in N}$. ■

Thus, let $\left\{P_1, P_2 ..., P_{|N|-1}, \left(\frac{1}{\sqrt{|N|}}\right)_{i\in N}\right\}$ be a set of linearly independent vectors in S_+^N. Label $P_N = \left(\frac{1}{\sqrt{|N|}}\right)_{i\in N}$. As the set $\{P_1, ..., P_N\}$ is linearly independent, we can construct an invertible matrix P so that the rows of P are exactly P_i's. The **weighted coalitional rule according to P** is defined as $f(B) = P^{-1}[\sup_{x\in B} P_k \cdot x]_{k=1}^n$. It is trivial to verify that the weighted coalitional rules are efficient, translation invariant, additive, and continuous. They are also recursively invariant.

Note that the Proposition establishes that for any set of $|N|$ linearly independent vectors, there is a *unique* rule whose measure μ has this set as its support. The unique such rule is the weighted coalitional rule according to any matrix whose rows are the elements in the support of μ. Moreover, the weighted coalitional rules are those rules whose support is *minimal.*

The weighted coalitional rules have a simple geometric interpretation, which leads to an interpretation in terms of weighted utilitarianism. Thus, given a matrix P, the weighted coalitional rule according to P works as follows. Given is a problem $B \in \mathcal{B}$. Fix a row of P, say P_k; this row gives a list of "weights," one for each agent in society. The maximal social weighted utility according to weights P_k that can be achieved by society before making transfers is simply $[\sup_{x\in B} P_k \cdot x]$. For each row of P, we get a maximal weighted utility of this form (for the row of equal coordinates, we actually get a maximal aggregate non-weighted utility). The vector $[\sup_{x\in B} P_k \cdot x]_{k=1}^n$ gives this profile of maximal weighted social utilities. Hence, the vector $P^{-1}[\sup_{x\in B} P_k \cdot x]_{k=1}^n$ gives the unique vector in $\mathbb{R}^N$ that achieves the same weighted social utilities as the maximal weighted social utilities attainable with problem B. Geometrically, this vector is the unique intersection of the tangent hyperplanes to B in the directions P_k.

Note that affine combinations of weighted coalitional rules also satisfy all of the axioms. In fact, Chambers [2] has shown that such affine combinations exhaust the family of all rules satisfying our axioms when there are two agents. It is not known if such a statement holds in the case of many agents.

5 On the possibility of advantageous transfers

An **advantageous reallocation** for a coalition $M \subset N$ exists for problem $B \in \mathcal{B}$ if there exists $B' \in \mathcal{B}$ such that

$$\left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B\right\} = \left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B'\right\}$$

and for all $i \in M$, $f_i(B') > f_i(B)$, with at least one inequality strict. An advantageous reallocation exists if it is possible for a group of agents to get together, and change the set of alternatives by promising ex-ante to make contingent monetary transfers among themselves upon the realization of a particular social alternative. In particular, our (strong) definition allows groups of agents to significantly expand the underlying bargaining set. Our definition appears to allow groups of agents to significantly expand the underlying problem. Of course our theory is welfarist. We only recognize the possibility that there exists an underlying set of alternatives and a preference profile generating B that generates B' when groups specify ex-ante which contingent monetary transfers will be made.

No advantageous reallocation For all $B \in \mathcal{B}$ and all $M \subset N$, there does not exist an advantageous reallocation of B for M.

The main result of this section is that for any rule satisfying our main axioms, there exists a problem B which gives some coalition M an advantageous reallocation. This is surprising, as our main axioms are satisfied by many rules.

We first begin with a simple lemma that discusses an implication of our primary axioms in two-agent environments. It states that, restricted to the class of problems for which there are exactly two decisions, each of which are efficient, the rule always recommends some weighted combination of the two decisions, where the weights are independent of the problem in question.

Lemma 1: Let f satisfy efficiency, translation invariance, additivity, and continuity. Suppose that $|N| = 2$, where $N = \{i, j\}$. Then there exists $\lambda \in \mathbb{R}$ such that the following is true: For all $x, y \in \mathbb{R}^N$ such that $x_i + x_j = y_i + y_j$ and $x_i \leq y_i$, $f(\mathcal{K}(\{x, y\})) = \lambda x + (1 - \lambda) y$.

Proof: We offer a proof that relies on an application of our general representation theorem, although the lemma can also be derived independently.

By the general representation theorem, there exists some $h : S_+^N \to \mathbb{R}^N$, as well as a measure μ defined on the Borel subsets of S_+^N which parametrize the rule. Define

$$\lambda \equiv \int_{\left\{u \in S_+^N : u_i < u_j\right\}} h_i(u)(u_i - u_j)\, d\mu(u).$$

We will show that for all $x, y \in \mathbb{R}^N$ such that $x_i + x_j = y_i + y_j$ and $x_i \leq y_j$, $f(\mathcal{K}(\{x, y\})) = \lambda x + (1 - \lambda) y$.

To this end, by translation invariance of f, it is enough to prove the statement for those x, y for which $x_i + x_j = y_i + y_j = 0$. Let $(x, -x), (y, -y) \in \mathbb{R}^N$, and suppose that $x \leq y$. By the representation of f,

$$f(\mathcal{K}(\{x, y\})) = \int_{S_+^N} h(u)(\max\{u_i x - u_j x, u_i y - u_j y\})\, d\mu(u).$$

For (u_i, u_j) such that $u_i < u_j$, $\max\{u_i x - u_j x, u_i y - u_j y\} = u_i x - u_j x$, and for (u_i, u_j) such that $u_j < u_i$, $\max\{u_i x - u_j x, u_i y - u_j y\} = u_i y - u_j y$. For $u_i = u_j$, $\max\{u_i x - u_j x, u_i y - u_j y\} = 0$. Therefore,

$$\begin{aligned} & f(\mathcal{K}(\{x, y\})) \\ = \; & \int_{\left\{u \in S_+^N : u_i < u_j\right\}} h(u)(u_i x - u_j x)\, d\mu(u) \\ & + \int_{\left\{u \in S_+^N : u_j < u_i\right\}} h(u)(u_i y - u_j y)\, d\mu(u). \end{aligned}$$

Factoring out x and y from the integrals obtains

$$\begin{aligned} &= x\int_{\left\{u\in S_+^N:u_i<u_j\right\}} h\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right) \\ &\quad +y\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right). \end{aligned}$$

As for all $u\neq\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $h_i\left(u\right)+h_j\left(u\right)=0$, we conclude

$$\begin{aligned} &= x\left(\int_{\left\{u\in S_+^N:u_i<u_j\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right),-\int_{\left\{u\in S_+^N:u_i<u_j\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right)\right) \\ &\quad +y\left(\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right),-\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} &= \left(x,-x\right)\int_{\left\{u\in S_+^N:u_i<u_j\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right) \\ &\quad +\left(y,-y\right)\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right). \end{aligned}$$

Lastly, we verify that $\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right)=1-\lambda$. To this end, we establish that

$$\begin{aligned} &\int_{\left\{u\in S_+^N:u_i<u_j\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right) \\ &+\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right)=1. \end{aligned}$$

The following equality is trivial:

$$\begin{aligned} &\int_{\left\{u\in S_+^N:u_i<u_j\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right) \\ &+\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right) \\ =\ &\int_{S_+^N\setminus\left\{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right\}} h_i\left(u\right)\left(u_i-u_j\right)d\mu\left(u\right), \end{aligned}$$

Moreover, if $u=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, $u_i-u_j=0$. Thus the preceding expression is equal to $\int_{S_+^N} h_i(u)(u_i-u_j)\,d\mu(u)$. Separating, we establish

$$\begin{aligned}&\int_{S_+^N} h_i(u)(u_i-u_j)\,d\mu(u)\\ =\ &\int_{S_+^N} h_i(u)\,u_i d\mu(u)\\ &-\int_{S_+^N} h_i(u)\,u_j d\mu(u).\end{aligned}$$

By conditions *ii*) and *iii*) in Theorem 1, this quantity is therefore equal to 1, so that $\int_{\left\{u\in S_+^N:u_j<u_i\right\}} h_i(u)(u_i-u_j)\,d\mu(u)=1-\lambda$.

Therefore, $f(\mathcal{K}(\{x,y\}))=\lambda(x,-x)+(1-\lambda)(y,-y)$. ■

Theorem: Suppose that $|N|\geq 3$. There does not exist a rule satisfying efficiency, translation invariance, continuity, additivity, and no advantageous reallocation.

Proof: Step 1: The rule chooses aggregate welfare levels for each group of agents independently

First, we claim that for all coalitions $M\subset N$, and all $B,B'\in\mathcal{B}$ such that

$$\left\{\left(\sum_{i\in M} x_i, x_{-M}\right):x\in B\right\}=\left\{\left(\sum_{i\in M} x_i, x_{-M}\right):x\in B'\right\},$$

$\sum_{i\in M} f_i(B)=\sum_{i\in M} f_i(B')$. The argument is due to Moulin [12]. Suppose, by means of contradiction, that there exists $M\subset N$, and $B,B'\in\mathcal{B}$ where

$$\left\{\left(\sum_{i\in M} x_i, x_{-M}\right):x\in B\right\}=\left\{\left(\sum_{i\in M} x_i, x_{-M}\right):x\in B'\right\}$$

and $\sum_{i\in M} f_i(B)<\sum_{i\in M} f_i(B')$. Let $z\in\mathbb{R}^N$ be defined as

$$z_i\equiv\begin{cases} f_i(B)-f_i(B')+\frac{\left(\sum_{i\in M} f_i(B')-\sum_{i\in M} f_i(B)\right)}{|M|} & \text{if } i\in M\\ 0 & \text{if } i\notin M\end{cases}.$$

Define $B'' \equiv B' + z$. By translation invariance, $f(B'') = f(B') + z$, so that for all $i \in M$, $f_i(B'') = f_i(B') + z_i = f_i(B) + \frac{\left(\sum_{i\in M} f_i(B') - \sum_{i\in M} f_i(B)\right)}{|M|} > f_i(B)$. Next,

$$\left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B''\right\} = \left\{\left(\sum_{i\in M} x_i + z_i, x_{-M}\right) : x \in B'\right\}$$

$$= \left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B'\right\} = \left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B\right\}.$$

Hence, we have constructed B'' which gives an advantageous transfer for M for the problem B.

Next, for all $M \subset N$, and all $B, B' \in \mathcal{B}$ such that

$$\left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B\right\} = \left\{\left(\sum_{i\in M} x_i, x_{-M}\right) : x \in B'\right\},$$

we claim that for all $i \notin M$, $f_i(B) = f_i(B')$. This follows trivially from the statement in the preceding paragraph, and by applying the no-advangtageous reallocation requirement to the problems B, B' and the coalition $M \cup \{i\}$.

Step 2: Constructing an induced rule for a partition of the agents into groups

Step 1 will be used in order to construct an "induced rule" which is defined on groups of the original agents. To this end, without loss of generality, label $N = \{1, ..., n\}$. Let $m < n$. We may partition N into m groups $\{N_j\}_{j=1}^m$, so that for all $j < m$, $N_j = \{j\}$, and $N_m = \{m, ..., n\}$. Label the partition $\mathcal{P} = \{N_j\}_{j=1}^m$. We show how to construct an induced rule on the partition, meaning that the agents are $\{N_j\}_{j=1}^m$.

To this end, let $\mathcal{B}^{\mathcal{P}}$ be the collection of closed, convex, comprehensive sets in $\mathbb{R}^{\mathcal{P}}$ that are bounded above. We claim that for all $B \in \mathcal{B}^{\mathcal{P}}$, there exists $B'(B) \in \mathcal{B}$ such that $B = \left\{\left(\sum_{i\in N_j} x_i\right)_{j=1}^m : x \in B'\right\}$. We define a function which carries elements of $\mathbb{R}^{\mathcal{P}}$ into elements of $\mathbb{R}^N$. Thus, define $X : \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^N$ by

$$X_i(x) = \left\{\begin{array}{c} x_{N_i} \text{ if } i \leq m \\ 0 \text{ otherwise} \end{array}\right\}.$$

For all $B \in \mathcal{B}^{\mathcal{P}}$, define $B'(B)$ as the comprehensive hull of $\{X(x) : x \in B\}$.

First, it is clear that $\{X(x) : x \in B\}$ is closed, convex, and bounded above. Therefore, $B'(B) \in \mathcal{B}$. Moreover, we claim that $B = \left\{\left(\sum_{i \in N_j} x_i\right)_{j=1}^{m} : x \in B'\right\}$. Thus, for all $x \in B$, $\left(\sum_{i \in N_j} X_i(x)\right)_{j=1}^{m} = x$. By definition of $B'(B)$, for all $x' \in B'(B)$, there exists $x \in B$ such that $x' \leq X(x)$. Hence $\left(\sum_{i \in N_j} X_i(x')\right)_{j=1}^{m} \leq x$, so that $\left(\sum_{i \in N_j} X_i(x')\right)_{j=1}^{m} \in B$.

We define an induced rule $f^{\mathcal{P}} : \mathcal{B}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}}$ by $f^{\mathcal{P}}(B) = \left(\sum_{i \in N_j} f_i(B'(B))\right)_{j=1}^{m}$. It is easy to see that for all $B, B^* \in \mathcal{B}^{\mathcal{P}}$, $B'(B + B^*) = B'(B) + B'(B^*)$, so that the rule $f^{\mathcal{P}}$ is additive. One can also similarly check its translation invariance. The efficiency and continuity of $f^{\mathcal{P}}$ follow immediately from the efficiency and continuity of f. Lastly, no advantageous reallocation is also trivially satisfied by $f^{\mathcal{P}}$.

Step 3: Construction of two problems leading to a contradiction

By Step 2, it is without loss of generality to assume that $|N| = 3$. We will establish that no three-agent rule can satisfy all of the axioms. Without loss of generality, label $N = \{1, 2, 3\}$.

By Step 2, f can be used to construct a collection of induced two-agent rules. In particular, for each agent $i \in N$, let $\mathcal{P}^i = \{\{i\}, \{j, k\}\}$ be a partition of N into a one-agent group containing agent i and a two-agent group containing the remaining agents. This induces a two-agent rule $f^{\mathcal{P}^i}$ as in Step 2, which satisfies all of the axioms. In particular, the Lemma establishes that for each such rule, there exists a corresponding $\lambda(i)$ associated with $\{i\} \in \mathcal{P}^i$.

We construct two problems in $\mathcal{B}$, each of which induces a two-agent problem that is the convex, comprehensive hull of two points. To this end, define

$$B \equiv \left\{x \in \mathbb{R}^N : x \leq 1 \text{ and } \sum_{i \in N} x_i \leq 2\right\}.$$

Clearly, this is a well-defined problem. For each partition P^i, B induces a problem $B^i \in \mathcal{B}^{P^i}$, where

$$B^i = \left\{(x, y) \in \mathbb{R}^{P^i} : x \leq 1, y \leq 2, x + y \leq 2\right\}.$$

By the Lemma, $f_{\{i\}}^{\mathcal{P}^i}(B^i) = \lambda(i)$. By Step 1, we conclude $f_i(B) = \lambda(i)$. Hence $\lambda(1)+\lambda(2)+\lambda(3) = 2$. Thus, there exists some i such that $\lambda(i) > 0$. Without loss of generality, we suppose that $\lambda(1) > 0$.

Let $B^* \in \mathcal{B}$ be defined as

$$\begin{aligned} B^* &\equiv \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i \leq 2, x_2 \leq 1, x_3 \leq 1 \right\} \\ &\cap \left\{ x \in \mathbb{R}^N : x_1 + x_3 \leq 2, x_1 + x_2 \leq 2, x_1 \leq 2 \right\}. \end{aligned}$$

In particular, for $i = 2, 3$, $B^{*i} = B^i$. Moreover,

$$B^{*1} = \left\{ x \in \mathbb{R}^{P^1} : x_{\{1\}} + x_{\{2,3\}} \leq 2, x_{\{1\}} \leq 2, x_{\{2,3\}} \leq 2 \right\}.$$

By the Lemma, $f_{\{1\}}^{\mathcal{P}^1}(B^{*1}) = 2\lambda(1)$, and by Step 1, $f_1(B^*) = 2\lambda(1)$. For $i = 1, 2$, $f_{\{i\}}^{\mathcal{P}^i}(B^{*i}) = \lambda(i)$, so that $f_i(B^*) = \lambda(i)$. Conclude

$$\begin{aligned} & f_1(B^*) + f_2(B^*) + f_3(B^*) \\ = \; & 2\lambda(1) + \lambda(2) + \lambda(3) \\ = \; & \lambda(1) + [\lambda(1) + \lambda(2) + \lambda(3)] \\ = \; & \lambda(1) + 2 > 2. \end{aligned}$$

Therefore, $f(B^*)$ is infeasible for B^*, a contradiction. ■

References

[1] C.D. Aliprantis and K. Border, "Infinite Dimensional Analysis: a Hitchhiker's Guide," 2nd edition, Springer-Verlag, New York, (1999).

[2] C.P. Chambers, "Multi-utilitarianism in two-agent quasilinear social choice," Caltech social science working paper #1177, California Institute of Technology, (2003).

[3] Y. Chun, The solidarity axiom for quasi-linear social choice problems, *Social Choice and Welfare* **3** (1986), 297-310.

[4] Y. Chun, Monotonicity and independence axioms for quasi-linear social choice problems, *Seoul Journal of Economics* **2** (1989), 225-244.

[5] Y. Chun, Agreement, separability, and other axioms for quasi-linear social choice problems, *Social Choice and Welfare* **17** (2000), 507-521.

[6] J.B. Conway, "A Course in Functional Analysis," second edition, Springer-Verlag, (1990).

[7] E. Dekel, B.L. Lipman, and A. Rustichini, Representing preferences with a unique subjective state space, *Econometrica* **69** (2001), 891-934.

[8] N. Dunford and J.T. Schwartz, "Linear Operators Part 1: General Theory," John Wiley & Sons, Inc. (1988).

[9] I. Gilboa and D. Schmeidler, Maxmin expected utility with nonunique prior, *Journal of Mathematical Economics* **18** (1989), 141-153.

[10] J. Green, "A Theory of Bargaining with Monetary Transfers," Discussion Paper Number 966, Harvard Institute of Economic Research, (1983).

[11] J. Green, Compensatory Transfers in Two-Player Decision Problems, *International Journal of Game Theory*, forthcoming.

[12] H. Moulin, Egalitarianism and utilitarianism in quasi-linear bargaining, *Econometrica* **53** (1985), 49-67.

[13] H. Moulin, The separability axiom and equal-sharing methods, *Journal of Economic Theory* **36** (1985), 120-148.

[14] H. Moulin, The pure compensation problem: Egalitarianism versus laissez-fairism, *Quarterly Journal of Economics* **102** (1987), 769-783.

[15] R. Myerson, Utilitarianism, egalitarianism, and the timing effect in social choice problems, *Econometrica* **49** 883-897, (1981).

[16] M. Perles and M. Maschler, The super-additive solution for the Nash bargaining game, *International Journal of Game Theory* **10** 163-193, (1981).

[17] R.T. Rockafellar, "Convex Analysis," Princeton University Press, Princeton, NJ, (1970).

[18] L.S. Shapley, A value of n-person games, *in* "Contributions to the Theory of Games II," H.E. Kuhn and W.A. Tuckers (eds.), Princeton University Press, Princeton, NJ, (1953), 307-317.